GENERALIZED QUASI–EINSTEIN MANIFOLDS WITH HARMONIC WEYL TENSOR

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Abstract. In this paper we introduce the notion of generalized quasi–Einstein manifold, which generalizes the concepts of Ricci soliton, Ricci almost soliton and quasi–Einstein manifolds. We prove that a complete generalized quasi–Einstein manifold with harmonic Weyl tensor and with zero radial Weyl curvature, is locally a warped product with $(n-1)$–dimensional Einstein fibers. In particular, this implies a local characterization for locally conformally flat gradient Ricci almost solitons, similar to the one proved for gradient Ricci solitons.

1. Introduction

In recent years, much attention has been given to the classification of Riemannian manifolds admitting an Einstein–like structure. In this paper we will define a class of Riemannian metrics which naturally generalizes the Einstein condition. More precisely, we say that a complete Riemannian manifold (M^n, g), $n \geq 3$, is a *generalized quasi–Einstein manifold*, if there exist three smooth functions f, μ, λ on M, such that

$$\mathrm{Ric} + \nabla^2 f - \mu\, df \otimes df = \lambda g\,. \tag{1.1}$$

Natural examples of GQE manifolds are given by Einstein manifolds (when f and λ are two constants), gradient Ricci solitons (when λ is constant and $\mu = 0$), gradient Ricci almost solitons (when $\mu = 0$, see [11]) and quasi–Einstein manifolds (when μ and λ are two constants, see [3] [5] [9]). We will call a GQE manifolds *trivial*, if the function f is constant. This will clearly imply that g is an Einstein metric.

The Riemann curvature operator of a Riemannian manifold (M^n, g) is defined as in [7] by

$$\mathrm{Riem}(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z\,.$$

In a local coordinate system the components of the $(3,1)$–Riemann curvature tensor are given by $\mathrm{R}^d_{abc}\frac{\partial}{\partial x^d} = \mathrm{Riem}\big(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\big)\frac{\partial}{\partial x^c}$ and we denote by $\mathrm{R}_{abcd} = g_{de}\mathrm{R}^e_{abc}$ its $(4,0)$–version.

In all the paper the Einstein convention of summing over the repeated indices will be adopted.

With this choice, for the sphere $\mathbb{S}^n$ we have $\mathrm{Riem}(v,w,v,w) = \mathrm{R}_{abcd}v^a w^b v^c w^d > 0$. The Ricci tensor is obtained by the contraction $\mathrm{R}_{ac} = g^{bd}\mathrm{R}_{abcd}$ and $\mathrm{R} = g^{ac}\mathrm{R}_{ac}$ will denote the scalar curvature. The so called Weyl tensor is then defined by the following decomposition formula (see [7, Chapter 3, Section K]) in dimension $n \geq 3$,

$$\mathrm{W}_{abcd} = \mathrm{R}_{abcd} + \tfrac{\mathrm{R}}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc}) - \tfrac{1}{n-2}(\mathrm{R}_{ac}g_{bd} - \mathrm{R}_{ad}g_{bc} + \mathrm{R}_{bd}g_{ac} - \mathrm{R}_{bc}g_{ad})\,.$$

We recall that a Riemannian metric has *harmonic Weyl tensor* if the divergence of W vanishes. In dimension three this condition is equivalent to local conformally flatness. Nevertheless, when $n \geq 4$, harmonic Weyl tensor is a weaker condition since locally conformally flatness is equivalent to the vanishing of the Weyl tensor.

In this paper we will give a local characterization of generalized quasi–Einstein manifolds with harmonic Weyl tensor and such that $\mathrm{W}(\nabla f, \cdot, \cdot, \cdot) = 0$. As we have seen, this class includes the case of locally conformally flat manifolds.

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Theorem 1.1. *Let (M^n, g), $n \geq 3$, be a generalized quasi–Einstein manifold with harmonic Weyl tensor and* $\mathrm{W}(\nabla f, \cdot, \cdot, \cdot) = 0$. *Then, around any regular point of f, the manifold (M^n, g) is locally a warped product with $(n-1)$–dimensional Einstein fibers.*

Remark 1.2. We would like to notice that the hypothesis $\mathrm{W}(\nabla f, \cdot, \cdot, \cdot) = 0$ cannot be removed. Indeed, if we consider the gradient shrinking soliton on $M = \mathbb{R}^k \times \mathbb{S}^{n-k}$, for $n \geq 4$ and $k \geq 2$, defined by the product metric $g = dx^1 \otimes \cdots \otimes dx^k + g_{\mathbb{S}^{n-k}}$ and the potential function

$$f = \frac{1}{2}\left(|x^1|^2 + \ldots |x^k|^2\right),$$

it is easy to verify that (M^n, g) has harmonic Weyl tensor, since it is the product of two Einstein metrics, whereas the radial part of the Weyl tensor $\mathrm{W}(\nabla f, \cdot, \cdot, \cdot)$ does not vanish.

Remark 1.3. Theorem 1.1 generalizes the results obtained for gradient Ricci solitons (see [2] and [4]) and, recently, for quasi–Einstein manifolds (see [5]).

As an immediate corollary, we have that a locally conformally flat generalized quasi–Einstein manifold is, locally, a warped product with $(n-1)$–dimensional fibers of constant sectional curvature. In particular, we can prove a local characterization for locally conformally flat Ricci almost solitons (which have been introduced in [11]), similar to the one for Ricci solitons ([2] [4]).

Corollary 1.4. *Let (M^n, g), $n \geq 3$, be a locally conformally flat gradient Ricci almost soliton. Then, around any regular point of f, the manifold (M^n, g) is locally a warped product with $(n-1)$–dimensional fibers of constant sectional curvature.*

If $n = 4$, since a three dimensional Einstein manifold has constant sectional curvature, we get the following

Corollary 1.5. *Let (M^4, g), be a four dimensional generalized quasi–Einstein manifold with harmonic Weyl tensor and* $\mathrm{W}(\nabla f, \cdot, \cdot, \cdot) = 0$. *Then, around any regular point of f, the manifold (M^4, g) is locally a warped product with three dimensional fibers of constant sectional curvature. In particular, if it is nontrivial, then (M^4, g) is locally conformally flat.*

Now, using the classification of locally conformally flat gradient steady Ricci solitons (see again [2] and [4]), we obtain

Corollary 1.6. *Let (M^4, g), be a four dimensional gradient steady Ricci soliton with harmonic Weyl tensor and* $\mathrm{W}(\nabla f, \cdot, \cdot, \cdot) = 0$. *Then (M^4, g) is either Ricci flat or isometric to the Bryant soliton.*

2. Proof of Theorem 1.1

Let (M^n, g), $n \geq 3$, be a generalized quasi–Einstein manifold with harmonic Weyl tensor and satisfying $\mathrm{W}(\nabla f, \cdot, \cdot, \cdot) = 0$. If $n = 3$, we have that g is locally conformally flat, while if $n \geq 4$, one has

$$
\begin{aligned}
0 &= \nabla^d \mathrm{W}_{abcd} \\
&= \nabla^d \Big(\mathrm{R}_{abcd} + \frac{\mathrm{R}}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc}) - \frac{1}{n-2}(\mathrm{R}_{ac}g_{bd} - \mathrm{R}_{ad}g_{bc} + \mathrm{R}_{bd}g_{ac} - \mathrm{R}_{bc}g_{ad}) \Big) \\
&= -\nabla_a \mathrm{R}_{bc} + \nabla_b \mathrm{R}_{ac} + \frac{\nabla_b \mathrm{R}}{(n-1)(n-2)} g_{ac} - \frac{\nabla_a \mathrm{R}}{(n-1)(n-2)} g_{bc} \\
&\quad - \frac{1}{n-2}(\nabla_b \mathrm{R}_{ac} - \nabla^d \mathrm{R}_{ad} g_{bc} + \nabla^d \mathrm{R}_{bd} g_{ac} - \nabla_a \mathrm{R}_{bc} g_{ad}) \\
&= -\frac{n-3}{n-2}(\nabla_a \mathrm{R}_{bc} - \nabla_b \mathrm{R}_{ac}) + \frac{\nabla_b \mathrm{R}}{(n-1)(n-2)} g_{ac} - \frac{\nabla_a \mathrm{R}}{(n-1)(n-2)} g_{bc} \\
&\quad + \frac{1}{2(n-2)}(\nabla_a \mathrm{R} g_{bc}/2 - \nabla_b \mathrm{R} g_{ac}/2) \\
&= -\frac{n-3}{n-2}\Big[\nabla_a \mathrm{R}_{bc} - \nabla_b \mathrm{R}_{ac} - \frac{(\nabla_a \mathrm{R} g_{bc} - \nabla_b \mathrm{R} g_{ac})}{2(n-1)} \Big] \\
&= -\frac{n-3}{n-2} \mathrm{C}_{cba} \\
&= -\frac{n-3}{n-2} \mathrm{C}_{abc} \,,
\end{aligned}
$$

where C is the Cotton tensor

$$
\mathrm{C}_{abc} \,=\, \nabla_c \mathrm{R}_{ab} - \nabla_b \mathrm{R}_{ac} - \tfrac{1}{2(n-1)} \big(\nabla_c \mathrm{R}\, g_{ab} - \nabla_b \mathrm{R}\, g_{ac} \big) \,.
$$

Hence, if $n \geq 3$, harmonic Weyl tensor is equivalent to the vanishing of the Cotton tensor.

Now, the condition $\mathrm{W}(\nabla f, \cdot, \cdot, \cdot) = 0$ implies that the conformal metric

$$
\widetilde{g} \,=\, e^{-\frac{2}{n-2} f} g
$$

has harmonic Weyl tensor. Indeed, from the conformal transformation law for the Cotton tensor (see the Appendix), one has that, if $n \geq 4$, then

$$
(n-2)\, \widetilde{\mathrm{C}}_{abc} \,=\, (n-2)\, \mathrm{C}_{abc} + \tfrac{1}{n-2} \mathrm{W}_{abcd} \nabla^d f = 0 \,,
$$

whereas $\widetilde{\mathrm{C}}_{abc} = \mathrm{C}_{abc} = 0$ in dimension three. Hence, from the definition of the Cotton tensor, we can observe that the Schouten tensor of $\widetilde{g}$ defined by

$$
\mathrm{S}_{\widetilde{g}} \,=\, \tfrac{1}{n-2} \big(\mathrm{Ric}_{\widetilde{g}} - \tfrac{1}{2(n-1)}\, \mathrm{R}_{\widetilde{g}}\, \widetilde{g} \big)
$$

is a Codazzi tensor, i.e. it satisfies the equation

$$
(\nabla_X \mathrm{S})\, Y \,=\, (\nabla_Y \mathrm{S})\, X \,, \qquad \text{for all } X, Y \in TM \,.
$$

(see [1, Chapter 16, Section C] for a general overview of Codazzi tensors).

Moreover, from the structural equation of generalized quasi–Einstein manifolds (1.1), the expression of the Ricci tensor of the conformal metric $\widetilde{g}$ takes the form

$$
\begin{aligned}
\mathrm{Ric}_{\widetilde{g}} &= \mathrm{Ric}_g + \nabla^2 f + \tfrac{1}{n-2} df \otimes df + \tfrac{1}{n-2} \big(\Delta f - |\nabla f|^2 \big) g \\
&= \big(\mu + \tfrac{1}{n-2} \big) df \otimes df + \tfrac{1}{n-2} \big(\Delta f - |\nabla f|^2 + (n-2)\lambda \big)\, e^{\frac{2}{n-2} f}\, \widetilde{g} \,.
\end{aligned}
$$

Then, at any regular point p of f, the Ricci tensor of $\widetilde{g}$ either has a unique eigenvalue or has two distinct eigenvalues η_1 and η_2 of multiplicity 1 and $(n-1)$ respectively. In both cases, $\nabla f / |\nabla f|_{\widetilde{g}}$ is an eigenvector of the Ricci tensor of $\widetilde{g}$. For every point in $\Omega = \{p \in M \,|\, p \text{ regular point}, \eta_1(p) \neq \eta_2(p)\}$ also the Schouten tensor $\mathrm{S}_{\widetilde{g}}$ has two distinct eigenvalues σ_1 of multiplicity one and σ_2 of multiplicity $(n-1)$, with the same eigenspaces of η_1 and η_2 respectively. Splitting results for Riemannian manifolds admitting a Codazzi tensor with only two distinct eigenvalues were obtained by Derdzinski [6] and Hiepko–Reckziegel [10] (see again [1, Chapter 16, Section C] for further discussion).

From Proposition 16.11 in [1] (see also [6]) we know that the tangent bundle of a neighborhood of p splits as the orthogonal direct sum of two integrable eigendistributions, a line field V_{σ_1}, and a codimension one distribution V_{σ_2} with totally umbilic leaves, in the sense that the second fundamental form $\widetilde{h}$ of each leaves is proportional to the metric $\widetilde{g}$ (with abuse of notation, we will call $\widetilde{g}$ also the induced metric on the leaves of V_{σ_2}). We will denote by $\widetilde{\nabla}$ the Levi–Civita connection of the metric $\widetilde{g}$ on M and by $\widetilde{\nabla}^{\sigma_2}$ the induced Levi–Civita connection of the induced metric $\widetilde{g}$ on the leaves of V_{σ_2}. In a suitable local chart $x^1, x^2, \dots, x^n$ with $\partial/\partial x^1 \in V_{\sigma_1}$, $\partial/\partial x^i \in V_{\sigma_2}$ (in the sequel i, j, k will range over $2, \dots, n$), we have $\widetilde{g}_{1i} = 0$. Since V_{σ_2} is totally umbilic, we have

$$\widetilde{h}_{ij} \;=\; -\big\langle \widetilde{\nabla}_{\frac{\partial}{\partial x^i}} \tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^1} \big\rangle = -\widetilde{\Gamma}^1_{ij}\,\widetilde{g}_{11} = \tfrac{\widetilde{\mathrm{H}}}{n-1} \widetilde{g}_{ij}\,, \tag{2.1}$$

where $\widetilde{\mathrm{H}}$ will denote the mean curvature function. We recall that, from the Codazzi-Mainardi equation (see Theorem 1.72 in [1]), one has

$$\big(\widetilde{\nabla}^{\sigma_2}_{\frac{\partial}{\partial x^i}} \widetilde{h}\big)\big(\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^k}\big) - \big(\widetilde{\nabla}^{\sigma_2}_{\frac{\partial}{\partial x^j}} \widetilde{h}\big)\big(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^k}\big) \;=\; \big\langle\, \widetilde{\mathrm{Rm}}\big(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\big) \tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^1} \,\big\rangle\,. \tag{2.2}$$

On the other hand, tracing with the metric $\widetilde{g}$, and using the umbilic property (2.1), we get

$$\big(\widetilde{\nabla}^{\sigma_2}_{\frac{\partial}{\partial x^i}} \widetilde{h}\big)\big(\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^i}\big) - \big(\widetilde{\nabla}^{\sigma_2}_{\frac{\partial}{\partial x^j}} \widetilde{h}\big)\big(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^i}\big) \;=\; \tfrac{1}{n-1}\,\partial_j \widetilde{\mathrm{H}} - \partial_j \widetilde{\mathrm{H}} \;=\; \tfrac{2-n}{n-1}\,\partial_j \widetilde{\mathrm{H}}\,.$$

Using equation (2.2), we get

$$\tfrac{2-n}{n-1}\,\partial_j \widetilde{\mathrm{H}} \;=\; \mathrm{Ric}_{\widetilde{g}}\big(\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^1}\big) \;=\; 0\,,$$

which implies that the mean curvature $\widetilde{\mathrm{H}}$ is constant on each leaves of V_{σ_2}. Now, from Proposition 16.11 (ii) in [1], one has that

$$\widetilde{\mathrm{H}} = \tfrac{1}{\sigma_1 - \sigma_2}\,\partial_1 \sigma_2\,.$$

The facts that both $\widetilde{\mathrm{H}}$ and σ_2 are constant on each leaves of V_{σ_2} imply that $\partial_j\,\sigma_1 = 0$, for every $j = 2, \dots, n$. This is equivalent to say that V_{σ_1} has to be a geodesic line distribution, which clearly implies $\Gamma^j_{00} = 0$, i.e. $\partial_j\, g_{11} = 0$. Equation (2.1) yields

$$\partial_1 \widetilde{g}_{ij} \;=\; -2\,\widetilde{\Gamma}^1_{ij} = 2\,\widetilde{g}_{11}^{-1}\,\tfrac{\widetilde{\mathrm{H}}}{n-1}\,\widetilde{g}_{ij}\,.$$

Since $\widetilde{\mathrm{H}}$ and g_{11} are constant along V_{σ_2}, one has

$$\partial_1 \widetilde{g}_{ij}(x^1, \dots, x^n) \;=\; \varphi(x^1)\,\widetilde{g}_{ij}(x^1, \dots, x^n)$$

for some function φ depending only on the x^1 variable. Choosing a function $\psi = \psi(x^1)$, such that $\frac{d\psi}{dx^1} = \varphi$, we have $\partial_1(e^{-\psi}\,\widetilde{g}_{ij}) = 0$, which means that

$$\widetilde{g}_{ij}(x^1, \dots, x^n) \;=\; e^{\psi(x^1)}\,G_{ij}(x^2, \dots, x^n)\,,$$

for some G_{ij}. This implies that the manifold $(M^n, \widetilde{g})$, locally around every regular point of f, has a warped product representation with $(n-1)$–dimensional fibers. By the structure of the conformal deformation, this conclusion also holds for the original Riemannian manifold (M^n, g). Now, the fact that g has harmonic Weyl tensor, implies that the $(n-1)$–dimensional fibers are Einstein manifolds (there are a lot of papers where this computation is done, for instance see [8]).

This complete the proof of Theorem (1.1).

Appendix

Lemma. *The Cotton tensor* C_{abc} *is pointwise conformally invariant in dimension three, whereas if* $n \geq 4$, *for* $\widetilde{g} = e^{-2u} g$, *we have*

$$(n-2)\,\widetilde{\mathrm{C}}_{abc} \;=\; (n-2)\,\mathrm{C}_{abc} + \mathrm{W}_{abcd} \nabla^d u\,.$$

Proof. The proof is a straightforward computation. Let $\widetilde{g} = e^{-2u}\, g$, then for the Schouten tensor $\mathrm{S} = \frac{1}{n-2}\big(\mathrm{Ric} - \frac{1}{2(n-1)}\mathrm{R}\, g\big)$ we have the conformal transformation rule

$$\widetilde{\mathrm{S}} \,=\, \mathrm{S} + \nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g\,. \tag{1.3}$$

The Cotton tensor of the metric $\widetilde{g}$ is defined by

$$(n-2)\,\widetilde{\mathrm{C}}_{abc} \,=\, \widetilde{\nabla}_c \widetilde{\mathrm{S}}_{ab} - \widetilde{\nabla}_b \widetilde{\mathrm{S}}_{ac}\,.$$

Moreover one can see that

$$\begin{aligned} \widetilde{\nabla}_c \widetilde{\mathrm{S}}_{ab} \quad &= \quad \nabla_c \mathrm{S}_{ab} + \nabla_c \nabla_a \nabla_b u + \nabla_c \nabla_a u\, \nabla_b u + \nabla_c \nabla_b u\, \nabla_a u - \nabla_c \nabla_d u\, \nabla_d u\, g_{ab} + \\ &\quad + \widetilde{\mathrm{S}}_{bc} \nabla_a u + \widetilde{\mathrm{S}}_{ac} \nabla_b u + \widetilde{\mathrm{S}}_{ab} \nabla_c u - \widetilde{\mathrm{S}}_{bd} \nabla_d u\, g_{ac} - \widetilde{\mathrm{S}}_{ad} \nabla_d u\, g_{bc}\,. \end{aligned}$$

Computing in the same way the term $\widetilde{\nabla}_b \widetilde{\mathrm{S}}_{ac}$, substituting in the previous formula $\widetilde{\mathrm{S}}$ with (1.3) and using the fact that

$$\begin{aligned} \nabla_c \nabla_b \nabla_a u - \nabla_b \nabla_c \nabla_a u \quad &= \quad \mathrm{R}_{cbad} \nabla^d u \;=\; \mathrm{R}_{abcd} \nabla^d u \\ &= \quad \mathrm{W}_{abcd} \nabla^d u + \mathrm{S}_{ac} \nabla_b u - \mathrm{S}_{cd} \nabla_d u\, g_{ab} + \mathrm{S}_{bd} \nabla_d u\, g_{ac} - \mathrm{S}_{ab} \nabla_c u\,, \end{aligned}$$

(we recall that W is zero in dimension three) one obtains the result. □

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