Invertibility of Orlicz-Sobolev maps

Giovanni Scilla and Bianca Stroffolini

Abstract We extend the global invertibility result [28] to a class of orientation-preserving Orlicz-Sobolev maps with an integrability just above $n-1$, whose traces on the boundary are also Orlicz-Sobolev and which do not present cavitation in the interior or at the boundary. As an application, we prove the existence of a.e. injective minimizers within this class for functionals in nonlinear elasticity.

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1 Introduction

We are interested in the invertibility of maps within the logarithmic scale of Orlicz–Sobolev spaces in view of the variational models in elasticity. In order to state our result, we will review the existing literature, focusing on the cavitation and fracture models.

Let Ω be a bounded open set of $\mathbb{R}^n$ and consider a map $\mathbf{u}\colon \Omega \to \mathbb{R}^n$. In nonlinear elasticity, the map $\mathbf{u}$ represents the deformation of a body that occupies the set Ω in the reference configuration. The first example of stored-energy functional for compressible materials is

$$E(\mathbf{u}) = \int_\Omega W(D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}\,, \tag{1}$$

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where $W(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^p + g(\mathrm{det}D\boldsymbol{\xi})$, $p > 1$. The function g is assumed to be convex and accounts for changes in volume, that is, it blows up as $\mathrm{det}D\mathbf{u} \to +\infty$ (expand the solid) and as $\mathrm{det}D\mathbf{u} \to 0^+$ (compress it). The lower semicontinuity of such functionals together with suitable coercivity conditions were addressed in the papers of Ball and Murat [6], Marcellini [39]. In the general case, namely, when $W = W(\boldsymbol{\xi}, \mathrm{det}D\boldsymbol{\xi}, \mathrm{cof}D\boldsymbol{\xi})$ is polyconvex, and under suitable coercivity conditions, existence of minimizers is well understood, see [2], [19].

From the physical point of view, the interpenetration of matter prescribes that two points cannot be mapped in the same one. Mathematically speaking, this problem can be formulated as the global invertibility of $\mathbf{u}$. A first global invertibility result was proven by Ball in the Sobolev space $W^{1,p}$, $p > n$ such that $\mathrm{det}D\mathbf{u} > 0$ and $\mathbf{u}$ coincides on $\partial\Omega$ with an invertible map $\mathbf{u}_0$, see [3].

In the case $p < n$ there are deformations in $W^{1,p}$ which present singularities and, in particular, are not continuous. One of such type of singularities is that of *cavitation*; that is, the formation of voids in solids. In this respect, Ball considered the minimization problem of a polyconvex energy among the restricted class of deformations that are radially symmetric; i.e.,

$$\mathbf{u}(\mathbf{x}) = \frac{r(|\mathbf{x}|)}{|\mathbf{x}|}\mathbf{x}$$

when Ω is the unit ball of $\mathbb{R}^n$. The prescribed boundary condition was the radial stretching $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$. He showed that there is a threshold $\lambda_c > 1$, such that the linear map is the unique minimizer for $1 \le \lambda \le \lambda_c$ whereas for $\lambda > \lambda_c$ there is a unique singular radial transformation with $r(0) > 0$. This means that it would be energetically favorable for the minimizer of the elastic energy to exhibit a cavitation.

Marcellini [39] revisited this example using the relaxation: the energy corresponding to a singular radial deformation must be defined through the lower semicontinuous envelope taken among all regular transformation $u_k(\mathbf{0}) = \mathbf{0}$ and with respect to the weak convergence. He derived a representation formula for the relaxed energy with an additional term proportional to the n-dimensional measure of the cavity, see also [13].

A further model was studied by Müller and Spector [42]. They were analyzing a counterexample which consists of a sequence of deformations that create more and more cavities. Consequently, they pointed out that such behaviour could be prevented by including an extra term in the energy that penalizes the creation of new surface. More precisely, their elastic energy consists of a bulk term plus a constant multiple of the perimeter of the geometric image of the deformation, see Definition 6. In proving the weak continuity of the determinants, it is required to know that not only deformations of the sequence have to be one-to-one almost everywhere, but also their limit. They constructed a counterexample where the weak limit of one-to-one almost everywhere maps with $\mathrm{det}D\mathbf{u}_j > 0$ satisfied $\mathrm{det}D\mathbf{u} > 0$ but it was not one-to-one almost everywhere. To overcome this difficulty, they introduced a new invertibility condition "INV" , formulated in terms of degree for maps in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n - 1$. This condition prevents the possibility of creating cavities which are subsequently filled with matter from elsewhere, is stable with respect to weak convergence in $W^{1,p}$, $p > n-1$, and implies invertibility almost everywhere. Its formulation relies on the topological degree. Later, Conti and De Lellis [18] were relaxing this condition to maps in $W^{1,n-1} \bigcap L^\infty$ obtaining some partial results.

An alternative theory for cavitation, which also includes fracture, was given by Henao and Mora-Corral. They replaced the perimeter with a surface energy, see equation (19) in Section 5, and proved that finite surface energy implies SBV-regularity of a suitable defined inverse of $\mathbf{u}$. In addition, they specified a notion of surface created by $\mathbf{u}$ and gave a precise meaning to the idea that $\mathcal{E}(\mathbf{u})$ measures the area of this surface $\Gamma(\mathbf{u})$. Looking deeply into the above counterexample of Müller and Spector, they decompose the surface energy into the $\mathcal{H}^{n-1}$-measure of a visible and invisible part, this latter created from elsewhere in the body. Henao and Mora-Corral extensively studied Lusin (N) condition in connection with local invertibility, [24, 25, 26, 27]. A key tool is the use of topological image $\mathrm{im_T}(\mathbf{u}, \Omega)$ of $\mathbf{u}$, see Definition 5, which is defined as the set of points for which $\mathbf{u}$ has nonzero degree, coincide a.e. with the image of $\mathbf{u}$ and the union of cavities created. If $\mathbf{u} \in W^{1,p}$, $p > n-1$, and $\det D\mathbf{u} > 0$, the condition INV is satisfied and the surface energy is finite, they were able to prove that the set $\mathrm{im_T}(\mathbf{u}, \Omega)$ is open and in this case, the generalized inverse belongs to $W^{1,1}(\mathrm{im_T}(\mathbf{u}, \Omega), \mathbb{R}^n)$. Whereas, in the limit case $p = n-1$, their result states a SBV regularity of the inverse and that the jump set does not intersect $\mathrm{im_T}(\mathbf{u}, \Omega)$.

Barchiesi, Henao and Mora-Corral [8], were focused in defining a class of orientation preserving maps that do not exhibit cavitation. The solely condition of equality between pointwise determinant and distributional determinant, $\det D\mathbf{u} = \mathrm{Det} D\mathbf{u}$, together with $\det D\mathbf{u} > 0$ were shown not to be sufficient for this requirement. In this case, the condition INV was not satisfied. In light of the aforementioned results, the surface energy (being 0) and the topological image were involved. We point out that the definition of degree is given only on selective "good" open subsets of Ω. The theorem states:

Theorem 1 *Let $p > n-1$ and suppose that $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ satisfies $\det D\mathbf{u} \in L^1_{loc}(\Omega)$. The following conditions are equivalent:*

- $\mathcal{E}(\mathbf{u}) = 0$ *and* $\det D\mathbf{u} > 0$ *a.e.;*
- $(\mathrm{adj}\, D\mathbf{u})\mathbf{u} \in L^1_{loc}(\Omega, \mathbb{R}^n)$, $\det D\mathbf{u} \neq 0$ *for a.e.* $x \in \Omega$, $\mathrm{Det} D\mathbf{u} = \det D\mathbf{u}$, *and* $\deg(\mathbf{u}, B, \cdot) \geq 0$ *for all balls B for which* $\deg(\mathbf{u}, B, \cdot)$ *is defined.*

In addition, they were applying their result to prove an existence theorem for minimizers of a variational model where the elastic energy has two terms: one written in Lagrangian coordinates and the mechanical one in Eulerian coordinates.

In a previous paper [29] the second author with Henao proved that many properties of orientation preserving maps, such as local invertibility and a.e. differentiability, can be pushed to a special class of Orlicz-Sobolev spaces, with an integrability exponent just above the space dimension minus one, in the logarithmic scale. In addition, they were showing that the maps considered in [8] were only weakly monotone. An important tool for relaxing the condition about integrability was the interplay between fine properties of Orlicz-Sobolev maps on manifolds of dimension

$n-1$ and the n-absolute continuity introduced by Malý. Indeed, this condition is satisfied by a function $u \in W^{1,1}(\Omega)$ whenever its weak derivatives belong to the Lorentz space $L^{n,1}(\Omega)$, that in turn is formulated via an Orlicz integrability condition, see [14]. The theorem reads as

Theorem 2 *Let* $A(t) = t^{n-1}\log^{\alpha}(e+t)$*,* $\alpha > n-2$*, and suppose that* $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$ *satisfies* $\det D\mathbf{u} \in L^1_{\mathrm{loc}}(\Omega)$*. The following conditions are equivalent:*

- $\mathcal{E}(\mathbf{u}) = 0$ *and* $\det D\mathbf{u} > 0$ *a.e.;*
- $(\operatorname{adj} D\mathbf{u})\mathbf{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$*,* $\det D\mathbf{u} \neq 0$ *for a.e.* $x \in \Omega$*,* $\mathrm{Det} D\mathbf{u} = \det D\mathbf{u}$*, and* $\deg(\mathbf{u}, B, \cdot) \geq 0$ *for all balls* B *for which* $\deg(\mathbf{u}, B, \cdot)$ *is defined.*

This kind of generalization could have not only a mathematical interest per se but it is also related to questions of integrability of Jacobian determinants and mappings of finite distortion (see, e.g., [23, 30, 31, 46]).

The drawback of this generalization is an existence theorem for models of magnetic elastomers, liquid crystals and magnetoelasticity, see, e.g., [7, 11, 34]. The existence theorems were proved in the scale of Sobolev spaces with $p > n-1$ in [8] and extended to our Sobolev-Orlicz class in [29]. Both the theorems were provided assuming polyconvexity in the mechanic energy and quadratic growth in the deformed configuration (nematic).

An existence theorem for the magnetoelastic model without any polyconvexity or quasiconvexity assumption was proven for the relaxed functional, the quasiconvex envelope in the same Sobolev-Orlicz class, see [44]. Actually, the quasiconvex envelope is the sum of the two envelopes: the quasiconvex for the mechanical, the tangential quasiconvexification for the nematic term (see [40] for the case t^p, $p > n-1$).

Our result: With this contribution, we are willing to push the previous result even further. To this aim, we got inspired by the paper of Henao, Mora-Corral and Oliva [28] where a global invertibility result was presented in the scale of Sobolev spaces. In particular, they were also revisiting the counterexamples of [42, Section 11], by showing that the creation of a cavitation or leakage to the boundary doesn't occur within their special class of maps: $\overline{\mathcal{A}}_p$. In our paper, we extend the global invertibility result [28] to a class $\bar{\mathcal{A}}$ of orientation-preserving Orlicz-Sobolev maps with an integrability just above $n-1$, whose traces on the boundary are also Orlicz-Sobolev and which do not present cavitation in the interior or on the boundary. Namely, we consider deformations belonging to the Orlicz-Sobolev space $W^{1,A}(\Omega;\mathbb{R}^n) \cap W^{1,A}(\partial\Omega;\mathbb{R}^n)$, generated by the N-function $A(t)$ as in Theorem 2. We then apply these results to prove the existence of minimizers within (a suitable subclass of) $\bar{\mathcal{A}}$ for functionals often used as models in nonlinear elasticity, of the form

$$\int_{\Omega} W(\mathbf{x}, \mathbf{u}(\mathbf{x}), D\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}\,,$$

where W is assumed to be polyconvex in the last variable.

We would like to mention the recent article of Krömer [33] where he raised the question of whether a continuous deformation is invertible on the boundary. He was working in the regime of $p \geq n$, and he was able to obtain the existence of homeomorphic minimizers under stronger assumptions. In our case, self-contact at the boundary is allowed, see [17].

Overview of the paper: The paper is organized as follows. In Section 2 we fix the main notation which will be used throughout the paper. Section 3 collects some basic definitions and results concerning N-functions and the Orlicz-Sobolev spaces. In particular, in Section 3.1 we define traces of Orlicz-Sobolev functions. Then, with Section 4 we recall the notions of topological degree for Orlicz-Sobolev maps (Definition 4), of topological image of a set (Definition 5), and the concept of geometric image (Definition 6). The class of admissible deformations $\bar{\mathcal{A}}$ is introduced in Section 5, where we prove their fine properties (Section 5.3); in particular, boundedness (Proposition 6) and global invertibility (Proposition 4). In the last Section 6, we exploit the results of Section 5 to prove the existence of minimizers in $\bar{\mathcal{A}}$ for a class of functionals in nonlinear elasticity.

2 Notation

In this section we fix the notation and introduce some definitions used in the paper.

Throughout the paper, we will assume $n \geq 3$, because our Orlicz class makes sense only for $n > 2$, see Section 5. In all the paper, Ω will be a non-empty open, bounded set of $\mathbb{R}^n$, which represents the body in its reference configuration. There, the coordinates will be denoted by $\mathbf{x}$, while in the deformed configuration by $\mathbf{y}$. Vector-valued and matrix-valued functions will be written in boldface. The closure of a set A is denoted by $\bar{A}$ and its topological boundary by ∂A. Given a square matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$, its determinant is denoted by $\det \mathbf{M}$. The adjugate matrix $\operatorname{adj} \mathbf{M} \in \mathbb{R}^{n\times n}$ satisfies $(\det \mathbf{M})\mathbf{I} = \mathbf{M} \operatorname{adj} \mathbf{M}$, where $\mathbf{I}$ denotes the identity matrix. The transpose of $\operatorname{adj} \mathbf{M}$ is the cofactor $\operatorname{cof} \mathbf{M}$. We recall the identity

$$\mathbf{M} \operatorname{adj} \mathbf{M} = \operatorname{cof} \mathbf{M}\mathbf{M}^T = (\det \mathbf{M})\mathbf{I} \,. \tag{2}$$

If $\mathbf{M}$ is invertible, its inverse is denoted by $\mathbf{M}^{-1}$. The inner product of vectors and of matrices will be denoted by $\cdot$ and their associated norms are denoted by $\|\cdot\|$. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the tensor product $\mathbf{a} \otimes \mathbf{b}$ is the $n \times n$ matrix whose component (i, j) is $a_i\, b_j$. The set $\mathbb{R}^{n\times n}_+$ denotes the subset of matrices in $\mathbb{R}^{n\times n}$ with positive determinant. The set $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$.

The Lebesgue measure in $\mathbb{R}^n$ is denoted by $|\cdot|$ or $\mathcal{L}^n$, and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. The abbreviation *a.e.* stands for *almost everywhere* or *almost every*; unless otherwise stated, it refers to $\mathcal{L}^n$. For Φ a Young function, L^Φ denotes the corresponding Orlicz space and $W^{1,\Phi}$, $W^{1,\Phi}_0$ the Orlicz-Sobolev spaces (see Section 3 for the precise definitions). The symbols C^1_c and C^∞_c stand for the spaces of C^1 and C^∞ functions, respectively, with compact support. The derivative of a Sobolev-Orlicz or a smooth vector-valued function $\mathbf{u}$ is written $D\mathbf{u}$.

The strong convergence in L^Φ or $W^{1,\Phi}$ and the a.e. convergence are denoted by $\to$, while the symbol for the weak convergence is $\rightharpoonup$, that for the weak* convergence in L^∞ is $\overset{*}{\rightharpoonup}$.

3 Orlicz-Sobolev spaces

We recall here few basic definitions and results concerning N-functions and Orlicz-Sobolev spaces. We refer the interested reader to [1, 10, 32, 35] for a detailed treatment of the topic.

An N-function A is a convex function from $[0,\infty)$ to $[0,\infty)$ which vanishes only at 0 and such that

$$\lim_{s\to 0^+}\frac{A(s)}{s}=0 \quad , \quad \lim_{s\to\infty}\frac{A(s)}{s}=\infty.$$

If A is an N-function, then we denote by A^* the *Young-Fenchel-Yosida* dual or conjugate transform of A; namely, the N-function defined as

$$A^*(s) := \sup\{sr - A(r) : \ 0 < r < +\infty\}.$$

In this paper, we restrict our analysis to functions A whose growth at infinity is at least such that

$$\int_{t_0}^{\infty}\left(\frac{t}{A(t)}\right)^{\frac{1}{n-2}} \mathrm{d}t < \infty. \tag{3}$$

for some $t_0 \geq 0$. The condition is satisfied, in particular, when $A(t) = t^p$ for $p > n-1$ and when $A(t) = t^{n-1}\log^\alpha(e+t)$ for every $\alpha > n-2$.

An N-function A is said to satisfy the Δ_2*-condition near infinity* if it is finite-valued and there exist constants $\mu > 2$ and $t_0 > 0$ such that

$$A(2t) \leq \mu A(t) \quad \text{for } t \geq t_0. \tag{4}$$

If (4) holds for every $t > 0$, we say that A satisfies the Δ_2*-condition globally*.

Remark 1 We notice that our function $A(t) = t^{n-1}\log^\alpha(e+t)$ for every $\alpha > n-2$ verifies the Δ_2 condition together with its conjugate. We will also being dealing with the function $B(t) = t\log^\beta(e+t)$ for a $\beta > 0$ (see Section 5): this function verifies the Δ_2-condition globally.

Let Ω be a measurable subset of $\mathbb{R}^n$. The Orlicz space $L^A(\Omega)$ built upon an N-function A is the Banach function space of those real-valued measurable functions u on Ω for which the Luxemburg norm

$$\|u\|_{L^A(\Omega)} := \inf\left\{\lambda > 0 : \ \int_\Omega A\left(\frac{|u(\mathbf{x})|}{\lambda}\right) \mathrm{d}\mathbf{x} \leq 1\right\}$$

is finite.

Since A is non-decreasing,

$$\int_\Omega A(|u(\mathbf{x})|)\mathrm{d}\mathbf{x} < \infty \ \Rightarrow\ \|u\|_{L^A(\Omega)} \leq 1. \tag{5}$$

If A satisfies the Δ_2-condition at infinity then

$$u \in L^A(\Omega) \ \Leftrightarrow\ \int_\Omega A(|u(\mathbf{x})|)\mathrm{d}\mathbf{x} < \infty. \tag{6}$$

Proposition 1 (generalized Hölder inequality) *Let A be an N-function and A^* its dual. Then it holds that*

$$\left|\int_\Omega u(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \leq 2\|u\|_{L^A(\Omega)}\|v\|_{L^{A^*}(\Omega)},$$

for every $u \in L^A(\Omega)$ and $v \in L^{A^*}(\Omega)$.

Note that we may introduce another norm on $L^A(\Omega)$, the *Orlicz norm* or *dual norm*, defined as

$$|u|_A := \sup\left\{\int_\Omega u(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x} :\ v \in L^{A^*}(\Omega), \|v\|_{L^{A^*}(\Omega)} \leq 1\right\}.$$

The norms $\|\cdot\|_{L^A(\Omega)}$ and $|\cdot|_A$ are equivalent, since it holds that

$$\|u\|_{L^A(\Omega)} \leq |u|_A \leq 2\|u\|_{L^A(\Omega)}, \quad u \in L^A(\Omega).$$

The Orlicz space $L^A(\Omega, \mathbb{R}^n)$ of vector-valued measurable functions on Ω is defined as $L^A(\Omega, \mathbb{R}^n) = (L^A(\Omega))^n$, and is equipped with the norm $\|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} = \|\,\|\mathbf{u}\|\,\|_{L^A(\Omega)}$ for $\mathbf{u} \in L^A(\Omega, \mathbb{R}^n)$. The Orlicz space $L^A(\Omega, \mathbb{R}^{n\times n})$ of matrix-valued measurable functions on Ω can be defined analogously.

We denote by $W^{1,A}(\Omega)$ the Orlicz-Sobolev space defined by

$$W^{1,A}(\Omega) := \{u \in L^A(\Omega) :\ u \text{ is weakly differentiable and } Du \in L^A(\Omega, \mathbb{R}^n)\}.$$

The space $W^{1,A}(\Omega)$, equipped with the norm

$$\|u\|_{W^{1,A}(\Omega)} := \|u\|_{L^A(\Omega)} + \|Du\|_{L^A(\Omega,\mathbb{R}^n)}$$

is a Banach space. The space $W_0^{1,A}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in the $W^{1,A}$ norm.

The Orlicz space $W^{1,A}(\Omega, \mathbb{R}^n)$ of vector-valued measurable functions on Ω is defined as $W^{1,A}(\Omega, \mathbb{R}^n) = (W^{1,A}(\Omega))^n$, and is equipped with the norm $\|\mathbf{u}\|_{W^{1,A}(\Omega,\mathbb{R}^n)} = \|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} + \|D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})}$ for $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$. The analogous spaces for matrix-valued functions are defined in the same way.

We now introduce a notion of ordering for Young functions (see, e.g., [35, Definition 3.5.6]).

Let A, B be Young functions. A is said to *dominate* B, and we write $B \prec A$, if there exists a positive constant c_0 such that

$$B(t) \leq A(c_0 t), \quad \text{for every } t > 0. \tag{7}$$

As customary, if there exists also $t_0 > 0$ such that (7) holds for every $t \geq t_0$, we say that A *dominates* B *near infinity*. If $A \prec B$ and $B \prec A$, the functions A and B are said to be *equivalent*, and we write $A \sim B$.

Let $p \geq 1$. The Orlicz-Sobolev space generated by the N-function $t^p \log^\alpha(e+t)$ is the *Zygmund space* $\mathrm{L}^p \mathrm{Log}^\alpha \mathrm{L}$. The space $\mathrm{L}^1 \mathrm{Log}^\alpha \mathrm{L}$ will be denoted by $\mathrm{L}\mathrm{Log}^\alpha \mathrm{L}$. If $\alpha < 0$, the equivalent notation $\dfrac{\mathrm{L}^p}{\mathrm{Log}^{-\alpha}\mathrm{L}}$ will be used. Since the conjugate N-function of $t^p \log^\alpha(e+t)$, $t \geq 0$, $p > 1$, is equivalent to $t^{p'} \log^{-\alpha \frac{p'}{p}}(e+t)$, where $p' := \frac{p}{p-1}$ (see, e.g., [32, Theorem 7.2]), we have that the dual space of $\mathrm{L}^{n-1}\mathrm{Log}^\alpha \mathrm{L}$, $\alpha > n-2$, is the Zygmund space $\dfrac{\mathrm{L}^{\frac{n-1}{n-2}}}{\mathrm{Log}^{\frac{\alpha}{n-2}}\mathrm{L}}$. Furthermore, by virtue of [9, Theorem 9.1], it holds that

$$L^q \subseteq \frac{\mathrm{L}^{\frac{n-1}{n-2}}}{\mathrm{Log}^{\frac{\alpha}{n-2}}\mathrm{L}}, \quad \text{for every } q > \frac{n-1}{n-2}. \tag{8}$$

We recall that a family of functions $\mathcal{F}$ has equi-absolutely continuous integrals if for every $\varepsilon > 0$ one can find $\delta > 0$ such that for all $u \in \mathcal{F}$ there holds $\int_E |u(\mathbf{x})| \, \mathrm{d}\mathbf{x} < \varepsilon$ provided $|E| < \delta$. A general criterion for the equi-absolute continuity of the integrals of a family of functions in $L^A(\Omega)$ is given by the following version of *De la Vallée Poussin's Theorem* (see, e.g., [32, Ch. II, §11.1]):

Theorem 3 *Let A be an N-function, and $\mathcal{F}$ be a family of functions in $L^A(\Omega)$. If there exists $C > 0$ such that*

$$\int_\Omega A(|u(\mathbf{x})|) \, \mathrm{d}\mathbf{x} \leq C, \quad u \in \mathcal{F},$$

then the family $\mathcal{F}$ has equi-absolutely continuous integrals.

Let $\{\mathbf{v}_j\}$ be a sequence of functions in $L^A(\Omega, \mathbb{R}^n)$ and let $\mathbf{v} \in L^A(\Omega, \mathbb{R}^n)$. If A is Δ_2 near infinity, then

$$\lim_{j \to +\infty} \|\mathbf{v}_j - \mathbf{v}\|_{L^A(\Omega,\mathbb{R}^n)} = 0 \iff \lim_{j \to +\infty} \int_\Omega A(\|\mathbf{v}_j - \mathbf{v}\|) \, \mathrm{d}\mathbf{x} = 0.$$

Note that, if A does not satisfy Δ_2-condition, the implication "$\Leftarrow$" fails. If $A \in \Delta_2$ near infinity, instead, we have

$$\lim_{j \to +\infty} \|\mathbf{v}_j - \mathbf{v}\|_{L^A(\Omega,\mathbb{R}^n)} = 0 \Rightarrow \lim_{j \to +\infty} \int_\Omega A(\|\mathbf{v}_j\|) \, \mathrm{d}\mathbf{x} = \int_\Omega A(\|\mathbf{v}\|) \, \mathrm{d}\mathbf{x}.$$

3.1 Traces

We define Orlicz-Sobolev functions on the (sufficiently smooth) boundary $\partial\Omega$ of Ω following the approach of [35, Section 6].

First, we recall the definition of open set of class $C^{k,\alpha}$. Since the minimum regularity for Ω will be Lipschitz, we are assuming that $k+\alpha \geq 1$.

Definition 1 Let $k \geq 0$ be an integer and $\alpha \in [0,1]$ be such that $k+\alpha \geq 1$. A bounded open set Ω is said to be of class $C^{k,\alpha}$ if there exist $r>0$, $b>0$, $m \in \mathbb{N}$, $a_1,\dots,a_m \in C^{k,\alpha}([0,r]^{n-1})$ and $\mathbf{M}_1,\dots,\mathbf{M}_m$ proper rigid transformations in $\mathbb{R}^n$ such that, setting

$$\begin{aligned}
\Gamma_i &:= \mathbf{M}_i^{-1}(\{(\hat{\mathbf{x}},x_n) \in (0,r)^{n-1}\times\mathbb{R} :\ x_n = a_i(\hat{\mathbf{x}})\}),\\
U_i^+ &:= \mathbf{M}_i^{-1}(\{(\hat{\mathbf{x}},x_n) \in (0,r)^{n-1}\times\mathbb{R} :\ a_i(\hat{\mathbf{x}}) < x_n < a_i(\hat{\mathbf{x}})+b\}),\\
U_i^- &:= \mathbf{M}_i^{-1}(\{(\hat{\mathbf{x}},x_n) \in (0,r)^{n-1}\times\mathbb{R} :\ a_i(\hat{\mathbf{x}})-b < x_n < a_i(\hat{\mathbf{x}})\}),
\end{aligned}$$

we have that

$$\partial\Omega = \bigcup_{i=1}^m \Gamma_i\,, \quad \bigcup_{i=1}^m U_i^+ \subset \Omega \quad \text{and} \quad \bigcup_{i=1}^m U_i^- \subset \mathbb{R}^n \setminus \bar{\Omega}\,.$$

For each $i=1,\dots,m$, the set Γ_i is relatively open in $\partial\Omega$ and the sets U_i^+, U_i^- are open. We denote by U_i the open set given by $U_i^+ \cup \Gamma_i \cup U_i^-$ for every $i=1,\dots,m$. Then, the family $\{U_i\}_{i=1}^m$ is an open cover of $\partial\Omega$. Furthermore, we consider an open set $U_0 \subset\subset \Omega$ such that $\bar{\Omega} \subset \bigcup_{i=0}^m U_i$.

For each $i=1,\dots,m$, we define $\mathbf{P}_i : [0,r]^{n-1}\times[-b,b] \to \mathbb{R}^n$ by $\mathbf{P}_i(\hat{\mathbf{x}},x_n) := (\hat{\mathbf{x}}, a_i(\hat{\mathbf{x}})+x_n)$, and $\mathbf{N}_i := \mathbf{M}_i^{-1}\circ\mathbf{P}_i$. Then, each $\mathbf{N}_i$ is injective and we have $\mathbf{N}_i((0,r)^{n-1}\times(-b,b)) = U_i$, $\mathbf{N}_i((0,r)^{n-1}\times(0,b)) = U_i^+$, $\mathbf{N}_i((0,r)^{n-1}\times[0,b)) = \Gamma_i \cup U_i^+$, and $\mathbf{N}_i((0,r)^{n-1}\times\{0\}) = \Gamma_i$.

Denoting by $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection onto the first $n-1$ coordinates, and by $\eta : \mathbb{R}^{n-1} \to \mathbb{R}^n$ the map $\eta(\hat{\mathbf{x}}) = (\hat{\mathbf{x}},0)$, for any function u defined on Γ_i we consider the map $L_i(u) := \pi(\mathbf{N}_i^{-1}(\Gamma_i)) \to \mathbb{R}$ defined by $L_i(u) := u\circ\mathbf{N}_i\circ\eta$.

We are now in position to give the definition of an Orlicz-Sobolev function defined on the boundary. Assume that Ω is a Lipschitz open set.

Definition 2 We denote by $W^{1,A}(\partial\Omega)$ the set of functions $u : \partial\Omega \to \mathbb{R}$ such that $L_i(u) \in W^{1,A}((0,r)^{n-1})$ for all $i \in \{1,\dots,m\}$, equipped with the norm

$$\|u\|_{W^{1,A}(\partial\Omega)} := \sum_{i=1}^m \|L_i(u)\|_{W^{1,A}((0,r)^{n-1})}\,.$$

An analogous definition can be given, with minor modifications, for the space $W^{1,A}(\Gamma_i)$ of Orlicz-Sobolev functions defined on the subset Γ_i. Moreover, it can be shown that $W^{1,A}(\partial\Omega)$ is a Banach space (see, e.g., [35, Section 6.3.6]), and this property does not depend on the description of the boundary considered in Definition 1.

Let $u \in W^{1,A}(\Omega)$. We denote by $u|_{\partial\Omega}$ the trace of u on $\partial\Omega$, which belongs to $L^A(\partial\Omega)$. With abuse of notation, we will write $u \in W^{1,A}(\partial\Omega)$ when the trace of u belongs to $W^{1,A}(\partial\Omega)$, following the definition given in 2. We then define the intersection space

$$W^{1,A}(\Omega) \cap W^{1,A}(\partial\Omega) := \{u \in W^{1,A}(\Omega) : \ u|_{\partial\Omega} \in W^{1,A}(\partial\Omega)\} .$$

It is equipped in a natural way with the norm of an intersection

$$\|u\|_{W^{1,A}(\Omega)\cap W^{1,A}(\partial\Omega)} := \|u\|_{W^{1,A}(\Omega)} + \|u\|_{W^{1,A}(\partial\Omega)} ,$$

and it can be easily shown that it is a Banach space. For Γ a relatively open subset of $\partial\Omega$, the notation $W^{1,A}(\Gamma)$ and $W^{1,A}(\Omega) \cap W^{1,A}(\Gamma)$ will be used. For vector-valued functions, the symbol $W^{1,A}(\Omega;\mathbb{R}^n) \cap W^{1,A}(\partial\Omega;\mathbb{R}^n)$ will be adopted.

In order to extend an Orlicz-Sobolev function defined on $(0,r)^{n-1} \times \{0\}$ to $(0,r)^{n-1} \times (0,b)$, we will extend putting the same value on the vertical fiber. Namely, first we will project onto the first $n-1$ coordinates and then we will compose with the map η that leaves the last coordinate fixed (equal to 0). The proof can be obtained as in [28, Lemma 4.1] dealing with the $W^{1,p}$ case.

Lemma 1 *Let $r, b > 0$ and set $D := (0,r)^{n-1} \times (0,b)$ and $\Gamma := (0,r)^{n-1} \times \{0\}$. Then the map $E : W^{1,A}(\Gamma) \to W^{1,A}(D)$ defined by $Eu := u \circ \eta \circ \pi$ is linear and bounded. Furthermore,*

$$\frac{\partial(Eu)}{\partial x_i} = \frac{\partial u}{\partial x_i} \circ \eta \circ \pi , \quad \textit{for } i = 1, \dots, n-1 , \quad \textit{and} \quad \frac{\partial(Eu)}{\partial x_n} = 0 . \tag{9}$$

In addition, $(Eu)_{|_\Gamma} = u$.

Proof Defining $\widetilde{u} := Eu$, by Fubini's Theorem we have

$$\int_D A(|\widetilde{u}(\mathbf{x})|)\, \mathrm{d}\mathbf{x} = b \int_{(0,r)^{n-1}} A(|u(\hat{\mathbf{x}}, 0)|)\, \mathrm{d}\hat{\mathbf{x}} \tag{10}$$

which implies $\widetilde{u} \in L^A(D)$. Now, choosing a test function $\varphi \in C^1_c(D)$, for each $i \in \{1, \dots, n-1\}$ a simple integration by parts gives

$$\int_D \widetilde{u}(\mathbf{x}) \frac{\partial\varphi}{\partial x_i}(\mathbf{x})\, \mathrm{d}\mathbf{x} = -\int_D \frac{\partial u}{\partial x_i}(\eta(\pi(\mathbf{x})))\varphi(\mathbf{x})\, \mathrm{d}\mathbf{x} , \tag{11}$$

while for $i = n$

$$\int_D \widetilde{u}(\mathbf{x}) \frac{\partial\varphi}{\partial x_n}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_{(0,r)^{n-1}} u(\hat{\mathbf{x}}, 0) \int_0^b \frac{\partial\varphi}{\partial x_n}(\hat{\mathbf{x}}, x_n)\, \mathrm{d}x_n\, \mathrm{d}\hat{\mathbf{x}} = 0 . \tag{12}$$

Thus, (9) holds. Since an analog of (10) holds also for $\frac{\partial \widetilde{u}}{\partial x_i}$, $i = 1, \dots, n-1$, we conclude that $\widetilde{u} \in W^{1,A}(D)$ and that the map E is linear and bounded. The last assertion $\widetilde{u}_{|\Gamma} = u$ follows from the continuity of the trace operator. □

The previous result is a tool for the proof of the following density result of smooth functions in $W^{1,A}(\Omega) \cap W^{1,A}(\partial\Omega)$.

Proposition 2 *Let $k \geq 0$ be an integer and $\alpha \in [0,1]$ be such that $k + \alpha \geq 1$. Let Ω be an open and bounded set with $C^{k,\alpha}$ boundary. Then $C^{k,\alpha}(\bar{\Omega})$ is dense in $W^{1,A}(\Omega) \cap W^{1,A}(\partial\Omega)$.*

Proof The strategy of [28, Proposition 4.2], based on the analogous of Lemma 1 and the result of Fonseca and Malý [22, Lemma 2.4], that allows to modify the boundary values of a function without increasing significantly its norm, can be performed in the Orlicz setting with minor modifications. The proof is based on a gluing lemma for functions defined on disjoint subsets, a partition of unity and triangle inequality for the norm. □

As a final remark, we notice that $u \in W^{1,A}(\partial\Omega)$, for an N-function A complying with assumption (3) and Ω of class C^1, admits a continuous representative (see [12, Remark 3.2]) on $n-1$ manifolds. If not stated otherwise, we will always assume that u itself is the continuous representative.

4 Some definitions and preliminary results

This section collects some basic definitions and preliminary results.

Let $\mathbf{u} : \Omega \longrightarrow \mathbb{R}^n$ be a measurable function and let $\mathbf{x}_0 \in \Omega$. If $\mathbf{u}$ is *approximately differentiable* at $\mathbf{x}_0$, we denote by $\nabla\mathbf{u}(\mathbf{x}_0)$ its approximate differential at $\mathbf{x}_0$. We denote the set of approximate differentiability points of $\mathbf{u}$ by Ω_d. If $\mathbf{u}$ is approximately differentiable a.e., for any $E \subset \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, we define

$$\mathcal{N}_{\mathbf{u},E}(\mathbf{y}) := \mathcal{H}^0(\{\mathbf{x} \in \Omega_d \cap E : \mathbf{u}(\mathbf{x}) = \mathbf{y}\}). \tag{13}$$

The number $\mathcal{N}_{\mathbf{u},\Omega}$ will be denoted by $\mathcal{N}_{\mathbf{u}}$.

Now, we recall the definition of *almost everywhere (a.e.) invertibility* for a vector-valued function.

Definition 3 A function $\mathbf{u} : \Omega \longrightarrow \mathbb{R}^n$ is said to be one-to-one a.e. in a subset $E \subset \Omega$ if there exists a subset $N \subset E$, with $\mathcal{L}^n(N) = 0$, such that $\mathbf{u}_{|E \setminus N}$ is one-to-one.

Since we are assuming Ω to be Lipschitz, for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x} \in \partial\Omega$ the tangent space of $\partial\Omega$ at $\mathbf{x}$, denoted by $T_{\mathbf{x}}\partial\Omega$, and the unit exterior normal $\boldsymbol{\nu}(\mathbf{x})$ to Ω at $\mathbf{x}$ are defined.

Let V be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ and let $\mathbf{L} : V \to \mathbb{R}^n$ be a linear map. We denote by $\Lambda_{n-1}V$ the space of all alternating $(n-1)$-tensors on V, and by $\Lambda_{n-1}\mathbf{L} : \Lambda_{n-1}V \to \mathbb{R}^n$ the transformation defined by

$$(\Lambda_{n-1}\mathbf{L})(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{n-1}) = \mathbf{L}\mathbf{a}_1 \wedge \cdots \wedge \mathbf{L}\mathbf{a}_{n-1}\,, \quad \mathbf{a}_1, \ldots, \mathbf{a}_{n-1} \in V\,,$$

where $\wedge$ indicates the exterior product between vectors in $\mathbb{R}^n$. It is well known that the one-dimensional space $\Lambda_{n-1}V$ can be identified in a canonical way with the subspace generated by $\mathbf{v}$, $\mathbf{v}$ being any of the two unit normal vectors to V. Thus, the linear transformation $\Lambda_{n-1}\mathbf{L}$ is determined by the value $(\Lambda_{n-1}\mathbf{L})\mathbf{v}$ and the identity

$$(\Lambda_{n-1}\mathbf{L})\mathbf{v} = (\operatorname{cof}\widetilde{\mathbf{L}})\mathbf{v} \tag{14}$$

holds whenever $\widetilde{\mathbf{L}} : \mathbb{R}^n \to \mathbb{R}^n$ is any linear map extending $\mathbf{L}$. Let $\mathbf{u} \in W^{1,A}(\partial\Omega;\mathbb{R}^n)$. Then the tangential derivative $D\mathbf{u}(\mathbf{x}) : T_{\mathbf{x}}\partial\Omega \to \mathbb{R}^n$ exists for a.e. $\mathbf{x} \in \partial\Omega$ and $|D\mathbf{u}| \in L^A(\partial\Omega)$. As a consequence, $(\Lambda_{n-1}D\mathbf{u}(\mathbf{x}))\boldsymbol{\nu}(\mathbf{x})$ exists for a.e. $\mathbf{x} \in \partial\Omega$, and $(\Lambda_{n-1}D\mathbf{u})\boldsymbol{\nu} \in \mathrm{LLog}^{\frac{\alpha}{n-1}}\mathrm{L}(\partial\Omega,\mathbb{R}^n)$.

4.1 Degree for Orlicz-Sobolev maps, topological image of a set and geometric image of a set

In order to introduce the concept of *topological image* (according to Šverák [45] (see also [42])), we need to recall the notion of topological degree for continuous functions (see, e.g., [20, 21]).

In the end of Section 3.1, we have recalled that every map $\mathbf{u} \in W^{1,A}(\partial\Omega,\mathbb{R}^n)$, where Ω is an open set of class C^1, and A satisfies (3) and the Δ_2-condition at infinity, admits a continuous representative $\bar{\mathbf{u}} : \partial\Omega \longrightarrow \mathbb{R}^n$. It can be extended to a continuous function $\tilde{\mathbf{u}} : \overline{\Omega} \longrightarrow \mathbb{R}^n$ (see, e.g., [43, Theorem 35.1]) and therefore, the following definition of degree can be given.

Definition 4 The degree $\deg(\bar{\mathbf{u}}, \Omega, \cdot) : \mathbb{R}^n \setminus \bar{\mathbf{u}}(\partial\Omega) \to \mathbb{Z}$ of $\bar{\mathbf{u}}$ on U is defined as the degree $\deg(\tilde{\mathbf{u}}, \Omega, \cdot) : \mathbb{R}^n \setminus \bar{\mathbf{u}}(\partial\Omega) \to \mathbb{Z}$ of $\tilde{\mathbf{u}}$ on Ω.

We will denote by $\deg(\mathbf{u}, \Omega, \cdot)$ the degree of $\mathbf{u} \in W^{1,A}(\partial\Omega,\mathbb{R}^n)$, with a slight abuse of notation, tacitly referring to the degree of its continuous representative.

We are now in position to define the concept of topological image.

Definition 5 Let A be an N-function satisfying (3) and let $\Omega \subset\subset \mathbb{R}^n$ be a nonempty open set with a C^1 boundary. If $\mathbf{u} \in W^{1,A}(\partial\Omega,\mathbb{R}^n)$, we define $\mathrm{im}_{\mathrm{T}}(\mathbf{u},\Omega)$, the topological image of Ω under $\mathbf{u}$, as the set of $\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(\partial\Omega)$ such that $\deg(\mathbf{u},\Omega,\mathbf{y}) \neq 0$.

The continuity of function $\deg(\mathbf{u},\Omega,\cdot)$ implies that the set $\mathrm{im}_{\mathrm{T}}(\mathbf{u},\Omega)$ is open and $\partial\,\mathrm{im}_{\mathrm{T}}(\mathbf{u},\Omega) \subset \mathbf{u}(\partial\Omega)$. Furthermore, as $\deg(\mathbf{u},\Omega,\cdot) = 0$ in the unbounded component of $\mathbb{R}^n \setminus \mathbf{u}(\partial\Omega)$ (see, e.g., [20, Sect. 5.1]), it follows that $\mathrm{im}_{\mathrm{T}}(\mathbf{u},\Omega)$ is bounded.

The following formula for the distributional derivative of the degree of Orlicz-Sobolev functions will be widely used (see [29, Proposition 2.12]).

Proposition 3 *Let A be an N-function satisfying* (3) *and the Δ_2-condition at infinity. Let $\Omega \subset \mathbb{R}^n$ be an open set of class C^1. Assume that $\mathbf{u}$ is the continuous representative of a function in $W^{1,A}(\Omega, \mathbb{R}^n)$. Then, for all $\mathbf{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$,*

$$\int_{\partial\Omega} \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot ((\Lambda_{n-1} D\mathbf{u}(\mathbf{x}))\boldsymbol{\nu}(\mathbf{x}))\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) = \int_{\mathbb{R}^n} \operatorname{div} \mathbf{g}(\mathbf{y})\ \deg(\mathbf{u}, \Omega, \mathbf{y})\, \mathrm{d}\mathbf{y}$$

where $\boldsymbol{\nu}$ is the unit outward normal to Ω.

The following is the notion of *geometric image* of a set adapted to the context of Orlicz spaces (see [29, Section 2.2]).

Definition 6 Let $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$ and assume that $\det D\mathbf{u}(\mathbf{x}) \neq 0$ for a.e. $\mathbf{x} \in \Omega$. Let Ω_0 be the subset of $\mathbf{x} \in \Omega$ where the following are satisfied:
i) $\mathbf{u}$ is approximately differentiable at $\mathbf{x}$ and $\det\nabla\mathbf{u}(\mathbf{x}) \neq 0$;
ii) there exist $\mathbf{w} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and a compact set $K \subset \Omega$ of density 1 at $\mathbf{x}$ such that $\mathbf{u}_{|K} = \mathbf{w}_{|K}$ and $\nabla\mathbf{u}_{|K} = D\mathbf{w}_{|K}$.

The geometric image of Ω under $\mathbf{u}$ is defined as

$$\operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega) := \mathbf{u}(\Omega_0). \tag{15}$$

It turns out that Ω_0 is a set of full measure in Ω (see the remarks after [29, Def. 2.4]).

5 The class of admissible functions

From now on, we fix as N-function A satisfying (3) and the Δ_2-condition at infinity (4) the function $A(t) := t^{n-1} \log^{\alpha}(e + t)$ for $\alpha > n - 2$.

We start introducing the following class $\mathcal{A}(\Omega)$.

Definition 7 Let $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$ and assume that $\det D\mathbf{u} \in L^1(\Omega)$. For every $\mathbf{f} \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$, we define

$$\bar{\mathcal{E}}_{\Omega}(\mathbf{u}, \mathbf{f}) := \int_{\Omega} [\operatorname{cof} D\mathbf{u}(\mathbf{x}) \cdot D\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \det D\mathbf{u}(\mathbf{x}) \operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))]\, \mathrm{d}\mathbf{x}\,. \tag{16}$$

We define $\mathcal{A}(\Omega)$ as the set of $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$ such that $\det D\mathbf{u} \in L^1(\Omega)$ and

$$\bar{\mathcal{E}}_{\Omega}(\mathbf{u}, \mathbf{f}) = 0 \quad \text{for all } \mathbf{f} \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)\,. \tag{17}$$

Remark 2 We notice that if $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$, then $D\mathbf{u} \in L^A(\Omega, \mathbb{R}^{n\times n})$, so $\operatorname{cof} D\mathbf{u} \in \mathrm{LLog}^{\frac{\alpha}{n-1}}\mathrm{L}(\Omega, \mathbb{R}^{n\times n})$. In particular, $\operatorname{cof} D\mathbf{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$. This implies that the energy (16) is finite.

In equation (16), $D\mathbf{f}(\mathbf{x}, \mathbf{y})$ denotes the derivative of $\mathbf{f}(\cdot, \mathbf{y})$ evaluated at $\mathbf{x}$, while $\operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{y})$ is the divergence of $\mathbf{f}(\mathbf{x}, \cdot)$ evaluated at $\mathbf{y}$.

The energy $\bar{\mathcal{E}}_\Omega(\mathbf{u})$ was introduced in [24] and measures the new surface in the deformed configuration created by $\mathbf{u}$. For our purposes, we are interested into deformations $\mathbf{u}$ such that $\bar{\mathcal{E}}_\Omega(\mathbf{u}) = 0$; i.e., that do not exhibit cavitation.

A global version of condition (17) is the following (20), leading to the introduction of the class $\bar{\mathcal{A}}(\Omega)$.

Definition 8 Given $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n) \cap W^{1,A}(\partial\Omega, \mathbb{R}^n)$ and $\mathbf{f} \in C^1_c(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$, we define

$$\mathcal{F}_{\partial\Omega}(\mathbf{u}, \mathbf{f}) := \int_{\partial\Omega} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \cdot ((\Lambda_{n-1} D\mathbf{u}(\mathbf{x}))\nu(\mathbf{x}))\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x})\,.$$

Then we define $\bar{\mathcal{A}}(\Omega)$ as the class of $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n) \cap W^{1,A}(\partial\Omega, \mathbb{R}^n)$ with $\det D\mathbf{u} \in L^1(\Omega)$ such that

$$\bar{\mathcal{E}}_\Omega(\mathbf{u}, \mathbf{f}) = \mathcal{F}_{\partial\Omega}(\mathbf{u}, \mathbf{f}) \quad \text{for all } \mathbf{f} \in C^1_c(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)\,. \tag{18}$$

Taking into account the density in $C^1_c(\Omega\times\mathbb{R}^n, \mathbb{R}^n)$ of sums of functions of separate variables (see [36, Corollary 1.6.5]), conditions (17) and (18) can be rephrased, respectively, as follows:

$$\mathcal{E}_\Omega(\mathbf{u}, \phi, \mathbf{g}) := \int_\Omega \left[\mathrm{cof}\, D\mathbf{u}(\mathbf{x})\cdot(\mathbf{g}(\mathbf{u}(\mathbf{x}))\otimes D\phi(\mathbf{x})) + \det D\mathbf{u}(\mathbf{x})\phi(\mathbf{x})\mathrm{div}\mathbf{g}(\mathbf{u}(\mathbf{x}))\right] \mathrm{d}\mathbf{x} = 0\,, \tag{19}$$

for all $\phi \in C^1_c(\Omega)$ and $\mathbf{g} \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, and

$$\bar{\mathcal{E}}_\Omega(\mathbf{u}, \phi\mathbf{g}) = \mathcal{F}_{\partial\Omega}(\mathbf{u}, \phi\mathbf{g}) \quad \text{for all } \phi \in C^1_c(\bar{\Omega}) \text{ and } \mathbf{g} \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)\,, \tag{20}$$

where $\phi\mathbf{g} \in C^1_c(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$ stands for the function $(\phi\mathbf{g})(\mathbf{x}, \mathbf{y}) := \phi(\mathbf{x})\mathbf{g}(\mathbf{y})$.

5.1 Extension properties

As a first feature of the class just introduced, we notice that every function in $\bar{\mathcal{A}}(\Omega)$ can be extended to an open set $\widetilde{\Omega} \supset \bar{\Omega}$ by a function in $\mathcal{A}(\widetilde{\Omega})$.

In fact, we need to assume that Ω is an *extendable domain*.

Definition 9 An open set Ω is said to be *extendable* if it is bounded, has a Lipschitz boundary and there exist a set N, with $\partial\Omega \subset N \subset \mathbb{R}^n\backslash\Omega$, a $\delta > 0$ and a bi-Lipschitz homeomorphism $\mathbf{w} : \partial\Omega\times(-\delta, 0] \to N$ onto N such that $\mathbf{w}(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Omega$.

It is easy to see that the assumption of piecewise $C^{1,1}$ implies Ω extendable. In addition, the set $\Omega \cup N$ is open (see the remarks below [28, Definition 6.2]).

We start by stating a technical lemma, whose proof can be easily obtained by Lemma 1 as in [28, Lemma 6.1].

Lemma 2 *Let $r, b > 0$ and set $D := (0, r)^{n-1} \times (0, b)$ and $\Gamma := (0, r)^{n-1} \times \{0\}$. Then the map $E : W^{1,A}(\Gamma, \mathbb{R}^n) \to W^{1,A}(D, \mathbb{R}^n)$ defined by $E\mathbf{u} := \mathbf{u} \circ \eta \circ \pi$ is* *linear and bounded. Moreover,* $\det D(E\mathbf{u}) = 0$ *and* $(E\mathbf{u})_{|\Gamma} = \mathbf{u}$*. If, in addition,* $(\Lambda_{n-1}D\mathbf{u})\mathbf{e}_n \in L^q(\Gamma, \mathbb{R}^n)$ *for some* $q \geq 1$*, then* $\mathrm{cof}D(E\mathbf{u}) \in L^q(D; \mathbb{R}^{n\times n})$ *and*

$$\|\mathrm{cof}D(E\mathbf{u})\|_{L^q(D;\mathbb{R}^{n\times n})} = b^{1/q}\|(\Lambda_{n-1}D\mathbf{u})\mathbf{e}_n\|_{L^q(\Gamma,\mathbb{R}^n)} .$$

The main result of extension is contained in the following proposition.

Proposition 4 *Let* Ω *be an extendable open set. Then there exist an open set* $\widetilde{\Omega} \supset \bar{\Omega}$ *and a linear bounded operator* $E : W^{1,A}(\Omega; \mathbb{R}^n) \cap W^{1,A}(\partial\Omega; \mathbb{R}^n) \to W^{1,A}(\widetilde{\Omega}; \mathbb{R}^n)$ *such that* $E\mathbf{u} = \mathbf{u}$ *a.e. in* Ω*,* $\det D(E\mathbf{u}) = 0$ *a.e. in* $\widetilde{\Omega}\setminus\Omega$ *and the following hold:*

(i) *if* $(\Lambda_{n-1}D\mathbf{u})\boldsymbol{\nu} \in L^q(\partial\Omega, \mathbb{R}^n)$ *for some* $q \geq 1$*, then* $\mathrm{cof}D(E\mathbf{u}) \in L^q(\widetilde{\Omega}\setminus\Omega; \mathbb{R}^{n\times n})$ *and*

$$\|\mathrm{cof}D(E\mathbf{u})\|_{L^q(\widetilde{\Omega}\setminus\Omega;\mathbb{R}^{n\times n})} \leq C\|(\Lambda_{n-1}D\mathbf{u})\boldsymbol{\nu}\|_{L^q(\partial\Omega,\mathbb{R}^n)}$$

for some constant $C > 0$ *independent of* $\mathbf{u}$*;*

(ii) $\mathbf{u} \in \bar{\mathcal{A}}(\Omega)$ *if and only if* $E\mathbf{u} \in \mathcal{A}(\widetilde{\Omega})$.

Proof We may follow the argument of [28, Proposition 6.3]. Therefore, we just provide a sketch of the proof.

Let $\mathbf{w}$ and N be as in Definition 9, and recall the notation of Section 3.1. We denote by $\widetilde{\pi} : \partial\Omega \times (-\delta, 0] \to \partial\Omega$ the projection onto $\partial\Omega$, and we set $\widetilde{\Omega} := \Omega \cup N$. We then define the function $\widetilde{\mathbf{u}} : \widetilde{\Omega} \to \mathbb{R}^n$ by

$$\widetilde{\mathbf{u}} := \begin{cases} \mathbf{u} & \text{in } \Omega\,, \\ \mathbf{u}_{|\partial\Omega} \circ \widetilde{\pi} \circ \mathbf{w}^{-1} & \text{in } N\,. \end{cases}$$

The first step is to prove that $\widetilde{\mathbf{u}} \in W^{1,A}(N\setminus\partial\Omega, \mathbb{R}^n)$. Then, one shows that the map $\mathbf{u} \to \widetilde{\mathbf{u}}$ is linear and, since

$$\|\widetilde{\mathbf{u}}\|_{W^{1,A}(N\setminus\partial\Omega,\mathbb{R}^n)} \leq c\|\mathbf{u}\|_{W^{1,A}(\partial\Omega,\mathbb{R}^n)}$$

for some constant $c > 0$, the map $\mathbf{u} \to \widetilde{\mathbf{u}}$ is bounded from $W^{1,A}(\partial\Omega, \mathbb{R}^n)$ to $W^{1,A}(N\setminus\partial\Omega, \mathbb{R}^n)$. Furthermore, from the continuity of the trace operator, the trace of $\widetilde{\mathbf{u}}_{|N\setminus\partial\Omega}$ on $\partial\Omega$ is $\mathbf{u}_{|\partial\Omega}$. Thus, $\widetilde{\mathbf{u}} \in W^{1,A}(\widetilde{\Omega}, \mathbb{R}^n)$ and the map $E : W^{1,A}(\Omega; \mathbb{R}^n) \cap W^{1,A}(\partial\Omega; \mathbb{R}^n) \to W^{1,A}(\widetilde{\Omega}; \mathbb{R}^n)$ defined setting $E\mathbf{u} := \widetilde{\mathbf{u}}$ is linear and bounded. The remaining assertions follow from Lemma 2 and explicit computations, that can be performed as in [28, Proposition 6.3]. A key ingredient therein is the stability of compositions with bi-Lipschitz homeomorphisms. □

5.2 Regular functions in $\bar{\mathcal{A}}(\Omega)$

As remarked in the introduction, a key question in the theory of existence in nonlinear elasticity is whether the distributional determinant $\mathrm{Det}D\mathbf{u}$ equals the pointwise determinant $\det D\mathbf{u}$. This can be rewritten equivalently as

$$\frac{1}{n}\mathrm{Div}[\mathrm{adj}\, D\mathbf{u}(\mathbf{x})\mathbf{u}(\mathbf{x})] = \det D\mathbf{u}(\mathbf{x}) \quad \mathbf{x} \in \Omega\,, \tag{21}$$

where Div in the left-hand side stands for the distributional divergence, and can be generalized to

$$\mathrm{Div}[\mathrm{adj}D\mathbf{u}(\mathbf{x})\mathbf{g}(\mathbf{u}(\mathbf{x}))] = \mathrm{div}\mathbf{g}(\mathbf{u}(\mathbf{x}))\det D\mathbf{u}(\mathbf{x})\,, \quad x \in \Omega\,, \tag{22}$$

for all $\mathbf{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$. It is easy to check that both the identities (21) and (22) hold true for smooth functions, say $\mathbf{u} \in C^2(\bar{\Omega}, \mathbb{R}^n)$, as a consequence of Piola's identity $\mathrm{Div}\,\mathrm{cof}\, D\mathbf{u} = \mathbf{0}$.

In a weaker setting, by using the definition of distributional divergence, (22) can be formulated as in (19), and in [29, 44] the class $\mathcal{A}$ of those orientation-preserving Orlicz-Sobolev functions satisfying (19) has been introduced and analyzed. The identity (20) underlying the definition of our class $\bar{\mathcal{A}}$ can be obtained from (22) multiplying by $\phi \in C^\infty(\bar{\Omega})$ and then integrating by parts.

The aim of this section is then to identify the "regular" functions included in the class $\bar{\mathcal{A}}$. As a first remark, we note that the same argument as in [28, Lemma 7.1] gives $C^1(\bar{\Omega}; \mathbb{R}^n) \subset \bar{\mathcal{A}}(\Omega)$. The main result is the following proposition, where we prove that a suitable subclass of $W^{1,A}(\Omega;\mathbb{R}^n) \cap W^{1,A}(\partial\Omega;\mathbb{R}^n)$ is contained in $\bar{\mathcal{A}}(\Omega)$. The idea is to combine the extension property of Proposition 4 with the adaptation to the Orlicz-Sobolev setting of the result by Müller, Qi and Yan [41].

Proposition 5 *Let Ω be an extendable open set. Let $q > \frac{n-1}{n-2}$. Then the class of maps $\mathbf{u} \in W^{1,A}(\Omega;\mathbb{R}^n) \cap W^{1,A}(\partial\Omega;\mathbb{R}^n)$ such that*

$$\mathrm{cof}D\mathbf{u} \in L^q(\Omega;\mathbb{R}^{n\times n})\,, \quad (\Lambda_{n-1}D\mathbf{u})\boldsymbol{\nu} \in L^q(\partial\Omega;\mathbb{R}^n)$$

is a subset of $\bar{\mathcal{A}}(\Omega)$.

Proof Let $\mathbf{u} \in W^{1,A}(\Omega;\mathbb{R}^n) \cap W^{1,A}(\partial\Omega;\mathbb{R}^n)$ be such that $\mathrm{cof}D\mathbf{u} \in L^q(\Omega;\mathbb{R}^{n\times n})$ and $(\Lambda_{n-1}D\mathbf{u})\boldsymbol{\nu} \in L^q(\partial\Omega;\mathbb{R}^n)$. Then, taking into account (2), (8) and Hölder's inequality, $\mathrm{cof}D\mathbf{u} \in L^q$, with q as above, implies $\det D\mathbf{u} \in L^1$. Since $\frac{n-1}{n-2} \geq \frac{n}{n-1}$ and L^A can be embedded into L^{n-1}, the proof can be deduced by [41, Theorem 3.2] applied to $\widetilde{\mathbf{u}}$ the extension of $\mathbf{u}$ to an open set $\widetilde{\Omega} \supset \bar{\Omega}$ obtained with Proposition 4, which complies with $\widetilde{\mathbf{u}} \in W^{1,A}(\widetilde{\Omega}, \mathbb{R}^n)$ and $\mathrm{cof}\, D\widetilde{\mathbf{u}} \in L^q(\widetilde{\Omega};\mathbb{R}^{n\times n})$. This gives $\widetilde{\mathbf{u}} \in \mathcal{A}(\widetilde{\Omega})$, whence, with Proposition 4(ii), we infer $\mathbf{u} \in \bar{\mathcal{A}}(\Omega)$. □

5.3 Some properties of orientation preserving functions in $\bar{\mathcal{A}}(\Omega)$: boundedness and global invertibility

In this section, we preliminarly prove that functions $\mathbf{u} \in \bar{\mathcal{A}}(\Omega)$ with $\det D\mathbf{u} \geq 0$ a.e. are bounded, thus proving the "global" counterpart of the local boundedness result [29, Proposition 4.2].

Proposition 6 *If* $\mathbf{u} \in \bar{\mathcal{A}}(\Omega)$ *with* $\det D\mathbf{u} \geq 0$ *a.e. then* $\deg(\mathbf{u}, \Omega, \cdot) = \mathcal{N}_{\mathbf{u}}$ *a.e.,* $\mathrm{im_T}(\mathbf{u}, \Omega) = \mathrm{im_G}(\mathbf{u}, \Omega)$ *a.e. and* $\mathbf{u} \in L^\infty(\Omega; \mathbb{R}^n)$.

Proof The proof can be obtained as in [28, Proposition 8.4] by using the formula for the distributional derivative of the degree, Proposition 3. We then omit the details.□

As a consequence of Proposition 6 we obtain the following global invertibility result.

Theorem 4 *Let* $\mathbf{u}, \mathbf{u}_0 \in \bar{\mathcal{A}}(\Omega)$ *be such that* $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega}$, $\det D\mathbf{u} > 0$ *a.e.,* $\det D\mathbf{u}_0 \geq 0$ *a.e. and* $\mathbf{u}_0$ *injective a.e. Then* $\mathbf{u}$ *is injective a.e. and* $\mathrm{im_G}(\mathbf{u}, \Omega) = \mathrm{im_G}(\mathbf{u}_0, \Omega)$ *a.e.*

Proof See [28, Theorem 9.1]. □

The a.e. injectivity of $\mathbf{u}$ allows to define a.e. its inverse (see [28, Definition 9.2]).

Definition 10 Let $\mathbf{u} \in \bar{\mathcal{A}}(\Omega)$ be injective a.e. Let Ω_0 be the set of Definition 6, and $\Omega_1 \subset \Omega_0$ such that $\mathcal{L}^n(\Omega \backslash \Omega_1) = 0$ and $\mathbf{u}|_{\Omega_1}$ be injective. The inverse $\mathbf{u}^{-1} : \mathrm{im_T}(\mathbf{u}, \Omega) \to \mathbb{R}^n$ is defined a.e. as $\mathbf{u}^{-1}(\mathbf{y}) = \mathbf{x}$, for every $\mathbf{y} \in \mathbf{u}(\Omega_1)$, where $\mathbf{x} \in \Omega_1$ satisfies $\mathbf{u}(\mathbf{x}) = \mathbf{y}$.

The a.e. inverse is a Sobolev function, as ensured by the following result which represents the global counterpart of [29, Proposition 4.11].

Theorem 5 *Let* $\mathbf{u} \in \bar{\mathcal{A}}(\Omega)$ *be injective a.e. with* $\det D\mathbf{u} > 0$ *a.e. Then* $\mathbf{u}^{-1} \in W^{1,1}(\mathrm{im_T}(\mathbf{u}, \Omega), \mathbb{R}^n)$ *and* $D\mathbf{u}^{-1}(\mathbf{y}) = D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))^{-1}$ *for a.e.* $\mathbf{y} \in \mathrm{im_T}(\mathbf{u}, \Omega)$.

Proof [28, Theorem 9.3]. □

6 Existence of minimizers

In this section we prove existence of minimizers for functionals of the form

$$I(\mathbf{u}) = \int_\Omega W(\mathbf{x}, \mathbf{u}(\mathbf{x}), D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \tag{23}$$

on the class $\bar{\mathcal{A}}(\Omega)$, under the assumption that W is *polyconvex* in the last variable (see, e.g., [19]).

The following result establishes the compactness in the class $\bar{\mathcal{A}}(\Omega)$.

Proposition 7 *Let* $\{\mathbf{u}_j\}_{j\in\mathbb{N}} \subset \bar{\mathcal{A}}(\Omega)$ *be a bounded sequence in* $W^{1,A}(\Omega; \mathbb{R}^n) \cap W^{1,A}(\partial\Omega; \mathbb{R}^n)$ *and such that* $\{\det D\mathbf{u}_j\}_{j\in\mathbb{N}}$ *is equiintegrable. Then, there exist a subsequence (not relabeled)* $\{\mathbf{u}_j\}$ *and a function* $\mathbf{u} \in \bar{\mathcal{A}}(\Omega)$ *such that*

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \ \ \textit{in } W^{1,A}(\Omega; \mathbb{R}^n) \cap W^{1,A}(\partial\Omega; \mathbb{R}^n) \ \ \textit{and} \ \ \det D\mathbf{u}_j \rightharpoonup \det D\mathbf{u} \ \ \textit{in } L^1(\Omega) \tag{24}$$

as $j \to +\infty$.

Proof The argument is quite standard, and we adapt the proof of [28, Proposition 10.2] to our setting.

By assumptions, we can find $\mathbf{u} \in W^{1,A}(\Omega;\mathbb{R}^n)$, $\mathbf{v} \in W^{1,A}(\partial\Omega;\mathbb{R}^n)$ and $w \in L^1(\Omega)$ such that, up to a (not relabeled) subsequence, we have

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \text{ in } W^{1,A}(\Omega;\mathbb{R}^n)\,, \; \mathbf{u}_j \rightharpoonup \mathbf{v} \text{ in } W^{1,A}(\partial\Omega;\mathbb{R}^n)\,, \; \det D\mathbf{u}_j \rightharpoonup w \text{ in } L^1(\Omega)\,.$$

Up to a further subsequence, we may assume that $\mathbf{u}_j \to \mathbf{u}$ a.e., and, taking into account the embedding result of [29, Proposition 2.6] and the weak continuity of the cofactors (see, e.g., [19, Theorem 8.20]), we get $\operatorname{cof} D\mathbf{u}_j \rightharpoonup \operatorname{cof} D\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^{n\times n})$. The continuity of the traces $\mathbf{u}_{j\,|\partial\Omega} \rightharpoonup \mathbf{u}_{|\partial\Omega}$ in $L^A(\partial\Omega;\mathbb{R}^n)$ implies $\mathbf{v} = \mathbf{u}_{|\partial\Omega}$, while (17) and [24, Theorem 3] give $w = \det D\mathbf{u}$ a.e. The rest of the proof leading to the identity (18) is based on standard computations and on the weak continuity result

$$(\Lambda_{n-1} D\mathbf{u}_j)\boldsymbol{\nu} \rightharpoonup (\Lambda_{n-1} D\mathbf{u})\boldsymbol{\nu} \quad \text{in } \mathrm{LLog}^{\frac{\alpha}{n-1}}\mathrm{L}(\partial\Omega,\mathbb{R}^n)$$

which can be inferred along the lines of [28, Proposition 10.1]. □

Now, we can prove the existence of minimizers for I on a suitable subclass of $\bar{\mathcal{A}}(\Omega)$.

Theorem 6 *Let* $\mathbf{u}_0 \in \bar{\mathcal{A}}(\Omega)$ *be injective a.e. and such that* $\det D\mathbf{u}_0 \geq 0$ *a.e. Let* $W : \Omega \times \mathrm{im_T}(\mathbf{u}_0,\Omega) \times \mathbb{R}^{n\times n}_+ \to \mathbb{R}$ *comply with the following assumptions:*

(i) *W is measurable;*
(ii) $W(\mathbf{x},\cdot,\cdot)$ *is lower semicontinuous for a.e.* $\mathbf{x} \in \Omega$*;*
(iii) $W(\mathbf{x},\mathbf{y},\cdot)$ *is polyconvex for a.e.* $\mathbf{x} \in \Omega$ *and for every* $\mathbf{y} \in \mathrm{im_T}(\mathbf{u}_0,\Omega)$*;*
(iiii) *there exist a constant* $c > 0$*, a function* $a \in L^1(\Omega)$ *and a Borel function* $h : (0,\infty) \to [0,\infty)$ *with*

$$\lim_{t\searrow 0} h(t) = \lim_{t\to\infty} \frac{h(t)}{t} = \infty \tag{25}$$

such that

$$W(\mathbf{x},\mathbf{y},\mathbf{F}) \geq a(\mathbf{x}) + c\,A(\|\mathbf{F}\|) + h(\det\mathbf{F}) \tag{26}$$

for a.e. $\mathbf{x} \in \Omega$*, every* $\mathbf{y} \in \mathrm{im_T}(\mathbf{u}_0,\Omega)$ *and* $\mathbf{F} \in \mathbb{R}^{n\times n}_+$.

Define

$$\mathcal{A} := \left\{\mathbf{u} \in \bar{\mathcal{A}}(\Omega) : \; \det D\mathbf{u} > 0 \textit{ a.e. } \textit{ and } \; \mathbf{u}_{|\partial\Omega} = \mathbf{u}_{0\,|\partial\Omega}\right\}$$

and assume that $\mathcal{A} \neq \emptyset$ *and* $I \not\equiv \infty$ *on* $\mathcal{A}$*. Then I admits a minimizer on* $\mathcal{A}$*, and any element in* $\mathcal{A}$ *is injective a.e.*

Proof Once a compactness result has been proved, the proof of the existence of minimizers is based on a well-known argument. We follow the scheme in [28, Theorem 10.3].

Under the coerciveness assumption (26), the lower semicontinuity of the functional I can be inferred from [5, Theorem 5.4]. The a.e. injectivity of each function in $\mathcal{A}$ is a consequence of Theorem 4.

Now, let $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ be a minimizing sequence of I in $\mathcal{A}$. Then, by Proposition 6 and Theorem 4 we get

$$\mathbf{u}_j(\mathbf{x}) \in \operatorname{im}_{\mathrm{T}}(\mathbf{u}_0, \Omega) \qquad \text{for a.e. } \mathbf{x} \in \Omega \text{ and all } j \in \mathbb{N}\,. \tag{27}$$

From assumption *(iiii)* and De la Vallée Poussin's Theorem (Theorem 3) we infer that $\{D\mathbf{u}_j\}_{j\in\mathbb{N}}$ is equibounded in $L^A(\Omega, \mathbb{R}^{n\times n})$ and $\{\det D\mathbf{u}_j\}_{j\in\mathbb{N}}$ is equiintegrable. Furthermore, (27) and the boundedness of topological image imply that $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ is bounded in L^∞ and then in $W^{1,A}(\Omega, \mathbb{R}^n)$. The compactness result of Proposition 7 provides a function $\mathbf{u} \in \bar{\mathcal{A}}(\Omega)$ such that, up to a subsequence,

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \;\text{ in }\; W^{1,A}(\Omega; \mathbb{R}^n)\,, \quad \det D\mathbf{u}_j \rightharpoonup \det D\mathbf{u} \;\text{ in }\; L^1(\Omega)\,.$$

Now, a simple argument by contradiction based on assumption *(iiii)* shows that $\det D\mathbf{u} > 0$ a.e. Then, since the boundary condition is preserved in the limit, $\mathbf{u}_{|\partial\Omega} = \mathbf{u}_{0\,|\partial\Omega}$, whence $\mathbf{u} \in \mathcal{A}$. This concludes the proof. □

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