NON-INTERPENETRATION CONDITIONS IN THE PASSAGE FROM NONLINEAR TO LINEARIZED GRIFFITH FRACTURE

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Abstract. We characterize the passage from nonlinear to linearized Griffith-fracture theories under non-interpenetration constraints. In particular, sequences of deformations satisfying a Ciarlet-Nečas condition in SBV^2 and for which a convergence of the energies is ensured, are shown to admit asymptotic representations in $GSBD^2$ satisfying a suitable contact condition. With an explicit counterexample, we prove that this result fails if convergence of the energies does not hold. We further prove that each limiting displacement satisfying the contact condition can be approximated by an energy-convergent sequence of deformations fulfilling a Ciarlet-Nečas condition. The proof relies on a piecewise Korn-Poincaré inequality in $GSBD^2$, on a careful blow-up analysis around jump points, as well as on a refined $GSBD^2$-density result guaranteeing enhanced contact conditions for the approximants.

1. Introduction

A crucial question in materials science is to provide an accurate description of phenomena exhibiting an intrinsic nonlinear nature, as well as to establish the range of validity of their linearized approximations. A key challenge in this direction consists in the mathematical modeling of impenetrability. In this paper we provide an analysis of impenetrability constraints for brittle hyperelastic materials and address the passage from nonlinear to linearized descriptions for Griffith-fracture theories.

To illustrate the main difficulties involved in the mathematics of impenetrability, consider the simplest modeling scenario in which finite strain deformations play a significant role, namely that of nonlinear elasticity. A standard constitutive assumption for large strain frameworks is the requirement that a body should not be allowed to interpenetrate itself during elastic deformations, and that extreme compressions should lead to a blow-up of the elastic energy, therefore being energetically unfavorable. Although the theory of nonlinear elasticity is by now quite classical (see, e.g., [5] for an introduction to the topic), necessary conditions on the stored-energy density guaranteeing existence of minimizers of nonlinearly elastic energy functionals encoding the behavior described above are not yet known, cf. [6] and [8].

The existence of injective energy minimizers for hyperelastic materials was pioneered by J. Ball in [7] (see also [53] for a regularity analysis of minimizing configurations, and [29] for a related local invertibility result). In the subsequent work [5], it was pointed out that requiring the positivity of the determinant of the gradient of deformations is neither enough to ensure local injectivity everywhere, nor sufficient to prevent a global loss of injectivity. In [20], P.G. Ciarlet and J. Nečas proposed a condition compatible with the existence theory of minimizers, ensuring frictionless contact, non-selfpenetrability, as well as injectivity almost everywhere when combined

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with a positivity constraint on the determinant of nonlinear strains. For an open bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, the Ciarlet-Nečas condition reads as follows:

$$\int_\Omega \det \nabla y(x)\, \mathrm{d}x \le \mathcal{L}^d(y(\Omega)), \tag{1.1}$$

where $y(\Omega)$ is the deformed set and $\mathcal{L}^d(y(\Omega))$ its d-dimensional Lebesgue measure.

Almost everywhere injectivity of deformations has been analyzed in [10] for limits of Sobolev homeomorphisms, in [47] in the presence of distorsion penalizations (see, e.g., [42, 49]), and in [40] for second-grade non-simple materials (cf. [54, 55]), whereas a first numerical implementation of the Ciarlet-Nečas condition as an energy penalization has been exploited in [46] in the setting of finite strain elastoplasticity. For completeness, we also mention the numerical analysis of a nonlocal alternative to the Ciarlet-Nečas condition for non-simple materials in [44], as well as [43] for a generalization of [5].

We focus here on impenetrability constraints in the setting of brittle hyperelastic bodies, and restrict ourselves to the planar case for simplicity. Following Griffith's theory of crack propagation [11, 30, 39], for a set $\Omega \subset \mathbb{R}^2$, the variational modeling of fracture mechanics relies on the competition between a frame-indifferent bulk energy and a surface term. This in turn rewrites as the minimization of a functional of the form:

$$\mathcal{E}(y) = \int_\Omega W(\nabla y(x))\, \mathrm{d}x + \kappa \mathcal{H}^1(J_y), \tag{1.2}$$

where $W\colon \mathbb{M}^{2\times 2} \to [0, +\infty)$ is a nonlinear elastic energy density, $\kappa > 0$ is a material constant, deformations $y\colon \Omega \to \mathbb{R}^2$ are meant to belong to the class $SBV(\Omega)$ of *special functions of bounded variation* [4], ∇y denotes the absolutely continuous part of their gradient, J_y is their jump set, and the latter energy-term, $\mathcal{H}^1(J_y)$, penalizes the crack length. See also [2, 28] for an introduction to the topic.

A generalization of (1.1) in this setting has been introduced and characterized in [37]. In the passage from nonlinear elasticity to large-strain Griffith theories, a first modeling difficulty is related to the fact that deformations do not admit, a priori, continuous representatives, so that the notion of volume of the deformed set in the right-hand side of (1.1) is no longer well-posed. In [37], this difficulty has been solved by means of the weaker notion of measure-theoretic image of the deformed set $[y(\Omega)]$. This, in turn, is defined by considering approximate-differentiability points of admissible deformations, cf. Definitions 3.1 and 3.2 below for the precise formulations. In the same paper, existence of minimizers of (1.2) inheriting the Ciarlet-Nečas condition is ensured, and alternative formulations of impenetrability are also discussed. In particular, in [37, Section 6.1] a contact condition of the form

$$[u](x) \cdot \nu_u(x) \ge 0 \text{ for } \mathcal{H}^1\text{-a.e. } x \in J_u \tag{1.3}$$

is proposed as a linearized counterpart to (1.1) for displacements $u\colon \Omega \to \mathbb{R}^2$, where J_u denotes the jump set of the displacement, $[u]$ its jump opening, and ν_u its approximate unit normal. We refer also to [30, Section 5.1]. A study of quasistatic crack growth under impenetrability constraints has been carried out in the series of works [25, 26, 45]. A variational model including both cavitation and fracture has been analyzed in [41]. Ambrosio-Tortorelli approximations of brittle fracture models under a non-interpenetration constraint are the subject of [17].

The goal of this paper is to provide a rigorous analysis of the connection between the Griffith-counterpart of (1.1) proposed in [37] and the contact condition in (1.3) by performing a nonlinear-to-linear passage. Before discussing our results, we briefly review the literature on linearization for brittle hyperelastic materials. A simultaneous discrete-to-continuum and nonlinear-to-linear study for general crack geometries and for deformations close to the identity is the subject of [34], whereas a linearization analysis for quasistatic evolution models and under additional assumptions on the admissible cracks has been performed in [48] (see also [50]). An effective linearized Griffith energy as Γ-limit of nonlinear and frame indifferent models in the small strain regime and under no assumptions on the crack has been identified in the planar setting in [31], and recently extended in dimension $d \geq 2$ in [32] for the framework of non-simple materials. We refer to Subsection 3.2 below for a precise description of this latter result. We only mention here that, since no a priori bounds are assumed on the deformations, the function spaces in which the analysis is developed are those of *generalized special functions of bounded variation*, $GSBV$, and *generalized special functions of bounded deformation*, $GSBD$, cf. [4, 23]. The topology in which the linearization in [31, 32] is performed is that of a *tripling of the variable*, in which to every sequence of deformations with equibounded rescaled energies, one associates a sequence of Caccioppoli partitions, corresponding piecewise rigid motions, and rescaled displacement fields which are defined separately on each component of the partitions, see Definition 3.8. The limiting displacement field obtained by means of this procedure is referred to as the *asymptotic representation* of the sequence of deformations.

The starting point of our analysis is the linearization result in [32]. The focus of our study is the asymptotic behavior of higher-order Griffith fracture energies under the $GSBV$-version of the impenetrability constraint in (1.1). Our contribution is threefold. Our first result is in the negative, for we give an example that, in the linearization process, sequences of deformations with equibounded nonlinear Griffith energies and satisfying a $GSBV$-formulation of (1.1) might lead to limiting displacements violating (1.3). We further show that, in the absence of additional conditions, a linearized version of (1.2) under (1.3) is not the variational limit of (1.2) complemented by (1.1). In particular, our construction suggests that the linearized counterpart of (1.2) contains an additional anisotropic surface term being positive when (1.3) is violated, which depends on the orientation and on the amplitude of the jump of the displacement u. This is shown in Examples 3.5 and 3.7, and motivates the remaining part of our analysis.

Our second contribution is to prove that adding further assumptions on the sequence of deformations under consideration and restricting the analysis to "energy-convergent sequences" leads in fact to a linearized Griffith model constrained by the contact condition (1.3). A simplified version of our result reads as follows, we refer to Theorem 3.6 and Theorem 3.11 for the precise statements.

Theorem 1.1. *Let $(y_\varepsilon)_\varepsilon$ be a sequence of deformations with equibounded Griffith energies, satisfying* (1.1)*, and such that their nonlinear energies converge to the linearized energy of their asymptotic representation u. Then u satisfies* (1.3).

Third, we prove that for each limiting displacement u fulfilling (1.3), an energy-convergent sequence satisfying the impenetrability condition (1.1) can be constructed (see Theorem 3.12).

Theorem 1.2. *Let u satisfy* (1.3)*. Then, there exists a sequence $(y_\varepsilon)_\varepsilon$ as in Theorem 1.1 having u as asymptotic representation.*

The proof of Theorem 1.1 is performed by contradiction: we postulate the existence of sets of positive measure where (1.3) is violated and show that this cannot be the case by means of a careful blow-up argument, cf. Proposition 4.1, which in turn essentially relies on a piecewise Korn-Poincaré inequality, see Proposition 2.2. Since this latter result is currently only available in dimension $d = 2$, this is the reason why our analysis is restricted to the planar setting. The main ingredient for establishing Theorem 1.2 is a density result in $GSBD$ (see Theorem 2.3) keeping track of boundary data, which in turn provides approximants of the given displacement satisfying a strengthened version of the contact condition in (1.3).

We remark once again that our analysis shows the following: In general, a linear Griffith energy under the contact condition (1.3) does not provide a linearized counterpart to the nonlinear model (1.2) under the Ciarlet-Nečas constraint (1.1), and convergence of minimizers of the nonlinear model to the linearized one is not ensured. In fact, as mentioned above, the compactness in $GSBD$ (see [32] and Definition 3.8 below) fails to guarantee (1.3) in the linearization process, unless there is convergence of the energies (cf. Examples 3.5 and 3.7), which cannot be proven a priori for sequences of minimizers. Further, motivated by Example 3.7, we conjecture that the effect of adding the Ciarlet-Nečas condition (1.1) to the functional (1.2) is given by the presence of an additional anisotropic surface term possibly depending on the orientation and on the amplitude of the jump of limiting displacements. The precise characterization of this surface term goes beyond the scope of this work, for it relies on the identification of a suitable cell-formula for the local limiting energy density around jump points. This will be the subject of a forthcoming analysis.

This paper is organized as follows. Section 2 collects some preliminary results, basic properties of the spaces $GSBV$ and $GSBD$, as well as Proposition 2.2 and Theorem 2.3. Section 3 contains the precise formulation of (1.1) and (1.3), the definition of nonlinear and linearized energy functionals, the description of our notion of convergence, Examples 3.5 and 3.7, and the statement of our main results. Section 4 is devoted to the blow-up argument, whereas Sections 5 and 6 tackle the proofs of Theorems 1.1 and 1.2.

2. Preliminaries and notation

In this section, we introduce the basic notation and define the function spaces we will use throughout the paper.

2.1. Basic notation.

We denote by Ω an open bounded subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. The symbols $\mathcal{L}^2$ and $\mathcal{H}^1$ represent the Lebesgue and the 1-dimensional Hausdorff measure in $\mathbb{R}^2$, respectively. We set $\mathbb{S}^1 := \{x \in \mathbb{R}^2 : |x| = 1\}$. The identity map on $\mathbb{R}^2$ is indicated by $\mathbf{id}$ and its gradient, the identity matrix, by $\mathbf{Id} \in \mathbb{R}^{2\times 2}$. The spaces of symmetric and skew symmetric matrices are denoted by $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and $\mathbb{R}^{2\times 2}_{\mathrm{skew}}$, respectively. We set $\mathrm{sym}(\mathrm{F}) := \frac{1}{2}(\mathrm{F}^T + \mathrm{F})$ for $\mathrm{F} \in \mathbb{R}^{2\times 2}$ and define $SO(2) := \{\mathrm{R} \in \mathbb{R}^{2\times 2} : \mathrm{R}^T\mathrm{R} = \mathbf{Id}, \det \mathrm{R} = 1\}$. For every $\mathrm{F} \in \mathbb{R}^{2\times 2}$ we denote by $\mathrm{dist}(\mathrm{F}, SO(2))$ the distance of F from the set $SO(2)$.

For an $\mathcal{L}^2$-measurable set $E \subset \mathbb{R}^2$, the symbol χ_E denotes its indicator function. For two sets $A, B \subset \mathbb{R}^2$, we define $A \triangle B := (A \setminus B) \cup (B \setminus A)$. By $B_\rho(x) \subset \mathbb{R}^2$ we denote the open ball with center $x \in \mathbb{R}^2$ and radius ρ. The symbol Q_ρ stands for the paraxial square centered in the origin and with side length ρ.

A mapping a of the form $a(x) = \mathrm{A}\,x + b$ for $\mathrm{A} \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $b \in \mathbb{R}^2$ is called an *infinitesimal rigid motion*. In the next sections, we will make use of the following elementary lemma on affine mappings in order to control the norm of infinitesimal rigid motions. We refer to [36, Lemma 3.4] or [21, Lemma 4.3] for similar statements (The proof relies on the equivalence of norms in finite dimensions).

Lemma 2.1. *Let $x_0 \in \mathbb{R}^2$, and let $R, \theta > 0$. Let $a\colon \mathbb{R}^2 \to \mathbb{R}^2$ be affine and let $E \subset B_R(x_0) \subset \mathbb{R}^2$ with $\mathcal{L}^2(E) \geq \theta R^2$. Then, there exists a constant $\bar{c}_\theta \geq 1$ only depending on θ such that*

$$\|a\|_{L^\infty(B_R(x_0))} \leq \bar{c}_\theta \|a\|_{L^\infty(E)}.$$

We conclude this subsection with the basic notation for the slicing technique. For $\xi \in \mathbb{S}^1$, we let

$$\Pi^\xi \coloneqq \{w \in \mathbb{R}^2 \colon w \cdot \xi = 0\}\,, \tag{2.1}$$

and for any $w \in \mathbb{R}^2$ and $B \subset \mathbb{R}^2$ we let

$$B^\xi_w \coloneqq \{t \in \mathbb{R} \colon w + t\xi \in B\}, \qquad \pi_\xi(B) = \{w \in \Pi^\xi \colon B^\xi_w \neq \emptyset\}\,. \tag{2.2}$$

We will use the abbreviation a.e. to indicate that a property holds almost everywhere, namely outside a set of zero measure.

2.2. **Area formula.** We recall below the area formula for a.e.-approximately differentiable maps and refer to [38, Chapter 3] for a complete treatment. For every measurable set $E \subset \Omega$, every map $y \colon \Omega \to \mathbb{R}^2$, and every $z \in \mathbb{R}^2$, let $m(y, z, E)$ be the number of preimages via y of z in the set E, that is,

$$m(y, z, E) \coloneqq \#\{x \in E \colon y(x) = z\}\,.$$

Let us assume that $y \colon \Omega \to \mathbb{R}^2$ is a.e.-approximately differentiable in Ω and let $\Omega_d \subseteq \Omega$ be the set of approximate differentiability of y. Then, the area formula (see e.g. [38, Theorem 1, Section 1.5, Chapter 3]) states that for every measurable set $E \subset \Omega$ the function $z \mapsto m(y, z, E \cap \Omega_d)$ is measurable and

$$\int_E |\det \nabla y(x)|\,\mathrm{d}x = \int_{\mathbb{R}^2} m(y, z, E \cap \Omega_d)\,\mathrm{d}z\,. \tag{2.3}$$

2.3. **Sets of finite perimeter.** For a set of finite perimeter E, we denote by $\partial^* E$ its essential boundary and by $(E)^1$ the points where E has density one, see [4, Definition 3.60]. A set of finite perimeter E is called *indecomposable* if it cannot be written as $E_\alpha \cup E_\beta$ with $E_\alpha \cap E_\beta = \emptyset$, $\mathcal{L}^2(E_\alpha), \mathcal{L}^2(E_\beta) > 0$, and $\mathcal{H}^1(\partial^* E) = \mathcal{H}^1(\partial^* E_\alpha) + \mathcal{H}^1(\partial^* E_\beta)$. Note that this notion generalizes the concept of connectedness to sets of finite perimeter. By [3, Theorem 1] for each set of finite perimeter E there exists a unique finite or countable family of pairwise disjoint indecomposable sets $(E_i)_i$ such that $E = \bigcup_i E_i$, $\mathcal{L}^2(E_i) > 0$ for every i, and $\mathcal{H}^1(\partial^* E) = \sum_i \mathcal{H}^1(\partial^* E_i)$. The sets $(E_i)_i$ are called the *connected components* of E. We call E *simple* if both E and $\mathbb{R}^2 \setminus E$ are indecomposable. For an indecomposable set E we define the *saturation* $\mathrm{sat}(E)$ of E as the union of E and its 'holes', i.e., the connected components of $\mathbb{R}^2 \setminus E$ with finite measure, see [3, Definition 2]. In a similar fashion, for general sets of finite perimeter E with connected components $(E_i)_i$, we define $\mathrm{sat}(E) = \bigcup_i \mathrm{sat}(E_i)$.

We also recall the structure theorem of the boundary of planar sets E of finite perimeter in [3, Corollary 1]: there exists a unique countable decomposition of $\partial^* E$ into pairwise disjoint rectifiable Jordan curves. We say that $\Gamma \subset \mathbb{R}^2$ is a rectifiable Jordan curve if $\Gamma = \gamma([a, b])$ for some $a < b$ and some Lipschitz continuous map γ, one-to-one on $[a, b)$, and such that $\gamma(a) = \gamma(b)$. According to the Jordan curve theorem, any Jordan curve splits $\mathbb{R}^2 \setminus \Gamma$ into exactly one bounded and one unbounded component, denoted by $\mathrm{int}(\Gamma)$ and $\mathrm{ext}(\Gamma)$, respectively, where $\mathrm{int}(\Gamma)$ denotes the bounded component.

For the definition and properties of Caccioppoli partitions we refer to [4, Section 4.4].

2.4. **Function spaces.** We use the standard notation $GSBV(\Omega; \mathbb{R}^2)$ for the space of *generalized special functions of bounded variation*, see [4, Section 4] and [24, Section 2]. We recall that a function $y \in GSBV(\Omega; \mathbb{R}^2)$ admits an approximate gradient ∇y a.e. in Ω. We denote by J_y the set of approximate jump points of $y \in GSBV(\Omega;\mathbb{R}^2)$, that is, the set of points $x \in \Omega$ for which there exist $\nu \in \mathbb{S}^1$ and $a, b \in \mathbb{R}^2$ such that $a \neq b$ and

$$\operatorname*{ap-lim}_{\substack{z\to x\\ (z-x)\cdot\nu>0}} y(z) = a \qquad \text{and} \qquad \operatorname*{ap-lim}_{\substack{z\to x\\ (z-x)\cdot\nu<0}} y(z) = b\,, \tag{2.4}$$

where the symbol ap- lim denotes the approximate limit. We recall that J_y is an $\mathcal{H}^1$-rectifiable set, and that the triple (a, b, ν) is uniquely defined, up to a permutation of a and b and a change of sign of ν. In particular, ν is the approximate unit normal to J_y and we denote it by ν_y from now on. The approximate limits a and b at $x \in J_y$ are indicated by y^+_x and y^-_x.

We further set

$$GSBV^2(\Omega;\mathbb{R}^2) = \{y \in GSBV(\Omega;\mathbb{R}^2) : \ \nabla y \in L^2(\Omega;\mathbb{R}^{2\times2}),\ \mathcal{H}^1(J_y) < +\infty\}. \tag{2.5}$$

We define the space

$$GSBV^2_2(\Omega;\mathbb{R}^2) := \big\{y \in GSBV^2(\Omega;\mathbb{R}^2) : \ \nabla y \in GSBV^2(\Omega;\mathbb{R}^{2\times2})\big\}. \tag{2.6}$$

The approximate differential and the jump set of ∇y will be denoted by $\nabla^2 y$ and $J_{\nabla y}$, respectively. (To avoid confusion, we point out that in [24] the notation $GSBV^2_2(\Omega;\mathbb{R}^2)$ was used for $GSBV^2(\Omega;\mathbb{R}^2) \cap L^2(\Omega;\mathbb{R}^2)$.)

We notice that spaces similar to (2.6) already appeared, for instance, in [13, 14] to treat second order free discontinuity functionals, e.g., a weak formulation of the Blake & Zissermann model [9] of image segmentation. Since a function in $GSBV^2_2(\Omega;\mathbb{R}^2)$ is allowed to exhibit discontinuities, our analysis is outside of the framework of the space of special functions with bounded Hessian $SBH(\Omega)$, considered for second order energies for elastic-perfectly plastic plates (see, e.g., [15]).

In order to treat linear models of fracture, we need the space $GSBD(\Omega)$ of *generalized special functions of bounded deformation*, introduced in [23]. We recall that a function $u \in GSBD(\Omega)$ admits an approximate symmetric gradient $e(u) \in L^1(\Omega;\mathbb{R}^{2\times2}_{\rm sym})$ and its jump set J_u, defined similarly to (2.4), is $\mathcal{H}^1$-rectifiable, so that the approximate unit normal ν_u to J_u is defined $\mathcal{H}^1$-a.e. on J_u together with the approximate limits u^+_x and u^-_x, $x \in J_u$. As usual, we set $[u] := u^+ - u^-$ as the jump of u through J_u. We further let

$$GSBD^2(\Omega) := \{u \in GSBD(\Omega) \colon\ e(u) \in L^2(\Omega;\mathbb{R}^{2\times2}_{\rm sym}),\ \mathcal{H}^1(J_u) < +\infty\}\,.$$

We conclude this section by recalling two technical results concerning $GSBD^2$-functions. The first one is a *piecewise Korn-Poincaré inequality* [35].

Proposition 2.2 (Piecewise Korn-Poincaré inequality). *Let $Q \subset \mathbb{R}^2$ be an open square and let $0 < \theta \leq \theta_0$ for some θ_0 sufficiently small. Then, there exists some $C_\theta = C_\theta(\theta) \geq 1$ such that the following holds: for each $u \in GSBD^2(Q)$ we find a (finite) Caccioppoli partition $Q = R \cup \bigcup_{j=1}^J P_j$,* *and corresponding rigid motions* $(a_j)_{j=1}^J$ *such that*

$$\sum_{j=1}^J \mathcal{H}^1\big((\partial^* P_j \cap Q) \setminus J_u\big) + \mathcal{H}^1\big((\partial^* R \cap Q) \setminus J_u\big) \le \theta(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q)), \tag{2.7a}$$

$$\mathcal{L}^2(R) \le \theta(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q))^2, \qquad \mathcal{L}^2(P_j) \ge \mathcal{L}^2(Q)\theta^3 \quad \text{for all } j = 1, \dots, J, \tag{2.7b}$$

$$\|u - a_j\|_{L^\infty(P_j)} \le C_\theta \|e(u)\|_{L^2(Q)} \qquad \text{for all } j = 1, \dots, J. \tag{2.7c}$$

Proof. The statement is a slightly simplified version of [35, Theorem 4.1]. We briefly explain how the result can be obtained therefrom. We first suppose that Q is the unit square. We define $\theta_0 \le 1/c$, where c is the constant from [35, Theorem 4.1] and apply [35, Theorem 4.1] for θ/c in place of θ. Then, (2.7a) follows from [35, (18)(i)], where we denote the component P_0 by R. Item (2.7b) follows from [35, (17)(i), (18)(ii)], choosing θ_0 sufficiently small such that $C_\Omega \ge \theta_0$. Finally, (2.7c) follows from [35, (18)(iii)], where also a corresponding Korn-type estimate has been proved. Eventually, if Q is not the unit square, the result follows by a standard rescaling argument, see [35, Remark 4.2]. □

In the next sections (see in particular Theorem 3.12) we will also deal with boundary conditions. As usual in BV and BD-like spaces, we impose a Dirichlet boundary condition by forcing a displacement $u \in GSBD(\Omega)$ to take a prescribed value on the set $\Omega' \setminus \overline{\Omega}$, where Ω' is an open bounded subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega'$ such that $\Omega \subseteq \Omega'$. Precisely, for a boundary datum $h \in W^{2,\infty}(\Omega';\mathbb{R}^2)$ we introduce the space

$$GSBD^2_h(\Omega') := \{u \in GSBD^2(\Omega') \colon\ u = h \text{ on } \Omega' \setminus \overline{\Omega}\}. \tag{2.8}$$

In what follows, we will make a geometrical assumption on the Dirichlet part of the boundary $\partial_D\Omega := \Omega' \cap \partial\Omega$, which will allow us to exploit a density result in $GSBD^2_h(\Omega')$ (see Theorem 2.3). Precisely, we assume that there exists a decomposition $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \cup N$ with

$$\partial_D\Omega, \partial_N\Omega \text{ relatively open}, \quad \mathcal{H}^{d-1}(N) = 0, \quad \partial_D\Omega \cap \partial_N\Omega = \emptyset, \quad \partial(\partial_D\Omega) = \partial(\partial_N\Omega), \tag{2.9}$$

and there exist $\bar\delta > 0$ small and $x_0 \in \mathbb{R}^d$ such that for all $\delta \in (0, \bar\delta)$ there holds

$$O_{\delta,x_0}(\partial_D\Omega) \subset \Omega, \tag{2.10}$$

where $O_{\delta,x_0}(x) := x_0 + (1-\delta)(x - x_0)$.

We conclude this section with the statement of a density result in $GSBD^2_h(\Omega')$. To shorten the notation, we introduce the space $\mathcal{W}(\Omega;\mathbb{R}^2)$ of all functions $u \in SBV(\Omega;\mathbb{R}^2)$ such that J_u is a finite union of disjoint segments and $u \in W^{k,\infty}(\Omega \setminus J_u;\mathbb{R}^2)$ for every $k \in \mathbb{N}$. The following theorem is essentially a consequence of results in [18] and [22]. The exact statement can be found in [32, Theorem 3.6].

Theorem 2.3 (Density with boundary data)**.** *Let $\Omega \subset \Omega' \subset \mathbb{R}^2$ be bounded Lipschitz domains satisfying (2.9)–(2.10). Let $h \in W^{r,\infty}(\Omega')$ for $r \in \mathbb{N}$ and let $u \in GSBD^2_h(\Omega')$. Then, there exists a sequence of functions $(u_n)_n$ in $SBV^2(\Omega;\mathbb{R}^2)$, a sequence of neighborhoods $(U_n)_n$ of $\Omega' \setminus \Omega$, and a sequence of neighborhoods $(\Omega_n)_n$ of $\Omega \setminus U_n$ such that $U_n \subset \Omega'$, $\Omega_n \subset \Omega$, $u_n = h$ on $\Omega' \setminus \overline{\Omega}$,*

$u_n|_{U_n} \in W^{r,\infty}(U_n;\mathbb{R}^2)$, and $u_n|_{\Omega_n} \in \mathcal{W}(\Omega_n;\mathbb{R}^2)$, and the following properties hold:

$$u_n \to u \text{ in measure on } \Omega', \tag{2.11a}$$

$$\lim_{n\to\infty} \|e(u_n) - e(u)\|_{L^2(\Omega')} = 0, \tag{2.11b}$$

$$\lim_{n\to\infty} \mathcal{H}^1(J_{u_n}) = \mathcal{H}^1(J_u). \tag{2.11c}$$

In particular, $u_n \in W^{r,\infty}(\Omega \setminus J_{u_n};\mathbb{R}^2)$ for every $n \in \mathbb{N}$.

3. Setting and main results

3.1. **Ciarlet-Nečas and contact conditions.** The aim of this paper is to characterize the relation between Ciarlet-Nečas and contact conditions in the passage from nonlinear to linearized brittle fracture. Following [37, Section 2], we first give a precise meaning to the Ciarlet-Nečas non-interpenetration condition, see [20]. To do this, we recall the definition of measure theoretic image of $GSBV$-functions.

Definition 3.1 (Measure theoretical image)**.** Let $y \in GSBV(\Omega;\mathbb{R}^2)$ and let $\Omega_d \subseteq \Omega$ be the set of points where y is approximate differentiable. We define y_d by

$$y_d(x) := \begin{cases} \tilde{y}(x) & \text{for } x \in \Omega_d, \\ 0 & \text{else,} \end{cases}$$

where $\tilde{y}(x)$ denotes the Lebesgue value of y at $x \in \Omega_d$. Given a measurable set $E \subseteq \Omega$, we say that $y_d(E)$ is the measure theoretic image of E under the map y, and we denote it by $[y(E)]$.

Definition 3.2 (Ciarlet-Nečas non-interpenetration condition for $GSBV$-maps)**.** We say that $y \in GSBV(\Omega;\mathbb{R}^2)$ satisfies the *Ciarlet-Nečas non-interpenetration condition* if $\det \nabla y(x) > 0$ for a.e. $x \in \Omega$ and

$$\int_\Omega \det \nabla y \,\mathrm{d}x \le \mathcal{L}^2([y(\Omega)])\,, \tag{CN}$$

where $[y(\Omega)]$ denotes the image of Ω under y according to Definition 3.1.

Remark 3.3. We recall that (CN) is equivalent to a.e. injectivity under the assumption $\det \nabla y(x) > 0$ for a.e. $x \in \Omega$, see [37, Proposition 2.5]. Here, we say that y is a.e.-injective if for every representative $\bar{y}$ of y there exists an $\mathcal{L}^2$-negligible set $E \subset \Omega$ such that the restriction of $\bar{y}$ on $\Omega \setminus E$ is injective.

We now define the linearized contact condition for functions in $GSBD^2(\Omega)$.

Definition 3.4 (Contact condition for $GSBD$-maps)**.** We say that $u \in GSBD^2(\Omega)$ satisfies the *contact condition* if

$$\nu_u(x) \cdot [u](x) \ge 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in J_u. \tag{CC}$$

Our first observation is the following: consider a sequence of deformations $(y_\varepsilon)_\varepsilon \subset GSBV^2(\Omega;\mathbb{R}^2)$ satisfying (CN) such that their associated rescaled displacements $(u_\varepsilon)_\varepsilon$, defined as

$$u_\varepsilon := \frac{1}{\varepsilon}(y_\varepsilon - \mathbf{id})\,, \tag{3.1}$$

have uniformly bounded linearized energies, i.e.,

$$\sup_{\varepsilon>0} \mathcal{F}(u_\varepsilon) < +\infty, \qquad \text{where } \mathcal{F}(u) := \|e(u_\varepsilon)\|^2_{L^2(\Omega)} + \mathcal{H}^1(J_{u_\varepsilon}) \,. \tag{3.2}$$

This is not enough to guarantee measure convergence of u_ε to a displacement u satisfying (CC).

Example 3.5. Let $\Omega = (-1,1)^2$ and let $u = (-1,0)\chi_{\{x_1>0\}}$. Clearly, we have $J_u = \{0\} \times (-1,1)$, $\nu_u = e_1$, and $[u] = -e_1$. Therefore, we have $[u] \cdot \nu_u = -1 < 0$ on J_u. We now construct a sequence $(y_\varepsilon)_\varepsilon \subset GSBV^2(\Omega;\mathbb{R}^2)$ satisfying (3.2), as well as (CN), and such that $u_\varepsilon \to u$ in measure on Ω, where $u_\varepsilon \in GSBV^2(\Omega;\mathbb{R}^2)$ is defined in (3.1). To this end, we let (see also Figure 1)

$$u_\varepsilon := (-1,0)\chi_{\{x_1>0\}} + \left(\frac{2}{\varepsilon}, 0\right)\chi_{\{-2\varepsilon<x_1<0\}}\,, \qquad y_\varepsilon = \mathbf{id} + \varepsilon u_\varepsilon\,. \tag{3.3}$$

Then, we see that $\nabla y_\varepsilon = \mathbf{Id}$ on Ω and $\mathcal{H}^1(J_{y_\varepsilon}) = 4$, i.e., (3.2) holds true. It is also easy to check that $u_\varepsilon \to u$ in measure on Ω. Finally, the functions y_ε satisfy the Ciarlet-Nečas non-interpenetration condition, since for ε sufficiently small the three sets

$$[y_\varepsilon(\{x_1 < -2\varepsilon\})], \quad [y_\varepsilon(\{-2\varepsilon < x_1 < 0\})], \quad [y_\varepsilon(\{x_1 > 0\})]$$

are pairwise disjoint.

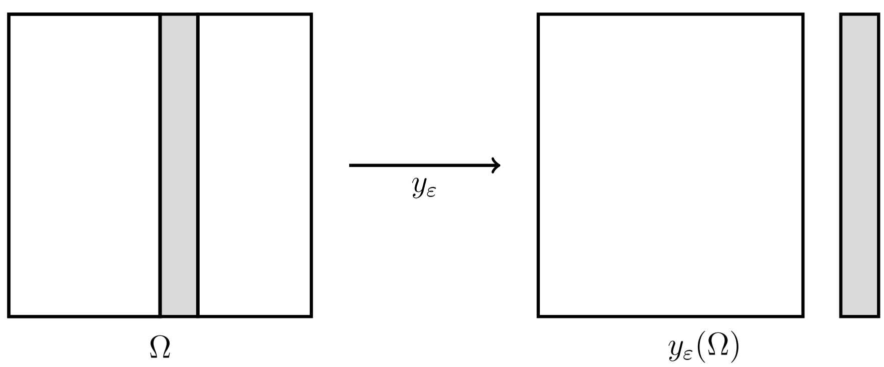

FIGURE 1. Graphic representation of the deformation y_ε in (3.3).

A crucial point in the example is that the length of the jump along the sequence has twice the size of the limiting jump. Our second result shows that, under a suitable energy convergence of the rescaled displacements and a slightly stronger control on elastic energies, the pathological situation in Example 3.5 can be avoided.

Theorem 3.6 (From Ciarlet-Nečas to contact condition)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open and bounded. Let $(y_\varepsilon)_\varepsilon \subset GSBV^2(\Omega;\mathbb{R}^2)$ be a sequence satisfying (CN). For every $\varepsilon > 0$, let u_ε be defined as in (3.1), and assume that there exists $u \in GSBD^2(\Omega)$ such that $u_\varepsilon \to u$ in measure on Ω. Suppose moreover that there exists $\gamma > \frac{1}{2}$ such that*

$$\sup_{\varepsilon>0} \varepsilon^{1-\gamma}\|\nabla u_\varepsilon\|_{L^2(\Omega)} < +\infty, \tag{3.4a}$$

$$\lim_{\varepsilon\to 0} \|e(u_\varepsilon)\|^2_{L^2(\Omega)} + \mathcal{H}^1(J_{u_\varepsilon}) = \|e(u)\|^2_{L^2(\Omega)} + \mathcal{H}^1(J_u). \tag{3.4b}$$

Then, u satisfies (CC).

Let us comment on the hypotheses of Theorem 3.6. By a compactness argument, see Proposition 3.10 and (3.11b) below, assumption (3.4a) holds for sequences $(u_\varepsilon)_\varepsilon$ such that the corresponding deformation fields $y_\varepsilon = \mathbf{id} + \varepsilon u_\varepsilon$ have bounded nonlinear Griffith energy $\mathcal{E}_\varepsilon$, defined in (3.6) below. Condition (3.4b) is instead stronger. In variational terms, it requires the rescaled displacements $(u_\varepsilon)_\varepsilon$ associated to $(y_\varepsilon)_\varepsilon$ to be an *energy-convergent sequence* for the limiting displacement u, in terms of the energy $\mathcal{F}$ defined in (3.2). As shown in Example 3.5, condition (3.4b) cannot be weakened to a more traditional energy bound of the form (3.2). Hence, Theorem 3.6 states that (CN) yields the contact condition (CC), provided $(u_\varepsilon)_\varepsilon$ is an energy-convergent sequence for $\mathcal{F}$.

We defer the proof of Theorem 3.6 to Section 5 and continue here with the presentation of our results. The second part of Section 3 is devoted to the definitions of linear and nonlinear Griffith energies and to the passage from nonlinear to linear models, under the additional Ciarlet-Nečas and contact conditions.

3.2. **From nonlinear to linear Griffith models with non-interpenetration.** We start by introducing the nonlinear Griffith energy for non-simple materials. We let $W\colon \mathbb{R}^{2\times 2} \to [0,+\infty)$ be a single well, frame indifferent stored energy density. To be precise, we suppose that there exists $c > 0$ such that

$$W \text{ continuous and } C^3 \text{ in a neighborhood of } SO(2), \tag{3.5a}$$

$$W(\mathrm{RF}) = W(\mathrm{F}) \text{ for all } \mathrm{F} \in \mathbb{R}^{2\times 2}, \mathrm{R} \in SO(2), \tag{3.5b}$$

$$W(\mathrm{F}) \geq c \operatorname{dist}^2(\mathrm{F}, SO(2)) \quad \text{for all } \mathrm{F} \in \mathbb{R}^{2\times 2},\ W(\mathrm{F}) = 0 \text{ iff } \mathrm{F} \in SO(2). \tag{3.5c}$$

Let us fix $\kappa > 0$, $\beta \in (\frac{2}{3}, 1)$, and two open bounded subsets $\Omega \subseteq \Omega'$ of $\mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega'$, respectively, such that (2.9)–(2.10) hold. Recalling the definition (2.6) of the space $GSBV^2_2(\Omega';\mathbb{R}^2)$, for $\varepsilon > 0$ we define the energy $\mathcal{E}_\varepsilon\colon GSBV^2_2(\Omega';\mathbb{R}^2) \to [0,+\infty]$ by

$$\mathcal{E}_\varepsilon(y) = \begin{cases} \varepsilon^{-2} \displaystyle\int_{\Omega'} W(\nabla y(x))\, dx + \varepsilon^{-2\beta} \int_{\Omega'} |\nabla^2 y(x)|^2\, dx + \kappa \mathcal{H}^1(J_y) & \text{if } J_{\nabla y} \subseteq J_y, \\ +\infty & \text{else.} \end{cases} \tag{3.6}$$

Here and in the following, the inclusion $J_{\nabla y} \subseteq J_y$ has to be understood up to an $\mathcal{H}^1$-negligible set. Since W grows quadratically around $SO(2)$, the parameter ε corresponds to the typical scaling of strains for configurations with finite energy. We further notice that the choice of two open sets Ω and Ω' is due to the fact that we are interested in boundary value problems, where a Dirichlet datum is to be imposed on $\Omega' \setminus \overline{\Omega}$ (see also Section 2.4).

Due to the presence of the second term in (3.6), we deal with a Griffith-type model for *nonsimple materials*. Elastic energies depending on the second gradient of the deformation were introduced by Toupin [54, 55] to enhance compactness and rigidity properties. In our context, we consider a second gradient term (describing the absolutely continuous part of the gradient of ∇y) for a material undergoing fracture, which has a regularization effect on the entire intact region $\Omega' \setminus J_y$ of the material. This is modeled by the condition $J_{\nabla y} \subseteq J_y$.

We finally remark that the condition $J_{\nabla y} \subseteq J_y$ in (3.6) is not closed under convergence in measure on Ω', and to guarantee the existence of minimizers one needs to pass to a suitable relaxation, see [32, Proposition 2.1 and Theorem 2.2].

The corresponding linearized Griffith model is represented by the functional $\mathcal{E}\colon GSBD^2(\Omega') \to [0,+\infty)$ given by

$$\mathcal{E}(u) := \int_{\Omega'} \frac{1}{2} Q(e(u))\,\mathrm{d}x + \kappa\mathcal{H}^1(J_u), \tag{3.7}$$

where $Q\colon \mathbb{R}^{2\times 2} \to [0,+\infty)$ is the quadratic form $Q(\mathrm{F}) = D^2W(\mathbf{Id})\mathrm{F} : \mathrm{F}$ for all $\mathrm{F} \in \mathbb{R}^{2\times 2}$. In view of (3.5), Q is positive definite on $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and vanishes on $\mathbb{R}^{2\times 2}_{\mathrm{skew}}$.

The Γ-convergence of $\mathcal{E}_\varepsilon$ to $\mathcal{E}$ has been studied in [32] in dimension $d \geq 2$, with neither non-interpenetration nor contact conditions. We also refer to [31] for a linearization result in dimension $d = 2$ without second order regularization in (3.6).

Justified by [37, Section 6.1], [25, Appendix A], or [30, Section 5.1], a natural conjecture would be that, in this limiting passage, the conditions (CN) and (CC) could simply be included on the nonlinear and linear level, respectively. Example 3.5 showed that this is not the case, as (CC) is not maintained for limits of sequences satisfying (CN). In the next example we further show that the variational limit of the functionals $\mathcal{E}_\varepsilon$ cannot expected to be expressed by means of the classical Griffith energy.

Example 3.7. Let $\Omega = (-1,1)^2$, let $\mu = (\mu_1,\mu_2) \in \mathbb{R}^2$ be such that $\mu_1 < 0$, and let $u = (\frac{\mu_1}{2}, \mu_2)\chi_{\{x_1>0\}}$. To fix the ideas, we also assume $\mu_2 < 0$. As in Example 3.5, $J_u = \{0\} \times (-1,1)$ has length $\mathcal{H}^1(J_u) = 2$ and normal vector $\nu_u = e_1$. Hence, $[u]\cdot e_1 = \frac{\mu_1}{2} < 0$ on J_u. For $\varepsilon > 0$ we set $n_\varepsilon := \lfloor \frac{1}{\varepsilon|\mu_2|} \rfloor$, where $\lfloor\cdot\rfloor$ denotes the integer part, and let

$$\begin{aligned}
R^k_\varepsilon &:= \Big(-\varepsilon\frac{|\mu_1|}{2}, \varepsilon\frac{|\mu_1|}{2}\Big) \times \big(-1 + 2k\varepsilon|\mu_2|, -1 + (2k+1)\varepsilon|\mu_2|\big) \qquad \text{for } k = 0,\dots,n_\varepsilon - 1,\\
R_\varepsilon &:= \bigcup_{k=0}^{n_\varepsilon-1} R^k_\varepsilon\,.
\end{aligned}$$

Then, we define $y_\varepsilon \in GSBV^2(\Omega;\mathbb{R}^2)$ as (see also Figure 2)

$$y_\varepsilon := \mathbf{id} + \varepsilon u_\varepsilon \qquad \text{with} \qquad u_\varepsilon := \Big(\frac{\mu_1}{2}, \mu_2\Big)\,\chi_{(\Omega\setminus R_\varepsilon)\cap\{x_1>0\}} + \Big(\frac{2}{\varepsilon}, 0\Big)\,\chi_{R_\varepsilon}\,, \tag{3.8}$$

so that $(y_\varepsilon)_\varepsilon$ satisfies (3.2), and (CN), as in Example 3.5, and $u_\varepsilon \to u$ in measure.

In particular, the jump set J_{y_ε} satisfies the inequalities

$$\begin{aligned}
\mathcal{H}^1(J_{y_\varepsilon}) &\leq \mathcal{H}^1(\partial R_\varepsilon) + (n_\varepsilon+1)\varepsilon|\mu_2| = 2n_\varepsilon\varepsilon|\mu_1| + 2n_\varepsilon\varepsilon|\mu_2| + (n_\varepsilon+1)\varepsilon|\mu_2|\\
&= 3n_\varepsilon\varepsilon|\mu_2| + 2n_\varepsilon\varepsilon|\mu_1| + \varepsilon|\mu_2| \leq 3 + 2\frac{|\mu_1|}{|\mu_2|} + \varepsilon|\mu_2|\,,\\
\mathcal{H}^1(J_{y_\varepsilon}) &\geq \mathcal{H}^1(\partial R_\varepsilon) - 2\varepsilon|\mu_1| + (n_\varepsilon-1)\varepsilon|\mu_2| = 2n_\varepsilon\varepsilon|\mu_1| + 2n_\varepsilon\varepsilon|\mu_2| - 2\varepsilon|\mu_1| + (n_\varepsilon-1)\varepsilon|\mu_2|\\
&= 3n_\varepsilon\varepsilon|\mu_2| + 2n_\varepsilon\varepsilon|\mu_1| - 2\varepsilon|\mu_1| - \varepsilon|\mu_2| \geq 3 - 4\varepsilon|\mu_2| + 2\frac{|\mu_1|}{|\mu_2|} - 4\varepsilon|\mu_1|\,.
\end{aligned}$$

Thus, we deduce that

$$\lim_{\varepsilon\to 0} \mathcal{H}^1(J_{y_\varepsilon}) = 3 + 2\frac{|\mu_1|}{|\mu_2|}\,,$$

whereas in comparison $\mathcal{H}^1(J_u) = 2$.

Examples 3.5 and 3.7 suggest that, besides $\kappa\mathcal{H}^1(J_u)$, the formulation of the variational limit of $\mathcal{E}_\varepsilon$ in (3.6) should account for an additional anisotropic surface term being positive whenever

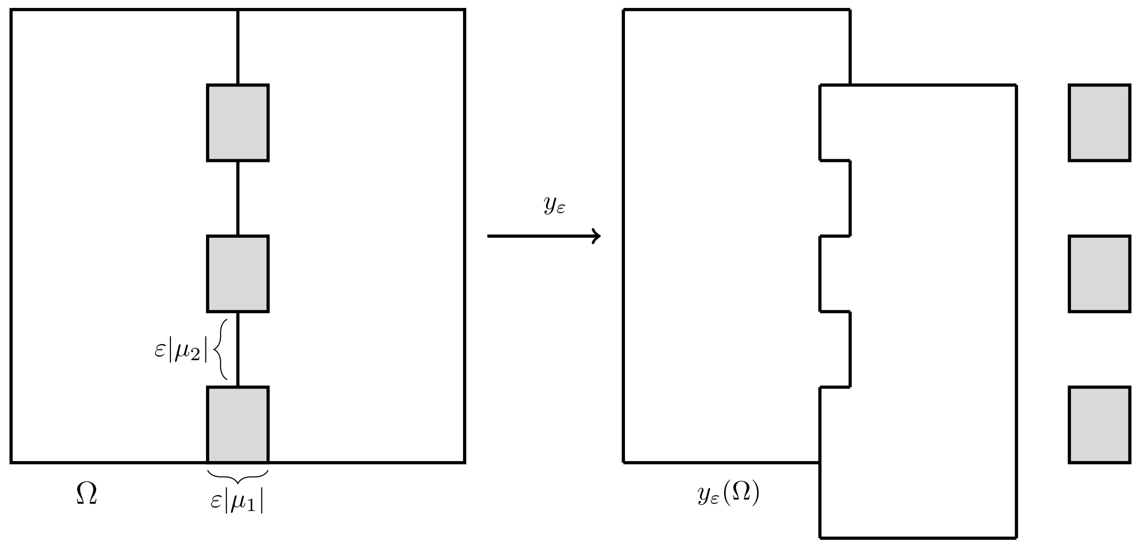

FIGURE 2. Graphic representation of the deformation y_ε in (3.8). The set R_ε is in gray.

(CC) is violated. This term should depend on the orientation and on the amplitude of the jump of the displacement u. The full characterization of this Γ-limit is beyond the scope of the present contribution, and in the following we restrict our attention to *energy-convergent sequences*. In order to comply with boundary conditions on $\Omega' \setminus \overline{\Omega}$, for $h \in W^{2,\infty}(\Omega';\mathbb{R}^2)$ and $\varepsilon > 0$ we set

$$\mathcal{S}_{\varepsilon,h} = \{y \in GSBV^2_2(\Omega';\mathbb{R}^d):\ y = \mathbf{id} + \varepsilon h \text{ on } \Omega' \setminus \overline{\Omega}\}, \tag{3.9}$$

and also recall the definition $GSBD^2_h(\Omega')$ in (2.8).

We start by clarifying the definition of convergence. The general idea in linearization results (see, e.g., [1, 12, 27, 33, 34, 48, 51, 52]) is to obtain compactness for the rescaled displacement fields $(u_\varepsilon)_\varepsilon$ associated to a sequence $(y_\varepsilon)_\varepsilon$ with $\sup_\varepsilon \mathcal{E}_\varepsilon(y_\varepsilon) < +\infty$, see (3.1). For bodies undergoing fracture, however, no compactness can be expected: consider, for instance, the functions $y_\varepsilon := \mathbf{id}\chi_{\Omega'\setminus B} + \mathrm{R}\,\mathbf{id}\chi_B$, for a small ball $B \subset \Omega$ and a rotation $\mathrm{R} \in SO(2)$, $\mathrm{R} \neq \mathbf{Id}$. Then $|u_\varepsilon|, |\nabla u_\varepsilon| \to \infty$ on B as $\varepsilon \to 0$. As observed in [32, Theorem 2.3], this phenomenon can be avoided if the deformation is *rotated back to the identity* on the set B. This justifies the following notion of convergence, see also [32, Definition 2.4].

Definition 3.8 (Asymptotic representation)**.** Fix $\gamma \in (\frac{2}{3}, \beta)$. We say that a sequence $(y_\varepsilon)_\varepsilon$ with $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$ is *asymptotically represented* by a limiting displacement $u \in GSBD^2_h(\Omega')$, and write $y_\varepsilon \rightsquigarrow u$, if there exist sequences of Caccioppoli partitions $(P^\varepsilon_j)_j$ of Ω' and corresponding rotations $(R^\varepsilon_j)_j \subset SO(2)$ such that, setting

$$y^{\mathrm{rot}}_\varepsilon := \sum_{j=1}^\infty R^\varepsilon_j\, y_\varepsilon\, \chi_{P^\varepsilon_j} \qquad \text{and} \qquad u_\varepsilon := \frac{1}{\varepsilon}(y^{\mathrm{rot}}_\varepsilon - \mathbf{id}), \tag{3.10}$$

the following conditions hold:

$$\|\mathrm{sym}(\nabla y_\varepsilon^{\mathrm{rot}}) - \mathbf{Id}\|_{L^2(\Omega')} \le C\varepsilon, \tag{3.11a}$$

$$\|\nabla y_\varepsilon^{\mathrm{rot}} - \mathbf{Id}\|_{L^2(\Omega')} \le C\varepsilon^\gamma, \tag{3.11b}$$

$$|\nabla y_\varepsilon^{\mathrm{rot}} - \mathbf{Id}| \le C\big(\varepsilon^\gamma + \mathrm{dist}(\nabla y_\varepsilon^{\mathrm{rot}}, SO(2))\big) \quad \text{a.e. on } \Omega' \tag{3.11c}$$

$$u_\varepsilon \to u \quad \text{a.e. in } \Omega' \setminus E_u, \tag{3.11d}$$

$$e(u_\varepsilon) \rightharpoonup e(u) \quad \text{weakly in } L^2(\Omega' \setminus E_u; \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \tag{3.11e}$$

$$\mathcal{H}^1(J_u) \le \liminf_{\varepsilon\to 0} \mathcal{H}^1(J_{u_\varepsilon}) \le \liminf_{\varepsilon\to 0} \mathcal{H}^1(J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon}), \tag{3.11f}$$

$$e(u) = 0 \quad \text{on} \quad E_u, \quad \mathcal{H}^1\big((\partial^* E_u \cap \Omega') \setminus J_u\big) = \mathcal{H}^1(J_u \cap (E_u)^1) = 0, \tag{3.11g}$$

where $E_u := \{x \in \Omega : |u_\varepsilon(x)| \to \infty\}$ is a set of finite perimeter.

Remark 3.9. The presence of the set E_u is due to the compactness result in $GSBD^2(\Omega')$, see [19]. We point out that the behavior of the sequence cannot be controlled on this set, but that this is not an issue for minimization problems of Griffith energies since a minimizer can be recovered by choosing u affine on E_u with $e(u) = 0$, cf. (3.11g). We also note that $E_u \subset \Omega$, i.e., $E_u \cap (\Omega' \setminus \overline{\Omega}) = \emptyset$.

We speak of asymptotic representation instead of convergence, and we use the symbol $\rightsquigarrow$, in order to emphasize that Definition 3.8 cannot be understood as a convergence with respect to a certain topology. Indeed, the limit u for a given (sub-)sequence $(y_\varepsilon)_\varepsilon$ is not uniquely determined, but rather depends on the choice of the sequences $(P_j^\varepsilon)_j$ and $(R_j^\varepsilon)_j$. For details in that direction, in particular concerning a characterization of limiting displacements, we refer to [32, Subsection 2.2].

We have the following compactness result for asymptotic representations.

Proposition 3.10 (Compactness)**.** *Let $h \in W^{2,\infty}(\Omega';\mathbb{R}^2)$, and assume that W satisfies (3.5). Let $\gamma \in (\frac{2}{3}, \beta)$. Let $(y_\varepsilon)_\varepsilon$ be a sequence satisfying $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$ and $\sup_\varepsilon \mathcal{E}_\varepsilon(y_\varepsilon) < +\infty$. Then there exists a subsequence (not relabeled) and $u \in GSBD^2_h(\Omega')$ such that $y_\varepsilon \rightsquigarrow u$.*

The statement has been shown in [32, Theorem 2.3]. Actually, property (3.11c) has not been stated there explicitly, but has been used in the proof, see [32, (4.11)]. We now present a consequence of Theorem 3.6 about the passage from the Ciarlet-Nečas to the contact condition, whose proof is also postponed to Section 5.

Theorem 3.11 (From Ciarlet-Nečas to contact condition in the asymptotic representation)**.** *Let $h \in W^{2,\infty}(\Omega';\mathbb{R}^2)$, and assume that W satisfies (3.5). Let $(y_\varepsilon)_\varepsilon$ be a sequence satisfying $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$ and (CN). Let $u \in GSBD^2_h(\Omega')$ be such that $y_\varepsilon \rightsquigarrow u$ and $\mathcal{E}_\varepsilon(y_\varepsilon) \to \mathcal{E}(u)$ as $\varepsilon \to 0$. Finally, assume that $y_\varepsilon^{\mathrm{rot}}$ in (3.10) also satisfies (CN). Then, u satisfies (CC) on $J_u \setminus \partial^* E_u$.*

The assumption that also $y_\varepsilon^{\mathrm{rot}}$ satisfies (CN) is not really restrictive since it would also be possible to consider modifications of the form $y_\varepsilon^{\mathrm{rot}} := \sum_{j=1}^\infty (R_j^\varepsilon\, y_\varepsilon - b_j^\varepsilon)\, \chi_{P_j^\varepsilon}$ for suitable $(b_j^\varepsilon)_j \subset \mathbb{R}^2$ (cf. [31, Theorem 2.2]) such that (CN) holds. The full proof of this statement would be quite technical. Since this is not the main focus of the paper, we would rather not dwell on this point and just explain the general idea behind it: for $\ell \in \mathbb{N}$, assume that the sum of the contributions on the first $\ell - 1$ sets is injective. If the local rotation of the contribution on the ℓ-th set creates some overlapping, a translation b_ℓ^ε is added to restore injectivity. An induction argument on the ordering of the partition then yields the claim.

We conclude this section by stating the third main contribution of this work, whose proof is postponed to Section 6. In particular, we assert that for every $u \in GSBD^2_h(\Omega')$ satisfying the contact condition (CC) there exists an energy-convergent sequence (in the sense of Definition 3.8) which fulfills the Ciarlet-Nečas condition (CN).

Theorem 3.12 (Existence of energy-convergent sequences)**.** *Let $\Omega \subset \Omega' \subset \mathbb{R}^2$ be bounded Lipschitz domains satisfying (2.9)–(2.10), let $h \in W^{2,\infty}(\Omega';\mathbb{R}^2)$, and assume that W satisfies (3.5). Then, for every $u \in GSBD^2_h(\Omega')$ satisfying (CC) there exists a sequence $(y_\varepsilon)_\varepsilon$ satisfying (CN) and such that $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$, $y_\varepsilon \rightsquigarrow u$, and*

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y_\varepsilon) = \mathcal{E}(u).$$

4. Structural result for blow up around jump points

This section is devoted to a preliminary result needed in the proofs of Theorems 3.6 and 3.12. For $\rho > 0$, we set $Q^\pm_\rho \coloneqq Q_\rho \cap \{\pm x\cdot e_1 > 0\}$. Here and in the following, $\pm$ is a placeholder for both $+$ and $-$.

Proposition 4.1. *Let $0 < \rho \le 1$, let $v \in GSBD^2(Q_\rho)$, let $\omega^+, \omega^- \in \mathbb{R}^2$, and let $0 < \eta \le \min\{\frac{1}{7}|\omega^+ - \omega^-|, \theta_0, 10^{-4}\}$, where θ_0 is the constant of Proposition 2.2. Assume that*

$$\mathcal{H}^1\big(J_v \cap Q_\rho\big) \le \rho(1+\eta), \tag{4.1a}$$

$$\mathcal{L}^2\Big(\Big\{x \in Q^+_\rho \colon\ |v-\omega^+| > \frac{\eta}{\bar c_{\eta^3/2}}\Big\}\Big) + \mathcal{L}^2\Big(\Big\{x \in Q^-_\rho \colon\ |v-\omega^-| > \frac{\eta}{\bar c_{\eta^3/2}}\Big\}\Big) \le \rho^2\eta^4, \tag{4.1b}$$

$$\int_{Q_\rho} |e(v)|^2 \,\mathrm{d}x \le \frac{\rho\eta^2}{C_\eta^2 \bar c^2_{\eta^3/2}}, \tag{4.1c}$$

where $C_\eta \ge 1$ denotes the constant of Proposition 2.2 applied for $\theta = \eta$, and $\bar c_{\eta^3/2} \ge 1$ denotes the constant of Lemma 2.1 applied for $\theta = \eta^3/2$. Then there exist two disjoint sets $D^+, D^- \subseteq Q_\rho$ such that

$$\|v-\omega^+\|_{L^\infty(D^+)} \le 3\eta \qquad \textit{and} \qquad \|v-\omega^-\|_{L^\infty(D^-)} \le 3\eta, \tag{4.2a}$$

$$\mathcal{H}^1\Big(\big((\partial^* D^+ \cup \partial^* D^-)\setminus J_v\big)\cap Q_\rho\Big) \le 6\eta\rho. \tag{4.2b}$$

Moreover, there exist two curves $\Gamma^\pm \subseteq \partial^* D^\pm \cap Q_\rho$ connecting $(-\frac{\rho}{2},\frac{\rho}{2})\times\{-\frac{\rho}{2}\}$ to $(-\frac{\rho}{2},\frac{\rho}{2})\times\{\frac{\rho}{2}\}$.

Remark 4.2. Later in the proofs of Theorem 3.6 and Theorem 3.12 we will show that (4.1) holds in the blow-up around jump points.

Proof of Proposition 4.1. We first apply Proposition 2.2 to construct the sets $D^\pm$. Afterwards, we prove the properties stated in (4.2).

Step 1: Application of the piecewise Korn inequality. We start by applying Proposition 2.2 for v and for $\theta = \eta$ on the set Q_ρ. (Note that $\eta \le \theta_0$ by assumption.) We obtain a (finite) Caccioppoli partition $Q_\rho = R \cup \bigcup_{j=1}^J P_j$, and corresponding rigid motions $(a_j)_{j=1}^J$ such that (2.7) holds. By assumptions (4.1a) and (4.1c) and the fact that $\mathcal{H}^1(\partial Q_\rho) = 4\rho$ we get

$$\sum_{j=1}^{J} \mathcal{H}^1\big((\partial^* P_j \cap Q_\rho) \setminus J_v\big) + \mathcal{H}^1\big((\partial^* R \cap Q_\rho) \setminus J_v\big) \le \eta(\rho(1+\eta) + 4\rho), \tag{4.3a}$$

$$\mathcal{L}^2(R) \le \eta(\rho(1+\eta) + 4\rho)^2, \qquad \mathcal{L}^2(P_j) \ge \rho^2\eta^3 \quad \text{for all } j = 1, \dots, J, \tag{4.3b}$$

$$\|v - a_j\|_{L^\infty(P_j)} \le C_\eta \|e(v)\|_{L^2(Q_\rho)} \le \frac{\sqrt{\rho}\eta}{\bar{c}_{\eta^3/2}} \qquad \text{for all } j = 1, \dots, J. \tag{4.3c}$$

We now show that for each $j = 1, \dots, J$ we have

$$\|v - \omega^+\|_{L^\infty(P_j)} \le 3\eta \qquad \text{or} \qquad \|v - \omega^-\|_{L^\infty(P_j)} \le 3\eta. \tag{4.4}$$

In fact, since (4.1b) and (4.3b) hold and $\eta \le \frac{1}{4}$, we find that

$$\mathcal{L}^2\Big(\Big\{|v - \omega^+| \le \frac{\eta}{\bar{c}_{\eta^3/2}}\Big\} \cap P_j\Big) \ge \frac{\rho^2\eta^3}{4} \qquad \text{or} \qquad \mathcal{L}^2\Big(\Big\{|v - \omega^-| \le \frac{\eta}{\bar{c}_{\eta^3/2}}\Big\} \cap P_j\Big) \ge \frac{\rho^2\eta^3}{4}. \tag{4.5}$$

Without loss of generality we may assume that (4.5) holds true for ω^+, and we write $S_j :=$ $\{|v - \omega^+| \le \eta/\bar{c}_{\eta^3/2}\} \cap P_j$. By (4.3c), the assumption that $\rho \le 1$, and the triangle inequality we get $\|\omega^+ - a_j\|_{L^\infty(S_j)} \le \frac{2\eta}{\bar{c}_{\eta^3/2}}$. By applying Lemma 2.1 for $\theta = \frac{\eta^3}{2}$ and $R = \frac{\sqrt{2}\rho}{2}$ we then get

$$\|a_j - \omega^+\|_{L^\infty(P_j)} \le \|a_j - \omega^+\|_{L^\infty(Q_\rho)} \le \bar{c}_{\eta^3/2} \|a_j - \omega^+\|_{L^\infty(S_j)} \le 2\eta,$$

where we used that $\mathcal{L}^2(S_j) \ge \frac{\eta^3}{2} R^2$ by (4.5). Another application of (4.3c) and using $\bar{c}_{\eta^3/2} \ge 1$ implies (4.4) for ω^+. In a similar fashion, we obtain the estimate for ω^-.

Since $|\omega^+ - \omega^-| \ge 7\eta$ by assumption on η, we observe that for each P_j estimate (4.4) either holds for ω^+ or for ω^-. We denote by $\mathcal{J}^+$ the set of indices such that (4.4) holds for ω^+, and set $\mathcal{J}^- = \{1, \dots, J\} \setminus \mathcal{J}^+$. We define the sets

$$D^+ := \bigcup_{j \in \mathcal{J}^+} P_j, \qquad D^- := \bigcup_{j \in \mathcal{J}^-} P_j.$$

Step 2: Proof of (4.2)*:* We start by observing that (4.4) implies $\|v - \omega^+\|_{L^\infty(D^+)} \le 3\eta$ and $\|v - \omega^-\|_{L^\infty(D^-)} \le 3\eta$, i.e., (4.2a) holds. By (4.3a) and since $\eta \le 1$ we also find that (4.2b) holds true. In particular, by (4.1a)

$$\mathcal{H}^1\big((\partial^* D^+ \cup \partial^* D^-) \cap Q_\rho\big) \le \rho + 7\eta\rho\,. \tag{4.6}$$

It remains to show the existence of the curves $\Gamma^\pm \subseteq \partial^* D^\pm$. First, we note that $D^\pm \supset \{|v - \omega^\pm| \le \eta/\bar{c}_{\eta^3/2}\} \cap (Q_\rho \setminus R)$ by construction and therefore we find by (4.1b) and (4.3b) that

$$\mathcal{L}^2\big(D^\pm \cap Q_\rho^\pm\big) \ge \frac{1}{2}\rho^2 - \rho^2\eta^4 - \mathcal{L}^2(R) \ge \frac{1}{2}\rho^2 - \rho^2\eta^4 - 36\rho^2\eta \ge \frac{1}{2}\rho^2 - C_0\rho^2\eta, \tag{4.7}$$

where we set $C_0 = 100$ for notational convenience. Thus, by (4.7) we get that

$$\mathcal{H}^1\Big(\big\{w \in \Pi^{e_1} :\ \mathcal{L}^1\big((D^\pm \cap Q_\rho^\pm)^{e_1}_w\big) > 0\big\}\Big) \ge \rho - 2C_0\eta\rho. \tag{4.8}$$

This in turn implies

$$\mathcal{H}^1\Big(\big\{w \in \Pi^{e_1} :\ \mathcal{H}^0\big((\partial^* D^\pm \cap Q_\rho)^{e_1}_w\big) \ge 1\big\}\Big) \ge \rho - 4C_0\eta\rho. \tag{4.9}$$

We further claim that

$$\mathcal{H}^1\Big(\big\{w\in\Pi^{e_1}\colon\ \mathcal{H}^0\big((\partial^*D^\pm\cap Q_\rho)^{e_1}_w\big)\geq 2\big\}\Big)\leq\frac{\rho}{2}+\frac{7}{2}\eta\rho\,. \tag{4.10}$$

Indeed, if (4.10) were not true, by the area formula (see, e.g., [4, Theorem 2.71]) and by (4.6) we would have that

$$\begin{aligned}\rho+7\eta\rho&<2\,\mathcal{H}^1\Big(\big\{w\in\Pi^{e_1}\colon\ \mathcal{H}^0\big((\partial^*D^\pm\cap Q_\rho)^{e_1}_w\big)\geq 2\big\}\Big)\leq\int_{\Pi^{e_1}}\mathcal{H}^0\big((\partial^*D^\pm\cap Q_\rho)^{e_1}_w\big)\,\mathrm{d}\mathcal{H}^1(w)\\&=\int_{\partial^*D^\pm\cap Q_\rho}|\nu_{D^\pm}\cdot e_1|\,\mathrm{d}\mathcal{H}^1\leq\mathcal{H}^1((\partial^*D^+\cup\partial^*D^-)\cap Q_\rho)\leq\rho+7\eta\rho\,,\end{aligned}$$

where by $\nu_{D^\pm}$ we denote the outer unit normal of $\partial^*D^\pm$. Thus, (4.10) holds true. Let us set $\delta\coloneqq 16C_0\eta$ and $K_\delta\coloneqq(-\frac{\delta\rho}{2},\frac{\delta\rho}{2})\times(-\frac{\rho}{2},\frac{\rho}{2})$. Then, by (4.7) we get

$$\begin{aligned}\mathcal{L}^2\big(D^\pm\cap Q^\pm_\rho\cap K_\delta\big)&\geq\mathcal{L}^2\big(D^\pm\cap Q^\pm_\rho\big)-\mathcal{L}^2(Q^\pm_\rho\setminus K_\delta)\\&\geq\frac{1}{2}\rho^2-C_0\rho^2\eta-\frac{1}{2}(1-\delta)\rho^2=\frac{1}{2}\delta\rho^2-C_0\rho^2\eta\,.\end{aligned}$$

Arguing as in (4.7)–(4.9) we deduce that

$$\mathcal{H}^1\Big(\big\{w\in\Pi^{e_1}\colon\ \mathcal{H}^0\big((\partial^*D^\pm\cap K_\delta)^{e_1}_w\big)\geq 1\big\}\Big)\geq\rho-4\frac{C_0\eta}{\delta}\rho=\frac{3}{4}\rho\,, \tag{4.11}$$

where in the last equality we have used the definition of δ. Hence, combining (4.10) and (4.11) we infer that

$$\mathcal{H}^1\Big(\big\{w\in\Pi^{e_1}\colon\ \mathcal{H}^0\big((\partial^*D^\pm\cap K_\delta)^{e_1}_w\big)=1\quad\text{and}\quad\mathcal{H}^0\big((\partial^*D^\pm\cap Q_\rho)^{e_1}_w\big)=1\big\}\Big)\geq\frac{\rho}{4}-\frac{7}{2}\eta\rho\,. \tag{4.12}$$

We now prove the existence of a curve $\Gamma^+\subseteq\partial^*D^+$ connecting $(-\frac{\rho}{2},\frac{\rho}{2})\times\{-\frac{\rho}{2}\}$ with $(-\frac{\rho}{2},\frac{\rho}{2})\times\{\frac{\rho}{2}\}$. The argument for $\Gamma^-\subseteq\partial^*D^-$ is the same, with a different notational realization. In view of (4.12), we can fix $w\in(-\frac{\rho}{2},\frac{\rho}{2})$ and $t\in(-\frac{\delta\rho}{2},\frac{\delta\rho}{2})$ such that

$$y\coloneqq(t,w)\in\partial^*D^+\cap K_\delta\qquad\text{and}\qquad(s,w)\notin\partial^*D^+\cap Q_\rho\quad\text{for } s\in(-\tfrac{\rho}{2},\tfrac{\rho}{2}),\ s\neq t. \tag{4.13}$$

Without loss of generality, we may assume that there exists the approximate unit normal $\nu_{D^+}(y)$ to ∂^*D^+ in y and that $\nu_{D^+}(y)\cdot e_1\neq 0$. By [3, Corollary 1], ∂^*D^+ can be decomposed uniquely into at most countably many pairwise disjoint rectifiable Jordan curves. Let us denote by $\Lambda\subseteq\partial^*D^+$ the Jordan curve containing y. Then, (4.13) and $\nu_{D^+}(y)\cdot e_1\neq 0$ imply that $\Lambda\cap Q_\rho\subsetneq\Lambda$. Thus, Λ must connect y to ∂Q_ρ. Let us denote by $\Gamma^+\subseteq\Lambda$ the sub-curve of Λ containing y and intersecting ∂Q_ρ only in its endpoints.

We now show that such endpoints lie in $(-\frac{\rho}{2},\frac{\rho}{2})\times\{-\frac{\rho}{2}\}$ and in $(-\frac{\rho}{2},\frac{\rho}{2})\times\{\frac{\rho}{2}\}$, respectively. By contradiction, let us assume that one of the endpoints is of the form $(\frac{\rho}{2},\overline{w})$ or $(-\frac{\rho}{2},\overline{w})$ with $\overline{w}\in(-\frac{\rho}{2},\frac{\rho}{2})$. Setting $|w-\overline{w}|=\zeta\rho$ for some $\zeta\in(0,1)$, by (4.9) and by definition of δ and η we estimate

$$\begin{aligned}\mathcal{H}^1((\partial^*D^+\cup\partial^*D^-)\cap Q_\rho)&\geq\sqrt{\Big(\frac{\rho(1-\delta)}{2}\Big)^2+|w-\overline{w}|^2}+(\rho-4C_0\eta\rho)-|w-\overline{w}|\\&=\rho\Big(\sqrt{\frac{(1-\delta)^2}{4}+\zeta^2}-\zeta\Big)+(\rho-4C_0\eta\rho)\\&\geq\rho\frac{(1-\delta)^2}{4+\sqrt{20}}+\rho-4C_0\eta\rho\geq\rho\frac{(1-\delta)^2}{9}+\rho-4C_0\eta\rho>\rho+7\eta\rho\,,\end{aligned}$$

which is in contradiction to (4.6). Hence, both endpoints of Γ^+ lie on $(-\frac{\rho}{2},\frac{\rho}{2})\times\{-\frac{\rho}{2}\}$ or on $(-\frac{\rho}{2},\frac{\rho}{2})\times\{\frac{\rho}{2}\}$. Since (4.13) holds, the endpoints can not both lie on the same side. Thus, Γ^+ connects $(-\frac{\rho}{2},\frac{\rho}{2})\times\{-\frac{\rho}{2}\}$ and $(-\frac{\rho}{2},\frac{\rho}{2})\times\{\frac{\rho}{2}\}$. This concludes the proof of the proposition. □

5. Proof of Theorem 3.6

This section is entirely devoted to the proofs of Theorems 3.6 and 3.11.

Proof of Theorem 3.6. We start by noting that the Ciarlet-Nečas non-interpenetration condition (CN) along with (2.3) implies

$$\int_E \det\nabla y_\varepsilon \,\mathrm{d}x = \mathcal{L}^2([y_\varepsilon(E)]) \tag{5.1}$$

for all measurable sets $E\subset\Omega$.

The proof of the theorem is performed by contradiction. We suppose that there exists a rectifiable set $J^{\rm int}\subset J_u$ with $\mathcal{H}^1(J^{\rm int})>0$ such that $[u](x)\cdot\nu_u(x)<0$ for all $x\in J^{\rm int}$. By a careful analysis of the blow-up around a point in $J^{\rm int}$, we will construct a sequence of subsets $E_\varepsilon\subseteq\Omega$ which violates (5.1), i.e., such that

$$\int_{E_\varepsilon} \det(\nabla y_\varepsilon)\,\mathrm{d}x > \mathcal{L}^2([y_\varepsilon(E_\varepsilon)]). \tag{5.2}$$

The argument is divided into several steps: in Step 1 we show by a blow-up argument that around a point in $J^{\rm int}$ the sequence $u_\varepsilon=\frac{1}{\varepsilon}(y_\varepsilon-\mathbf{id})$ satisfies the assumptions (4.1) of Proposition 4.1. In Steps 2 and 3 we estimate the two sides of (5.2) separately, assuming that a sequence $E_\varepsilon=G^+_\varepsilon\cup G^-_\varepsilon$ of subsets of Ω exists such that (5.11) below is satisfied. The remaining part of the proof (Steps 4–6) is devoted to the construction of such a sequence.

Step 1: Blow-up. Up to a translation and rotation, it is not restrictive to assume that $0\in J^{\rm int}$, that $\nu_u(0)=e_1$, and that there exist $u^+,u^-\in\mathbb{R}^2$ with $(u^+-u^-)\cdot e_1<0$ such that

$$\lim_{\rho\to0}\rho^{-1}\,\mathcal{H}^1\big(J_u\cap Q_\rho\big)=1, \tag{5.3a}$$

$$\lim_{\rho\to0}\rho^{-2}\Big(\mathcal{L}^2\big(\{x\in Q^+_\rho:\ |u-u^+|>\epsilon\}\big)+\mathcal{L}^2\big(\{x\in Q^-_\rho:\ |u-u^-|>\epsilon\}\big)\Big)=0\quad\forall\epsilon>0, \tag{5.3b}$$

$$\lim_{\rho\to0}\rho^{-1}\int_{Q_\rho}|e(u)|^2\,\mathrm{d}x=0, \tag{5.3c}$$

where we recall that $Q^\pm_\rho=Q_\rho\cap\{\pm x\cdot e_1>0\}$. Indeed, (5.3) holds true for $\mathcal{H}^1$-a.e. $x\in J^{\rm int}$: property (5.3a) follows from the countably $\mathcal{H}^1$-rectifiability of $J^{\rm int}$, (5.3b) follows directly from the definition of J_u, and (5.3c) holds due to $|e(u)|^2\in L^1(\Omega)$.

For convenience, we denote the direction of the jump by $\mu:=(u^+-u^-)/|u^+-u^-|$. Recall that $\mu\cdot e_1<0$. We now pick a constant η sufficiently small whose choice will become clear along the proof. To this end, we first choose $\delta\in(0,1)$ sufficiently small such that

$$16\delta\le|\mu\cdot e_1|\qquad\text{and}\qquad 1\ge(1-\delta)\sqrt{1+9\delta^2}, \tag{5.4}$$

and then choose $\lambda\in(0,1)$ sufficiently small such that

$$\text{for each }\nu\in\mathbb{S}^1\text{ with }|\nu\cdot e_1|\ge1-\lambda,\text{ we have }\big||\mu\cdot e_1|-|\mu\cdot\nu|\big|\le\delta, \tag{5.5a}$$

$$2\sqrt{1-(1-\lambda)^2}\le\delta. \tag{5.5b}$$

For notational convenience, we indicate by C_0 a fixed constant with $C_0 \geq 10^3$. Eventually, we define $\eta \in (0,1)$ such that

$$\eta < \min\Big\{\theta_0,\ \frac{|u^+ - u^-||\mu \cdot e_1|}{16(C_0 N_2 + 1)}, \frac{1}{C_0^2}, \ \frac{\lambda\delta}{21}\Big\}, \tag{5.6}$$

where N_2 denotes the dimensional constant appearing in Besicovitch's covering theorem (see, e.g., [4, Theorem 2.18]), and θ_0 denotes the constant from Proposition 2.2.

By [23, Theorem 11.3] for every open subset A of Ω it holds that

$$\liminf_{\varepsilon\to 0} \|e(u_\varepsilon)\|^2_{L^2(A)} \geq \|e(u)\|^2_{L^2(A)} \qquad \text{and} \qquad \liminf_{\varepsilon\to 0} \mathcal{H}^1(J_{u_\varepsilon}\cap A) \geq \mathcal{H}^1(J_u \cap A).$$

Therefore, hypothesis (3.4b) implies

$$\lim_{\varepsilon\to 0} \|e(u_\varepsilon)\|_{L^2(A)} = \|e(u)\|_{L^2(A)} \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \mathcal{H}^1(J_{u_\varepsilon}\cap A) = \mathcal{H}^1(J_u \cap A) \tag{5.7}$$

for all $A \subset \Omega$ open with $\mathcal{H}^1(\partial A \cap J_u) = 0$. In view of (5.7) applied for $A = Q_\rho$, of (5.3) with $\epsilon = \frac{\eta}{\bar{c}_{\eta^3/2}}$, and of the fact that $u_\varepsilon \to u$ in measure, we can fix a particular $0 < \rho \leq 1$ with $\mathcal{H}^1(\partial Q_\rho \cap J_u) = 0$ such that for all ε sufficiently small we have

$$\mathcal{H}^1\big(J_{u_\varepsilon} \cap Q_\rho\big) \leq \rho(1+\eta), \tag{5.8a}$$

$$\mathcal{L}^2\Big(\Big\{x \in Q^+_\rho\colon\ |u_\varepsilon - u^+| > \frac{\eta}{\bar{c}_{\eta^3/2}}\Big\}\Big) + \mathcal{L}^2\Big(\Big\{x \in Q^-_\rho\colon\ |u_\varepsilon - u^-| > \frac{\eta}{\bar{c}_{\eta^3/2}}\Big\}\Big) \leq \rho^2\eta^4, \tag{5.8b}$$

$$\int_{Q_\rho} |e(u_\varepsilon)|^2 \,\mathrm{d}x \leq \frac{\rho\eta^2}{C_\eta^2 \bar{c}^2_{\eta^3/2}}, \tag{5.8c}$$

where $C_\eta \geq 1$ denotes the constant of Proposition 2.2 applied for $\theta = \eta$, and $\bar{c}_{\eta^3/2} \geq 1$ denotes the constant of Lemma 2.1 applied for $\theta = \eta^3/2$. In the following, without further notice, ε will always be chosen sufficiently small such that (5.8) holds. The strategy of the proof is to construct a measurable subset $E_\varepsilon \subset Q_\rho$ such that (5.2) holds, which is a contradiction to (5.1). To show (5.2), we now estimate separately its left- and right-hand side.

Step 2: Estimate on the determinant. In dimension two, a Taylor expansion implies that

$$\big|\det(\mathbf{Id} + F) - (1 + \mathrm{tr}(F))\big| \leq c_0|F|^2$$

for a universal constant $c_0 > 0$. For all measurable $E \subset Q_\rho$ this implies by (5.8c), the fact that $C_\eta \bar{c}_{\eta^3/2} \geq 1$, and Hölder's inequality that

$$\begin{aligned}\int_E \det(\nabla y_\varepsilon)\,\mathrm{d}x &\geq \mathcal{L}^2(E) - \int_E \sqrt{2}\varepsilon|e(u_\varepsilon)|\,\mathrm{d}x - c_0\int_E \varepsilon^2|\nabla u_\varepsilon|^2\,\mathrm{d}x \\ &\geq \mathcal{L}^2(E) - \sqrt{2}\varepsilon\big(\mathcal{L}^2(E)\big)^{1/2}\|e(u_\varepsilon)\|_{L^2(Q_\rho)} - c_0\int_\Omega \varepsilon^2|\nabla u_\varepsilon|^2\,\mathrm{d}x \\ &\geq \mathcal{L}^2(E) - \sqrt{2}\varepsilon\rho\sqrt{\rho}\eta - c_0\int_\Omega \varepsilon^2|\nabla u_\varepsilon|^2\,\mathrm{d}x.\end{aligned} \tag{5.9}$$

Note that $\varepsilon\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \to 0$ as $\varepsilon \to 0$ by (3.4a) and the fact that $\gamma > \frac{1}{2}$. Therefore, for all $\varepsilon > 0$ sufficiently small (depending on ρ and η), we deduce from (5.9) that

$$\int_E \det(\nabla y_\varepsilon)\,\mathrm{d}x \geq \mathcal{L}^2(E) - 2\varepsilon\rho^{3/2}\eta \geq \mathcal{L}^2(E) - 2\varepsilon\rho\eta, \tag{5.10}$$

where the last step follows from observing that $\rho \leq 1$.

Step 3: Estimate on the measure of the image and conclusion. By $\rho \le 1$, (5.6), and (5.8), we can apply Proposition 4.1 to $v = u_\varepsilon$ and $\omega^\pm = u^\pm$. We find two sets $D^\pm_\varepsilon$ satisfying (4.2) and the curves $\Gamma^\pm_\varepsilon$. Based on the definition of $D^\pm_\varepsilon$, we construct in Steps 3–6 two disjoint sets $G^\pm_\varepsilon \subset D^\pm_\varepsilon$ satisfying

$$\mathcal{L}^2\big(\{x \in \mathbb{R}^2 \setminus G^\pm_\varepsilon \colon \operatorname{dist}(x, G^\pm_\varepsilon) \le 3\eta\varepsilon\}\big) \le C_0 N_2 \eta\varepsilon\rho, \tag{5.11a}$$

$$\mathcal{L}^2\big((\varepsilon u^- + G^-_\varepsilon) \cup (\varepsilon u^+ + G^+_\varepsilon)\big) \le \mathcal{L}^2(G^+_\varepsilon) + \mathcal{L}^2(G^-_\varepsilon) - \frac{\rho}{8}\varepsilon|u^+ - u^-||\mu \cdot e_1|. \tag{5.11b}$$

Let us assume for the moment that such sets exist and let us explain how to conclude the contradiction. By (4.2a), (5.11), and the fact that $G^\pm_\varepsilon \subset D^\pm_\varepsilon$ we find that, for ε sufficiently small, the functions $y_\varepsilon = \mathbf{id} + \varepsilon u_\varepsilon$ satisfy

$$\begin{aligned} &\mathcal{L}^2([y_\varepsilon(G^+_\varepsilon \cup G^-_\varepsilon)]) \\ &\quad\le \mathcal{L}^2\Big(\{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, \varepsilon u^+ + G^+_\varepsilon) \le 3\eta\varepsilon\} \cup \{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, \varepsilon u^- + G^-_\varepsilon) \le 3\eta\varepsilon\}\Big) \\ &\quad\le \mathcal{L}^2\big((\varepsilon u^- + G^-_\varepsilon) \cup (\varepsilon u^+ + G^+_\varepsilon)\big) + 2C_0 N_2 \eta\varepsilon\rho \\ &\quad\le \mathcal{L}^2(G^+_\varepsilon) + \mathcal{L}^2(G^-_\varepsilon) - \frac{\rho}{8}\varepsilon|u^+ - u^-||\mu \cdot e_1| + 2C_0 N_2 \eta\varepsilon\rho. \end{aligned} \tag{5.12}$$

On the other hand, since (5.10) holds and $G^+_\varepsilon \cap G^-_\varepsilon = \emptyset$, we have

$$\int_{G^+_\varepsilon \cup G^-_\varepsilon} \det(\nabla y_\varepsilon)\,\mathrm{d}x \ge \mathcal{L}^2(G^+_\varepsilon) + \mathcal{L}^2(G^-_\varepsilon) - 2\varepsilon\rho\eta. \tag{5.13}$$

Combining (5.6) and (5.12)–(5.13) we infer that

$$\begin{aligned} \int_{G^+_\varepsilon \cup G^-_\varepsilon} \det(\nabla y_\varepsilon)\,\mathrm{d}x &\ge \mathcal{L}^2([y_\varepsilon(G^+_\varepsilon \cup G^-_\varepsilon)]) + \frac{\rho}{8}\varepsilon|u^+ - u^-||\mu \cdot e_1| - 2C_0 N_2 \eta\varepsilon\rho - 2\varepsilon\rho\eta \\ &> \mathcal{L}^2([y_\varepsilon(G^+_\varepsilon \cup G^-_\varepsilon)]). \end{aligned}$$

This shows (5.2) for $E_\varepsilon = G^+_\varepsilon \cup G^-_\varepsilon$ and the argument by contradiction is concluded. To conclude the proof, it remains to give the construction of the sets $G^\pm_\varepsilon$ and to prove the properties (5.11).

Step 4: Definition of $G^\pm_\varepsilon$. By Proposition 4.1 there exists a curve $\Gamma_\varepsilon \subseteq \partial^* D^+_\varepsilon$ which connects $(-\frac{\rho}{2}, \frac{\rho}{2}) \times \{\frac{\rho}{2}\}$ with $(-\frac{\rho}{2}, \frac{\rho}{2}) \times \{-\frac{\rho}{2}\}$. In particular, we assume that there exists a continuous curve $\gamma_\varepsilon \colon [a,b] \to \mathbb{R}^2$ with $\Gamma_\varepsilon = \gamma_\varepsilon([a,b])$, $\gamma_\varepsilon((a,b)) \subset Q_\rho$, $\gamma_\varepsilon(a) \in (-\frac{\rho}{2}, \frac{\rho}{2}) \times \{\frac{\rho}{2}\}$, and $\gamma_\varepsilon(b) \in (-\frac{\rho}{2}, \frac{\rho}{2}) \times \{-\frac{\rho}{2}\}$.

We define $F^+_\varepsilon := \operatorname{Int}(\Psi_\varepsilon)$ and $F^-_\varepsilon := Q_\rho \setminus F^+_\varepsilon$, where $\operatorname{Int}(\cdot)$ stands for the interior of a Jordan curve. We denote the connected components of $\operatorname{sat}(F^\pm_\varepsilon \setminus D^\pm_\varepsilon)$ by $(S^\pm_{i,\varepsilon})_i$, where $\operatorname{sat}(\cdot)$ indicates the saturation of a set. Note that each of these sets is simple, i.e., $\partial^* S^\pm_{i,\varepsilon}$ is equivalent to a rectifiable Jordan curve up to an $\mathcal{H}^1$-negligible set. We define

$$G^+_\varepsilon := F^+_\varepsilon \setminus \bigcup_i S^+_{i,\varepsilon}, \qquad G^-_\varepsilon := F^-_\varepsilon \setminus \bigcup_i S^-_{i,\varepsilon}. \tag{5.14}$$

By construction, we note that $\Gamma_\varepsilon \subset \partial^* G^+_\varepsilon \cap Q_\rho$ up to an $\mathcal{H}^1$-negligible set. Moreover, we have

$$\mathcal{H}^1(\Gamma_\varepsilon) \le \mathcal{H}^1\big((\partial^* G^+_\varepsilon \cup \partial^* G^-_\varepsilon) \cap Q_\rho\big) \le \rho + 7\rho\eta \le 2\rho, \tag{5.15a}$$

$$\mathcal{H}^1\big(\partial^* G^+_\varepsilon \cup \partial^* G^-_\varepsilon\big) \le 6\rho. \tag{5.15b}$$

Indeed, by [3, Proposition 6(ii)] and by (4.2b) and (5.8a) one can check that

$$\begin{aligned} \mathcal{H}^1\big((\partial^* G^+_\varepsilon \cup \partial^* G^-_\varepsilon) \cap Q_\rho\big) &\le \mathcal{H}^1\big((\partial^* D^+_\varepsilon \cup \partial^* D^-_\varepsilon) \cap Q_\rho\big) \\ &\le \mathcal{H}^1(J_{u_\varepsilon} \cap Q_\rho) + \mathcal{H}^1\big(((\partial^* D^+_\varepsilon \cup \partial^* D^-_\varepsilon) \setminus J_{u_\varepsilon}) \cap Q_\rho\big) \le \rho(1 + 7\eta). \end{aligned}$$

Then, (5.15a) follows from the fact that $\Gamma_\varepsilon \subset \partial^* G^+_\varepsilon \cap Q_\rho$ up to an $\mathcal{H}^1$-negligible set, and the fact that $7\eta \le 1$, see (5.6). To get (5.15b), we simply note that $\mathcal{H}^1(\partial Q_\rho) = 4\rho$.

Step 5: Proof of (5.11a). Without loss of generality, we show (5.11a) only for G^+_ε. The proof for G^-_ε works in the same way, up to a different notational realization. Let $\mathcal{S}^{\rm small}_\varepsilon := \{i\colon \mathcal{H}^1(\partial^* S^+_{i,\varepsilon}) \le 3\varepsilon\eta\}$ and $\mathcal{S}^{\rm big}_\varepsilon := \{i\colon \mathcal{H}^1(\partial^* S^+_{i,\varepsilon}) > 3\varepsilon\eta\}$. By (5.14) and the fact that $\partial^* F^+_\varepsilon$ and $(\partial^* S^+_{i,\varepsilon})_i$ are equivalent to rectifiable Jordan curves we get

$$\mathcal{L}^2\big(\{x \in \mathbb{R}^2 \setminus G^+_\varepsilon\colon \operatorname{dist}(x, G^+_\varepsilon) \le 3\eta\varepsilon\}\big) \le \sum_{i\in\mathcal{S}^{\rm small}_\varepsilon} \mathcal{L}^2(S^+_{i,\varepsilon}) + \sum_{i\in\mathcal{S}^{\rm big}_\varepsilon} \mathcal{L}^2\big(\{x\colon \operatorname{dist}(x, \partial^* S^+_{i,\varepsilon}) \le 3\varepsilon\eta\}\big) + \mathcal{L}^2\big(\{x\colon \operatorname{dist}(x, \partial^* F^+_\varepsilon) \le 3\varepsilon\eta\}\big). \tag{5.16}$$

We now estimate the terms on the right-hand side of (5.16) separately. For the first term, we recall the definition of $\mathcal{S}^{\rm small}_\varepsilon$ and use the isoperimetric inequality to get

$$\sum_{i\in\mathcal{S}^{\rm small}_\varepsilon} \mathcal{L}^2(S^+_{i,\varepsilon}) \le \frac{1}{4\pi} \sum_{i\in\mathcal{S}^{\rm small}_\varepsilon} \big(\mathcal{H}^1(\partial^* S^+_{i,\varepsilon})\big)^2 \le \frac{3\varepsilon\eta}{4\pi} \sum_{i\in\mathcal{S}^{\rm small}_\varepsilon} \mathcal{H}^1(\partial^* S^+_{i,\varepsilon}). \tag{5.17}$$

For the other two terms, we show that

$$\mathcal{L}^2\big(\{x\colon \operatorname{dist}(x, \partial^* S^+_{i,\varepsilon}) \le 3\varepsilon\eta\}\big) \le 40 N_2 \varepsilon\eta\, \mathcal{H}^1(\partial^* S^+_{i,\varepsilon}), \tag{5.18a}$$

$$\mathcal{L}^2\big(\{x\colon \operatorname{dist}(x, \partial^* F^+_\varepsilon) \le 3\varepsilon\eta\}\big) \le 40 N_2 \varepsilon\eta\, \mathcal{H}^1(\partial^* F^+_\varepsilon), \tag{5.18b}$$

where N_2 denotes the constant in the Besicovitch covering theorem. We first perform the proof for the sets $S^+_{i,\varepsilon}$. For notational simplicity, we set $\tilde{S}^+_{i,\varepsilon} := \{x\colon \operatorname{dist}(x, \partial^* S^+_{i,\varepsilon}) \le 3\varepsilon\eta\}$. We cover $\tilde{S}^+_{i,\varepsilon}$ by balls $B_{6\varepsilon\eta}(x)$ with $x \in \partial^* S^+_{i,\varepsilon}$. In particular, since $\partial^* S^+_{i,\varepsilon}$ is a rectifiable Jordan curve and $i \in \mathcal{S}^{\rm big}_\varepsilon$ we have that

$$\mathcal{H}^1(\partial^* S^+_{i,\varepsilon} \cap B_{3\varepsilon\eta}(x)) \ge 3\varepsilon\eta. \tag{5.19}$$

Then, by the Besicovitch covering theorem there exists a countable subcollection of balls $B_{3\varepsilon\eta}(x)$, $x \in \mathcal{X}_{i,\varepsilon}$, which cover $\partial^* \tilde{S}^+_{i,\varepsilon}$ up to an $\mathcal{H}^1$-negligible set such that each $y \in \mathbb{R}^2$ is contained in at most N_2 different balls. Clearly, $B_{6\varepsilon\eta}(x)$, $x \in \mathcal{X}_{i,\varepsilon}$, then covers $\tilde{S}^+_{i,\varepsilon}$ up to a set of negligible $\mathcal{L}^2$-measure. Therefore, by (5.19) we compute

$$\mathcal{L}^2\big(\tilde{S}^+_{i,\varepsilon}\big) \le \sum_{x\in\mathcal{X}_{i,\varepsilon}} \mathcal{L}^2(B_{6\varepsilon\eta}(x)) \le \sum_{x\in\mathcal{X}_{i,\varepsilon}} 12\pi\varepsilon\eta\mathcal{H}^1(B_{3\varepsilon\eta}(x) \cap \partial^* S^+_{i,\varepsilon}) \le 12 N_2 \pi\varepsilon\eta\mathcal{H}^1(\partial^* S^+_{i,\varepsilon}).$$

This shows (5.18) for the sets $S^+_{i,\varepsilon}$. The proof of (5.18b) for the set F^+_ε works in the same way once we notice that $\mathcal{H}^1(\partial^* F^+_\varepsilon) \ge \rho > 3\varepsilon\eta$ for ε sufficiently small.

Eventually, we conclude the proof of (5.11a) by combining (5.15)–(5.18).

Step 6: Proof of (5.11b). We recall that $\mu = (u^+ - u^-)/|u^+ - u^-|$ satisfies $\mu \cdot e_1 < 0$ and let $\tau := |u^+ - u^-|$ for brevity. We also write $\Lambda_\varepsilon := (\partial^* G^+_\varepsilon \cup \partial^* G^-_\varepsilon) \cap Q_\rho$ and recall that the curve Γ_ε (without its endpoints) is contained in Λ_ε. We also recall the notation in (2.1)–(2.2). The main point of this step is to prove the estimate

$$\mathcal{H}^1\big(V^\mu_\varepsilon\big) \ge \frac{\rho}{8}|\mu \cdot e_1|, \text{ where } V^\mu_\varepsilon := \big\{w \in \Pi^\mu\colon \mathcal{H}^0((\Gamma_\varepsilon \cap Q_{\rho-2\tau\varepsilon})^\mu_w) = 1 \text{ and } \mathcal{H}^0((\Lambda_\varepsilon)^\mu_w) = 1\big\}. \tag{5.20}$$

The set V^μ_ε (see Figure 3) corresponds to the vectors $w \in \Pi^\mu$ such that there exists a unique $t_w \in \mathbb{R}$ with $w + t\mu \in \Gamma_\varepsilon$, $w + t\mu \in G^+_\varepsilon$ for $t \in (t_w - \tau\varepsilon, t_w)$, and $w + t_w\mu \in G^-_\varepsilon$ for $t \in (t_w, t_w + \tau\varepsilon)$. This estimate implies (5.11b). In fact, for $w \in \Pi^\mu$, the two sets

$$\{w + t\mu + \varepsilon u^+ \colon t \in (t_w - \tau\varepsilon, t_w)\} \quad \text{and} \quad \{w + t\mu + \varepsilon u^- \colon t \in (t_w, t_w + \tau\varepsilon)\}$$

coincide, where we used that $u^+ - u^- = \tau\mu$. This implies

$$\mathcal{L}^2\big((\varepsilon u^- + G_\varepsilon^-) \cup (\varepsilon u^+ + G_\varepsilon^+)\big) \le \mathcal{L}^2(G_\varepsilon^+) + \mathcal{L}^2(G_\varepsilon^-) - \varepsilon\tau\mathcal{H}^1(V_\varepsilon^\mu)\,.$$

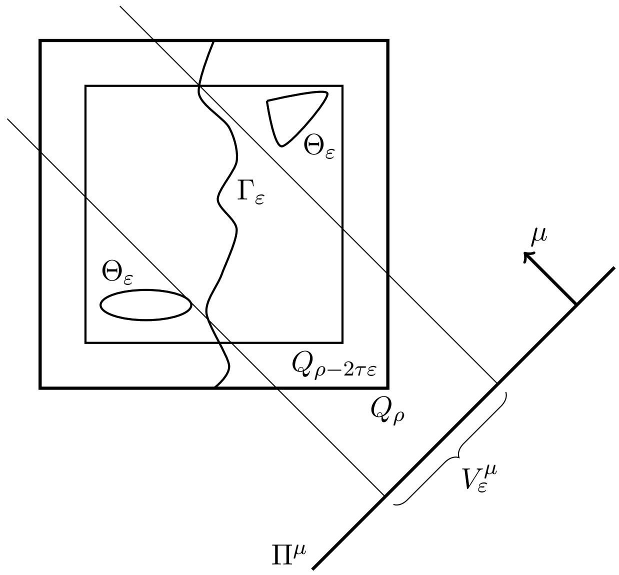

FIGURE 3. Visualization of V_ε^μ. Here, $\Theta_\varepsilon = \Lambda_\varepsilon \setminus \Gamma_\varepsilon$.

We are hence left to prove (5.20). To this end, we recall the definition of δ and λ in (5.4)–(5.5). We define $\Gamma'_\varepsilon := \{x \in \Gamma_\varepsilon\colon |\nu_{\Gamma_\varepsilon}(x)\cdot e_1| \ge 1-\lambda\}$, where ν_{Γ_ε} denotes a unit normal vector of the curve Γ_ε. We start by observing that

$$\mathcal{H}^1\big(\Gamma_\varepsilon \setminus \Gamma'_\varepsilon\big) \le 7\rho\eta/\lambda \le \rho\delta. \tag{5.21}$$

Indeed, by the area formula, by (5.15a), and by the fact that Γ_ε connects $(-\frac{\rho}{2},\frac{\rho}{2})\times\{\frac{\rho}{2}\}$ with $(-\frac{\rho}{2},\frac{\rho}{2})\times\{-\frac{\rho}{2}\}$ we calculate

$$\begin{aligned}\rho \le \int_{\Pi^{e_1}} \mathcal{H}^0((\Gamma_\varepsilon)^{e_1}_w)\,\mathrm{d}\mathcal{H}^1(w) &= \int_{\Gamma_\varepsilon} |\nu_{\Gamma_\varepsilon}\cdot e_1|\,\mathrm{d}\mathcal{H}^1 \le \mathcal{H}^1(\Gamma'_\varepsilon) + (1-\lambda)\mathcal{H}^1(\Gamma_\varepsilon\setminus\Gamma'_\varepsilon)\\ &= \mathcal{H}^1(\Gamma_\varepsilon) - \lambda\mathcal{H}^1(\Gamma_\varepsilon\setminus\Gamma'_\varepsilon) \le \rho + 7\rho\eta - \lambda\mathcal{H}^1(\Gamma_\varepsilon\setminus\Gamma'_\varepsilon).\end{aligned}$$

This along with $7\eta/\lambda \le \delta$ (see (5.6)) shows (5.21). Now, (5.21) and the definition of Γ'_ε particularly imply that

$$\sup\nolimits_{x_1,x_2\in\Gamma_\varepsilon} |(x_1-x_2)\cdot e_1| \le \mathcal{H}^1(\Gamma'_\varepsilon)\sqrt{1-(1-\lambda)^2} + \mathcal{H}^1(\Gamma_\varepsilon\setminus\Gamma'_\varepsilon) \le 2\delta\rho, \tag{5.22}$$

where in the last step we also used (5.5b) and (5.15a). Thus, we can choose two points $z_+^\varepsilon, z_-^\varepsilon \in \Gamma_\varepsilon$ with $z_\pm^\varepsilon \cdot e_2 = \pm(\frac{\rho}{2} - \tau\varepsilon)$ such that the segment connecting z_+^ε and z_-^ε, denoted by σ^ε, satisfies $\pi_\mu(\sigma^\varepsilon) \le \pi_\mu(\Gamma_\varepsilon \cap Q_{\rho-2\tau\varepsilon})$. (Recall notation (2.2).) Clearly, $\rho - 2\tau\varepsilon \le \mathcal{H}^1(\sigma^\varepsilon) \le \mathcal{H}^1(\Gamma_\varepsilon) \le \rho + 7\eta\rho$ by (5.15a) and $\frac{|\nu_{\sigma^\varepsilon}\cdot e_2|}{|\nu_{\sigma^\varepsilon}\cdot e_1|} \le \frac{2\delta\rho}{\rho-2\tau\varepsilon} \le 3\delta$ by (5.22), provided that ε is small enough. By (5.4), the latter also yields $|\nu_{\sigma^\varepsilon}\cdot e_1| \ge \frac{1}{\sqrt{1+9\delta^2}} \ge 1-\delta$. For ε sufficiently small, this gives

$$\begin{aligned}\mathcal{L}^1\big(\pi_\mu(\Gamma_\varepsilon\cap Q_{\rho-2\tau\varepsilon})\big) &\ge \mathcal{L}^1\big(\pi_\mu(\sigma^\varepsilon)\big) = \mathcal{L}^1(\sigma^\varepsilon)|\mu\cdot\nu_{\sigma^\varepsilon}| \ge (\rho-2\tau\varepsilon)\big(|\mu\cdot e_1||\nu_{\sigma^\varepsilon}\cdot e_1| - |\nu_{\sigma^\varepsilon}\cdot e_2|\big)\\ &\ge (\rho-2\tau\varepsilon)|\nu_{\sigma^\varepsilon}\cdot e_1|\big(|\mu\cdot e_1| - 3\delta\big) \ge \rho|\mu\cdot e_1| - 4\rho\delta \ge \frac{3\rho}{4}|\mu\cdot e_1|,\end{aligned} \tag{5.23}$$

where the last step follows from (5.4). By (5.15a), (5.21), the inclusion $\Gamma_\varepsilon \subset \Lambda_\varepsilon$ (up to the endpoints), and the fact that $\mathcal{H}^1(\Gamma_\varepsilon) \geq \rho$ we find $\mathcal{H}^1(\Lambda_\varepsilon \setminus \Gamma'_\varepsilon) \leq (7\eta + \delta)\rho$, where $\Lambda_\varepsilon = (\partial^* G^+_\varepsilon \cup \partial^* G^-_\varepsilon) \cap Q_\rho$. Recalling again the definition of Γ'_ε and using $\mathcal{H}^1(\Gamma'_\varepsilon) \leq \rho + 7\eta\rho$ (see (5.15a)), we get by (5.5a) and the area formula

$$\begin{aligned}\int_{\Pi^\mu} \mathcal{H}^0\big((\Lambda_\varepsilon)^\mu_w\big)\,\mathrm{d}\mathcal{H}^1(w) &= \int_{\Lambda_\varepsilon} |\nu_{\Lambda_\varepsilon}\cdot\mu|\,\mathrm{d}\mathcal{H}^1 \leq (|\mu\cdot e_1| + \delta)\mathcal{H}^1(\Gamma'_\varepsilon) + \mathcal{H}^1(\Lambda_\varepsilon\setminus\Gamma'_\varepsilon)\\ &\leq \rho|\mu\cdot e_1| + 21\eta\rho + 2\rho\delta \leq \rho|\mu\cdot e_1| + 4\rho\delta \leq \frac{5\rho}{4}|\mu\cdot e_1|,\end{aligned} \tag{5.24}$$

where in the last steps we used (5.4) and (5.6). Here, $\nu_{\Lambda_\varepsilon}$ denotes a unit normal of Λ_ε. Consequently, by (5.23)–(5.24) and the fact that $\Gamma_\varepsilon \subset \Lambda_\varepsilon$ (up to the endpoints) we conclude

$$\mathcal{L}^1\Big(\big\{w\colon \mathcal{H}^0((\Gamma_\varepsilon\cap Q_{\rho-2\tau\varepsilon})^\mu_w) = 1 \text{ and } \mathcal{H}^0((\Lambda_\varepsilon)^\mu_w) = 1\big\}\Big) \geq \frac{\rho}{8}|\mu\cdot e_1|.$$

This shows (5.20) and concludes the proof of the theorem. □

Proof of Theorem 3.11. As (CC) is a local condition, it is enough to prove the statement on any Lipschitz set $\Omega_u \subset\subset \Omega'\setminus E_u$ with $\mathcal{H}^1(J_u\cap\partial\Omega_u) = 0$. In view of Theorem 3.6 (applied on Ω_u) and Definition 3.8, in particular (3.11b), it suffices to prove that (3.4b) holds for the rescaled displacements u_ε defined in (3.10) (for Ω_u in place of Ω'). To this end, we recall that in the proof of the Γ-liminf inequality in [32], see particularly [32, (4.16)ff.], it has been shown that

$$\liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon)\,\mathrm{d}x \geq \liminf_{\varepsilon\to0}\int_{\Omega'}\chi_\varepsilon\frac12 Q(e(u_\varepsilon))\,\mathrm{d}x \geq \int_{\Omega'}\frac12 Q(e(u))\,\mathrm{d}x\,,$$

where $(\chi_\varepsilon)_\varepsilon$ is a sequence of indicator functions satisfying $\chi_\varepsilon \to 1$ in measure on Ω'. More specifically, $\chi_\varepsilon(x) = \chi_{[0,\eta_\varepsilon)}(|\nabla u_\varepsilon|(x))$ for a sequence $\eta_\varepsilon\to+\infty$ with $\varepsilon^{1-\gamma}\eta_\varepsilon\to+\infty$. By the same argument, for any open $A\subset\Omega'$ and any second sequence $(\bar\chi_\varepsilon)_\varepsilon$ of indicator functions with $\bar\chi_\varepsilon\to1$ in measure on A it still holds

$$\liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2}\int_A\bar\chi_\varepsilon W(\nabla y_\varepsilon)\,dx \geq \liminf_{\varepsilon\to0}\int_A\chi_\varepsilon\bar\chi_\varepsilon\frac12 Q(e(u_\varepsilon))\,\mathrm{d}x \geq \int_A\frac12 Q(e(u))\,\mathrm{d}x\,. \tag{5.25}$$

By (5.25) for $\bar\chi_\varepsilon\equiv1$ and $A\in\{\Omega_u,\Omega'\setminus\overline{\Omega_u}\}$, since $\mathcal{E}_\varepsilon(y_\varepsilon)\to\mathcal{E}(u)$, and $\mathcal{H}^1(J_u\cap\partial\Omega_u)=0$, by (3.11f) we have

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^2}\int_{\Omega_u}W(\nabla y_\varepsilon)\,\mathrm{d}x = \int_{\Omega_u}\frac12 Q(e(u))\,\mathrm{d}x\,,\qquad \lim_{\varepsilon\to0}\mathcal{H}^1(J_{u_\varepsilon}\cap\Omega_u) = \mathcal{H}^1(J_u\cap\Omega_u). \tag{5.26}$$

It remains to show $\|e(u_\varepsilon)\|_{L^2(\Omega_u)}\to\|e(u)\|_{L^2(\Omega_u)}$. To see this, we will use an argument based on equiintegrability, related to the one in [51, Proof of Theorem 2.3].

As a preliminary step, we check that the sequence $g_\varepsilon\colon\Omega_u\to\mathbb{R}$ given by $g_\varepsilon := \frac{1}{\varepsilon^2}\operatorname{dist}^2(\nabla y_\varepsilon, SO(2))$ is equiintegrable. In fact, if the statement were wrong, we would get

$$\lim_{M\to\infty}\limsup_{\varepsilon\to0}\int_{\{g_\varepsilon>M\}} g_\varepsilon\,\mathrm{d}x \geq \kappa$$

for some $\kappa>0$. Then, by a diagonal argument we can choose a sequence $(M_\varepsilon)_\varepsilon$ with $M_\varepsilon\to+\infty$ such that

$$\liminf_{\varepsilon\to0}\int_{\{g_\varepsilon>M_\varepsilon\}} g_\varepsilon\,\mathrm{d}x \geq \kappa. \tag{5.27}$$

Define $\bar{\chi}_\varepsilon := \chi_{\{g_\varepsilon \le M_\varepsilon\}}$, and note that $\bar{\chi}_\varepsilon \to 1$ in measure on Ω_u by $\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(y_\varepsilon) < +\infty$, (3.5c), and $M_\varepsilon \to +\infty$. Thus, by using (3.5c), (5.25) for $A = \Omega_u$, and (5.27) we calculate

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega_u} W(\nabla y_\varepsilon)\,\mathrm{d}x \ge \liminf_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon^2}\int_{\Omega_u} \bar{\chi}_\varepsilon W(\nabla y_\varepsilon)\,\mathrm{d}x + \int_{\Omega_u}(1-\bar{\chi}_\varepsilon) c g_\varepsilon\,\mathrm{d}x\Big)$$
$$\ge \int_{\Omega_u} \frac{1}{2}Q(e(u))\,\mathrm{d}x + c\kappa.$$

This, however, contradicts (5.26), and shows that g_ε is equiintegrable on Ω_u.

Next, we show that $|e(u_\varepsilon)|^2$ is equiintegrable on Ω_u. To this end, by using the linearization formula $|\mathrm{sym}(F - \mathbf{Id})| = \mathrm{dist}(F, SO(d)) + \mathrm{O}(|F-\mathbf{Id}|^2)$ (see [32, (4.12)]) and (3.11c), we get for each $x \in \Omega_u$ satisfying $\mathrm{dist}(\nabla y_\varepsilon(x), SO(2)) \le 1$ that

$$|e(u_\varepsilon)(x)|^2 \le C g_\varepsilon(x) + C\varepsilon^{-2}|\nabla y^{\mathrm{rot}}_\varepsilon(x) - \mathbf{Id}|^4 \le Cg_\varepsilon(x) + C\varepsilon^{-2}\big(\varepsilon^{4\gamma} + \mathrm{dist}^4(\nabla y_\varepsilon(x), SO(2))\big)$$
$$\le Cg_\varepsilon(x) + C\varepsilon^{-2}\,\mathrm{dist}^2(\nabla y_\varepsilon(x), SO(2)) + C \le Cg_\varepsilon(x) + C,$$

where we used $\gamma \ge \frac{1}{2}$. On the other hand, if $\mathrm{dist}(\nabla y_\varepsilon(x), SO(2)) > 1$, we easily find

$$|e(u_\varepsilon)(x)|^2 \le \varepsilon^{-2}|\nabla y^{\mathrm{rot}}_\varepsilon(x) - \mathbf{Id}|^2 \le C\varepsilon^{-2}\,\mathrm{dist}^2(\nabla y_\varepsilon(x), SO(2)) = Cg_\varepsilon$$

for a sufficiently large universal constant $C > 0$. Combining both estimates, we get that $|e(u_\varepsilon)|^2$ is equiintegrable since g_ε is equiintegrable.

Moreover, (3.11b) implies

$$\lim_{\varepsilon\to 0}\mathcal{L}^2(\{|\nabla u_\varepsilon| \ge \eta_\varepsilon\}\cap\Omega_u) = \lim_{\varepsilon\to 0}\mathcal{L}^2(\{|\nabla y^{\mathrm{rot}}_\varepsilon - \mathbf{Id}| \ge \varepsilon\eta_\varepsilon\}\cap\Omega_u) = 0, \tag{5.28}$$

where we used that $\varepsilon^{1-\gamma}\eta_\varepsilon \to +\infty$. We now conclude as follows. By (5.25) for $A = \Omega_u$ and $\bar{\chi}_\varepsilon \equiv 1$, by the equiintegrability of $|e(u_\varepsilon)|^2$, and (5.28) we get

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega_u} W(\nabla y_\varepsilon)\,\mathrm{d}x \ge \liminf_{\varepsilon\to 0}\int_{\Omega_u\cap\{|\nabla u_\varepsilon|<\eta_\varepsilon\}}\frac{1}{2}Q(e(u_\varepsilon))\,\mathrm{d}x = \liminf_{\varepsilon\to 0}\int_{\Omega_u}\frac{1}{2}Q(e(u_\varepsilon))\,\mathrm{d}x\,.$$

This along with (3.11e), (5.26), and the fact that Q is positive definite on $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ implies

$$\int_{\Omega_u}\frac{1}{2}Q(e(u))\,\mathrm{d}x = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega_u} W(\nabla y_\varepsilon)\,\mathrm{d}x \ge \liminf_{\varepsilon\to 0}\int_{\Omega_u}\frac{1}{2}Q(e(u_\varepsilon))\,\mathrm{d}x \ge \int_{\Omega_u}\frac{1}{2}Q(e(u))\,\mathrm{d}x\,.$$

This yields convergence of the linearized energies which together with weak convergence shows the strong convergence $\|e(u_\varepsilon)\|_{L^2(\Omega_u)} \to \|e(u)\|_{L^2(\Omega_u)}$. This concludes the proof. □

6. Proof of Theorem 3.12

This section is devoted to the proof of Theorem 3.12. We start by a preliminary approximation result which allows us to strengthen the contact condition.

Lemma 6.1 (Stronger contact condition)**.** *Let $\Omega \subset \Omega' \subset \mathbb{R}^2$ be bounded Lipschitz domains satisfying (2.9)–(2.10). Given $h \in W^{r,\infty}(\Omega;\mathbb{R}^2)$ for $r \in \mathbb{N}$, let $u \in GSBD^2_h(\Omega')$ satisfy (CC). Then,* *there exist sequences* $(\tau_n)_n$ *in* $(0,+\infty)$ *and* $(u_n)_n$ *in* $GSBD^2_h(\Omega')$ *such that*

$$u_n \to u \text{ in measure on } \Omega', \tag{6.1a}$$

$$\lim_{n\to\infty} \|e(u_n)-e(u)\|_{L^2(\Omega')} = 0, \tag{6.1b}$$

$$\lim_{n\to\infty} \mathcal{H}^1(J_{u_n}) = \mathcal{H}^1(J_u), \tag{6.1c}$$

$$\lim_{n\to\infty} \mathcal{H}^1\big(\{x\in J_{u_n}\colon [u_n](x)\cdot\nu_{u_n}(x)\le 2\tau_n\}\big) = 0. \tag{6.1d}$$

Proof. Fix $0<\theta\le\frac{1}{2}$. It suffices to construct a function $\bar u\in GSBD^2_h(\Omega')$ such that

$$\|u-\bar u\|_{L^\infty(\Omega')} + \|e(u)-e(\bar u)\|_{L^2(\Omega')} + \mathcal{H}^1(J_u\triangle J_{\bar u}) \le c(1+\mathcal{H}^1(J_u))\theta \tag{6.2}$$

for some universal $c>0$, and such that for some $\bar\tau>0$ we have

$$\mathcal{H}^1\big(\{x\in J_{\bar u}\colon [\bar u](x)\cdot\nu_{\bar u}(x)\le 2\bar\tau\}\big) \le c(1+\mathcal{H}^1(J_u))\theta. \tag{6.3}$$

Then the result follows by considering a sequence $(\theta_n)_n$ converging to 0.

We start by using the fact that J_u is countably $\mathcal{H}^1$-rectifiable: arguing as in, e.g., [16, Proof of Theorem 2] or [18, Proof of Theorem 1.1], we infer that for $\mathcal{H}^1$-a.e. $x_0\in J_u$ there exist the approximate unit normal $\nu_u(x_0)\in\mathbb{S}^1$ to J_u at x_0, a positive number $\overline\rho(x_0)\in(0,\theta^3)$, and a curve Γ_{x_0} such that the following properties hold: Γ_{x_0} is the graph of a C^1 and Lipschitz function with $x_0\in\Gamma_{x_0}$, for every $\rho<\overline\rho(x_0)$ the curve $\Gamma_{x_0}\cap B_\rho(x_0)$ separates $B_\rho(x_0)$ in two open connected components $B^{\Gamma,\pm}_\rho(x_0)$, and

$$\mathcal{H}^1\big(J_u\cap B_\rho(x_0)\big)\ge(1-\theta)2\rho, \tag{6.4a}$$

$$\mathcal{H}^1\big((J_u\triangle\Gamma_{x_0})\cap B_\rho(x_0)\big)\le\theta\rho, \tag{6.4b}$$

$$\mathcal{H}^1\big(J_u\cap(B_\rho(x_0)\setminus B_{(1-\theta)\rho}(x_0))\big)\le 3\theta\rho,\quad \mathcal{H}^1\big(\Gamma_{x_0}\cap(B_\rho(x_0)\setminus B_{(1-\theta)\rho}(x_0))\big)\le 3\theta\rho \tag{6.4c}$$

$$\nu_{\Gamma_{x_0}}\cdot\nu_u(x_0)>1-\theta \quad \text{ on } \Gamma_{x_0}\cap B_\rho(x_0), \tag{6.4d}$$

where $\nu_{\Gamma_{x_0}}(x)$ denotes the outer normal to $\partial B^{\Gamma,-}_\rho(x_0)\cap\Gamma_{x_0}$ at x. Moreover, for each $\rho<\overline\rho(x_0)$, we have

$$B_\rho(x_0)\subset\Omega \quad \text{ if } x_0\in J_u\cap\Omega, \qquad B^{\Gamma,+}_\rho(x_0)\subset\Omega \quad \text{ if } x_0\in J_u\cap\partial\Omega, \tag{6.5}$$

where in the latter case $\nu_u(x_0)$ corresponds to the inner unit normal at $x_0\in\partial\Omega$.

For $x\in J_u$ and $\rho\in(0,\overline\rho(x))$, the balls $B_\rho(x)$ are a fine cover of J_u up to a set of negligible $\mathcal{H}^1$-measure. By applying Besicovitch's covering theorem to this fine cover, we find a finite number of pairwise disjoint balls $B_{\rho_i}(x_i)$, $i=1,\dots,m$, such that $x_i\in J_u$, (6.4)–(6.5) hold, and

$$\mathcal{H}^1\Big(J_u\setminus\bigcup_{i=1}^m B_{\rho_i}(x_i)\Big)\le\theta. \tag{6.6}$$

We consider $\varphi\in C^\infty_c(B_1(0))$ with $\varphi\equiv1$ on $B_{1-\theta}(0)$, $\|\varphi\|_\infty=1$, and $\|\nabla\varphi\|_\infty\le c\theta^{-1}$ for some $c>0$. We define the function

$$\bar u(x) := u(x)+\sum_{i=1}^m \rho_i\,\varphi\big((x-x_i)/\rho_i\big)\,\nu_u(x_i)\,\chi_{B^{\Gamma,+}_{\rho_i}(x_i)}(x) \quad \text{ for } x\in\Omega'. \tag{6.7}$$

We start by observing that (6.5) implies $\bar u=u$ on $\Omega'\setminus\overline\Omega$, and thus $\bar u\in GSBD^2_h(\Omega')$. Let us now check (6.2)–(6.3). First, since $\|\varphi\|_\infty\le1$ and the balls $B_{\rho_i}(x_i)$ are pairwise disjoint, we clearly have

$$\|u-\bar u\|_{L^\infty(\Omega')}\le\max_i\rho_i\le\theta^3\le\theta, \tag{6.8}$$

where we used $\max_i \rho_i \leq \theta^3$. By a change of variables, (6.4a), and $\|\nabla\varphi\|_\infty \leq c\theta^{-1}$ we further get

$$\begin{aligned}\|e(\bar{u}) - e(u)\|^2_{L^2(\Omega')} &\leq \sum_{i=1}^m \int_{B^{\Gamma_i,+}_{\rho_i}(x_i)} |\nabla\varphi\big((x-x_i)/\rho_i\big)|^2 \,\mathrm{d}x \\ &\leq \sum_{i=1}^m \rho_i^2 \int_{B_1(0)} |\nabla\varphi(x)|^2 \,\mathrm{d}x \\ &\leq \frac{\theta^3}{2(1-\theta)} \|\nabla\varphi\|^2_{L^2(B_1(0))} \textstyle\sum_{i=1}^m \mathcal{H}^1(J_u \cap B_{\rho_i}(x_i)) \\ &\leq c\theta\mathcal{H}^1(J_u)\end{aligned} \tag{6.9}$$

for a universal constant $c > 0$. Up to slightly altering the values of ρ_i, we can suppose that $\mathcal{H}^1(J_u \setminus J_{\bar{u}}) = 0$. As $J_{\bar{u}} \setminus J_u \subset \bigcup_{i=1}^m (\Gamma_{x_i} \setminus J_u) \cap B_{\rho_i}(x_i)$, (6.4a)–(6.4b) imply

$$\mathcal{H}^1(J_u \triangle J_{\bar{u}}) \leq \sum_{i=1}^m \theta\rho_i \leq c\theta\mathcal{H}^1(J_u). \tag{6.10}$$

Combining (6.8)–(6.10) we conclude (6.2).

We now show (6.3). First, for $i = 1, \dots, m$ and for $\mathcal{H}^1$-a.e. $x \in \big(J_{\bar{u}} \cap J_u \cap \Gamma_{x_i}\big) \cap B_{(1-\theta)\rho_i}(x_i)$ we find $\nu_{\bar{u}}(x) = \nu_u(x) = \nu_{\Gamma_{x_i}}(x)$. Thus, by (6.4d), by (6.7), and by (CC), we get for $\mathcal{H}^1$-a.e. $x \in \bigcup_{i=1}^m \big(J_{\bar{u}} \cap J_u \cap \Gamma_{x_i}\big) \cap B_{(1-\theta)\rho_i}(x_i)$

$$[\bar{u}](x) \cdot \nu_{\bar{u}}(x) = [u](x) \cdot \nu_u(x) + \big([\bar{u}](x) - [u](x)\big) \cdot \nu_{\Gamma_{x_i}}(x) \geq \rho_i \nu_u(x_i) \cdot \nu_{\Gamma_{x_i}}(x) > (1-\theta)\rho_i. \tag{6.11}$$

On the other hand, we have

$$\begin{aligned}\mathcal{H}^1\Big(J_{\bar{u}} \setminus \bigcup_{i=1}^m \big(J_u \cap \Gamma_{x_i} \cap B_{(1-\theta)\rho_i}(x_i)\big)\Big) &\leq \mathcal{H}^1\Big(J_u \setminus \bigcup_{i=1}^m B_{\rho_i}(x_i)\Big) + \sum_{i=1}^m \mathcal{H}^1\big((J_u \triangle \Gamma_{x_i}) \cap B_{(1-\theta)\rho_i}(x_i)\big) \\ &\quad + \sum_{i=1}^m \mathcal{H}^1\big((J_u \cup \Gamma_{x_i}) \cap \big(B_{\rho_i}(x_i) \setminus B_{(1-\theta)\rho_i}(x_i)\big)\big).\end{aligned}$$

Then, by (6.4a)–(6.4c) and (6.6) we conclude

$$\mathcal{H}^1\Big(J_{\bar{u}} \setminus \bigcup_{i=1}^m \big(J_u \cap \Gamma_{x_i} \cap B_{(1-\theta)\rho_i}(x_i)\big)\Big) \leq \theta + \sum_{i=1}^m 7\theta\rho_i \leq \theta + c\theta\mathcal{H}^1(J_u).$$

This along with (6.11) shows that (6.3) holds for $\bar{\tau} = \frac{1}{2}(1-\theta)\min_i \rho_i$. □

We now provide an adaption of the $GSBD^2$-density result stated in Theorem 2.3 which guarantees the contact condition up to a part of the jump set with small $\mathcal{H}^1$-measure.

Theorem 6.2 (Density with boundary data and contact condition)**.** *Let $\Omega \subset \Omega' \subset \mathbb{R}^2$ be bounded Lipschitz domains satisfying (2.9)–(2.10). Let $\theta > 0$, $\tau > 0$, $h \in W^{r,\infty}(\Omega')$ for $r \in \mathbb{N}$, and let $u \in GSBD^2_h(\Omega')$ satisfy*

$$\mathcal{H}^1\big(\{x \in J_u \colon\ [u](x) \cdot \nu_u(x) \leq 2\tau\}\big) \leq \theta. \tag{6.12}$$

Then, there exist a sequence of functions $(u_n)_n$ in $SBV^2(\Omega;\mathbb{R}^2)$, a sequence of neighborhoods $(U_n)_n$ of $\Omega' \setminus \Omega$, and a sequence of neighborhoods $(\Omega_n)_n$ of $\Omega \setminus U_n$ such that $U_n \subset \Omega'$, $\Omega_n \subset \Omega$, $u_n = h$

on $\Omega' \setminus \overline{\Omega}$, $u_n|_{U_n} \in W^{r,\infty}(U_n;\mathbb{R}^2)$, $u_n|_{\Omega_n} \in \mathcal{W}(\Omega_n;\mathbb{R}^2)$, *and*

$$u_n \to u \text{ in measure on } \Omega' \text{ as } n \to \infty, \tag{6.13a}$$

$$\lim_{n\to\infty} \|e(u_n) - e(u)\|_{L^2(\Omega')} = 0, \tag{6.13b}$$

$$\lim_{n\to\infty} \mathcal{H}^1(J_{u_n}) = \mathcal{H}^1(J_u), \tag{6.13c}$$

$$\limsup_{n\to\infty} \mathcal{H}^1\big(\{x \in J_{u_n}\colon\ [u_n](x)\cdot\nu_{u_n}(x) \le \tau\}\big) \le 3\theta. \tag{6.13d}$$

In particular, $u_n \in W^{r,\infty}(\Omega \setminus J_{u_n};\mathbb{R}^2)$ *for all* $n \in \mathbb{N}$.

Proof. Given $\theta > 0$ and $u \in GSBD^2_h(\Omega')$ as in the statement, we apply Theorem 2.3 to u to obtain an approximating sequence $(u_n)_n \subset SBV^2(\Omega;\mathbb{R}^2)$ satisfying the properties (6.13a)–(6.13c). Note also that it is not restrictive to assume that $\mathcal{H}^1(J_u) > 0$. Otherwise, (6.13d) would follow directly from (6.13c). By defining $J^{\rm bad}_{u_n} := \{x \in J_{u_n}\colon\ [u_n](x)\cdot\nu_{u_n}(x) \le \tau\}$, we see that to conclude (6.13d) we need to show that

$$\limsup_{n\to\infty} \mathcal{H}^1(J^{\rm bad}_{u_n}) \le 3\theta. \tag{6.14}$$

Let us fix $\zeta > 0$ sufficiently large such that $\mathcal{H}^1(\{x \in J_u\colon\ |[u](x)| < \zeta^{-1} \text{ or } |[u](x)| > \zeta) \le \theta$ and let us set $J^{\rm good}_u := \{x \in J_u\colon\ [u](x)\cdot\nu_u(x) > 2\tau,\ \zeta^{-1} \le |[u](x)| \le \zeta\}$. By (6.12) we get

$$\mathcal{H}^1(J_u \setminus J^{\rm good}_u) \le 2\theta. \tag{6.15}$$

Let us also fix $\lambda \in (0,1)$ such that

$$\lambda \le \frac{\tau}{2\zeta} \tag{6.16}$$

and $\eta \in (0,1)$ such that

$$\eta < \min\Big\{\frac{\lambda^2\theta}{56(\lambda^2+1)\mathcal{H}^1(J_u)}, \frac{\tau}{12}, \frac{1}{7\zeta}, \theta_0, 10^{-4}\Big\}, \tag{6.17}$$

with θ_0 from Proposition 2.2. The choice of λ and η will become clear along the proof.

Step 1: Blow-up. We now introduce a covering of $J^{\rm good}_u$: for $\mathcal{H}^1$-a.e. $x \in J^{\rm good}_u$, we find $\nu_u(x) \in \mathbb{S}^1$, $u^+_x, u^-_x \in \mathbb{R}^2$, and $0 < \bar\rho(x) \le 1$ such that for all $0 < \rho < \bar\rho(x)$ it holds that

$$|\mathcal{H}^1\big(J_u \cap Q^x_\rho\big) - \rho| \le \frac{\rho\eta}{2}, \tag{6.18a}$$

$$\mathcal{L}^2\Big(\Big\{y \in Q^{x,+}_\rho\colon |u(y) - u^+_x| > \frac{\eta}{\bar c_{\eta^3/2}}\Big\}\Big) + \mathcal{L}^2\Big(\Big\{y \in Q^{x,-}_\rho\colon |u(y) - u^-_x| > \frac{\eta}{\bar c_{\eta^3/2}}\Big\}\Big) \le \frac{\rho^2\eta^4}{2}, \tag{6.18b}$$

$$\int_{Q^x_\rho} |e(u)|^2\,{\rm d}y \le \frac{\rho\eta^2}{2C_\eta \bar c^2_{\eta^3/2}}, \tag{6.18c}$$

where Q^x_ρ denotes the square with sidelength ρ centered at x with two sides parallel to $\nu_u(x)$, and $Q^{x,\pm}_\rho := Q^x_\rho \cap \{y\colon\ \pm(y-x)\cdot\nu_u(x) > 0\}$. Moreover, $C_\eta \ge 1$ and $\bar c_{\eta^3/2} \ge 1$ denote the constants of Proposition 2.2 applied for $\theta = \eta$ and of Lemma 2.1 applied for $\theta = \eta^3/2$, respectively. We refer to (5.3) and (5.8) above for an analogous argument.

For $x \in J^{\rm good}_u$ and for $\rho < \bar\rho(x)$ such that $\mathcal{H}^1\big(J_u \cap \partial Q^x_\rho\big) = 0$, the squares Q^x_ρ form a fine cover of $J^{\rm good}_u$ up to a set of negligible $\mathcal{H}^1$-measure. By applying Besicovitch's covering theorem to this fine cover, we find a finite number of pairwise disjoint squares $Q^{x_i}_{\rho_i}$, $i = 1, \ldots, m$, such that the centers x_i belong to $J^{\rm good}_u$, (6.18) holds, $\mathcal{H}^1\big(J_u \cap \partial Q^{x_i}_{\rho_i}\big) = 0$, and

$$\mathcal{H}^1\Big(J^{\rm good}_u \setminus \bigcup_{i=1}^m Q^{x_i}_{\rho_i}\Big) \le \frac{\theta}{2}. \tag{6.19}$$

Arguing as in (5.7), by (6.13b)–(6.13c) and the fact that $\mathcal{H}^1\big(J_u \cap \partial Q^{x_i}_{\rho_i}\big) = 0$ we deduce that

$$\lim_{n\to\infty} \|e(u_n)\|_{L^2(Q^{x_i}_{\rho_i})} = \|e(u)\|_{L^2(Q^{x_i}_{\rho_i})} \qquad \text{for all } i = 1, \ldots, m, \tag{6.20a}$$

$$\lim_{n\to\infty} \mathcal{H}^1(J_{u_n} \cap Q^{x_i}_{\rho_i}) = \mathcal{H}^1(J_u \cap Q^{x_i}_{\rho_i}) \qquad \text{for all } i = 1, \ldots, m, \tag{6.20b}$$

$$\lim_{n\to\infty} \mathcal{H}^1\Big(J_{u_n} \setminus \bigcup_{i=1}^m Q^{x_i}_{\rho_i}\Big) = \mathcal{H}^1\Big(J_u \setminus \bigcup_{i=1}^m Q^{x_i}_{\rho_i}\Big). \tag{6.20c}$$

The convergence in (6.20) along with (6.13a) and (6.18) shows that for n large enough we have

$$\mathcal{H}^1\big(J_{u_n} \cap Q^{x_i}_{\rho_i}\big) \le \rho_i(1+\eta), \tag{6.21a}$$

$$\mathcal{L}^2\Big(\Big\{x \in Q^{x_i,+}_{\rho_i} \colon |u_n - u^+_{x_i}| > \frac{\eta}{\bar{c}_{\eta^3/2}}\Big\}\Big) + \mathcal{L}^2\Big(\Big\{x \in Q^{x_i,-}_{\rho_i} \colon |u_n - u^-_{x_i}| > \frac{\eta}{\bar{c}_{\eta^3/2}}\Big\}\Big) \le \rho_i^2\eta^4, \tag{6.21b}$$

$$\int_{Q^{x_i}_{\rho_i}} |e(u_n)|^2 \,{\rm d}x \le \frac{\rho_i \eta^2}{C^2_\eta \bar{c}^2_{\eta^3/2}}. \tag{6.21c}$$

In the following, without further notice, n will always be chosen sufficiently large such that (6.21) holds for all $i = 1, \ldots, m$.

Step 2: Conclusion. The main step of the proof consists in showing that

$$\mathcal{H}^1\Big(J^{\rm bad}_{u_n} \cap Q^{x_i}_{\rho_i}\Big) \le \frac{\theta}{4\mathcal{H}^1(J_u)}\rho_i \quad \text{for } i = 1, \ldots, m. \tag{6.22}$$

Once we have proved (6.22), the claim (6.14) is achieved as follows: by applying (6.15), (6.19), and (6.20c) we find

$$\limsup_{n\to\infty} \mathcal{H}^1(J^{\rm bad}_{u_n}) \le \limsup_{n\to\infty} \mathcal{H}^1\Big(J_{u_n} \setminus \bigcup_{i=1}^m Q^{x_i}_{\rho_i}\Big) + \limsup_{n\to\infty} \sum_{i=1}^m \mathcal{H}^1(J^{\rm bad}_{u_n} \cap Q^{x_i}_{\rho_i}) \le \frac{5}{2}\theta + \sum_{i=1}^m \frac{\theta}{4\mathcal{H}^1(J_u)}\rho_i.$$

Then, in view of (6.18a), the assumption $\eta \le 1$, and the fact that the squares $(Q^{x_i}_{\rho_i})^m_{i=1}$ are pairwise disjoint, we get

$$\limsup_{n\to\infty} \mathcal{H}^1(J^{\rm bad}_{u_n}) \le \frac{5}{2}\theta + \frac{\theta}{4\mathcal{H}^1(J_u)} \frac{1}{1-\eta/2} \sum_{i=1}^m \mathcal{H}^1(J_u \cap Q^{x_i}_{\rho_i}) \le 3\theta,$$

and the proof of (6.14) is thus concluded.

Step 3: Proof of (6.22). It remains to prove (6.22). Let us fix $i \in \{1, \ldots, m\}$. After possible rotation and translation, we may suppose that $x_i = 0$ and $\nu_u(x_i) = e_1$, and we write Q_{ρ_i} in place of $Q^{x_i}_{\rho_i}$.

By the choice of η in (6.17) and the fact that $x_i = 0 \in J^{\rm good}_u$ we particularly have $\eta \le 1/(7\zeta) \le |u^+_{x_i} - u^-_{x_i}|/7$. Moreover, (6.17) implies $\eta < \min\{\theta_0, 10^{-4}\}$. Thus, in view of (6.21), we can apply

Proposition 4.1 to $v = u_n$ and $\omega^\pm = u^\pm_{x_i}$ to find two subsets $D^+_n, D^-_n \subset Q_{\rho_i}$ such that

$$\|u_n - u^+_{x_i}\|_{L^\infty(D^+_n)} \le 3\eta \qquad \text{and} \qquad \|u_n - u^-_{x_i}\|_{L^\infty(D^-_n)} \le 3\eta, \tag{6.23a}$$

$$\mathcal{H}^1\Big(\big((\partial^* D^+_n \cup \partial^* D^-_n) \setminus J_{u_n}\big) \cap Q_{\rho_i}\Big) \le 6\eta\rho_i, \tag{6.23b}$$

and two curves $\Gamma^\pm_n \subseteq \partial^* D^\pm_n \cap Q_{\rho_i}$ connecting $(-\frac{\varrho}{2}, \frac{\varrho}{2}) \times \{-\frac{\varrho}{2}\}$ to $(-\frac{\varrho}{2}, \frac{\varrho}{2}) \times \{\frac{\varrho}{2}\}$. For simplicity of notation, let us set $\Psi_n \coloneqq \Gamma^-_n$. As observed above, we have $\eta \le |u^+_{x_i} - u^-_{x_i}|/7$. Then (6.23a) implies $J_{u_n} \supset \Psi_n$ up to an $\mathcal{H}^1$-negligible set. We now show that

$$\rho_i \le \mathcal{H}^1(\Psi_n) \le \rho_i(1 + 7\eta), \tag{6.24a}$$

$$\mathcal{H}^1\big(\Psi'_n\big) \le 7\rho_i\eta, \quad \text{where } \Psi'_n \coloneqq \{x \in \Psi_n \colon\ \nu_{u_n}(x) \cdot e_1 < 0\}, \tag{6.24b}$$

and where we choose the orientation of ν_{u_n} such that ν_{u_n} coincides with the outer normal to D^-_n. Inequality (6.24a) follows from (6.21a) and (6.23b). As for (6.24b), by the fact Ψ_n is a curve connecting $(-\frac{\varrho}{2}, \frac{\varrho}{2}) \times \{-\frac{\varrho}{2}\}$ to $(-\frac{\varrho}{2}, \frac{\varrho}{2}) \times \{\frac{\varrho}{2}\}$ we find that

$$\Psi'_n \subset \{x \in \Psi_n \colon\ t_{\Psi_n} \cdot e_2 < 0\},$$

where t_{Ψ_n} denotes the tangent vector of the curve Ψ_n. This along with (6.24a) shows (6.24b).

We recall the definition of λ in (6.16) and define

$$\Psi''_n \coloneqq \{x \in \Psi_n \colon\ |\nu_{u_n}(x) - e_1| \le \lambda\}.$$

Since $x_i = 0 \in J^{\rm good}_u$, we have $|u^+_{x_i} - u^-_{x_i}| \le \zeta$ as well as $(u^+_{x_i} - u^-_{x_i}) \cdot e_1 > 2\tau$, and thus for each $x \in \Psi''_n$ we deduce from (6.23a), (6.16), and the fact that $\eta < \tau/12$ (see (6.17)) that

$$\begin{aligned} [u_n](x) \cdot \nu_{u_n}(x) &\ge (u^+_{x_i} - u^-_{x_i}) \cdot \nu_{u_n}(x) - 6\eta \\ &\ge (u^+_{x_i} - u^-_{x_i}) \cdot e_1 - |u^+_{x_i} - u^-_{x_i}|\lambda - 6\eta \ge 2\tau - \zeta\lambda - 6\eta > \tau. \end{aligned}$$

This implies that $\Psi''_n \cap J^{\rm bad}_{u_n} = \emptyset$. Based on this, we now derive (6.22). First, we observe that

$$\mathcal{H}^1\big(\Psi_n \setminus (\Psi'_n \cup \Psi''_n)\big) \le \frac{14\rho_i\eta}{\lambda^2}. \tag{6.25}$$

Indeed, for $x \in \Psi_n \setminus (\Psi'_n \cup \Psi''_n)$ we find $0 \le e_1 \cdot \nu_{u_n}(x) = 1 - |e_1 - \nu_{u_n}(x)|^2/2 \le 1 - \lambda^2/2$ by a simple expansion. Then, by the area formula and by (6.24a) we estimate

$$\begin{aligned} \rho_i &\le \int_{\Pi^{e_1}} \mathcal{H}^0((\Psi_n)^{e_1}_w)\, \mathrm{d}\mathcal{H}^1(w) = \int_{\Psi_n} |\nu_{u_n} \cdot e_1|\, \mathrm{d}\mathcal{H}^1 \le \mathcal{H}^1(\Psi'_n \cup \Psi''_n) + (1 - \tfrac{\lambda^2}{2})\mathcal{H}^1(\Psi_n \setminus (\Psi'_n \cup \Psi''_n)) \\ &= \mathcal{H}^1(\Psi_n) - \tfrac{\lambda^2}{2}\mathcal{H}^1(\Psi_n \setminus (\Psi'_n \cup \Psi''_n)) \le \rho_i + 7\rho_i\eta - \tfrac{\lambda^2}{2}\mathcal{H}^1(\Psi_n \setminus (\Psi'_n \cup \Psi''_n)), \end{aligned}$$

which yields (6.25).

Since $\Psi''_n \cap J^{\rm bad}_{u_n} = \emptyset$ and $J_{u_n} \cap Q_{\rho_i} \supset \Psi_n$, we conclude by (6.21a), (6.24), and (6.25) that

$$\begin{aligned} \mathcal{H}^1\big(J^{\rm bad}_{u_n} \cap Q_{\rho_i}\big) &\le \mathcal{H}^1\Big((J_{u_n} \setminus \Psi''_n) \cap Q_{\rho_i}\Big) \le \mathcal{H}^1\Big((J_{u_n} \setminus \Psi_n) \cap Q_{\rho_i}\Big) + \mathcal{H}^1\big(\Psi_n \setminus \Psi''_n\big) \\ &= \mathcal{H}^1\big(J_{u_n} \cap Q_{\rho_i}\big) - \mathcal{H}^1(\Psi_n \cap Q_{\rho_i}) + \mathcal{H}^1\big(\Psi_n \setminus \Psi''_n\big) \\ &\le \rho_i(1 + \eta) - \rho_i + \frac{14\rho_i\eta}{\lambda^2} + 7\rho_i\eta. \end{aligned}$$

Therefore, $\mathcal{H}^1(J^{\rm bad}_{u_n} \cap Q_{\rho_i}) \le 14\eta\rho_i(1 + 1/\lambda^2)$ which by (6.17) implies (6.22). This concludes the proof. □

We close this section with the proof of Theorem 3.12. Recall the definition in (3.9).

Proof of Theorem 3.12. Consider $u \in GSBD^2_h(\Omega')$ with $h \in W^{2,\infty}(\Omega';\mathbb{R}^d)$ satisfying (CC). Let $\gamma \in (\frac{2}{3}, \beta)$. By Lemma 6.1 and the definition of the energy in (3.7) we obtain sequences $(\tau_n)_n \subset (0,+\infty)$ and $(u_n)_n \subset GSBD^2_h(\Omega')$ such that $u_n \to u$ in measure on Ω', $\mathcal{E}(u_n) \to \mathcal{E}(u)$, and

$$\theta_n := \mathcal{H}^1\big(\{x \in J_{u_n} \colon [u_n](x) \cdot \nu_{u_n}(x) \le 2\tau_n\}\big) \to 0 \quad \text{ as } n \to \infty.$$

Since the convergence in Definition 3.8 allows for diagonal arguments (measure convergence and weak convergence on bounded sets are metrizable), it suffices to construct for every u_n a sequence $(y^n_\varepsilon)_\varepsilon$ such that $y^n_\varepsilon \rightsquigarrow u_n$ and $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^n_\varepsilon) \le \mathcal{E}(u_n) + 12\theta_n$. Then, since $u_n \to u$ in measure on Ω' and $\mathcal{E}(u_n) \to \mathcal{E}(u)$, by a diagonal argument and [32, Theorem 2.7(ii)] we obtain an energy-convergent sequence for u.

To simplify notation, in what follows we drop the index n, so that we consider a function $u \in GSBD^2_h(\Omega')$ and two positive parameters θ and τ such that

$$\theta := \mathcal{H}^1\big(\{x \in J_u \colon [u](x) \cdot \nu_u(x) \le 2\tau\}\big), \tag{6.26}$$

and we construct a sequence $(y_\varepsilon)_\varepsilon$, $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$, satisfying (CN) and such that $y_\varepsilon \rightsquigarrow u$ and

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y_\varepsilon) \le \mathcal{E}(u) + 12\theta. \tag{6.27}$$

Step 1: Construction of $(y_\varepsilon)_\varepsilon$. In view of (6.26), we can apply Theorem 6.2 to find a sequence $(v_\varepsilon)_\varepsilon \subset GSBV^2_2(\Omega';\mathbb{R}^2)$ such that $v_\varepsilon = h$ on $\Omega' \setminus \overline{\Omega}$, the jump set J_{v_ε} of v_ε is a finite union of disjoint segments $(S^i_\varepsilon)_{i=1}^{m_\varepsilon}$, $v_\varepsilon \in W^{2,\infty}(\Omega' \setminus J_{v_\varepsilon};\mathbb{R}^2)$, and the following conditions hold:

$$v_\varepsilon \to u \text{ in measure on } \Omega' \text{ as } \varepsilon \to 0, \tag{6.28a}$$

$$\lim_{\varepsilon\to 0} \|e(v_\varepsilon) - e(u)\|_{L^2(\Omega')} = 0, \tag{6.28b}$$

$$\lim_{\varepsilon\to 0} \mathcal{H}^1(J_{v_\varepsilon}) = \mathcal{H}^1(J_u), \tag{6.28c}$$

$$\|v_\varepsilon\|_{L^\infty(\Omega')} + \|\nabla v_\varepsilon\|_{L^\infty(\Omega')} + \|\nabla^2 v_\varepsilon\|_{L^\infty(\Omega')} \le \varepsilon^{(\beta-1)/2} \le \varepsilon^{\gamma-1}, \tag{6.28d}$$

$$\mathrm{dist}(S^i_\varepsilon, S^j_\varepsilon) \ge 4\sqrt{\varepsilon} \text{ for all } 1 \le i < j \le m_\varepsilon, \tag{6.28e}$$

$$\mathrm{dist}(S^i_\varepsilon, \Omega' \setminus \Omega) \ge 4\sqrt{\varepsilon} \text{ for all } 1 \le i \le m_\varepsilon, \tag{6.28f}$$

$$\limsup_{\varepsilon\to 0} \mathcal{H}^1\big(\{x \in J_{v_\varepsilon} \colon [v_\varepsilon](x) \cdot \nu_{v_\varepsilon}(x) \le \tau\}\big) \le 3\theta. \tag{6.28g}$$

Indeed, properties (6.28a)–(6.28c) and (6.28g) follow directly from Theorem 6.2. Property (6.28d) can be achieved by a diagonal argument since the approximations satisfy $v_\varepsilon \in W^{2,\infty}(\Omega' \setminus J_{v_\varepsilon};\mathbb{R}^2)$. (Recall $\gamma < \beta < 1$.) Eventually, properties (6.28e) and (6.28f) can again be guaranteed by a diagonal argument since the segments $(S^i_\varepsilon)_{i=1}^{m_\varepsilon}$ are closed, pairwise disjoint, and do not intersect a neighborhood of $\Omega' \setminus \Omega$. Moreover, $v_\varepsilon \in W^{2,\infty}(\Omega' \setminus J_{v_\varepsilon};\mathbb{R}^2)$ also implies $J_{\nabla v_\varepsilon} \subset J_{v_\varepsilon}$.

Since $v_\varepsilon \in W^{2,\infty}(\Omega' \setminus J_{v_\varepsilon};\mathbb{R}^2)$ and J_{v_ε} consists of a finite number of segments, by the coarea formula applied on $x \mapsto [v_\varepsilon](x) \cdot \nu_{v_\varepsilon}(x)$ we find $\tau_\varepsilon \in (\tau/2, \tau)$ such that

$$J^{\mathrm{bad}}_{v_\varepsilon} := \{x \in J_{v_\varepsilon} \colon [v_\varepsilon](x) \cdot \nu_{v_\varepsilon}(x) \le \tau_\varepsilon\} \tag{6.29}$$

consists of a finite number of segments $(T^i_\varepsilon)_{i=1}^{n_\varepsilon}$. We cover these segments by pairwise disjoint rectangles R^i_ε, $i = 1,\dots,n_\varepsilon$, of length $\mathcal{H}^1(T^i_\varepsilon)$ and height $\min\{\mathcal{H}^1(T^i_\varepsilon), \sqrt{\varepsilon}\}$ such that T^i_ε separates R^i_ε into two rectangles of length $\mathcal{H}^1(T^i_\varepsilon)$ and height $\min\{\mathcal{H}^1(T^i_\varepsilon), \sqrt{\varepsilon}\}/2$, as in Figure 4.

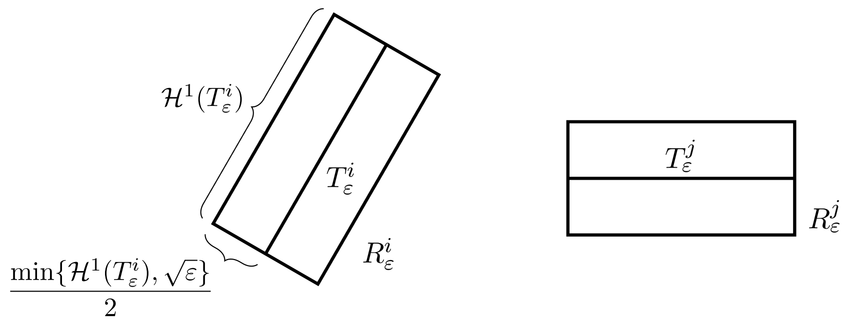

FIGURE 4. The rectangles R^i_ε and R^j_ε

Clearly, by (6.28g) and (6.29) we obtain

$$\sum_{i=1}^{n_\varepsilon} \mathcal{H}^1(\partial R^i_\varepsilon) \le \sum_{i=1}^{n_\varepsilon} 4\mathcal{H}^1(T^i_\varepsilon) \le 4\mathcal{H}^1\big(J^{\rm bad}_{v_\varepsilon}\big) \le 12\theta. \tag{6.30}$$

We define

$$w_\varepsilon := v_\varepsilon \chi_{\Omega'\setminus\bigcup_{i=1}^{n_\varepsilon} R^i_\varepsilon} + \sum_{i=1}^{n_\varepsilon} s^i_\varepsilon \chi_{R^i_\varepsilon} \tag{6.31}$$

for suitable constants $(s^i_\varepsilon)_i \subset \mathbb{R}^2$ for which the functions $y_\varepsilon := \mathbf{id} + \varepsilon w_\varepsilon$ are such that the sets

$$\big[y_\varepsilon\big(\Omega' \setminus \bigcup_{i=1}^{n_\varepsilon} R^i_\varepsilon\big)\big], \quad [y_\varepsilon(R^i_\varepsilon)],\ i=1,\dots,n_\varepsilon, \quad \text{are pairwise disjoint.} \tag{6.32}$$

Note that this is possible since $v_\varepsilon \in L^\infty(\Omega';\mathbb{R}^2)$. By construction and by (6.28f) we see that the rectangles $(R^i_\varepsilon)_i$ do not intersect $\Omega'\setminus\overline{\Omega}$. As $v_\varepsilon \in GSBV^2_2(\Omega';\mathbb{R}^2)$ and $v_\varepsilon = h$ on $\Omega'\setminus\overline{\Omega}$, we get $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$, see (3.9).

Step 2: Ciarlet-Nečas condition. We now check that y_ε is injective. Clearly, y_ε is injective on each R^i_ε, $i=1,\dots,n_\varepsilon$. In view of (6.32), it suffices to check that y_ε is also injective on $\Omega'\setminus\bigcup_{i=1}^{n_\varepsilon} R^i_\varepsilon$. To this end, fix arbitrary $x_1, x_2 \in \Omega'\setminus\bigcup_{i=1}^{n_\varepsilon} R^i_\varepsilon$, $x_1 \neq x_2$, and recall that $y_\varepsilon = \mathbf{id}+\varepsilon v_\varepsilon$ on $\Omega'\setminus\bigcup_{i=1}^{n_\varepsilon} R^i_\varepsilon$. We distinguish between two cases according to the distance between x_1 and x_2.

Case 1. $|x_1 - x_2| \ge \sqrt{\varepsilon}$. By (6.28d) and $\gamma > \frac{2}{3}$ we get

$$|y_\varepsilon(x_1) - y_\varepsilon(x_2)| \ge |x_1 - x_2| - 2\varepsilon\|v_\varepsilon\|_{L^\infty(\Omega')} \ge \sqrt{\varepsilon} - 2\varepsilon\varepsilon^{\gamma-1} > 0.$$

Case 2. $|x_1 - x_2| < \sqrt{\varepsilon}$. Inequality (6.28e) implies that the segment between x_1 and x_2, denoted by $[x_1;x_2]$, intersects at most one segment S^i_ε. We subdivide this case in two subcases, distinguishing between $[x_1;x_2]\cap S^i_\varepsilon = \emptyset$ and $[x_1;x_2]\cap S^i_\varepsilon \neq \emptyset$.

Case 2(i). If $[x_1;x_2]$ does not intersect one of the segments S^i_ε, v_ε is Lipschitz in a neighborhood of $[x_1;x_2]$, and we get by (6.28d) and $\gamma > \frac{2}{3}$ that

$$|y_\varepsilon(x_1) - y_\varepsilon(x_2)| \ge |x_1 - x_2| - \varepsilon|x_1 - x_2|\|\nabla v_\varepsilon\|_{L^\infty(\Omega')} \ge |x_1 - x_2|(1 - \varepsilon\varepsilon^{\gamma-1}) > 0.$$

Case 2(ii). Let us now suppose that $[x_1;x_2]$ intersects S^i_ε. By construction of R^i_ε, we can find a piecewise affine curve $\Gamma\colon [0,l_\Gamma] \to \Omega'\setminus\bigcup_{i=1}^{n_\varepsilon} R^i_\varepsilon$ with $\Gamma(0) = x_1$, $\Gamma(l_\Gamma) = x_2$, parametrized by arc-length, such that

$$\text{(a)}\ \ l_\Gamma = |x_1 - x_2| \qquad \text{or} \qquad \text{(b)}\ \ l_\Gamma \le |x_1 - x_2| + \mathcal{H}^1(\partial R^j_\varepsilon), \tag{6.33}$$

where case (b) holds if $[x_1; x_2]$ intersects some $T^j_\varepsilon \subset J^{\mathrm{bad}}_{v_\varepsilon}$. Moreover, we have that $\Gamma(t) \in S^i_\varepsilon$ for at most one $t \in (0, l_\Gamma)$, where in this case we have $\nu_{S^i_\varepsilon} \cdot \Gamma'(t) \geq 0$, where $\nu_{S^i_\varepsilon}$ denotes the normal vector to S^i_ε oriented such that $\nu_{S^i_\varepsilon} \cdot (x_2 - x_1) \geq 0$ (see Figure 5).

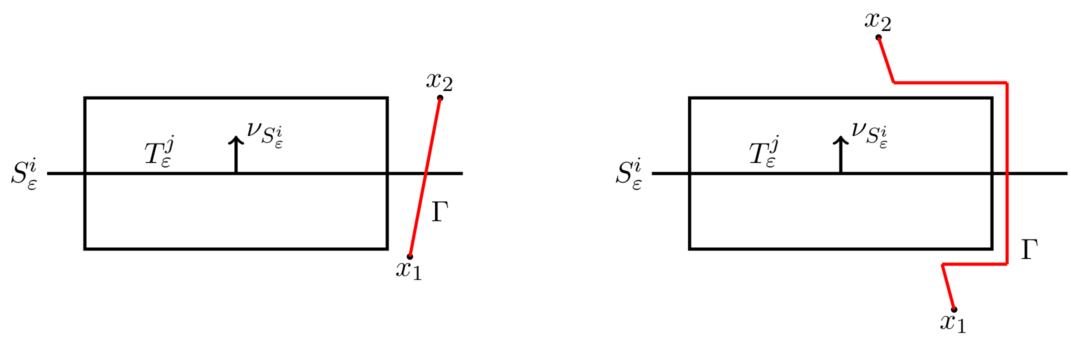

FIGURE 5. Visualization of the curve Γ in the case (a) (left) and (b) (right) of (6.33).

If (a) of (6.33) holds, then

$$\text{(a)}\quad l_\Gamma \leq \sqrt{\varepsilon}\,. \tag{6.34}$$

If (b) of (6.33) holds, we further distinguish two cases, namely

$$(\mathrm{b}_1)\quad \mathcal{H}^1(T^j_\varepsilon) \geq \sqrt{\varepsilon} \qquad \text{or} \qquad (\mathrm{b}_2)\quad \mathcal{H}^1(T^j_\varepsilon) < \sqrt{\varepsilon}\,. \tag{6.35}$$

If (b_2) holds, we immediately infer that

$$(\mathrm{b}_2)\quad l_\Gamma \leq 5\sqrt{\varepsilon}\,. \tag{6.36}$$

In the case (b_1), instead, we get that $x_k + \mathbb{R}\nu_{S^i_\varepsilon}$ intersects T^j_ε for some $k = 1, 2$, and since R^j_ε has height $\min\{\mathcal{H}^1(T^j_\varepsilon), \sqrt{\varepsilon}\} = \sqrt{\varepsilon}$, we get that

$$(\mathrm{b}_1)\quad (x_2 - x_1) \cdot \nu_{S^i_\varepsilon} \geq \sqrt{\varepsilon}/2 \qquad \text{and} \qquad l_\Gamma \leq \sqrt{\varepsilon} + 4\mathcal{H}^1(T^j_\varepsilon) \leq 5\mathcal{H}^1(T^j_\varepsilon)\,. \tag{6.37}$$

In the following, we will only treat the cases (b_1) and (b_2), as the argument for (a) is easier. By the fundamental theorem of calculus we compute

$$y_\varepsilon(x_2) - y_\varepsilon(x_1) = x_2 - x_1 + \int_0^t \varepsilon \nabla v_\varepsilon(s) \cdot \Gamma'(s)\,\mathrm{d}s + \varepsilon[v_\varepsilon](\Gamma(t)) + \int_t^{l_\Gamma} \varepsilon \nabla v_\varepsilon(s) \cdot \Gamma'(s)\,\mathrm{d}s\,,$$

where t is chosen uniquely such that $\Gamma(t) \in S^i_\varepsilon$. Since $\Gamma(t) \in J_{v_\varepsilon} \setminus J^{\mathrm{bad}}_{v_\varepsilon}$, we get $[v_\varepsilon] \cdot \nu_{S^i_\varepsilon} \geq \tau_\varepsilon \geq \tau/2$. If (b_1) of (6.35) holds, by using (6.28d), property (b_1) in (6.37), and the arc-length parametrization of Γ, giving $|\Gamma'| \equiv 1$, we find

$$\begin{aligned}\big(y_\varepsilon(x_2) - y_\varepsilon(x_1)\big) \cdot \nu_{S^i_\varepsilon} &\geq (x_2 - x_1) \cdot \nu_{S^i_\varepsilon} - l_\Gamma \varepsilon^\gamma + \varepsilon[v_\varepsilon](\Gamma(t)) \cdot \nu_{S^i_\varepsilon} \\ &\geq \frac{1}{2}\sqrt{\varepsilon} - l_\Gamma \varepsilon^\gamma + \frac{\tau}{2}\varepsilon \geq \frac{1}{2}\sqrt{\varepsilon} + \frac{\tau}{2}\varepsilon - 5\varepsilon^\gamma \mathcal{H}^1(T^j_\varepsilon)\,.\end{aligned}$$

If (b_2) of (6.35) holds, arguing in a similar way and using (b_2) in (6.36) we obtain

$$\big(y_\varepsilon(x_2) - y_\varepsilon(x_1)\big) \cdot \nu_{S^i_\varepsilon} \geq (x_2 - x_1) \cdot \nu_{S^i_\varepsilon} - l_\Gamma \varepsilon^\gamma + \varepsilon[v_\varepsilon](\Gamma(t)) \cdot \nu_{S^i_\varepsilon} \geq 0 - 5\varepsilon^{(\gamma + \frac{1}{2})} + \frac{\tau}{2}\varepsilon.$$

In both cases, since $\gamma > \frac{2}{3}$ we find that $(y_\varepsilon(x_2) - y_\varepsilon(x_1)) \cdot \nu_{S^i_\varepsilon} > 0$ for ε sufficiently small, depending only on $\mathcal{H}^1(J_u)$ and τ. This shows $y_\varepsilon(x_1) \neq y_\varepsilon(x_2)$ and yields that y_ε is injective.

By (6.28d) and (6.31) we further get $\det(\nabla y_\varepsilon) > 0$ for a.e. $x \in \Omega'$, provided that ε is sufficiently small. Therefore, y_ε satisfies the Ciarlet-Nečas non-interpenetration condition.

Step 3: Convergence of functions and energies. We now check that $y_\varepsilon \rightsquigarrow u$ in the sense of Definition 3.8. We define $y_\varepsilon^{\rm rot} = y_\varepsilon$, i.e., the Caccioppoli partition in (3.10) consists of the set Ω' only with corresponding rotation $\mathbf{Id}$. As $\nabla y_\varepsilon^{\rm rot} - \mathbf{Id} = \varepsilon \nabla v_\varepsilon \chi_{\Omega' \setminus \bigcup_{i=1}^{n_\varepsilon} R_\varepsilon^i}$, (3.11a)–(3.11c) follow from (6.28b) and (6.28d). The rescaled displacement fields u_ε defined in (3.10) satisfy $u_\varepsilon = v_\varepsilon \chi_{\Omega' \setminus \bigcup_{i=1}^{n_\varepsilon} R_\varepsilon^i}$. Then, (3.11d)–(3.11g) for $E_u = \emptyset$ follows from (6.28a)–(6.28b), the lower semicontinuity result in [23, Theorem 11.3], and the fact that

$$\sum_{i=1}^{n_\varepsilon} \mathcal{L}^2(R_\varepsilon^i) \le \sqrt{\varepsilon} \sum_{i=1}^{n_\varepsilon} \mathcal{H}^1(T_\varepsilon^i) \le 3\sqrt{\varepsilon}\theta,$$

where in the last step we used (6.30).

Finally, we confirm (6.27). Since $J_{\nabla y_\varepsilon} \subset J_{y_\varepsilon}$ and (6.28c), (6.30), and (6.31) hold, we get

$$\limsup_{\varepsilon \to 0} \mathcal{H}^1(J_{y_\varepsilon}) \le \limsup_{\varepsilon \to 0} \mathcal{H}^1(J_{v_\varepsilon}) + \limsup_{\varepsilon \to 0} \sum_{i=1}^{n_\varepsilon} \mathcal{H}^1(\partial R_\varepsilon^i) \le \mathcal{H}^1(J_u) + 12\theta.$$

Consequently, by the definition of the energies in (3.6) and (3.7), it suffices to show

$$\lim_{\varepsilon \to 0} \Big(\frac{1}{\varepsilon^2} \int_{\Omega'} W(\nabla y_\varepsilon)\, \mathrm{d}x + \frac{1}{\varepsilon^{2\beta}} \int_{\Omega'} |\nabla^2 y_\varepsilon|^2 \, \mathrm{d}x \Big) = \int_{\Omega'} \frac{1}{2} Q(e(u))\, \mathrm{d}x\,. \tag{6.38}$$

The second term in (6.38) vanishes by (6.28d), $\beta < 1$, and the fact that $\nabla^2 y_\varepsilon = \varepsilon \nabla^2 v_\varepsilon$. For the first term in (6.38), we use that $W(\mathbf{Id}+F) = \frac{1}{2}Q(\mathrm{sym}(F)) + \omega(F)$ with $|\omega(F)| \le C|F|^3$ for $|F| \le 1$, and compute by (6.28b) and (6.28d)

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{\Omega'} W(\nabla y_\varepsilon)\, \mathrm{d}x &\le \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{\Omega'} W(\mathbf{Id} + \varepsilon \nabla v_\varepsilon)\, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_{\Omega'} \Big(\frac{1}{2} Q(e(v_\varepsilon))\, \mathrm{d}x + \frac{1}{\varepsilon^2} \omega(\varepsilon \nabla v_\varepsilon)\, \mathrm{d}x \Big) \\ &= \int_{\Omega'} \frac{1}{2} Q(e(u))\, \mathrm{d}x + \lim_{\varepsilon \to 0} \int_{\Omega'} \mathrm{O}\big(\varepsilon |\nabla v_\varepsilon|^3\big)\, \mathrm{d}x = \int_{\Omega'} \frac{1}{2} Q(e(u))\, \mathrm{d}x\,, \end{aligned}$$

where in the last step we have used that $\|\nabla v_\varepsilon\|_{L^\infty(\Omega')} \le C\varepsilon^{\gamma - 1}$ for some $\gamma > 2/3$. This concludes the proof of (6.38) and of the theorem. □

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References

[1] V. Agostiniani, G. Dal Maso, A. DeSimone. Linearized elasticity obtained from finite elasticity by Γ-convergence under weak coerciveness conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **29** (2012), 715–735.

[2] L. Ambrosio, A. Braides. Energies in *SBV* and variational models in fracture mechanics. In: *Homogenization and applications to material sciences. GAKUTO International Series in Mathematical Sciences and Applications*, **9** (1995), 1–22.

[3] L. Ambrosio, V. Caselles, S. Masnou, J. Morel. Connected components of sets of finite perimeter and applications to image processing. *J. Eur. Math. Soc. (JEMS)* **3** (2001), 39–92.

[4] L. Ambrosio, N. Fusco, D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford University Press, Oxford 2000.

[5] J. M. Ball. Global invertibility of Sobolev functions and the interpenetration of matter. *Proc. R. Soc. Edinburgh Sect. A*, **88** (1981), 315–328.
[6] J. M. Ball. Some open problems in elasticity. In: *Geometry, mechanics, and dynamics*, 3–59. Springer, New York, 2002.
[7] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **63** (1977), 337–403.
[8] B. Benešová, M. Kružík. Weak lower semicontinuity of integral functionals and applications. *SIAM Rev.* **59** (2017), 703–766.
[9] A. Blake, A. Zisserman. *Visual Reconstruction*. The MIT Press, Cambridge, 1987.
[10] O. Bouchala, S. Hencl, A. Molchanova. Injectivity almost everywhere for weak limits of Sobolev homeomorphisms. *J. Funct. Anal.*, **279** (2020), 108658, 32.
[11] B. Bourdin, G. A. Francfort, J. J. Marigo. The variational approach to fracture. *J. Elasticity* **91** (2008), 5–148.
[12] A. Braides, M. Solci, E. Vitali. A derivation of linear elastic energies from pair-interaction atomistic systems. *Netw. Heterog. Media*, **2** (2007), 551–567.
[13] M. Carriero, A. Leaci, F. Tomarelli. A second order model in image segmentation: Blake & Zisserman functional, in: *Variational Methods for Discontinuous Structures (Como, 1994), Progr. Nonlinear Differential Equations Appl.* **25**, Birkhäuser, Basel, (1996), 57–72.
[14] M. Carriero, A. Leaci, F. Tomarelli. *Second Order Variational Problems with Free Discontinuity and Free Gradient Discontinuity*, in: Calculus of Variations: Topics from the Mathematical Heritage of Ennio De Giorgi, Quad. Mat., 14, 135–186, Dept. Math., Seconda Univ. Napoli, Caserta, (2004).
[15] M. Carriero, A. Leaci, F. Tomarelli. Plastic free discontinuities and special bounded hessian. *C. R. Acad. Sci. Paris* **314** (1992), 595–600.
[16] A. Chambolle. An approximation result for special functions with bounded deformation. *J. Math. Pures Appl.* **83** (2004), 929–954.
[17] A. Chambolle, S. Conti, G. Francfort. Approximation of a brittle fracture energy with a constraint of non-interpenetration. *Arch. Ration. Mech. Anal.* **228** (2018), 867–889.
[18] A. Chambolle, V. Crismale. A density result in $GSBD^p$ with applications to the approximation of brittle fracture energies. *Arch. Ration. Mech. Anal.* **232** (2019), 1329–1378.
[19] A. Chambolle, V. Crismale. Compactness and lower semicontinuity in $GSBD$. *J. Eur. Math. Soc. (JEMS)* **23** (2021), 701–719.
[20] P. G. Ciarlet, J. Nečas. Injectivity and self-contact in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **97** (1987), 171–188.
[21] S. Conti, M. Focardi, F. Iurlano. Which special functions of bounded deformation have bounded variation? *Proc. Roy. Soc. Edinb. A* **148** (2018), 33–50.
[22] G. Cortesani, R. Toader. A density result in SBV with respect to non-isotropic energies. *Nonlinear Analysis*, **38** (1999), 585–604.
[23] G. Dal Maso. Generalized functions of bounded deformation. *J. Eur. Math. Soc. (JEMS)*, **15** (2013), 1943–1997.
[24] G. Dal Maso, G. A. Francfort, R. Toader. Quasistatic crack growth in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **176** (2005), 165–225.
[25] G. Dal Maso, G. Lazzaroni. Quasistatic crack growth in finite elasticity with non-interpenetration. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), 257–290.
[26] G. Dal Maso, G. Lazzaroni. Crack growth with non-interpenetration: a simplified proof for the pure Neumann problem. *Discrete Contin. Dyn. Syst.* **31** (2011), 1219–1231.
[27] G. Dal Maso, M. Negri, D. Percivale. Linearized elasticity as Γ-limit of finite elasticity. *Set-valued Anal.* **10** (2002), 165–183.
[28] E. De Giorgi, L. Ambrosio. Un nuovo funzionale del calcolo delle variazioni. *Acc. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199–210.
[29] I. Fonseca, W. Gangbo. Local invertibility of Sobolev functions. *SIAM J. Math. Anal.* **26** (1995), 280–304.
[30] G. A. Francfort, J. J. Marigo. *Revisiting brittle fracture as an energy minimization problem.* J. Mech. Phys. Solids **46** (1998), 1319–1342.
[31] M. Friedrich. A derivation of linearized Griffith energies from nonlinear models. *Arch. Ration. Mech. Anal.* **225** (2017), 425–467.
[32] M. Friedrich. Griffith energies as small strain limit of nonlinear models for nonsimple brittle materials. *Mathematics in Engineering* **2** (2020), 75–100.
[33] M. Friedrich, M. Kružík. On the passage from nonlinear to linearized viscoelasticity. *SIAM J. Math. Anal.* **50** (2018), 4426–4456.

[34] M. Friedrich, B. Schmidt. On a discrete-to-continuum convergence result for a two dimensional brittle material in the small displacement regime. *Netw. Heterog. Media* **10** (2015), 321–342.
[35] M. Friedrich, F. Solombrino. Quasistatic crack growth in 2d-linearized elasticity. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **35** (2018), 27–64.
[36] M. Friedrich, F. Solombrino. Functionals defined on piecewise rigid funtions: integral representation and Γ-convergence. *Arch. Ration. Mech. Anal.* **236** (2020), 1325–1387.
[37] A. Giacomini, M. Ponsiglione. *Non-interpenetration of matter for SBV deformations of hyperelastic brittle materials.* Proc. Roy. Soc. Edinburgh Sect. A, **138** (2008), 1019–1041.
[38] M. Giaquinta, G. Modica, J. Soucek. *Cartesian currents in the calculus of variations*, Springer-Verlag, Berlin, 1998.
[39] A. A. Griffith. The phenomena of rupture and flow in solids. *Philos. Trans. R. Soc. London*, **221** (1921), 163–198.
[40] T.J. Healey, S. Krömer. Injective weak solutions in second-gradient nonlinear elasticity. *ESAIM Control Optim. Calc. Var.* **15** (2009), 863–871.
[41] D. Henao, C. Mora Corral. Invertibility and weak continuity of the determinant for the modelling of cavitation and fracture in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **197** (2010), 619–655.
[42] S. Hencl, P. Koskela. *Lectures on mappings of finite distortion.* Lecture Notes in Mathematics, Springer, Cham, 2014.
[43] S. Krömer. Global invertibility for orientation-preserving Sobolev maps via invertibility on or near the boundary. *Arch. Ration. Mech. Anal.* **238** (2020), 1113–1155.
[44] S. Krömer, J. Valdman. Global injectivity in second-gradient nonlinear elasticity and its approximation with penalty terms. *Mathematics and Mechanics of Solids* **24** (2019), 3644–3673.
[45] G. Lazzaroni. Quasistatic crack growth in finite elasticity with Lipschitz data. *Ann. Mat. Pura Appl. (4)* **190** (2011), 165–194.
[46] A. Mielke, T. Roubíček. Rate-independent elastoplasticity at finite strains and its numerical approximation. *Math. Models Methods Appl. Sci.* **26** (2016), 2203–2236.
[47] A. Molchanova, S. Vodopyanov. Injectivity almost everywhere and mappings with finite distortion in nonlinear elasticity. *Calc. Var. Partial Differential Equations*, **59** (2020), Paper No. 17, 25.
[48] M. Negri, R. Toader. Scaling in fracture mechanics by Bažant's law: from finite to linearized elasticity. *Math. Models Methods Appl. Sci.* **25** (2015), 1389–1420.
[49] Yu. G. Reshetnyak. *Space mappings with bounded distortion.* Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1989.
[50] M. Negri, C. Zanini. From finite to linear elastic fracture mechanics by scaling. *Calc. Var. Partial Differential Equations* **50** (2014), 525–548.
[51] B. Schmidt. Linear Γ-limits of multiwell energies in nonlinear elasticity theory. *Continuum Mech. Thermodyn.* **20** (2008), 375–396.
[52] B. Schmidt. On the derivation of linear elasticity from atomistic models. *Netw. Heterog. Media* **4** (2009), 789–812.
[53] V. Šverák. Regularity properties of deformations with finite energy. *Arch. Ration. Mech. Anal.* **100** (1988), 105–127.
[54] R.A. Toupin. Elastic materials with couple stresses. *Arch. Ration. Mech. Anal.* **11** (1962), 385–414.
[55] R.A. Toupin. Theory of elasticity with couple stress. *Arch. Ration. Mech. Anal.* **17** (1964), 85–112.

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