Maps into projective spaces: liquid crystal and conformal energies

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Abstract. *Variational problems for the liquid crystal energy of mappings from three-dimensional domains into the real projective plane are studied. More generally, we study the dipole problem, the relaxed energy, and density properties concerning the conformal p-energy of mappings from n-dimensional domains that are constrained to take values into the p-dimensional real projective space, for any positive integer p. Furthermore, a notion of optimally connecting measure for the singular set of such class of maps is given.*

A *liquid crystal* is a state of a matter, called mesomorphic, intermediate between a crystalline solid and a normal isotropic liquid, in which long rod-shaped molecules display orientational order.

According to the *continuum* description in the *Ericksen-Leslie theory* [10, 29], a configuration of a liquid crystal which occupies a domain Ω in $\mathbb{R}^3$ is described mathematically as a unitary vector field $u(x)$ in Ω. The bulk energy associated to the configuration u is given by

$$\mathcal{E}(u,\Omega) := \int_\Omega W(u,Du)\,dx\,. \tag{0.1}$$

The form of the energy was derived by Oseen [32] on the basis of a molecular theory, and by Frank [15] as a consequence of Galilean invariance. This means that the energy density satisfies the *invariance properties*

$$\begin{array}{l} W(u,Du) = w(-u,-Du) \\ W(Qu,Q\,Du\,Q^T) = W(u,Du) \qquad \forall\, Q\in O(3)\,, \end{array} \tag{0.2}$$

so that the functional (0.1) is well-defined on the class of vector fields in Ω, *regardless of the orientation.*

We address to [7, 8, 9, 11] for further information on the general theory.

For *nematic* vector fields, the liquid crystal appears to have low viscosity and a thread-like structure. In this case, the energy density can be written as

$$W(u,Du) := \alpha\,|Du|^2 + (k_1-\alpha)\,(\operatorname{div} u)^2 + (k_2-\alpha)\,(u\cdot\operatorname{curl} u)^2 + (k_3-\alpha)\,|u\times\operatorname{curl} u|^2\,, \tag{0.3}$$

where the constants depend on the specific material under consideration at a fixed temperature, and satisfy $\alpha>0$ and $k_i\geq\alpha$ for every i.

Formula (0.3) is obtained from the general form of the Oseen-Frank energy density by means of a change of parameter in the divergence-free term, compare e.g. [20, Vol. II, Sec. 5].

Taking $k_i=\alpha=1/2$ for every i, the liquid crystal energy (0.1) agrees with the *Dirichlet energy*

$$\mathbf{D}(u,\Omega) := \frac{1}{2}\int_\Omega |Du|^2\,dx\,. \tag{0.4}$$

Therefore, in this case the theory of liquid crystals reduces to the by now classical theory of *harmonic maps* from Ω into $\mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$.

Physical evidence shows that in general *the ends of the molecules of a nematic liquid cannot be distinguished.* This means that the vector field u should actually take values into the *projective plane* $\mathbb{RP}^2$, obtained by identification of antipodal points in $\mathbb{S}^2$. But the *lack of orientability* of $\mathbb{RP}^2$ causes a lot of trouble in the analysis of a variational theory.

In their celebrated paper [5] of 1986, Brezis-Coron-Lieb consider the *Dipole problem* for harmonic maps with values into $\mathbb{RP}^2$. One of the aim of this paper is to recover and extend some of the results from [5]. To this purpose, we shall see the projective plane $\mathbb{RP}^2$ as an embedded submanifold RP^2 of $\mathbb{R}^6$, and we shall work with the corresponding class of Sobolev maps

$$W^{1,2}(B^n, \mathrm{RP}^2) := \{u \in W^{1,2}(B^n, \mathbb{R}^6) \mid u(x) \in \mathrm{RP}^2 \quad \text{for a.e. } x \in B^n\},$$

where B^n is the unit ball in $\mathbb{R}^n$, the physical dimension being $n = 3$.

In the setting of Sobolev maps into $\mathbb{S}^2$,

$$W^{1,2}(B^n, \mathbb{S}^2) := \{u \in W^{1,2}(B^n, \mathbb{R}^3) \,:\, |u(x)| = 1 \quad \text{for a.e. } x \in B^n\},$$

both the variational theories of harmonic mappings (0.4), in any dimension n, and of the liquid crystal energy (0.1) are quite understood. The stable equilibrium for liquid crystals has been studied in [23, 24, 25, 26, 27, 28, 30, 39].

It is well-known that in the classical Sobolev approach to the theory of harmonic maps, the weak limit process destroys energy concentration, the so called bubbling-off phenomenon, and does not preserve geometric properties such as the degree, showing e.g. creation of cavitations.

Using tools from *Geometric measure theory* [12, 14, 35], variational results (such as e.g. the representation formula for the *relaxed energy*) have been tackled in a satisfactory way in any dimension n by means of the theory of *Cartesian currents* of Giaquinta-Modica-Souček [17, 20], see also [22]. Roughly speaking, given a sequence of smooth maps u_k from B^n into $\mathbb{S}^2$ with equibounded Dirichlet energies, and denoted by G_{u_k} the current integration of n-forms on the naturally oriented graph of u_k, possibly passing to a subsequence the currents G_{u_k} weakly converge to a Cartesian current in $B^n \times \mathbb{S}^2$. Now, the weak convergence as currents preserves geometric properties such as the absence of fractures, the orientation, and the degree. Therefore, by Federer-Fleming's closure theorem [14], it turns out that a Cartesian current T in the so called class $\mathrm{cart}^{2,1}$ is an integer multiplicity (say i.m.) rectifiable n-current in $B^n \times \mathbb{S}^2$ given by

$$T = G_u + L \times [\![\, \mathbb{S}^2 \,]\!] \,. \tag{0.5}$$

In this formula, G_u is the current carried by the graph of a Sobolev map $v \in W^{1,2}(B^n, \mathbb{S}^2)$, and L is an i.m. rectifiable current of codimension two in B^n that "encloses the singularity" of u. For example, in the physical dimension $n = 3$, if the sequence $\{u_k\}$ weakly converges in $W^{1,2}$ to the Sobolev function $\overline{u}(x) = x/|x|$, then (possibly passing to a subsequence) the currents G_{u_k} weakly converge to a Cartesian current as in (0.5), where $u = \overline{u}$ and L is the current integration of 1-forms on an oriented line with initial point at the boundary and final point at the center of the ball B^3.

Using the same geometric approach, an exhaustive variational theory of liquid crystals has been developed in [18], see also [20, Vol. II, Sec. 5.1].

In a similar way, one may consider for any integer exponent $\mathfrak{p} \geq 2$ the *conformal* $\mathfrak{p}$*-energy*

$$\mathbf{D}_{\mathfrak{p}}(u, B^n) := \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{B^n} |Du|^{\mathfrak{p}} \, dx$$

of $W^{1,\mathfrak{p}}$-mappings that take values into the unit $\mathfrak{p}$-sphere $\mathbb{S}^{\mathfrak{p}}$, i.e., in the class

$$W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}}) := \{u \in W^{1,\mathfrak{p}}(B^n, \mathbb{R}^{\mathfrak{p}+1}) \,:\, |u(x)| = 1 \quad \text{for a.e. } x \in B^n\}.$$

Paralleling the theory of harmonic maps into the sphere, variational problems for the $\mathfrak{p}$-energy of mappings with values into the $\mathfrak{p}$-sphere have been dealt in any dimension n, see e.g. [19, 20] and Secs. 5 and 6 below.

However, when one tries to attack similar problems in the case of mappings with values into the $\mathfrak{p}$-*dimensional projective space* $\mathbb{RP}^{\mathfrak{p}}$, a further difficulty occurs in the case $\mathfrak{p} \geq 2$ even, as $\mathbb{RP}^{\mathfrak{p}}$ is orientable if and only if $\mathfrak{p}$ is odd.

As mentioned above, in this paper we shall see the projective $\mathfrak{p}$-space $\mathbb{RP}^{\mathfrak{p}}$ as an embedded submanifold $\mathrm{RP}^{\mathfrak{p}}$ of some Euclidean space, i.e., see Sec. 1,

$$\mathrm{RP}^{\mathfrak{p}} := g_{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}), \qquad g_{\mathfrak{p}} : \mathbb{S}^{\mathfrak{p}} \to \mathbb{R}^{N(\mathfrak{p})}, \qquad N(\mathfrak{p}) := \frac{(\mathfrak{p}+1)(\mathfrak{p}+2)}{2} \tag{0.6}$$

and we shall correspondingly work with the Sobolev class

$$W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}}) := \{u \in W^{1,\mathfrak{p}}(B^n, \mathbb{R}^{N(\mathfrak{p})}) \mid u(x) \in \mathrm{RP}^{\mathfrak{p}} \quad \text{for a.e. } x \in B^n\}.$$

Notice that we have $g_{\mathfrak{p}}(-y) = g_{\mathfrak{p}}(y)$, whereas

$$|Du| = |Dv| \qquad \text{if } \; u = g_{\mathfrak{p}} \circ v \; \text{ for some } \; v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}}) \,.$$

In the case $\mathfrak{p} \geq 3$ odd, since $\mathrm{RP}^{\mathfrak{p}}$ is oriented, a homological theory based on Cartesian currents can be performed, see Sec. 5 below. However, this cannot be done for $\mathfrak{p} \geq 2$ even.

In fact, when analyzing e.g. the weak limits of sequences of currents integration of forms on graphs of smooth maps in $W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$ with equibounded $\mathfrak{p}$-energies, no concentration phenomenon can be seen by testing with forms, for $\mathfrak{p} \geq 2$ even.

To avoid cancellation in the limit process, one may try to work with measures associated to graphs, and settle the problem in the framework e.g. of *rectifiable varifolds*, see [35]. However, we shall not pursue this direction.

MAIN RESULTS. In Sec. 1, we collect some preliminary facts about continuous maps into the projective space $\mathbb{RP}^{\mathfrak{p}}$. According to (0.6), in Sec. 2 we shall then prove in any dimension n the following:

Theorem 0.1 *Let $\mathfrak{p} \geq 2$. For every $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$, there exist exactly two Sobolev maps $v_1, v_2 \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v_i = u$ a.e. in B^n. Moreover, $v_2 = -v_1$ and $\mathbf{D}_{\mathfrak{p}}(v_i, B^n) = \mathbf{D}_{\mathfrak{p}}(u, B^n)$.*

This allows us to speak in a consistent way of *singularity, degree, D-fields*, Sec. 3, *flat norm* and *minimal connections*, Sec. 4, for $W^{1,\mathfrak{p}}$-maps with values in $\mathrm{RP}^{\mathfrak{p}}$, for any integer exponent $\mathfrak{p} \geq 2$ and in any dimension.

In fact, making use of the corresponding features concerning Sobolev maps in $W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$, in Sec. 4 we shall analyze the Dipole problem as in [5, VIII-B], giving detailed proofs.

Notice that in the notation from [5, VIII-B] we have $\mathfrak{p} = N - 1$, hence in our paper we e.g. have $\mathfrak{p}$ odd if N is even in [5].

Trivial examples show that Theorem 0.1 is false for $\mathfrak{p} = 1$. Related questions about $W^{1,1}$-maps into the projective line RP^1 are discussed in [31], where we also compare the case $\mathfrak{p} = 1$ with the case $\mathfrak{p} \geq 3$ odd.

In Sec. 5, we shall then study the *weak limit points* of sequences of smooth maps in $W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$ with equibounded $\mathfrak{p}$-energies, and *density properties* such as in Theorem 0.2 below.

For $\mathfrak{p} \geq 3$ odd, we shall also see that the Dipole problem analyzed in Sec. 4 can be reformulated in terms of Cartesian currents in $\mathbb{R}^{\mathfrak{p}+1} \times \mathrm{RP}^{\mathfrak{p}}$. Moreover, in this framework the minimum is attained, see Proposition 5.13, and is given by the energy of a current of the type $G_P + L \times [\![\mathrm{RP}^{\mathfrak{p}}]\!]$, where G_P is the current carried by the graph of a constant map, and L is an oriented line connecting in an optimal way the prescribed singularities.

Therefore, in such a geometric approach, *point defects connected by lines of concentration* occur, whereas in the classical Sobolev approach only point defects with total degree equal to zero come into the play.

In Sec. 7, we shall discuss a notion of *optimally connecting measure* of the *singular set* of Sobolev maps with values into the non-orientable projective space $\mathrm{RP}^{\mathfrak{p}}$, i.e., for $\mathfrak{p} \geq 2$ even.

Taking e.g., $\mathfrak{p} = 2$, for every Sobolev map $u \in W^{1,2}(B^n, \mathrm{RP}^2)$, where $n \geq 3$, we can define a countably $(n-2)$-rectifiable set $\mathcal{L}_u$ in B^n, equipped with an $\mathcal{H}^{n-2} \llcorner \mathcal{L}_u$-summable and non-negative integer valued multiplicity function $\theta_u : \mathcal{L} \to \mathbb{N}^+$, such that *the measure*

$$\mu_u := \theta_u \, \mathcal{H}^{n-2} \llcorner \mathcal{L}_u$$

encloses the singularity of u in an optimal sense. Roughly speaking, the *total variation*

$$|\mu_u|(B^n) = \int_{\mathcal{L}_u} \theta_u \, d\mathcal{H}^{n-2}$$

agrees with the *integral minimal connection* of the $(n-3)$-dimensional current that describes the homological singularity of *any* Sobolev map $v \in W^{1,2}(B^n, \mathbb{S}^2)$ that is linked to u by the relation $g_2 \circ v = u$, according to Theorem 0.1.

In the physical dimension $n = 3$, it turns out that

$$4\pi \cdot |\mu_u|(B^3) = \mathbf{L}(u, B^3) \qquad \forall \, u \in W^{1,2}(B^3, \mathrm{RP}^2) \,,$$

where $\mathbf{L}(u, B^3)$ is the *flat norm*

$$\mathbf{L}(u, B^3) := \sup \left\{ \int_{\Omega} D(v) \cdot D\phi \; dx \mid \phi \in C_c^\infty(\Omega) \,, \; \|d\phi\| \leq 1 \;\text{ in }\; B^3 \right\}.$$

In this formula, $D(v)$ is the classical D-field of *any* Sobolev map $v \in W^{1,2}(B^n, \mathbb{S}^2)$ satisfying $g_2 \circ v = u$. Theorem 0.1, in fact, yields that *the definition of* $\mathbf{L}(u, B^3)$ *does not depend on the choice of* v, as $D(-v) = -D(v)$, compare [5].

Moreover, we shall obtain a density property for the class $W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$, for any $n \geq \mathfrak{p}+1$, that we state here in the case $\mathfrak{p} = 2$.

Theorem 0.2 *Let* $u \in W^{1,2}(B^n, \mathrm{RP}^2)$*, where* $n \geq 3$*. There exists a sequence of* smooth *maps* $\{u_k\} \subset W^{1,2}(B^n, \mathrm{RP}^2)$ *satisfying the following properties:*

i) $u_k \rightharpoonup u$ *weakly in* $W^{1,2}$ *as* $k \to \infty$*;*

ii) $\mathbf{D}(u_k, B^n) \to \mathbf{D}(u, B^n) + 4\pi \cdot |\mu_u|(B^n)$ *as* $k \to \infty$*;*

iii) $\dfrac{1}{2}|Du_k|^2\, \mathcal{L}^n \llcorner B^n \rightharpoonup \dfrac{1}{2}|Du|^2\, \mathcal{L}^n \llcorner B^n + 4\pi \cdot \mu_u$ *weakly as measures;*

iv) *for any open set* A *contained in* $B^n \setminus \operatorname{spt}\mu_u$*, we have strong* $W^{1,2}$*-convergence of* $u_{k|A}$ *to* $u_{|A}$*.*

In Sec. 6, in the same spirit as Lebesgue's relaxed area, we shall also introduce the *relaxed energy* of maps $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$, defined by

$$\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^n) := \inf \left\{ \liminf_{k\to\infty} \mathbf{D}_{\mathfrak{p}}(u_k, B^n) \mid \{u_k\} \subset C^\infty(B^n, \mathrm{RP}^{\mathfrak{p}}),\ u_k \rightharpoonup u \text{ weakly in } W^{1,\mathfrak{p}}(B^n, \mathbb{R}^{N(\mathfrak{p})}) \right\} .$$

In low dimension $n = \mathfrak{p}$, by Schoen-Uhlenbeck density theorem [34] we clearly have

$$\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^{\mathfrak{p}}) = \mathbf{D}_{\mathfrak{p}}(u, B^{\mathfrak{p}}) \qquad \forall\, u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}}) .$$

In higher dimension $n \geq \mathfrak{p}+1$, we shall prove that the relaxed energy $\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^n)$ is finite, and actually

$$\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^n) \leq 2\, \mathbf{D}_{\mathfrak{p}}(u, B^n) \quad \forall\, u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}}) .$$

We shall also obtain an explicit formula for the relaxed energy, that in the case $\mathfrak{p} = 2$ reads as

$$\overline{\mathbf{D}}(u, B^n) = \mathbf{D}(u, B^n) + 4\pi \cdot |\mu_u|(B^n) . \tag{0.7}$$

Therefore, in dimension $n = 3$ we have

$$\overline{\mathbf{D}}(u, B^3) = \mathbf{D}(u, B^3) + \mathbf{L}(u, B^3) , \tag{0.8}$$

a formula that goes back to the analogous one proved by Bethuel-Brezis-Coron [3] for the relaxed Dirichlet energy of maps in $W^{1,2}(B^3, \mathbb{S}^2)$.

Notice that a formula similar to (0.7), and to (0.8) for $n = 3$, holds true if we replace B^n with any bounded domain $\Omega \subset \mathbb{R}^n$, or with e.g. $\Omega = \mathbb{S}^n$, the n-sphere in $\mathbb{R}^{n+1}$. This clearly yields that the relaxed energy is a *non-local functional*, for $n \geq 3$.

Moreover, one may similarly consider the analogous problem for Sobolev mappings satisfying a prescribed *Dirichlet-type condition* at the boundary.

Theorem 0.2, as well as the formula (0.7) for the relaxed energy, shows that in the limit process the measure corresponding to the energy density concentrates on a measure the singular part of which agrees with μ_u. This will be discussed with an example at the end of the introduction.

In Sec. 8, using the same approach as above, we shall finally analyze the problem of the liquid crystal energy (0.1) for Sobolev maps u in $W^{1,2}(B^3, \mathrm{RP}^2)$. By Theorem 0.1, and by the invariance properties (0.2), it turns out that the energy $\mathcal{E}(u, B^3)$ is well-defined by the energy $\mathcal{E}(v, B^3)$ of any Sobolev map v in $W^{1,2}(B^3, \mathbb{S}^2)$ such that $g_2 \circ v = u$. As we have seen, the unit vector field $u(x)$ describes a *liquid such that the ends of the molecules cannot be distinguished.*

We shall restrict to the physical model of a nematic liquid crystal, with energy density given by (0.3), where, for the sake of simplicity, we normalize the constant α to $\alpha = 1$. The case of *cholesteric liquid crystals*, compare e.g. [20, Vol. II, Sec. 5.1], can be treated in a similar way.

The relevant quantity corresponding to the energy density (0.3), where $\alpha = 1$, is the physical constant

$$\Gamma(k_1,k_2,k_3) := \sqrt{k\,k_3}\int_0^1 \sqrt{1+\Big(\frac{k}{k_3}-1\Big)s^2}\,ds\,, \qquad k:=\min\{k_1,k_2\}\,,$$

compare [20, Vol. II, Sec. 5.1.2]. Notice that if $k_i = 1$ for every i, we have $\mathcal{E}(u,B^3) = 2\,\mathbf{D}(u,B^3)$ and $\Gamma(1,1,1) = 1$, whereas in general $\Gamma(k_1,k_2,k_3) \geq 1$.

Using results taken from [18] about the liquid crystal energy of maps in $W^{1,2}(B^3,\mathbb{S}^2)$, we shall obtain the following density result, that extends Theorem 0.2:

Theorem 0.3 *Let $u \in W^{1,2}(B^3,\mathbb{RP}^2)$ and let $\mu_u := \theta_u\,\mathcal{H}^1 \llcorner \mathcal{L}_u$ be an* optimally connecting measure*. Then there exists a sequence of* smooth *maps $\{u_k\} \subset W^{1,2}(B^3,\mathrm{RP}^2)$ satisfying the following properties:*

i) *$u_k \rightharpoonup u$ weakly in $W^{1,2}$ as $k \to \infty$;*

ii) *$\mathcal{E}(u_k,B^3) \to \mathcal{E}(u,B^3) + 8\pi\,\Gamma(k_1,k_2,k_3)\cdot|\mu_u|(B^3)$ as $k\to\infty$;*

iii) *$W(u_k,Du_k)\,\mathcal{L}^3 \llcorner B^3 \rightharpoonup W(u,Du)\,\mathcal{L}^3 \llcorner B^3 + 8\pi\,\Gamma(k_1,k_2,k_3)\,\mu_u$ weakly as measures;*

iv) *for any open set A contained in $B^3 \setminus \operatorname{spt}\mu_u$, we have strong $W^{1,2}$-convergence of $u_{k|A}$ to $u_{|A}$.*

In a similar way, we shall also discuss the Dipole problem for the liquid crystal energy in $W^{1,2}(B^3,\mathrm{RP}^2)$.

As to the relaxed energy of maps u in $W^{1,2}(B^3,\mathrm{RP}^2)$, defined by

$$\overline{\mathcal{E}}(u,B^3) := \inf\left\{\liminf_{k\to\infty}\mathcal{E}(u_k,B^3) \mid \{u_k\}\subset C^\infty(B^3,\mathrm{RP}^2)\,,\ u_k \rightharpoonup u \text{ weakly in } W^{1,2}(B^3,\mathbb{R}^6)\right\}\,,$$

we shall then obtain the following representation formula, that extends (0.7) and (0.8):

Theorem 0.4 *For every $u\in W^{1,2}(B^3,\mathrm{RP}^2)$ the relaxed energy $\overline{\mathcal{E}}(u,B^3)$ is finite. Moreover,*

$$\overline{\mathcal{E}}(u,B^3) = \mathcal{E}(u,B^3) + 8\pi\,\Gamma(k_1,k_2,k_3)\cdot|\mu_u|(B^3) = \mathcal{E}(u,B^3) + 2\,\Gamma(k_1,k_2,k_3)\cdot\mathbf{L}(u,B^3)\,.$$

Theorem 0.4 holds true if we replace B^3 with any bounded domain $\Omega\subset\mathbb{R}^3$, or with e.g. $\Omega = \mathbb{S}^3$. Therefore, again the relaxed energy is a non-local functional. Moreover, one may similarly consider the analogous problem for Sobolev mappings satisfying a prescribed Dirichlet-type condition at the boundary.

We finally mention here that the interpretation of defects as singularities is meaningful only when the degree of order is the same everywhere in the body Ω. This is the case of the theory of liquid crystals of maps into the real projective plane that we consider here. When the degree of order may vary, a manifold richer in states than $\mathbb{RP}^2$ has to be considered in the attempt to describe transitions between differently ordered phases which take different places in space, compare e.g. [6].

An example. In the physical dimension $n = 3$, let $v\in W^{1,2}(B^3,\mathbb{S}^2)$ a Sobolev map with e.g. two point singularities of *degree* ±1 at the points $a_\pm = (\pm a,0,0)$, where $0<a<1$, and let $u := g_2\circ v$ the corresponding map in $W^{1,2}(B^3,\mathrm{RP}^2)$. In this case, we set $\mu_u = \mathcal{H}^1\llcorner\mathcal{L}$, where $\mathcal{L}$ is the line segment connecting a_+ and a_-, so that $|\mu_u|(B^3) = |a_+ - a_-|$.

Theorems 0.3 and 0.4 yield that the energy concentrates on $\mathcal{L}$, as the limit energy contains the extra term given by the factor $8\pi\,\Gamma(k_1,k_2,k_3)$ times *the length* $|a_+ - a_-|$ *of the Dipole* $\mathcal{L}$. Correspondingly, for the Dirichlet energy we have the extra term $4\pi\,|a_+ - a_-|$. On the other hand, the area of the projective space RP^2 is 2π.

From a phenomenological point of view, this means that in the process of concentration, at each point of the Dipole $\mathcal{L}$ the crystalline character disappears, and the vector field of the nematic liquid crystal covers *twice* each direction of the projective space RP^2. This is coherent with well-known topological facts that are collected in Sec. 1, and with the Dipole problem described by Brezis-Coron-Lieb in [5, Sec. VIII-B-c].

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1 Maps into projective spaces

In this section we collect some well-known facts about maps taking values into real projective spaces, focusing on the case of maps into the real projective plane.

MAPS INTO $\mathbb{RP}^{\mathfrak{p}}$. Let $\mathfrak{p} \geq 2$ integer and $\mathbb{S}^{\mathfrak{p}}$ denote the unit $\mathfrak{p}$-sphere in $\mathbb{R}^{\mathfrak{p}+1}$

$$\mathbb{S}^{\mathfrak{p}} := \{y \in \mathbb{R}^{\mathfrak{p}+1} \, : \, |y| = 1\} \, .$$

We recall that the *real projective* $\mathfrak{p}$*-dimensional space* $\mathbb{RP}^{\mathfrak{p}}$ is defined by the quotient space $\mathbb{RP}^{\mathfrak{p}} := \mathbb{S}^{\mathfrak{p}} / \sim_{\mathfrak{p}}$, the equivalence relation being $y \sim_{\mathfrak{p}} \widetilde{y} \iff y = \widetilde{y}$ or $y = -\widetilde{y}$, and we denote by $[y]_{\mathfrak{p}}$ the elements of $\mathbb{RP}^{\mathfrak{p}}$. Every function $v : A \subset \mathbb{R}^n \to \mathbb{S}^{\mathfrak{p}}$ yields a function $[v]_{\mathfrak{p}} : A \to \mathbb{RP}^{\mathfrak{p}}$ defined by $[v]_{\mathfrak{p}} := P_{\mathfrak{p}} \circ v$, where $P_{\mathfrak{p}} : \mathbb{S}^{\mathfrak{p}} \to \mathbb{RP}^{\mathfrak{p}}$ is the canonical projection map $P_{\mathfrak{p}}(y) = [y]_{\mathfrak{p}}$, i.e., $[v]_{\mathfrak{p}}(x) := [v(x)]_{\mathfrak{p}}$ for $x \in A$.

We equip $\mathbb{RP}^{\mathfrak{p}}$ with the induced metric

$$d_{\mathbb{RP}^{\mathfrak{p}}}([y]_{\mathfrak{p}}, [\widetilde{y}]_{\mathfrak{p}}) := \min\{d_{\mathbb{S}^{\mathfrak{p}}}(y, \widetilde{y}), d_{\mathbb{S}^{\mathfrak{p}}}(y, -\widetilde{y})\} \, ,$$

where $d_{\mathbb{S}^{\mathfrak{p}}}$ denotes the geodesic distance on $\mathbb{S}^{\mathfrak{p}}$. The metric space $(\mathbb{RP}^{\mathfrak{p}}, d_{\mathbb{RP}^{\mathfrak{p}}})$ is complete, and the projection map $P_{\mathfrak{p}}$ is continuous. Therefore, $[v]_{\mathfrak{p}} := P_{\mathfrak{p}} \circ v$ is continuous, if $v : A \to \mathbb{S}^{\mathfrak{p}}$ is continuous. By the *lifting theorem*, see e.g. [36, p. 34], if A is *simply connected*, for any continuous map $U : A \to \mathbb{RP}^{\mathfrak{p}}$ there are exactly two continuous functions $v_i : A \to \mathbb{S}^{\mathfrak{p}}$ such that $[v_i]_{\mathfrak{p}} = U$, for $i = 1, 2$, with $v_2(x) = -v_1(x)$ for every $x \in A$.

The manifold $\mathbb{RP}^{\mathfrak{p}}$ is orientable if and only if $\mathfrak{p}$ is odd. This yields that the *degree* of a continuous map $v : \Sigma^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$, where $\Sigma^{\mathfrak{p}}$ is a copy of $\mathbb{S}^{\mathfrak{p}}$, satisfies $\deg_{\mathbb{S}^{\mathfrak{p}}}(-v) = \deg_{\mathbb{S}^{\mathfrak{p}}}(v)$ if $\mathfrak{p}$ is odd, whereas $\deg_{\mathbb{S}^{\mathfrak{p}}}(-v) = -\deg_{\mathbb{S}^{\mathfrak{p}}}(v)$ if $\mathfrak{p}$ is even. For this reason, for $\mathfrak{p} \geq 3$ odd the degree of a continuous map $U : \Sigma^{\mathfrak{p}} \to \mathbb{RP}^{\mathfrak{p}}$ is well-defined, compare [5, Sec. VIII-B-a)], where $\mathfrak{p} = N - 1$, and Sec. 3 below.

We also recall that the *free homotopy groups* of $\mathbb{RP}^{\mathfrak{p}}$ are

$$\pi_k(\mathbb{RP}^{\mathfrak{p}}) \simeq \begin{cases} 0 & \text{if } k = 0 \text{ or } 1 < k < \mathfrak{p} \\ \mathbb{Z}_2 & \text{if } k = 1 \\ \mathbb{Z} & \text{if } k = \mathfrak{p} \end{cases}$$

whereas $\pi_k(\mathbb{RP}^{\mathfrak{p}}) = \pi_k(\mathbb{S}^{\mathfrak{p}})$ for $k > \mathfrak{p}$. Moreover, the *integral homology groups* are

$$H_k(\mathbb{RP}^{\mathfrak{p}}; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = \mathfrak{p} \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } 0 < k < \mathfrak{p} \text{ and } k \text{ is odd} \\ 0 & \text{otherwise}\,. \end{cases}$$

HOMOTOPICALLY NON-TRIVIAL MAPS. The following result implies that if a continuous map from $\Sigma^{\mathfrak{p}}$ into $\mathbb{RP}^{\mathfrak{p}}$ is homotopically non-trivial, then its "winding number" is a positive even integer. This is coherent with the discussion from [5, Sec. VIII-B-c)] for the case $\mathfrak{p}$ even.

Proposition 1.1 *Let $\mathfrak{p} \geq 2$. Let $U : \Sigma^{\mathfrak{p}} \to \mathbb{RP}^{\mathfrak{p}}$ be a continuous map such that $U^{-1}(\{[y]_{\mathfrak{p}}\})$ contains at most one point, for some $[y]_{\mathfrak{p}} \in \mathbb{RP}^{\mathfrak{p}}$. Then there exists a continuous map $h : [0,1] \times \Sigma^{\mathfrak{p}} \to \mathbb{RP}^{\mathfrak{p}}$ such that $h(0, x) = U(x)$ for every $x \in \Sigma^{\mathfrak{p}}$ and $x \mapsto h(1, x)$ is constant.*

PROOF: Since $\Sigma^{\mathfrak{p}}$ is simply-connected, there exists $v \in C^0(\Sigma^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ such that $P_{\mathfrak{p}} \circ v = U$. By the assumption, v is not onto $\mathbb{S}^{\mathfrak{p}}$, otherwise $\mathcal{H}^0(U^{-1}(\{[y]_{\mathfrak{p}}\})) \geq 2$ for every $[y]_{\mathfrak{p}} \in \mathbb{RP}^{\mathfrak{p}}$, whence v takes values in $\mathbb{S}^{\mathfrak{p}} \setminus \{y\}$ for some point $y \in \mathbb{S}^{\mathfrak{p}}$. Since $\mathbb{S}^{\mathfrak{p}} \setminus \{y\}$ is contractible, we can find a continuous map $\widetilde{h} : [0,1] \times \Sigma^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ such that $\widetilde{h}(0, x) = v(x)$ and $x \mapsto \widetilde{h}(1, x)$ is constant. Define $h := P_{\mathfrak{p}} \circ \widetilde{h}$. □

EMBEDDING OF $\mathbb{RP}^2$. Assume now $\mathfrak{p} = 2$. Following [33, Sec. 3.1], the mapping $g : \mathbb{S}^2 \to \mathbb{R}^6$

$$g(y_1, y_2, y_3) = \Big(\frac{\sqrt{2}}{2} {y_1}^2, \, \frac{\sqrt{2}}{2} {y_2}^2, \, \frac{\sqrt{2}}{2} {y_3}^2, \, y_1 y_2, \, y_2 y_3, \, y_3 y_1\Big) \tag{1.1}$$

induces an embedding

$$\widetilde{g} : \mathbb{RP}^2 \to \mathrm{RP}^2 \, , \quad \mathrm{RP}^2 := g(\mathbb{S}^2) \subset \mathbb{R}^6 \, , \quad \widetilde{g}([y]_2) := g(y) \, .$$

Notice that RP^2 is a *non-orientable*, smooth, compact, connected submanifold of $\mathbb{R}^6$ without boundary, such that $|z| = \sqrt{2}/2$ for every $z \in \mathrm{RP}^2$. Also, g maps the equator $\mathbb{S}^2 \cap \{y^3 = 0\}$ into a circle C of radius $1/2$, covered twice, with constant velocity equal to one. The circle C is a minimum length generator of the first homotopy group $\pi_1(\mathrm{RP}^2) \simeq \mathbb{Z}_2$.

Remark 1.2 Let $[\![\, \mathbb{S}^2 \,]\!]$ denote the standard integer multiplicity (say i.m.) rectifiable current integration of 2-forms in $\mathbb{S}^2$. Since RP^2 is covered twice with opposite orientation by g, we infer that the image current is zero, $g_\# [\![\, \mathbb{S}^2 \,]\!] = 0$.

Proposition 1.3 *We have* $\mathcal{H}^2(\mathrm{RP}^2) = 2\pi$.

Proof: By the *area formula*, and since $\mathcal{H}^0(g^{-1}(z)) = 2$ for every $z \in \mathrm{RP}^2$, we find that

$$2\,\mathcal{H}^2(\mathrm{RP}^2) = \int_{\mathrm{RP}^2} \mathcal{H}^0(g^{-1}(z))\, d\mathcal{H}^2(z) = \int_{\mathbb{S}^2} J_g^{\mathbb{S}^2}(y)\, d\mathcal{H}^2(y)\,,$$

where $J_g^{\mathbb{S}^2}$ is the *Jacobian* of g. Since $\mathcal{H}^2(\mathbb{S}^2) = 4\pi$, it suffices to observe that

$$J_g^{\mathbb{S}^2}(y) = 1 \qquad \forall\, y \in \mathbb{S}^2\,.$$

This can be checked by recalling that

$$J_g^{\mathbb{S}^2}(y) = \sqrt{\det(dg_y)^* \circ (dg_y)}\,,$$

where $dg_y : T_y\mathbb{S}^2 \to \mathbb{R}^6$ is the induced linear map and $(dg_y)^* : \mathbb{R}^6 \to \mathbb{S}^2$ is the adjoint transformation, compare [35, Sec. 7]. In polar coordinates, if $y = (\cos\alpha \sin\beta, \sin\alpha\sin\beta, \cos\beta)$, setting

$$\tau_1 := (\sin\alpha, -\cos\alpha, 0)\,, \qquad \tau_2 := (\cos\alpha\cos\beta, \sin\alpha\cos\beta, -\sin\beta)\,,$$

then $\{\tau_1, \tau_2\}$ is an orthonormal basis to $T_y\mathbb{S}^2$, with $\tau_1 \wedge \tau_2 = (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)$. For each component g^j we have

$$dg_y^j(\tau) = \tau \cdot \nabla g^j(y) \qquad \forall\, \tau \in T_y\mathbb{S}^2\,,$$

whence the (6×2) matrix corresponding to dg_y is

$$M := \begin{pmatrix} \sqrt{2}\sin\alpha\cos\alpha\sin\beta & \sqrt{2}\cos^2\alpha\sin\beta\cos\beta \\ -\sqrt{2}\sin\alpha\cos\alpha\sin\beta & \sqrt{2}\sin^2\alpha\sin\beta\cos\beta \\ 0 & -\sqrt{2}\sin\beta\cos\beta \\ \sin\beta(\sin^2\alpha - \cos^2\alpha) & 2\sin\alpha\cos\alpha\sin\beta\cos\beta \\ -\cos\alpha\cos\beta & \sin\alpha(\cos^2\beta - \sin^2\beta) \\ \sin\alpha\cos\beta & \cos\alpha(\cos^2\beta - \sin^2\beta) \end{pmatrix}.$$

Denoting by $a, b \in \mathbb{R}^6$ the columns of the matrix M, we have $\det(M^T \cdot M) = |a|^2|b|^2 - (a \cdot b)^2$. Since $|a| = |b| = 1$ and $a \cdot b = 0$, we infer that $\sqrt{\det(dg_y)^* \circ (dg_y)} = 1$ for every $y \in \mathbb{S}^2$, as required. $\square$

In the sequel we let $B^n(x, r)$ denote the ball in $\mathbb{R}^n$ centered at x and with radius r. We also set $B^n_r := B^n(0, r)$, and $B^n = B^n(0, 1)$, the unit ball centered at the origin.

For $X = C^0, C^\infty, L^2, W^{1,2}$, and $B \subset \mathbb{R}^n$ a Borel set, we define the classes

$$X(B, \mathbb{S}^2) := \{v \in X(B, \mathbb{R}^3) \,:\, |v(x)| = 1 \text{ for a.e. } x \in B\}\,,$$
$$X(B, \mathrm{RP}^2) := \{u \in X(B, \mathbb{R}^6) \,:\, u(x) \in \mathrm{RP}^2 \text{ for a.e. } x \in B\}\,,$$

where RP^2 is equipped with the induced metric from $\mathbb{R}^6$. We also denote by

$$\mathbf{D}(w, B) := \frac{1}{2} \int_B |Dw(x)|^2\, dx$$

the *Dirichlet energy* of a map w in $W^{1,2}(B,\mathbb{S}^2)$ or in $W^{1,2}(B,\mathrm{RP}^2)$. For $B=B^n$, we finally set

$$\mathbf{D}(w):=\mathbf{D}(w,B^n)\,.$$

If $u:B\to\mathrm{RP}^2$ is given by $u=g\circ v$ for some map $v\in W^{1,2}(B,\mathbb{S}^2)$, we have $u\in W^{1,2}(B,\mathrm{RP}^2)$ and

$$|D_iu|^2=|v|^2\cdot|D_iv|^2+(v\cdot D_iv)^2$$

for each partial derivative D_i. Therefore, since $|v|=1$ and $2\,(v\cdot D_iv)=D_i|v|^2=0$ a.e. for every i, we infer that

$$|Du|=|Dv|\qquad\text{if } u=g\circ v\,. \tag{1.2}$$

In particular, for every $v\in W^{1,2}(B,\mathbb{S}^2)$ we have

$$\mathbf{D}(g\circ v,B)=\mathbf{D}(v,B)\,.$$

Embedding of $\mathbb{RP}^{\mathfrak{p}}$. In a similar way, for every $\mathfrak{p}\geq 2$ integer we can find a smooth map

$$g_{\mathfrak{p}}:\mathbb{S}^{\mathfrak{p}}\to\mathbb{R}^{N(\mathfrak{p})}\,,\qquad N(\mathfrak{p}):=\frac{(\mathfrak{p}+1)(\mathfrak{p}+2)}{2}\,,$$

with $g_2=g$ in (1.1), that induces an embedding

$$\widetilde{g}_{\mathfrak{p}}:\mathbb{RP}^{\mathfrak{p}}\to\mathrm{RP}^{\mathfrak{p}}\,,\quad \mathrm{RP}^{\mathfrak{p}}:=g_{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})\subset\mathbb{R}^{N(\mathfrak{p})}\,,\quad \widetilde{g}_{\mathfrak{p}}([y]_{\mathfrak{p}}):=g_{\mathfrak{p}}(y)\,.$$

More precisely, the first $\mathfrak{p}+1$ components of $g_{\mathfrak{p}}$ are the functions $y\mapsto(\sqrt{2}/2)y_i^2$, for $i=1,\dots,\mathfrak{p}+1$, and the other ones are the functions $y\mapsto y_iy_j$, for $1\leq i<j\leq\mathfrak{p}+1$.

For $X=C^0,C^\infty,L^{\mathfrak{p}},W^{1,\mathfrak{p}}$, and for $B\subset\mathbb{R}^n$ a Borel set, we similarly define the classes

$$\begin{aligned}X(B,\mathbb{S}^{\mathfrak{p}})&:=\{v\in X(B,\mathbb{R}^{\mathfrak{p}+1})\,:\,|v(x)|=1\text{ for a.e. }x\in B\}\,,\\ X(B,\mathrm{RP}^{\mathfrak{p}})&:=\{u\in X(B,\mathbb{R}^{N(\mathfrak{p})})\,:\,u(x)\in\mathrm{RP}^{\mathfrak{p}}\text{ for a.e. }x\in B\}\,,\end{aligned}$$

where $\mathrm{RP}^{\mathfrak{p}}$ is equipped with the induced metric from $\mathbb{R}^{N(\mathfrak{p})}$. We also denote by

$$\mathbf{D}_{\mathfrak{p}}(w,B):=\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\int_B|Dw(x)|^{\mathfrak{p}}\,dx$$

the *conformal* $\mathfrak{p}$*-energy* of a map w in $W^{1,\mathfrak{p}}(B,\mathbb{S}^{\mathfrak{p}})$ or in $W^{1,\mathfrak{p}}(B,\mathrm{RP}^{\mathfrak{p}})$, so that $\mathbf{D}_2=\mathbf{D}$, and for $B=B^n$

$$\mathbf{D}_{\mathfrak{p}}(w):=\mathbf{D}_{\mathfrak{p}}(w,B^n)\,.$$

Proposition 1.4 *The following properties hold true for every* $\mathfrak{p}\geq 2$:

i) $\mathrm{RP}^{\mathfrak{p}}$ *is a smooth, compact, connected submanifold without boundary, orientable if and only if* $\mathfrak{p}$ *is odd;*

ii) $|z|=\sqrt{2}/2$ *for every* $z\in\mathrm{RP}^{\mathfrak{p}}$*;*

iii) *if* $\mathfrak{p}$ *is even, the image current* $g_{\mathfrak{p}\#}[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!]=0$*;*

iv) *if* $\mathfrak{p}$ *is odd, we can equip* $\mathrm{RP}^{\mathfrak{p}}$ *with an orientation in such a way that* $g_{\mathfrak{p}\#}[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!]=2\,[\![\,\mathrm{RP}^{\mathfrak{p}}\,]\!]$*;*

v) $\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}})=\alpha_{\mathfrak{p}}/2$*, where* $\alpha_{\mathfrak{p}}:=\mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})$*;*

vi) *if* $u:B\to\mathrm{RP}^{\mathfrak{p}}$ *is given by* $u=g_{\mathfrak{p}}\circ v$ *for some* $v\in W^{1,\mathfrak{p}}(B,\mathbb{S}^{\mathfrak{p}})$*, then* $u\in W^{1,\mathfrak{p}}(B,\mathrm{RP}^{\mathfrak{p}})$ *and*

$$|Du|=|Dv|\qquad\text{if}\quad u=g_{\mathfrak{p}}\circ v\,. \tag{1.3}$$

In particular, for every $v\in W^{1,\mathfrak{p}}(B,\mathbb{S}^{\mathfrak{p}})$ *we have*

$$\mathbf{D}_{\mathfrak{p}}(g_{\mathfrak{p}}\circ v,B)=\mathbf{D}_{\mathfrak{p}}(v,B)\,.$$

Proof: For $\mathfrak{p}\geq 3$, it is a technical adaptation of the argument given above for the case $\mathfrak{p}=2$. For this reason, it is omitted. □

2 Sobolev maps into projective spaces

In this section we analyze some crucial properties concerning $W^{1,\mathfrak{p}}$-functions that take values into real projective $\mathfrak{p}$-spaces. This properties hold true for every $\mathfrak{p} \geq 2$. We then collect from [20] some features about the *modified of the inverse of the stereographic projection.* We finally consider a minimum problem for continuous maps into the projective $\mathfrak{p}$-space.

Clearly, if $u : B \to \mathrm{RP}^{\mathfrak{p}}$ is continuous (or Lipschitz), the corresponding function $U = \widetilde{g}_{\mathfrak{p}}^{-1} \circ u : B \to \mathbb{RP}^{\mathfrak{p}}$ is continuous (or Lipschitz). Moreover, we have:

Proposition 2.1 *Let $u \in W^{1,\mathfrak{p}}(A, \mathrm{RP}^{\mathfrak{p}})$ a continuous map, where $A \subset \mathbb{R}^n$ is open and simply-connected. Denote $U := \widetilde{g}_{\mathfrak{p}}^{-1} \circ u : A \to \mathbb{RP}^{\mathfrak{p}}$, and let $v : A \to \mathbb{S}^{\mathfrak{p}}$ a continuous map such that $P_{\mathfrak{p}} \circ v = U$. Then v belongs to $W^{1,\mathfrak{p}}(A, \mathbb{S}^{\mathfrak{p}})$ and $\mathbf{D}_{\mathfrak{p}}(v, A) = \mathbf{D}_{\mathfrak{p}}(u, A)$.*

Proof: Let $x_0 \in A$. By continuity, there exists a positive number $\delta > 0$ such that $B^n(x_0, \delta) \subset A$ and $d_{\mathbb{RP}^{\mathfrak{p}}}(U(x), U(x_0)) < 1$ for every $x \in B^n(x_0, \delta)$. Setting

$$\Omega := \widetilde{g}_{\mathfrak{p}}(\{[y]_{\mathfrak{p}} \in \mathbb{RP}^{\mathfrak{p}} \mid d_{\mathbb{RP}^{\mathfrak{p}}}([y]_{\mathfrak{p}}, U(x_0)) < 1\}),$$

the pull-back $g_{\mathfrak{p}}^{-1}(\Omega)$ is given by the union of two disjoint open connected subsets $\Omega_1, \Omega_2 \in \mathbb{S}^{\mathfrak{p}}$ such that $\Omega_2 = -\Omega_1$. Since v is continuous, then $v(B^n(x_0, \delta))$ is connected, with

$$P_{\mathfrak{p}}(v(B^n(x_0, \delta))) = U(B^n(x_0, \delta)) \subset \Omega .$$

We then may and do assume that $v(B^n(x_0, \delta)) \subset \Omega_1$. This yields that

$$v_{|B^n(x_0,\delta)} = (g_{\mathfrak{p}\,|\Omega_1})^{-1} \circ u_{|B^n(x_0,\delta)} ,$$

where $g_{\mathfrak{p}\,|\Omega_1} : \Omega_1 \to \Omega$ is bi-Lipschitz, whence $v_{|B^n(x_0,\delta)} \in W^{1,\mathfrak{p}}(B^n(x_0, \delta), \mathbb{S}^{\mathfrak{p}})$. Moreover, we have

$$u_{|B^n(x_0,\delta)} = g_{\mathfrak{p}} \circ v_{|B^n(x_0,\delta)} ,$$

so that (1.3) yields that $|Dv(x_0)| = |Du(x_0)|$. The claims follow. □

Density results. Let $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}}$ or $\mathrm{RP}^{\mathfrak{p}}$. In dimension $n = \mathfrak{p}$, Schoen-Uhlenbeck density theorem [34] yields that the class of smooth maps in $W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathcal{Y})$ is strongly dense in $W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathcal{Y})$. This is false in the case of higher dimension $n \geq \mathfrak{p} + 1$. For this reason, Bethuel [2] introduced the classes $R_{\mathfrak{p}}^{\infty}(B^n, \mathcal{Y})$ and $R_{\mathfrak{p}}^{0}(B^n, \mathcal{Y})$ of maps $w \in W^{1,\mathfrak{p}}(B^n, \mathcal{Y})$ that are smooth, respectively continuous, outside a smooth closed singular subset $\Sigma(w)$ of B^n of dimension $(n - \mathfrak{p} - 1)$, e.g., a discrete set for $n = \mathfrak{p} + 1$. He also proved:

Theorem 2.2 (Bethuel [2]) *For any $n \geq \mathfrak{p}+1$, the classes $R_{\mathfrak{p}}^{\infty}(B^n, \mathcal{Y})$ and $R_{\mathfrak{p}}^{0}(B^n, \mathcal{Y})$ are strongly dense in $W^{1,\mathfrak{p}}(B^n, \mathcal{Y})$.*

Now, in any dimension $n \geq \mathfrak{p} + 1$, and for any $\mathfrak{p} \geq 2$, it turns out that the open set $A := B^n \setminus \Sigma(w)$ is simply-connected and with full measure $|A| = |B^n|$. Notice that this is false for $\mathfrak{p} = 1$. Therefore, from the above facts we obtain:

Proposition 2.3 *Let $n \geq \mathfrak{p} + 1$ and $\mathfrak{p} \geq 2$. For every map $u \in R_{\mathfrak{p}}^{0}(B^n, \mathrm{RP}^{\mathfrak{p}})$ there exist exactly two maps $v_1, v_2 \in R_{\mathfrak{p}}^{0}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v_i = u$ a.e. in B^n.*

Proof: Letting $A = B^{\mathfrak{p}} \setminus \Sigma(u)$, by Proposition 2.1 we find a continuous map $v \in W^{1,\mathfrak{p}}(A, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v = u$ in A. Actually v belongs to the class $R_{\mathfrak{p}}^{0}(B^n, \mathbb{S}^{\mathfrak{p}})$, as $\Sigma(v) = \Sigma(u)$ is a smooth closed subset of B^n of dimension $(n - \mathfrak{p} - 1)$. Since $U := \widetilde{g}_{\mathfrak{p}}^{-1} \circ u_{|A}$ is continuous, the lifting theorem yields the claim. □

Remark 2.4 Of course, Proposition 2.3 holds true in dimension $n = \mathfrak{p} \geq 2$, replacing $R_{\mathfrak{p}}^{0}$ with $W^{1,\mathfrak{p}} \cap C^0$.

We now prove that a similar property holds true for $W^{1,\mathfrak{p}}$-maps into $\mathrm{RP}^{\mathfrak{p}}$, in any dimension n.

Theorem 2.5 *Let $\mathfrak{p} \geq 2$ integer. For every $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$, there exist exactly two Sobolev maps $v_1, v_2 \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v_i = u$ a.e. in B^n. Moreover, $v_2 = -v_1$ and $\mathbf{D}_{\mathfrak{p}}(v_i, B^n) = \mathbf{D}_{\mathfrak{p}}(u, B^n)$.*

Proof: Assume first $n \geq \mathfrak{p}+1$, and let $\{u_k\} \subset R^0_{\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$ be a sequence that strongly converges to u in $W^{1,\mathfrak{p}}(B^n, \mathbb{R}^{N(\mathfrak{p})})$, Theorem 2.2. By Proposition 2.3, we find a sequence $\{v_k\} \subset R^0_{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v_k = u_k$ for every k. Since $\mathbf{D}_{\mathfrak{p}}(v_k) = \mathbf{D}_{\mathfrak{p}}(u_k)$, possibly passing to a subsequence, we find a Sobolev map $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $v_k \rightharpoonup v$ weakly in $W^{1,\mathfrak{p}}$ and $g_{\mathfrak{p}} \circ v = u$, so that $\mathbf{D}_{\mathfrak{p}}(u) = \mathbf{D}_{\mathfrak{p}}(v)$. By uniform convexity, the convergence $\mathbf{D}_{\mathfrak{p}}(u_k) \to \mathbf{D}_{\mathfrak{p}}(u)$ yields that $v_k \to v$ strongly in $W^{1,\mathfrak{p}}$. This gives that v and $-v$ satisfy the required properties. Assume now by contradiction that there exists a third Sobolev map $w \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ w = u$. Setting

$$A := \{x \in B^n \mid w(x) = v(x)\}, \qquad B := \{x \in B^n \mid w(x) = -v(x)\},$$

since $|v(x)| = 1$ and $g_{\mathfrak{p}}(-y) = g_{\mathfrak{p}}(y)$, we have that $|A \cap B| = 0$ and $|A \cup B| = |B^n|$, whereas $|A| > 0$ and $|B| > 0$. By a slicing argument, this property should hold true also in dimension $n = \mathfrak{p} - 1$, a contradiction, as $W^{1,\mathfrak{p}}$-maps defined on $(\mathfrak{p}-1)$-dimensional domains are continuous, whereas $|v(x)| = 1$ for a.e. x. In the case $n = \mathfrak{p}$ we argue in a similar way, on account of Schoen-Uhlenbeck density theorem [34]. The case $n < \mathfrak{p}$ is an immediate consequence of the cited lifting theorem. □

Remark 2.6 Notice that in dimension $n \geq \mathfrak{p}$, Theorem 2.5 continues to hold if we replace B^n with any bounded domain $\Omega \subset \mathbb{R}^n$, or with e.g. $\Omega = \mathbb{S}^n$, the n-sphere in $\mathbb{R}^{n+1}$. In fact, for $n \geq \mathfrak{p} \geq 2$, any such set Ω is simply-connected, hence we can apply again Propositions 2.1 and 2.3.

The modified stereographic projection. We recall that for every $\mathfrak{p} \geq 2$ integer, the stereographic projection σ of the unit $\mathfrak{p}$-sphere $\mathbb{S}^{\mathfrak{p}}$ onto $\mathbb{R}^{\mathfrak{p}}$, from the south pole $P_S := (0_{\mathbb{R}^{\mathfrak{p}}}, -1)$, maps $(y,z) \in \mathbb{S}^{\mathfrak{p}} \subset \mathbb{R}^{\mathfrak{p}} \times \mathbb{R}$, with $|y|^2 + |z|^2 = 1$, to $y/(1+z) \in \mathbb{R}^{\mathfrak{p}}$, whereas its inverse $\sigma^{-1} : \mathbb{R}^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ sends $x \in \mathbb{R}^{\mathfrak{p}}$ to

$$\sigma^{-1}(x) = \left(\frac{2}{1+|x|^2}\, x,\ \frac{1-|x|^2}{1+|x|^2} \right).$$

Since the Jacobian $J_{\sigma^{-1}}$ is equal to $\mathfrak{p}^{-\mathfrak{p}/2}\, |D\sigma^{-1}|^{\mathfrak{p}}$, by the area formula we have

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\mathbb{R}^{\mathfrak{p}}} |D\sigma^{-1}|^{\mathfrak{p}}\, dx = \int_{\mathbb{R}^{\mathfrak{p}}} J_{\sigma^{-1}}\, dx = \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}).$$

Recall also that the map $(-1)^{\mathfrak{p}}\, \sigma^{-1}$ is an orientation preserving *conformal* diffeomorphism from $\mathbb{R}^{\mathfrak{p}}$ into $\mathbb{S}^{\mathfrak{p}} \setminus \{(-1)^{\mathfrak{p}} P_S\}$, where $\mathbb{S}^{\mathfrak{p}}$ is equipped with the natural orientation induced from the outward unit normal; in particular, $(-1)^{\mathfrak{p}}\, \sigma^{-1}_{\#}[\![\mathbb{R}^{\mathfrak{p}}]\!] = [\![\mathbb{S}^{\mathfrak{p}}]\!]$. We modify σ^{-1} as follows. We first write

$$\sigma^{-1}(x) = \left(\frac{x}{|x|} \sin\theta(|x|), -\cos\theta(|x|) \right), \qquad x \in \mathbb{R}^{\mathfrak{p}},$$

where $\theta(r)$, for $r > 0$, is the *angular distance*, i.e. the geodesic distance of $\sigma^{-1}(\partial B^{\mathfrak{p}}_r)$ from the south pole P_S. For $\varepsilon > 0$ we set

$$\theta_\varepsilon(r) := \begin{cases} \theta(r) & \text{if} \quad r < R_\varepsilon \\ \varepsilon\,(2R_\varepsilon - r)/R_\varepsilon & \text{if} \quad R_\varepsilon \leq r \leq 2R_\varepsilon \\ 0 & \text{if} \quad r > 2R_\varepsilon, \end{cases}$$

where $R_\varepsilon := \theta^{-1}(\varepsilon)$, and we define $\varphi_\varepsilon : \mathbb{R}^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ by

$$\varphi_\varepsilon(x) := (-1)^{\mathfrak{p}} \left(\frac{x}{|x|} \sin\theta_\varepsilon(|x|), -\cos\theta_\varepsilon(|x|) \right), \qquad x \in \mathbb{R}^{\mathfrak{p}}.$$

Clearly φ_ε is Lipschitz-continuous, with $\varphi_\varepsilon(x) = (-1)^{\mathfrak{p}} \sigma^{-1}(x)$ for $|x| < R_\varepsilon$ and $\varphi_\varepsilon(x) \equiv (-1)^{\mathfrak{p}}\, P_S$ for $|x| > 2R_\varepsilon$. Moreover, see [20, Vol. II, Sec. 4.1.1], it can be shown that the $\mathfrak{p}$-energy satisfies

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\mathbb{R}^{\mathfrak{p}}} |D\varphi_\varepsilon|^{\mathfrak{p}}\, dx \leq \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) + c\,\varepsilon,$$

where $c > 0$ is an absolute constant, and that the image current

$$\varphi_{\varepsilon\#}[\![\mathbb{R}^{\mathfrak{p}}]\!] = \varphi_{\varepsilon\#}[\![B^{\mathfrak{p}}_{2R_\varepsilon}]\!] = [\![\mathbb{S}^{\mathfrak{p}}]\!].$$

Finally, by considering the mapping $\varphi_{\varepsilon,\delta}(x) := \varphi_\varepsilon(R_\varepsilon x/\delta)$, where the positive parameter δ can be chosen independently of ε, one can even shrink the set $\{x \in \mathbb{R}^{\mathfrak{p}} \mid \varphi_\varepsilon(x) \neq (-1)^{\mathfrak{p}} P_S\}$ to $\{0_{\mathbb{R}^{\mathfrak{p}}}\}$, without affecting the $\mathfrak{p}$-energy, and state the following.

Proposition 2.7 *For any $\varepsilon, \delta > 0$ there exists a smooth map $\varphi_{\varepsilon,\delta} : \mathbb{R}^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ such that*

i) $\varphi_{\varepsilon,\delta} \equiv (-1)^{\mathfrak{p}} P_S$ *outside* $B^{\mathfrak{p}}_\delta$ *;*

ii) $\mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) \leq \dfrac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \displaystyle\int_{\mathbb{R}^{\mathfrak{p}}} |D\varphi_{\varepsilon,\delta}|^{\mathfrak{p}}\, dx \leq \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) + \varepsilon$ *;*

iii) $\varphi_{\varepsilon,\delta\#}[\![\mathbb{R}^{\mathfrak{p}}]\!] = [\![\mathbb{S}^{\mathfrak{p}}]\!]$*, i.e., $\varphi_{\varepsilon,\delta}$ has* degree *one;*

iv) $\varphi_{\varepsilon,\delta}$ *is conformal on* $B^{\mathfrak{p}}_{\delta/2}$ *.*

A MINIMUM PROBLEM. According to Proposition 1.1, we now consider a minimum problem concerning homotopically non-trivial maps. To this purpose, we introduce the class

$$\mathcal{F}_{\mathfrak{p}} := \{u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}}) \cap C^0 \mid u \text{ is constant on } \partial B^{\mathfrak{p}} \text{ and homotopically non-trivial}\},$$

the homotopy to be intended with *fixed boundary datum on $\partial B^{\mathfrak{p}}$.*

Proposition 2.8 *For every $\mathfrak{p} \geq 2$, we have:*

$$\inf\{\mathbf{D}_{\mathfrak{p}}(u) \mid u \in \mathcal{F}_{\mathfrak{p}}\} = 2\,\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}}) .$$

Remark 2.9 Recall that $2\,\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}}) = \alpha_{\mathfrak{p}}$, where $\alpha_{\mathfrak{p}} := \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})$. Proposition 2.8 is the counterpart of a well-known fact concerning homotopically non-trivial maps in $W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}}) \cap C^0$, where the corresponding infimum is equal to $\mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})$. This time, in fact, Proposition 1.1 yields that maps in $\mathcal{F}_{\mathfrak{p}}$ have to cover at least twice the target space $\mathrm{RP}^{\mathfrak{p}}$, see Remark 3.5 below.

PROOF OF PROPOSITION 2.8: Let $u \in \mathcal{F}_{\mathfrak{p}}$ and $U := \widetilde{g}_{\mathfrak{p}}^{-1} \circ u$. By Proposition 1.1, we deduce that $\mathcal{H}^0(U^{-1}(z)) \geq 2$ for every $z \in \mathbb{RP}^{\mathfrak{p}}$, whence $\mathcal{H}^0(u^{-1}(z)) \geq 2$ for every $z \in \mathrm{RP}^{\mathfrak{p}}$. Denote by J_u the Jacobian of u, so that by the parallelogram inequality

$$J_u(x) \leq \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |Du(x)|^{\mathfrak{p}} \qquad \text{for a.e. } x \in B^{\mathfrak{p}}.$$

Using the area formula, as in [1] we obtain

$$\mathbf{D}_{\mathfrak{p}}(u, B^{\mathfrak{p}}) \geq \int_{B^{\mathfrak{p}}} J_u\, dx = \int_{\mathrm{RP}^{\mathfrak{p}}} \mathcal{H}^0(u^{-1}(z))\, d\mathcal{H}^{\mathfrak{p}}(z) \geq 2\,\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}}) = \alpha_{\mathfrak{p}} .$$

On the other hand, by Proposition 2.7 we obtain for every $\varepsilon > 0$ a (homotopically non-trivial) map $v : B^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ that is constant on $\partial B^{\mathfrak{p}}$ and satisfies

$$\mathbf{D}_{\mathfrak{p}}(v, B^{\mathfrak{p}}) \leq \alpha_{\mathfrak{p}} + \varepsilon .$$

Taking $u = g_{\mathfrak{p}} \circ v$, on account of (1.3) we find a map $u \in \mathcal{F}_{\mathfrak{p}}$ satisfying $\mathbf{D}_{\mathfrak{p}}(u, B^{\mathfrak{p}}) \leq \alpha_{\mathfrak{p}} + \varepsilon$, that proves the assertion. □

Remark 2.10 Notice that the infimum in Proposition 2.8 is not attained. Otherwise, the infimum in the corresponding problem about the class $W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ would be attained too, a contradiction.

3 Singularity and degree

In this section we collect the definition of singularity and degree of $W^{1,\mathfrak{p}}$-mappings with values into the projective $\mathfrak{p}$-space, recovering the notation from [20] and [5], respectively.

SINGULARITY OF MAPPINGS INTO $\mathbb{S}^{\mathfrak{p}}$. For $\mathfrak{p}\geq 2$, let $\omega_{\mathbb{S}^{\mathfrak{p}}}$ denote the *volume $\mathfrak{p}$-form on* $\mathbb{S}^{\mathfrak{p}}$,

$$\omega_{\mathbb{S}^{\mathfrak{p}}} := \sum_{j=1}^{\mathfrak{p}+1} (-1)^{j-1} y^j \widehat{dy^j}\,, \qquad y=(y^1,\dots,y^{\mathfrak{p}+1})$$

where $\widehat{dy^j} := dy^1\wedge\cdots\wedge dy^{j-1}\wedge dy^{j+1}\wedge\cdots\wedge dy^{\mathfrak{p}+1}$, so that $d\omega_{\mathbb{S}^{\mathfrak{p}}} = (\mathfrak{p}+1)\cdot dy^1\wedge\cdots\wedge dy^{\mathfrak{p}+1}$ and

$$[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!](\omega_{\mathbb{S}^{\mathfrak{p}}}) := \int_{\mathbb{S}^{\mathfrak{p}}} \omega_{\mathbb{S}^{\mathfrak{p}}} = \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) =: \alpha_{\mathfrak{p}}\,.$$

Let $\Omega\subset\mathbb{R}^n$ open, where $n\geq \mathfrak{p}+1$. To every Sobolev function $v\in W^{1,\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}})$ it corresponds an $(n-\mathfrak{p}-1)$-dimensional current $\mathbb{P}(v)\in\mathcal{D}_{n-\mathfrak{p}-1}(\Omega)$ acting on compactly supported smooth $(n-\mathfrak{p}-1)$-forms $\phi\in\mathcal{D}^{n-\mathfrak{p}-1}(\Omega)$ as

$$\langle \mathbb{P}(v),\phi\rangle := \frac{1}{\alpha_{\mathfrak{p}}}\int_\Omega d\phi\wedge v^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}}\,. \tag{3.1}$$

For future use, we notice that

$$(-v)^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}} = (-1)^{\mathfrak{p}+1} v^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}} \qquad \forall\, v\in W^{1,\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}})\,. \tag{3.2}$$

Remark 3.1 The current $\mathbb{P}(v)$ in (3.1) describes the *singularity* of $v\in W^{1,\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}})$. In fact, if e.g. $n=\mathfrak{p}+1$ and $v\in R^0_{\mathfrak{p}}(B^{\mathfrak{p}+1},\mathbb{S}^{\mathfrak{p}})$, denoting by $\Sigma(v):=\{a_j\mid j=1,\dots m\}$ the discrete set of $B^{\mathfrak{p}+1}$ of singular points of v, we have, see Remark 5.1 below,

$$\mathbb{P}(v) = -\sum_{j=1}^m \Delta_j\,\delta_{a_j}\,, \tag{3.3}$$

where $\Delta_j\in\mathbb{Z}$ and δ_{a_j} is the unit Dirac mass centered at a_j. For example, if $v_0(x)=x/|x|$ we get

$$\mathbb{P}(v_0) = -\delta_a\,, \qquad a=0_{\mathbb{R}^{\mathfrak{p}+1}}\,.$$

In [20], the authors also defined the $(n-\mathfrak{p})$-current $\mathbb{D}(v)\in\mathcal{D}_{n-\mathfrak{p}}(\Omega)$ given by

$$\langle \mathbb{D}(v),\gamma\rangle := \frac{1}{\alpha_{\mathfrak{p}}}\int_\Omega \gamma\wedge v^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}}$$

for every $\gamma\in\mathcal{D}^{n-\mathfrak{p}}(\Omega)$, so that clearly

$$\mathbb{P}(v) = \partial\,\mathbb{D}(v) \qquad \text{on} \quad \mathcal{D}^{n-\mathfrak{p}-1}(\Omega)\,. \tag{3.4}$$

In the particular case $n=\mathfrak{p}+1$, the above can be stated in terms of the so called *D-field* of Brezis-Coron-Lieb. For $\mathfrak{p}=2$, the vector field $D(v)$ is defined for every $v\in W^{1,2}(\Omega,\mathbb{S}^2)$, where $\Omega\subset\mathbb{R}^3$, by

$$D(v) := (v\cdot v_{x_2}\times v_{x_3},\, v\cdot v_{x_3}\times v_{x_1},\, v\cdot v_{x_1}\times v_{x_2})$$

where, setting $v=(v^1,v^2,v^3)$,

$$v\cdot v_{x_i}\times v_{x_j} := \det\begin{pmatrix} v^1 & v^2 & v^3\\ v^1_{x_i} & v^2_{x_i} & v^3_{x_i}\\ v^1_{x_j} & v^2_{x_j} & v^3_{x_j}\end{pmatrix}\,, \qquad v^h_{x_k} := \frac{\partial v^h}{\partial x_k}.$$

More generally, see [5, App. B], for $\mathfrak{p}\geq 3$ the D-field of a map $v\in W^{1,\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}})$, where $\Omega\subset\mathbb{R}^{\mathfrak{p}+1}$, is the vector field $D(v)\in L^1(\Omega,\mathbb{R}^{\mathfrak{p}+1})$ defined in components by $D(v)=(D^1(v),\dots,D^{\mathfrak{p}+1}(v))$, where

$$D^i(v) := \det\Big(\frac{\partial v}{\partial x_1},\dots,\frac{\partial v}{\partial x_{i-1}},v,\frac{\partial v}{\partial x_{i+1}},\dots,\frac{\partial v}{\partial x_{\mathfrak{p}+1}}\Big)\,.$$

If e.g. $v \in R^\infty_{\mathfrak{p}}(B^{\mathfrak{p}+1},\mathbb{S}^{\mathfrak{p}})$, it turns out that for a.e. $x \in B^{\mathfrak{p}+1}$ the vector $D(v)(x) \in \mathbb{R}^{\mathfrak{p}+1}$ is tangent to the naturally oriented level lines $\{z \in B^{\mathfrak{p}+1} \mid v(z) = v(x)\}$. More precisely, when normalized, the vector $D(v)(x)$ orients the slices of the current $[\![B^{\mathfrak{p}+1}]\!]$ by the map v at $v(x) \in \mathbb{S}^{\mathfrak{p}}$. Moreover, for every $v \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1},\mathbb{S}^{\mathfrak{p}})$ we have

$$\langle \mathbb{D}(v), \gamma\rangle = \frac{1}{\alpha_{\mathfrak{p}}} \int_{B^{\mathfrak{p}+1}} \langle \gamma, D(v)\rangle\, dx \qquad \forall\, \gamma \in \mathcal{D}^1(B^{\mathfrak{p}+1}),$$

so that (3.4) yields to:

$$\mathbb{P}(v) = 0 \qquad \Longleftrightarrow \qquad \operatorname{Div} D(v) = 0 \quad \text{on } B^{\mathfrak{p}+1},$$

where Div denotes the distributional divergence.

In higher dimension $n \geq \mathfrak{p}+2$, the $(n-\mathfrak{p})$-vector field $D(v)$ can be defined as the dual to $v^\# \omega_{\mathbb{S}^{\mathfrak{p}}}$,

$$\langle \eta, D(v)(x)\rangle\, dx^1 \wedge \cdots \wedge dx^n := \eta \wedge v^\# \omega_{\mathbb{S}^{\mathfrak{p}}}(x) \qquad \forall\, \eta \in \Lambda^{n-\mathfrak{p}}(\mathbb{R}^n),$$

see [20, Vol. II, Sec. 5.2.1]. More precisely, $D(v)$ may be identified with $* v^\# \omega_{\mathbb{S}^{\mathfrak{p}}}$, where $*$ is the *Hodge operator*. For maps $v \in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$ we thus have

$$\langle \mathbb{D}(v), \gamma\rangle = \frac{1}{\alpha_{\mathfrak{p}}} \int_{B^n} \langle \gamma, D(v)\rangle\, dx \qquad \forall\, \gamma \in \mathcal{D}^{n-\mathfrak{p}}(B^n). \tag{3.5}$$

Also, for a.e. $x \in B^n$ the $(n-\mathfrak{p})$-vector $D(v)(x) \in \Lambda_{n-\mathfrak{p}}\mathbb{R}^n$ is tangent to the naturally oriented level $(n-\mathfrak{p})$-surfaces $\{z \in B^n \mid v(z) = v(x)\}$, if v belongs to $R^\infty_{\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$.

Singularity of mappings into $\mathrm{RP}^{\mathfrak{p}}$. If $\mathfrak{p} \geq 3$ is odd, due to (3.2) we may and do define the *singularity* of functions in $W^{1,\mathfrak{p}}(\Omega,\mathrm{RP}^{\mathfrak{p}})$ by means of homological arguments.

The case $\mathfrak{p}$ odd. The projective space $\mathrm{RP}^{\mathfrak{p}}$ being orientable for $\mathfrak{p} \geq 3$ odd, there exists a (normalized) volume form $\omega_{\mathrm{RP}^{\mathfrak{p}}}$ on $\mathrm{RP}^{\mathfrak{p}}$, i.e., a closed $\mathfrak{p}$-form $\omega_{\mathrm{RP}^{\mathfrak{p}}} \in \mathcal{D}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}})$ such that $\omega_{\mathrm{RP}^{\mathfrak{p}}}(z) \neq 0$ for every $z \in \mathrm{RP}^{\mathfrak{p}}$ and $[\![\mathrm{RP}^{\mathfrak{p}}]\!](\omega_{\mathrm{RP}^{\mathfrak{p}}}) = 1$, compare iv) in Proposition 1.4. To our purposes, we define $\omega_{\mathrm{RP}^{\mathfrak{p}}}$ by taking the pull-back

$$\omega_{\mathrm{RP}^{\mathfrak{p}}} := \frac{2}{\alpha_{\mathfrak{p}}}\, (\widehat{g}_{\mathfrak{p}}^{-1})^\# \omega_{\mathbb{S}^{\mathfrak{p}}}, \tag{3.6}$$

where $\widehat{g}_{\mathfrak{p}}$ is the one-to-one map given by the restriction of $g_{\mathfrak{p}}$ to the upper semi-sphere $\mathbb{S}^{\mathfrak{p}}_+ := \{y \in \mathbb{S}^{\mathfrak{p}} \mid y^{\mathfrak{p}+1} > 0\}$. In fact, since $(\widehat{g}_{\mathfrak{p}}^{-1})_\# [\![\mathrm{RP}^{\mathfrak{p}}]\!] = [\![\mathbb{S}^{\mathfrak{p}}_+]\!]$, we check

$$[\![\mathrm{RP}^{\mathfrak{p}}]\!](\omega_{\mathrm{RP}^{\mathfrak{p}}}) = \frac{2}{\alpha_{\mathfrak{p}}}\, [\![\mathrm{RP}^{\mathfrak{p}}]\!]((\widehat{g}_{\mathfrak{p}}^{-1})^\# \omega_{\mathbb{S}^{\mathfrak{p}}}) = \frac{2}{\alpha_{\mathfrak{p}}}\, [\![\mathbb{S}^{\mathfrak{p}}_+]\!](\omega_{\mathbb{S}^{\mathfrak{p}}}) = 1.$$

Notice that $\omega_{\mathrm{RP}^{\mathfrak{p}}}$ can be smoothly extended to the whole of $\mathrm{RP}^{\mathfrak{p}}$, as $\mathfrak{p}$ is odd.

According to (3.1), to every map $u \in W^{1,\mathfrak{p}}(\Omega,\mathrm{RP}^{\mathfrak{p}})$, where $\Omega \subset \mathbb{R}^n$ is open and $n \geq \mathfrak{p}+1$, we associate the current $\mathbf{P}(u) \in \mathcal{D}_{n-\mathfrak{p}-1}(\Omega)$ acting on forms $\phi \in \mathcal{D}^{n-\mathfrak{p}-1}(\Omega)$ as

$$\langle \mathbf{P}(u), \phi\rangle := \int_\Omega d\phi \wedge u^\# \omega_{\mathrm{RP}^{\mathfrak{p}}}. \tag{3.7}$$

Due to Theorem 2.5, the following result clarifies the situation:

Proposition 3.2 *Let $\mathfrak{p} \geq 3$ odd and $\Omega \subset \mathbb{R}^n$ open and simply-connected, with $n \geq \mathfrak{p}+1$. Let $u \in W^{1,\mathfrak{p}}(\Omega,\mathrm{RP}^{\mathfrak{p}})$ and $v \in W^{1,\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}})$ be such that $g_{\mathfrak{p}} \circ v = u$. Then we have*

$$\frac{1}{2}\, \mathbf{P}(u) = \mathbb{P}(v).$$

Proof: By (3.6), we can write

$$u^\# \omega_{\mathrm{RP}^{\mathfrak{p}}} = \frac{2}{\alpha_{\mathfrak{p}}}\, (g_{\mathfrak{p}} \circ v)^\# ((\widehat{g}_{\mathfrak{p}}^{-1})^\# \omega_{\mathbb{S}^{\mathfrak{p}}}) = \frac{2}{\alpha_{\mathfrak{p}}}\, v^\# ((\widehat{g}_{\mathfrak{p}}^{-1} \circ g_{\mathfrak{p}})^\# \omega_{\mathbb{S}^{\mathfrak{p}}}).$$

Moreover,

$$\widehat{g}_{\mathfrak{p}}^{-1} \circ g_{\mathfrak{p}}(y) = h(y) := \begin{cases} y & \text{if} \quad y \in \mathbb{S}^{\mathfrak{p}}_{+} \\ -y & \text{if} \quad -y \in \mathbb{S}^{\mathfrak{p}}_{+} \end{cases}$$

and $h^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}} = \omega_{\mathbb{S}^{\mathfrak{p}}}$ if $\mathfrak{p}$ is odd, see (3.2), so that

$$\frac{1}{2}\, u^{\#}\omega_{\mathrm{RP}^{\mathfrak{p}}} = \frac{1}{\alpha_{\mathfrak{p}}}\, v^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}} \quad \forall\, v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}}) \quad \text{such that} \quad g_{\mathfrak{p}} \circ v = u\,. \tag{3.8}$$

The assertion follows from the definitions (3.1) and (3.7). □

THE CASE $\mathfrak{p}$ EVEN. The above does not hold for $\mathfrak{p} \geq 2$ even. In this case, in fact, $\mathrm{RP}^{\mathfrak{p}}$ being non-orientable, according to (3.2) the form $\omega_{\mathrm{RP}^{\mathfrak{p}}}$ defined in (3.6) cannot be extended to a smooth form in the whole of $\mathrm{RP}^{\mathfrak{p}}$. On the other hand, if we define the i.m. rectifiable current $[\![\, \mathrm{RP}^{\mathfrak{p}} \,]\!]$ as $[\![\, \mathrm{RP}^{\mathfrak{p}} \,]\!] := \widehat{g}_{\mathfrak{p}\#}[\![\, \mathbb{S}^{\mathfrak{p}}_{+} \,]\!]$, for $\mathfrak{p}$ even it turns out that its boundary is non-zero, $\partial[\![\, \mathrm{RP}^{\mathfrak{p}} \,]\!] \neq 0$.

Example 3.3 If e.g. $\mathfrak{p} = 2$, it is readily checked that $\partial[\![\, \mathrm{RP}^2 \,]\!] = g_{2\#}[\![\, \partial\mathbb{S}^2_{+} \,]\!] = 2\,[\![\, C \,]\!]$, where the 1-cycle C is a generator of $\pi_1(\mathrm{RP}^2)$, see Sec. 1.

However, taking e.g. $\Omega = B^n$, by Theorem 2.5, for $\mathfrak{p}$ even, the singularity of a Sobolev map $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$ is actually identified (up to the sign) by the current $\mathbb{P}(v)$ in (3.1), for any Sobolev map $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v = u$, compare Proposition 6.6 below. In fact, (3.2) yields $\mathbb{P}(-v) = -\,\mathbb{P}(v)$.

DEGREE OF MAPPINGS INTO $\mathbb{S}^{\mathfrak{p}}$. Let $\Sigma^{\mathfrak{p}}$ a copy of $\mathbb{S}^{\mathfrak{p}}$ in $\mathbb{R}^{\mathfrak{p}+1}$. We recall that the degree of a smooth map $v \in W^{1,\mathfrak{p}}(\Sigma^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ is given by the integer

$$\deg_{\mathbb{S}^{\mathfrak{p}}}(v) := \frac{1}{\alpha_{\mathfrak{p}}} \int_{\Sigma^{\mathfrak{p}}} D(v) \cdot \nu \, d\mathcal{H}^{\mathfrak{p}} \in \mathbb{Z}\,,$$

where $D(v)$ is the D-field of a smooth extension of v to a neighbor of $\Sigma^{\mathfrak{p}}$ and ν is the outward unit normal to $\Sigma^{\mathfrak{p}}$ in $\mathbb{R}^{\mathfrak{p}+1}$, so that $D(v) \cdot \nu$ agrees with the Jacobian $J_v^{\Sigma^{\mathfrak{p}}}$, compare [5]. According to (3.2), we thus have:

$$\deg_{\mathbb{S}^{\mathfrak{p}}}(v) = \frac{1}{\alpha_{\mathfrak{p}}} \int_{\Sigma^{\mathfrak{p}}} v^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}}\,. \tag{3.9}$$

In fact, if e.g. $\mathfrak{p} = 2$, we check:

$$\begin{aligned} v^{\#}\omega_{\mathbb{S}^2} &= v^1 dv^2 \wedge dv^3 + v^2 dv^3 \wedge dv^1 + v^3 dv^1 \wedge dv^2 \\ &= (v \cdot v_{x_2} \times v_{x_3})\, dx_2 \wedge dx_3 + (v \cdot v_{x_3} \times v_{x_1})\, dx_3 \wedge dx_1 + (v \cdot v_{x_1} \times v_{x_2})\, dx_1 \wedge dx_2 \\ &= D(v) \cdot \nu\, \mathcal{H}^2\,. \end{aligned}$$

Similarly, the degree of a continuous map $v \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ that is constant at the boundary $\partial B^{\mathfrak{p}}$ is defined by

$$\deg_{\mathbb{S}^{\mathfrak{p}}}(v) := \frac{1}{\alpha_{\mathfrak{p}}} \int_{B^{\mathfrak{p}}} v^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}} \in \mathbb{Z}\,. \tag{3.10}$$

In dimension $n = \mathfrak{p} + 1$, the *degree of a map* $v \in R^0_{\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$ *at a singular point* $a_j \in \Sigma(v)$ is:

$$\deg_{\mathbb{S}^{\mathfrak{p}}}(v, a_j) := \frac{1}{\alpha_{\mathfrak{p}}} \int_{\partial B^{\mathfrak{p}+1}(a_j, r)} D(v) \cdot \nu_{a_j, r}\, d\mathcal{H}^{\mathfrak{p}} \in \mathbb{Z}\,,$$

where $\nu_{a_j,r}$ is the outward unit normal to $\partial B^{\mathfrak{p}+1}(a_j, r)$ and the radius $r > 0$ is smaller than the distance of a_j from $\Sigma(v) \setminus \{a_j\}$, see Remark 3.1. The definition does not depend on the choice of r small. Moreover, if the current of the singularity $\mathbb{P}(v)$ satisfies (3.3), we have

$$\deg_{\mathbb{S}^{\mathfrak{p}}}(v, a_j) = \Delta_j \qquad \forall\, j = 1, \ldots, m\,. \tag{3.11}$$

Recall that if v has zero degree at a_j, then the singularity at a_j can be removed by paying an arbitrary small amount of energy. More precisely, Bethuel-Zheng [4] proved the following:

Proposition 3.4 *Let* $v \in R^0_{\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$ *satisfy* $\deg_{\mathbb{S}^{\mathfrak{p}}}(v, a_j) = \Delta_j$ *for some* $a_j \in \Sigma(v)$. *Then for every* $\varepsilon > 0$ *there exists a Sobolev map* $v_\varepsilon \in R^0_{\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$ *smooth in* $B^{\mathfrak{p}+1}(a_j, r)$, *with* $r = r(\varepsilon) > 0$ *small, and equal to* v *outside* $B^{\mathfrak{p}+1}(a_j, r)$, *such that*

$$\mathbf{D}_{\mathfrak{p}}(v_\varepsilon, B^{\mathfrak{p}+1}) \le \mathbf{D}_{\mathfrak{p}}(v, B^{\mathfrak{p}+1}) + |\Delta_j|\,\alpha_{\mathfrak{p}} + \varepsilon\,, \qquad \alpha_{\mathfrak{p}} := \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})\,.$$

Degree of mappings into $\mathrm{RP}^{\mathfrak{p}}$. Due to (3.2), we distinguish between $\mathfrak{p} \ge 3$ odd and $\mathfrak{p} \ge 2$ even. Actually, our Proposition 2.3 and Theorem 2.5 clarify the situation.

The case $\mathfrak{p}$ odd. As in [5, Sec. VIII-B-a)], for $\mathfrak{p} \ge 3$ odd the D-field $D(u)$ is well-defined for maps $u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathrm{RP}^{\mathfrak{p}})$ by

$$D(u) := D(v) \quad \text{for some } v \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}}) \text{ such that } g_{\mathfrak{p}} \circ v = u\,. \tag{3.12}$$

In fact, on account of Theorem 2.5, it suffices to recall that for $\mathfrak{p}$ odd we have $D(-v) = D(v)$.

Therefore, the degree of a smooth map $u \in W^{1,\mathfrak{p}}(\Sigma^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}})$ is well-defined by

$$\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) := \frac{1}{\alpha_{\mathfrak{p}}} \int_{\Sigma^{\mathfrak{p}}} D(u) \cdot \nu \, d\mathcal{H}^{\mathfrak{p}}\,.$$

In principle $\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) \in \mathbb{Z}/2$, as $\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}}) = \alpha_{\mathfrak{p}}/2$. However, by (3.12) we have $\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) = \deg_{\mathbb{S}^{\mathfrak{p}}}(v)$ for any Sobolev map $v \in W^{1,\mathfrak{p}}(\Sigma^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v = u$, whence $\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) \in \mathbb{Z}$. Moreover, due to (3.8) and (3.9), we infer that

$$\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) = \frac{1}{2} \int_{\Sigma^{\mathfrak{p}}} u^{\#}\omega_{\mathrm{RP}^{\mathfrak{p}}}\,.$$

If e.g. $\overline{u} = g_{\mathfrak{p}} \circ \overline{v}$, where $\overline{v}(x) = x/|x| \in W^{1,\mathfrak{p}}(\Sigma^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$, we have $\deg_{\mathrm{RP}^{\mathfrak{p}}}(\overline{u}) = 1$. According to [5, Sec. VIII-B-a)], this means that the *double* of the degree, $2\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) \in 2\mathbb{Z}$, tells the times the function $u : \Sigma^{\mathfrak{p}} \to \mathrm{RP}^{\mathfrak{p}}$ winds around $\mathrm{RP}^{\mathfrak{p}}$, with orientation prescribed by the sign.

Similarly, according to (3.8) and (3.10), the degree of a continuous map $u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}})$ that is constant at the boundary $\partial B^{\mathfrak{p}}$ is defined by

$$\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) := \frac{1}{2} \int_{B^{\mathfrak{p}}} u^{\#}\omega_{\mathrm{RP}^{\mathfrak{p}}} \in \mathbb{Z}\,.$$

In dimension $n = \mathfrak{p} + 1$, the *degree of a map* u *in* $R^0_{\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathrm{RP}^{\mathfrak{p}})$ *at a singular point* $a_j \in \Sigma(u)$ is well-defined by

$$\deg_{\mathrm{RP}^{\mathfrak{p}}}(u, a_j) := \frac{1}{\alpha_{\mathfrak{p}}} \int_{\partial B^{\mathfrak{p}+1}(a_j, r)} D(u) \cdot \nu_{a_j, r} \, d\mathcal{H}^{\mathfrak{p}} \in \mathbb{Z}\,,$$

for $r > 0$ small. We thus have

$$\deg_{\mathrm{RP}^{\mathfrak{p}}}(u, a_j) = \deg_{\mathbb{S}^{\mathfrak{p}}}(v, a_j) \in \mathbb{Z} \tag{3.13}$$

for any Sobolev map $v \in R^0_{\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v = u$. In fact, on account of Proposition 2.3, one has

$$\deg_{\mathbb{S}^{\mathfrak{p}}}(v, a_j) = \deg_{\mathbb{S}^{\mathfrak{p}}}(-v, a_j) \quad \text{if } \mathfrak{p} \text{ is odd}\,,$$

compare (3.2). If e.g. $\widetilde{u} = g_{\mathfrak{p}} \circ \widetilde{v}$, where $\widetilde{v}(x) = x/|x| \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$, then

$$\frac{1}{2}\mathbf{P}(\widetilde{u}) = -\delta_a\,, \qquad \deg_{\mathrm{RP}^{\mathfrak{p}}}(\widetilde{u}, a) = 1\,, \qquad a = 0_{\mathbb{R}^{\mathfrak{p}+1}}\,.$$

More generally, by Proposition 3.2 and (3.11), if $\Sigma(u) = \{a_j \mid j = 1, \dots, m\}$, we infer that

$$\frac{1}{2}\mathbf{P}(u) = -\sum_{j=1}^{m} \Delta_j\, \delta_{a_j} \iff \deg_{\mathrm{RP}^{\mathfrak{p}}}(u, a_j) = \Delta_j \in \mathbb{Z} \quad \forall j\,. \tag{3.14}$$

Again, the *double* of the degree, $2\deg_{\mathrm{RP}^{\mathfrak{p}}}(u, a_i) \in 2\mathbb{Z}$, tells the times the function $u_{|\partial B^{\mathfrak{p}+1}(a_j, r)}$, for r small, winds around $\mathrm{RP}^{\mathfrak{p}}$, with orientation prescribed by the sign, compare Proposition 3.6 below.

THE CASE $\mathfrak{p}$ EVEN. Following [5, Sec. VIII-B-c)], by (3.2) this time for every $v \in W^{1,\mathfrak{p}}(\Sigma^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ we have

$$\deg_{\mathbb{S}^{\mathfrak{p}}}(v) = -\deg_{\mathbb{S}^{\mathfrak{p}}}(-v) \quad \text{if } \mathfrak{p} \geq 2 \text{ is even}.$$

Therefore, arguing as above, in dimension $n = \mathfrak{p}$, the *degree* of a smooth map $u \in W^{1,\mathfrak{p}}(\Sigma^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}})$ is well-defined by

$$\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) := |\deg_{\mathbb{S}^{\mathfrak{p}}}(v)| \in \mathbb{N}$$

for any Sobolev map $v \in W^{1,\mathfrak{p}}(\Sigma^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v = u$. In a similar way one defines the degree of a continuous map $u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}})$ that is constant at the boundary $\partial B^{\mathfrak{p}}$.

Remark 3.5 If $u \in W^{1,\mathfrak{p}}(\Sigma^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}}) \cap C^0$, the degree is equal to the *topological degree* of the mapping $U \in C^0(\Sigma^{\mathfrak{p}}, \mathbb{RP}^{\mathfrak{p}})$ that corresponds to u by the formula

$$U = \widetilde{g}_{\mathfrak{p}}^{-1} \circ u,$$

so that both U *and* u *are homotopically trivial if and only if* $\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) = 0$, compare Proposition 2.8.

Notice that for $\mathfrak{p}$ even, we cannot define the degree of e.g. smooth maps $u \in W^{1,\mathfrak{p}}(\Sigma^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}})$ by using the form $\omega_{\mathrm{RP}^{\mathfrak{p}}}$ in definition (3.6). In fact, arguing as in Proposition 3.2, we have

$$(\widehat{g}_{\mathfrak{p}}^{-1} \circ g_{\mathfrak{p}})^{\#} \omega_{\mathbb{S}^{\mathfrak{p}}}(y) = \begin{cases} \omega_{\mathbb{S}^{\mathfrak{p}}}(y) & \text{if } \; y \in \mathbb{S}^{\mathfrak{p}}_+ \\ -\omega_{\mathbb{S}^{\mathfrak{p}}}(y) & \text{if } \; -y \in \mathbb{S}^{\mathfrak{p}}_+ \end{cases}$$

and hence $\displaystyle\frac{1}{2}\int_{\Sigma^{\mathfrak{p}}} u^{\#} \omega_{\mathrm{RP}^{\mathfrak{p}}} = 0$.

Similarly, in dimension $n = \mathfrak{p}+1$, the degree of a map $u \in R^0_{\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$ at a singular point $a_j \in \Sigma(u)$ is well-defined by

$$\deg_{\mathrm{RP}^{\mathfrak{p}}}(u, a_j) := |\deg_{\mathbb{S}^{\mathfrak{p}}}(v, a_j)| \in \mathbb{N},$$

for any Sobolev map $v \in R^0_{\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v = u$. In fact, by (3.2) we again have

$$\deg_{\mathbb{S}^{\mathfrak{p}}}(v, a_j) = -\deg_{\mathbb{S}^{\mathfrak{p}}}(-v, a_j) \quad \text{if } \mathfrak{p} \text{ is even}.$$

For the same reason, the D-field $D(u)$ is not well-defined for maps $u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathrm{RP}^{\mathfrak{p}})$, as

$$D(-v) = (-1)^{\mathfrak{p}+1} D(v) \quad \forall\, v \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}}). \tag{3.15}$$

A MINIMUM PROBLEM. As a consequence, setting for every $k \in \mathbb{N}^+$

$$\mathcal{F}_{\mathfrak{p},k} := \{u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}}) \cap C^0 \mid u \text{ is constant on } \partial B^{\mathfrak{p}} \text{ and } |\deg_{\mathrm{RP}^{\mathfrak{p}}}(u)| = k\}$$

on account of Remark 3.5 we extend Proposition 2.8 as follows:

Proposition 3.6 *For every* $\mathfrak{p} \geq 2$ *we have*

$$\inf\{\mathbf{D}_{\mathfrak{p}}(u, B^{\mathfrak{p}}) \mid u \in \mathcal{F}_{\mathfrak{p},k}\} = 2\,k\,\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}}).$$

PROOF: Let $u \in \mathcal{F}_{\mathfrak{p},k}$ and $v \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}}) \cap C^0$ be such that $g_{\mathfrak{p}} \circ v = u$, see Remark 2.4. By the area formula, as in [1] we have

$$\mathbf{D}_{\mathfrak{p}}(v, B^{\mathfrak{p}}) \geq \int_{B^{\mathfrak{p}}} J_v \, dx = \int_{\mathbb{S}^{\mathfrak{p}}} \mathcal{H}^0(v^{-1}(y))\, d\mathcal{H}^{\mathfrak{p}}(y).$$

Property $|\deg_{\mathbb{S}^{\mathfrak{p}}}(v)| = |\deg_{\mathrm{RP}^{\mathfrak{p}}}(u)| = k$ yields that $\mathcal{H}^0(v^{-1}(y)) \geq k$ for every $y \in \mathbb{S}^{\mathfrak{p}}$, whence

$$\mathbf{D}_{\mathfrak{p}}(v, B^{\mathfrak{p}}) \geq k\,\mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) = k\,\alpha_{\mathfrak{p}}.$$

Since $\mathbf{D}_{\mathfrak{p}}(u, B^{\mathfrak{p}}) = \mathbf{D}_{\mathfrak{p}}(v, B^{\mathfrak{p}})$, and $\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}}) = \alpha_{\mathfrak{p}}/2$, we obtain the inequality "$\geq$".

Conversely, by Proposition 2.7, for every $\varepsilon > 0$ we can easily find a continuous map $v \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ constant on $\partial B^{\mathfrak{p}}$, with degree $|\deg_{\mathbb{S}^{\mathfrak{p}}}(v)| = k$, such that $\mathbf{D}_{\mathfrak{p}}(v, B^{\mathfrak{p}}) \leq k\,\alpha_{\mathfrak{p}} + \varepsilon$. Taking $u = g_{\mathfrak{p}} \circ v$, we obtain a map $u \in \mathcal{F}_{\mathfrak{p},k}$ satisfying $\mathbf{D}_{\mathfrak{p}}(u, B^{\mathfrak{p}}) \leq k\,\alpha_{\mathfrak{p}} + \varepsilon$, that proves the assertion. □

4 The dipole problem

In this section we recover the results about the classical dipole problem that have been stated in [5, Sec. VIII]. We only consider the problem in dimension $\mathfrak{p}+1$, where the singularities of $W^{1,\mathfrak{p}}$-maps into $\mathrm{RP}^{\mathfrak{p}}$ are assumed to be zero-dimensional, with a prescribed constant condition at infinity. More general dipole problems as in [5] can be treated in a similar way. To this purpose, we first recall the definition of real and integral mass.

Real and integral mass. Let $\Omega\subset\mathbb{R}^n$ open, where $n\geq\mathfrak{p}+1$. For every current $\Gamma\in\mathcal{D}_{n-\mathfrak{p}-1}(\Omega)$ we denote by

$$\begin{array}{rcl} m_{r,\Omega}(\Gamma) & := & \inf\{\mathbf{M}(D)\mid D\in\mathcal{D}_{n-\mathfrak{p}}(\Omega)\,,\quad (\partial D)\llcorner\Omega=\Gamma\} \\ m_{i,\Omega}(\Gamma) & := & \inf\{\mathbf{M}(L)\mid L\in\mathcal{R}_{n-\mathfrak{p}}(\Omega)\,,\quad (\partial L)\llcorner\Omega=\Gamma\} \end{array} \tag{4.1}$$

the *real* and *integral mass* of Γ *relative to* Ω, respectively.

Integral flat chains and minimal connections. The current $\Gamma\in\mathcal{D}_{n-\mathfrak{p}-1}(\Omega)$ is said to be an *integral flat chain* if there exists an i.m. rectifiable current $L\in\mathcal{R}_{n-\mathfrak{p}}(\Omega)$ such that $(\partial L)\llcorner\Omega=\Gamma$ or, equivalently, if $m_{i,\Omega}(\Gamma)<\infty$. In this case, moreover, Federer-Fleming's closure-compactness theorem [14] yields that the minimum is always attained. Therefore, an i.m. rectifiable current $L\in\mathcal{R}_{n-\mathfrak{p}}(\Omega)$ is an *integral minimal connection* of Γ *allowing connections to the boundary* of Ω if $(\partial L)\llcorner\Omega=\Gamma$ and $\mathbf{M}(L)=m_{i,\Omega}(\Gamma)$, see [20, Vol. II, Sec. 4.2.6].

For example, the current $\mathbb{P}(v)\in\mathcal{D}_{n-\mathfrak{p}-1}(\Omega)$ of the singularity of a Sobolev map $v\in W^{1,\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}})$, see (3.1), is an integral flat chain:

Proposition 4.1 *Let $\Omega\subset\mathbb{R}^n$ open, where $n\geq\mathfrak{p}+1$. Then for every $v\in W^{1,\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}})$, the integral mass $m_{i,\Omega}(\mathbb{P}(v))$ of $\mathbb{P}(v)$ relative to Ω is finite, and actually*

$$\alpha_{\mathfrak{p}}\cdot m_{i,\Omega}(\mathbb{P}(v))\leq\mathbf{D}_{\mathfrak{p}}(v,\Omega)\,.$$

Proof: By the parallelogram inequality and the *coarea formula* [1], we have

$$\mathbf{D}_{\mathfrak{p}}(v,\Omega)\geq\int_\Omega J_v\,dx=\int_{\mathbb{S}^{\mathfrak{p}}}\mathcal{H}^{n-\mathfrak{p}}(v^{-1}(y))\,d\mathcal{H}^{\mathfrak{p}}(y)\,.$$

We then find $y\in\mathbb{S}^{\mathfrak{p}}$ such that the i.m. rectifiable current $L\in\mathcal{R}_{n-\mathfrak{p}}(\Omega)$

$$L:=\tau(v^{-1}(y),1,\overrightarrow{L})\,,\qquad\overrightarrow{L}(x):=\frac{D(v(x))}{|D(v(x))|}\,,\quad x\in v^{-1}(y)\,,$$

acting on forms $\gamma\in\mathcal{D}^{n-\mathfrak{p}}(\Omega)$ as

$$\langle L,\gamma\rangle=\int_{v^{-1}(y)}\langle\gamma(x),\overrightarrow{L}(x)\rangle\,d\mathcal{H}^{n-\mathfrak{p}}(x)\,,$$

has finite mass

$$\mathbf{M}(L)=\mathcal{H}^{n-\mathfrak{p}}(v^{-1}(y))\leq\frac{1}{\alpha_{\mathfrak{p}}}\,\mathbf{D}_{\mathfrak{p}}(v,\Omega)<\infty$$

whereas by (3.4) and (3.5) it also bounds the singularity of v, i.e., $(\partial L)\llcorner\Omega=\mathbb{P}(v)$. □

Remark 4.2 Therefore, *for every $v\in W^{1,\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}})$ there exists an integral minimal connection of* $\mathbb{P}(v)$, i.e., an i.m. rectifiable current $L\in\mathcal{R}_{n-\mathfrak{p}}(\Omega)$ such that $(\partial L)\llcorner\Omega=\mathbb{P}(v)$ and

$$\mathbf{M}(L)=m_{i,\Omega}(\mathbb{P}(u))\,.$$

As a consequence, by Theorem 2.5, Remark 2.6, and Propositions 3.2 and 4.1, we immediately deduce:

Corollary 4.3 *Let $\mathfrak{p}\geq 3$ odd and $\Omega\subset\mathbb{R}^n$ open and simply connected, where $n\geq\mathfrak{p}+1$. Then, for every $u\in W^{1,\mathfrak{p}}(\Omega,\mathrm{RP}^{\mathfrak{p}})$ the current $\frac{1}{2}\,\mathbf{P}(u)$ is an integral flat chain, and actually*

$$\alpha_{\mathfrak{p}}\cdot m_{i,\Omega}\Big(\frac{1}{2}\,\mathbf{P}(u)\Big)=\frac{\alpha_{\mathfrak{p}}}{2}\cdot m_{i,\Omega}(\mathbf{P}(u))\leq\mathbf{D}_{\mathfrak{p}}(u,\Omega)\,.$$

FLAT NORM. Let now $\Omega \subset \mathbb{R}^{\mathfrak{p}+1}$ open. The *flat norm* of a function $v \in W^{1,\mathfrak{p}}(\Omega, \mathbb{S}^{\mathfrak{p}})$ is defined by

$$\mathbb{L}(v,\Omega) := \frac{1}{\alpha_{\mathfrak{p}}} \sup\Big\{ \int_{\Omega} D(v) \cdot D\phi \, dx \mid \phi \in C_c^{\infty}(\Omega)\,,\ \|d\phi\| \leq 1 \ \text{ in } \ \Omega \Big\}\,, \tag{4.2}$$

compare [5]. Since the integral mass $m_{i,\Omega}(\mathbb{P}(v))$ is finite, by Federer's theorem [13] it agrees with the real mass $m_{r,\Omega}(\mathbb{P}(v))$. On the other hand, by a duality argument, see [20, Vol. II, Sec. 4.2.6], the real mass $m_{r,\Omega}(\mathbb{P}(v))$ agrees with the *flat norm* $\mathbb{L}(v,\Omega)$. We then obtain:

Proposition 4.4 *If $n = \mathfrak{p}+1$, for every $v \in W^{1,\mathfrak{p}}(\Omega, \mathbb{S}^{\mathfrak{p}})$ we have $m_{i,\Omega}(\mathbb{P}(v)) = \mathbb{L}(v,\Omega)$.*

By Theorem 2.5 and (3.15), we may and do give the following

Definition 4.5 *Let $\Omega \subset \mathbb{R}^{\mathfrak{p}+1}$ open and simply-connected. The (non-normalized)* flat norm *of a function $u \in W^{1,\mathfrak{p}}(\Omega, \mathrm{RP}^{\mathfrak{p}})$ is well-defined by*

$$\mathbf{L}(u,\Omega) := \alpha_{\mathfrak{p}} \cdot \mathbb{L}(v,\Omega)\,,$$

where $\mathbb{L}(v,\Omega)$ is the flat norm (4.2) of any Sobolev map $v \in W^{1,\mathfrak{p}}(\Omega, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v = u$.

Notice that for $\mathfrak{p} \geq 3$ odd, by (3.12) we deduce that

$$\mathbf{L}(u,\Omega) = \sup\Big\{ \int_{\Omega} D(u) \cdot D\phi \, dx \mid \phi \in C_c^{\infty}(\Omega)\,,\ \|d\phi\| \leq 1 \ \text{ in } \ \Omega \Big\}. \tag{4.3}$$

THE DIPOLE PROBLEM. Let now $\mathfrak{p} \geq 2$ integer and $n = \mathfrak{p}+1$. We choose a finite number m of points $a_i \subset \mathbb{R}^{\mathfrak{p}+1}$, for $i = 1, \ldots, m$, and to each point a_i we assign a non-zero integer number Δ_i, that corresponds to the degree at a_i. Similarly to [5, Sec. VIII-B-a)], the classical dipole problem is formulated as

$$\inf\{\mathbf{D}_{\mathfrak{p}}(u, \mathbb{R}^{\mathfrak{p}+1}) \mid u \in \widetilde{\mathcal{F}}_{\mathfrak{p}}\}\,,$$

where $\widetilde{\mathcal{F}}_{\mathfrak{p}}$ denotes the class

$$\begin{aligned} \widetilde{\mathcal{F}}_{\mathfrak{p}} := \{u \in W^{1,\mathfrak{p}}(\mathbb{R}^{\mathfrak{p}+1}, \mathrm{RP}^{\mathfrak{p}}) \quad | \quad & u \in C^{\infty}(\mathbb{R}^{\mathfrak{p}+1} \setminus \{a_i \mid i = 1, \ldots, m\})\,, \\ & u \text{ is constant at infinity, } \deg_{\mathrm{RP}^{\mathfrak{p}}}(u, a_i) = \Delta_i \quad \forall\, i\}\,. \end{aligned} \tag{4.4}$$

THE CASE $\mathfrak{p}$ ODD. By the definition of degree $\deg_{\mathrm{RP}^{\mathfrak{p}}}$ from Sec. 3, we assume $\Delta_i \in \mathbb{Z} \setminus \{0\}$ for every i.

Proposition 4.6 *Let $\mathfrak{p} \geq 3$ odd. The class $\widetilde{\mathcal{F}}_{\mathfrak{p}}$ is non-empty if and only if the compatibility condition*

$$\sum_{i=1}^{m} \Delta_i = 0 \tag{4.5}$$

on the non-zero integers Δ_i is satisfied. If (4.5) holds, moreover, we have

$$\inf\{\mathbf{D}_{\mathfrak{p}}(u, \mathbb{R}^{\mathfrak{p}+1}) \mid u \in \widetilde{\mathcal{F}}_{\mathfrak{p}}\} = 2\,\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}}) \cdot m_{i,\mathbb{R}^{\mathfrak{p}+1}}(\Gamma_0)\,, \quad \textit{where} \quad \Gamma_0 := -\sum_{i=1}^{m} \Delta_i \, \delta_{a_i}\,.$$

PROOF: Assume that $\widetilde{\mathcal{F}}_{\mathfrak{p}}$ is non-empty, and let $v \in R^0_{\mathfrak{p}}(\mathbb{R}^{\mathfrak{p}+1}, \mathrm{RP}^{\mathfrak{p}})$ be such that $g_{\mathfrak{p}} \circ v = u \in \widetilde{\mathcal{F}}_{\mathfrak{p}}$, see Proposition 2.3. Then $\deg_{\mathbb{S}^{\mathfrak{p}}}(v, a_i) = \deg_{\mathrm{RP}^{\mathfrak{p}}}(u, a_i)$ for every i. Therefore, since v is constant at infinity, condition (4.5) holds. The converse holds true, too. Moreover, if (4.5) holds, by a density argument, and on account of Theorem 2.5 and (3.13), we deduce that

$$\inf\{\mathbf{D}_{\mathfrak{p}}(u, \mathbb{R}^{\mathfrak{p}+1}) \mid u \in \widetilde{\mathcal{F}}_{\mathfrak{p}}\} = \inf\{\mathbf{D}_{\mathfrak{p}}(v, \mathbb{R}^{\mathfrak{p}+1}) \mid v \in \widetilde{\mathcal{G}}_{\mathfrak{p}}\}\,,$$

where

$$\begin{aligned} \widetilde{\mathcal{G}}_{\mathfrak{p}} := \{v \in W^{1,\mathfrak{p}}(\mathbb{R}^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}}) \quad | \quad & v \in C^{\infty}(\mathbb{R}^{\mathfrak{p}+1} \setminus \{a_i \mid i = 1, \ldots, m\})\,, \\ & v \text{ is constant at infinity, } \deg_{\mathbb{S}^{\mathfrak{p}}}(v, a_i) = \Delta_i \quad \forall\, i\}\,. \end{aligned} \tag{4.6}$$

Finally, condition $\deg_{\mathbb{S}^{\mathfrak{p}}}(v,a_i)=\Delta_i$ yields that $\mathbb{P}(v)=\Gamma_0$ for every $v\in\widetilde{\mathcal{G}}_{\mathfrak{p}}$, see (3.3) and (3.11), so that

$$\inf\{\mathbf{D}_{\mathfrak{p}}(v,\mathbb{R}^{\mathfrak{p}+1})\mid v\in\widetilde{\mathcal{G}}_{\mathfrak{p}}\}=\alpha_{\mathfrak{p}}\cdot m_{i,\mathbb{R}^{\mathfrak{p}+1}}(\Gamma_0)\,,$$

compare e.g. [20, Vol. II, Sec. 4.2.10], whereas $\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}})=\alpha_{\mathfrak{p}}/2$. □

As a consequence, we readily obtain:

Corollary 4.7 *Let $\mathfrak{p}\geq 3$ odd and assume that (4.5) holds. Then we have*

$$\inf\{\mathbf{D}_{\mathfrak{p}}(u,\mathbb{R}^{\mathfrak{p}+1})\mid u\in\widetilde{\mathcal{F}}_{\mathfrak{p}}\}=\sup\left\{\int_{\mathbb{R}^{\mathfrak{p}+1}}D(u_0)\cdot D\phi\,dx\mid \phi\in C^\infty_c(\mathbb{R}^{\mathfrak{p}+1})\,,\ \|d\phi\|\leq 1\ \text{ in }\ \mathbb{R}^{\mathfrak{p}+1}\right\}$$

for any $u_0\in\widetilde{\mathcal{F}}_{\mathfrak{p}}$, where the D-field $D(u_0)$ is defined by (3.12).

Proof: If $v_0\in\widetilde{\mathcal{G}}_{\mathfrak{p}}$ is such that $g_{\mathfrak{p}}\circ v_0=u_0$, we have $\mathbb{P}(v_0)=\Gamma_0$, hence $m_{i,\mathbb{R}^{\mathfrak{p}+1}}(\Gamma_0)=m_{i,\mathbb{R}^{\mathfrak{p}+1}}(\mathbb{P}(v_0))$. The assertion follows from Propositions 4.4 and 4.6, on account of Definition 4.5 and (4.3). □

The case $\mathfrak{p}$ even. We now recover the statement from [5, Sec. VIII-B-c)]. To this purpose, by the definition of degree from Sec. 3, we first observe that in the notation of $\widetilde{\mathcal{F}}_{\mathfrak{p}}$ from (4.4) this time we *assume that $\Delta_i\in\mathbb{N}\setminus\{0\}$ for every i.*

Proposition 4.8 *Let $\mathfrak{p}\geq 2$ even. The class $\widetilde{\mathcal{F}}_{\mathfrak{p}}$ is non-empty if and only if we can find some signs $\varepsilon_i=\pm1$, for $i=1,\dots,m$, such that the compatibility condition on the natural numbers $\Delta_i\in\mathbb{N}\setminus\{0\}$*

$$\sum_{i=1}^m\varepsilon_i\,\Delta_i=0\,,\qquad \varepsilon_i\in\{+1,-1\}\,,\tag{4.7}$$

is satisfied. If (4.7) holds, moreover, we have

$$\inf\{\mathbf{D}_{\mathfrak{p}}(u,\mathbb{R}^{\mathfrak{p}+1})\mid u\in\widetilde{\mathcal{F}}_{\mathfrak{p}}\}=2\,\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}})\cdot\inf_{\varepsilon_i}\Big\{m_{i,\mathbb{R}^{\mathfrak{p}+1}}\Big(\sum_{i=1}^m\varepsilon_i\Delta_i\,\delta_{a_i}\Big)\mid (4.7)\ \textit{holds}\ \Big\}\,.$$

Proof: Assume again that $\widetilde{\mathcal{F}}_{\mathfrak{p}}$ is non-empty, and let $v\in R^0_{\mathfrak{p}}(\mathbb{R}^{\mathfrak{p}+1},\mathrm{RP}^{\mathfrak{p}})$ be such that $g_{\mathfrak{p}}\circ v=u\in\widetilde{\mathcal{F}}_{\mathfrak{p}}$. Then for every i there exists $\varepsilon_i\in\{+1,-1\}$ such that $\deg_{\mathbb{S}^{\mathfrak{p}}}(v,a_i)=\varepsilon_i\deg_{\mathrm{RP}^{\mathfrak{p}}}(u,a_i)$. Therefore, since v is constant at infinity, condition (4.7) holds. The converse holds true, too. Moreover, if (4.7) holds, we now deduce that

$$\inf\{\mathbf{D}_{\mathfrak{p}}(u,\mathbb{R}^{\mathfrak{p}+1})\mid u\in\widetilde{\mathcal{F}}_{\mathfrak{p}}\}=\inf\{\mathbf{D}_{\mathfrak{p}}(v,\mathbb{R}^{\mathfrak{p}+1})\mid v\in\widehat{\mathcal{G}}_{\mathfrak{p}}\}\,,$$

where

$$\begin{aligned}\widehat{\mathcal{G}}_{\mathfrak{p}}:=\{v\in W^{1,\mathfrak{p}}(\mathbb{R}^{\mathfrak{p}+1},\mathbb{S}^{\mathfrak{p}})\mid v\in C^0(\mathbb{R}^{\mathfrak{p}+1}\setminus\{a_i\mid i=1,\dots,m\})\,,\\ v\text{ is constant at infinity, }\deg_{\mathbb{S}^{\mathfrak{p}}}(v,a_i)=\varepsilon_i\Delta_i\ \forall\,i\,,\text{ and (4.7) holds}\}\,.\end{aligned}\tag{4.8}$$

Finally, this time we have

$$\inf\{\mathbf{D}_{\mathfrak{p}}(v,\mathbb{R}^{\mathfrak{p}+1})\mid v\in\widehat{\mathcal{G}}_{\mathfrak{p}}\}=\alpha_{\mathfrak{p}}\cdot\inf_{\varepsilon_i}\Big\{m_{i,\mathbb{R}^{\mathfrak{p}+1}}\Big(\sum_{i=1}^m\varepsilon_i\Delta_i\,\delta_{a_i}\Big)\mid (4.7)\text{ holds}\ \Big\}\,,$$

as required. □

Example 4.9 For $\mathfrak{p}\geq 3$ odd, taking e.g. $m=2$, $\Delta_1=1$, and $\Delta_2=-1$, we have $\Gamma_0=\delta_{a_2}-\delta_{a_1}$ and $m_{i,\mathbb{R}^{\mathfrak{p}+1}}(\Gamma_0)=|a_1-a_2|$, whence

$$\inf\{\mathbf{D}_{\mathfrak{p}}(u,\mathbb{R}^{\mathfrak{p}+1})\mid u\in\widetilde{\mathcal{F}}_{\mathfrak{p}}\}=2\,\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}})\cdot|a_1-a_2|\,.\tag{4.9}$$

For $\mathfrak{p}\geq 2$ even, taking this time $m=2$, $\Delta_1=1$, and $\Delta_2=1$, we obtain again the formula (4.9).

This means that in both cases, the energy of minimizing sequences $\{u_k\}_k\subset\widetilde{\mathcal{F}}_{\mathfrak{p}}$ for the Dipole problem concentrates along the minimal connection of the singularities, but the *double* of the degrees Δ_i tells how many times the u_k's have to "cover" the target manifold $\mathrm{RP}^{\mathfrak{p}}$ near the lines of concentration of energy.

5 Weak limits, currents, and dipoles

In this section we analyze the weak limit points of sequences of smooth maps with equibounded $\mathfrak{p}$-energies that are constrained to take values into the projective space $\mathrm{RP}^{\mathfrak{p}}$.

Our approach relies on the results from the previous sections and on well-known facts for the analogous problem concerning maps that take values into the sphere $\mathbb{S}^{\mathfrak{p}}$. For this reason, we briefly recall some facts from the theory of *Cartesian currents*, for further details of which we refer to [20] and [22].

CARTESIAN CURRENTS. Let $n \geq \mathfrak{p} \geq 2$. If $v : B^n \to \mathbb{S}^{\mathfrak{p}}$ is a smooth map, the n-current G_v is defined by the integration of compactly supported smooth n-forms ω in $B^n \times \mathbb{S}^{\mathfrak{p}}$ over the naturally oriented n-manifold given by the graph $\mathcal{G}_v$ of v, i.e.,

$$G_v(\omega) := \int_{\mathcal{G}_v} \omega\,, \qquad \omega \in \mathcal{D}^n(B^n \times \mathbb{S}^{\mathfrak{p}})\,.$$

We thus have

$$G_v(\omega) = \int_{B^n} (\mathrm{Id} \bowtie v)^{\#}\omega \qquad \forall\, \omega \in \mathcal{D}^n(B^n \times \mathbb{S}^{\mathfrak{p}})\,, \tag{5.1}$$

where $(\mathrm{Id} \bowtie v)(x) := (x, v(x))$. More generally, to every Sobolev map v in $W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ we associate an i.m. rectifiable current $G_v \in \mathcal{R}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$ by means of definition (5.1), where this time the pull-back involves the distributional gradient of v. More precisely, G_v acts on forms in $\mathcal{D}^n(B^n \times \mathbb{S}^{\mathfrak{p}})$ by integration on the *rectifiable graph* $\mathcal{G}_v$ of v, and the *mass* agrees with the *area* of $\mathcal{G}_v$, i.e.,

$$\mathbf{M}(G_v) = \mathcal{H}^n(\mathcal{G}_v) \leq c\,\mathbf{D}_{\mathfrak{p}}(v, B^n) < \infty\,.$$

Remark 5.1 For $n \geq \mathfrak{p}+1$, if $\mathbb{P}(v) \in \mathcal{D}_{n-\mathfrak{p}-1}(B^n)$ is the current of the singularity of a Sobolev map $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, by (3.1) and (5.1) we find that

$$\alpha_{\mathfrak{p}} \cdot \langle \mathbb{P}(v), \phi \rangle = G_v(d\phi \wedge \omega_{\mathbb{S}^{\mathfrak{p}}}) = \partial G_v(\phi \wedge \omega_{\mathbb{S}^{\mathfrak{p}}}) \tag{5.2}$$

for every $\phi \in \mathcal{D}^{n-\mathfrak{p}-1}(B^n)$, as $G_v(\phi \wedge d\omega_{\mathbb{S}^{\mathfrak{p}}}) = 0$. More precisely, from the proof of Proposition 5.5 below we deduce that for every $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$

$$\partial G_v = \mathbb{P}(v) \times [\![\, \mathbb{S}^{\mathfrak{p}} \,]\!] \qquad \text{on} \quad \mathcal{D}^{n-1}(B^n \times \mathbb{S}^{\mathfrak{p}})\,.$$

For example, if $n = \mathfrak{p}+1$ and $v_0 = x/|x|$, we have

$$\partial G_{v_0} = -\delta_0 \times [\![\, \mathbb{S}^{\mathfrak{p}} \,]\!] \qquad \text{on} \quad \mathcal{D}^{\mathfrak{p}}(B^{\mathfrak{p}+1} \times \mathbb{S}^{\mathfrak{p}})\,,$$

compare [20, Vol. I, Sec. 3.2.2], whence $\mathbb{P}(v_0) = -\delta_0$, see Remark 3.1. More generally, if $v \in R^0_{\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$ and $\Sigma(v) := \{a_j \mid j = 1, \dots m\}$ is the discrete set of singular points of v, we recall that the current $\mathbb{P}(v) \in \mathcal{D}_0(B^{\mathfrak{p}+1})$ and the degree of v at the a_j's are related by (3.3) and (3.11).

If $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ is smooth, by Stokes' theorem the current G_v has null boundary inside $B^n \times \mathbb{S}^{\mathfrak{p}}$, as for every compactly supported smooth $(n-1)$-form η in $B^n \times \mathbb{S}^{\mathfrak{p}}$

$$\partial G_v(\eta) := G_v(d\eta) = \int_{\mathcal{G}_v} d\eta = \int_{\partial \mathcal{G}_v} \eta = 0\,. \tag{5.3}$$

We also recall that a sequence of currents $\{T_k\} \subset \mathcal{D}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$ is said to converge weakly in $\mathcal{D}_n$ to some $T \in \mathcal{D}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$, say $T_k \rightharpoonup T$, if $T_k(\omega) \to T(\omega)$ for every test form $\omega \in \mathcal{D}^n(B^n \times \mathbb{S}^{\mathfrak{p}})$. Moreover, the mass is lower semicontinuous along sequences of weakly converging currents.

For this reasons, taking into account Federer-Fleming's closure theorem [14], one has:

Theorem 5.2 (Giaquinta-Modica-Souček) *Let $\{v_k\}$ be a sequence of smooth maps in $W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $\sup_k \mathbf{D}_{\mathfrak{p}}(v_k, B^n) < \infty$. Then, possibly passing to a subsequence, the currents G_{v_k} weakly converge in $\mathcal{D}_n$ to some current $T \in \mathcal{D}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$ satisfying the following properties:*

i) *T is i.m. rectifiable in $\mathcal{R}_n(B^n\times\mathbb{S}^{\mathfrak{p}})$;*

ii) *there exist a function $v_T\in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$ and an i.m. rectifiable current $L_T\in\mathcal{R}_{n-\mathfrak{p}}(B^n)$ such that*

$$T=G_{v_T}+L_T\times[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!]\,; \tag{5.4}$$

iii) *T has finite mass, $\mathbf{M}(T)=\mathbf{M}(G_{v_T})+\alpha_{\mathfrak{p}}\cdot\mathbf{M}(L_T)<\infty$;*

iv) *T has no interior boundary, i.e.,*

$$\partial T(\eta):=T(d\eta)=0\quad\forall\,\eta\in\mathcal{D}^{n-1}(B^n\times\mathbb{S}^{\mathfrak{p}})\,. \tag{5.5}$$

PROOF: Compare [20, Vol. II], Sec. 5.2.3 for $\mathfrak{p}=2$, and Note 6 in Ch. 5 for $\mathfrak{p}\geq 3$. □

Notice that the sequence $\{v_k\}$ weakly converges in $W^{1,\mathfrak{p}}(B^n,\mathbb{R}^{\mathfrak{p}+1})$ to the Sobolev function v_T in (5.4).

Theorem 5.2 motivates the following definition, that agrees with the one in [20, Vol. II], as cited above.

Definition 5.3 *We denote by* $\text{cart}^{\mathfrak{p},1}(B^n\times\mathbb{S}^{\mathfrak{p}})$ *the class of n-currents in $B^n\times\mathbb{S}^{\mathfrak{p}}$ satisfying the properties* i)–iv) *in Theorem* 5.2.

Therefore, G_v belongs to $\text{cart}^{\mathfrak{p},1}(B^n\times\mathbb{S}^{\mathfrak{p}})$ for every smooth map $v\in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$ or, more generally, for every Sobolev map $v\in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$ satisfying the null-boundary condition

$$\partial G_v(\eta):=G_v(d\eta)=0\quad\forall\,\eta\in\mathcal{D}^{n-1}(B^n\times\mathbb{S}^{\mathfrak{p}})\,. \tag{5.6}$$

Remark 5.4 In dimension $n=\mathfrak{p}$, property (5.6) is always satisfied. In fact, by Schoen-Uhlenbeck density theorem, for every $v\in W^{1,\mathfrak{p}}(B^{\mathfrak{p}},\mathbb{S}^{\mathfrak{p}})$ we find a smooth sequence $\{v_k\}\subset C^\infty(B^{\mathfrak{p}},\mathbb{S}^{\mathfrak{p}})$ that strongly converges to v in $W^{1,\mathfrak{p}}$. By Lebesgue's theorem, this yields the weak convergence $G_{v_k}(\omega)\to G_v(\omega)$ for every form $\omega\in\mathcal{D}^{\mathfrak{p}}(B^{\mathfrak{p}}\times\mathbb{S}^{\mathfrak{p}})$. Since $G_{v_k}(d\eta)=0$ for every $\eta\in\mathcal{D}^{\mathfrak{p}-1}(B^{\mathfrak{p}}\times\mathbb{S}^{\mathfrak{p}})$, see (5.3), letting $k\to\infty$ we get (5.6).

More generally, we obtain:

Proposition 5.5 *Let $n\geq\mathfrak{p}+1$ and $T\in\mathcal{R}_n(B^n\times\mathbb{S}^{\mathfrak{p}})$ satisfy* (5.4)*, where $v_T\in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$ and $L_T\in\mathcal{R}_{n-\mathfrak{p}}(B^n)$. Then the null-boundary condition* (5.5) *is equivalent to*

$$(\partial L_T)\llcorner B^n=-\,\mathbb{P}(v_T)\,, \tag{5.7}$$

where $\mathbb{P}(v_T)$ is given by (3.1).

PROOF: In order to prove that (5.7) implies (5.5), we decompose any form $\omega\in\mathcal{D}^k(B^n\times\mathbb{S}^{\mathfrak{p}})$ as $\omega=\sum_{j=0}^k\omega^{(j)}$, where $\omega^{(j)}$ is the (possibly zero) component with exactly j differentials in the "vertical" y-directions. Moreover, we split the differential $d=d_x+d_y$.

Since $v_T\in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$, arguing as e.g. in [22, Prop. 4.2.10] we get:

(a) $\partial G_{v_T}(\eta^{(j)})=0$ for every $j=0,\dots,\mathfrak{p}-1$ and $\eta\in\mathcal{D}^{n-1}(B^n\times\mathbb{S}^{\mathfrak{p}})$;

(b) $\partial G_{v_T}(d_y\gamma^{(j)})=0$ for every $j=0,\dots,\mathfrak{p}-1$ and $\gamma\in\mathcal{D}^{n-2}(B^n\times\mathbb{S}^{\mathfrak{p}})$.

Since $\partial(L_T\times[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!])(\eta^{(j)})=0$ for every $j=0,\dots,\mathfrak{p}-1$ and $\eta\in\mathcal{D}^{n-1}(B^n\times\mathbb{S}^{\mathfrak{p}})$, by (5.4) and (a) we deduce that the null-boundary condition (5.5) is equivalent to the property

$$\partial(L_T\times[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!])(\eta^{(\mathfrak{p})})=-\partial G_{v_T}(\eta^{(\mathfrak{p})})\qquad\forall\,\eta\in\mathcal{D}^{n-1}(B^n\times\mathbb{S}^{\mathfrak{p}})\,. \tag{5.8}$$

By a density argument we reduce to show that (5.8) holds for every η such that $\eta^{(\mathfrak{p})}=\phi\wedge\alpha$ for some $\phi\in\mathcal{D}^{n-\mathfrak{p}-1}(B^n)$ and $\alpha\in\mathcal{D}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})$. By Hodge decomposition theorem, we can write $\alpha=\lambda\,\omega_{\mathbb{S}^{\mathfrak{p}}}+d\beta$ for some $\lambda\in\mathbb{R}$ and $\beta\in\mathcal{D}^{\mathfrak{p}-1}(\mathbb{S}^{\mathfrak{p}})$, so that

$$\eta^{(\mathfrak{p})}=\phi\wedge\alpha=\lambda\,\phi\wedge\omega_{\mathbb{S}^{\mathfrak{p}}}+\phi\wedge d\beta\,.$$

Using (5.2) and (b), we have

$$\partial G_{v_T}(\phi\wedge\alpha)=\partial G_{v_T}(\lambda\,\phi\wedge\omega_{\mathbb{S}^{\mathfrak{p}}})+\partial G_{v_T}(\phi\wedge d\beta)=\lambda\,\alpha_{\mathfrak{p}}\cdot\langle\mathbb{P}(v_T),\phi\rangle+0\,.$$

Moreover, by definition of Cartesian product of currents we obtain

$$\begin{aligned}\partial(L_T\times[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!])(\phi\wedge\alpha) &= \partial(L_T\times[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!])(\lambda\,\phi\wedge\omega_{\mathbb{S}^{\mathfrak{p}}})+\partial(L_T\times[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!])(\phi\wedge d\beta)\\ &= \lambda\,(L_T\times[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!])(d\phi\wedge\omega_{\mathbb{S}^{\mathfrak{p}}})+(L_T\times[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!])(d\phi\wedge d\beta)\\ &= \lambda\,L_T(d\phi)\cdot[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!](\omega_{\mathbb{S}^{\mathfrak{p}}})+L_T(d\phi)\cdot[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!](d\beta)\\ &= \lambda\,\alpha_{\mathfrak{p}}\cdot\partial L_T(\phi)+0\,,\end{aligned}$$

as $[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!](d\beta)=\partial[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!](\beta)=0$, so that (5.7) implies (5.8), hence (5.5). The converse implication follows from the previous computation, where we take $\eta=\eta^{(\mathfrak{p})}=\phi\wedge\omega_{\mathbb{S}^{\mathfrak{p}}}$, i.e., $\lambda=1$ and $\beta=0$. □

In [17] and [19], by means of the *parametric lower semicontinuous extension* of the conformal $\mathfrak{p}$-energy integrand, Giaquinta-Modica-Souček defined a non-negative functional $T\mapsto\mathbf{D}_{\mathfrak{p}}(T)$ on the class $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathbb{S}^{\mathfrak{p}})$, called the $\mathfrak{p}$-*energy*, satisfying the following properties:

Proposition 5.6 *We have:*

(a) $T\mapsto\mathbf{D}_{\mathfrak{p}}(T)$ *is lower semicontinuous with respect to the weak convergence in* $\mathcal{D}_n$*;*

(b) if T *satisfies* (5.4)*, then* $\mathbf{D}_{\mathfrak{p}}(T)=\mathbf{D}_{\mathfrak{p}}(v_T,B^n)+\alpha_{\mathfrak{p}}\cdot\mathbf{M}(L_T)$*;*

(c) the class $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathbb{S}^{\mathfrak{p}})$ *is closed under the weak* $\mathcal{D}_n$*-convergence of sequences* $\{T_k\}$ *of currents with equibounded* $\mathfrak{p}$*-energies,* $\sup_k\mathbf{D}_{\mathfrak{p}}(T_k)<\infty$*;*

(d) if $\{T_k\}\subset\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathbb{S}^{\mathfrak{p}})$ *satisfies* $\sup_k\mathbf{D}_{\mathfrak{p}}(T_k)<\infty$*, possibly passing to a subsequence* T_k *weakly converges to some current* T *in* $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathbb{S}^{\mathfrak{p}})$*;*

(e) for every $T\in\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathbb{S}^{\mathfrak{p}})$*, there exists a sequence of* smooth *maps* $\{v_k\}\subset W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$ *such that* $G_{v_k}\rightharpoonup T$ *in* $\mathcal{D}_n$ *and* $\mathbf{D}_{\mathfrak{p}}(v_k,B^n)\to\mathbf{D}_{\mathfrak{p}}(T)$ *as* $k\to\infty$*.*

Proof: As to the properties (a) and (b), see [20, Vol. II, Sec. 1.2.4] and also [22, Sec. 4.9]. Properties (c) and (d) follow by arguing as in [16], where they were proved for the case $\mathfrak{p}=2$ in any dimension n. The density property (e) is obtained by using the same argument as for the case $\mathfrak{p}=2$ in [21], see also [22, Ch. 5], on account of Proposition 2.7. For this reason, we omit any further detail. □

We finally notice that the weak convergence $T_k\rightharpoonup T$ of currents in $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathbb{S}^{\mathfrak{p}})$ with equibounded $\mathfrak{p}$-energies yields the weak $W^{1,\mathfrak{p}}$-convergence $v_{T_k}\rightharpoonup v_T$ of the corresponding functions in $W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$.

Cartesian currents in $B^n\times\mathrm{RP}^{\mathfrak{p}}$. In the case $\mathfrak{p}\geq 3$ odd, we are able to characterize the weak limits of sequences of smooth maps in $W^{1,\mathfrak{p}}(B^n,\mathrm{RP}^{\mathfrak{p}})$ with equibounded $\mathfrak{p}$-energies by means of homological arguments as above.

We first recall that $\mathrm{RP}^{\mathfrak{p}}\subset\mathbb{R}^{N(\mathfrak{p})}$ and that by Federer's flatness theorem [13], every i.m. rectifiable n-current in $B^n\times\mathbb{R}^{N(\mathfrak{p})}$ with support in $\overline{B}^n\times\mathrm{RP}^{\mathfrak{p}}$ gives rise to an i.m. rectifiable current in $\mathcal{R}_n(B^n\times\mathrm{RP}^{\mathfrak{p}})$. This holds true if e.g. $T=G_u$ for some Sobolev map $u\in W^{1,\mathfrak{p}}(B^n,\mathrm{RP}^{\mathfrak{p}})$, where the current $G_u\in\mathcal{R}_n(B^n\times\mathrm{RP}^{\mathfrak{p}})$ is defined in a way similar to (5.1), but with $\mathbb{S}^{\mathfrak{p}}$ replaced by $\mathrm{RP}^{\mathfrak{p}}$.

The case $\mathfrak{p}\geq 3$ odd. In this case the i.m. rectifiable current $[\![\,\mathrm{RP}^{\mathfrak{p}}\,]\!]$ has been defined in Sec. 1 so that

$$g_{\mathfrak{p}\#}[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!]=2\,[\![\,\mathrm{RP}^{\mathfrak{p}}\,]\!]\,,\qquad \mathbf{M}([\![\,\mathrm{RP}^{\mathfrak{p}}\,]\!])=\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}})=\frac{\alpha_{\mathfrak{p}}}{2}\,. \tag{5.9}$$

Moreover, in dimension $n\geq\mathfrak{p}+1$, for every $u\in W^{1,\mathfrak{p}}(B^n,\mathrm{RP}^{\mathfrak{p}})$, by definition (3.7) we have

$$\langle\mathbf{P}(u),\phi\rangle=G_u(d\phi\wedge\omega_{\mathrm{RP}^{\mathfrak{p}}})\,. \tag{5.10}$$

Therefore, we introduce the following.

Definition 5.7 *For $\mathfrak{p} \geq 3$ odd, and $n \geq \mathfrak{p}$, we denote by $\text{cart}^{\mathfrak{p},1}(B^n \times \text{RP}^{\mathfrak{p}})$ the subclass of currents $T \in \mathcal{D}_n(B^n \times \text{RP}^{\mathfrak{p}})$ satisfying the following properties:*

i) *T is i.m. rectifiable in $\mathcal{R}_n(B^n \times \text{RP}^{\mathfrak{p}})$;*

ii) *we have*

$$T = G_{u_T} + 2\,L_T \times [\![\,\text{RP}^{\mathfrak{p}}\,]\!] \tag{5.11}$$

for some $u_T \in W^{1,\mathfrak{p}}(B^n, \text{RP}^{\mathfrak{p}})$ and some i.m. rectifiable current $L_T \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$;

iii) *T has finite mass, $\mathbf{M}(T) = \mathbf{M}(G_{u_T}) + \alpha_{\mathfrak{p}} \cdot \mathbf{M}(L_T) < \infty$;*

iv) *T has no interior boundary, i.e.,*

$$\partial T(\eta) := T(d\eta) = 0 \quad \forall\, \eta \in \mathcal{D}^{n-1}(B^n \times \text{RP}^{\mathfrak{p}})\,. \tag{5.12}$$

Notice that the weak convergence $T_k \rightharpoonup T$ of currents in $\text{cart}^{\mathfrak{p},1}(B^n \times \text{RP}^{\mathfrak{p}})$ with equibounded $\mathfrak{p}$-energies yields again the weak $W^{1,\mathfrak{p}}$-convergence $u_{T_k} \rightharpoonup u_T$ of the corresponding functions in $W^{1,\mathfrak{p}}(B^n, \text{RP}^{\mathfrak{p}})$. Moreover, as in Remark 5.4, we infer that in dimension $n = \mathfrak{p}$, the current G_u belongs to $\text{cart}^{\mathfrak{p},1}(B^{\mathfrak{p}} \times \text{RP}^{\mathfrak{p}})$ for every $u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \text{RP}^{\mathfrak{p}})$. In higher dimension, we have:

Proposition 5.8 *Let $n \geq \mathfrak{p}+1$ and $T \in \mathcal{R}_n(B^n \times \text{RP}^{\mathfrak{p}})$ satisfy (5.11) for some $u_T \in W^{1,\mathfrak{p}}(B^n, \text{RP}^{\mathfrak{p}})$ and $L_T \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$. Then the null-boundary condition (5.12) is equivalent to*

$$(\partial L_T) \llcorner B^n = -\frac{1}{2}\,\mathbf{P}(u_T)\,, \tag{5.13}$$

where $\mathbf{P}(u_T)$ is given by (3.7). Moreover, for every $u \in W^{1,\mathfrak{p}}(B^n, \text{RP}^{\mathfrak{p}})$ we have:

$$\partial G_u = \mathbf{P}(u) \times [\![\,\text{RP}^{\mathfrak{p}}\,]\!] \qquad on \quad \mathcal{D}^{n-1}(B^n \times \text{RP}^{\mathfrak{p}})\,.$$

PROOF: Since the $\mathfrak{p}^{th}$ de Rham group $\mathcal{H}^{\mathfrak{p}}_{dR}(\text{RP}^{\mathfrak{p}}) \simeq \mathbb{Z}$, by Hodge theorem, for every form $\alpha \in \mathcal{D}^{\mathfrak{p}}(\text{RP}^{\mathfrak{p}})$ we have $\alpha = \lambda\,\omega_{\text{RP}^{\mathfrak{p}}} + d\beta$ for some $\lambda \in \mathbb{R}$ and $\beta \in \mathcal{D}^{\mathfrak{p}-1}(\text{RP}^{\mathfrak{p}})$. Therefore, the proof is similar to the one of Proposition 5.7, using this time (5.10). $\square$

As a consequence, we obtain the following relation:

Proposition 5.9 *If $\mathfrak{p} \geq 3$ is odd, and $n \geq \mathfrak{p}$, we have*

$$\{h_{\mathfrak{p}\#}T \mid T \in \text{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})\} = \text{cart}^{\mathfrak{p},1}(B^n \times \text{RP}^{\mathfrak{p}})\,,$$

where $h_{\mathfrak{p}}(x,y) := (x, g_{\mathfrak{p}}(y)) \in B^n \times \text{RP}^{\mathfrak{p}}$, for any $(x,y) \in B^n \times \mathbb{S}^{\mathfrak{p}}$.

PROOF: If $u_T = g_{\mathfrak{p}} \circ v_T$ for some $v_T \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, since $h_{\mathfrak{p}} \circ (Id \bowtie v_T) = Id \bowtie (g_{\mathfrak{p}} \circ v_T)$, for every form $\omega \in \mathcal{D}^n(B^n \times \text{RP}^{\mathfrak{p}})$, by (5.1) we get

$$\begin{aligned} \langle h_{\mathfrak{p}\#}G_{v_T}, \omega\rangle \quad &:= \quad \langle G_{v_T}, h_{\mathfrak{p}}^{\#}\omega\rangle = \int_{B^n} (Id \bowtie v_T)^{\#}\,(h_{\mathfrak{p}}^{\#}\omega) \\ &= \quad \int_{B^n} \big(Id \bowtie (g_{\mathfrak{p}} \circ v_T)\big)^{\#}\omega = \int_{B^n} (Id \bowtie u_T)^{\#}\omega =: \langle G_{u_T}, \omega\rangle\,. \end{aligned}$$

Moreover, by (5.9) we have

$$h_{\mathfrak{p}\#}(L_T \times [\![\,\mathbb{S}^{\mathfrak{p}}\,]\!]) = L_T \times g_{\mathfrak{p}\#}[\![\,\mathbb{S}^{\mathfrak{p}}\,]\!] = 2\,L_T \times [\![\,\text{RP}^{\mathfrak{p}}\,]\!]\,.$$

Therefore, if $T \in \text{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ satisfies (5.4), the image current $h_{\mathfrak{p}\#}T$ satisfies the structure property (5.11). Propositions 3.2, 5.5, and 5.8 yield the inclusion "$\subset$". The equality follows from Theorem 2.5. $\square$

Finally, setting for every $T \in \text{cart}^{\mathfrak{p},1}(B^n \times \text{RP}^{\mathfrak{p}})$ as in (5.11)

$$\mathbf{D}_{\mathfrak{p}}(T) := \mathbf{D}_{\mathfrak{p}}(u_T, B^n) + \frac{\alpha_{\mathfrak{p}}}{2} \cdot \mathbf{M}(2\,L_T)\,, \tag{5.14}$$

by Theorem 2.5 and Propositions 5.6 and 5.9 we readily obtain:

Proposition 5.10 *For* $\mathfrak{p}\geq 3$ *odd, and* $n\geq\mathfrak{p}$*, the* $\mathfrak{p}$-energy *functional* $T\mapsto\mathbf{D}_{\mathfrak{p}}(T)$ *on* $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathrm{RP}^{\mathfrak{p}})$ *satisfies the following properties:*

(a) $T\mapsto\mathbf{D}_{\mathfrak{p}}(T)$ *is lower semicontinuous with respect to the weak convergence in* $\mathcal{D}_n(B^n\times\mathrm{RP}^{\mathfrak{p}})$*;*

(b) the class $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathrm{RP}^{\mathfrak{p}})$ *is closed under the weak* $\mathcal{D}_n$*-convergence of sequences* $\{T_k\}$ *of currents with equibounded* $\mathfrak{p}$*-energies,* $\sup_k\mathbf{D}_{\mathfrak{p}}(T_k)<\infty$*;*

(c) if $\{T_k\}\subset\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathrm{RP}^{\mathfrak{p}})$ *satisfies* $\sup_k\mathbf{D}_{\mathfrak{p}}(T_k)<\infty$*, possibly passing to a subsequence* T_k *weakly converges to some current* T *in* $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathrm{RP}^{\mathfrak{p}})$*;*

(d) for every $T\in\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathrm{RP}^{\mathfrak{p}})$*, there exists a sequence of* smooth *maps* $\{u_k\}\subset W^{1,\mathfrak{p}}(B^n,\mathrm{RP}^{\mathfrak{p}})$ *such that* $G_{u_k}\rightharpoonup T$ *in* $\mathcal{D}_n$ *and* $\mathbf{D}_{\mathfrak{p}}(u_k,B^n)\to\mathbf{D}_{\mathfrak{p}}(T)$ *as* $k\to\infty$*.*

The case $\mathfrak{p}\geq 2$ even. Since $\mathrm{RP}^{\mathfrak{p}}$ is not orientable for $\mathfrak{p}\geq 2$ even, the above arguments fail the attempt to define the class $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathrm{RP}^{\mathfrak{p}})$. In fact, this time we have $g_{\mathfrak{p}\#}[\![\mathbb{S}^{\mathfrak{p}}]\!]=0$, hence *the concentration phenomenon cannot be seen by means of a homological theory.*

More precisely, if G_{u_k} is a sequence of currents in $B^n\times\mathrm{RP}^{\mathfrak{p}}$ carried by the graph of smooth maps $u_k\in W^{1,\mathfrak{p}}(B^n,\mathrm{RP}^{\mathfrak{p}})$ satisfying $\sup_k\mathbf{D}_{\mathfrak{p}}(u_k,B^n)<\infty$, possibly passing to a subsequence, the G_{u_k}'s weakly converge to the current G_{u_T} carried by the graph of the Sobolev map $u_T\in W^{1,\mathfrak{p}}(B^n,\mathrm{RP}^{\mathfrak{p}})$ given by the weak limit $u_k\rightharpoonup u_T$ in $W^{1,\mathfrak{p}}$.

We also recall from Sec. 3 that if $\mathfrak{p}\geq 2$ is even, the i.m. rectifiable current $[\![\mathrm{RP}^{\mathfrak{p}}]\!]:=\widehat{g}_{\mathfrak{p}\#}[\![\mathbb{S}^{\mathfrak{p}}_+]\!]$ has a non-zero boundary, $\partial[\![\mathrm{RP}^{\mathfrak{p}}]\!]\neq 0$, see Example 3.3.

Weak limits of maps with values in $\mathrm{RP}^{\mathfrak{p}}$. For the above reasons, if $\mathfrak{p}\geq 2$ is even, one may attack the problem of identifying the weak limit points by means of a measure-theoretic approach, based e.g. on the theory of *rectifiable varifolds*, see [35]. We shall not pursue this direction. In fact, by Theorem 2.5 and Propositions 5.5 and 5.6, we readily obtain the following result, that holds true for all $n\geq\mathfrak{p}\geq 2$.

Theorem 5.11 *Let* $\{u_k\}\subset W^{1,\mathfrak{p}}(B^n,\mathrm{RP}^{\mathfrak{p}})$ *be a sequence of smooth maps satisfying* $\sup_k\mathbf{D}_{\mathfrak{p}}(u_k,B^n)<\infty$*. Let* $\{v_k\}\subset W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$ *be such that* $g_{\mathfrak{p}}\circ v_k=u_k$*. Then, possibly passing to a subsequence,* $G_{v_k}\rightharpoonup T$ *weakly in* $\mathcal{D}_n(B^n\times\mathbb{S}^{\mathfrak{p}})$ *to some current* $T=G_v+L\times[\![\mathbb{S}^{\mathfrak{p}}]\!]$ *in* $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathbb{S}^{\mathfrak{p}})$*, i.e.,* $v\in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$*,* $L\in\mathcal{R}_{n-\mathfrak{p}}(B^n)$*. For* $n\geq\mathfrak{p}+1$*, we also have* $(\partial L)\llcorner B^n=-\mathbb{P}(v)$*. Moreover, the sequence* $\{u_k\}$ *weakly converges in* $W^{1,\mathfrak{p}}$ *to the Sobolev function* $u:=g_{\mathfrak{p}}\circ v\in W^{1,\mathfrak{p}}(B^n,\mathrm{RP}^{\mathfrak{p}})$*, and*

$$\mathbf{D}_{\mathfrak{p}}(u,B^n)+\alpha_{\mathfrak{p}}\cdot\mathbf{M}(L)\leq\liminf_{k\to\infty}\mathbf{D}_{\mathfrak{p}}(u_k,B^n)\,.$$

On account of Proposition 5.8, for $\mathfrak{p}\geq 3$ odd we also obtain:

Corollary 5.12 *Under the hypotheses of Theorem* 5.11*, if* $\mathfrak{p}\geq 3$ *is odd we also have that* $G_{u_k}\rightharpoonup T$ *weakly in* $\mathcal{D}_n(B^n\times\mathrm{RP}^{\mathfrak{p}})$ *to the current* $T=G_u+2\,L\times[\![\mathrm{RP}^{\mathfrak{p}}]\!]$ *in* $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathrm{RP}^{\mathfrak{p}})$*, where* $(\partial L)\llcorner B^n=-\frac{1}{2}\,\mathbf{P}(u)$ *if* $n\geq\mathfrak{p}+1$*.*

We shall see in the next section that Theorem 2.5 allows us to describe the *relaxed energy* in the case $\mathfrak{p}\geq 2$ even, too.

The Dipole problem. We finally observe that in the case $\mathfrak{p}\geq 3$ odd, the Dipole problem from Proposition 4.6 can be reformulated in terms of Cartesian currents. In this framework, moreover, *the minimum is attained.* More precisely, according to the notation in (4.4), for $n=\mathfrak{p}+1$ we denote

$$\widehat{\mathcal{F}}_{\mathfrak{p}}:=\{T=G_u+2L\times[\![\mathrm{RP}^{\mathfrak{p}}]\!]\quad|\quad \begin{array}{l}u\in W^{1,\mathfrak{p}}(\mathbb{R}^{\mathfrak{p}+1},\mathrm{RP}^{\mathfrak{p}})\,,\ L\in\mathcal{R}_1(\mathbb{R}^{\mathfrak{p}+1})\,,\\ u\text{ is constant at infinity, }\partial T=2\,\Gamma_0\times[\![\mathrm{RP}^{\mathfrak{p}}]\!]\}\end{array} \tag{5.15}$$

where, we recall,

$$\Gamma_0:=-\sum_{i=1}^m\Delta_i\,\delta_{a_i}\,,\qquad\Delta_i\in\mathbb{Z}\setminus\{0\}\,.$$

If the compatibility condition (4.5) holds, we have $m_{i,\mathbb{R}^{\mathfrak{p}+1}}(\Gamma_0) < \infty$, see (4.1), and we can find an integral minimal connection for Γ_0, i.e., an i.m. rectifiable current $L_0 \in \mathcal{R}_1(\mathbb{R}^{\mathfrak{p}+1})$ such that

$$\partial L_0 = \Gamma_0 \quad \text{and} \quad \mathbf{M}(L_0) = m_{i,\mathbb{R}^{\mathfrak{p}+1}}(\Gamma_0)\,.$$

Therefore, the locally i.m. rectifiable $(\mathfrak{p}+1)$-current in $\mathbb{R}^{\mathfrak{p}+1} \times \mathrm{RP}^{\mathfrak{p}}$

$$T_0 = G_P + 2\,L_0 \times [\![\, \mathrm{RP}^{\mathfrak{p}} \,]\!]\,, \tag{5.16}$$

where G_P is the current carried by the graph of a constant map $P \in \mathrm{RP}^{\mathfrak{p}}$, satisfies

$$T_0 \in \widehat{\mathcal{F}}_{\mathfrak{p}} \quad \text{and} \quad \mathbf{D}_{\mathfrak{p}}(T_0) = 2\,\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}}) \cdot m_{i,\mathbb{R}^{\mathfrak{p}+1}}(\Gamma_0)\,. \tag{5.17}$$

Moreover, (3.14) and Proposition 5.8 imply that the currents carried by graphs of maps in $\widetilde{\mathcal{F}}_{\mathfrak{p}}$, see (4.4), belong to the class $\widehat{\mathcal{F}}_{\mathfrak{p}}$ in (5.15), i.e.,

$$\{G_u \mid u \in \widetilde{\mathcal{F}}_{\mathfrak{p}}\} \subset \widehat{\mathcal{F}}_{\mathfrak{p}}\,. \tag{5.18}$$

Proposition 5.13 *Let $\mathfrak{p} \geq 3$ odd, and assume that* (4.5) *holds. Then we have*

$$\begin{aligned} \inf\{\mathbf{D}_{\mathfrak{p}}(u,\mathbb{R}^{\mathfrak{p}+1}) \mid u \in \widetilde{\mathcal{F}}_{\mathfrak{p}}\} &= \inf\{\mathbf{D}_{\mathfrak{p}}(T) \mid T \in \widehat{\mathcal{F}}_{\mathfrak{p}}\} \\ &= \min\{\mathbf{D}_{\mathfrak{p}}(T) \mid T \in \widehat{\mathcal{F}}_{\mathfrak{p}}\} = 2\,\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}}) \cdot m_{i,\mathbb{R}^{\mathfrak{p}+1}}(\Gamma_0)\,. \end{aligned}$$

PROOF: The inclusion (5.18) yields the inequality "$\geq$" in the first line of the assertion. To prove the converse inequality, it suffices to show that for every $T \in \widehat{\mathcal{F}}_{\mathfrak{p}}$ and $\varepsilon > 0$, we can find a map $u_\varepsilon \in \widetilde{\mathcal{F}}_{\mathfrak{p}}$ such that $\mathbf{D}_{\mathfrak{p}}(u_\varepsilon, \mathbb{R}^{\mathfrak{p}+1}) \leq \mathbf{D}_{\mathfrak{p}}(T) + \varepsilon$. On account of (1.3), the map u_ε can be defined by $u_\varepsilon := g_{\mathfrak{p}} \circ v_\varepsilon$ for a suitable map $v_\varepsilon \in W^{1,\mathfrak{p}}(\mathbb{R}^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$ that actually belongs to the class $\widetilde{\mathcal{G}}_{\mathfrak{p}}$ in (4.6). The map v_ε can be obtained as in the proof of the density property (e) in Proposition 5.6 for the case $n = \mathfrak{p}+1$, compare e.g. [17], using a dipole-type construction based on Proposition 2.7. The claims follow from Proposition 4.6 and (5.17). □

As for the case of maps into $\mathbb{S}^{\mathfrak{p}}$, Proposition 5.13 says that by formulating the Dipole problem for maps into $\mathrm{RP}^{\mathfrak{p}}$ in the framework of Cartesian currents, point defects connected by lines of concentration occur. Moreover, according to Example 4.9, taking $m = 2$, $\Delta_1 = 1$, and $\Delta_2 = -1$, since $\Gamma_0 = \delta_{a_2} - \delta_{a_1}$, we have that the infimum of the Dipole problem agrees with the energy $\mathbf{D}_{\mathfrak{p}}(T_0)$, where T_0 is given by (5.16) with L_0 equal to the current integration on the oriented line segments with initial point a_1 and final point a_2. Therefore, differently to what happens in the case of maps into $\mathbb{S}^{\mathfrak{p}}$, the multiplicity of the minimal connection $2L_0$ is dictated by the *double* of the degrees Δ_i.

6 Relaxed energy

In this section we study the relaxation problem concerning $W^{1,\mathfrak{p}}$-maps with values in $\mathrm{RP}^{\mathfrak{p}}$. We shall *assume* $\mathfrak{p} \geq 2$, and we first recall how the analogous problem about $W^{1,\mathfrak{p}}$-maps into $\mathbb{S}^{\mathfrak{p}}$ is solved.

THE CASE OF MAPS INTO $\mathbb{S}^{\mathfrak{p}}$. The *relaxed energy* of maps $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ is defined for every integers $n \geq \mathfrak{p} \geq 2$ by

$$\widetilde{\mathbf{D}}_{\mathfrak{p}}(v, B^n) := \inf \left\{ \liminf_{k\to\infty} \mathbf{D}_{\mathfrak{p}}(v_k, B^n) \mid \{v_k\} \subset C^\infty(B^n, \mathbb{S}^{\mathfrak{p}})\,,\ v_k \rightharpoonup v \text{ weakly in } W^{1,\mathfrak{p}}(B^n, \mathbb{R}^{\mathfrak{p}+1}) \right\}.$$

For any $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, we denote by $\mathcal{T}_v^{\mathfrak{p},1}$ the class of Cartesian currents with corresponding function v_T equal to v, i.e.,

$$\mathcal{T}_v^{\mathfrak{p},1} := \{T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}}) \mid v_T = v \text{ in } (5.4)\}\,.$$

In dimension $n = \mathfrak{p}$, by Schoen-Uhlenbeck density theorem [34] we have

$$\widetilde{\mathbf{D}}_{\mathfrak{p}}(v, B^{\mathfrak{p}}) = \mathbf{D}_{\mathfrak{p}}(v, B^{\mathfrak{p}}) \qquad \forall\, v \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})\,.$$

On the other hand, G_v belongs to $\mathcal{T}_v^{1,\mathfrak{p}}$ for every $v \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$, see Remark 5.4.

In higher dimension n, by Propositions 4.1 and 5.5 we readily obtain:

Proposition 6.1 *Let $n \geq \mathfrak{p}+1$. For every $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^\mathfrak{p})$ the class $\mathcal{T}_v^{\mathfrak{p},1}$ is non-empty, and we have*

$$\mathcal{T}_v^{\mathfrak{p},1} = \{G_v + L \times [\![\mathbb{S}^\mathfrak{p}]\!] \mid L \in \mathcal{R}_{n-\mathfrak{p}}(B^n)\,,\ (\partial L) \llcorner B^n = -\,\mathbb{P}(v)\}\,, \tag{6.1}$$

where $\mathbb{P}(v) \in \mathcal{D}_{n-\mathfrak{p}-1}(B^n)$ is given by (3.1).

As a consequence, we deduce the following representation formula, first proved for $\mathfrak{p}=2$ in [17] and [3], in dimension $n=3$, and in [38], in higher dimension n.

Theorem 6.2 *Let $n \geq \mathfrak{p}+1$, where $\mathfrak{p} \geq 2$. For every $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^\mathfrak{p})$ the relaxed energy $\widetilde{\mathbf{D}}_\mathfrak{p}(v, B^n)$ is finite. Moreover,*

$$\begin{array}{rcl} \widetilde{\mathbf{D}}_\mathfrak{p}(v, B^n) & = & \inf\{\mathbf{D}_\mathfrak{p}(T) \mid T \in \mathcal{T}_v^{\mathfrak{p},1}\} \\ & = & \mathbf{D}_\mathfrak{p}(v, B^n) + \alpha_\mathfrak{p} \cdot m_{i,B^n}(\mathbb{P}(v)) < \infty\,. \end{array} \tag{6.2}$$

Proof: The below cited properties always refer to Proposition 5.6. Let $T \in \mathcal{T}_v^{\mathfrak{p},1}$, see Proposition 6.1, and apply the density property (e). Since the convergence $G_{v_k} \rightharpoonup T$ with $\mathbf{D}_\mathfrak{p}(v_k, B^n) \to \mathbf{D}_\mathfrak{p}(T)$ yields the weak convergence $v_k \rightharpoonup v_T$ in $W^{1,\mathfrak{p}}$, and $v_T = v$, we deduce that the inequality "$\leq$" holds in the first line of (6.2). Therefore, the relaxed energy $\widetilde{\mathbf{D}}_\mathfrak{p}(v, B^n)$ is finite. On the other hand, for any smooth sequence $\{v_k\} \subset C^\infty(B^n, \mathbb{S}^\mathfrak{p})$ such that $v_k \rightharpoonup v$ weakly in $W^{1,\mathfrak{p}}$, since $\sup_k \mathbf{D}_\mathfrak{p}(G_{v_k}) = \sup_k \mathbf{D}_\mathfrak{p}(v_k, B^n) < \infty$, by the closure-compactness property (d), and possibly passing to a subsequence, we have $G_{v_k} \rightharpoonup T$ to some $T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^\mathfrak{p})$ with $v_T = v$, i.e., $T \in \mathcal{T}_v^{\mathfrak{p},1}$. The lower semicontinuity property (a) yields $\mathbf{D}_\mathfrak{p}(T) \leq \liminf_k \widetilde{\mathbf{D}}_\mathfrak{p}(v_k, B^n)$, hence the inequality "$\geq$" holds in the first line of (6.2). The second equality then follows from (4.1), (6.1), and from the representation property (b). $\square$

In dimension $n = \mathfrak{p}+1$, by Proposition 4.4 we then infer:

Corollary 6.3 *For every $\mathfrak{p} \geq 2$ and $v \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^\mathfrak{p})$ we have*

$$\widetilde{\mathbf{D}}_\mathfrak{p}(v, B^{\mathfrak{p}+1}) = \mathbf{D}_\mathfrak{p}(v, B^{\mathfrak{p}+1}) + \alpha_\mathfrak{p} \cdot \mathbb{L}(v, B^{\mathfrak{p}+1})\,,$$

where the flat norm $\mathbb{L}(v, B^{\mathfrak{p}+1})$ is given by (4.2).

Finally, by Proposition 4.1 we obtain in any dimension $n \geq \mathfrak{p}+1$:

Corollary 6.4 *For every $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^\mathfrak{p})$ we have*

$$\widetilde{\mathbf{D}}_\mathfrak{p}(v, B^n) \leq 2\,\mathbf{D}_\mathfrak{p}(v, B^n)\,.$$

The case of maps into $\mathrm{RP}^\mathfrak{p}$. We now similarly introduce the *relaxed energy* of maps $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^\mathfrak{p})$, defined by

$$\overline{\mathbf{D}}_\mathfrak{p}(u, B^n) := \inf\left\{\liminf_{k\to\infty} \mathbf{D}_\mathfrak{p}(u_k, B^n) \mid \{u_k\} \subset C^\infty(B^n, \mathrm{RP}^\mathfrak{p})\,,\ u_k \rightharpoonup u \text{ weakly in } W^{1,\mathfrak{p}}(B^n, \mathbb{R}^{N(\mathfrak{p})})\right\}. \tag{6.3}$$

On account of Theorem 2.5, we deduce:

Theorem 6.5 *Let $n \geq \mathfrak{p} \geq 2$. The relaxed energy $\overline{\mathbf{D}}_\mathfrak{p}(u, B^n)$ of a Sobolev map $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^\mathfrak{p})$ agrees with the relaxed energy $\widetilde{\mathbf{D}}_\mathfrak{p}(v, B^n)$ of any Sobolev map $v \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^\mathfrak{p})$ such that $g_\mathfrak{p} \circ v = u$.*

Proof: Let u and v be as in the claim, see Theorem 2.5. Since $u_k := g_\mathfrak{p} \circ v_k$ is smooth in $W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^\mathfrak{p})$ if $v_k \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^\mathfrak{p})$ is smooth, and $\mathbf{D}_\mathfrak{p}(u_k, B^n) = \mathbf{D}_\mathfrak{p}(v_k, B^n)$, the weak convergence $v_k \rightharpoonup v$ in $W^{1,\mathfrak{p}}$ yields the weak convergence $u_k \rightharpoonup u$. Therefore, the inequality

$$\overline{\mathbf{D}}_\mathfrak{p}(u, B^n) \leq \widetilde{\mathbf{D}}_\mathfrak{p}(v, B^n)$$

holds. On the other hand, if u_k is smooth, by Theorem 2.5 we find $v_k \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^\mathfrak{p})$ continuous such that $u_k = g_\mathfrak{p} \circ v_k$. A standard convolution arguments yields a smooth sequence $\{v_h^{(k)}\} \subset W^{1,\mathfrak{p}}(B^n, \mathbb{S}^\mathfrak{p})$ that strongly converges to v_k in $W^{1,\mathfrak{p}}$ as $h \to \infty$. A diagonal argument yields the assertion. $\square$

As a consequence, and by (1.3), in low dimension $n = \mathfrak{p}$ we again have

$$\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^{\mathfrak{p}}) = \mathbf{D}_{\mathfrak{p}}(u, B^{\mathfrak{p}}) \qquad \forall\, u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}})\,.$$

In higher dimension $n \geq \mathfrak{p}+1$, we obtain:

Proposition 6.6 *Let $n \geq \mathfrak{p}+1$, where $\mathfrak{p} \geq 2$. For every $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$ the relaxed energy $\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^n)$ is finite. Moreover,*

$$\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^n) = \mathbf{D}_{\mathfrak{p}}(u, B^n) + \alpha_{\mathfrak{p}} \cdot m_{i,B^n}(\mathbb{P}(v)) < \infty\,, \tag{6.4}$$

where v is any function in $W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $u = g_{\mathfrak{p}} \circ v$. Finally,

$$\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^n) \leq 2\,\mathbf{D}_{\mathfrak{p}}(u, B^n)\,.$$

PROOF: The assertion follows from (1.3), Theorem 6.2, Corollary 6.4, and Theorem 6.5. □

In dimension $n = \mathfrak{p}+1$, by Corollary 6.3, Theorem 6.5, and Definition 4.5, we then infer:

Corollary 6.7 *For every $\mathfrak{p} \geq 2$ and $u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathrm{RP}^{\mathfrak{p}})$ we have*

$$\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^{\mathfrak{p}+1}) = \mathbf{D}_{\mathfrak{p}}(u, B^{\mathfrak{p}+1}) + \mathbf{L}(u, B^{\mathfrak{p}+1})\,.$$

Remark 6.8 Theorems 6.2 and 6.5, Proposition 6.6, and therefore Corollaries 6.3 and 6.7, hold true if we replace B^n with any bounded domain $\Omega \subset \mathbb{R}^n$, or with e.g. $\Omega = \mathbb{S}^n$, the n-sphere in $\mathbb{R}^{n+1}$. In fact, Theorem 2.5 continues to hold, see Remark 2.6. This clearly yields that the relaxed energy is a *non-local functional*, for $n \geq \mathfrak{p}+1$.

THE CASE $\mathfrak{p}$ ODD. If $\mathfrak{p} \geq 3$ is odd, according to Definition 5.7, we denote for any $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$

$$\widetilde{\mathcal{T}}_u^{\mathfrak{p},1} := \{T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathrm{RP}^{\mathfrak{p}}) \mid u_T = u \text{ in } (5.11)\}\,,$$

compare (6.1), so that by Proposition 5.8, for $n \geq \mathfrak{p}+1$ we have

$$\widetilde{\mathcal{T}}_u^{\mathfrak{p},1} = \Big\{G_u + 2\,L \times [\![\, \mathrm{RP}^{\mathfrak{p}}\,]\!] \mid L \in \mathcal{R}_{n-\mathfrak{p}}(B^n)\,,\, (\partial L) \llcorner B^n = -\frac{1}{2}\,\mathbf{P}(u)\Big\}\,,$$

where $\mathbf{P}(u) \in \mathcal{D}_{n-\mathfrak{p}-1}(B^n)$ is given by (3.7). We finally obtain:

Proposition 6.9 *Let $n \geq \mathfrak{p}+1$, where $\mathfrak{p} \geq 3$ is odd. For every $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$ we have*

$$\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^n) = \inf\{\mathbf{D}_{\mathfrak{p}}(T) \mid T \in \widetilde{\mathcal{T}}_u^{\mathfrak{p},1}\} = \mathbf{D}_{\mathfrak{p}}(u, B^n) + \frac{\alpha_{\mathfrak{p}}}{2} \cdot m_{i,B^n}(\mathbf{P}(u)) < \infty\,.$$

PROOF: The first equality is obtained by arguing as in the proof of Theorem 6.2, but this time making use of the properties from Proposition 5.10. The second equality follows from (5.13) and (5.14). □

7 Optimally connecting measure

In this section we discuss a notion of *optimally connecting measure* of the *singular set* of Sobolev maps with values into the projective space $\mathrm{RP}^{\mathfrak{p}}$, for any $\mathfrak{p} \geq 2$ and $n \geq \mathfrak{p}+1$.

If $\mathfrak{p} \geq 3$ is odd, by Corollary 4.3 we infer that *for every $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$ there exists an integral minimal connection of $\frac{1}{2}\,\mathbf{P}(u)$*, see (3.7), i.e., an i.m. rectifiable current $L_u \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$ such that

$$(\partial L_u) \llcorner B^n = \frac{1}{2}\,\mathbf{P}(u) \qquad \text{and} \qquad \mathbf{M}(L_u) = m_{i,B^n}\Big(\frac{1}{2}\,\mathbf{P}(u)\Big)\,.$$

Therefore, there exist a countably $(n-\mathfrak{p})$-rectifiable set $\mathcal{L}_u$ in B^n, an $\mathcal{H}^{n-\mathfrak{p}} \llcorner \mathcal{L}_u$-summable and non-negative integer valued multiplicity function $\theta_u : \mathcal{L}_u \to \mathbb{N}^+$, and an $\mathcal{H}^{n-\mathfrak{p}} \llcorner \mathcal{L}_u$-measurable unit $(n-\mathfrak{p})$-vector field $\overrightarrow{\mathcal{L}}_u : \mathcal{L}_u \to \Lambda_{n-\mathfrak{p}}\mathbb{R}^n$, orienting the approximate tangent space to $\mathcal{L}_u$ at $\mathcal{H}^{n-\mathfrak{p}}$-a.e. point, such that

$$\langle L_u, \gamma\rangle = \int_{\mathcal{L}_u} \theta_u\, \langle \gamma, \overrightarrow{\mathcal{L}}_u\rangle\, d\mathcal{H}^{n-\mathfrak{p}} \qquad \forall\, \gamma \in \mathcal{D}^{n-\mathfrak{p}}(B^n)\,.$$

In this case, one writes $L_u = \tau(\mathcal{L}_u, \theta_u, \overrightarrow{\mathcal{L}}_u)$, and we can say that *the measure*

$$\mu_u := \theta_u \, \mathcal{H}^{n-\mathfrak{p}} \llcorner \mathcal{L}_u \tag{7.1}$$

encloses the singularity of u in an optimal sense.

In order to extend the definition (7.1) to the case $\mathfrak{p} \geq 2$ even, we let $v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ satisfy $g_{\mathfrak{p}} \circ v = u$, see Theorem 2.5. By Remark 4.2, there exists an i.m. rectifiable current $L_v \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$ such that $(\partial L_v) \llcorner B^n = \mathbb{P}(v)$ and $\mathbf{M}(L_v) = m_{i,B^n}(\mathbb{P}(v))$. Write again $L_v = \tau(\mathcal{L}_v, \theta_v, \overrightarrow{\mathcal{L}}_v)$.

Due to (3.2), we deduce that the i.m. rectifiable current $(-1)^{\mathfrak{p}+1} L_v := \tau(\mathcal{L}_v, \theta_v, (-1)^{\mathfrak{p}+1} \overrightarrow{\mathcal{L}}_v)$ is a minimal integral connection for $\mathbb{P}(-v)$. Therefore, by Theorem 2.5 we can write $\mathcal{L}_u = \mathcal{L}_v$ and $\theta_u = \theta_v$ in the definition (7.1). Notice that for $\mathfrak{p} \geq 3$ odd, by Proposition 3.2 we have $\frac{1}{2} \mathbf{P}(u) = \mathbb{P}(v)$, as required.

Formula (7.1) defines an *optimally connecting measure* μ_u, the *total variation* of which satisfies

$$|\mu_u|(B^n) = \int_{\mathcal{L}_u} \theta_u \, d\mathcal{H}^{n-\mathfrak{p}} = m_{i,B^n}(\mathbb{P}(v)) \qquad \forall\, v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}}) \quad \text{such that} \quad g_{\mathfrak{p}} \circ v = u \,.$$

Therefore, by Proposition 6.6 we have

$$\overline{\mathbf{D}}_{\mathfrak{p}}(u, B^n) = \mathbf{D}_{\mathfrak{p}}(u, B^n) + \alpha_{\mathfrak{p}} \cdot |\mu_u|(B^n) \qquad \forall\, u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}}) \,.$$

In dimension $n = \mathfrak{p} + 1$, by Corollary 6.7 we also deduce that

$$\alpha_{\mathfrak{p}} \cdot |\mu_u|(B^{\mathfrak{p}+1}) = \mathbf{L}(u, B^{\mathfrak{p}+1}) \qquad \forall\, u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathrm{RP}^{\mathfrak{p}}) \,,$$

where the flat norm $\mathbf{L}(u, B^{\mathfrak{p}+1})$ is given by Definition 4.5.

Example 7.1 If e.g. $\widetilde{u} = g_{\mathfrak{p}} \circ \widetilde{v}$, where $\widetilde{v}(x) = x/|x| \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$, we have $\mu_{\widetilde{u}} = \mathcal{H}^1 \llcorner \widetilde{\mathcal{L}}$, where $\widetilde{\mathcal{L}}$ is any line segment connecting the origin $0_{\mathbb{R}^{\mathfrak{p}+1}}$ to the boundary of $B^{\mathfrak{p}+1}$. Therefore, for every $\mathfrak{p} \geq 2$ we have

$$|\mu_{\widetilde{u}}|(B^{\mathfrak{p}+1}) = 1 \quad \text{and} \quad \overline{\mathbf{D}}_{\mathfrak{p}}(\widetilde{u}, B^{\mathfrak{p}+1}) = \mathbf{D}_{\mathfrak{p}}(\widetilde{u}, B^{\mathfrak{p}+1}) + \alpha_{\mathfrak{p}} \,, \quad \alpha_{\mathfrak{p}} = 2\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}}) \,.$$

Finally, we deduce:

Theorem 7.2 *Let $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$ and $\mu_u := \theta_u \, \mathcal{H}^{n-\mathfrak{p}} \llcorner \mathcal{L}_u$ an* optimally connecting measure *for the singular set of u. Then there exists a sequence of* smooth *maps $\{u_k\} \subset W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$ satisfying the following properties:*

i) $u_k \rightharpoonup u$ *weakly in* $W^{1,\mathfrak{p}}$ *as* $k \to \infty$*;*

ii) $\mathbf{D}_{\mathfrak{p}}(u_k, B^n) \to \mathbf{D}_{\mathfrak{p}}(u, B^n) + \alpha_{\mathfrak{p}} \cdot |\mu_u|(B^n)$ *as* $k \to \infty$*;*

iii) $\dfrac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |Du_k|^{\mathfrak{p}} \, \mathcal{L}^n \llcorner B^n \rightharpoonup \dfrac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |Du|^{\mathfrak{p}} \, \mathcal{L}^n \llcorner B^n + \alpha_{\mathfrak{p}} \, \mu_u$ *weakly as measures;*

iv) *for any open set A contained in $B^n \setminus \operatorname{spt} \mu_u$, we have strong $W^{1,\mathfrak{p}}$-convergence of $u_{k|A}$ to $u_{|A}$.*

PROOF: The first three assertions follow from the density property (e) in Proposition 5.6 and from the results previously obtained. The last assertion is given by the strict convexity of the energy density. □

8 The liquid crystal energy

In this section we analyze the liquid crystal energy

$$\mathcal{E}(v, B^3) = \int_{B^3} W(v, Dv) \, dx \,. \tag{8.1}$$

The Oseen-Frank energy density of nematic vector fields v is defined on mappings v in $W^{1,2}(B^3, \mathbb{S}^2)$ by

$$W(v, Dv) := |Dv|^2 + (k_1 - 1)\,(\operatorname{div} v)^2 + (k_2 - 1)\,(v \cdot \operatorname{curl} v)^2 + (k_3 - 1)\,|v \times \operatorname{curl} v|^2 \,, \tag{8.2}$$

where $k_i \geq 1$ for every i, see the discussion in the introduction.

The *parametric extension* of the liquid crystal energy over the class of Cartesian currents $T \in \mathrm{cart}^{2,1}(B^3 \times \mathbb{S}^2)$ has been computed in [18], see also [20, Vol. II, Sec. 1.2.4]. We recall that

$$T = G_{v_T} + L_T \times [\![\mathbb{S}^2]\!]$$

for some $v_T \in W^{1,2}(B^3, \mathbb{S}^2)$ and $L_T \in \mathcal{R}_1(B^3)$ satisfying $(\partial L_T) \llcorner B^3 = -\mathbb{P}(v_T)$, see Sec. 5. The liquid crystal energy functional $T \mapsto \mathcal{E}(T)$ satisfies

$$\mathcal{E}(T) = \mathcal{E}(v_T, B^3) + 8\pi\, \Gamma(k_1, k_2, k_3)\, \mathbf{M}(L_T)$$

where, compare [20, Vol. II, Sec. 5.1.2],

$$\Gamma(k_1, k_2, k_3) := \sqrt{k\, k_3} \int_0^1 \sqrt{1 + \Big(\frac{k}{k_3} - 1\Big) s^2}\, ds\,, \qquad k := \min\{k_1, k_2\}\,.$$

If $k_i = 1$ for every i, we have $W(v, Dv) = |Dv|^2$ and $\Gamma(1,1,1) = 1$, so that $\mathcal{E}(T) = 2\,\mathbf{D}(T)$. In general, the functional $\mathcal{E}(T)$ it is controlled by the Dirichlet energy, as

$$2\,\mathbf{D}(T) \leq \mathcal{E}(T) \leq c\,\mathbf{D}(T) \qquad \forall\, T \in \mathrm{cart}^{2,1}(B^3 \times \mathbb{S}^2)\,,$$

where the positive constant c only depends on the choice of the k_i's. Notice that $\Gamma(k_1, k_2, k_3) \geq 1$.

Moreover, by the construction, it turns out that $\mathcal{E}(T)$ is lower semicontinuous with respect to the weak convergence of sequences of currents in $\mathrm{cart}^{2,1}(B^3 \times \mathbb{S}^2)$.

Also, the following density property was proved in [18]:

Theorem 8.1 (Giaquinta-Modica-Souček) *For every $T \in \mathrm{cart}^{2,1}(B^3 \times \mathbb{S}^2)$, there exists a sequence of* smooth *maps $\{v_k\} \subset W^{1,2}(B^3, \mathbb{S}^2)$ such that $G_{v_k} \rightharpoonup T$ in $\mathcal{D}_3(B^3 \times \mathbb{S}^2)$ and $\mathcal{E}(v_k, B^3) \to \mathcal{E}(T)$ as $k \to \infty$.*

The proof of Theorem 8.1 is similar to the one from [17] for the Dirichlet energy in dimension $n = 3$, where this time the so called *irrotational* and *solenoidal dipoles* are used, when $k = k_1$ and $k = k_2$, respectively, compare [20, Vol. II, Sec. 5.1.3].

Finally, consider the relaxed energy of the liquid crystal functional (8.1), with energy density (8.2), defined by

$$\widetilde{\mathcal{E}}(v, B^3) := \inf \left\{ \liminf_{k\to\infty} \mathcal{E}(v_k, B^3) \mid \{v_k\} \subset C^\infty(B^3, \mathbb{S}^2)\,,\ v_k \rightharpoonup v \text{ weakly in } W^{1,2}(B^3, \mathbb{R}^3) \right\}.$$

In [20, Vol. II, Sec. 5.1.2], it is shown that for every Sobolev map $v \in W^{1,2}(B^3, \mathbb{S}^2)$ one has:

$$\widetilde{\mathcal{E}}(v, B^3) = \mathcal{E}(v, B^3) + 8\pi\, \Gamma(k_1, k_2, k_3) \cdot m_{i,B^3}(\mathbb{P}(v)) < \infty\,. \tag{8.3}$$

This representation formula can be recovered by arguing as in the proof of Theorem 6.2, taking advantage of Theorem 8.1.

THE LIQUID CRYSTAL ENERGY OF MAPS INTO RP^2. We first recall that if $k_i = 1$ for every i, then $W(u, Du) = |Du|$, hence by (1.2) we deduce that the energy density of a map $v \in W^{1,2}(B^3, \mathbb{S}^2)$ has the same structure as the energy density of the map $u \in W^{1,2}(B^3, \mathrm{RP}^2)$ given by $u = g \circ v$, where $g : \mathbb{S}^2 \to \mathbb{R}^6$ is the embedding (1.1).

Of course, this is not true in general, i.e., when $k_i > 1$ for some i in the energy density (8.2). In the appendix below, we shall see how the non-linear terms in (8.2) can be written in terms of the components of $u = g \circ v$. However, by the invariance properties (0.2), on account of Theorem 2.5 we may and do give the following

Definition 8.2 *The liquid crystal energy $\mathcal{E}(u, B^3)$ of a Sobolev map $u \in W^{1,2}(B^3, \mathrm{RP}^2)$ is defined by the energy $\mathcal{E}(v, B^3)$ of* any *function $v \in W^{1,2}(B^3, \mathbb{S}^2)$ such that $u = g \circ v$.*

THE DIPOLE PROBLEM. Similarly to the analogous problem for the Dirichlet energy in the case $\mathfrak{p} \geq 2$ even, see Sec. 4, according to the previous definition we obtain:

Theorem 8.3 *Let* $\widetilde{\mathcal{F}}_2$ *be the class of functions in* (4.4)*, where* $\mathfrak{p} = 2$*. Assume that the compatibility condition* (4.7) *on the natural numbers* $\Delta_i \in \mathbb{N} \setminus \{0\}$ *is satisfied. Then we have*

$$\inf\{\mathcal{E}(u, \mathbb{R}^3) \mid u \in \widetilde{\mathcal{F}}_2\} = 8\pi\,\Gamma(k_1, k_2, k_3) \cdot \inf_{\varepsilon_i}\Big\{ m_{i,\mathbb{R}^3}\Big(\sum_{i=1}^m \varepsilon_i \Delta_i\, \delta_{a_i}\Big) \mid (4.7) \text{ holds } \Big\}. \tag{8.4}$$

Proof: As in Proposition 4.8, we infer that

$$\inf\{\mathcal{E}(u, \mathbb{R}^3) \mid u \in \widetilde{\mathcal{F}}_2\} = \inf\{\mathcal{E}(v, \mathbb{R}^3) \mid v \in \widehat{\mathcal{G}}_2\},$$

where $\widehat{\mathcal{G}}_2$ is defined by (4.8), with $\mathfrak{p} = 2$. Arguing as in [20, Vol. II, Sec. 5.1.2], we deduce that the infimum $\inf\{\mathcal{E}(v, \mathbb{R}^3) \mid v \in \widehat{\mathcal{G}}_2\}$ agrees with the right-hand side of eq. (8.4), as required. □

A density theorem. In Sec. 7, we have defined an *optimally connecting measure* μ_u of the singular set of Sobolev maps $u \in W^{1,2}(B^3, \mathrm{RP}^2)$. We recall that $\mu_u := \theta_u\, \mathcal{H}^1 \llcorner \mathcal{L}_u$ for some countably 1-rectifiable set $\mathcal{L}_u$ in B^3 and some $\mathcal{H}^1 \llcorner \mathcal{L}_u$-summable multiplicity function $\theta_u : \mathcal{L} \to \mathbb{N}^+$ such that the following properties hold:

(a) $|\mu_u|(B^3) = \displaystyle\int_{\mathcal{L}_u} \theta_u\, d\mathcal{H}^1$;

(b) $|\mu_u|(B^3) = m_{i,B^3}(\mathbb{P}(v))$ for every $v \in W^{1,2}(B^3, \mathbb{S}^2)$ such that $g \circ v = u$;

(c) $\overline{\mathbf{D}}(u, B^3) = \mathbf{D}(u, B^3) + 4\pi \cdot |\mu_u|(B^3)$;

(d) $4\pi \cdot |\mu_u|(B^3) = \mathbf{L}(u, B^3)$, where the flat norm $\mathbf{L}(u, B^3)$ is given by Definition 4.5, with $\mathfrak{p} = 2$.

Similarly to Theorem 7.2, the following density property holds true:

Theorem 8.4 *Let* $u \in W^{1,2}(B^3, \mathbb{RP}^2)$ *and* $\mu_u := \theta_u\, \mathcal{H}^1 \llcorner \mathcal{L}_u$ *an* optimally connecting measure. *Then there exists a sequence of* smooth *maps* $\{u_k\} \subset W^{1,2}(B^3, \mathrm{RP}^2)$ *satisfying the following properties:*

i) $u_k \rightharpoonup u$ *weakly in* $W^{1,2}$ *as* $k \to \infty$;

ii) $\mathcal{E}(u_k, B^3) \to \mathcal{E}(u, B^3) + 8\pi\,\Gamma(k_1, k_2, k_3) \cdot |\mu_u|(B^3)$ *as* $k \to \infty$;

iii) $W(u_k, Du_k)\, \mathcal{L}^3 \llcorner B^3 \rightharpoonup W(u, Du)\, \mathcal{L}^3 \llcorner B^3 + 8\pi\,\Gamma(k_1, k_2, k_3)\, \mu_u$ *weakly as measures;*

iv) *for any open set* A *contained in* $B^3 \setminus \operatorname{spt} \mu_u$*, we have strong* $W^{1,2}$*-convergence of* $u_{k|A}$ *to* $u_{|A}$.

Proof: The first three assertions follow from Theorem 8.1 and the results from the previous sections. As to the last assertion, since $|Du|^2 \le W(u, Du)$, and the energy density $W(u, Du)$ is strictly convex in Du, using a continuity theorem by Reshetnyak [37, p. 329], compare Thm. 2 in [20, Vol. II, Sec. 1.3.4], we infer that $\mathbf{D}(u_k, A) \to \mathbf{D}(u, A)$, that gives iv), again by strict convexity. □

Remark 8.5 By the above property (d), in the previous formula ii) we have

$$8\pi\,\Gamma(k_1, k_2, k_3) \cdot |\mu_u|(B^3) = 2\,\Gamma(k_1, k_2, k_3) \sup\Big\{ \int_{B^3} D(v) \cdot D\phi\, dx \mid \phi \in C_c^\infty(B^3),\ \|d\phi\| \le 1 \text{ in } B^3 \Big\}$$

for *any* $v \in W^{1,2}(B^3, \mathbb{S}^2)$ such that $g \circ v = u$.

Relaxed energy. We finally introduce the relaxed energy of maps $u \in W^{1,2}(B^3, \mathrm{RP}^2)$, defined by

$$\overline{\mathcal{E}}(u, B^3) := \inf\left\{ \liminf_{k\to\infty} \mathcal{E}(u_k, B^3) \mid \{u_k\} \subset C^\infty(B^3, \mathrm{RP}^2),\ u_k \rightharpoonup u \text{ weakly in } W^{1,2}(B^3, \mathbb{R}^6) \right\}.$$

The following representation formula holds:

Theorem 8.6 *For every* $u \in W^{1,2}(B^3, \mathrm{RP}^2)$ *the relaxed energy* $\overline{\mathcal{E}}(u, B^3)$ *is finite. Moreover,*

$$\overline{\mathcal{E}}(u, B^3) = \mathcal{E}(u, B^3) + 8\pi\, \Gamma(k_1, k_2, k_3) \cdot |\mu_u|(B^3) = \mathcal{E}(u, B^3) + 2\, \Gamma(k_1, k_2, k_3) \cdot \mathbf{L}(u, B^3)\,, \tag{8.5}$$

where μ_u *is an optimally connecting measure of the singular set and* $\mathbf{L}(u, B^3)$ *is the flat norm of* u*, see Definition* 4.5*. Finally,*

$$\overline{\mathcal{E}}(u, B^3) \le \mathcal{E}(u, B^3) + 2\, \Gamma(k_1, k_2, k_3) \cdot \mathbf{D}(u, B^3)\,.$$

Proof: As in the proof of Theorem 6.5, using Theorem 8.1 we deduce that

$$\overline{\mathcal{E}}(u, B^3) = \widetilde{\mathcal{E}}(v, B^3) \qquad \forall\, v \in W^{1,2}(B^3, \mathbb{S}^2) \quad \text{such that} \quad g \circ v = u\,.$$

By the representation formula (8.3), on account of the above properties (b) and (d), we obtain (8.5). Moreover, property (b), Proposition 4.1, where $\alpha_2 = 4\pi$, and (1.2) yield

$$4\pi \cdot |\mu_u|(B^3) = 4\pi \cdot m_{i,B^3}(\mathbb{P}(v)) \le \mathbf{D}(v, B^3) = \mathbf{D}(u, B^3)\,.$$

The last assertion then follows from (8.5). □

Finally, the representation formulas (8.3) and (8.5) hold true if we replace B^3 with any bounded domain $\Omega \subset \mathbb{R}^3$, or with e.g. $\Omega = \mathbb{S}^3$, see Remark 6.8. This yields again that the relaxed energy is a non-local functional.

A Appendix

We compute the nonlinear terms of the energy density (8.2) of a map $v \in W^{1,2}(B^3, \mathbb{S}^2)$ with respect to the components of the Sobolev map $u := g \circ v \in W^{1,2}(B^3, \mathrm{RP}^2)$.

To this purpose, we consider the vector field $U = (U_1, U_2, U_3) : B^3 \to \mathbb{R}^3$, where $U_1 = v^2 v^3$, $U_2 = v^3 v^1$, $U_3 = v^1 v^2$ are the last three components of $u := g \circ v$, see (1.1). We make use of a *cyclic notation* on the indexes $i, j, k \in \{1, 2, 3\}$, so that $j = i + 1$, $k = j + 1$, and $i = k + 1$. Assume e.g. that $U_h > 0$ and $v^h > 0$ for every h. We thus have

$$v^i = \sqrt{\frac{U_j U_k}{U_i}} \qquad \text{if} \quad u = g \circ v\,, \quad v = (v^1, v^2, v^3)\,,$$

so that

$$D_\alpha v^i = \frac{1}{2\sqrt{U_i U_j U_k}} \Big(D_\alpha (U_j U_k) - \frac{U_j U_k}{U_i}\, D_\alpha U_i \Big)\,.$$

We then compute

$$\operatorname{div} v = \frac{1}{2\sqrt{U_1 U_2 U_3}} \sum_{i=1}^{3} \Big(D_i (U_j U_k) - \frac{U_j U_k}{U_i}\, D_i U_i \Big)\,.$$

We also recall that $\operatorname{curl} v := \sum_{i=1}^3 (D_j v^k - D_k v^j)\, \mathbf{e_i}$, where $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ denotes the canonical basis on $\mathbb{R}^3$. We thus have

$$\operatorname{curl} v = \frac{1}{2\sqrt{U_1 U_2 U_3}} \sum_{i=1}^{3} \Big(D_j (U_i U_j) - \frac{U_i U_j}{U_k}\, D_j U_k - D_k (U_k U_i) + \frac{U_k U_i}{U_j}\, D_k U_j \Big)\, \mathbf{e_i}\,.$$

Since $v \cdot \operatorname{curl} v = \sum_{i=1}^3 v_i\, (D_j v^k - D_k v^j)$, this gives

$$v \cdot \operatorname{curl} v = \frac{1}{2\sqrt{U_1 U_2 U_3}} \sum_{i=1}^{3} \left(\sqrt{\frac{U_j U_k}{U_i}}\, \big(U_i (D_j U_j - D_k U_k) + U_j D_j U_i - U_k D_k U_i \big) \right.$$
$$\left. - \sqrt{\frac{U_i U_j}{U_k}}\, U_j D_j U_k + \sqrt{\frac{U_k U_i}{U_j}}\, U_k D_k U_j \right)\,.$$

Therefore, by simplifying the terms D_hU_h we get

$$\begin{aligned} v\cdot \operatorname{curl} v &= \frac{1}{2}\sum_{i=1}^{3}\left(\frac{1}{U_i}\big(U_jD_jU_i-U_kD_kU_i\big)-\frac{U_j}{U_k}D_jU_k+\frac{U_k}{U_j}D_kU_j\right)\\ &= \sum_{i=1}^{3}\left(\frac{U_j}{U_k}\,D_jU_k-\frac{U_k}{U_j}\,D_kU_j\right)\,. \end{aligned}$$

Finally, we have $v\times \operatorname{curl} v=\sum_{i=1}^{3}\Phi_i\,\mathbf{e_i}$, where $\Phi_i := v^j(D_iv^j-D_jv^i)-v^k(D_kv^i-D_iv^k)$, so that

$$\begin{aligned} 2\,\Phi_i &= \frac{1}{U_jU_k}\big(U_j^2\,(D_iU_i-D_kU_k)+U_k^2\,(D_iU_i-D_jU_j)\big)\\ &\quad+\frac{U_i}{U_j^2U_k}\,(U_j^2-U_k^2)\,D_iU_j+\frac{U_i}{U_jU_k^2}\,(U_k^2-U_j^2)\,D_iU_k\\ &\quad+\frac{U_k}{U_i}\,D_jU_i+\frac{U_j}{U_i}\,D_kU_i-D_jU_k-D_kU_j\,. \end{aligned}$$

References

[1] F.J. ALMGREN, W. BROWDER, E.H. LIEB: Co-area, liquid crystals, and minimal surfaces, In: *Partial Differential equations*, Springer Lecture Notes in Math. **1306** (1988), 1–22.

[2] F. BETHUEL: The approximation problem for Sobolev maps between manifolds, *Acta Math.* **167** (1992), 153–206.

[3] F. BETHUEL, H. BREZIS, J. M. CORON: Relaxed energies for harmonic maps, In: *Variational methods*, H. Berestycki, J.M. Coron, J. Ekeland (eds.), Birkauser, Basel (1990), 37–52.

[4] F. BETHUEL, X. M. ZHENG: Density of smooth functions between two manifolds in Sobolev spaces, *J. Funct. Anal.* **80** (1988), 60–75.

[5] H. BREZIS, J.M. CORON, E.H. LIEB: Harmonic maps with defects, *Commun. Math. Phys.* **107** (1986), 649–705.

[6] P. BISCARI, G. GUIDONE PEIROLI: A hierarchy of defects in biaxial nematics, *Commun. Math. Phys.* **186** (1997), 381–392.

[7] S. CHANDRESEKHAR: *Liquid crystals*, Cambridge Univ. Press, Cambridge, 1977.

[8] J.M. CORON, J.M. GHIDAGLIA, F. HELEIN (eds.): *Nematics*, Nato Series **332**, Kluwer Academic Publ., Dodrecht, 1991.

[9] P.-G. DE GENNES: *The physics of liquid crystals*, Oxford Univ. Press, Oxford, 1974.

[10] J.L. ERICKSEN: Equilibrium theory of liquid crystals, In: *Advances in liquid crystals* **2**, G.H. Brown (ed.), Academic Press, New York (1976), 233–299.

[11] J.L. ERICKSEN, D. KINDERLEHRER (eds.): *Theory and applications of liquid crystals*, Springer, New York, 1987.

[12] H. FEDERER: *Geometric measure theory*, Grundlehren math. Wissen. **153**, Springer, New York, 1969.

[13] H. FEDERER: Real flat chains, cochains and variational problems, *Indiana Univ. Math. J.* **24** (1974), 351–407.

[14] H. FEDERER, W.H. FLEMING: Normal and integral currents, *Ann. of Math.* **72** (1960), 458–520.

[15] F.C. FRANK: On the theory of liquid crystals, *Discuss. Faraday Soc.* **28** (1938), 19–28.

[16] M. GIAQUINTA, G. MODICA: On sequences of maps with equibounded energies, *Calc. Var. Partial Differential Equations* **12** (2001), 213–222.

[17] M. GIAQUINTA, G. MODICA, J. SOUČEK: The Dirichlet energy of mappings with values into the sphere, *Manuscr. Math.* **65** (1989), 489–507.

[18] M. GIAQUINTA, G. MODICA, J. SOUČEK: Liquid crystals: relaxed energies, dipoles, singular lines and singular points, *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* (4) **17** (1990), 415–437.

[19] M. GIAQUINTA, G. MODICA, J. SOUČEK: Variational problems for the conformally invariant integral $\int |du|^n$, In: *Progress in partial differential equations: calculus of variations, applications*, C. Bandle, J.M.C. Bemelmans, M. Gr uter, J.S.J. Paulin (eds.), Pitman Research Notes in Math. Ser. **267**, Longman Harlow (1992), 27–47.

[20] M. GIAQUINTA, G. MODICA, J. SOUČEK: *Cartesian currents in the calculus of variations,* I, II. Ergebnisse Math. Grenzgebiete (III Ser), **37**, **38**, Springer, Berlin, 1998.

[21] M. GIAQUINTA, D. MUCCI: Density results relative to the Dirichlet energy of mappings into a manifold, *Commun. Pure Appl. Math.* **59** (2006), 1791–1810.

[22] M. GIAQUINTA, D. MUCCI: *Maps into manifolds and currents: area and* $W^{1,2}$*-,* $W^{1/2}$*-,* BV*-energies*, CRM Series, Edizioni della Normale, Pisa, 2006.

[23] R. HARDT, D. KINDERLEHRER: Mathematical questions of liquid crystals theory, In: *Theory and applications of liquid crystals*, J.L. Ericksen, D. Kinderlehrer (eds.), Springer, New York (1987), 151–184.

[24] R. HARDT, D. KINDERLEHRER, F. LIN: Existence and partial regularity of static liquid cristals configurations, *Commun. Math. Phys.* **105** (1986), 547–570.

[25] R. HARDT, D. KINDERLEHRER, F. LIN: Stable defects of minimizers of constrained variational principles, *Ann. Inst. H. Poincaré Anal. Nonlinéaire* **5** (1988), 297–322.

[26] R. HARDT, D. KINDERLEHRER, F. LIN: The variety of configurations of liquid crystals, In: *Variational methods*, H. Beresticky, J.M. Coron, I. Ekeland (eds.), PNDE, Birkäuser, Basel (1990), 115–131.

[27] R. HARDT, D. KINDERLEHRER, M. LUSKIN: Remarks about the variational theory of liquid crystals, In: *Calculus of variations and partial differential equations*, S. Hildebrandt, D. Kinderlehrer, M. Miranda (eds.), Springer Lecture Notes in Math. **1340**, Springer, New York (1988), 123–138.

[28] D. KINDERLEHRER: Recent developments in liquid crystals theory, In: *Frontiers in pure and applied mathemathics*, North-Holland, Amsterdam (1991), 151–178.

[29] F.C. LESLIE: Theory of flow phenomena in liquid crystals, In: *Advances in liquid crystals* **2**, G.H. Brown (ed.), Academic Press, New York (1976), 1–81.

[30] F. LIN, C. LIU: Static and dynamic theories of liquid crystals, *J. Partial Differential Equations* **14** (2001), 289–330.

[31] D. MUCCI: Sobolev maps into the projective line with bounded total variation. *Preprint* (2010).

[32] C.W. OSEEN: The theory of liquid crystals, *Trans. Faraday Soc.* **29** (1933), 883–889.

[33] M.R. PAKZAD, T. RIVIÈRE: Weak density of smooth maps for the Dirichlet energy between manifolds, *Geom. Funct. Anal.* **13** (2003), 223–257.

[34] R. SCHOEN, K. UHLENBECK: Boundary regularity and the Dirichlet problem for harmonic maps, *J. Diff. Geom.* **18** (1983), 253–268.

[35] L. SIMON: *Lectures on geometric measure theory,* Proc. of the Centre for Math. Analysis **3**, Australian National University, Canberra, 1983.

[36] E.H. SPANIER: *Algebraic topology,* Mac Graw-Hill, New York, 1986.

[37] Y.G. RESHETNYAK: *Space mappings with bounded distortion*, Translation of Mathematical Monographs **73**, American Math. Soc., Providence, RI, 1989.

[38] U. TARP-FICENC: On the minimizers of the relaxed energy functionals of mappings from higher dimensional balls into $\mathbb{S}^2$, *Calc. Var. Partial Differential Equations* **23** (2005), 451–467.

[39] E.G. VIRGA: *Variational theories for liquid crystals*, Applied Mathematics and Mathematical Computation **8**, Chapman & Hall, London, 1994.

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