RENORMALIZED ENERGY BETWEEN FRACTIONAL VORTICES WITH TOPOLOGICALLY INDUCED FREE DISCONTINUITIES ON 2-DIMENSIONAL RIEMANNIAN MANIFOLDS

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Abstract. On a two-dimensional Riemannian manifold without boundary we consider the variational limit of a family of functionals given by the sum of two terms: a Ginzburg-Landau and a perimeter term. Our scaling allows low-energy states to be described by an order parameter which can have finitely many point singularities (vortex-like defects) of (possibly) fractional-degree connected by line discontinuities (string defects) of finite length. Our main result is a compactness and Γ-convergence theorem which shows how the coarse grained limit energy depends on the geometry of the manifold in driving the interaction between vortices and string defects.

Contents

1. Introduction

In many physical and biological systems low energy states form complex patterns. The latter result from the necessary coexistence of different and often incompatible geometries that characterize the ground states of those systems. Explaining the emergence of such a complexity is a fascinating task which in the last decades has attracted the attention of the mathematical community. The variational methods, combined with ad-hoc rigorous coarse graining procedures, have proved to be very successful tools to obtain detailed information in several cases of interest. They have lead, for instance, to a satisfactory understanding of the energetic mechanism at the basis of phase coexistence. It is worth mentioning the formation of microstructures in austenite-martensite mechanical transformations, micromagnetics, the theory of liquid crystals, fracture mechanics or plasticity theory, to cite a few examples.

In the present paper we are interested in the variational analysis of some energy functionals that can drive the emergence and the coexistence of point and line singularities of a vector-valued order parameter defined on a two-dimensional manifold. In their simplest form this type of functionals consist of the sum of a Ginzburg-Landau term, which penalizes point defects, and a perimeter term which penalizes line defects. Functionals of this kind have been rigorously investigated from a mathematical point of view in connection with models of ripple phases coexistence in biological systems in [25]. Slightly different energy functionals leading to point and line defects have been recently investigated in the context of discrete systems to model chirality phase separations in geometrically frustrated spin systems [9, 10], the dependence of the energy concentration phenomenon on the rate of divergence of n in the n-clock model [20, 21, 22, 23] or the formation of partial vortices and line defects in modified xy models [11]. The latter analysis is also connected to the investigation of orientability issues of the director field of some liquid crystals model as first discussed in [12, 13]. In all such cases the analysis has been carried out in a Euclidean setting. Here, instead, we aim at beginning the extension of the analysis done in the flat setting to the general two-dimensional Riemannian setting. We start this program with the investigation of the energetic model for ripple phases in biological matter (see for instance [14, 30, 34, 35]) and extend some of the results first obtained in [25]. It is our opinion that some of the results obtained here will help advancing the variational theory of spin systems on planar networks recently investigated in [1, 3, 17, 18] (see also [2] and the references therein) to the case of discrete systems on manifolds.

Let (S, g) denote a 2-dimensional, compact Riemannian manifold without boundary endowed with a metric tensor g. We denote by $SBV(TS)$ the space of special sections of bounded variation of the tangent bundle TS, i.e., those vector fields which are tangent to S, have bounded variation, and vanishing Cantor part of the distributional derivative. For $m \in \mathbb{N}$ we define the space of admissible vector fields $\mathcal{AS}^{(m)}(S)$ as those $u \in SBV(TS)$ with square integrable approximate gradient ∇u, jump set $\mathcal{J}_u$ of finite length, and such that for a.e. point on $\mathcal{J}_u$ the traces satisfy $(u^+)^m = (u^-)^m$ (here the product is taken in the sense of complex numbers, see Section 2.2). Roughly speaking, the latter condition can be understood as the angle between u^+ and u^- being equal to $\frac{2\pi}{m} \mod 2\pi$. A rigorous definition of the above spaces can be found in Subsection 4.1 and Appendix A. Given $\varepsilon > 0$ we consider the generalized Ginzburg-Landau functional $GL_\varepsilon \colon \mathcal{AS}^{(m)}(S) \to [0, +\infty)$ defined as

$$GL_\varepsilon(u) \coloneqq \frac{1}{2} \int_S |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \operatorname{vol} + \mathcal{H}^1_g(\mathcal{J}_u). \tag{1}$$

Here vol denotes the volume form on S and $\mathcal{H}^1_g$ the one-dimensional Hausdorff measure induced by the geodesic distance on S. The main goal of this paper is to analyse the asymptotic behavior as $\varepsilon \to 0$ of a renormalization of the above functionals in the spirit of the first order Γ-convergence (see e.g. Theorem 6.1 in [4]). In the Euclidean setting such an analysis has been carried out in [25] for a slightly modified version of (1) and in [11] for a related lattice spin model.

Furthermore, notice that for $m = 1$ the space of admissible spins $\mathcal{AS}^{(1)}(S)$ simplifies to $W^{1,2}(TS)$ and that the functional GL_ε restricted to $\mathcal{AS}^{(1)}(S)$ coincides with the one considered in [27], where a similar asymptotic variational analysis is one of the core results of the paper. An analogous lattice spin version on a Riemannian manifold was investigated in [19].

In order to explain the main result of this paper we start with some heuristic arguments. Let us fix $m = 1$ and let (u_ε) be a sequence of admissible fields with equi-bounded energy, i.e., such that $\sup_\varepsilon GL_\varepsilon(u_\varepsilon) \leq C$. On one hand, for small ε, by the definition of GL_ε, the penalization term in (1) forces the vector field u_ε to have length close to one while having square integrable gradient. On the other hand, it is a standard result (see [16]) that the space of maps in $W^{1,2}(TS)$ with unit length is empty if the Euler characteristic $\chi(S)$ of S is different from zero. In such a case the minimal Ginzburg-Landau energy diverges as $|\log\varepsilon|$ as $\varepsilon\to 0$. Consequently, it is natural to assume a logarithmic energy bound for (u_ε), namely $GL_\varepsilon(u_\varepsilon)\le C|\log\varepsilon|$ for all ε. Under this bound it has been proved (see [15, 28, 36, 37]) that the family of vector fields (u_ε) can have $K\in\mathbb{N}$ many vortex-like singularities around which u_ε winds an integer amount of times. The winding numbers $d_1,\dots,d_K$ of the K singularities are related to the topology of S by the following formula

$$d_1+\cdots+d_K=\chi(S). \tag{2}$$

The case of general $m\in\mathbb{N}$ is similar. Under the same logarithmic energy bound as above the vector field u_ε creates $K\in\mathbb{N}$ fractional vortices at locations $x_1,\dots,x_K\in S$ with degree $d_1,\dots,d_K\in\frac{\mathbb{Z}}{m}$, respectively. Roughly speaking, a vortex of fractional degree $\frac{k}{m}$ for some $k\in\mathbb{Z}$ is a point around which the vector field rotates by an angle of $\frac{2\pi k}{m}$. As before, the formula (2) remains true. Furthermore, for any vortex center x_k and $r>0$ small enough the following energy lower-bound holds true (see Lemma 14):

$$\liminf_{\varepsilon\to 0}\left(GL_\varepsilon(u_\varepsilon,B_r(x_k))-\frac{|d_k|}{m^2}\pi\log\left(\frac{r}{\varepsilon}\right)\right)\ge\tilde{C}, \tag{3}$$

where $B_r(x_k)$ is the geodesic ball around x_k with radius r and $\tilde{C}>-\infty$.

In this paper we are interested in the energetic behavior of sequences $(u_\varepsilon)\subset\mathcal{AS}^{(m)}(S)$ whose Ginzburg-Landau energy satisfies

$$GL_\varepsilon(u_\varepsilon)\le\frac{N}{m}\pi|\log\varepsilon|+C,$$

for some fixed $N\in\mathbb{N}$. By (3) the energy bound above allows for the creation of vortices of degrees satisfying the bound $|d_1|+\cdots+|d_K|\le N$. This heuristic picture is stated and proved in the first part of the compactness result in Theorem 6. More precisely, as it is customary in the framework of Ginzburg-Landau energies, to every u_ε we associate a two-form $\omega(u_\varepsilon)$ (see (15)) keeping track of the energy concentration of u_ε around vortices. In the planar setting this agrees with a multiple of the distributional jacobian of u_ε. At this stage, in Theorem 6 the compactness properties of u_ε can only be described via the ones of $\omega(u_\varepsilon)$. We show that, up to subsequences, $\omega(u_\varepsilon)$ converges in the flat sense towards a finite sum of weighted Dirac deltas $\mu=\sum_k^K d_k\,\delta_{x_k}$ with $d_k\in\frac{\mathbb{Z}}{m}$ and satisfying the same degree bound. In the same theorem we also prove a more refined compactness result when the energy of the vector fields (u_ε) diverges like $\frac{N}{m}|\log\varepsilon|$. In this case we can find a vector field $u\in SBV(TS)$ such that $u_\varepsilon\to u$ weakly in $SBV^2(S\setminus\{x_1,\dots,x_K\};TS)$. The limit vector field u has unitary length, Nm fractional vortices, each of degree $\pm\frac{1}{m}$, and it is such that $(u^+)^m=(u^-)^m$ at a.e. point of the jump set. Under the same assumptions as in the refined part of the compactness result the following asymptotic lower and upper bound are shown in Theorem 6:

$$\begin{aligned}\liminf_{\varepsilon\to 0}GL_\varepsilon(u_\varepsilon)-\frac{N}{m}\pi|\log\varepsilon|&\ge\mathcal{W}^{(m)}(u)+\mathcal{H}^1_g(\mathcal{J}_u)+Nm\gamma_m,\\ \limsup_{\varepsilon\to 0}GL_\varepsilon(u_\varepsilon)-\frac{N}{m}\pi|\log\varepsilon|&\le\mathcal{W}^{(m)}(u)+\mathcal{H}^1_g(\mathcal{J}_u)+Nm\gamma_m,\end{aligned} \tag{4}$$

where $\gamma_m\in\mathbb{R}$ is the so called core energy defined in (38). This is to be understood as the local energy contribution due to the presence of a vortex of degree $\pm\frac{1}{m}$. In Lemma 6 it is show that γ_m does not depend on the geometry of S. In fact, it coincides with the Euclidean analog obtained in [25]. The other two terms in the limit energy depend on the geometry of S. The term $\mathcal{H}^1_g(\mathcal{J}_u)$ penalizes the length of the jump set of u in terms of its one-dimensional Hausdorff measure induced by the geodesic distance on

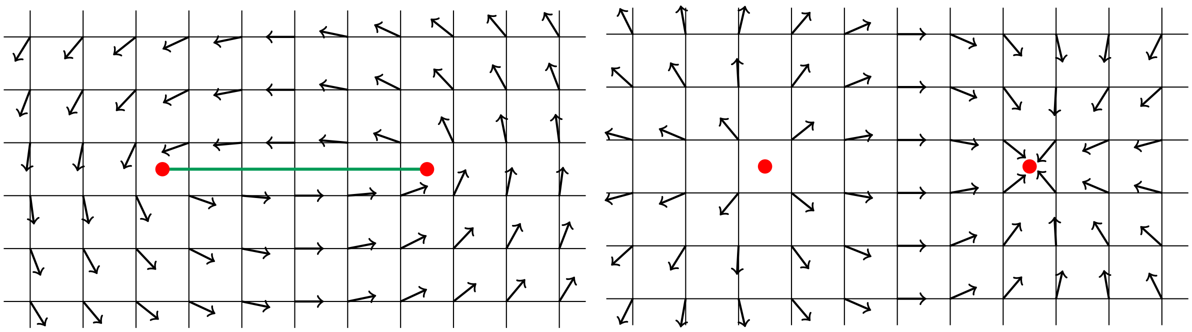
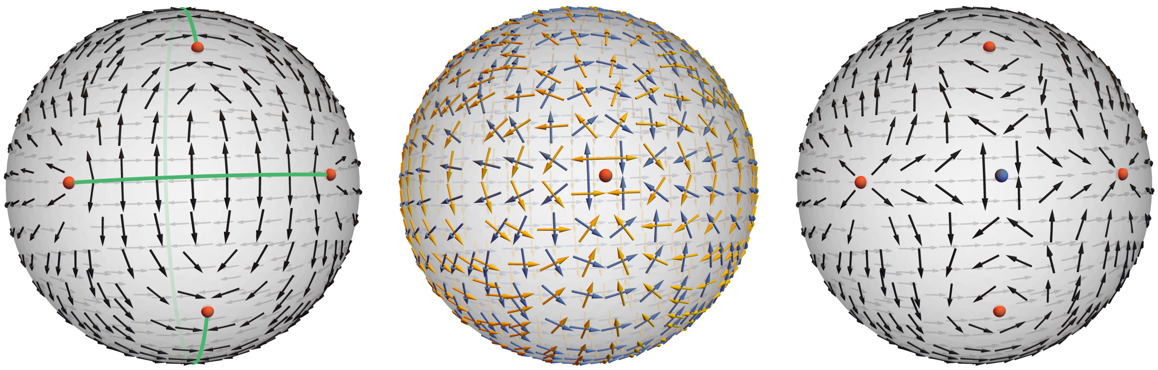


FIGURE 1. Doubling angles of a stadium configuration. On the left the vector field u which has 2 fractional vortices of degrees $1/2$ and jumps by π along a segment. On the right the vector field $v = u^2$ which has 2 integer vortices of degrees 1 and does not jump.

S. The term $\mathcal{W}^{(m)}(u)$ stands for the renormalized energy that weights the interaction between vortices and, according to its definition, pairs of vortices attract or repel each other (as usual depending on their sign) with a force which scales linearly with the inverse of their geodesic distance.

In what follows we highlight the major obstacles we need to overcome in order to prove our result. The main source of difficulties arises from the possible nontrivial topology as well as geometry of S. A first sign of a nontrivial interplay between the topology and the energy concentration phenomenon appears in the constraint (2). A consequence of this condition is that, in the case $\chi(S) \neq 0$, there exists no global smooth orthonormal frame on S. This induces many technical difficulty in most of our proofs. We show here that the 'power map tool' described below cannot be easily exported from the Euclidean to the Riemannian setting.

Roughly speaking, in the flat setting, in which a global frame is available, in [25] and [11] the authors could (in part) simplify some of the arguments of their proofs by exploiting the power map $p : \mathbb{R}^2 \to \mathbb{R}^2$, $p(x) := x^m$ (here we identify $\mathbb{R}^2$ with $\mathbb{C}$) as a simple tool to show in the fractional setting the analog version of some of the results proved in the classical Ginzburg-Landau theory. To understand how the power map is used one can look at the action of p on a prototype vector field u describing the energy concentration in the flat model (see e.g. Figure 1). Such a vector field u has K fractional vortices of degree $d_1, \ldots, d_K \in \frac{\mathbb{Z}}{m}$ and it jumps on a segment with its traces on both sides of the jump set having an angular difference of $\frac{2\pi}{m}\mathbb{Z}$. The vector $v := p(u)$ does not jump and has vortices at same locations as u but of degrees $md_1, \ldots, md_K \in \mathbb{Z}$. Moreover, given $N \in \mathbb{N}$ and a sequence of vector fields (u_ε) such that $GL_\varepsilon(u_\varepsilon) \leq \frac{N}{m}\pi|\log\varepsilon| + C$ the corresponding transformed sequence $(v_\varepsilon) = (p(u_\varepsilon))$ satisfies the energy bound $GL_\varepsilon(v_\varepsilon) \leq N\pi|\log\varepsilon| + C$. Of course these two properties enable one to transfer to the fractional setting many of the results developed in the framework of the integer-degree Ginzburg-Landau theory. The generalization of such an idea to the manifold is not straightforward. In order to generalize the map p to the manifold setting one needs a global choice of frame $\{\tau, i\tau\}$ on TS, where $i\tau$ is the rotation of τ by $\frac{\pi}{2}$. But any such frame is forced to develop singularities if and only if S has a non-trivial Euler characteristic. Globally applying the map p defined according to such a singular frame would induce additional "spurious" singularities, further complicating the analysis. For instance, in Figure 2 doubling the angles of the vector field in 2a with respect to the singular frame in 2b results not only in changing half-vortices into integer vortices, but also in creating an additional vorticity at the blue point in 2c. (We refer to [24] for a numerical computation of such global frames.) On the one hand, this forces us



(A) Initial vector field with 4 half-vortices (red dots) jumping along geodesic segments (green lines)

(B) Singular frame with a double-vortex singularity (red dot)

(C) Vector field after doubling with single-vortices (red dots) and a spurious (-2)-vortex (blue dot)

FIGURE 2. Globally doubling angles on a sphere.

to localize many of the Euclidean results and eventually to apply a partition of unity argument. On the other hand, even in a coordinate neighborhood O where (u_ε) is such that $GL_\varepsilon(u_\varepsilon) \leq \frac{\pi}{m}|\log \varepsilon| + C$, it is a non-trivial task to show that the sequence of vector fields $v_\varepsilon := p(u_\varepsilon)$ satisfies $GL_\varepsilon(v_\varepsilon) \leq \pi|\log\varepsilon| + \tilde{C}$. The main difficulties in pursuing this task are already evident when one considers how the gradient term in the Ginzburg-Landau energy transforms under the map p. By the chain rule formula (see Proposition 4) it holds that

$$|\nabla v_\varepsilon|^2 = m^2|\nabla u_\varepsilon|^2 + (1-m^2)|\mathrm{d}|u_\varepsilon||^2 + (m-1)^2|u_\varepsilon|^2|\mathrm{j}(\tau)|^2 - 2m(m-1)\langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle.$$

In the equation above $\{\tau, i\tau\}$ is a smooth (up to the boundary) frame in O and j denotes the pre-jacobian, which for any vector field w satisfies $\mathrm{j}(w) := \langle \nabla w, iw\rangle$. Notice the last two terms above are peculiar of the manifold setting and need to be uniformly controlled in order to derive the needed energy upper bound for (v_ε). In particular, it is non-trivial to show the boundedness of the last term since $\mathrm{j}(\tau) \neq 0$ as the manifold in general is curved and (∇u_ε) is not a priori bounded in L^1. To tackle this problem we need to combine a ball construction argument together with a specific choice of frame τ, namely a frame having least Dirichlet energy $\int_O |\nabla\tau|^2 \,\mathrm{vol}$.

Another source of difficulties is the characterization of the core energy γ_m in (4). Let us recall the definition of γ_m in the Euclidean setting (see for instance [25]). For any $r > 0$ we have that

$$\gamma_m := \lim_{\varepsilon\to 0}\left(\bar{\gamma}_\varepsilon^{(m)}(r) - \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right)\right).$$

Here $\gamma_\varepsilon^{(m)}(r)$ is given by the following minimization problem:

$$\bar{\gamma}_\varepsilon^{(m)}(r) := \min\left\{\overline{GL}_\varepsilon^{(m)}(v, B_r(0)) \colon v \in W^{1,2}(B_r(0);\mathbb{R}^2),\ v = \frac{x}{|x|} \text{ on } \partial B_r(0)\right\},$$

where

$$\overline{GL}_\varepsilon^{(m)}(v, B_r(0)) := \frac{1}{2m^2}\int_{B_r(0)} |\nabla v|^2 + (m^2-1)|\nabla|v||^2 + \frac{m^2}{2\varepsilon^2}(1-|v|^2)^2 \,\mathrm{d}x.$$

The core energy in the manifold setting can be obtained by comparing $\overline{GL}_\varepsilon^{(m)}$ with its manifold analog $GL_\varepsilon^{(m)}$ defined as

$$GL_\varepsilon^{(m)}(v, B_r(x_0)) \coloneqq \frac{1}{2m^2} \int_{B_r(x_0)} |\nabla v|^2 + (m^2-1)|\mathrm{d}|v||^2 + \frac{m^2}{2\varepsilon^2}(1-|v|^2)^2 \,\mathrm{vol}$$

for any $v \in W^{1,2}(TB_r(x_0))$. More precisely, the two functionals can be compared by choosing centered normal coordinates at x_0 and an orthonormal frame which results in a coordinate representation $\bar{v}$ of v satisfying

$$\left| GL_\varepsilon^{(m)}(v, B_r(x_0)) - \overline{GL}_\varepsilon^{(m)}(\bar{v}, B_r(0)) \right| \leq Cr \left(1 + \overline{GL}_\varepsilon^{(m)}(\bar{v}, B_r(0)) \right). \tag{5}$$

Notice that for a sequence of minimizers (in the Euclidean setting) and fixed $r > 0$ the right-hand side can diverge as $|\log \varepsilon|$, making our comparison strategy inefficient. A possible way out already appeared in [27] and consists in considering a properly chosen sequence of radii (r_ε) such that $r_\varepsilon |\log \varepsilon| \to 0$ for which the error in (5) vanishes. For each $\varepsilon > 0$ one then needs to find Nm disjoint balls of radius r_ε (cores) such that the Ginzburg-Landau energy in each ball scales like $\frac{\pi}{m^2} \log\left(\frac{r_\varepsilon}{\varepsilon}\right)$. In the Euclidean setting $r_\varepsilon = r > 0$ is an admissible choice and the task is rather straightforward as one can simply take balls around the limit vortex centers. On the manifold, instead, one must resort once again to the ball construction in order to find an appropriate choice of ball centers around which a sufficient amount of energy concentrates. Moreover, in contrast to [27] we need to take the Γ-limit with respect to the L^1-convergence which does not only track the defects but also the asymptotic behavior of the vector fields realizing them. Because of that, compared to [27], further work is invested to assure that the strategy in the Euclidean setting (see e.g. [25]) generalizes to the case of small cores (see in particular Step 3-5 in the proof of our Γ-lim inf).

To finish we would like to mention that even the definition of special sections of bounded variation that we roughly introduced at the beginning of this introduction requires some care. More precisely, most of the notions in the statements of this paper benefit from an intrinsic definition of $SBV(TS)$, whose most important properties are stated in Section 2.1, while their proofs are contained in Appendix A. Although the notion and the main properties of BV functions are well-understood even in the more abstract setting of metric spaces (see [5, 6, 7, 29, 31]) and the translation of their finer properties to the Riemannian manifold setting is possible, we have found it more convenient for the reader to derive these results directly. It is worth mentioning that the definition and some of the properties of BV functions on manifolds can be found in other papers as for instance in [26, 33]. To the best of our knowledge however, the derivation of the finer properties of BV functions in the manifold setting was lacking. It is now contained in the Appendix A where in particular we prove the decomposition theorem 3. The key ideas behind its proof is an intrinsic definition of blow-up quantities, the investigation of their relation to the Euclidean ones, and a partition of unity argument. For the purposes of the variational analysis contained in this paper it would be sufficient to prove the decomposition theorem only for tangent vector fields or scalar maps on S. Instead, since the main argument remains unchanged, we have decided to extend the theorem to the general case of sections of an arbitrary Riemannian vector bundle.

2. Preliminaries

2.1. **Tangent vector fields of bounded variation.** We wish to provide a definition of *(special) functions* and *(special) tangent vector fields* of bounded variation on a manifold. Furthermore, we will state several important results concerning this function spaces. The missing proofs can be found in Appendix A. (In the same appendix we will also deal with case of a general vector bundle.)

Let $n \in \mathbb{N} \coloneqq \{1, 2, \dots\}$. We denote by M an n-dimensional, oriented Riemannian manifold (with or without boundary) with metric tensor g. The open geodesic ball of radius r centered at $x \in M$ will be written as $B_r(x)$. If no confusion is possible, given $x \in \mathbb{R}^n$ we use the same notation $B_r(x)$ to denote the Euclidean open ball. We will write $r^* = r^*(M)$ for the injectivity radius of M and $\exp_x \colon T_xM \to M$, where T_xM denotes the tangent space to M at x, for the exponential map at $x \in M$. The volume form on M will be denoted by vol. We will write TM for the tangent bundle and T^*M for the co-tangent bundle of M, respectively. For the covariant derivative we use the symbol ∇. Whenever possible, Einstein summation convention will be used. Herewith, we will implicitly assume that any index such as i, j, ... we encounter is ranging in $\{1, \dots, n\}$.

Functions spaces $X(M;TM)$ of tangent vector fields $u \colon M \to TM$ with regularity prescribed by X will be shortly written as $X(TM)$. We further use the notation $X_{\mathrm{loc}}(TM)$ to denote those tangent vector fields u that belong to $X(TK)$ for any compact $K \subset M$, where given any set $A \subset M$ we denote by $TA \coloneqq \bigsqcup_{x \in A} T_xM$. For example, $C_c^\infty(TM)$ will be the space of smooth compactly supported sections of TM. For any $\alpha \in [0, n]$, we will denote by $\mathcal{H}_g^\alpha$ the α-dimensional Hausdorff measure on M defined through the metric distance induced by g. Note that for $\alpha = n$ we recover the usual notion of integration on the manifold M. More precisely, for any $f \in C_c^\infty(M)$ we have

$$\int_M f \,\mathrm{vol} = \int_M f \,\mathrm{d}\,\mathcal{H}_g^n \,.$$

From now on, we will shortly write a.e. in place of $\mathcal{H}_g^n$-a.e., when no confusion is possible. Let u be a $\mathcal{H}_g^n$-measurable tangent vector field and $p \in [1, \infty]$. If $p \in [1, \infty)$, we define the space $L^p(TM)$ of those $\mathcal{H}_g^n$-measurable functions $u \colon M \to TM$ with finite $\|u\|_{L^p}$, namely

$$\|u\|_{L^p} \coloneqq \int_M |u|^p \,\mathrm{vol} < \infty,$$

where $|\cdot|$ denotes the norm induced by g. For $p = \infty$ we require the following L^∞-norm to be finite:

$$\|u\|_{L^\infty} \coloneqq \inf\{C \in \mathbb{R} \colon |u(x)| \le C \text{ for a.e. } x \in M\} < \infty.$$

The spaces $L^p(M)$ for $p \in [1, \infty]$ are defined similarly. Given $\Omega \subset \mathbb{R}^n$ and $O \subset M$ open sets, we denote by $\Psi \colon \Omega \times \mathbb{R}^n \to TO$ a local trivialization of TM. We remark that any such Ψ induces a unique coordinate chart as well as a unique frame which, without further mention, will be denoted by $\Phi \colon \Omega \to O$ and $\{\tau_1, \dots, \tau_n\}$, respectively. Ψ, Φ, and $\{\tau_1, \dots, \tau_n\}$ are implicitly assumed to be smooth up to the boundary.

We single out tangent vector fields of *bounded variation* as precisely those elements of $L^1(TM)$ whose *total variation* is finite. In order to define the total variation in our present setting we will first need to introduce a classical object from differential geometry: the *adjoint covariant derivative* ∇^*. The latter is the unique operator $\nabla^* \colon C^\infty(TM \otimes T^*M) \to C^\infty(TM)$ such that for all $u \in C_c^\infty(TM)$ and $v \in C_c^\infty(TM \otimes T^*M)$ the following integration-by-parts formula holds true (see also Proposition 10.1.30 in [32] for further details):

$$\int_M \langle u, \nabla^* v \rangle \,\mathrm{vol} = \int_M \langle \nabla u, v \rangle \,\mathrm{vol}\,. \tag{6}$$

Here, $\otimes$ denotes the tensor product. The total variation of a section $u \in L^1(TM)$ is then defined as the supremum of the left-hand side of (6) over test-functions bounded by 1:

$$\mathrm{var}(u) \coloneqq \sup\left\{ \int_M \langle u, \nabla^* v \rangle \,\mathrm{vol} \colon v \in C_c^\infty(TM \otimes T^*M),\ \|v\|_{L^\infty} \le 1 \right\}. \tag{7}$$

Notice that in the special case of $M = \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open set, equipped with the Euclidean distance, the adjoint gradient ∇^* satisfies for any $i \in \{1, \dots, n\}$

$$-(\nabla^* v)_i = (\operatorname{div} v)_i := \sum_{j=1}^n \frac{\partial v_{ij}}{\partial x^i},$$

where $v \in C_c^\infty(\Omega; \mathbb{R}^{n\times n})$. Hence, the general definition in (7) of total variation in the manifold setting agrees with the usual one in Euclidean space (see also (3.4) in [8]). We now introduce a local definition of total variation. Given an $O \subset M$ open we define the total variation of u in O as

$$\operatorname{var}(u, O) := \sup\left\{ \int_M \langle u, \nabla^* v\rangle \operatorname{vol} \colon v \in C_c^\infty(TO \otimes T^*O),\ \|v\|_{L^\infty} \le 1 \right\}, \tag{8}$$

where $T^*O := \bigsqcup_{x\in O} T_x^* M$.

We are now ready to define *tangent vector fields of bounded variation.*

Definition 1 (Tangent vector fields of bounded variation)**.** A section $u \in L^1(TM)$ is of bounded variation if and only if $\operatorname{var}(u) < \infty$. The set of all such sections wll be denoted by $BV(TM)$. It is equipped with the norm

$$\|u\|_{BV} := \|u\|_{L^1} + \operatorname{var}(u).$$

With this norm $BV(TM)$ turns out to be a Banach space.

We will now introduce the Riesz representation theorem and Radon-Nikodym theorem in our setting. The former provides a representation of linear bounded functionals on the space of compactly supported continuous sections of TM via $TM\otimes T^*M$-valued Radon measures. These measures are defined as follows:

Definition 2 ($TM \otimes T^*M$-valued Radon measures)**.** Let $\mathcal{M}_+(M)$ denote the set of positive (finite) Radon measures on M. Given $\mu \in \mathcal{M}_+(M)$ we define as $L^1(TM\otimes T^*M; \mu)$ the set of measurable sections σ of $TM \otimes T^*M$ such that

$$\int_M |\sigma| \,\mathrm{d}\mu < \infty.$$

Then, the set $\mathcal{M}(TM \otimes T^*M)$ of $TM \otimes T^*M$-valued Radon-measures is defined as

$$\mathcal{M}(TM \otimes T^*M) := \{(\sigma, \mu) \colon \mu \in \mathcal{M}_+(M),\ \sigma \in L^1(TM \otimes T^*M; \mu)\}.$$

Note that the pair (σ, μ) will be usually written as $\sigma\mu$. Furthermore, for a given $\nu = \sigma\mu \in \mathcal{M}(TM\otimes T^*M)$ such that $|\sigma| = 1$ at μ-a.e. in M we will call μ the *total variation* of ν (written as $|\nu|$) and σ its *polar density* (written as σ or σ_ν if confusion is possible). Two measures $\nu, \tilde\nu \in \mathcal{M}(TM \otimes T^*M)$ are said to be equal if and only if $|\nu| = |\tilde\nu|$ in the sense of measures and $\sigma_\nu = \sigma_{\tilde\nu}$ at $|\nu|$-a.e. point in M. Given $\nu \in \mathcal{M}(TM \otimes T^*M)$ and $\mu \in \mathcal{M}_+(M)$ we use the notation $\nu << \mu$ if ν is *absolutely continuous* and $\nu \perp \mu$ if ν is singular with respect to μ. A sequence $(\nu_h) \subset \mathcal{M}(TM \otimes T^*M)$ weakly* converges towards $\nu \in \mathcal{M}(TM\otimes T^*M)$ (shortly written as $\nu_h \overset{*}{\rightharpoonup} \nu$) if and only if for all continuous and compactly supported $v \in C_c(TM \otimes T^*M)$ it holds that

$$\lim_{h\to\infty} \int_M \langle \sigma_{\nu_h}, v\rangle \,\mathrm{d}|\nu_h| \to \int_M \langle \sigma_\nu, v\rangle \,\mathrm{d}|\nu|.$$

Theorem 1. *(Riesz representation for tangent vector fields) Let $T \colon C_c(TM \otimes T^*M) \to \mathbb{R}$ be a bounded linear functional, then there exists a unique $TM \otimes T^*M$-valued Radon-measure $\nu \in \mathcal{M}(TM \otimes T^*M)$ such that*

$$T(v) = \int_M \langle v, \sigma_\nu\rangle \,\mathrm{d}|\nu|.$$

Proof. We refer the reader to Theorem [26] for the proof of the statement in the scalar case. The same proof works also for the case of the tangent bundle or, more generally, the case of an arbitrary vector bundle with minor modifications. □

Given $u \in BV(TM)$ we can define a linear functional $T_u \colon C_c^\infty(TM \otimes T^*M) \to \mathbb{R}$ as follows:

$$T_u(v) = \int_M \langle u, \nabla^* v\rangle \operatorname{vol}.$$

By the definition of total variation in (7) it turns out that T_u is bounded since $\|T_u\| = \operatorname{var}(u)$. Due to Theorem 1 there exists a unique measure in $\mathcal{M}(TM \otimes T^*M)$ which we will from now on denote by Du such that for all $v \in C_c^\infty(TM \otimes T^*M)$ the following integration-by-parts formula holds true

$$\int_M \langle u, \nabla^* v\rangle \operatorname{vol} = \int_M \langle v, \sigma_u\rangle \, \mathrm{d}|Du|,$$

where $\sigma_u \coloneqq \sigma_{Du}$ is the polar density of Du.

The following Radon-Nikodym decomposition holds true:

Theorem 2 (Radon-Nikodym). *For any $\nu \in \mathcal{M}(TM \otimes T^*M)$ and $\mu \in \mathcal{M}_+(M)$ there exist only two measures ν^a, $\nu^s \in \mathcal{M}(TM \otimes T^*M)$ such that $\nu^a << \mu$, $\nu^s \perp \mu$ and $\nu = \nu^a + \nu^s$.*

Furthermore, there exists a unique $\sigma^a \in L^1(TM \otimes T^*M; \mu)$ such that $\nu^a = \sigma^a \mu$.

In the special case of $\nu = Du$ for some $u \in BV(TM)$ and $\mu = \mathcal{H}^n_g$ in Theorem 2 we will denote ν^a by $D^a u$ and ν^s by $D^s u$.

We will now define intrinsic blow-ups of a section $u \in L^1_{\mathrm{loc}}(TM)$. An intrinsic definition will involve comparing a vector in the bundle T_xM with another vector in the bundle T_yM for two different points x, $y \in M$. In order to assure invariance under a change of coordinates we will employ *parallel transport* on TM, which can be briefly described as follows: Given a smooth curve $\gamma \colon [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$ and a vector $v_0 \in T_{\gamma(0)}M$, there exists a unique family $\{P_t^{(\gamma)}\}_{t\in[0,1]}$ of linear isomorphisms $P_t^{(\gamma)} \colon T_{\gamma(0)}M \to T_{\gamma(t)M}$ such that $v(t) \coloneqq P_t^{(\gamma)}(v_0)$ satisfies:

$$\begin{cases} \nabla_{\dot\gamma(t)} v(t) = 0, & t \in [0,1], \\ \qquad v(0) = v_0. \end{cases}$$

This notion of transport between T_xM and T_yM depends on the choice of curve γ. Nevertheless, for points close enough (more precisely strictly closer than the injectivity radius of M) we can make the transport unique by taking γ as the geodesic between x and y. More precisely, for any $x \in M$ we define the *transport map* $\mathcal{T}_x \colon B_{r^*}(x) \times T_xM \to TB_r(x)$ from $x \in M$ as:

$$\mathcal{T}_x(y, v) \coloneqq P_1^{(\gamma_y)}(v), \tag{9}$$

where $\gamma_y \colon [0,1] \to M$ is the unique geodesic starting at x and ending at y with constant speed equal to the geodesic distance $\operatorname{dist}_g(x,y)$ between x and y.

Definition 3 (Approximate limit). Let $u \in L^1_{\mathrm{loc}}(TM)$ and let $\mathcal{T}_x$ be the transport map from $x \in M$ defined in (9). We say that u has an *approximate limit* $z \in T_xM$ at x if

$$\lim_{r\to 0} \fint_{B_r(x)} |u(y) - \mathcal{T}_x(y,z)| \operatorname{vol}(y) \coloneqq \lim_{r\to 0} \frac{1}{\mathcal{H}^n_g(B_r(x))} \int_{B_r(x)} |u(y) - \mathcal{T}_x(y,z)| \operatorname{vol}(y) = 0. \tag{10}$$

The set $\mathcal{S}_u$ where this property does not hold is called the *approximate discontinuity set* of u. For any $x \in M \setminus \mathcal{S}_u$ the approximate limit z in (10) is uniquely determined and will be denoted by $\tilde u(x)$. Finally, we say that u is *approximately continuous* at x if $x \in M \setminus \mathcal{S}_u$ and $u(x) = \tilde u(x)$.

It is sometimes useful to resort to coordinates. In this regard we wish to define the pull-back of a section of TM through a local trivialization.

Definition 4. Given a local trivialization $\Psi\colon \Omega\times\mathbb{R}^n \to TO$ and a section u of TO we define $\Psi^*u\colon \Omega \to \mathbb{R}^n$ at $x \in \Omega$ through

$$(\Psi^* u(\Phi(x)))^\alpha \tau_\alpha(\Phi(x)) = u(\Phi(x)).$$

The following proposition investigates the relationship between approximate limit points and their coordinate representations in Euclidean space.

Proposition 1 (Approximate limits and coordinates)**.** *Let $\Psi\colon \Omega \times \mathbb{R}^n \to TO$ be a local trivialization. Then, a section $u \in L^1(TO)$ has approximate limit z at $x \in O$ if and only if its coordinate representation Ψ^*u has approximate limit $\Psi^*z \in \mathbb{R}^n$ at $\Phi^{-1}(x)$.*

Definition 5 (Approximate jump points)**.** Let $u \in L^1_{\text{loc}}(TM)$ and let $\mathcal{T}_x$ be the transport map from $x \in M$ defined in (9). We say that x is an *approximate jump* point of u if there exist a, $b \in T_xM$ with $a \neq b$ and a unit vector $\nu \in T_xM$ such that

$$\lim_{r\to 0} \fint_{B_r^+(x,\nu)} |u(y) - \mathcal{T}_x(y,a)| \operatorname{vol}(y) = 0, \quad \lim_{r\to 0} \fint_{B_r^-(x,\nu)} |u(y) - \mathcal{T}_x(y,b)| \operatorname{vol}(y) = 0, \tag{11}$$

where $B_r^+(x,\nu)$ and $B_r^-(x,\nu)$ are the geodesic half balls defined by

$$B_r^+(x,\nu) \coloneqq \exp_x(\{X \in T_xM\colon |X| < r,\, \langle X,\nu\rangle > 0\}),$$
$$B_r^-(x,\nu) \coloneqq \exp_x(\{X \in T_xM\colon |X| < r,\, \langle X,\nu\rangle < 0\}).$$

The triplet (a, b, ν) is uniquely determined by 11 up to switching a and b as well as changing the sign of ν. The triplet in the definition will be denoted by $(u^+(x), u^-(x), \nu(x))$ and the set of approximate jump points will be denoted by $\mathcal{J}_u$.

The relation between approximate jump points and their coordinate representations in Euclidean space is as follows:

Proposition 2 (Approximate jumps and coordinates)**.** *Let $\Psi\colon \Omega \times \mathbb{R}^n \to TO$ be a local trivialization. Then, a section $u \in L^1_{\text{loc}}(TO)$ has an approximate jump at $x \in O$ with triplet (a, b, ν) if and only if Ψ^*u has an approximate jump at $\Phi^{-1}(x)$ in the usual Euclidean sense with triplet $(\Psi^*a, \Psi^*b, \bar\nu)$, such that*

$$\nu^k = \frac{1}{\sqrt{g^{ij}\bar\nu^i\bar\nu^j}} g^{kl}\bar\nu^l \qquad \text{for } k \in \{1, \dots, n\}$$

and (g^{ij}) denotes the inverse of the metric tensor (g_{ij}).

Definition 6 (Approximate differentiability)**.** Let $u \in L^1_{\text{loc}}(TM)$ and let $\mathcal{T}_x$ be the transport map from $x \in M$ defined in (9). We say that x is an *approximate differentiability* point of u if $x \in M \setminus \mathcal{S}_u$ and if there exists $L \in T_xM \otimes T_x^*M$ such that

$$\lim_{r\to 0} \fint_{B_r(x)} r^{-1} |u(y) - \mathcal{T}_x(y, \tilde u(x)) - \mathcal{T}_x(y, L(X))| \operatorname{vol}(y) = 0, \quad X \coloneqq \exp_x^{-1}(y), \tag{12}$$

where we identified $T_xM \otimes T_x^*M$ with the space of linear maps from T_xM to T_xM. The tensor L is uniquely determined by (12) and will be denoted by $\nabla u(x)$. The set of approximate differentiability points of u will be written as $\mathcal{D}_u$.

The next proposition clarifies the relationship between approximate differentiability points and their coordinate representations.

Proposition 3. *Let $\Psi\colon \Omega\times\mathbb{R}^n \to TO$ be a local trivialization with induced frame $\{\tau_1,\dots,\tau_n\}$. Then, any section $u\in L^1(TO)$ is approximately differentiable at $x\in O$ with approximate gradient $L\in T_xM\otimes T^*_xM$ if and only if Ψ^*u is approximately differentiable at $\Phi^{-1}(x)$ in the usual Euclidean sense with approximate gradient $\bar L$ and approximate limit $\bar z\in\mathbb{R}^n$ such that*

$$L = (\bar L_i^\alpha + \Gamma^\alpha_{i\beta}\bar z^\beta)\,\tau_\alpha\otimes \mathrm{d}x^i, \tag{13}$$

where $(\Gamma^\alpha_{i\beta})$ denotes the Christoffel symbols at x.

In the next definition we recall the notion of *rectifiability* on a Riemannian manifold M.

Definition 7 ($\mathcal{H}^{n-1}_g$-rectifiable)**.** A set $N\subset M$ is $\mathcal{H}^{n-1}_g$-rectifiable if and only if there exists a countable family $\{N_h\}_h$ of C^1-regular $(n-1)$-dimensional submanifolds of M such that

$$\mathcal{H}^{n-1}_g(M\setminus \cup_h N_h) = 0.$$

We are ready to state a fundamental theorem for tangent vector fields of bounded variation.

Theorem 3 (Decomposition of tangent vector fields of bounded variation)**.** *Let $u\in BV(TM)$, then the discontinuity set $\mathcal{S}_u$ is $\mathcal{H}^{n-1}_g$-rectifiable, $\mathcal{H}^{n-1}_g(\mathcal{S}_u\setminus\mathcal{J}_u)=0$, and the restriction $D^ju := D^su \llcorner \mathcal{J}_u$ of the singular part of Du to $\mathcal{J}_u$ can be represented as*

$$D^ju = (u^+ - u^-)\otimes \nu^\flat\, \mathcal{H}^{n-1}_g \llcorner \mathcal{J}_u,$$

where the triplet $(u^+,\, u^-,\, \nu)$ is as in Definition 5 and $\nu^\flat$ is the 1-form given by $\nu^\flat(X)=\langle \nu, X\rangle$ for any $X\in T_xM$.

Furthermore, u is approximately differentiable at a.e. point of M and the absolutely continuous part of Du can be written as

$$D^au = \nabla u\ \mathcal{H}^n_g,$$

∇u being the approximate gradient of u.

Remark 1*.* To summarize the above theorem, we end up with the following decomposition of Du:

$$Du = (u^+ - u^-)\otimes\nu^\flat\,\mathcal{H}^{n-1}_g \llcorner \mathcal{J}_u + \nabla u\,\mathcal{H}^n_g + D^cu,$$

where $D^cu := D^su\llcorner (M\setminus\mathcal{S}_u)$ is the so called *Cantor part* of u.

At this point we wish to shortly comment on the decomposition theorem the scalar case. A scalar function $f\in L^1(M)$ has bounded variation if and only if

$$\mathrm{var}(f) := \sup\left\{\int_M f\,\mathrm{d}^*v\,\mathrm{vol}\colon v\in C^\infty_c(T^*M), \|v\|_{L^\infty}\le 1\right\} < \infty,$$

where d^* is the adjoint exterior derivative. Similar definitions (to the vector-valued case) hold true for the blow-up quantities. Instead of ∇f we will usually write $\mathrm{d}f$ for the approximate gradient fo f. Notice that given an approximate differentiability point x of f and a local chart Φ in the vicinity of x we have

$$\mathrm{d}f(x) = \frac{\partial}{\partial x^i}(f\circ\Phi)(\Phi^{-1}(x))\,\mathrm{d}x^i.$$

Finally, the following decomposition holds true in the scalar setting for the distributional derivative Df of f:

$$Df = (f^+ - f^-)\nu^\flat\,\mathcal{H}^{n-1}_g \llcorner \mathcal{J}_f + \mathrm{d}f\,\mathcal{H}^{n-1}_g + D^cf.$$

The definition of *special* tangent vector fields of bounded variation then naturally follows:

Definition 8 (Special sections of bounded variation)**.** The set of *special sections of bounded variation* consists of $u \in BV(TM)$ with vanishing Cantor part. More precisely we set

$$SBV(TM) \coloneqq \{u \in BV(TM) \colon D^c u = 0\},$$

For any $p \in (1,\infty)$ we also define the space

$$SBV^p(TM) \coloneqq \{u \in SBV(TM) \colon \nabla u \in L^p(TM \otimes T^*M),\, \mathcal{H}^{n-1}_g(\mathcal{J}_u) < \infty\}.$$

Let $u \in SBV^p(TM)$, then a sequence $(u_h) \subset SBV^p(TM)$ is said to converges weakly towards u in $SBV^p(TM)$ (shortly written as $u_h \rightharpoonup u$) if and only if

$$(i) \qquad \nabla u_h \rightharpoonup \nabla u \text{ weakly in } L^p(TM \otimes T^*M),$$

$$(ii) \qquad D^j u_h \overset{*}{\rightharpoonup} D^j u \text{ weakly* in } \mathcal{M}(TM \otimes T^*M).$$

In order to show that a section u is in $SBV_{\mathrm{loc}}(TM)$ we can equivalently resort to coordinates. More precisely:

Lemma 1. *A section $u \in L^1_{\mathrm{loc}}(TM)$ is in $BV_{\mathrm{loc}}(TM)$ ($SBV_{\mathrm{loc}}(TM)$, $L^p_{\mathrm{loc}}(TM) \cap SBV^p_{\mathrm{loc}}(TM)$) if and only if for any local trivialization $\Psi \colon \Omega \times \mathbb{R}^n \to TO$ the pull-back $\Psi^* u$ is in $BV_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ ($SBV_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, $L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \cap SBV^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$) in the usual Euclidean sense.*

The last result of this section concerns a compactness result in the spaces $SBV^p(TM)$.

Theorem 4 (Compactness in SBV^p)**.** *Let M be a compact manifold (with or without boundary), $p \in (1,\infty)$, and $(u_h) \subset SBV^p(TM)$ be a sequence satisfying the following bound:*

$$\sup_h \left(\|u_h\|_{L^\infty} + \|\nabla u_h\|_{L^p} + \mathcal{H}^{n-1}_g(\mathcal{J}_{u_h})\right) < \infty.$$

Then, up to taking a subsequence, $u_h \rightharpoonup u$ weakly in $SBV^p(TM)$ and $u_h \to u$ in $L^1(TM)$.

2.2. **Vorticity.** From this point on, we restrict ourselves to the case of a closed, oriented 2-dimensional Riemannian manifold S with metric tensor and volume form still denote by g and vol, respectively.

For the moment, let $u \in C^\infty(TS)$ be a smooth tangent vector-field. The *pre-jacobian* of u is the 1-form $\mathrm{j}(u) \in C^\infty(T^*S)$ defined by

$$\mathrm{j}(u)(X) \coloneqq \langle \nabla_X u, iu\rangle, \qquad \text{for all } X \in TS.$$

Here, $i \colon TS \to TS$ is the isometry of TS onto itself characterized by

$$i^2 v = -v, \quad \langle iv, w\rangle = -\langle v, iw\rangle = \mathrm{vol}(v,w) \qquad \text{for all } v,w \in TS,$$

where $\langle \cdot,\cdot\rangle$ is the scalar product on TS induced by the metric tensor g.

Given an open subset $O \subset S$ with Lipschitz boundary such that $|u| \geq c$ on ∂O for some $c > 0$ we can define the *degree* of u on ∂O as

$$\deg(u,\partial O) \coloneqq \frac{1}{2\pi}\left(\int_{\partial O} \frac{\mathrm{j}(u)}{|u|^2} + \int_O \kappa \,\mathrm{vol}\right), \tag{14}$$

where κ is the Gauss curvature. It can be shown that the degree is valued in $\mathbb{Z}$.

If u is of unit length on ∂O, by Stokes' theorem it holds that

$$\deg(u,\partial O) = \int_O \omega(u), \qquad \omega(u) \coloneqq \mathrm{d}\,\mathrm{j}(u) + \kappa\,\mathrm{vol}\,.$$

The 2-form $\omega(u)$ is called the *vorticity* of u.

The pre-jacobian and the vorticity can be extended to a more general setting. Given $S \subset M$, in this paper we are mainly interested in the case of $u \in SBV(TS)$ such that $|\nabla u| \in L^p(S)$ and $|u| \in L^q(S)$ for p, $q \in [1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Note that ∇u is the approximate gradient of u (see also Definition 6). By an application of Hölder's inequality we see that $\mathrm{j}(u) \in L^1(T^*S)$. It is then possible to define the vorticity of u in distributional sense. In fact, given $\alpha \in C^\infty(T^*S)$ and $\beta \in C^\infty(\Lambda^2 S)$, where $\Lambda^2(S) := T^*S \wedge T^*S$, the adjoint exterior derivative d^* satisfies:

$$\int_S \langle \mathrm{d}\alpha, \beta \rangle \,\mathrm{vol} = \int_S \langle \alpha, \mathrm{d}^*\beta \rangle \,\mathrm{vol}\,.$$

Furthermore, for $\Phi = \varphi \,\mathrm{vol}$ with $\varphi \in C^\infty(S)$, we have that

$$\mathrm{d}^*\Phi = \star\, \mathrm{d}(\star\Phi) = \star(\mathrm{d}\varphi),$$

where $\star$ is the Hodge star. Consequently, as $\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle \,\mathrm{vol}$ for α, β as before, for smooth u the following integration-by-parts formula holds true:

$$\int_S \langle \mathrm{d}\,\mathrm{j}(u), \Phi \rangle \,\mathrm{vol} = \int_S \langle \mathrm{j}(u), \star(\mathrm{d}\varphi) \rangle \,\mathrm{vol} = \int_S \mathrm{j}(u) \wedge (\star\star\, \mathrm{d}\varphi) = -\int_S \mathrm{j}(u) \wedge \mathrm{d}\varphi,$$

where $\wedge$ is the wedge product. This allows us to define $\mathrm{d}\,\mathrm{j}(u)$ in distributional sense through its action on smooth 2-forms $\Phi = \varphi \,\mathrm{vol} \in C^\infty(\Lambda^2 S)$ as follows:

$$\mathrm{d}\,\mathrm{j}(u)(\Phi) := -\int_S \mathrm{j}(u) \wedge \mathrm{d}\varphi.$$

For any such Φ, we can then define the vorticity of $u \in SBV(TS)$ in distributional sense via

$$\omega(u)(\Phi) = -\int_S \Big(\mathrm{j}(u) \wedge \mathrm{d}\varphi + \kappa\varphi \,\mathrm{vol} \Big). \tag{15}$$

With this definition the validity of *Morse's index formula* in our present function setting, easily follows.

Theorem 5 (Morse's index formula)**.** *For any $u \in SBV(TS) \cap L^\infty(TS)$ it holds that*

$$\int_S \omega(u) = \chi(S),$$

where $\chi(S)$ is the Euler characteristic of S.

Proof. Testing (15) with $\varphi \equiv 1$ and using the Gauss-Bonnet theorem we conclude that

$$\int_S \omega(u) = \omega(u)(\mathrm{vol}) = \int_S \mathrm{j}(u) \wedge \mathrm{d}1 + \int_S \kappa \,\mathrm{vol} = \int_S \kappa \,\mathrm{vol} = \chi(S).$$

□

In the absence of jumps the distributional jacobian satisfies the following two useful properties:

Lemma 2. *Let $O \subset S$ be an open subset and $u \in W^{1,2}(TO)$. Then, the distributional jacobian $\mathrm{d}\,\mathrm{j}(u)$ is in $L^1(\Lambda^2 O)$ and for a.e. point in O we have that*

$$|\mathrm{d}\,\mathrm{j}(u)| \le |\nabla u|^2. \tag{16}$$

Furthermore, given another vector field $v \in W^{1,2}(TO)$ it holds that

$$\|\omega(u) - \omega(v)\|_{W_0^{-1,\infty}(\Lambda^2 O)} \le \|u - v\|_{L^2}(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}). \tag{17}$$

Proof. A proof of (16) can be found in [27] (see Lemma 5.3), while a proof of (17) is contained in [19] (see (3.27) in Lemma 3.7). □

In the remainder of this section we will investigate how a vector-field u and associated quantities such as its pre-jacobian $\mathrm{j}(u)$ change under local "doubling" of the angles. Let us first precisely define what we mean by "doubling" or, more generally, multiplying all angles by $m \in \{2, 3, \ldots\}$, which will be shortly written as "m-pling". For this purpose, let O be a coordinate neighborhood of S. This guarantees that we can find a smooth (up to the boundary) unit-length vector field $\tau \in C^\infty(TO)$. For each $x \in O$ we will see $\tau(x)$ as the unique unit-length vector in T_xS having zero angle and represent a tangent vector X through its *polar coordinates* $r = r(X)$, $\alpha = \alpha(X)$, which are characterized by

$$X = r\cos(\alpha)\,\tau + r\sin(\alpha)\,i\tau.$$

The map $p_\tau^{(m)} \colon TO \to TO$ multiplying the angles by m (shortly written as m-pling the anlges) is then given by

$$p_\tau^{(m)}(X) \coloneqq r\cos(m\alpha)\,\tau + r\sin(m\alpha)\,i\tau. \tag{18}$$

The next proposition shows how the derivative of a tangent vector field in $SBV(TO)$ changes after m-pling the angles. From this point on, we will denote the approximate gradient of a scalar map $f \in SBV(O)$ by $\mathrm{d}f$ instead of ∇f.

Proposition 4 (Derivative and related quantities after m-pling)**.** *Given a simply connected open set $O \subset S$, let $u \in SBV(TO)$, and let $v \coloneqq p_\tau^{(m)}(u)$ with $p_\tau^{(m)}$ as in (18) for some smooth unit-length vector field $\tau \in C^\infty(TO)$. Then, $|u| \in SBV(O)$ and $v \in SBV(TO)$. The approximate gradient and jump part of v are*

$$\nabla v = |u|^{-1} v \otimes \mathrm{d}|u| + iv \otimes (m|u|^{-2}\,\mathrm{j}(u) - (m-1)\,\mathrm{j}(\tau)), \tag{19}$$

$$D^j v = (p_\tau^{(m)}(u^+) - p_\tau^{(m)}(u^-)) \otimes \nu_u^\flat\ \mathcal{H}_g^1 \llcorner \mathcal{J}_u, \tag{20}$$

where the right-hand side of (19) is implicitly set to be 0 in $\{u = 0\}$. Furthermore, the squared approximate gradients and pre-jacobians transform in the following way:

$$|\nabla v|^2 = m^2|\nabla u|^2 + (1-m^2)|\mathrm{d}|u||^2 + (m-1)^2|u|^2|\mathrm{j}(\tau)|^2 - 2m(m-1)\langle \mathrm{j}(u), \mathrm{j}(\tau)\rangle, \tag{21}$$

$$\mathrm{j}(v) = m\,\mathrm{j}(u) - (m-1)|u|^2\,\mathrm{j}(\tau). \tag{22}$$

Additionally assuming that $u \in L^\infty(TO)$ we have the following relation between the vorticities of u and v:

$$\omega(v) = m\,\omega(u) - (m-1)\,\omega(|u|\tau), \tag{23}$$

with $\omega(\cdot)$ defined distributionally as in (15).

Before proving Proposition 4 we will derive several helpful lemmas.

Lemma 3. *Let $u \in SBV(TO)$ for some open subset $O \subset S$; then, $|u| \in SBV(TO)$. Furthermore, given coordinates $\{x^1, x^2\}$ and an orthonormal frame $\{\tau, i\tau\}$, we have for any $k \in \{1, 2\}$*

$$\mathrm{d}|u|\Big(\frac{\partial}{\partial x^k}\Big) = \frac{1}{|u|}\left(u^1\frac{\partial u^1}{\partial x^k} + u^2\frac{\partial u^2}{\partial x^k}\right) \qquad \text{a.e. in } O, \tag{24}$$

where $u^1 = \langle u, \tau\rangle$, $u^2 = \langle u, i\tau\rangle$, and the expression on the right-hand side of the equality sign is implicitly defined to be 0 in $\{u = 0\}$.

Proof. Without loss of generality we can assume that the coordinates and the frame can be globally defined in O. Let w denote the coordinate representation of $|u|$. As we chose an orthonormal frame we have

$$w = f(u^1, u^2) := \sqrt{(u^1)^2 + (u^2)^2}.$$

Then by Theorem 3.92 (a) from [8] the approximate gradient satisfies $\mathrm{d}u = 0$ a.e. in $\{w = 0\}$. At a.e. point in the remaining set $\{w \neq 0\}$ we derive by the Euclidean chain rule in BV (see also Theorem 3.96 in [8]) that

$$\frac{\partial w}{\partial x^k} = \frac{\partial f}{\partial u^1}(u)\frac{\partial u^1}{\partial x^k} + \frac{\partial f}{\partial u^2}(u)\frac{\partial u^2}{\partial x^k} = \frac{1}{\sqrt{(u^1)^2 + (u^2)^2}}\left(u^1\frac{\partial u^1}{\partial x^k} + u^2\frac{\partial u^2}{\partial x^k}\right).$$

With (13) this directly leads to (24). □

Lemma 4. *Given $m \in \{2, 3, \ldots\}$, the map $p\colon \mathbb{C} \to \mathbb{C}$ defined by $p(z) := \frac{z^m}{|z|^{m-1}}$ for $z \neq 0$ and $p(0) := 0$ is Lipschitz continuous with Lipschitz constant bounded by $2m - 1$.*

Proof. Let z, $w \in \mathbb{C}$. Without loss of generality we can assume that $|z| \geq |w| > 0$. We then compute:

$$\begin{aligned}
|p(z) - p(w)| &= \left|\frac{z^m}{|z|^{m-1}} - \frac{w^m}{|z|^{m-1}} + \frac{w^m}{|z|^{m-1}} - \frac{w^m}{|w|^{m-1}}\right| \\
&\leq \frac{1}{|z|^{m-1}}|z^m - w^m| + \frac{|w|^m}{|z|^{m-1}|w|^{m-1}}||z|^{m-1} - |w|^{m-1}| \\
&= \frac{|z-w|}{|z|^{m-1}}\left|\sum_{k=0}^{m-1} z^k w^{m-1-k}\right| + \frac{1}{|z|^{m-2}}||z| - |w||\left|\sum_{k=0}^{m-2}|z|^k|w|^{m-2-k}\right| \\
&\leq m|z - w| + (m-1)|z - w| = (2m-1)|z - w|,
\end{aligned}$$

as desired. □

Lemma 5 (Product rule in BV)**.** *Let $f \in BV(O)$ for some open subset $O \subset S$ and $u \in C^\infty(T\bar{O})$; then, $fv \in BV(TO)$ with its approximate gradient satisfying*

$$\nabla(fu) = u \otimes \mathrm{d}f + f\nabla u. \tag{25}$$

Proof. Without loss of generality we can assume that O is a coordinate neighborhood. By linearity, the distributional derivative $D(fu)$ is uniquely determined by the values $\int_O \langle fu, \nabla^* v\rangle \,\mathrm{vol}$ for $v \in C_c^\infty(TO \otimes T^*O)$ of the special form $v = X \otimes \alpha$ with $X \in C_c^\infty(TO)$ and $\alpha \in C_c^\infty(T^*O)$. Assume for the moment that $f \in C_c^\infty(O)$. Then, the product rule in (25) holds pointwise. Consequently, taking the scalar product of both sides of (25) with v and integrating by parts leads to

$$\int_O \langle fu, \nabla^* v\rangle \,\mathrm{vol} = \int_O \langle u, X\rangle\langle \mathrm{d}f, \alpha\rangle \,\mathrm{vol} + \int_O \langle f\nabla u, v\rangle \,\mathrm{vol} = \int_O \langle f, \mathrm{d}^*(\langle u, X\rangle\alpha)\rangle + \int_O \langle f\nabla u, v\rangle \,\mathrm{vol}\,.$$

Note that we used that

$$\langle u \otimes \mathrm{d}f, X \otimes \alpha\rangle := \langle u, X\rangle\langle \mathrm{d}f, \alpha\rangle.$$

The second equality above can be extended to any $f \in BV(O)$ by approximation in $L^1(O)$. By the very definition of the total variation in (7) the formula above proves that fv belongs to $BV(O)$. Furthermore, by Riesz representation (see Theorem 1) it follows that

$$\int_O \langle \sigma_{fu}, v\rangle \,\mathrm{d}|D(fu)| = \int_O \langle u \otimes \sigma_f, v\rangle \,\mathrm{d}|Df| + \int_O \langle f\nabla u, v\rangle \,\mathrm{vol}\,.$$

Using the decomposition of Df (see Theorem 3) and the uniqueness of the respective decomposition of $D(fu)$, we derive that the absolutely continuous part of $D(fu)$ with respect to $\mathcal{H}^2_g$ is given by

$$(u \otimes \mathrm{d}f + f\nabla u)\, \mathcal{H}^2_g,$$

as desired. □

Proof of Proposition 4. We can assume that O is a coordinate neighborhood with coordinates $\{x^1, x^2\}$ and an orthonormal frame $\{\tau, i\tau\}$. Let us denote by $\Psi\colon \Omega \times \mathbb{R}^2 \to TO$ the induced local trivialization and let $p_\tau^{(m)}$ be the power map defined in (18). To shorten notation, we will write $\bar{u} := \Psi^* u$ and $\bar{v} := \Psi^* v$.

Step 1 ($v \in SBV(TO)$ and chain rule for the jump part): We can represent $p_\tau^{(m)}$ in coordinates as

$$\bar{p}_\tau^{(m)}(u^1, u^2) = \begin{pmatrix} r\cos(m\alpha) \\ r\sin(m\alpha) \end{pmatrix}, \qquad \text{where} \quad r = \sqrt{(u^1)^2 + (u^2)^2}, \quad \alpha = \arg(u).$$

Here, u^1 and u^2 are the components of u with respect to the frame $\{\tau, i\tau\}$. Furthermore, $\arg(u)$ is the argument of $u^1 + iu^2$. By Lemma 4, the map $\bar{p}_\tau^{(m)}$ is Lipschitz continuous. Since $u \in SBV(TO)$, the Euclidean chain rule in BV and Lemma 1 imply that $v \in SBV(TO)$.

We now wish to prove (20). Fix $x \in \mathcal{J}_u$. By Proposition 2, $\Phi^{-1}(x) \in \mathcal{J}_{\bar{u}}$. Consequently, by the Euclidean chain rule and Lemma 4, $\bar{v}$ has approximate upper and lower limits at $\Phi^{-1}(x)$ given by $\bar{v}^{\pm} = \bar{p}_\tau^{(m)}(\bar{u}^{\pm})$ and approximate normal $\bar{\nu}_{\bar{v}} = \bar{\nu}_{\bar{u}}$. Using Proposition 2 again it follows that v has approximate limits $v^{\pm} = p_\tau^{(m)}(u^{\pm})$ and approximate normal $\nu_v = \nu_u$. By the arbitrariness of x we see that

$$D^j v \llcorner \mathcal{J}_u = (p_\tau^{(m)}(u^+) - p_\tau^{(m)}(u^-)) \otimes \nu_u^\flat\, \mathcal{H}^1_g \llcorner \mathcal{J}_u.$$

Using again the chain rule and the relations between approximate quantities on the manifold and in the Euclidean setting given by Proposition 1 and Proposition 2, from $\mathcal{J}_{\bar{v}} \subset \mathcal{S}_{\bar{u}}$ and $\mathcal{H}^1(\mathcal{S}_{\bar{u}} \setminus \mathcal{J}_{\bar{u}}) = 0$ it follows that $\mathcal{H}^1_g(\mathcal{J}_v \setminus \mathcal{J}_u) = 0$, which together with the equality above shows (20).

Step 2 (Chain rule for the approximate gradient): As $\nabla \bar{u} = 0$ a.e in $\{\bar{u} = 0\}$ (see also Proposition 3.92 a) in [8]) and $\bar{u}$ has approximate limit 0 at a.e. point in $\{\bar{u} = 0\}$, by (13), it follows that $\nabla u = 0$ a.e. in $\{u = 0\}$. Therefore, (19) is satisfied at a.e. point in $\{u = 0\}$.

By $\mathcal{H}^n_g(\mathcal{S}_u) = 0$ (see Theorem 3), and therefore $\mathcal{H}^n(\mathcal{S}_u \setminus \mathcal{S}_v)$, it remains to investigate points $x \in O \setminus \{u = 0\}$ at which v and u are approximately differentiable. Using

$$\frac{\partial}{\partial u^1}(r\cos(m\alpha)) = \frac{u^1}{r}\cos(m\alpha) + m\frac{u^2}{r}\sin(m\alpha), \qquad \frac{\partial}{\partial u^1}(r\sin(m\alpha)) = \frac{u^1}{r}\sin(m\alpha) - m\frac{u^2}{r}\cos(m\alpha),$$
$$\frac{\partial}{\partial u^2}(r\cos(m\alpha)) = \frac{u^2}{r}\cos(m\alpha) - m\frac{u^1}{r}\sin(m\alpha), \qquad \frac{\partial}{\partial u^2}(r\sin(m\alpha)) = \frac{u^2}{r}\sin(m\alpha) + m\frac{u^1}{r}\cos(m\alpha),$$

we see that

$$\begin{aligned} \frac{\partial}{\partial u^1}\bar{p}_\tau^{(m)} &= \frac{u^1}{(u^1)^2 + (u^2)^2}\bar{p}_\tau^{(m)} - m\frac{u^2}{(u^1)^2 + (u^2)^2} i\bar{p}_\tau^{(m)}, \\ \frac{\partial}{\partial u^2}\bar{p}_\tau^{(m)} &= \frac{u^2}{(u^1)^2 + (u^2)^2}\bar{p}_\tau^{(m)} + m\frac{u^1}{(u^1)^2 + (u^2)^2} i\bar{p}_\tau^{(m)}, \end{aligned} \tag{26}$$

where for every $p \in \mathbb{R}^2$ we have written ip for its anticlockwise rotation by $\frac{\pi}{2}$. By the Euclidean chain rule it follows that

$$\frac{\partial \bar{v}}{\partial x^k} = \frac{\partial}{\partial u^1}\bar{p}_\tau^{(m)}(\bar{u})\frac{\partial u^1}{\partial x^k} + \frac{\partial}{\partial u^2}\bar{p}_\tau^{(m)}(\bar{u})\frac{\partial u^2}{\partial x^k}.$$

With the help of (13) and (26) this implies

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^k}} v &= \frac{1}{(u^1)^2+(u^2)^2}\left((u^1 v - m u^2 iv)\frac{\partial u^1}{\partial x^k} + (u^2 v + m u^1 iv)\frac{\partial u^2}{\partial x^k}\right) + v^1 \nabla_{\frac{\partial}{\partial x^k}}\tau + v^2 \nabla_{\frac{\partial}{\partial x^k}}(i\tau)\\ &= \frac{1}{(u^1)^2+(u^2)^2}\left(u^1\frac{\partial u^1}{\partial x^k} + u^2\frac{\partial u^2}{\partial x^k}\right) v + m\frac{1}{(u^1)^2+(u^2)^2}\left(u^1\frac{\partial u^2}{\partial x^k} - u^2\frac{\partial u^1}{\partial x^k}\right) iv\\ &\quad + v^1\nabla_{\frac{\partial}{\partial x^k}}\tau + v^2\nabla_{\frac{\partial}{\partial x^k}}(i\tau),\end{aligned} \tag{27}$$

where we have used

$$v^1(\Gamma^1_{k1}\tau + \Gamma^2_{k1} i\tau) + v^2(\Gamma^1_{k2}\tau + \Gamma^2_{k2} i\tau) = v^1\nabla_{\frac{\partial}{\partial x^k}}\tau + v^2\nabla_{\frac{\partial}{\partial x^k}}(i\tau). \tag{28}$$

In order to find an intrinsic expression of (27), we need to make use of several additional formulas that we derive below. By differentiating the identity $\langle \tau, \tau\rangle = 1$ as well as $\langle i\tau, \tau\rangle = 0$ we have $\langle \nabla_{\frac{\partial}{\partial x^k}}\tau, \tau\rangle = \langle \nabla_{\frac{\partial}{\partial x^k}} i\tau, i\tau\rangle = 0$ and $\langle \nabla_{\frac{\partial}{\partial x^k}}(i\tau), \tau\rangle = -\langle \nabla_{\frac{\partial}{\partial x^k}}\tau, i\tau\rangle$. Hence,

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^k}}\tau &= \langle \nabla_{\frac{\partial}{\partial x^k}}\tau, i\tau\rangle i\tau = \mathrm{j}(\tau)\left(\frac{\partial}{\partial x^k}\right) i\tau,\\ \nabla_{\frac{\partial}{\partial x^k}} i\tau &= -\langle \nabla_{\frac{\partial}{\partial x^k}}\tau, i\tau\rangle \tau = -\,\mathrm{j}(\tau)\left(\frac{\partial}{\partial x^k}\right)\tau.\end{aligned} \tag{29}$$

Hence, we can express the pre-jacobian in coordinates as

$$\begin{aligned}\mathrm{j}(u)\left(\frac{\partial}{\partial x^k}\right) = \langle \nabla_{\frac{\partial}{\partial x^k}} u, iu\rangle &= \left\langle \frac{\partial u^1}{\partial x^k}\tau + \frac{\partial u^2}{\partial x^k} i\tau + u^1\nabla_{\frac{\partial}{\partial x^k}}\tau + u^2\nabla_{\frac{\partial}{\partial x^k}} i\tau, -u^2\tau + u^1 i\tau\right\rangle\\ &= u^1\frac{\partial u^2}{\partial x^k} - u^2\frac{\partial u^1}{\partial x^k} + |u|^2\,\mathrm{j}(\tau)\left(\frac{\partial}{\partial x^k}\right).\end{aligned}$$

Substituting (27) into (24), by the above equality and (29) we have

$$\begin{aligned}\nabla v &= |u|^{-1} v\otimes \mathrm{d}|u| + m|u|^{-2} iv\otimes(\mathrm{j}(u) - |u|^2\,\mathrm{j}(\tau)) + iv\otimes \mathrm{j}(\tau)\\ &= |u|^{-1} v\otimes \mathrm{d}|u| + iv\otimes(m|u|^{-2}\,\mathrm{j}(u) - (m-1)\,\mathrm{j}(\tau)),\end{aligned}$$

as desired.

Step 3 (Chain rule for the squared gradient and pre-jacobian): As already discussed above, we can restrict ourselves to points in $O\setminus\{u=0\}$ at which v, u, and $|u|$ are approximately differentiable. Using (19), $\langle v, iv\rangle = 0$, and $|u| = |v|$ we have that

$$\begin{aligned}|\nabla v|^2 &= |u|^{-2}|v|^2|\mathrm{d}|u||^2 + m^2|u|^{-4}|v|^2|\mathrm{j}(u)|^2 + (m-1)^2|v|^2|\mathrm{j}(\tau)|^2 - 2m(m-1)|u|^{-2}|v|^2\langle \mathrm{j}(u), \mathrm{j}(\tau)\rangle\\ &= |\mathrm{d}|u||^2 + m^2|u|^{-2}|\mathrm{j}(u)|^2 + (m-1)^2|u|^2|\mathrm{j}(\tau)|^2 - 2m(m-1)\langle \mathrm{j}(u), \mathrm{j}(\tau)\rangle,\end{aligned} \tag{30}$$

Let us decompose ∇u into the components parallel and orthogonal to u:

$$\nabla u = |u|^{-2}(u\otimes\langle \nabla u, u\rangle + (iu)\otimes \mathrm{j}(u)). \tag{31}$$

Employing the coordinate representation of ∇u in (13), (28) with v replaced by u, (29), and (24) leads to

$$\begin{aligned}\langle \nabla_{\frac{\partial}{\partial x^k}} u, u\rangle &= \left\langle \frac{\partial u^1}{\partial x^k}\tau + \frac{\partial u^2}{\partial x^k} i\tau + \mathrm{j}(\tau)\left(\frac{\partial}{\partial x^k}\right) u^1 i\tau - \mathrm{j}(\tau)\left(\frac{\partial}{\partial x^k}\right) u^2\tau, u^1\tau + u^2 i\tau\right\rangle\\ &= u^1\frac{\partial u^1}{\partial x^k} + u^2\frac{\partial u^2}{\partial x^k} = |u|\,\mathrm{d}|u|\left(\frac{\partial}{\partial_{x^k}}\right).\end{aligned}$$

Using the above identity in (31), we derive that

$$|\nabla u|^2 = |u|^{-4}(|u|^2|\mathrm{d}|u||^2|u|^2 + |\mathrm{j}(u)|^2|u|^2) = |\mathrm{d}|u||^2 + |u|^{-2}|\mathrm{j}(u)|^2. \tag{32}$$

Hence, (30) implies

$$|\nabla v|^2 = m^2|\nabla u|^2 + (1-m^2)|\mathrm{d}|u||^2 + (m-1)^2|u|^2|\mathrm{j}(\tau)|^2 - 2m(m-1)\langle \mathrm{j}(u), \mathrm{j}(\tau)\rangle,$$

which shows (21).

From the very definition of the pre-jacobian by a direct computation using (19) and $|u| = |v|$ we obtain (22).

Step 4 (chain rule for the vorticity): From this point on, we additionally assume that $u \in L^\infty(TO)$. Using the smoothness of τ, $m \geq 1$, and (21) we can estimate

$$\|\nabla v\|_{L^1} \leq m\|\nabla u\|_{L^1} + (m-1)\|u\|_{L^1}\|\mathrm{j}(\tau)\|_{L^\infty} + 2m(m-1)\|\nabla u\|_{L^1}\|\mathrm{j}(\tau)\|_{L^\infty} < \infty,$$

by which we can define $\omega(v)$ in distributional sense. Let now $\Phi = \varphi\,\mathrm{vol}$ with $\varphi \in C_c^\infty(O)$, by (15) and (22) it follows that

$$\begin{aligned}\omega(v)(\Phi) = \int_O -\mathrm{j}(v)\wedge \mathrm{d}\varphi + \kappa\,\mathrm{vol} &= m\int_O (-\mathrm{j}(u)\wedge\mathrm{d}\varphi + \kappa\,\mathrm{vol}) - (m-1)\int_O \left(-|u|^2\,\mathrm{j}(\tau)\wedge\mathrm{d}\varphi + \kappa\,\mathrm{vol}\right)\\ &= m\,\omega(u)(\Phi) - (m-1)\int_O\left(-|u|^2\,\mathrm{j}(\tau)\wedge\mathrm{d}\varphi + \kappa\,\mathrm{vol}\right). \end{aligned}\tag{33}$$

By the product rule from Lemma 5 applied to $|u|\tau$ and $|\mathrm{d}u|u|| \leq |\nabla u|$ which follows from (32) we can estimate

$$\|\nabla(|u|\tau)\|_{L^1} \leq \|\nabla u\|_{L^1} + \|u\|_{L^1}\|\nabla\tau\|_{L^\infty} < \infty.$$

Hence, $\omega(|u|\tau)$ can be defined in distributional sense. Furthermore, by the same product rule we get that

$$\mathrm{j}(|u|\tau) = \langle \tau\otimes\mathrm{d}|u| + |u|\nabla\tau, i|u|\tau\rangle = |u|^2\,\mathrm{j}(\tau).$$

The above equality combined with (33) gives (23). □

3. Statement of the main result

This section is devoted to the statement of our Γ-convergence result. Let us first define all required objects. Given $m \in \mathbb{N}$, the set of the *admissible spin fields* is

$$\mathcal{AS}^{(m)}(S) := \left\{u \in SBV^2(TS)\colon (u^+)^m = (u^-)^m\ \mathcal{H}^1_g\text{-a.e. on } \mathcal{J}_u\right\}.$$

Here, u^+, u^-, and $\mathcal{J}_u$ are the approximate upper-value, the approximate lower-value, and the approximate jump-set of u (see also Definition 5). Furthermore, for $x \in \mathcal{J}_u$ the expression $(u^+(x))^m$ stands for $p_\tau^{(m)}(u^+(x))$ for some local frame $\{e, ie\}$ in the vicinity of x (see (18) for a definition of $p_\tau^{(m)}$). In this context it is important to note that the condition $(u^+)^m = (u^-)^m$ is independent of the choice of local frame. Note also that, in the case $m = 1$ the spin field u satisfies $u^+ = u^-$ at $\mathcal{H}^1_g$-a.e. point on $\mathcal{J}_u$ and therefore

$$\mathcal{AS}^{(1)}(S) = W^{1,2}(TS).$$

Given $\varepsilon > 0$, the Ginzburg-Landau energy of $u \in \mathcal{AS}^{(m)}$ is defined as

$$GL_\varepsilon(u) := \frac{1}{2}\int_S |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\,\mathrm{vol} + \mathcal{H}^1_g(\mathcal{J}_u). \tag{34}$$

Given $O \subset S$ open, the following two sets of Dirac measures on O will be relevant for the compactness result:

$$\tilde{X}^{(m)}(O) \coloneqq \left\{ \mu = \sum_{k=1}^{K} \frac{d_k}{m}\, \delta_{x_k} \colon K \in \mathbb{N},\, d_k \in \mathbb{Z},\, x_k \in O,\, \mu(S) = \chi(S) \right\},$$
$$X^{(m)}(O) \coloneqq \left\{ \mu = \sum_{k=1}^{K} \frac{d_k}{m}\, \delta_{x_k} \in \tilde{X}^{(m)}(O) \colon d_k \in \{-1, 1\},\, x_k \neq x_l \text{ for } k \neq l \right\},$$

where $\chi(S)$ is the *Euler characteristic* of S. For any Dirac measure $\mu = \sum_{k=1}^{K} d_k\, \delta_{x_k}$ we will from now on implicitly assume that $x_k \neq x_l$ for $k \neq l$. Furthermore, we will denote by $|\mu|(S) \coloneqq \sum_{k=1}^{K} |d_k|$ its total variation in S.

The set $\mathcal{LS}^{(m)}(S)$ of the *limit spin fields* consists exactly of those $u \in SBV(TS)$ such that

(i) $|u| = 1$ a.e. in S;
(ii) $(u^+)^m = (u^-)^m$ $\mathcal{H}^1_g$-a.e. on $\mathcal{J}_u$ and $\mathcal{H}^1_g(\mathcal{J}_u) < \infty$;
(iii) $\omega(u) \in X^{(m)}$ and $u \in SBV^2_{\mathrm{loc}}(S \setminus \operatorname{spt}(\omega(u)); TS)$;
(iv) $|\nabla u| \in L^p(S)$ for all $p \in [1, 2)$.

Let $u \in \mathcal{LS}^{(m)}(S)$ with $\omega(u) = \sum_{k=1}^{K} \frac{d_k}{m}\, \delta_{x_k}$. We define the *renormalized energy* of such u as

$$\mathcal{W}^{(m)}(u) \coloneqq \lim_{r \to 0} \left(\frac{1}{2} \int_{S_r(\omega(u))} |\nabla u|^2 \,\mathrm{vol} - \frac{|\omega(u)|(S)}{m} \pi |\log r| \right), \tag{35}$$

where we have introduced the notation

$$S_r(\omega(u)) \coloneqq S \setminus \bigcup_{k=1}^{K} \bar{B}_r(x_k).$$

Note that we will show in Lemma 15 that the renormalized energy is well-defined, i.e. the limit in (35) exists and belongs to $[-\infty, \infty)$ for all $u \in \mathcal{LS}^{(m)}(S)$.

Let us continue by introducing a minimum problem on Euclidean balls. Given $r > 0$, let $B_r(0)$ denote the Euclidean open ball centered at the origin. For any $v \in W^{1,2}(B_r(0); \mathbb{R}^2)$ we define

$$\overline{GL}_{\varepsilon}^{(m)}(v, B_r(0)) \coloneqq \frac{1}{2m^2} \int_{B_r(0)} |\nabla v|^2 + (m^2 - 1)|\nabla |v||^2 + \frac{m^2}{2\varepsilon^2}(1 - |v|^2)^2 \,\mathrm{d}x. \tag{36}$$

Then, for $\lambda \in \mathbb{S}^1$ let

$$\bar{\gamma}_{\varepsilon}^{(m)}(r, \lambda) \coloneqq \min\left\{ \overline{GL}_{\varepsilon}^{(m)}(v, B_r(0)) \colon v \in W^{1,2}(B_r(0); \mathbb{R}^2),\, v = \lambda \frac{x}{|x|} \text{ on } \partial B_r(0) \right\}, \tag{37}$$

where the product $\lambda \frac{x}{|x|}$ is meant as a product in $\mathbb{C}$. Note that by direct methods, we can show that the minimum in (37) exists. As $\overline{GL}_{\varepsilon}^{(m)}(v, B_r(0)) = \overline{GL}_{\varepsilon}^{(m)}(\tilde{v}, B_r(0))$ for $\tilde{v}(x) \coloneqq \lambda^{-1} v(\varepsilon x)$ and $\tilde{v}$ is admissible for the minimum problem in the definition of $\bar{\gamma}_1^{(m)}(\varepsilon^{-1} r, 1)$ we see that for any $r > 0$, $\varepsilon > 0$, and $\lambda \in \mathbb{S}^1$

$$\bar{\gamma}^{(m)}(\varepsilon^{-1} r) \coloneqq \bar{\gamma}_1^{(m)}(\varepsilon^{-1} r, 1) = \bar{\gamma}_{\varepsilon}^{(m)}(r, \lambda).$$

The following convergence result was proved in [25] (see Lemma 3.9):

Lemma 6. *There exists $\gamma_m \in \mathbb{R}$ such that:*

$$\lim_{R \to \infty} \left(\bar{\gamma}^{(m)}(R) - \frac{\pi}{m^2} |\log(R)| \right) = \gamma_m. \tag{38}$$

Consequently, given $(\lambda_\varepsilon) \subset \mathbb{S}^1$ *and* $(r_\varepsilon) \subset \mathbb{R}_+$ *such that* $\lim_{\varepsilon\to 0} \varepsilon^{-1} r_\varepsilon = \infty$*, we have that*

$$\lim_{\varepsilon\to 0} \left(\bar{\gamma}^{(m)}_\varepsilon(r_\varepsilon, \lambda_\varepsilon) - \frac{\pi}{m^2} \log\left(\frac{r_\varepsilon}{\varepsilon}\right)\right) = \gamma_m. \tag{39}$$

We are ready to state our Γ-convergence result:

Theorem 6 (Γ-convergence)**.** *The following Gamma-convergence result holds true for the sequence of functionals* (GL_ε) *in* (34)*.*

(i) Compactness: Let $(u_\varepsilon) \subset \mathcal{AS}^{(m)}(S)$ *be a bounded sequence in* $L^\infty(TS)$ *such that for all* $\varepsilon > 0$

$$GL_\varepsilon(u_\varepsilon) \le \frac{N}{m}\pi|\log\varepsilon| + C, \tag{40}$$

where $N \in \mathbb{N}$ *and* $C > 0$ *are constants independent of* ε*. Then, there exists* $\mu \in \tilde{X}^{(m)}$ *with* $|\mu| \le N$ *such that, up to subsequences, it holds that*

$$\omega(u_\varepsilon) \rightharpoonup \mu \text{ weakly in } W^{-1,\infty}(S). \tag{41}$$

If $|\mu| = N$ *we can find* $u \in \mathcal{LS}^{(m)}(S)$ *satisfying* $\omega(u) = \mu$ *such that, up to subsequences,*

$$u_\varepsilon \rightharpoonup u \text{ weakly in } SBV^2_{\mathrm{loc}}(S \setminus \operatorname{spt}(\mu); TS) \tag{42}$$

and for all $p \in [1, 2)$

$$u_\varepsilon \rightharpoonup u \text{ weakly in } SBV^p(TS). \tag{43}$$

(ii) Γ*-liminf: Let* $(u_\varepsilon) \subset \mathcal{AS}^{(m)}(S)$ *and* $u \in \mathcal{LS}^{(m)}(S)$ *be such that* $u_\varepsilon \to u$ *in* $L^1(TS)$*. Then,*

$$\liminf_{\varepsilon\to 0} GL_\varepsilon(u_\varepsilon) - \frac{|\omega(u)|}{m}\pi|\log\varepsilon| \ge \mathcal{W}^{(m)}(u) + \mathcal{H}^1_g(\mathcal{J}_u) + m|\omega(u)|\gamma_m. \tag{44}$$

(iii) Γ*-limsup: For any* $u \in \mathcal{LS}^{(m)}(S)$ *there exists a sequence* $(u_\varepsilon) \subset \mathcal{AS}^{(m)}(S)$ *such that* $u_\varepsilon \to u$ *in* $L^1(TS)$ *and*

$$\limsup_{\varepsilon\to 0} GL_\varepsilon(u_\varepsilon) - \frac{|\omega(u)|}{m}\pi|\log\varepsilon| \le \mathcal{W}^{(m)}(u) + \mathcal{H}^1_g(u) + m|\omega(u)|\gamma_m. \tag{45}$$

4. Proof of Γ-Convergence

All constants appearing in this paper are implicitly assumed to be independent of ε and may change from line to line. We also employ standard asymptotic notation. For example $\mathcal{O}_r(1)$ stands for a bounded term as $r \to 0$. Furthermore, we will not explicitly write out ε in our asymptotic notation. For example $\mathrm{o}(1)$ is shorthand for $\mathrm{o}_\varepsilon(1)$.

4.1. Compactness.

In this subsection we will prove the compactness result for our sequence of vector fields with equibounded energy. Given an open set $O \subset S$ we define

$$\mathcal{AS}^{(m)}(O) \coloneqq \left\{u \in SBV^2(TO) \colon (u^+)^m = (u^-)^m \ \mathcal{H}^1_g\text{-a.e. on } \mathcal{J}_u\right\}.$$

and for $u \in \mathcal{AS}^{(m)}(O)$ the localized generalized Ginzburg-Landau energy as

$$GL_\varepsilon(u, O) \coloneqq \frac{1}{2}\int_O |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \operatorname{vol} + \mathcal{H}^1_g(\mathcal{J}_u \cap O).$$

Furthermore, given $v \in C^\infty(TO)$ and a geodesic ball $B \subset S$ the localized degree of v around ∂B in O is given by

$$\mathrm{dg}(v, \partial B; O) := \begin{cases} \deg(v, \partial B) & \text{if } B \subset O \text{ and } |v| \geq \frac{1}{2} \text{ on } \partial B, \\ 0 & \text{else,} \end{cases} \tag{46}$$

where $\deg(v, \partial B)$ is as in (14).

In the proofs of this subsection we will employ the following ball-construction result:

Theorem 7 (Ball construction in an open subset)**.** *For every T, $C > 0$, every integer $n > T - 1$ and every $q \in (0, 1 - T(n+1)^{-1})$, there exist constants ε_0, σ_0, $\tilde{C} > 0$ such that the following holds true: if $\varepsilon \in (0, \varepsilon_0)$, $\sigma \in [\varepsilon^q, \sigma_0]$, and $v \in C^\infty(TO)$, for an open set $O \subset S$ with Lipschitz boundary satisfying the energy upper bound*

$$GL_\varepsilon(v, O) \leq T\pi|\log \varepsilon| + C, \tag{47}$$

there exists $K_\sigma \in \mathbb{N}$ and a finite collection of pairwise disjoint geodesic balls $\mathcal{B}^{(\sigma)} = \{B_k^{(\sigma)}\}_{k=1}^{K_\sigma}$, each one with radius denoted by $r_k^{(\sigma)}$, such that the following properties are satisfied:

- *(i)* $\{x \in O : |v(x)| \leq \frac{1}{2}\} \subset \bigcup_{k=1}^{K_\sigma} B_k^{(\sigma)}$;
- *(ii)* $D_\sigma := \sum_{k=1}^{K_\sigma} |d_k^{(\sigma)}| \leq n$, *where* $d_k^{(\sigma)} := \mathrm{dg}(v, \partial B_k^{(\sigma)}; O)$;
- *(iii)* $\sum_{k=1}^{K_\sigma} r_k^{(\sigma)} \leq (n+1)\sigma$;
- *(iv)* $GL_\varepsilon(v_\varepsilon, B_k^{(\sigma)} \cap O) \geq |d_k^{(\sigma)}|(\pi \log\left(\frac{\sigma}{\varepsilon}\right) - \tilde{C})$ *for* $k = 1, \dots, K_\sigma$.

Note that the above result is a generalization of Proposition 8.2 in [27]. The necessary modifications can be found in Appendix B.

For the readers convenience we recall here an important result from [27] (see also Lemma A.1):

Lemma 7 (Energy lower bound on circles)**.** *Let $u \in C^\infty(TS)$, $\varepsilon > 0$, $x \in O$, and $r \in (\varepsilon, r^*)$. Then,*

$$\frac{1}{2}\int_{\partial B_r(x)} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \,\mathrm{vol} \geq \lambda_\varepsilon\left(\frac{r}{d}\right) \tag{48}$$

for

$$d := \begin{cases} \deg(u, \partial B_r(x)) & \textit{if } \min_{\partial B_r(x)}|u| \geq \frac{1}{2}, \\ \textit{any positive integer} & \textit{else.} \end{cases}$$

Here, $\lambda_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\lambda_\varepsilon(r) \geq \frac{\pi(1 - Cr^2)}{r + C\varepsilon},$$

where C is a universal constant only depending on S.

Lemma 8. *Given an open set $O \subset S$ and $v \in W^{1,1}(TO)$, the following holds true at a.e. point in O:*

$$\nabla v = \begin{cases} |v|^{-1} v \otimes \mathrm{d}|v| + |v|^{-2}(iv) \otimes \mathrm{j}(v), & \textit{if } v \neq 0, \\ 0 & \textit{else.} \end{cases} \tag{49}$$

Proof. By an approximation argument we can reduce ourselves to the case of $v \in C^\infty(TO)$. Furthermore for a.e. $x \in \{x \in O : v(x) = 0\}$ it holds that $\nabla v(x) = 0$. It remains to investigate a points in O where $v \neq 0$. Using the product rule, we have that

$$\nabla v = \nabla(|v||v|^{-1}v) = |v|^{-1} v \otimes \mathrm{d}|v| + |v|\nabla(|v|^{-1}v).$$

Differentiating both sides of $\langle \frac{v}{|v|}, \frac{v}{|v|} \rangle = 1$ we see that $\langle \nabla(|v|^{-1}v), v\rangle = 0$. Hence, by yet another product rule, it follows that

$$\nabla(|v|^{-1}v) = |v|^{-1}(iv) \otimes \langle \nabla(|v|^{-1}v), |v|^{-1}iv\rangle = |v|^{-2}(iv) \otimes \langle \mathrm{d}(|v|^{-1}) \otimes v + |v|^{-1}\nabla v, iv\rangle = |v|^{-3}(iv) \otimes \mathrm{j}(v),$$

which leads to (49). □

From this point on, given a coordinate neighborhood $O \subset S$, we will shortly write

$$C^\infty(\bar{O}; \mathbb{S}^1) = \{\tau \in C^\infty(T\bar{O}) \colon |\tau| = 1 \text{ on } O\}$$

for smooth (up to the boundary) unit-length tangent vector fields on O. We also define the truncation $\mathrm{Tr}_{\frac{1}{2}} \colon TS \to TS$ by

$$\mathrm{Tr}_{\frac{1}{2}}(X) \coloneqq \begin{cases} \frac{X}{|X|}, & |X| \geq \frac{1}{2}, \\ 2X, & \text{else}, \end{cases}$$

for any $X \in TS$.

Lemma 9. *Given an open coordinate neighborhood O, let $\tau \in C^\infty(\bar{O}; \mathbb{S}^1)$ and $(v_\varepsilon) \subset C^\infty(TO)$ be a bounded sequence in $L^\infty(TO)$ such that for all ε*

$$GL_\varepsilon(v_\varepsilon, O) \leq C|\log \varepsilon|. \tag{50}$$

Then,

$$\omega(|v_\varepsilon|\tau) \rightharpoonup 0 \textit{ weakly in } W_0^{-1,\infty}(O). \tag{51}$$

Proof. Step 1: Let $n > \frac{C}{\pi} - 1$ and $q \in (0, 1 - \frac{C}{\pi(n+1)})$, where C is as in (50). For each εwe then apply Theorem 7 to v_ε, where n, q are as above and $\sigma = \sigma_\varepsilon = \varepsilon^q$. Hence, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a finite collection of disjoint closed geodesic balls $\mathcal{B}_\varepsilon = \{B_k^{(\varepsilon)}\}_{k=1}^{K_\varepsilon}$ such that

$$\{x \in O \colon |v_\varepsilon| < \tfrac{1}{2}\} \subset \bigcup_{k=1}^{K_\varepsilon} B_k^{(\varepsilon)}, \tag{52}$$

$$\sum_{k=1}^{K_\varepsilon} r_k^{(\varepsilon)} \leq (n+1)\varepsilon^q, \tag{53}$$

where $r_k^{(\varepsilon)}$ is the radius of $B_k^{(\varepsilon)}$.

Step 2: In O we define $w_\varepsilon \coloneqq |v_\varepsilon|\tau$ and $\tilde{w}_\varepsilon \coloneqq \mathrm{Tr}_{\frac{1}{2}} w_\varepsilon$. Our aim now is to show

$$\omega(w_\varepsilon) - \omega(\tilde{w}_\varepsilon) \rightharpoonup 0 \text{ weakly in } W_0^{-1,\infty}(O). \tag{54}$$

Note that there exists a constant $C > 0$ such that $\mathcal{H}_g^2(B_r(x)) \leq Cr^2$ for all $x \in S$ and $r \in (0, r^*)$. Hence, (51), (52), and $|v_\varepsilon| = |u_\varepsilon|$ we derive that

$$\int_{\{|w_\varepsilon| < \frac{1}{2}\}} |w_\varepsilon|^2 \,\mathrm{vol} \leq \frac{1}{4} \sum_{k=1}^{K_\varepsilon} \mathcal{H}_g^2(B_k^{(\varepsilon)}) \leq C \sum_{k=1}^{K_\varepsilon} (r_k^{(\varepsilon)})^2 \leq C \left(\sum_{k=1}^{K_\varepsilon} r_k^{(\varepsilon)} \right)^2 \leq C(n+1)^2 \varepsilon^{2q}.$$

On the set $\{|w_\varepsilon| \geq 1/2\}$, by the definition of w_ε and $\tilde{w}_\varepsilon$ it follows that

$$|w_\varepsilon - \tilde{w}_\varepsilon|^2 = (1 - |w_\varepsilon|)^2 \leq (1 - |v_\varepsilon|)^2 (1 + |v_\varepsilon|)^2,$$

and therefore, with (50) and the definition of GL_ε

$$\int_{\{|w_\varepsilon|\geq\frac{1}{2}\}} |w_\varepsilon - \tilde{w}_\varepsilon|^2 \operatorname{vol} \leq 4GL_\varepsilon(v_\varepsilon, O)\varepsilon^2 \leq C|\log\varepsilon|\varepsilon^2.$$

Combining the aforementioned estimates it follows that

$$\|w_\varepsilon - \tilde{w}_\varepsilon\|_{L^2}^2 \leq C|\log\varepsilon|\varepsilon^2 + C(n+1)^2\varepsilon^{2q} \leq C\varepsilon^{2q}, \tag{55}$$

where we have used $q < 1$. We next estimate $\|\nabla w_\varepsilon\|_{L^2}$. By the product rule,

$$\nabla w_\varepsilon = \nabla(|v_\varepsilon|\tau) = \tau \otimes \mathrm{d}|v_\varepsilon| + |v_\varepsilon|\nabla\tau.$$

Hence, due to the boundedness of (v_ε) in $L^\infty(TO)$, the smoothness of τ , and $|\mathrm{d}|v_\varepsilon|| \leq |\nabla v_\varepsilon|$ which follows from (49) it holds that $|\nabla w_\varepsilon| \leq |\nabla v_\varepsilon| + C$. With (50) this shows

$$\|\nabla w_\varepsilon\|_{L^2}^2 \leq \int_O |\nabla v_\varepsilon|^2 \operatorname{vol} + C \leq 2GL_\varepsilon(v_\varepsilon, O) + C \leq C|\log\varepsilon|. \tag{56}$$

We will now derive a similar estimate for $\|\nabla\tilde{w}_\varepsilon\|_{L^2}$. At points in O for which $|v_\varepsilon| > \frac{1}{2}$, by the product rule, it holds that

$$\nabla\tilde{w}_\varepsilon = \nabla(|w_\varepsilon|^{-1}w_\varepsilon) = -|w_\varepsilon|^{-2}w_\varepsilon \otimes \mathrm{d}|v_\varepsilon| + |w_\varepsilon|^{-1}\nabla w_\varepsilon.$$

Taking the norm on both sides of the above equation leads to the following bound:

$$|\nabla\tilde{w}_\varepsilon| \leq 2|\mathrm{d}|v_\varepsilon|| + 2|\nabla w_\varepsilon|.$$

At points in O with $|v_\varepsilon| < \frac{1}{2}$ we have $|\nabla\tilde{w}_\varepsilon| = 2|\nabla w_\varepsilon|$. The last two estimates combined with (49) and (56) result in

$$\|\nabla\tilde{w}_\varepsilon\|_{L^2}^2 \leq C|\log\varepsilon|, \tag{57}$$

which eventually gives (54) by using (17) together with (55) and (56).

Step 3: By the previous step, (51) is proved if we show that

$$\omega(\tilde{w}_\varepsilon) \rightharpoonup 0 \text{ weakly in } W_0^{-1,\infty}(O).$$

Fix an arbitrary test-function $\varphi \in W_0^{1,\infty}(O)$ with $\|\mathrm{d}\varphi\|_{L^\infty} \leq 1$. Let $B = B_r(x) \in \mathcal{B}_\varepsilon$ such that $B \subset O$. As $\tilde{w}_\varepsilon = \tau$ on ∂B, by Stoke's theorem and the smoothness of τ

$$\int_B \omega(\tilde{w}_\varepsilon) = \int_{\partial B} \mathrm{j}(\tilde{w}_\varepsilon) = \int_{\partial B} \mathrm{j}(\tau) = \deg(\tau, \partial B) = 0.$$

Hence, by $\|\mathrm{d}\varphi\|_{L^\infty} \leq 1$, (16), (50), and (57) it follows that

$$\begin{aligned}\left|\int_B \varphi\,\omega(\tilde{w}_\varepsilon)\right| &= \left|\varphi(x)\int_B \omega(\tilde{w}_\varepsilon) + \int_B (\varphi(y) - \varphi(x))\,\omega(\tilde{w}_\varepsilon(y))\right| \\ &\leq 2r\int_B |\omega(\tilde{w}_\varepsilon)| \operatorname{vol} \leq 2r\int_B |\nabla\tilde{w}_\varepsilon|^2 + |\kappa| \operatorname{vol} \leq Cr|\log\varepsilon|.\end{aligned}$$

Let us now consider $B = B_r(x) \in \mathcal{B}_\varepsilon$ such that $B \setminus O \neq \emptyset$, instead. Take $x' \in B \cap \partial O$, using that $\varphi(x') = 0$ we can derive , similarly to the previous case, that

$$\left|\int_B \varphi\,\omega(\tilde{w}_\varepsilon)\right| \leq \int_B |\varphi(y) - \varphi(x')||\omega(\tilde{w}_\varepsilon(y))| \operatorname{vol}(y) \leq 2r\int_B |\nabla\tilde{w}_\varepsilon|^2 + |\kappa| \operatorname{vol} \leq Cr|\log\varepsilon|.$$

In $O \setminus \cup_{k=1}^{K_\varepsilon} B_k^{(\varepsilon)}$ we have that $|\tilde{w}_\varepsilon| = 1$ and therefore $\omega(\tilde{w}_\varepsilon) = 0$. Consequently, by (16), (50), and (53) we conclude that

$$\left|\int_O \varphi\, \omega(\tilde{w}_\varepsilon)\right| = \sum_{k=1}^{K_\varepsilon}\left|\int_{B_k^{(\varepsilon)}} \varphi\, \omega(\tilde{w}_\varepsilon)\right| \leq C\left(\sum_{k=1}^{K_\varepsilon} r_k^{(\varepsilon)}\right)|\log\varepsilon| \leq C(n+1)\varepsilon^q|\log\varepsilon|.$$

The weak convergence in (51) follows by the arbitrariness of φ. □

In the next lemma we will derive our initial compactness result for the vorticities.

Lemma 10 (Initial vorticity compactness)**.** *Let $(u_\varepsilon) \subset \mathcal{AS}^{(m)}(S)$ be a bounded sequence in $L^\infty(TS)$ such that for all ε:*

$$GL_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|. \tag{58}$$

Then, there exists a measure $\mu \in \tilde{X}^{(m)}$ such that, up to subsequences,

$$\omega(u_\varepsilon) \rightharpoonup \mu \text{ weakly in } W^{-1,\infty}(S).$$

Proof. Step 1 (locally m-pling the angles): We start by localizing the problem. Let $O \subset S$ be a coordinate neighborhood with smooth boundary. Furthermore, choose an arbitrary $\tau \in C^\infty(\bar{O};\mathbb{S}^1)$ and set $v_\varepsilon := p_\tau^{(m)}(u_\varepsilon)$ in O. By (21) it holds that

$$\begin{aligned}\left|\int_O |\nabla v_\varepsilon|^2 \operatorname{vol}\right| = m^2 \int_O |\nabla u_\varepsilon|^2 \operatorname{vol} + (1-m^2)\int_O |\mathrm{d}|u_\varepsilon||^2 \operatorname{vol} + (m-1)^2\int_O |u_\varepsilon|^2|\mathrm{j}(\tau)|^2 \operatorname{vol}\\ - 2m(m-1)\int_O \langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \operatorname{vol}.\end{aligned}$$

Using Young's inequality we derive

$$\int_O \langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \operatorname{vol} \leq \int_O |u_\varepsilon||\nabla u_\varepsilon||\mathrm{j}(\tau)| \operatorname{vol} \leq \frac{1}{2}\int_O |u_\varepsilon|^2|\nabla u_\varepsilon|^2 \operatorname{vol} + \frac{1}{2}\int_O |\mathrm{j}(\tau)|^2 \operatorname{vol}.$$

Consequently, by the energy bound (58), the boundedness of (u_ε) in $L^\infty(TS)$, and the smoothness of τ it follows that

$$\begin{aligned}GL_\varepsilon(v_\varepsilon, O) &= \frac{1}{2}\int_O |\nabla v_\varepsilon|^2 \operatorname{vol} + \frac{1}{4\varepsilon^2}\int_O (1-|v_\varepsilon|^2)^2 \operatorname{vol}\\ &\leq \frac{(m-1)^2}{2}\|u_\varepsilon\|_{L^\infty}\int_O |\mathrm{j}(\tau)|^2 \operatorname{vol} + \left(\frac{m^2}{2} + m(m-1)\|u_\varepsilon\|_{L^\infty}\right)\int_O |\nabla u_\varepsilon|^2 \operatorname{vol} + \frac{1}{4\varepsilon^2}\int_O (1-|u_\varepsilon|^2)^2 \operatorname{vol}\\ &\leq C(1+GL_\varepsilon(u_\varepsilon)) \leq C|\log\varepsilon|.\end{aligned} \tag{59}$$

Using a standard approximation argument in $W^{1,2}(TO)$ we can assume that $v_\varepsilon \in C^\infty(TO)$ for all ε and (59) holds true with a possibly larger constant C. We now apply Theorem 7 to the sequence (v_ε), where $n > \frac{C}{\pi} - 1$, $q \in (0, 1 - C(\pi(n+1))^{-1})$, and $\sigma = \sigma_\varepsilon = \varepsilon^q$. Hence, there exists $\varepsilon_0 > 0$ such that for all

$\varepsilon \in (0, \varepsilon_0)$ we can find a finite collection of disjoint geodesic balls $\mathcal{B}_\varepsilon = \{B_k^{(\varepsilon)}\}_{k=1}^{K_\varepsilon}$ such that

$$\{x \in O \colon |v_\varepsilon| \leq \tfrac{1}{2}\} \subset \bigcup_{k=1}^{K_\varepsilon} B_k^{(\varepsilon)}, \tag{60}$$

$$D_\varepsilon = \sum_{k=1}^{K_\varepsilon} |d_k^{(\varepsilon)}| \leq n, \quad \text{where } d_k^{(\varepsilon)} \coloneqq \mathrm{dg}(v_\varepsilon, \partial B_k^{(\varepsilon)}, O), \tag{61}$$

$$\sum_{k=1}^{K_\varepsilon} r_k^{(\varepsilon)} \leq (n+1)\varepsilon^q, \tag{62}$$

where $r_k^{(\varepsilon)}$ is the radius of $B_k^{(\varepsilon)}$.

Step 2 (Compactness locally): We will prove the compactness of $(\omega(u_\varepsilon))$ in $W_0^{-1,\infty}(O)$. With the notation from the first step, for $\varepsilon \in (0, \varepsilon_0)$ let $\nu_\varepsilon \in \tilde{X}^{(1)}(O)$ be the measure

$$\nu_\varepsilon \coloneqq \sum_{k=1}^{K_\varepsilon} d_k^{(\varepsilon)} \, \delta_{x_k^{(\varepsilon)}},$$

where $x_k^{(\varepsilon)}$ is the center of the ball $B_k^{(\varepsilon)}$. Using (61) we see that $|\nu_\varepsilon| = D_\varepsilon \leq n < \infty$. Consequently, there exists $\nu \in \tilde{X}(O)$ and a (not relabeled) subsequence (ν_ε) such that $\nu_\varepsilon \overset{*}{\rightharpoonup} \nu$ weakly* in $\mathcal{M}(O)$, in particular

$$\nu_\varepsilon \rightharpoonup \nu \text{ weakly in } W_0^{-1,\infty}(O). \tag{63}$$

Setting $\tilde{v}_\varepsilon \coloneqq \mathrm{Tr}_{\frac{1}{2}} v_\varepsilon$, by (60) and (62), following the proof of Step 2 in Lemma 9 we have that

$$\omega(v_\varepsilon) - \omega(\tilde{v}_\varepsilon) \rightharpoonup 0 \text{ weakly in } W_0^{-1,\infty}(O). \tag{64}$$

Let $\varphi \in W_0^{1,\infty}(O)$ such that $\|\nabla\varphi\|_{L^\infty} \leq 1$. For a ball $B = B_k^{(\varepsilon)} \in \mathcal{B}_\varepsilon$ that is contained in O we see by (60) and (60) that $d_k^{(\varepsilon)} = \deg(v_\varepsilon, \partial B)$. Hence, reasoning as in the proof of Step 3 in Lemma 9, we have

$$\begin{aligned}
\left| \int_B \varphi\, \omega(\tilde{v}_\varepsilon) - \int_B \varphi \, \mathrm{d}\nu_\varepsilon \right| &= \left| \int_B \varphi\, \omega(\tilde{v}_\varepsilon) - d_k^{(\varepsilon)} \varphi(x_k^{(\varepsilon)}) \right| \\
&= \left| \int_B \varphi\, \omega(\tilde{v}_\varepsilon) - \varphi(x_k^{(\varepsilon)}) \int_{\partial B} \mathrm{j}(\tilde{v}_\varepsilon) \right| \\
&\leq \int_B |\varphi(y) - \varphi(x_k^{(\varepsilon)})| |\omega(\tilde{v}_\varepsilon)(y)| \,\mathrm{vol}(y) \\
&\leq r_k^{(\varepsilon)} \int_B |\omega(\tilde{v}_\varepsilon)| \,\mathrm{vol} \leq C r_k^{(\varepsilon)} |\log\varepsilon|.
\end{aligned}$$

Let us now consider a ball $B = B_k^{(\varepsilon)} \in \mathcal{B}_\varepsilon$ such that $B \setminus O \neq \emptyset$. Taking $x' \in \partial O \cap B$ and using $\varphi(x') = d_k^{(\varepsilon)} = 0$ we obtain the same estimate as above:

$$\begin{aligned}
\left| \int_{B\cap O} \varphi\, \omega(\tilde{v}_\varepsilon) - \int_{B\cap O} \varphi \, \mathrm{d}\nu_\varepsilon \right| = \left| \int_{B\cap O} \varphi\, \omega(\tilde{v}_\varepsilon) \right| &\leq \int_{B\cap O} |\varphi(y) - \varphi(x')| |\omega(\tilde{v}_\varepsilon)(y)| \,\mathrm{vol}(y) \\
&\leq 2 r_k^{(\varepsilon)} \int_{B\cap O} |\omega(\tilde{v}_\varepsilon)| \,\mathrm{vol} \leq C r_k^{(\varepsilon)} |\log\varepsilon|.
\end{aligned}$$

Consequently, as $\omega(\tilde{v}_\varepsilon) = 0$ outside $\cup_{k=1}^{K_\varepsilon} B_k^{(\varepsilon)}$, by the arbitrariness of φ and (62) it follows that

$$\omega(\tilde{v}_\varepsilon) - \nu_\varepsilon \rightharpoonup 0 \text{ weakly in } W_0^{-1,\infty}(O).$$

By (63) and (64) $\omega(v_\varepsilon) \rightharpoonup \nu$ weakly in $W_0^{-1,\infty}(O)$. Hence, using (23), Lemma (9), and possibly extracting a further subsequence, we see that

$$\omega(u_\varepsilon) = \frac{1}{m}\,\omega(v_\varepsilon) + \frac{m-1}{m}\,\omega(|v_\varepsilon|\tau) \rightharpoonup \frac{1}{m}\nu =: \mu \in \tilde{X}^{(m)}(O) \text{ weakly in } W_0^{-1,\infty}(O).$$

Step 3 (Partition of unity): The global compactness result then follows by a partition of unity argument. Let $K \in \mathbb{N}$ and let $\{O_k\}_{k=1}^K$ be a finite familyof coordinate neighborhoods with smooth boundary such that $S = \cup_{k=1}^K O_k$. Furthermore, let $\{\rho_k\}_{k=1}^K$ be a smooth partition of unity subordinate to the cover $\{O_k\}_{k=1}^K$. Due to Step 2 we can find for each k a measure $\mu_k \in \tilde{X}^{(m)}(O_k)$ and a (not relabeled) subsequence such that $\omega(u_\varepsilon) \rightharpoonup \mu_k$ weakly in $W_0^{-1,\infty}(O_k)$. Then, for the measure $\mu \coloneqq \sum_{k=1}^K \rho_k\mu_k$ and $\varphi \in W^{1,\infty}(TS)$ we have

$$\int_S \varphi\,\omega(u_\varepsilon) = \sum_{k=1}^K \int_S \rho_k\varphi\,\omega(u_\varepsilon) \to \sum_{k=1}^K \langle \rho_k\mu_k, \varphi\rangle = \langle \mu, \varphi\rangle,$$

where we have used $\rho_k\varphi \in W_0^{1,\infty}(O_k)$. Let us check that $\mu \in \tilde{X}^{(m)}$. For this purpose consider $x \in \operatorname{spt}(\mu_k) \cap O_l$ for $k, l \in \{1, \dots, K\}$ with $k \neq l$. Take $\varphi \in C_c^\infty(B_r(x))$ with $\varphi(x) = 1$, where $r > 0$ is sufficiently small such that $B_r(x) \cap \operatorname{spt}(\mu_k) = \{x\}$, $B_r(x) \cap \operatorname{spt}(\mu_l) \setminus \{x\} = \emptyset$, and $B_r(x) \subset O_k \cap O_l$. As $\varphi \in W_0^{1,\infty}(O_k)$ we have

$$\int_S \varphi\,\omega(u_\varepsilon) = \int_{O_k} \varphi\,\omega(u_\varepsilon) \to \varphi(x)\mu_k(x) = \mu_k(x).$$

As we also have $\varphi \in W_0^{1,\infty}(O_l)$ we can similarly derive that

$$\int_S \varphi\,\omega(u_\varepsilon) \to \mu_l(x),$$

and therefore $\mu_k(x) = \mu_l(x)$. By the arbitrariness of x it follows that $(\mu_k - \mu_l) \llcorner O_k \cap O_l = 0$. By the arbitrariness of k and l, in order to prove $\mu \in \tilde{X}^{(m)}(S)$ we only need to check that $\mu(S) = \chi(S)$. This follows by the $W^{-1,\infty}$-convergence of $(\omega(u_\varepsilon))$ and Morse's index formula (see Theorem 5) since

$$\mu(S) = \lim_{\varepsilon\to 0} \int_S 1\,\omega(u_\varepsilon) = \chi(S).$$

□

In order to improve the above compactness result we will need to employ harmonic vector fields.

Definition 9 (Harmonic vector fields)**.** Let $O \subset S$ be a coordinate neighborhood with Lipschitz boundary. We call a vector field $\tau \in C^\infty(O, \mathbb{S}^1)$ *harmonic* (on O) if and only if $\mathrm{j}(\tau) = \mathrm{d}^*\Phi$ in O, where Φ is the 2-form solving

$$\begin{cases} \Delta\Phi = -\kappa\,\mathrm{vol} & \text{in } O, \\ \Phi = 0 & \text{on } \partial O, \end{cases} \tag{65}$$

$\Delta = \mathrm{d}\,\mathrm{d}^*$ being the *Laplace-Beltrami* operator.

In the next lemma we show the existence of such vector fields.

Lemma 11 (Existence of harmonic unit-length vector fields)**.** *On any coordinate neighborhood $O \subset S$ with Lipschitz boundary, there exists a harmonic vector field τ.*

Proof. Step 1: Let $\Phi \in C^\infty(\Lambda^2 \bar{O})$ solve (65). Fix an arbitrary point $x_0 \in O$ and a unit-length vector $\tau_0 \in T_{x_0}O$. We then define τ at a point $x \in O$ as follows: Let $\gamma\colon [0,1] \to U$ be a smooth curve with $\gamma(0) = x_0$ and $\gamma(1) = x$. Then by classic ODE theory there exists a smooth vector-field $X \in C^\infty(TO)$, such that

$$\begin{cases} \nabla_{\gamma'(s)} X(\gamma(s)) = \mathrm{d}^*\Phi(\gamma'(s)) X^\perp(\gamma(s)) & \text{for } s \in [0,1], \\ X(x_0) = \tau_0. \end{cases} \tag{66}$$

We set $\tau(x) \coloneqq X(x)$. By construction for all $s \in [0,1]$ it follows that

$$\mathrm{d}_{\gamma'(s)} |X(\gamma(s))|^2 = 2\langle \nabla_{\gamma'(s)} X(\gamma(s)), X(\gamma(s))\rangle = 2\,\mathrm{d}^*\Phi(\gamma'(s))\langle X^\perp(\gamma(s)), X(\gamma(s))\rangle = 0.$$

As $|X(x_0)| = |\tau_0| = 1$ this proves $|X(\gamma(s))| = 1$ for all $s \in [0,1]$. In particular, we see that $|\tau(x)| = 1$.

Step 2: Let us now check that this definition of $\tau(x)$ does not depend on the path γ. To this end, consider another smooth curve $\zeta\colon [0,1] \to O$ with $\zeta(0) = x_0$, $\zeta(1) = x$, and let $Y \in C^\infty(TO)$ be a solution of

$$\begin{cases} \nabla_{\zeta'(s)} Y(\zeta(s)) = \mathrm{d}^*\Phi(\zeta'(s)) Y^\perp(\zeta(s)) & \text{for } s \in [0,1], \\ Y(x_0) = \tau_0, \end{cases}$$

As, both, $X(1)$ and $Y(1)$ are of unit length it is sufficient to show that the angle between $X(1)$ and $Y(1)$ is a multiple of 2π. Fixing $\eta \in C^\infty(O; \mathbb{S}^1)$ we can find α, $\beta \in C^\infty([0,1])$ such that

$$X(\gamma(s)) = e^{i\alpha(s)}\eta(\gamma(s)), \qquad Y(\zeta(s)) = e^{i\beta(s)}\eta(\zeta(s)).$$

We then have for any $s \in [0,1]$ that

$$\begin{aligned} \langle \nabla_{\gamma'(s)} X(\gamma(s)), X^\perp(\gamma(s))\rangle &= \langle i e^{i\alpha(s)}\alpha'(s)\eta(\gamma(s)) + e^{i\alpha(s)}\,\mathrm{j}(\eta)(\gamma'(s)) i\eta(\gamma(s)), e^{i\alpha(s)} i\eta(\gamma(s))\rangle \\ &= \alpha'(s) + \mathrm{j}(\eta)(\gamma'(s)), \end{aligned}$$

and similarly

$$\langle \nabla_{\zeta'(s)} Y(\zeta(s)), Y^\perp(\zeta(s))\rangle = \beta'(s) + \mathrm{j}(\eta)(\zeta'(s)).$$

Let $\omega\colon [0,1] \to O$ be the curve:

$$\omega(s) \coloneqq \begin{cases} \gamma(2s) & \text{if } s \in [0, \frac{1}{2}), \\ \zeta(1-2s) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

By the simple connectedness of O the curve η is homologous to 0. Therefore, there exists an integrable function $f\colon O \to \mathbb{Z}$ such that for any 1-form φ on O we have (see Section 5.4 in [27])

$$\int_\eta \varphi = \int_O f\,\mathrm{d}\varphi.$$

Applying the above result for $\varphi = \mathrm{d}^*\Phi + A$ leads to

$$\begin{aligned} \alpha(1) - \beta(1) &= \int_0^1 \langle \nabla_{\gamma'(s)} X(s), iX(s)\rangle\,\mathrm{d}s - \int_0^1 \langle \nabla_{\zeta'(s)} Y(s), iY(s)\rangle\,\mathrm{d}s - \int_\gamma \mathrm{j}(\eta) + \int_\mu \mathrm{j}(\eta) \\ &= \int_\eta \mathrm{d}^*\Phi - \mathrm{j}(\eta) = \int_O f(\Delta\Phi + \kappa\,\mathrm{vol}) = 0 \mod 2\pi. \end{aligned}$$

Consequently, τ is a well-defined unit-length tangent vector field on O. Its smoothness follows from the smoothness of Φ. Lastly, the fact that $\mathrm{j}(\tau) = \mathrm{d}^*\Phi$ is directly implied by (66). □

By possibly decreasing O we can from now on assume without loss of generality that any harmonic vector field we encounter is smooth up to the boundary.

Lemma 12. *Let O be a coordinate neighborhood with smooth boundary, τ a harmonic unit-length vector field in O, and $(v_\varepsilon) \subset \mathcal{AS}^{(1)}(O)$ such that*

$$\omega(v_\varepsilon) \rightharpoonup k\,\delta_x \ \text{ weakly in } W_0^{-1,\infty}(O),$$

where $k \in \mathbb{Z}$, and $x \in O$. Let Φ be the 2-form solving (65) and $\mathrm{j}(\tau) = \mathrm{d}^*\Phi$, then,

$$\int_O \langle \mathrm{j}(v_\varepsilon), \mathrm{j}(\tau)\rangle \,\mathrm{vol} \to k\,(\star\Phi)(x) + \int_O |\mathrm{j}(\tau)|^2 \,\mathrm{vol}\,. \tag{67}$$

Proof. By (65), integration by parts, and the definition of $\omega(v_\varepsilon)$ we derive that

$$\begin{aligned}\int_O \langle \mathrm{j}(v_\varepsilon), \mathrm{j}(\tau)\rangle \,\mathrm{vol} = \int_O \langle \mathrm{j}(v_\varepsilon), \mathrm{d}^*\Phi\rangle \,\mathrm{vol} &= \int_O \langle \omega(v_\varepsilon) - \kappa\,\mathrm{vol}, \Phi\rangle \,\mathrm{vol} + \int_{\partial O} \star\Phi\,\mathrm{j}(v_\varepsilon)\\ &= \int_O (\star\Phi)\,\omega(v_\varepsilon) - \int_O \kappa\,\Phi \to k\,(\star\Phi)(x) - \int_O \kappa\,\Phi\end{aligned}$$

as $\varepsilon \to 0$. To show (67) it is enough to rewrite the last integral on the right-hand side above as follows:

$$\int_O -\kappa\,\Phi = \int_O \langle \mathrm{d}\,\mathrm{d}^*\Phi, \Phi\rangle \,\mathrm{vol} = \int_O |\,\mathrm{d}^*\Phi|^2 \,\mathrm{vol} = \int_O |\mathrm{j}(\tau)|^2 \,\mathrm{vol}\,.$$

□

Lemma 13. *For any $r_0 \in (0, r^*)$ there exists a constant $C > 0$ such that for all $x \in S$, $r \in (0, r_0]$, 2-form Φ solving (65) on $B_r(x)$, and the corresponding harmonic unit-length vector field τ it holds that*

$$\|\Phi\|_{L^\infty} \le C. \tag{68}$$

and

$$\int_{B_r(x)} |\mathrm{j}(\tau)|^2 \,\mathrm{vol} \le C \int_S |\kappa| \,\mathrm{vol}\,. \tag{69}$$

Proof. The bound in (68) follows by standard elliptic theory. Furthermore, by $\mathrm{j}(\tau) = \mathrm{d}^*\Phi$ it holds that

$$\int_{B_r(x)} |\mathrm{j}(\tau)|^2 \,\mathrm{vol} = \int_{B_r(x)} |\mathrm{d}^*\Phi|^2 \,\mathrm{vol} = -\int_{B_r(x)} \kappa\Phi \le C \int_S |\kappa| \,\mathrm{vol},$$

which is (69). □

Lemma 14 (Localized Γ-liminf inequality)**.** *Let $(u_\varepsilon) \subset \mathcal{AS}^{(m)}(S)$ be a bounded sequence in $L^\infty(TS)$ such that*

$$GL_\varepsilon(u_\varepsilon) \le C|\log\varepsilon| \text{ for all } \varepsilon > 0 \quad \text{and} \tag{70}$$

$$\omega(u_\varepsilon) \rightharpoonup \mu \coloneqq \sum_{k=1}^K \frac{d_k}{m}\,\delta_{x_k} \in \tilde{X}^{(m)}(S) \text{ weakly in } W^{-1,\infty}(S). \tag{71}$$

Furthermore, let $r_0 \in (0, r^*)$ be small enough such that the balls $\{B_{r_0}(x_k)\}$ are disjoint. Then, there exist $C \in \mathbb{R}$ such that for every $k \in \{1, \dots, K\}$ and $r \in (0, r_0]$ it holds that

$$\liminf_{\varepsilon\to 0} \left(GL_\varepsilon(u_\varepsilon, B_r(x_k)) - \frac{\pi|d_k|}{m^2} \log\left(\frac{r}{\varepsilon}\right)\right) \ge C. \tag{72}$$

Proof. Fix k and $r \in (0, r_0]$ and shortly write $B := B_r(x_k)$. Furthermore, let $v_\varepsilon := p_\tau^{(m)}(u_\varepsilon)$, where τ is a harmonic unit-length vector field on B. By (21), the boundedness of (u_ε) in L^∞, and (69) we derive that

$$\begin{aligned} GL_\varepsilon(u_\varepsilon, B) &\geq \frac{1}{4\varepsilon^2}\int_B (1-|v_\varepsilon|^2)^2 \operatorname{vol} + \frac{1}{2m^2}\int_B |\nabla v_\varepsilon|^2 \operatorname{vol} + \frac{m^2-1}{2m^2}\int_B |\mathrm{d}|u_\varepsilon||^2 \operatorname{vol} \\ &\quad - \frac{(m-1)^2}{2m^2}\int_B |u_\varepsilon|^2 |\mathrm{j}(\tau)|^2 \operatorname{vol} - \frac{m-1}{m}\int_B \langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \operatorname{vol} \\ &\geq \frac{1}{m^2} GL_\varepsilon(v_\varepsilon, B) - \|u_\varepsilon\|_{L^\infty}^2 \int_B |\mathrm{j}(\tau)|^2 \operatorname{vol} - \frac{m-1}{m}\int_B \langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \operatorname{vol} \\ &\geq \frac{1}{m^2} GL_\varepsilon(v_\varepsilon, B) - C - \frac{m-1}{m}\int_B \langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \operatorname{vol} \end{aligned} \tag{73}$$

for some constant C independent of ε and r. By (22) and (69) we also have that

$$\begin{aligned} -\frac{m-1}{m}\int_B \langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \operatorname{vol} &= -\frac{m-1}{m^2}\int_B \langle \mathrm{j}(v_\varepsilon), \mathrm{j}(\tau)\rangle \operatorname{vol} - \frac{(m-1)^2}{m^2}\int_B |u_\varepsilon|^2 |\mathrm{j}(\tau)|^2 \operatorname{vol} \\ &\geq -\frac{m-1}{m^2}\int_B \langle \mathrm{j}(v_\varepsilon), \mathrm{j}(\tau)\rangle \operatorname{vol} - C. \end{aligned} \tag{74}$$

Note that by (70), (71), Lemma 9, and (23)

$$\omega(v_\varepsilon) = m\,\omega(u_\varepsilon) - (m-1)\,\omega(|v_\varepsilon|\tau) \rightharpoonup d_k\,\delta_{x_k}$$

weakly in $W_0^{-1,\infty}(B)$ and hence by Lemma 12, (74), (67), (68), and (69) it follows that

$$\liminf_{\varepsilon\to 0} -\frac{m-1}{m}\int_B \langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \operatorname{vol} \geq -\frac{m-1}{m^2} d_k\,(\star\Phi)(x_k) - \frac{m-1}{m^2}\int_B |\mathrm{j}(\tau)|^2 - C \geq -C. \tag{75}$$

Recalling Proposition 4(ii) in [19] stating that under our assumptions

$$\liminf_{\varepsilon\to 0}\Big(GL_\varepsilon(v_\varepsilon, B_r(x_k)) - \pi|d_k|\log\Big(\frac{r}{\varepsilon}\Big)\Big) \geq -C,$$

for a constant C independent of r, by (73) and (75) we eventually obtain (72). □

Proposition 5 (Improved vorticity compactness)**.** *Let $(u_\varepsilon) \subset \mathcal{AS}^{(m)}$ be a bounded sequence in $L^\infty(TS)$ such that for all ε*

$$GL_\varepsilon(u_\varepsilon) \leq \frac{N}{m}\pi|\log\varepsilon| + C, \tag{76}$$

where $N \in \mathbb{N}$ and assume that $\omega(u_\varepsilon) \rightharpoonup \mu \in \tilde{X}^{(m)}(S)$ weakly in $W^{-1,\infty}(S)$. Then, $|\mu| \leq N$. Moreover, if $|\mu| = N$, then $\mu \in X^{(m)}(S)$.

Proof. Step 1 (Upper bound on $|\omega(u)|$): Let us write $\mu = \sum_{k=1}^K \frac{d_k}{m}\delta_{x_k}$. Furthermore, let $r_0 \in (0, r^*)$ be chosen sufficiently small that the balls $\{B_{r_0}(x_k)\}$ are disjoint. We will shortly write B_k for the ball $B_{r_0}(x_k)$. By (76) and Lemma 14 there exist $\tilde{C}$and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\frac{\pi N}{m}|\log\varepsilon| + C \geq \sum_{k=1}^K GL_\varepsilon(u_\varepsilon, B_k) \geq \sum_{k=1}^K \frac{\pi|d_k|}{m^2}\log\Big(\frac{r_0}{\varepsilon}\Big) - K\tilde{C} \geq \frac{\pi|\mu|}{m}|\log\varepsilon| - K\tilde{C} - \frac{\pi|\mu|}{m}|\log r_0|.$$

Taking $\varepsilon \to 0$, this can only hold true if $|\mu| \leq N$.

Step 2 (Case $|\omega(u)| = N$) : Let us assume that $|\mu| = N$. We will show that this implies $|d_k| = 1$ for all k and hence $\mu \in X^{(m)}(S)$. For fixed k let $v_\varepsilon := p_\tau^{(m)}(u_\varepsilon)$ in B_k for a harmonic unit-length vector field τ on B_k. From (21), the boundedness of (u_ε) in L^∞, and (69) we derive that

$$\begin{aligned} GL_\varepsilon(v_\varepsilon, B_k) &= \frac{1}{4\varepsilon^2}\int_{B_k}(1-|u_\varepsilon|^2)^2 \,\mathrm{vol} + \frac{m^2}{2}\int_{B_k}|\nabla u_\varepsilon|^2 \,\mathrm{vol} + \frac{1-m^2}{2}\int_{B_k}|\mathrm{d}|u_\varepsilon||^2 \,\mathrm{vol} \\ &\quad + \frac{(m-1)^2}{2}\int_{B_k}|u_\varepsilon|^2|\mathrm{j}(\tau)|^2 \,\mathrm{vol} - m(m-1)\int_{B_k}\langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \,\mathrm{vol} \\ &\le m^2 GL_\varepsilon(u_\varepsilon, B_k) - m(m-1)\int_{B_k}\langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \,\mathrm{vol} + C. \end{aligned} \tag{77}$$

Furthermore, using (22), we can write

$$\begin{aligned} -m(m-1)\int_{B_k}\langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \,\mathrm{vol} &= -(m-1)\int_{B_k}\langle \mathrm{j}(v_\varepsilon), \mathrm{j}(\tau)\rangle \,\mathrm{vol} - (m-1)^2\int_{B_k}|u_\varepsilon|^2|\mathrm{j}(\tau)|^2 \,\mathrm{vol} \\ &\le -(m-1)\int_{B_k}\langle \mathrm{j}(v_\varepsilon), \mathrm{j}(\tau)\rangle \,\mathrm{vol}\,. \end{aligned}$$

Hence, by Lemma 12 it holds that

$$\limsup_{\varepsilon\to 0} -m(m-1)\int_{B_k}\langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \,\mathrm{vol} \le -(m-1)\left(\star\Phi(x_k) + \int_{B_k}|\mathrm{j}(\tau)|^2 \,\mathrm{vol}\right).$$

By (77) and Lemma 13 we have that

$$GL_\varepsilon(v_\varepsilon, B_k) \le m^2 GL_\varepsilon(u_\varepsilon, B_k) + C. \tag{78}$$

As the balls $\{B_l\}$ are disjoint we derive using $|\mu| = N$ and Lemma 14 the following upper bound on $GL_\varepsilon(u_\varepsilon, B_k)$ for ε small enough:

$$\begin{aligned} GL_\varepsilon(u_\varepsilon, B_k) &\le GL_\varepsilon(u_\varepsilon) - \sum_{l\neq k} GL_\varepsilon(u_\varepsilon, B_l) \\ &\le \pi\frac{N}{m}|\log\varepsilon| - \pi\sum_{l\neq k}\frac{|d_l|}{m^2}\log\left(\frac{r_0}{\varepsilon}\right) + C \\ &\le \pi\frac{N}{m}|\log\varepsilon| - \pi\frac{|\mu| - \frac{|d_k|}{m}}{m}|\log\varepsilon| + C = \pi\frac{|d_k|}{m^2}|\log\varepsilon| + C. \end{aligned} \tag{79}$$

Using this bound in (78) we see that

$$GL_\varepsilon(v_\varepsilon) \le \pi|d_k||\log\varepsilon| + C. \tag{80}$$

Following the by now standard arguments in the proof of Theorem B (i) in [19] (see also Theorem 5.3 (i) in [4]) this bound can only hold true for $\varepsilon \to 0$ if $|d_k| = 1$, as desired. □

From now on we define $A_{r,r'}(x) := B_{r'}(x) \setminus B_r(x)$ for any $x \in S$ and $0 < r < r' < r^*$.

Lemma 15 (Well-definedness of the renormalized energy)**.** *Let $u \in \mathcal{LS}^{(m)}(S)$ be such that $\omega(u) = \sum_{k=1}^K \frac{d_k}{m}\delta_{x_k} \in X^{(m)}$. Then, the limit in (35) exists and belongs to $(-\infty, \infty]$. Moreover, for $r_0 \in (0, r^*)$*

sufficiently small that the balls $\{B_{r_0}(x_k)\}_k$ *are disjoint we have that*

$$\mathcal{W}^{(m)}(u) = \frac{1}{2}\int_{S_{r_0}(\omega(u))} |\nabla u|^2 \operatorname{vol} + \sum_{k=1}^{K} \frac{1}{m^2}\mathcal{W}(v^{(k)}, B_{r_0}(x_k)) - \sum_{k=1}^{K} \frac{(m-1)^2}{2m^2}\int_{B_{r_0}(x_k)} |\mathrm{j}(\tau^{(k)})|^2 \operatorname{vol} + \sum_{k=1}^{K} \frac{m-1}{m}\int_{B_{r_0}(x_k)} \langle \mathrm{j}(v^{(k)}), \mathrm{j}(\tau^{(k)})\rangle \operatorname{vol},$$

where $\tau^{(k)} \in C^\infty(\bar{B}; \mathbb{S}^1)$*,* $v^{(k)} \coloneqq p_{\tau^{(k)}}(u)$*, and*

$$\mathcal{W}(v^{(k)}, B_{r_0}(x_k)) = \lim_{r\to 0}\left(\frac{1}{2}\int_{A_{r,r_0}(x_k)} |\nabla v^{(k)}|^2 \operatorname{vol} - \pi|\log r|\right). \tag{81}$$

Proof. For any $0 < r < r_0$ we can write

$$\begin{aligned} \frac{1}{2}\int_{S_r(\omega(u))} |\nabla u|^2 \operatorname{vol} - \pi\frac{|\omega(u)|(S)}{m}|\log r| &= \frac{1}{2}\int_{S_r(\omega(u))} |\nabla u|^2 \operatorname{vol} - \pi\frac{K}{m^2}|\log r| \\ &= \frac{1}{2}\int_{S_{r_0}(\omega(u))} |\nabla u|^2 \operatorname{vol} + \sum_{k=1}^{K}\left(\frac{1}{2}\int_{A_{r,r_0}(x_k)} |\nabla u|^2 \operatorname{vol} - \frac{\pi}{m^2}|\log r|\right). \end{aligned}$$

Therefore, to prove the existence of the limit in (35), it is enough to show that for every k

$$\lim_{r\to 0}\frac{1}{2}\int_{A_{r,r_0}(x_k)} |\nabla u|^2 \operatorname{vol} - \frac{\pi}{m^2}|\log r| \in (-\infty, \infty]. \tag{82}$$

Let us fix k and set $B \coloneqq B_{r_0}(x_k)$ and $A_r \coloneqq A_{r,r_0}(x_k)$. Furthermore, let $\tau \in C^\infty(\bar{B}; \mathbb{S}^1)$ and $v \coloneqq p_\tau^{(m)}(u)$ in B. Then, by (21) and $|u| = 1$ a.e. in S we get that

$$\begin{aligned} \frac{1}{2}\int_{A_r} |\nabla u|^2 \operatorname{vol} - \frac{\pi}{m^2}|\log r| = \frac{1}{m^2}\left(\frac{1}{2}\int_{A_r} |\nabla v|^2 \operatorname{vol} - \pi|\log r|\right) - \frac{(m-1)^2}{2m^2}\int_{A_r} |\mathrm{j}(\tau)|^2 \operatorname{vol} \\ + \frac{m-1}{m}\int_{A_r} \langle \mathrm{j}(v), \mathrm{j}(\tau)\rangle \operatorname{vol}. \end{aligned}$$

Since $u \in \mathcal{LS}^{(m)}(S)$ is such that $\nabla u \in L^1(TS)$, by (21)we have that $\nabla v \in L^1(TB)$, which together with the smoothness of τ leads to

$$\begin{aligned} &\lim_{r\to 0}\left(\frac{m-1}{m}\int_{A_r} \langle \mathrm{j}(v), \mathrm{j}(\tau)\rangle \operatorname{vol} - \frac{(m-1)^2}{2m^2}\int_{A_r} |\mathrm{j}(\tau)|^2 \operatorname{vol}\right) \\ &\quad = \frac{m-1}{m}\int_{B} \langle \mathrm{j}(v), \mathrm{j}(\tau)\rangle \operatorname{vol} - \frac{(m-1)^2}{2m^2}\int_{B} |\mathrm{j}(\tau)|^2 \operatorname{vol}. \end{aligned}$$

Finally, (82) follows thanks to

$$\lim_{r\to 0}\left(\frac{1}{2}\int_{A_r} |\nabla v|^2 \operatorname{vol} - \pi|\log r|\right) \in (-\infty, \infty]$$

whose proof can be found in [19] (see Subsection 6.1). □

Lemma 16 (Localization of a unit vortex)**.** *Let $O \subset S$ be an open set and $(v_\varepsilon) \subset C^\infty(TO)$ a bounded sequence in $L^\infty(TO)$ such that for all ε:*

$$GL_\varepsilon(v_\varepsilon, O) \le \pi|\log\varepsilon| + C.$$

Furthermore, assume that for $x_0 \in O$ and $d \in \{-1, 1\}$

$$\omega(v_\varepsilon) \rightharpoonup d\,\delta_{x_0} \text{ weakly in } W_0^{-1,\infty}(O).$$

Then, there exists $\varepsilon_0 > 0$ and $\tilde{C} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we can find $x_\varepsilon \in O$ such that

(i) $\lim_{\varepsilon\to 0} x_\varepsilon = x_0$*;*
(ii) $\deg(v_\varepsilon, \partial B_r(x_\varepsilon)) = d$ *for $r > 0$ such that $\partial B_r(x_\varepsilon) \subset O$ and $|v_\varepsilon| \ge \frac{1}{2}$ on $\partial B_r(x_\varepsilon)$;*
(iii) $GL_\varepsilon(v_\varepsilon, B_{r_\varepsilon}(x_\varepsilon)) \ge \pi\log(\frac{r_\varepsilon}{\varepsilon}) - \tilde{C}$*, where $r_\varepsilon = 2\varepsilon^{\frac{1}{3}}$.*

Proof. We start by applying Theorem 7 with $T = 1$, $n = 1$, and $q = \frac{1}{3}$. Hence, taking ε_0 small enough, we can find for each $\varepsilon \in (0, \varepsilon_0)$ a finite family of closed geodesic balls $\mathcal{B}^{(\varepsilon)} = \{B_k^{(\varepsilon)}\}_{k=1}^{K_\varepsilon}$ satisfying condition (i)-(iv) from Theorem 7. Let us now define the measure

$$\nu_\varepsilon := \sum_k d_k^{(\varepsilon)}\,\delta_{x_k^{(\varepsilon)}},$$

where $x_k^{(\varepsilon)}$ is the center of $B_k^{(\varepsilon)}$ and $d_k^{(\varepsilon)} := \deg(v_\varepsilon, \partial B_k^{(\varepsilon)})$. By Theorem 7 (ii) we have that $|\nu_\varepsilon| \le 1$. Hence, up to taking a subsequence, $\nu_\varepsilon \overset{*}{\rightharpoonup} \mu$ weakly* in $\mathcal{M}(O)$. Arguing as in the proof of Lemma 10 the whole sequence (ν_ε) weakly* converges towards $\mu = d\,\delta_{x_0}$. By possibly decreasing ε_0 this assures that for any $\varepsilon \in (0, \varepsilon_0)$ we have that $B_{r_\varepsilon}(x_\varepsilon) \subset O$ and we can find a unique $k_\varepsilon \in \{1, \dots, K_\varepsilon\}$ such that $d_{k_\varepsilon}^{(\varepsilon)} = 1$ while $d_k^{(\varepsilon)} = 0$ for all $k \in \{1, \dots, K_\varepsilon\} \setminus \{k_\varepsilon\}$. With Theorem 7 (i) this shows (ii) in the statement. Furthermore, by Theorem 7 (iv) we see that

$$GL_\varepsilon(v_\varepsilon, B_{r_\varepsilon}(x_\varepsilon)) \ge \pi\log\frac{r_\varepsilon}{2\varepsilon} - \tilde{C} = \pi\log\frac{r_\varepsilon}{\varepsilon} - \pi\log(2) - \tilde{C},$$

which shows (iii) in the statement, after redefining $\tilde{C}$. □

Lemma 17. *Let $(u_\varepsilon) \subset \mathcal{AS}^{(m)}(S)$ be a bounded sequence in $L^\infty(TS)$ such that for all ε*

$$GL_\varepsilon(u_\varepsilon) \le \frac{N}{m}\pi|\log\varepsilon| + C \tag{83}$$

for constants $N \in \mathbb{N}$, $C > 0$ independent of ε. Suppose that

$$\omega(u_\varepsilon) \rightharpoonup \mu := \sum_{k=1}^{mN} \frac{d_k}{m}\,\delta_{x_k} \in X^{(m)} \text{ weakly in } W^{-1,\infty}(S).$$

Furthermore, set $r_0 \in (0, r^*)$ to be sufficiently small that the balls $\{B_{r_0}(x_k)\}_k$ are disjoint. Then, for any $p \in [1, 2)$ and $r \in (0, r_0)$ it holds that

$$\sup_\varepsilon \big(\|\nabla u_\varepsilon\|_{L^2(T^*S_r(\mu)\otimes TS_r(\mu))} + \mathcal{H}_g^1(\mathcal{J}_{u_\varepsilon} \cap S_r(\mu)\big) < \infty, \tag{84}$$

$$\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^p(T^*S\otimes TS)} < \infty. \tag{85}$$

Proof. Step 1 (L^2-bound outside vortices): Let $r \in (0, r_0)$. By Lemma 14 and the energy bound in (83) we have that for ε small enough

$$
\begin{aligned}
GL_\varepsilon(u_\varepsilon, S_r(\mu)) &\le GL_\varepsilon(u_\varepsilon) - \sum_{k=1}^{mN} GL_\varepsilon(u_\varepsilon, B_r(x_k)) \\
&\le \frac{N}{m}\pi|\log\varepsilon| - \sum_{k=1}^{mN} \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) + C \le \frac{N}{m}\pi|\log r| + C \le C|\log r|.
\end{aligned}
$$

By the definition of GL_ε this leads to (84).

Step 2 (Partitioning of $B_{r_0}(x_k)$): For (85) it now suffices to prove for all k that

$$\sup_\varepsilon \int_{B_{r_0}(x_k)} |\nabla u_\varepsilon|^p \,\mathrm{vol} < \infty. \tag{86}$$

For this purpose, let us fix k and shortly write $B \coloneqq B_{r_0}(x_k)$. Furthermore, let τ be a harmonic unit-length vector field on B and $v_\varepsilon \coloneqq p_\tau^{(m)}(u_\varepsilon)$ in B. By (19), (20), and the definition of $\mathcal{AS}^{(m)}(S)$, we have that $v_\varepsilon \in W^{1,2}(TB)$. Our goal now is to show that the energy bound on (v_ε) and the assumption $|d_k| = 1$ gives

$$\sup_\varepsilon \int_B |\nabla v_\varepsilon|^p \,\mathrm{vol} < \infty. \tag{87}$$

The above estimate will be achieved by partitioning B and eventually estimating the L^p-norm of ∇v_ε on each component of the partition, separately. Given $\tilde r_0 \coloneqq \frac{3}{4} r_0$ we define $J_\varepsilon \in \mathbb{N}$ as the largest natural number satisfying $2^{-J_\varepsilon}\tilde r_0 > 2\varepsilon^{\frac{1}{3}}$. By this choice we have that

$$J_\varepsilon \le -1 + \frac{1}{\log(2)}\left(\frac{1}{3}|\log\varepsilon| - \log\tilde r_0\right) \le C|\log\varepsilon|. \tag{88}$$

As in the proof of Proposition (5) we can show that

$$GL_\varepsilon(v_\varepsilon, B) \le \pi|\log\varepsilon| + C. \tag{89}$$

By the weak convergence of $(\omega(u_\varepsilon))$, (23), and Lemma 9 it follows that

$$\omega(v_\varepsilon) \rightharpoonup d_k\,\delta_{x_k} \ \text{ weakly in } W_0^{-1,\infty}(B). \tag{90}$$

Since, using a standard approximation argument, we can assume that v_ε is smooth. We are now in a position to apply Lemma 16 for the sequence (v_ε) and for $O = B$. Let $(x_k^{(\varepsilon)})_\varepsilon$ denote the resulting sequence of vortex centers. For $j \in \{0, \dots, J_\varepsilon\}$ we then set

$$B_j^{(\varepsilon)} \coloneqq B_{2^{-j}\tilde r_0}(x_k^{(\varepsilon)})$$

and for $j \in \{0, \dots, J_\varepsilon - 1\}$

$$A_j^{(\varepsilon)} = B_j^{(\varepsilon)} \setminus B_{j+1}^{(\varepsilon)}.$$

With the above notation we partition B as follows:

$$B = B_{J_\varepsilon}^{(\varepsilon)} \cup \bigcup_{j=0}^{J_\varepsilon - 1} A_j^{(\varepsilon)} \cup (B \setminus B_0^{(\varepsilon)}).$$

Step 3 (Bound on $B^{(\varepsilon)}_{J_\varepsilon}$): We estimate the L^p-norm of ∇v_ε on the set $B^{(\varepsilon)}_{J_\varepsilon}$. By the maximality of J_ε we must have $2^{-(J_\varepsilon+1)}\tilde r_0 \le 2\varepsilon^{\frac13}$, and therefore $\mathcal{H}^2_g(B^{(\varepsilon)}_{J_\varepsilon}) \le C\varepsilon^{\frac23}$. Hölder's inequality and the energy bound (83) then lead to

$$\int_{B^{(\varepsilon)}_{J_\varepsilon}} |\nabla v_\varepsilon|^p \operatorname{vol} \le \mathcal{H}^2_g(B^{(\varepsilon)}_{J_\varepsilon})^{\frac{2}{2-p}} \int_B |\nabla v_\varepsilon|^2 \operatorname{vol} \le C\varepsilon^{\frac{4}{3(2-p)}}|\log\varepsilon| = \mathrm{o}(1).$$

Step 4 (Bound on $B\setminus B^{(\varepsilon)}_0$): By Lemma 16 (i) we have $|x_k - x^{(\varepsilon)}_k| \le \frac14 r_0$ for sufficiently small ε. Then, by our choice of $\tilde r_0$ for all such ε it holds that $B_{\frac{r_0}{2}}(x_k) \subset B^{(\varepsilon)}_0 \subset B$. By Hölder's inequality and (84) it follows that

$$\int_{B\setminus B^{(\varepsilon)}_0} |\nabla v_\varepsilon|^p \operatorname{vol} \le \int_{B\setminus B_{\frac{r_0}{2}}(x_k)} |\nabla v_\varepsilon|^p \operatorname{vol} \le \mathcal{H}^2_g(B)^{\frac{2}{2-p}} \int_{B\setminus B_{\frac{r_0}{2}}(x_k)} |\nabla v_\varepsilon|^2 \operatorname{vol} = \mathcal{O}(1).$$

Step 5 (Bound on $\bigcup_{j=0}^{J_\varepsilon-1} A^{(\varepsilon)}_j$): Using (89), Lemma 16 (iii) , and the maximality of J_ε we have that

$$\begin{aligned} GL_\varepsilon(v_\varepsilon, B\setminus B^{(\varepsilon)}_{J_\varepsilon}) &= GL_\varepsilon(v_\varepsilon, B) - GL_\varepsilon(v_\varepsilon, B^{(\varepsilon)}_{J_\varepsilon}) \\ &\le \pi|\log\varepsilon| - \pi\log\left(\frac{2\varepsilon^{\frac13}}{\varepsilon}\right) + C \le -\pi\log(2^{-(J_\varepsilon+1)}\tilde r_0) + C \le \pi J_\varepsilon \log(2) + C. \end{aligned} \tag{91}$$

Furthermore, by Lemma 16 (ii) , (48), and $2^{-j}\tilde r_0 \ge 2^{-J_\varepsilon}\tilde r_0 \ge 2\varepsilon^{\frac13}$ for any $j \in \{0,\dots,J_\varepsilon-1\}$ we can estimate

$$\begin{aligned} GL_\varepsilon(v_\varepsilon, A^{(\varepsilon)}_j) &\ge \int_{2^{-(j+1)}\tilde r_0}^{2^{-j}\tilde r_0} \frac{\pi(1-Cr^2)}{r+C\varepsilon}\,\mathrm{d}r \\ &\ge \int_{2^{-(j+1)}\tilde r_0}^{2^{-j}\tilde r_0} \frac{\pi}{r+C\varepsilon} - Cr\,\mathrm{d}r \\ &\ge \pi\log\left(\frac{2^{-j}\tilde r_0 + C\varepsilon}{2^{-(j+1)}\tilde r_0 + C\varepsilon}\right) - C2^{-2j}\tilde r_0^2 \\ &\ge \pi\log(2) - \pi\log\left(1 + C\frac{\varepsilon}{2^{-j}\tilde r_0}\right) - C2^{-2j}\tilde r_0^2 \ge \pi\log(2) - C(\varepsilon^{\frac23} + 2^{-2j}\tilde r_0^2) \end{aligned}$$

Fix $j' \in \{0,\dots,J_\varepsilon-1\}$. We are now able to prove boundedness of $GL_\varepsilon(v_\varepsilon, A^{(\varepsilon)}_{j'})$ independently of ε. By (91) and the above estimate it follows that

$$\begin{aligned} (J_\varepsilon-1)\pi\log(2) - C\Big((J_\varepsilon-1)\varepsilon^{\frac23} + \tilde r_0^2\sum_{j\ne j'} 2^{-2j}\Big) &\le \sum_{j\ne j^*} GL_\varepsilon(v_\varepsilon, A^{(\varepsilon)}_j) \\ &\le GL_\varepsilon(v_\varepsilon, B\setminus B^{(\varepsilon)}_{J_\varepsilon}) - GL_\varepsilon(v_\varepsilon, A^{(\varepsilon)}_{j'}) \\ &\le J_\varepsilon\pi\log(2) + C - GL_\varepsilon(v_\varepsilon, A^{(\varepsilon)}_{j'}). \end{aligned}$$

Solving for $GL_\varepsilon(v_\varepsilon, A^{(\varepsilon)}_{j'})$ and using (88) then leads to

$$\begin{aligned} GL_\varepsilon(v_\varepsilon, A^{(\varepsilon)}_{j'}) &\le \pi\log(2) + C\Big(1 + J_\varepsilon \varepsilon^{\frac{2}{3}} + \tilde r_0^2 \sum_{j\neq j'} 2^{-2j}\Big) \\ &\le \pi\log(2) + C\Big(1 + \varepsilon^{\frac{2}{3}}|\log\varepsilon| + \tilde r_0^2 \sum_{j=0}^{\infty} 2^{-2j}\Big) = \mathcal{O}(1). \end{aligned}$$

Consequently, by the arbitrariness of j' and Hölder's inequality we derive for $p\in[1,2)$ that

$$\begin{aligned} \sum_{j=0}^{J_\varepsilon-1}\int_{A^{(\varepsilon)}_j} |\nabla v_\varepsilon|^p \operatorname{vol} &\le \sum_{j=0}^{J_\varepsilon-1} \mathcal{H}^2_g(A^{(\varepsilon)}_j)^{\frac{2}{2-p}} \int_{A^{(\varepsilon)}_j} |\nabla v_\varepsilon|^2 \operatorname{vol} \\ &\le 2\sum_{j=0}^{J_\varepsilon-1} \mathcal{H}^2_g(B^{(\varepsilon)}_j)^{\frac{2}{2-p}} GL_\varepsilon(v_\varepsilon, A^{(\varepsilon)}_j) \le C\tilde r_0^{\frac{4}{2-p}} \sum_{j=0}^{\infty} (2^{-\frac{4}{2-p}})^j = \mathcal{O}(1). \end{aligned}$$

Finally (87) holds true by combining the estimates from the previous three steps.

Step 6 (Proof of (86)): Let us first show that

$$\sup_\varepsilon \int_B |\mathrm{d}|u_\varepsilon||^2 \operatorname{vol} < \infty. \tag{92}$$

Thanks to (89) and (90) we can apply Lemma 14 to (v_ε) with $m=1$. Exploiting also (21), Lemma 12, (83), the boundedness of (u_ε) in L^∞, and repeating the argument for the proof of (79) we obtain the estimate

$$\begin{aligned} \pi|\log\varepsilon| - C &\le \int_B \frac{1}{2}|\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2 \operatorname{vol} \\ &= \int_B \frac{m^2}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \operatorname{vol} + \frac{1-m^2}{2}\int_B |\mathrm{d}|u_\varepsilon||^2 \operatorname{vol} \\ &\quad + \frac{(m-1)^2}{2}\int_B |u_\varepsilon|^2 |\mathrm{j}(\tau)|^2 \operatorname{vol} - m(m-1)\int_B \langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \operatorname{vol} \\ &\le m^2 GL_\varepsilon(u_\varepsilon, B) + \tilde C - \frac{m^2-1}{2}\int_B |\mathrm{d}|u_\varepsilon||^2 \operatorname{vol} \\ &\le \pi|\log\varepsilon| + \tilde C - \frac{m^2-1}{2}\int_B |\mathrm{d}|u_\varepsilon||^2 \operatorname{vol}. \end{aligned}$$

Consequently, (92) follows. Furthermore, by (19)we see that

$$|u_\varepsilon|^{-2}\,\mathrm{j}(u_\varepsilon)\otimes i v_\varepsilon = \frac{1}{m}(\nabla v_\varepsilon - |u|^{-1}\,\mathrm{d}|u_\varepsilon|\otimes v_\varepsilon + (m-1)\,\mathrm{j}(\tau)\otimes(i v_\varepsilon)).$$

Hence, by the triangular inequality, (87), and the boundedness of (v_ε) in L^∞ it follows for any $p\in[1,2)$ that

$$\||u_\varepsilon|^{-1}\,\mathrm{j}(u_\varepsilon)\|_{L^p(T^*B)} \le \frac{1}{m}(\|\nabla v_\varepsilon\|_{L^p(T^*B\otimes TB)} + \|\mathrm{d}u_\varepsilon\|_{L^p(T^*B)} + C) = \mathcal{O}(1). \tag{93}$$

Finally, combining (92), (93), and (31) leads to (86). □

We are ready to prove our main compactness result.

Proof of Theorem 6 (i). Let $r_0 \in (0, r^*)$ be chosen small enough that the balls $\{B_{r_0}(x_k)\}_k$ are disjoint. The existence of $\mu \in \tilde{X}^{(m)}(S)$ such that (41) holds true follows by combining Lemma 10 and Proposition 5. Let us assume that $|\mu| = N$. Thanks to Proposition 5 we know that μ belongs to $X^{(m)}(S)$.

Step 1 (SBV^2_{loc}-compactness): For any $r \in (0, r_0)$, by (84), the boundedness of (u_ε) in L^∞, and Theorem 4 there exists $u \in SBV^2(S_r(\mu))$ such that, up to taking a subsequence, $u_\varepsilon \rightharpoonup u$ weakly in $SBV^2(S_r(\mu))$. Then, via a standard diagonal argument (42) follows. Suppose from this point on that we have extracted a subsequence, without relabeling, such that (42) holds true. Given $p \in [1, 2)$, by (85) we have that $\nabla u_\varepsilon \rightharpoonup G$ weakly in $L^p(T^*S \otimes TS)$. As by (42) $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^p(T^*S_r(\mu) \otimes TS_r(\mu))$ for $r \in (0, r_0)$, we derive that $\nabla u = G$ a.e. in S. This shows that $\nabla u \in L^p(T^*S \otimes TS)$ for any $p \in [1, 2)$.

Step 2 (Pointwise properties of u): We continue by showing that u has unit length. By the energy-bound in (40) and the definition of GL_ε we derive for any $r \in (0, r_0)$

$$\int_{S_r(\mu)} (1 - |u_\varepsilon|)^2 \,\text{vol} \leq 4\varepsilon^2 GL_\varepsilon(u_\varepsilon, S) \leq C\varepsilon^2 |\log \varepsilon| = \text{o}(1).$$

By the strong convergence of (u_ε) in $L^2(TS_r(\mu))$, up to subsequences (not relabeled) we have that $u_\varepsilon \to u$ pointwise a.e. in $S_r(\mu)$. Then, by the previous estimate, the boundedness of (u_ε) in $L^\infty(TS)$, and the dominated convergence theorem, it follow that

$$\int_{S_r(\mu)} (1 - |u|)^2 \,\text{vol} = \lim_{\varepsilon \to 0} \int_{S_r(\mu)} (1 - |u_\varepsilon|)^2 \,\text{vol} = 0.$$

and therefore $|u| = 1$ a.e. in $S_r(\mu)$, and in S due to the arbitrariness of r. Let us now consider a coordinate neighborhood $O \subset S_r(\mu)$ and arbitrary $\tau \in C^\infty(\bar{O}; \mathbb{S}^1)$. We set $v_\varepsilon := p_\tau^{(m)}(u_\varepsilon)$. By the definition of $\mathcal{AS}^{(m)}(S)$ and (20) applied to u_ε we have

$$v_\varepsilon^+ = p_\tau^{(m)}(u_\varepsilon^+) = p_\tau^{(m)}(u_\varepsilon^-) = v_\varepsilon^- \qquad \mathcal{H}^1_g\text{-a.e. on } \mathcal{J}_{u_\varepsilon} \cap O$$

and, therefore, $v_\varepsilon \in W^{1,1}(TO)$. Using the boundedness of (u_ε) in $L^\infty(TS)$, (84), and (21) applied to u_ε we derive that $\sup_\varepsilon \|v_\varepsilon\|_{W^{1,2}(TO)} < \infty$. Consequently, we can find $v \in W^{1,2}(TO)$ such that, up to taking a subsequence, $v_\varepsilon \to v$ pointwise a.e. in O. We have already shown that $u_\varepsilon \to u$ pointwise a.e. in $S_r(\mu)$, up to subsequences. By the continuity of $p_\tau^{(m)}$ this leads to $v = p_\tau^{(m)}(u)$ a.e. in O. Hence, applying (20) to u, and using $v \in W^{1,2}(TO)$ we see that

$$p_\tau^{(m)}(u^+) = v^+ = v^- = p_\tau^{(m)}(u^-) \qquad \mathcal{H}^1_g\text{-a.e. on } \mathcal{J}_u \cap O.$$

By the arbitrariness of O and r the above property extends to S.

Step 3 ($\omega(u) = \mu$): We will now relate the vorticity of u with the limit μ in 41. Note that by $\|\nabla u\|_{L^1} < \infty$ and $|u| = 1$ it follows that $\omega(u)$ is well defined in distributional sense. By (85) and the fact that $|\text{j}(u_\varepsilon)| \leq |\nabla u_\varepsilon||u_\varepsilon|$ there exists $j \in L^1(T^*S)$ such that, up to taking a subsequence, $\text{j}(u_\varepsilon) \rightharpoonup j$ weakly in $L^1(T^*S)$. We will now show that $j = \text{j}(u)$ a.e. in S. Let $r \in (0, r_0)$ and $\varphi \in L^\infty(T^*S_r(\mu))$, then

$$\begin{aligned}
\int_{S_r(\mu)} \langle \varphi, \text{j}(u_\varepsilon) - \text{j}(u) \rangle \,\text{vol} &= \int_{S_r(\mu)} \langle \varphi, \langle \nabla u_\varepsilon, i(u_\varepsilon - u) \rangle \rangle \,\text{vol} + \int_{S_r(\mu)} \langle \varphi, \langle \nabla u_\varepsilon - \nabla u, iu \rangle \rangle \,\text{vol} \\
&= \int_{S_r(\mu)} \langle \nabla u_\varepsilon, i(u_\varepsilon - u) \otimes \varphi \rangle \,\text{vol} + \int_{S_r(\mu)} \langle \nabla u_\varepsilon - \nabla u, iu \otimes \varphi \rangle \,\text{vol}\,.
\end{aligned}$$

By (42) we have that $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(T^*S_r(\mu) \otimes TS_r(\mu))$ and $u_\varepsilon \to u$ in $L^2(TS_r(\mu))$. Hence, by weak-strong convergence, both integrals in the last line above converge to 0 as $\varepsilon \to 0$. Furthermore, the weak convergence of $(j(u_\varepsilon))$ in $L^1(T^*S)$ implies that

$$\lim_{\varepsilon\to 0}\int_{S_r(\mu)}\langle\varphi, \mathrm{j}(u_\varepsilon)-\mathrm{j}(u)\rangle \,\mathrm{vol} = \int_{S_r(\mu)}\langle\varphi, j-\mathrm{j}(u)\rangle \,\mathrm{vol}\,.$$

The arbitrariness of φ and r shows that $j=\mathrm{j}(u)$ a.e. in S. Using the weak convergence of $(\mathrm{j}(u_\varepsilon))$ we derive for any $\varphi\in W^{1,\infty}(TS)$ that

$$\langle\omega(u),\varphi\rangle = \int_S \mathrm{d}\varphi\wedge\mathrm{j}(u) = \lim_{\varepsilon\to 0}\int_S \mathrm{d}\varphi\wedge\mathrm{j}(u_\varepsilon) = \lim_{\varepsilon\to 0}\langle\omega(u_\varepsilon),\varphi\rangle = \langle\mu,\varphi\rangle$$

which gives $\omega(u)=\mu$ by the arbitrariness of φ.

Step 4 (Finite jump): In order to prove that $u\in\mathcal{LS}^{(m)}$ it remains to show $\mathcal{H}^1_g(\mathcal{J}_u)<\infty$. By the same argument as in the proof of Proposition 5 we can show that for any $r\in(0,r_0)$

$$GL_\varepsilon(u_\varepsilon, S_r(\mu)) \le \pi\frac{N}{m}|\log r| + C$$

for some constant C independent of r and ε. Solving the above inequality for $\mathcal{H}^1_g(\mathcal{J}_\varepsilon)$ we derive that

$$\mathcal{H}^1_g(\mathcal{J}_{u_\varepsilon}\cap S_r(\mu)) \le \pi\frac{N}{m}|\log r| + C - \frac{1}{2}\int_{S_r(\mu)}|\nabla u_\varepsilon|^2 \,\mathrm{vol}\,.$$

Then, by (42)

$$\begin{aligned}\mathcal{H}^1_g(\mathcal{J}_u\cap S_r(\mu)) &\le \liminf_{\varepsilon\to 0}\mathcal{H}^1_g(\mathcal{J}_{u_\varepsilon}\cap S_r(\mu)) \le C + \pi\frac{N}{m}|\log r| + \limsup_{\varepsilon\to 0} -\frac{1}{2}\int_{S_r(\mu)}|\nabla u_\varepsilon|^2 \,\mathrm{vol}\\ &\le C - \left(\frac{1}{2}\int_{S_r(\mu)}|\nabla u|^2 \,\mathrm{vol} - \pi\frac{N}{m}|\log r|\right).\end{aligned}$$

With $\mathcal{W}^{(m)}(u)>-\infty$ (see Lemma 15), it then follows:

$$\mathcal{H}^1_g(\mathcal{J}_u) = \limsup_{r\to 0}\mathcal{H}^1(\mathcal{J}_u\cap S_r) \le C - \mathcal{W}^{(m)}(u) < \infty,$$

as desired. From the inequality above we can also derive that

$$\limsup_{\varepsilon\to 0}\mathcal{H}^1_g(\mathcal{J}_{u_\varepsilon}) \le C - \mathcal{W}^{(m)}(u) < \infty.$$

Hence, by possibly selecting a subsequence we can assume that $\sup_\varepsilon \mathcal{H}^1_g(\mathcal{J}_{u_\varepsilon})<\infty$, which, combined with (85) and the boundedness of (u_ε) in L^∞ leads to (43) thanks to Theorem 4. □

4.2. **Γ-liminf.** In this section we will prove the lim inf-inequality of Theorem 6 (ii). For any open set $O\subset S$ we define the modified Ginzburg-Landau energy $GL_\varepsilon^{(m)}\colon \mathcal{AS}^{(1)}(O)\to\mathbb{R}$ as:

$$GL_\varepsilon^{(m)}(v) = GL_\varepsilon^{(m)}(v,O) \coloneqq \frac{1}{2m^2}\int_O |\nabla v|^2 + (m^2-1)|\mathrm{d}|v||^2 + \frac{m^2}{2\varepsilon^2}(1-|v|^2)^2 \,\mathrm{vol}\,. \tag{94}$$

Note that, the functional above is the natural candidate to keep track of the energy concentration in our setting. More precisely, let $O=B$ be a geodesic ball with radius $r\in(0,r^*)$, $u\in\mathcal{AS}^{(m)}(B)$, $\tau\in C^\infty(\bar{B};\mathbb{S}^1)$, and set $v\coloneqq p_\tau^{(m)}(u)\in\mathcal{AS}^{(1)}(B)$. Then, by (21)

$$GL_\varepsilon(u) = GL_\varepsilon^{(m)}(v,B) - \frac{m^2-1}{m^2}\int_B |u_\varepsilon|^2|\mathrm{j}(\tau)|^2 \,\mathrm{vol} + \frac{2m(m-1)}{m^2}\int_B\langle\mathrm{j}(u_\varepsilon),\mathrm{j}(\tau)\rangle \,\mathrm{vol}, \tag{95}$$

where the latter two terms will turn out to be negligible for small balls.

Remark 2. Throughout this subsection, given $x_0 \in S$ and $r \in (0, r^*)$, Ψ will stand for a local trivialization of TS corresponding to centered (at x_0) normal coordinates on $B_r(x_0)$ with chart denoted by Φ and an auxiliary orthonormal frame $\{\tau_1, \tau_2\}$ on $TB_r(x_0)$ (smooth up to the boundary). Objects such as g^{ij}, $\sqrt{|g|}$, Γ_{ij}^k, etc. will always correspond to the above choice of coordinates. For an arbitrary section v of $TB_r(x_0)$ we will write $\Psi^* v$ for its coordinate representation.

Note that, under this assumptions, the following holds true:

$$g^{ij} = \delta^{ij} + \mathcal{O}(r), \quad \Gamma_{ij}^k = \mathcal{O}(r), \quad \sqrt{|g|} = 1 + \mathcal{O}(r). \tag{96}$$

Lemma 18. *Let $v \in W^{1,2}(TB_{r_0}(x_0)) \cap L^\infty(TB_{r_0}(x_0))$ for $x_0 \in S$ and $r_0 \in (0, r^*)$. Furthermore, let Ψ be a local trivialization of $TB_{r_0}(x_0)$ as described in Remark 2. Then, for any $r \in (0, r_0)$ it holds that*

$$\int_{B_r(x_0)} |\nabla v|^2 \operatorname{vol} = (1 + \mathcal{O}(r)) \int_{B_r(0)} |\nabla(\Psi^* v)|^2 \, \mathrm{d}x + \mathcal{O}(r) \|v\|_{L^\infty}^2, \tag{97}$$

$$GL_\varepsilon^{(m)}(v, B_r(x_0)) = (1 + \mathcal{O}(r)) \overline{GL}_\varepsilon^{(m)}(\Psi^* v, B_r(0)) + \mathcal{O}(r) \|v\|_{L^\infty}^2, \tag{98}$$

where all $\mathcal{O}(r)$-terms are independent of v and $\overline{GL}_\varepsilon^{(m)}$ is as in (36).

Proof. For the sake of shorter notation we will write $\bar{v}$ instead of $\Psi^* v$. By a standard approximation procedure we can assume without loss of generality that v is smooth. Furthermore, by the equivalence of norms, we have that $\|\bar{v}\|_{L^\infty} \le C \|v\|_{L^\infty}$. Hence, using (96) and Young's inequality it follows that

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla v|^2 \operatorname{vol} &= \int_{B_r(0)} \sum_{k=1}^2 \left(\frac{\partial \bar{v}^k}{\partial x^i} + \Gamma_{il}^k \bar{v}^l \right) \left(\frac{\partial \bar{v}^k}{\partial x^j} + \Gamma_{jl'}^k \bar{v}^{l'} \right) g^{ij} \sqrt{|g|} \, \mathrm{d}x \\
&= \int_{B_r(0)} |\nabla \bar{v}|^2 + \mathcal{O}(1) |\bar{v}| (|\bar{v}| + |\nabla \bar{v}|) \, \mathrm{d}x + \mathcal{O}(r) \int_{B_r(0)} |\bar{v}|^2 + |\nabla \bar{v}|^2 \, \mathrm{d}x \\
&= \int_{B_r(0)} |\nabla \bar{v}|^2 + \mathcal{O}(r^{-1}) |\bar{v}|^2 + \mathcal{O}(r) |\nabla \bar{v}|^2 \, \mathrm{d}x + \mathcal{O}(r) \int_{B_r(0)} |\bar{v}|^2 + |\nabla \bar{v}|^2 \, \mathrm{d}x \\
&= (1 + \mathcal{O}(r)) \int_{B_r(0)} |\nabla \bar{v}|^2 \, \mathrm{d}x + \mathcal{O}(r^{-1}) \int_{B_r(0)} |\bar{v}|^2 \, \mathrm{d}x = (1 + \mathcal{O}(r)) \int_{B_r(0)} |\nabla \bar{v}|^2 \, \mathrm{d}x + \mathcal{O}(r) \|v\|_{L^\infty}^2,
\end{aligned}$$

which shows (97). Using the above result together with (96) we can similarly show (98). □

Let us now consider $v \in W^{1,2}(TB_r(x_0))$ for some $x_0 \in S$ and $r_0 \in (0, r^*)$. Then, given any $r \in (0, r_0)$, we define

$$\mathrm{d}_{\bar{\mathcal{H}}}(v, r) \coloneqq \inf_{z \in \bar{\mathcal{H}}} \left\{ r^{-1} \|\Psi^* v - z\|_{L^2(B_r(0); \mathbb{R}^2)} + \|\nabla(\Psi^* v) - \nabla z\|_{L^2(B_r(0); \mathbb{R}^{2 \times 2})} \right\},$$

where

$$\bar{\mathcal{H}} \coloneqq \left\{ v \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \colon v(x) = \lambda \frac{x}{|x|} \text{ for some } \lambda \in \mathbb{S}^1 \right\}.$$

Lemma 19. *Let $(r_\varepsilon) \subset (0, r^*)$ with $\lim_{\varepsilon \to 0} r_\varepsilon = 0$, $(x_\varepsilon) \subset S$ and $(v_\varepsilon) \subset W^{1,2}(TA_\varepsilon)$, where $A_\varepsilon \coloneqq A_{\frac{r_\varepsilon}{2}, r_\varepsilon}(x_\varepsilon)$. We assume that $\sup_\varepsilon \|v_\varepsilon\|_{L^\infty(A_\varepsilon)} < \infty$, $\deg(v_\varepsilon, \partial B_{r_\varepsilon}(x_\varepsilon)) = s \in \{-1, 1\}$,*

$$\lim_{\varepsilon \to 0} r_\varepsilon^{-2} \int_{A_\varepsilon} (1 - |v_\varepsilon|)^2 \operatorname{vol} = 0, \tag{99}$$

and

$$\mathrm{d}_{\bar{\mathcal{H}}}(v_\varepsilon, r_\varepsilon) \ge \delta \tag{100}$$

for some $\delta > 0$. Then, there exists $\omega(\delta) > 0$ such that

$$\frac{1}{2}\int_{A_\varepsilon} |\nabla v_\varepsilon|^2 \,\mathrm{vol} \geq \pi\log(2) + \omega(\delta) + \mathrm{o}(1).$$

Proof. Without loss of generality we can assume that $s = 1$, as the other case follows by a similar argument. Suppose, by contradiction, that up to a subsequence

$$\limsup_{\varepsilon\to 0} \frac{1}{2}\int_{A_\varepsilon} |\nabla v_\varepsilon|^2 \,\mathrm{vol} \leq \pi\log(2). \tag{101}$$

Let $\bar{A}_\varepsilon := A_{\frac{r_\varepsilon}{2}, r_\varepsilon}(0)$ be the Euclidean annulus corresponding to A_ε and $\bar{v}_\varepsilon := \Psi^* v_\varepsilon$. Note that by the equivalence of norms and our assumptions on (v_ε)

$$\sup_\varepsilon \Big\{ \|\bar{v}_\varepsilon\|_{L^\infty(\bar{A}_\varepsilon)} + \int_{\bar{A}_\varepsilon} |\nabla \bar{v}_\varepsilon|^2 \,\mathrm{d}x \Big\} < \infty.$$

As $\bar{r}_\varepsilon \to 0$, with (97) and the contradiction assumption (101) we have

$$\limsup_{\varepsilon\to 0} \int_{\bar{A}_\varepsilon} |\nabla \bar{v}_\varepsilon|^2 \,\mathrm{d}x \leq \pi\log(2). \tag{102}$$

We rescale each $\bar{v}_\varepsilon$ to a vector-field $\bar{w}_\varepsilon$ defined on the unit annulus $\bar{A}_1$. More precisely, we set $\bar{w}_\varepsilon(x) := \bar{v}_\varepsilon(r_\varepsilon x)$ for $x \in \bar{A}_1$. From (102) and a change of coordinates it follows that

$$\limsup_{\varepsilon\to 0} \int_{\bar{A}_1} |\nabla \bar{w}_\varepsilon|^2 \,\mathrm{d}x = \limsup_{\varepsilon\to 0} \int_{\bar{A}_\varepsilon} |\nabla \bar{v}_\varepsilon|^2 \,\mathrm{d}x \leq \pi\log(2).$$

Together with the boundedness of $(\bar{w}_\varepsilon)$ in L^∞, this implies that, up to selecting a subsequence, $\bar{w}_\varepsilon \rightharpoonup \bar{w}$ weakly in $W^{1,2}(\bar{A}_1;\mathbb{R}^2)$ with $\bar{w}$ satisfying

$$\frac{1}{2}\int_{\bar{A}_1} |\nabla \bar{w}|^2 \,\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int_{\bar{A}_1} |\nabla \bar{w}_\varepsilon|^2 \,\mathrm{d}x \leq \pi\log(2). \tag{103}$$

Furthermore, by (99) and a change of coordinates we also have

$$\int_{\bar{A}_1} (1-|\bar{w}_\varepsilon|)^2 \,\mathrm{d}x = r_\varepsilon^{-2}\int_{\bar{A}_\varepsilon} (1-|\bar{v}_\varepsilon|)^2 \,\mathrm{d}x \leq C r_\varepsilon^{-2}\int_{A_\varepsilon} (1-|v_\varepsilon|)^2 \,\mathrm{vol} \to 0,$$

as $\varepsilon \to 0$, and therefore $|\bar{w}| = 1$ a.e. in $\bar{A}_1$. Finally, by the continuity of the degree with respect to weak convergence in $W^{1,2}$, it follows that $\deg(\bar{w}, \partial B_1(0)) = 1$. Combining this result with (103) gives that $\bar{w} \in \bar{\mathcal{H}}$ thanks to Remark 5.2 in [4]. By (103), recalling that $\frac{1}{2}\int |\nabla v| \,\mathrm{d}x \geq \pi\log(2)$ for any $v \in \bar{\mathcal{H}}$, we have

$$\lim_{\varepsilon\to 0} \int_{\bar{A}_1} |\nabla \bar{w}_\varepsilon|^2 \,\mathrm{d}x = \int_{\bar{A}_1} |\nabla \bar{w}|^2 \,\mathrm{d}x,$$

By the weak convergence of $(\bar{w}_\varepsilon)$ this leads to $\bar{w}_\varepsilon \to \bar{w}$ strongly in $W^{1,2}(\bar{A}_1;\mathbb{R}^2)$. Hence, changing coordinates and using the definition of $\mathrm{d}_{\bar{\mathcal{H}}}$ we derive that

$$\begin{aligned}
\mathrm{d}^2_{\bar{\mathcal{H}}}(v_\varepsilon, r_\varepsilon) &\leq r_\varepsilon^{-2}\int_{\bar{A}_\varepsilon} |\bar{v}_\varepsilon(x) - \bar{w}(x)|^2 \,\mathrm{d}x + \int_{\bar{A}_\varepsilon} |\nabla \bar{v}_\varepsilon(x) - \nabla \bar{w}(x)|^2 \,\mathrm{d}x \\
&= \int_{\bar{A}_1} |\bar{w}_\varepsilon(x) - \bar{w}(x)|^2 \,\mathrm{d}x + \int_{\bar{A}_1} |\nabla \bar{w}_\varepsilon(x) - \nabla \bar{w}(x)|^2 \,\mathrm{d}x \to 0
\end{aligned}$$

as $\varepsilon \to 0$, which is a contradiction to (100) for ε small enough. □

Proof of Theorem 6 (ii). Let us first select a subsequence (without relabeling) such that

$$\liminf_{\varepsilon\to 0}\left(GL_\varepsilon(u_\varepsilon)-\pi\frac{N}{m}|\log\varepsilon|\right)=\lim_{\varepsilon\to 0}\left(GL_\varepsilon(u_\varepsilon)-\pi\frac{N}{m}|\log\varepsilon|\right).$$

Note that given any $w\in\mathcal{AS}^{(m)}$, the truncation $\hat{w}:=\min\{1,|w|^{-1}\}w$ has lower energy: $GL_\varepsilon(\hat{w})\le GL_\varepsilon(w)$. Consequently, without loss of generality, we can assume that $\sup_\varepsilon\|u_\varepsilon\|_{L^\infty}\le 1$. Furthermore, it is not restrictive to suppose that

$$GL_\varepsilon(v_\varepsilon)\le\pi\frac{N}{m}|\log\varepsilon|+C$$

for some constant C independent of ε since otherwise (44) trivially follows. By Theorem 6 (i), we can select a subsequence, again without relabeling, such that

$$u_\varepsilon\rightharpoonup u \text{ weakly in } SBV^2_{\mathrm{loc}}(S\setminus\operatorname{spt}(\mu);TS), \tag{104}$$

where $\mu=\omega(u)=\sum_{k=1}^{mN}\frac{s_k}{m}\,\delta_{x_k}$ with $s_k\in\{-1,1\}$ for all k. Let $r_0\in(0,r^*)$ be small enough such that the balls in $\{B_{r_0}(x_k)\}_k$ are pairwise disjoint and for $r\in(0,r_0)$ recall the definition of $S_r(\mu):=S\setminus\bigcup_k B_{r_0}(x_k)$.

Step 1 (Lower bound in $S_r(\mu)$): We first wish to derive the Γ-$\liminf$ inequality in $S_r(\mu)$ for $r\in(0,r_0)$. By (104), standard lower semicontinuity arguments, the definition of $\mathcal{W}^{(m)}(u)$, and the fact that $\mathcal{H}^1(J_u)<\infty$, it holds that

$$\begin{aligned}\liminf_{\varepsilon\to 0}GL_\varepsilon(u_\varepsilon,S_r(\mu))&\ge\liminf_{\varepsilon\to 0}\left(\frac{1}{2}\int_{S_r(\mu)}|\nabla u_\varepsilon|^2\operatorname{vol}+\mathcal{H}^1_g(\mathcal{J}_{u_\varepsilon}\cap S_r)\right)\\&\ge\frac{1}{2}\int_{S_r(\mu)}|\nabla u|^2\operatorname{vol}+\mathcal{H}^1_g(\mathcal{J}_u\cap S_r(\mu))=\mathcal{W}^{(m)}(u)+\pi\frac{N}{m}|\log r|+\mathcal{H}^1(\mathcal{J}_u)+\mathrm{o}_r(1).\end{aligned}$$

Step 2 (From u_ε to v_ε): It remains to show

$$\liminf_{\varepsilon\to 0}\left(GL_\varepsilon(u_\varepsilon,B_r(x_k))-\frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right)\right)\ge\gamma_m+\mathrm{o}_r(1) \tag{105}$$

for any vortex center x_k of u. Let us from now on fix k and shortly write $B_r:=B_r(x_k)$. Furthermore, let $\tau\in C^\infty(\bar{B}_{r_0};\mathbb{S}^1)$, $v_\varepsilon:=p^{(m)}_\tau(u_\varepsilon)$ (with $p^{(m)}_\tau$ as in (18)) and let $GL^{(m)}_\varepsilon$ be the energy functional in (94). As it was done in the proof of Proposition 5 one can show that (see (80))

$$GL_\varepsilon(v_\varepsilon,B_{r_0})\le m^2GL_\varepsilon(u_\varepsilon,B_{r_0})\le\pi|\log\varepsilon|+C, \tag{106}$$

for a constant C independent of ε. Using (85), the boundedness of (u_ε) in L^∞, the smoothness of τ, and Hölder's inequality we derive that

$$\sup_\varepsilon\int_{B_r(x_0)}|u_\varepsilon|^2|\mathrm{j}(\tau)|^2+|\langle\mathrm{j}(u_\varepsilon),\mathrm{j}(\tau)\rangle|\operatorname{vol}\le C\|\mathrm{j}(\tau)\|^2_{L^\infty}r^2+C\|\mathrm{j}(\tau)\|_{L^\infty}r^{\frac{2}{3}}\sup_\varepsilon\|\nabla u_\varepsilon\|_{L^{\frac{3}{2}}}\le Cr^{\frac{2}{3}}. \tag{107}$$

Hence, by 95, instead of (105), we can equivalently show that

$$\liminf_{\varepsilon\to 0}\left(GL^{(m)}_\varepsilon(v_\varepsilon,B_r(x_0))-\frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right)\right)\ge\gamma_m+\mathrm{o}_r(1). \tag{108}$$

Step 3 (Lower bound outside dyadic annuli): By a standard approximation argument we can assume that v_ε is smooth for every ε. Thanks to (106), we can exploit Lemma 16 with $O=B_r$. Let (x_ε) be the sequence in Lemma 16 (i). We further set $r_\varepsilon:=2\varepsilon^{\frac{1}{3}}$ and $R_\varepsilon:=\varepsilon^{\frac{1}{4}}$. Note that, as (x_ε) converges to the center of B_{r_0}, for $t \in (0,r)$ and ε small enough it holds that $B_t(x_\varepsilon) \subset B_{r_0}$. Consequently, by (48) in Lemma 7 there exists a constant C only depending on S such that for any $r' \in (0,r)$ we have that

$$\begin{aligned} GL_\varepsilon^{(m)}(v_\varepsilon, A_{R_\varepsilon,r'}(x_\varepsilon)) &\geq \frac{1}{m^2} GL_\varepsilon(v_\varepsilon, A_{R_\varepsilon,r'}(x_\varepsilon)) \\ &= \frac{1}{m^2}\int_{R_\varepsilon}^{r'} \frac{1}{2}\int_{\partial B_t(x_\varepsilon)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t \\ &\geq \frac{1}{m^2}\int_{R_\varepsilon}^{r'} \frac{\pi(1-Ct^2)}{t+C\varepsilon}\,\mathrm{d}t \\ &\geq \frac{\pi}{m^2}\left(\log\left(\frac{r'}{R_\varepsilon}\right) - \log(1+C\varepsilon^{\frac{3}{4}}) - \frac{C}{2}((r')^2 - R_\varepsilon^2)\right) \\ &\geq \frac{\pi}{m^2}\log\left(\frac{r}{R_\varepsilon}\right) - C\left(\log\left(\frac{r}{r'}\right) + r\right) + \mathrm{o}(1). \end{aligned} \tag{109}$$

In the same fashion we can also show that for any $K \in \mathbb{N}$ and $\varepsilon > 0$ small enough,

$$GL_\varepsilon^{(m)}(v_\varepsilon, A_{r_\varepsilon,2^{-K}R_\varepsilon}(x_\varepsilon)) \geq \frac{\pi}{m^2}\log\left(\frac{R_\varepsilon}{r_\varepsilon}\right) - K\frac{\pi}{m^2}\log(2) + \mathrm{o}(1). \tag{110}$$

Lastly, by Lemma 16 (iii) it holds that

$$GL_\varepsilon^{(m)}(v_\varepsilon, B_{r_\varepsilon}(x_\varepsilon)) \geq \frac{1}{m^2}GL_\varepsilon(v_\varepsilon, B_{r_\varepsilon}(x_\varepsilon)) \geq \frac{\pi}{m^2}\log\left(\frac{r_\varepsilon}{\varepsilon}\right) - \tilde{C}. \tag{111}$$

Combining (109), (110), and (111) leads to

$$\begin{aligned} &GL_\varepsilon^{(m)}(v_\varepsilon, B_{r'}(x_\varepsilon) \setminus A_{2^{-K}R_\varepsilon,R_\varepsilon}(x_\varepsilon)) \\ &\quad\geq \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) - K\frac{\pi}{m^2}\log(2) - \tilde{C} - C\left(\log\left(\frac{r}{r'}\right) + r\right) + \mathrm{o}(1). \end{aligned} \tag{112}$$

Given $\delta > 0$, let $K = K(\delta) \in \mathbb{N}$ be chosen (independently of ε and r) big enough such that

$$K\omega(\delta) \geq \gamma_m + \tilde{C}, \tag{113}$$

where $\omega(\delta) > 0$ is as in Lemma 19. We need to discern between two cases.

Step 4 ((v_ε) away from rotations): In the first case we assume that, up to taking a subsequence, $\mathrm{d}_{\bar{\mathcal{H}}}(v_\varepsilon, 2^{-k}R_\varepsilon) \geq \delta$ for all $k \in \{0,\dots,K-1\}$. Let us observe that, thanks to (106) we have that

$$(2^{-k}R_\varepsilon)^{-2}\int_{A_k^{(\varepsilon)}} (1-|v_\varepsilon|)^2 \,\mathrm{vol} \leq 2^{K+2}\varepsilon^{\frac{3}{2}} GL_\varepsilon(v_\varepsilon) = \mathrm{o}(1).$$

where $A_k^{(\varepsilon)} \coloneqq A_{2^{-(k+1)}R_\varepsilon,2^{-k}R_\varepsilon}(x_\varepsilon)$. Hence the assumptions of Lemma 19 are satisfied and we obtain that

$$\int_{A_k^{(\varepsilon)}} |\nabla v_\varepsilon|^2 \,\mathrm{vol} \geq \frac{\pi}{m^2}\log(2) + \omega(\delta) + \mathrm{o}(1), \tag{114}$$

As $B_{r'}(x_\varepsilon) \subset B_r(x_0)$ for sufficiently small ε, for all such ε, by (112), (113), and (114) it follows that

$$\begin{aligned} GL_\varepsilon^{(m)}(v_\varepsilon, B_r) &\geq GL_\varepsilon^{(m)}(v_\varepsilon, B_{r'}(x_\varepsilon)) \\ &\geq GL_\varepsilon^{(m)}(v_\varepsilon, B_{r'}(x_\varepsilon) \setminus A_{2^{-K}R_\varepsilon, R_\varepsilon}(x_\varepsilon)) + GL_\varepsilon^{(m)}(v_\varepsilon, A_{2^{-K}R_\varepsilon, R_\varepsilon}(x_\varepsilon)) \\ &\geq \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) + K\omega(\delta) - \tilde{C} - C\left(\log\left(\frac{r}{r'}\right) + r\right) + \mathrm{o}(1) \\ &\geq \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) + \gamma_m - C\left(\log\left(\frac{r}{r'}\right) + r^2\right) + \mathrm{o}(1). \end{aligned}$$

Letting first $\varepsilon \to 0$ and then $r' \to r$, (108) follows from the previous estimate.

Step 5 ((v_ε) close to rotations): We will now deal with the second case. Suppose that, up to taking a subsequence, we can find $k_0 \in \{0, \dots, K-1\}$ such that

$$\mathrm{d}_{\bar{\mathcal{H}}}(v_\varepsilon, \sigma_\varepsilon) < \delta \qquad \text{for all } \varepsilon \tag{115}$$

where $\sigma_\varepsilon \coloneqq 2^{-k_0}R_\varepsilon$. We will now show that the following inequality (116) leads to the conclusion:

$$GL_\varepsilon^{(m)}(v_\varepsilon, B_{\sigma_\varepsilon}(x_\varepsilon)) \geq \frac{\pi}{m^2}\log\left(\frac{\sigma_\varepsilon}{\varepsilon}\right) + \gamma_m - C\delta + \mathrm{o}(1), \tag{116}$$

where C is a constant independent of ε, δ, and r. Given $r' \in (0, r)$, where $r \in (0, r_0)$, by the same argument as in the third step, we have that

$$GL_\varepsilon^{(m)}(v_\varepsilon, A_{\sigma_\varepsilon, r'}(x_\varepsilon)) \geq \frac{\pi}{m^2}\log\left(\frac{r}{\sigma_\varepsilon}\right) - C\left(\log\left(\frac{r}{r'}\right) + r^2\right) + \mathrm{o}(1).$$

Consequently, by (116)

$$\begin{aligned} &\liminf_{\varepsilon\to 0} GL_\varepsilon^{(m)}(v_\varepsilon, B_r(x_0)) - \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) \\ &\quad\geq \liminf_{\varepsilon\to 0} GL_\varepsilon^{(m)}(v_\varepsilon, A_{\sigma_\varepsilon, r'}(x_\varepsilon)) - \frac{\pi}{m^2}\log\left(\frac{r}{\sigma_\varepsilon}\right) + \liminf_{\varepsilon\to 0} GL_\varepsilon^{(m)}(v_\varepsilon, B_{\sigma_\varepsilon}(x_\varepsilon)) - \frac{\pi}{m^2}\log\left(\frac{\sigma_\varepsilon}{\varepsilon}\right) \\ &\quad\geq \gamma_m - C\left(\log\left(\frac{r}{r'}\right) + r^2 + \delta\right), \end{aligned}$$

which shows (108) after sending first $r' \to r$, then $r \to 0$, and eventually $\delta \to 0$. It remains to prove (116). Let $\bar{v}_\varepsilon \coloneqq \Psi_\varepsilon^* v_\varepsilon$ for a sequence of local trivializations (Ψ_ε) of $TB_{\sigma_\varepsilon}(x_\varepsilon)$ as described in Remark 2. [1] Using (98) and (106) we derive that

$$\left| GL_\varepsilon^{(m)}(v_\varepsilon, B_{\sigma_\varepsilon}(x_\varepsilon)) - \overline{GL}_\varepsilon^{(m)}(\bar{v}_\varepsilon, B_{\sigma_\varepsilon}(0)) \right| \leq \mathcal{O}(\sigma_\varepsilon) GL_\varepsilon(v_\varepsilon, B_{\sigma_\varepsilon}(x_\varepsilon)) + \mathcal{O}(\sigma_\varepsilon) = \mathrm{o}(1).$$

According to the above estimate, instead of (116), it suffices to prove

$$\overline{GL}_\varepsilon^{(m)}(\bar{v}_\varepsilon, B_{\sigma_\varepsilon}(0)) \geq \frac{\pi}{m^2}\log\left(\frac{\sigma_\varepsilon}{\varepsilon}\right) + \gamma_m - C\delta + \mathrm{o}(1). \tag{117}$$

Note that by the definition of $\mathrm{d}_{\bar{\mathcal{H}}}$ and (115) we can find $\bar{z}_\varepsilon = \lambda_\varepsilon \frac{x}{|x|} \in \bar{\mathcal{H}}$ such that

$$\int_{A_{\frac{\sigma_\varepsilon}{2}, \sigma_\varepsilon}(0)} \frac{|\bar{v}_\varepsilon - \bar{z}_\varepsilon|^2}{\sigma_\varepsilon^2} + |\nabla \bar{v}_\varepsilon - \nabla \bar{z}_\varepsilon|^2 \,\mathrm{d}x \leq \delta^2. \tag{118}$$

[1] Let Ψ be a local trivialization of $B_{r_0}(x_0)$ as described in Remark 2, then $\Psi_\varepsilon(x) \coloneqq \Psi(x - \Phi^{-1}(x_\varepsilon))$.

Through an interpolation procedure we will now modify $\bar{v}_\varepsilon$ into a vector-field $\hat{v}_\varepsilon$ such that $\bar{v}_\varepsilon = \hat{v}_\varepsilon$ in $B_{\frac{\sigma_\varepsilon}{2}}(0)$ and $\hat{v}_\varepsilon = \bar{z}_\varepsilon$ on $\partial B_{\sigma_\varepsilon}(0)$. As $B_{\frac{r_0}{2}}(x_\varepsilon) \subset B_{r_0}(x_0)$ for ε small enough, using (106) and (107) we then see by a similar argument as in Step 3 that

$$\begin{aligned}
&GL_\varepsilon^{(m)}(v_\varepsilon, A_{\frac{\sigma_\varepsilon}{2},\sigma_\varepsilon(x_\varepsilon)}) \\
&\leq GL_\varepsilon(u_\varepsilon, B_{r_0}(x_0)) + Cr_0^{\frac{2}{3}} - GL_\varepsilon^{(m)}(v_\varepsilon, A_{\sigma_\varepsilon,\frac{r_0}{2}}(x_\varepsilon)) - GL_\varepsilon^{(m)}(v_\varepsilon, B_{\frac{\sigma_\varepsilon}{2}}(x_\varepsilon)) \\
&\leq \frac{\pi}{m^2}|\log\varepsilon| + Cr_0^{\frac{2}{3}} - \frac{\pi}{m^2}\log\left(\frac{r_0}{2\sigma_\varepsilon}\right) + Cr_0^2 - \frac{\pi}{m^2}\log\left(\frac{\sigma_\varepsilon}{2\varepsilon}\right) + C \\
&\leq \frac{\pi}{m^2}(2\log(2) + |\log r_0|) + C\left(1 + r_0^{\frac{2}{3}}\right).
\end{aligned}$$

Passing to coordinates, it follows by (98) that $\overline{GL}_\varepsilon^{(m)}(\bar{v}_\varepsilon, A_{\frac{\sigma_\varepsilon}{2},\sigma_\varepsilon}(0)) \leq C$. Consequently, by Fubini's theorem and (118) we can find $\tilde{\sigma}_\varepsilon \in (\frac{\sigma_\varepsilon}{2}, \frac{3\sigma_\varepsilon}{4})$ such that

$$\int_{\partial B_{\tilde{\sigma}_\varepsilon}(0)} \frac{|\bar{v}_\varepsilon - \bar{z}_\varepsilon|^2}{\sigma_\varepsilon^2} + |\nabla\bar{v}_\varepsilon - \nabla\bar{z}_\varepsilon|^2 \,\mathrm{d}\,\mathcal{H}^1 \leq \frac{C\delta^2}{\sigma_\varepsilon} \tag{119}$$

$$\int_{\partial B_{\tilde{\sigma}_\varepsilon}(0)} |\nabla\bar{v}_\varepsilon|^2 + |\nabla|\bar{v}_\varepsilon||^2 + \frac{1}{\varepsilon^2}(1 - |\bar{v}_\varepsilon|^2)^2 \,\mathrm{d}\,\mathcal{H}^1 \leq \frac{C}{\sigma_\varepsilon}. \tag{120}$$

Let $\theta(x)$ be the argument of $\frac{x}{|x|}$ and let $\alpha_\varepsilon \in \mathbb{R}$ be such that $\lambda_\varepsilon = e^{i\alpha_\varepsilon}$. Note that $\bar{z}_\varepsilon = e^{i(\theta+\alpha_\varepsilon)}$. By Young's inequality we also obtain that

$$\||\bar{v}_\varepsilon| - 1\|^2_{L^\infty(\partial B_{\tilde{\sigma}_\varepsilon}(0))} \leq C\varepsilon \int_{\partial B_{\tilde{\sigma}_\varepsilon}} |\nabla\bar{v}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |\bar{v}_\varepsilon|^2)^2 \,\mathrm{d}x \leq C\varepsilon^{\frac{3}{4}}. \tag{121}$$

Consequently, for ε small enough we have that $\rho_\varepsilon := |\bar{v}_\varepsilon| \geq \frac{1}{2}$. By Lemma 16 (ii) we have that $\deg(v_\varepsilon, \partial B_{\tilde{\sigma}_\varepsilon}(0)) = 1$. Therefore, since also $\deg(z_\varepsilon, \partial B_{\sigma_\varepsilon}) = 1$, by a standard lifting argument we can write $\bar{v}_\varepsilon = \rho_\varepsilon e^{i\theta_\varepsilon}$, where $\theta_\varepsilon - \theta \in H^1(\partial B_{\tilde{\sigma}_\varepsilon}(0))$ and from (119) obtain

$$\int_{\partial B_{\tilde{\sigma}_\varepsilon}(0)} \frac{|\bar{\theta}_\varepsilon - (\theta + \alpha_\varepsilon)|^2}{\sigma_\varepsilon^2} + |\nabla\theta_\varepsilon - \nabla\theta|^2 \,\mathrm{d}\,\mathcal{H}^1 \leq C\int_{\partial B_{\tilde{\sigma}_\varepsilon}(0)} \frac{|\bar{v}_\varepsilon - \bar{z}_\varepsilon|^2}{\sigma_\varepsilon^2} + |\nabla\bar{v}_\varepsilon - \nabla\bar{z}_\varepsilon|^2 \,\mathrm{d}\,\mathcal{H}^1 \leq \frac{C\delta^2}{\sigma_\varepsilon}. \tag{122}$$

Let us extend ρ_ε and θ_ε by zero homogeneity outside of $B_{\tilde{\sigma}_\varepsilon}(0)$. Setting $\hat{\sigma}_\varepsilon := \tilde{\sigma}_\varepsilon + \varepsilon^{\frac{3}{8}}$ we define $\hat{v}_\varepsilon$ in $B_{\hat{\sigma}_\varepsilon}(0)$ through:

$$\hat{v}_\varepsilon(x) := \begin{cases} \bar{v}_\varepsilon(x) & \text{if } x \in B_{\tilde{\sigma}_\varepsilon}(0), \\ \left(\rho_\varepsilon(x)\dfrac{\hat{\sigma}_\varepsilon - |x|}{\hat{\sigma}_\varepsilon - \tilde{\sigma}_\varepsilon} + \dfrac{|x| - \tilde{\sigma}_\varepsilon}{\hat{\sigma}_\varepsilon - \tilde{\sigma}_\varepsilon}\right) e^{i\theta_\varepsilon(x)} & \text{if } x \in A_{\tilde{\sigma}_\varepsilon,\hat{\sigma}_\varepsilon}(0). \end{cases}$$

By (121) and the definition $\hat{\sigma}_\varepsilon$ we have for any $x \in A_{\tilde{\sigma}_\varepsilon,\hat{\sigma}_\varepsilon}(0)$:

$$|\nabla|\hat{v}_\varepsilon||^2 = \left|\nabla\rho_\varepsilon\frac{\hat{\sigma}_\varepsilon - |x|}{\hat{\sigma}_\varepsilon - \tilde{\sigma}_\varepsilon} + \frac{1 - \rho_\varepsilon}{\hat{\sigma}_\varepsilon - \tilde{\sigma}_\varepsilon}\right|^2 \leq \left(|\nabla\rho_\varepsilon| + \frac{|1 - \rho_\varepsilon|}{\varepsilon^{\frac{3}{8}}}\right)^2 \leq C(|\nabla\rho_\varepsilon|^2 + 1).$$

For the same x as before we can similarly compute that

$$|\nabla\hat{v}_\varepsilon|^2 \leq C(|\nabla(\rho_\varepsilon e^{i\theta_\varepsilon})|^2 + |\nabla\rho_\varepsilon|^2 + 1).$$

By the last two estimates, a change of coordinates, and (120) we derive that

$$\begin{aligned}&\overline{GL}_\varepsilon^{(m)}(\hat v_\varepsilon, A_{\tilde\sigma_\varepsilon,\hat\sigma_\varepsilon}(0))\\&\quad= C\int_{\tilde\sigma_\varepsilon}^{\hat\sigma_\varepsilon}\frac{r}{\tilde\sigma_\varepsilon}\,\mathrm{d}r\int_{\partial B_{\tilde\sigma_\varepsilon}(0)}\frac{1}{2m^2}|\nabla\bar v_\varepsilon|^2+\frac{m^2-1}{2m^2}|\nabla|\bar v_\varepsilon||^2+\frac{1}{4\varepsilon^2}(1-|\bar v_\varepsilon|^2)^2\,\mathrm{d}\,\mathcal{H}^1+C\varepsilon^{\frac38}\le C\frac{\varepsilon^{\frac38}}{\sigma_\varepsilon}\le C\varepsilon^{\frac18}.\end{aligned}\tag{123}$$

Lastly, we extend $\hat v_\varepsilon$ into $A_{\hat\sigma_\varepsilon,\sigma_\varepsilon}(0)$ by linearly interpolating between θ_ε and $\theta+\alpha_\varepsilon$. More precisely, we set for $x\in A_{\hat\sigma_\varepsilon,\sigma_\varepsilon}(0)$

$$\hat v_\varepsilon(x)=e^{i\hat\theta_\varepsilon(x)},\qquad \text{where } \hat\theta_\varepsilon(x):=\frac{\sigma_\varepsilon-|x|}{\sigma_\varepsilon-\hat\sigma_\varepsilon}\theta_\varepsilon(x)+\frac{|x|-\hat\sigma_\varepsilon}{\sigma_\varepsilon-\hat\sigma_\varepsilon}(\theta(x)+\alpha_\varepsilon).$$

Let $x\in A_{\hat\sigma_\varepsilon,\sigma_\varepsilon}(0)$, as $|\hat v_\varepsilon(x)|=1$ we derive by Young's inequality and the definition of θ

$$\begin{aligned}|\nabla\hat v_\varepsilon(x)|^2=|\nabla\hat\theta_\varepsilon(x)|^2&=\left|\nabla\theta(x)+\frac{\sigma_\varepsilon-|x|}{\sigma_\varepsilon-\hat\sigma_\varepsilon}(\nabla\theta_\varepsilon(x)-\nabla\theta(x))+\frac{\theta(x)+\alpha_\varepsilon-\theta_\varepsilon(x)}{\sigma_\varepsilon-\hat\sigma_\varepsilon}\right|^2\\&\le(1+2\delta)|\nabla\theta(x)|^2+\left(2+\frac1\delta\right)|\nabla\theta_\varepsilon(x)-\nabla\theta(x)|^2+\left(2+\frac1\delta\right)\frac{|\theta_\varepsilon(x)-(\theta(x)+\alpha_\varepsilon)|^2}{(\sigma_\varepsilon-\hat\sigma_\varepsilon)^2}\\&\le\frac{1}{|x|^2}+\frac{C}{\delta}\left(|\nabla\theta_\varepsilon(x)-\nabla\theta(x)|^2+\frac{|\theta_\varepsilon(x)-(\theta(x)+\alpha_\varepsilon)|^2}{\sigma_\varepsilon^2}\right)+\frac{C\delta}{|x|^2}.\end{aligned}$$

As a result, by Fubini's theorem, a change of coordinates, and (122) we get that

$$\begin{aligned}&\int_{A_{\hat\sigma_\varepsilon,\sigma_\varepsilon}(0)}|\nabla\hat v_\varepsilon|^2\,\mathrm{d}x=\int_{\hat\sigma_\varepsilon}^{\sigma_\varepsilon}\int_{\partial B_r(0)}|\nabla\hat v_\varepsilon|^2\,\mathrm{d}\,\mathcal{H}^1\,\mathrm{d}r\\&\quad\le\int_{\hat\sigma_\varepsilon}^{\sigma_\varepsilon}\frac{r}{\tilde\sigma_\varepsilon}\,\mathrm{d}r\cdot\frac{C}{\delta}\int_{\partial B_{\tilde\sigma_\varepsilon}(0)}|\nabla\theta_\varepsilon(x)-\nabla\theta(x)|^2+\frac{|\theta_\varepsilon(x)-(\theta(x)+\alpha_\varepsilon)|^2}{\sigma_\varepsilon^2}\,\mathrm{d}\,\mathcal{H}^1+2\pi\log(\frac{\sigma_\varepsilon}{\hat\sigma_\varepsilon})\\&\quad\le2\pi\log\left(\frac{\sigma_\varepsilon}{\hat\sigma_\varepsilon}\right)+C\delta.\end{aligned}\tag{124}$$

Using (119) we can show in a similar fashion that

$$\begin{aligned}\int_{A_{\hat\sigma_\varepsilon,\sigma_\varepsilon}(0)}|\nabla\bar v_\varepsilon|^2\,\mathrm{d}x&=\int_{A_{\hat\sigma_\varepsilon,\sigma_\varepsilon}(0)}|\nabla\bar z_\varepsilon+\nabla\bar v_\varepsilon-\nabla\bar z_\varepsilon|^2\,\mathrm{d}x\\&\ge(1-\delta)\int_{A_{\hat\sigma_\varepsilon,\sigma_\varepsilon}(0)}|\nabla\bar z_\varepsilon|^2\,\mathrm{d}x-\frac1\delta\int_{A_{\hat\sigma_\varepsilon,\sigma_\varepsilon}(0)}|\nabla\bar v_\varepsilon-\nabla\bar z_\varepsilon|^2\,\mathrm{d}x\\&\ge2\pi\log\left(\frac{\sigma_\varepsilon}{\hat\sigma_\varepsilon}\right)-C\delta.\end{aligned}$$

Note that our construction assures that $\hat v_\varepsilon=z_\varepsilon$ on $\partial B_{\sigma_\varepsilon}(0)$. Hence, by (123), (124), the above estimate, and (39) we derive that

$$\begin{aligned}\overline{GL}_\varepsilon^{(m)}(\bar v_\varepsilon,B_{\sigma_\varepsilon}(0))&=\overline{GL}_\varepsilon^{(m)}(\hat v_\varepsilon,B_{\sigma_\varepsilon}(0))+\overline{GL}_\varepsilon^{(m)}(\bar v_\varepsilon,A_{\tilde\sigma_\varepsilon,\sigma_\varepsilon}(0))-\overline{GL}_\varepsilon^{(m)}(\hat v_\varepsilon,A_{\tilde\sigma_\varepsilon,\sigma_\varepsilon}(0))\\&\ge\bar\gamma_\varepsilon^{(m)}(\sigma_\varepsilon,\lambda_\varepsilon)-\overline{GL}_\varepsilon^{(m)}(\hat v_\varepsilon,A_{\tilde\sigma_\varepsilon,\hat\sigma_\varepsilon}(0))+\frac{1}{2m^2}\int_{A_{\hat\sigma_\varepsilon,\sigma_\varepsilon}(0)}|\nabla\bar v_\varepsilon|^2-|\nabla\hat v_\varepsilon|^2\,\mathrm{d}x\\&\ge\frac{\pi}{m^2}\log(\frac{\sigma_\varepsilon}{\varepsilon})+\gamma_m-C(\delta+\varepsilon^{\frac18})+\mathrm{o}(1),\end{aligned}$$

which is (117). □

4.3. **Γ-limsup.** The goal in this section is the construction of the recovery sequence in Theorem 6 (iii).

In the next lemma we relate the non-fractional renormalized energy on a surface to the Euclidean one:

Lemma 20. *Given $x_0 \in S$, $r_0 \in (0, \min\{1, r^*\})$, and $v \in W^{1,2}_{\mathrm{loc}}(B_{r_0}(x_0) \setminus \{x_0\}; \mathbb{S}^1) \cap W^{1,1}(TB_{r_0}(x_0))$ such that $\omega(v) = d\,\delta_{x_0}$, where $d \in \{-1, 1\}$, and $\mathcal{W}(v, B_{r_0}(x_0)) < \infty$, with $\mathcal{W}(v, B_{r_0}(x_0))$ as in (81). Then,*

$$\lim_{k\to\infty} \frac{1}{2}\int_{A_k} |\nabla v|^2 \,\mathrm{vol} = \pi \log(2), \tag{125}$$

where $A_k \coloneqq A_{2^{-(k+1)}r_0, 2^{-k}r_0}(x_0)$. Further, given a trivialization Ψ of $TB_{r_0}(x_0)$ as described in Remark 2 and $\bar{v} \coloneqq \Psi^* v$ we have that the Euclidean renormalized energy of $\bar{v}$ in $B_{r_0}(0)$ is finite:

$$\bar{\mathcal{W}}(\bar{v}, B_{r_0}(0)) \coloneqq \lim_{r\to 0} \left(\frac{1}{2}\int_{A_{r,r_0}(0)} |\nabla \bar{v}|^2 \,\mathrm{d}x - \pi|\log(r)| \right) < \infty.$$

Proof. Step 1 (Proof of (125)): Note that we can write the renormalized energy as a series as follows:

$$\mathcal{W}(v, B_{r_0}) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\int_{A_k} |\nabla v|^2 \,\mathrm{vol} - \pi \log\left(\frac{2^{-k} r_0}{2^{-(k+1)} r_0} \right) \right) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\int_{A_k} |\nabla v|^2 \,\mathrm{vol} - \pi \log(2) \right),$$

where we set $B_{r_0} \coloneqq B_{r_0}(x_0)$. Using $|v| = 1$ a.e. in $B_{r_0}(x_0)$ and $\omega(v) = d\,\delta_{x_0}$ with $|d| = 1$, by a standard convolution argument and (48) we have that for every $0 < r < R \le r_0$ and $\varepsilon \in (0, r)$

$$\frac{1}{2}\int_{A_{r,R}(x_0)} |\nabla v|^2 \,\mathrm{vol} \ge \int_r^R \frac{\pi(1 - Ct^2)}{t + C\varepsilon} \,\mathrm{d}t \ge \pi \log\left(\frac{R + C\varepsilon}{r + C\varepsilon} \right) - C(R^2 - r^2),$$

where C is a constant independent of ε, r, and R. Passing to the limit $\varepsilon \to 0$ then leads to

$$\frac{1}{2}\int_{A_{r,R}(x_0)} |\nabla v|^2 \,\mathrm{vol} \ge \pi \log\left(\frac{R}{r} \right) - C(R^2 - r^2).$$

As a consequence, we have that for every $k \in \mathbb{N}$

$$\frac{1}{2}\int_{A_k} |\nabla v|^2 \,\mathrm{vol} \ge \pi \log\left(\frac{2^{-k} r_0}{2^{-(k+1)} r_0} \right) - C(2^{-2k} r_0^2 - 2^{-2(k+1)} r_0^2) = \pi \log(2) - C 2^{-2(k+1)} r_0^2.$$

Hence,

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}|\nabla v|^2 \,\mathrm{vol} - \pi \log(2) + C 2^{-2(k+1)} r_0^2 \right) \le \mathcal{W}(v, B_{r_0}(x_0)) + C r_0^2 < \infty.$$

Since each term of the series is nonnegative we get (125).

Step 2 (Finiteness of the Euclidean renormalized energy): From the properties of v in the statement and our choice of Ψ we derive that $\bar{v} \in W^{1,2}_{\mathrm{loc}}(B_{r_0}(0) \setminus \{0\}; \mathbb{S}^1)$ and $\omega(v) = d\,\delta_0$. By 125 there exists $K_0 \in \mathbb{N}$ big enough such that for any $k \ge K_0$

$$\frac{1}{2}\int_{A_k} |\nabla v|^2 \,\mathrm{vol} \le 3\pi \log(2).$$

Setting $\bar{A}_k \coloneqq A_{2^{-(k+1)}r_0, 2^{-k}r_0}(0)$ and using (97) we therefore obtain that for every $k \ge K_0$

$$\begin{aligned} \frac{1}{2}\int_{\bar{A}_k} |\nabla \bar{v}|^2 \,\mathrm{d}x - \pi \log(2) &\le \frac{1}{2}\int_{A_k} |\nabla v|^2 \,\mathrm{vol} - \pi \log(2) + C 2^{-k} r_0 \left(1 + \int_{A_k} |\nabla v|^2 \,\mathrm{vol} \right) \\ &\le \frac{1}{2}\int_{A_k} |\nabla v|^2 \,\mathrm{vol} - \pi \log(2) + C 2^{-k} r_0 (1 + 3\pi \log(2)). \end{aligned}$$

As a consequence, we have that

$$\begin{aligned}\mathcal{W}(\bar{v}, B_{r_0}(0)) &= \frac{1}{2}\int_{A_{2^{-K_0}r_0,r_0}(0)} |\nabla \bar{v}|^2 \,\mathrm{d}x - K\pi\log(2) + \sum_{k=K_0}^{\infty}\left(\frac{1}{2}\int_{\bar{A}_k}|\nabla\bar{v}|^2\,\mathrm{d}x - \pi\log(2)\right)\\ &\le \frac{1}{2}\int_{A_{2^{-K_0}r_0,r_0}(0)} |\nabla \bar{v}|^2 \,\mathrm{d}x - K\pi\log(2) + \mathcal{W}(v, B_{2^{-K_0}r_0}(x_0)) + Cr_0 < \infty,\end{aligned}$$

as desired. □

The next lemma will be useful for the construction of a recovery sequence outside of vortices.

Lemma 21 (Approximation outside cores)**.** *Let $u \in \mathcal{LS}^{(m)}(S)$ with $\mathcal{W}(u, B_{r_0}(x_0) < \infty$ where $x_0 \in S$ is one of the vortex centers of u and $r_0 \in (0, r^*)$ is chosen sufficiently small that $B_{r_0}(x_0) \cap \mathrm{spt}(\omega(u)) = \{x_0\}$. Furthermore, given an orthonormal frame $\{\tau, i\tau\}$ with $\tau \in C^\infty(B_{r_0}(x_0))$, let Ψ be a local trivialization of $TB_{r_0}(x_0)$ as described in Remark 2. Then, for any $\delta > 0$ we can find $r \in (0, r_0)$, $\lambda_r \in \mathbb{S}^1$, and $u_* \in SBV^2(A_{\frac{r}{2},r_0}(x_0); \mathbb{S}^1)$ such that*

(i) $p_\tau^{(m)}(u_*^+) = p_\tau^{(m)}(u_*^-)$ *at $\mathcal{H}_g^1$-a.e. point on $\mathcal{J}_{u_*}$;*
(ii) $u_* = u$ *in $A_{r,r_0}(x_0)$;*
(iii) $\bar{u}_*^m = \lambda_r \frac{x}{|x|}$ *on $\partial B_{\frac{r}{2}}(0)$, where $\bar{u}_* := \Psi^* u_*$ and we identified $\mathbb{R}^2$ with $\mathbb{C}$;*
(iv) $\|u - u_*\|_{SBV^2(TA_{\frac{r}{2},r}(x_0))} \le \delta$.

Proof. In $B_{r_0}(x_0)$ we define $v := p_\tau^{(m)}(u)$. Furthermore, let us set $\bar{u} := \Psi^* u$ and $\bar{v} := \Psi^* v$. Note that by our choice of coordinates we have that $\bar{v} = \bar{u}^m \in W^{1,2}_{\mathrm{loc}}(B_{r_0}(0) \setminus \{0\}; \mathbb{S}^1)$. By Lemma 20 it holds that $\mathcal{W}(\bar{v}, B_{r_0}(0)) < \infty$. Given $\delta' > 0$, by a standard cut-off argument in the Euclidean setting (see e.g. the proof of Lemma 3.15 in [25], we can find by $r \in (0, r_0)$, $\lambda_r \in \mathbb{S}^1$, and $\bar{u}_* \in SBV^2(A_{\frac{r}{2},r_0}(0); \mathbb{S}^1)$ such that

(a) $(\bar{u}_*^+)^m = (\bar{u}_*^-)^m$ at $\mathcal{H}^1$-a.e. point on $\mathcal{J}_{\bar{u}_*}$;
(b) $\bar{u}_* = \bar{u}$ in $A_{r,r_0}(0)$;
(c) $\bar{u}_*^m = \lambda_r \frac{x}{|x|}$ on $\partial B_{\frac{r}{2}}(0)$;
(d) $\|\bar{u}_* - \bar{u}\|_{SBV^2(A_{\frac{r}{2},r}(0))} \le \delta'$.

Let u_* be the vector field on $B_{r_0}(x_0)$ such that $\Psi^* u_* = \bar{u}_*$. By (b) and (c) this choice of u_* trivially satisfies (ii) and (iii) from the statement, respectively. As we chose an orthonormal frame, which preserves angles we also see that (i) follows from (a). Lastly, by Lemma 1

$$\|u - u^*\|_{SBV^2(TA_{\frac{r}{2},r}(x_0))} \le C\delta',$$

for a constant only depending on S. Hence, choosing δ' small enough (iv) follows. □

The next lemma will be employed in the construction of the recovery sequence in the vicinity of vortices.

Lemma 22 (Approximation inside cores)**.** *Let $x_0 \in S$, $r \in (0, r^*)$, $\lambda \in \mathbb{S}^1$, $\delta > 0$, and Ψ be a local trivialization of $TB_r(x_0)$ as described in Remark 2. Then, there exists sequence $(u_\varepsilon) \subset \mathcal{AS}^{(m)}(B_r(x_0))$ such that*

(i) $\Psi^* u_\varepsilon = \lambda \frac{x}{|x|}$ *on $\partial B_r(0)$;*
(ii) $GL_\varepsilon(u_\varepsilon, B_r(x_0)) - \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) \le \gamma_m + \delta + \mathrm{o}_r(1) + \mathrm{o}(1)$*;*

Proof. By Corollary 3.11 in [25], we can find $\varepsilon_0 \in (0,1)$ and $\bar{z} \in SBV^2(B_1(0);\mathbb{S}^1)\cap L^\infty(B_1(0))$ such that $(\bar{z}^+)^m = (\bar{z}^-)^m$ at $\mathcal{H}^1$-a.e. point in $\mathcal{J}_{\bar{z}}$, $\bar{z}^m = x$ on $\partial B_1(0)$, and

$$\overline{GL}^{(m)}_{\varepsilon_0}(\bar{z}^m, B_1(0)) \leq \frac{\pi}{m^2}|\log\varepsilon_0| + \gamma_m + \delta. \tag{126}$$

For any $\varepsilon \in (0, r\varepsilon_0)$ we then define

$$\bar{u}_\varepsilon(x) := \begin{cases} \lambda\bar{z}(\frac{\varepsilon_0}{\varepsilon}x) & \text{if } |x| \leq \frac{\varepsilon}{\varepsilon_0} \\ e^{i\frac{\arg(x)+\alpha}{m}} & \text{if } \frac{\varepsilon}{\varepsilon_0} < |x| \leq r, \end{cases}$$

where α is such that $\lambda = e^{i\alpha}$. Let u_ε be the vector field on $B_r(x_0)$ such that $\Psi^* u_\varepsilon = \bar{u}_\varepsilon$, hence (i) follows. By our choice of trivialization it also follows that $u_\varepsilon \in \mathcal{AS}^{(m)}(B_r(x_0))$. It remains to show (ii). Setting $\bar{v}_\varepsilon := \bar{u}_\varepsilon^m$ a change of coordinates gives for $p \in [1,2)$ that

$$\begin{aligned}\int_{B_r(0)} |\nabla \bar{v}_\varepsilon|^p \,\mathrm{d}x &= \int_{B_{\frac{\varepsilon}{\varepsilon_0}}(0)} \left(\frac{\varepsilon_0}{\varepsilon}\right)^p \left|\nabla \bar{z}^m\left(\frac{\varepsilon_0}{\varepsilon}x\right)\right|^p \mathrm{d}x + \int_{A_{\frac{\varepsilon}{\varepsilon_0},r}(0)} \left|\nabla\frac{x}{|x|}\right|^p \mathrm{d}x \\ &\leq \int_{B_1(0)} |\nabla\bar{z}^m|^p\,\mathrm{d}x + \frac{2\pi}{2-p}r^{2-p} < \infty. \end{aligned}\tag{127}$$

In order to show (ii) we claim that it suffices to prove

$$GL^{(m)}_\varepsilon(v_\varepsilon, B_r(x_0)) - \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) + \mathcal{H}^1_g(\mathcal{J}_{u_\varepsilon}) \leq \gamma_m + \delta + \mathrm{o}_r(1) + \mathrm{o}(1), \tag{128}$$

where $v_\varepsilon = p^{(m)}_\tau(u_\varepsilon)$. In fact, assume that the claim (128) is proved, then (ii) follows by (95) provided we show that $\int_{B_r}\langle \mathrm{j}(u_\varepsilon), \mathrm{j}(\tau)\rangle \,\mathrm{vol} = \mathcal{O}_r(1)$. This last inequality follows by the definition of $\mathrm{j}(u_\varepsilon)$ and (127). Let us now prove the claim. By construction we have that

$$\mathcal{J}_{\bar{u}_\varepsilon} \subset \frac{\varepsilon}{\varepsilon_0}\mathcal{J}_{\bar{z}} \cup \partial B_{\frac{\varepsilon}{\varepsilon_0}}(0) \cup \{(t,0)\colon t \in [-r, -\varepsilon/\varepsilon_0]\}.$$

Therefore,

$$\mathcal{H}^1(\mathcal{J}_{\bar{u}_\varepsilon}) \leq \frac{\varepsilon}{\varepsilon_0}\,\mathcal{H}^1(\mathcal{J}_{\bar{z}}) + \frac{2\pi\varepsilon}{\varepsilon_0} + r,$$

which by the equivalence of norms shows that

$$\mathcal{H}^1_g(\mathcal{J}_{u_\varepsilon}) \leq \mathrm{o}_r(1) + \mathrm{o}(1). \tag{129}$$

Furthermore, by changing coordinates and (126) we derive that

$$\begin{aligned}\overline{GL}^{(m)}_\varepsilon(\bar{v}_\varepsilon, B_r(0)) &= \frac{1}{2m^2}\int_{B_{\frac{\varepsilon}{\varepsilon_0}}(0)} |\nabla\bar{v}_\varepsilon|^2 + (m^2-1)|\nabla|\bar{v}_\varepsilon||^2 + \frac{m^2}{2\varepsilon^2}(1-|\bar{v}_\varepsilon|^2)^2\,\mathrm{d}x + \frac{\pi}{m^2}\log\left(\frac{r\varepsilon_0}{\varepsilon}\right) \\ &= \overline{GL}^{(m)}_{\varepsilon_0}(\bar{z}^m, B_1(0)) - \frac{\pi}{m^2}|\log\varepsilon_0| + \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) \leq \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) + \gamma_m + \delta. \end{aligned}\tag{130}$$

Denoting by K_0 the largest natural number satisfying $2^{-K_0}r \geq \frac{\varepsilon}{\varepsilon_0}$ we define for any $k \in \{0,\dots,K_0\}$:

$$\bar{A}_k := \begin{cases} A_{2^{-(k+1)}r, 2^{-k}r}(0) & \text{if } k < K_0, \\ A_{\frac{\varepsilon}{\varepsilon_0}, 2^{-K_0}r}(x_0) & \text{if } k = K_0. \end{cases}$$

With this notation, (130), and (18) it follows that

$$
\begin{aligned}
GL_\varepsilon^{(m)}(v_\varepsilon, B_r(x_0)) &\le \left(1 + C\frac{\varepsilon}{\varepsilon_0}\right)\overline{GL}_\varepsilon^{(m)}(\bar v_\varepsilon, B_{\frac{\varepsilon}{\varepsilon_0}}(0)) + C\frac{\varepsilon}{\varepsilon_0} + \sum_{k=0}^{K_0}(1 + C2^{-k}r)\overline{GL}_\varepsilon^{(m)}(\bar v_\varepsilon, \bar A_k) + C2^{-k}r \\
&= \overline{GL}_\varepsilon^{(m)}(\bar v_\varepsilon, B_r(0)) + C\varepsilon|\log\varepsilon| + \sum_{k=0}^{K_0} C2^{-k}r\pi\log(2) \\
&\le \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) + \gamma_m + \delta + \mathrm{o}_r(1) + \mathrm{o}(1),
\end{aligned}
$$

which proves the claim once combined with (129). □

The Γ-lim sup is then a straightforward consequence of Lemma 21 and Lemma 22.

Proof of Theorem 6 (iii) . Let $N := |\omega(u)|$ and $\mu := \omega(u) = \sum_{k=1}^{Nm} s_k\,\delta_{x_k}$. Let us further take $\delta > 0$ and $r \in (0, r_0)$, where $r_0 \in (0, r^*)$ is small enough such that the balls $\{B_{r_0}(x_k)\}_k$ are disjoint. Fix k, applying Lemma 21 in the ball $B_{r_0}(x_k)$ for δ and r as above shows the existence of $u_*^{(k)} \in \mathcal{AS}^{(m)}(B_{r_0}(x_k))$ satisfying

$$
u_*^{(k)} = u \text{ in } A_{r,r_0}(x_k), \tag{131}
$$

$$
(\bar u_*^{(k)})^m = \lambda_r^{(k)}\frac{x}{|x|} \text{ on } \partial B_{\frac{r}{2}}(0), \tag{132}
$$

$$
\|u - u_*^{(k)}\|_{SBV^2(TA_{\frac{r}{2},r}(x_0))} \le \delta,
$$

where $\lambda_r^{(k)} \in \mathbb{S}^1$ and $\bar u_*^{(k)} := \Psi^* u_*^{(k)}$ where Ψ is the trivialization of $TB_r(x_k)$ from Lemma 21. Furthermore, using Lemma 22 in $B_r(x_k)$ for $\lambda = \lambda_r^k$ we can find a sequence $(u_\varepsilon^{(k)})_\varepsilon \subset \mathcal{AS}^{(m)}(B_r(x_k))$ such that $\sup_{\varepsilon>0}\|u_\varepsilon^{(k)}\|_{L^\infty(TB_r(x_k))} \le C$ and

$$
\bar u_\varepsilon^{(k)} = \lambda_r^{(k)}\frac{x}{|x|} \text{ on } \partial B_r(0), \tag{133}
$$

$$
GL_\varepsilon(u_\varepsilon^{(k)}, B_r(x_k)) - \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) \le \gamma_m + \delta + \mathrm{o}_r(1) + \mathrm{o}(1), \tag{134}
$$

where $\bar u_\varepsilon^{(k)} := \Psi^* u_\varepsilon^{(k)}$. We then define

$$
u_\varepsilon(x) := \begin{cases} u(x) & \text{if } x \in S_{r_0}(\mu), \\ u_*^{(k)} & \text{if } x \in A_{\frac{r}{2},r_0}(x_k), \\ u_\varepsilon^{(k)} & \text{if } x \in B_{\frac{r}{2}}(x_k). \end{cases}
$$

Note that from (131), (132), and (133) we derive that $u_\varepsilon \in \mathcal{AS}^{(m)}(S)$. Due to (131) and $|u_\varepsilon| \le 1$ a.e. in Ω we also have that

$$
\sup_\varepsilon \|u - u_\varepsilon\|_{L^1(TS)} \le C\sum_{k=1}^{Nm}\mathcal{H}_g^2(B_r(x_k)) \le Cr^2.
$$

Finally, by (131), the definition of $\mathcal{W}^{(m)}(u)$, and (134) it holds for any r that

$$\begin{aligned} GL_\varepsilon^{(m)}(u_\varepsilon, S) - \frac{N}{m}\pi|\log\varepsilon| &= \frac{1}{2}\int_{S_r(\mu)} |\nabla u|^2 \operatorname{vol} - \frac{N}{m}|\log r| + \mathcal{H}^1_g(\mathcal{J}_u \cap S_r(\mu)) + \sum_{k=1}^{Nm} \mathcal{H}^1_g(\mathcal{J}_{u_\varepsilon} \cap \partial B_r(x_k)) \\ &\quad + GL_\varepsilon(u_\varepsilon^{(k)}, B_r(x_k)) - \frac{\pi}{m^2}\log\left(\frac{r}{\varepsilon}\right) \\ &\le \mathcal{W}^{(m)}(u) + \mathcal{H}^1_g(\mathcal{J}_u) + Nm\gamma_m + Nm\delta + \mathrm{o}_r(1) + \mathrm{o}(1). \end{aligned}$$

Hence, a desired recovery sequence can be found by a standard diagonal sequence argument. □

Appendix

Appendix A. Proof of the decomposition theorem

In this appendix we provide the missing proofs of the statements in Section 2.1 using the same notation provided therein. While the results in Section 2.1 are concerned with the special case of the tangent bundle, for the sake of generality, we will deal in this appendix with a general metric vector bundle E over M with rank m. The induced covariant derivative on E will still be denoted by ∇ and its adjoint by ∇^*. As previously we assume Einstein summation convention where latin indices such as i, j, $k, \dots$ and Greek indices such as α, β, $\gamma, \dots$ that appear multiple times are implicitly summed over $\{1,\dots,n\}$ and $\{1,\dots,m\}$, respectively. Notice that an analog of (6) remains true in the present setting for $u \in C_c^\infty(E)$ and $v \in C_c^\infty(E \otimes T^*M)$. Given $u \in L^1_{\mathrm{loc}}(E)$ and $O \subset M$ open we define the total variation of u in O as follows:

$$\operatorname{var}(u, O) \coloneqq \sup\left\{ \int_M \langle u, \nabla^* v\rangle \operatorname{vol} \colon v \in C_c^\infty(E|_O \otimes T^*O) \right\}.$$

A section $u \in L^1(E)$ is then said to have bounded variation, shortly writing $u \in BV(E)$, if and only if $\operatorname{var}(u) \coloneqq \operatorname{var}(u, M) < \infty$. Riesz representation also holds in the more general setting. In fact, for a bounded linear functional $T\colon C_c(E) \to \mathbb{R}$ there exist a unique $E \otimes T^*M$-valued Radon measure ν such that

$$T(v) = \int_M \langle v, \sigma_\nu\rangle \,\mathrm{d}|\nu|,$$

where $|\nu|$ and σ_ν are the polar density and total variation of ν, respectively.

The following theorem is a generalization of Theorem 2 to the case of general vector bundles.

Theorem 8 (Radon-Nikodym)**.** *For any $\nu \in \mathcal{M}(E \otimes T^*M)$ and $\mu \in \mathcal{M}_+(M)$ there exist only two measures ν^a, $\nu^s \in \mathcal{M}(E \otimes T^*M)$ such that $\nu^a << \mu$, $\nu^s \perp \mu$ and $\nu = \nu^a + \nu^s$.*

Furthermore, there exists a unique $\sigma^a \in L^1(E \otimes T^*M; \mu)$ such that $\nu^a = \sigma^a\mu$.

Proof of Theorem 8. Step 1 (Existence): We start by showing existence of ν^a and ν^s. Let $|\nu|$ be the total variation of ν and σ_ν its polar density. As $|\nu|$ is a scalar Radon measure we can apply the classical Radon-Nikodym theorem (see for example Theorem 1.28 in [8]) to the pair $|\nu|$, μ. Therefore, we can find positive Radon measures $|\nu|^a$, $|\nu|^s$ such that $|\nu| = |\nu|^a + |\nu|^s$, $|\nu|^a << \mu$, and $|\nu|^s \perp \mu$. Furthermore, there exists $f \in L^1(M;\mu)$ such that $|\nu|^a = f\mu$. Let us now set $\nu_1 \coloneqq \sigma_\nu|\nu|^a$ and $\nu_2 \coloneqq \sigma_\nu|\nu|^s$; then,

$$\nu = \sigma_\nu|\nu| = \sigma_\nu(|\nu|^a + |\nu|^s) = \nu_1 + \nu_2.$$

Furthermore, by $|\sigma_\nu| = 1$ $|\nu|$-a.e., it follows that $|\nu_1| = |\nu|^a << \mu$ and $|\nu_2| = |\nu|^s \perp \mu$. With the boundedness of σ_ν we also derive that $f\sigma_\nu \in L^1(E;\mu)$. Thanks to that, the measures ν^1 and ν^2 are admissible candidates for ν^a and ν^s, respectively.

Step 2 (Uniqueness): It remains to prove the uniqueness of ν_1 and ν_2 found in the previous step. Let $\tilde\nu_1\,\tilde\nu_2 \in \mathcal{M}(E)$ be such that $\nu = \tilde\nu_1 + \tilde\nu_2$, $\tilde\nu_1 << \mu$, and $\tilde\nu_2 \perp \mu$. Our first aim is to show $|\nu| = |\tilde\nu_1| + |\tilde\nu_2|$. Given an open bounded set $O \subset M$ we consider a sequence $(v_h) \subset C_c^\infty(E|_O)$ with $\|v_h\|_{L^\infty} \leq 1$ that converges in $L^1(E|_O; |\nu|)$ towards $\sigma_\nu \mathbb{1}_O$. Then, by the dominated convergence theorem we have that

$$|\tilde\nu_1|(O) + |\tilde\nu_2|(O) \geq \int_M \langle v_h, \sigma_{\tilde\nu_1}\rangle \,\mathrm{d}|\tilde\nu_1| + \int_M \langle v_h, \sigma_{\tilde\nu_2}\rangle \,\mathrm{d}|\tilde\nu_2| = \int_M \langle v_h, \sigma_\nu\rangle \,\mathrm{d}|\nu| \to |\nu|(O).$$

By the arbitrariness of O it follows that $|\nu| \leq |\tilde\nu_1| + |\tilde\nu_2|$. Let us us investigate the reverse inequality. As $|\tilde\nu_2| \perp \mu$ there exists a Borel-set B such that $|\tilde\nu_2|(M \setminus B) = \mu(B) = 0$ and as $|\tilde\nu_1| << \mu$ we also have that $|\tilde\nu_1|(B) = 0$. Let O be as before and $(v_h) \subset C_c^\infty(E|_O)$ with $\|v_h\|_{L^\infty} \leq 1$ converging in $L^1(E|_O; |\nu|)$ towards $\sigma_{\tilde\nu_1}\mathbb{1}_{O\setminus B} + \sigma_{\tilde\nu_2}\mathbb{1}_B$, then

$$\begin{aligned}|\nu|(O) \geq \int_M \langle v_h, \sigma_\nu\rangle \,\mathrm{d}|\nu| = \int_M \langle v_h, \sigma_{\tilde\nu_1}\rangle \,\mathrm{d}|\tilde\nu_1| + \int_M \langle v_h, \sigma_{\tilde\nu_2}\rangle \,\mathrm{d}|\tilde\nu_2| &\to |\tilde\nu_1|(O \setminus B) + |\tilde\nu_2|(B)\\ &= |\tilde\nu_1|(O) + |\tilde\nu_2|(O),\end{aligned}$$

By the arbitrariness of O leads to $|\tilde\nu_1| + |\tilde\nu_2| \leq |\nu|$. With the uniqueness of the absolutely continuous and singular part in the classical setting we see that $|\nu_1| = |\tilde\nu_1|$ and $|\nu_2| = |\tilde\nu_2|$ in the sense of measures. It remains to show the uniqueness of σ_{ν_1} and σ_{ν_2}. Let O be an open bounded set and B a Borel set such that $|\nu_1|(B) = |\nu_2|(O \setminus B) = 0$. Furthermore, let us take a sequence $(v_h) \subset C_c^\infty(E|_O)$ converging towards $\sigma_{\tilde\nu_1}\mathbb{1}_{O\setminus B}$ in $L^1(E|_O; |\nu|)$, then

$$\begin{aligned}|\nu_1|(O \setminus B) \geq \int_{O\setminus B} \langle \sigma_{\tilde\nu_1}, \sigma_{\nu_1}\rangle \,\mathrm{d}|\nu_1| &= \lim_{h\to\infty} \int_M \langle v_h, \sigma_{\nu_1}\rangle \,\mathrm{d}|\nu_1| + \int_M \langle v_h, \sigma_{\nu_2}\rangle \,\mathrm{d}|\nu_2|\\ &= \lim_{h\to\infty} \int_M \langle v_h, \sigma_{\tilde\nu_1}\rangle \,\mathrm{d}|\tilde\nu_1| + \int_M \langle v_h, \sigma_{\tilde\nu_2}\rangle \,\mathrm{d}|\tilde\nu_2|\\ &\geq |\tilde\nu_1|(O \setminus B) - |\tilde\nu_2|(O \setminus B) = |\nu_1|(O \setminus B),\end{aligned}$$

where in the last equality we have used the uniqueness of the decomposition of $|\nu|$ proved in the previous step. Hence, the first inequality above must be an equality, which can only hold true if $\sigma_{\tilde\nu_1} = \sigma_{\nu_1}$ at $|\nu_1|$-a.e. point. The uniqueness of σ_{ν_2} follows similarly. □

Our next goal is the proof of Proposition 6, Proposition 7, and Proposition 8 which are generalizations of Proposition 1, Proposition 2, and Proposition 3 to the case of a general vector bundle, respectively. We remark that Definition 3, 5, and 6 can be easily generalized to sections using the parallel transport induced by the metric structure on E. The next lemma provides a Taylor expansion of parallel transport in a coordinate domain, which will turn up to be useful in this task.

Lemma 23 (First order Taylor expansion of parallel transport)**.** *Let $\Psi\colon \Omega \times \mathbb{R}^m \to E|_O$ be a local trivialization, $x \in O$, and $r_0 < r^*$ sufficiently small so that $B_{r_0}(x) \subset O$. Then, there exists $T = T_{\Phi^{-1}(x)}\colon \Phi^{-1}(B_{r_0}(x)) \to \mathbb{R}^{m\times m}$ such that*

$$\Psi^*(\mathcal{T}_x(y, w)) = T(\Phi^{-1}(y))\Psi^* w.$$

Furthermore, T enjoys the following Taylor expansion

$$T^\alpha_\beta(z) = \delta^\alpha_\beta - X^k\Gamma^\alpha_{k\beta} + O(\mathrm{dist}(x, \Phi(z))^2), \qquad X \coloneqq \exp_x^{-1}(\Phi(z)), \tag{135}$$

where $(\Gamma^\alpha_{k\beta})$ are the Christoffel symbols at x.

Proof. For a proof we refer to Section 3.3.2 in [32]. □

Let $O \subset M$ be a bounded open set and let $\Phi\colon \Omega \to O$ and $\{\tau_1, \ldots, \tau_m\}$ denote a chart and a frame (smooth up to the boundary), respectively. We define

$$\Lambda = \Lambda(\Omega) := \sup\Big\{\sqrt{\tilde\lambda} : \tilde\lambda \text{ eigenvalue of } (g_{ij}(x)) \text{ or } (\tilde g_{\alpha\beta}(x)),\, x \in \Omega\Big\},$$
$$\lambda = \lambda(\Omega) := \inf\Big\{\sqrt{\tilde\lambda} : \tilde\lambda \text{ eigenvalue of } (g_{ij}(x)) \text{ or } (\tilde g_{\alpha\beta}(x)),\, x \in \Omega\Big\}.$$

As the chart and the frame are smooth in Ω and g and $\tilde g$ are pointwise positive definite in $\bar\Omega$ it follows that $0 < \lambda \le \Lambda < \infty$. Then, for any $x \in \Omega$, $v \in \mathbb{R}^n$, and $w \in \mathbb{R}^m$ it holds that

$$\lambda|v| \le |v|_{g(x)} \le \Lambda|v|, \qquad \lambda|w| \le |w|_{\tilde g(x)} \le \Lambda|w|, \qquad \lambda^n \le \sqrt{|g(x)|} \le \Lambda^n, \tag{136}$$

where $|\cdot|$ denotes the Euclidean norm in both $\mathbb{R}^n$ and $\mathbb{R}^m$, $|v|_{g(x)} := \sqrt{g_{ij}(x)v^iv^j}$, $|w|_{\tilde g(x)} := \sqrt{\tilde g_{ij}(x)w^iw^j}$, and $|g(x)|$ is the determinant of $(g_{ij}(x))$. In particular, the first estimate in (136) implies the following relation between Euclidean balls an the pre-image of a geodesic ball in coordinates:

$$B_{\lambda r}(x) \subset \Phi^{-1}(B_r(\Phi(x))) \subset B_{\Lambda r}(x), \tag{137}$$

where $x \in \tilde\Omega$ and $r > 0$ is chosen sufficiently small so that $B_r(\Phi(x)) \subset \Phi(\tilde\Omega)$. Furthermore, changing coordinates and using (136) we can also derive that

$$\lambda^n\, \mathcal{H}^n(B_{\lambda r}(\Phi^{-1}(x))) \le \mathcal{H}^n_g(B_r(x)) \le \Lambda^n\, \mathcal{H}^n(B_{\Lambda r}(\Phi^{-1}(x))). \tag{138}$$

We are ready to prove the relation of approximate limits on the manifold and in Euclidean space.

Proposition 6 (Approximate limits and coordinates)**.** *Let $\Psi\colon \Omega \times \mathbb{R}^m \to E|_O$ be a local trivialization. Then, a section $u \in L^1(E|_O)$ has approximate limit z at $x \in O$ if and only if its coordinate representation Ψ^*u has approximate limit $\Psi^*z \in \mathbb{R}^m$ at $\Phi^{-1}(x)$.*

Proof of Proposition 6. Suppose that Ψ^*w is the approximate limit of Ψ^*u at the point $\Phi^{-1}(x)$. Then, by changing coordinates $z = \Phi^{-1}(y)$, (136), (137), (138), and (135)

$$\begin{aligned}
\fint_{B_r(x)} |u(y) - \mathcal{T}(y,w)|\,\mathrm{d}\,\mathcal{H}^n_g &= \frac{1}{\mathcal{H}^n_g(B_r(x))}\int_{\Phi^{-1}(B_r(x))} |\Psi^*u(z) - T(z)\Psi^*w|_{\tilde g(z)}\sqrt{|g(z)|}\,\mathrm{d}z\\
&\le \frac{1}{\lambda^n\,\mathcal{H}^n(B_{\lambda r}(\Phi^{-1}(x)))}\int_{B_{\Lambda r}(\Phi^{-1}(x))} \Lambda|\Psi^*u(z) - T(z)\Psi^*w|\Lambda^n\,\mathrm{d}z\\
&\le \frac{\Lambda^{2n+1}}{\lambda^n}\fint_{B_{\Lambda r}(\Phi^{-1}(x))} |\Psi^*u(z) - \Psi^*w| + |T(z)\Psi^*w - \Psi^*w|\,\mathrm{d}z\\
&\le C\fint_{B_{\Lambda r}(\Phi^{-1}(x))} |\Psi^*u(z) - \Psi^*w|\,\mathrm{d}z + Cr
\end{aligned}$$

for $C > 0$ independent of r and T is as in (135). Letting $r \to 0$ in the inequality above shows that z is an approximate limit of u at x.

The reverse implication in the statement follows similarly. ☐

Before coming to the proof of Proposition 7 we introduce the following helpful result:

Lemma 24. *Let $N \subset M$ be an $(n-1)$-dimensional C^1-submanifold of M and ν a unit normal on N at some point $x \in N$. Further, let $\Phi\colon \Omega \to O$ be a chart of M with $x \in O$. Then, the following relation holds true between ν and the Euclidean normal $\bar\nu$ on $\Phi^{-1}(N \cap O)$ at $\Phi^{-1}(x)$ with orientation induced by ν:*

$$\nu^i = \frac{1}{\sqrt{g^{kl}\bar\nu^k\bar\nu^l}}\, g^{ij}\bar\nu^j \qquad \textit{for } k \in \{1, \ldots, n\}. \tag{139}$$

Proof. Note that the coordinate representation (X^k) for any $X \in T_xN$ is tangential to $\Phi^{-1}(N \cap O)$ at $\Phi^{-1}(x)$ in the Euclidean sense. Let us define $\mu^i := g^{ij}\bar{\nu}^j$ for $i \in \{1, \dots, n\}$. Then, as $\bar{\nu}$ is orthogonal to the coordinate representation of any $X \in T_xN$ we derive that

$$g_{ij}\mu^i X^j = g_{ij}g^{ik}\bar{\nu}^k X^j = \delta^k_j \bar{\nu}^k X^j = \bar{\nu}^j X^j = 0.$$

Consequently, μ must be parallel to the coordinate representation of ν. Furthermore, by positive definiteness of (g^{ij}) it follows that

$$\mu^i\bar{\nu}^i = g^{ik}\bar{\nu}^k\bar{\nu}^i = \bar{\nu}^i\nu^i > 0,$$

where we have used in the last inequality that $\bar{\nu}$ and ν have the same orientation. Eventually the coordinate representation of ν follows by renormalizing μ, that is

$$\nu^i = \frac{1}{|\mu|_g}\mu^i = \frac{1}{\sqrt{g_{k'l'}g^{k'l}\bar{\nu}^l g^{l'k}\bar{\nu}^k}}\mu^i = \frac{1}{\sqrt{\delta^l_{l'}g^{l'k}\bar{\nu}^k\bar{\nu}^l}}\mu^i = \frac{1}{\sqrt{g^{kl}\bar{\nu}^k\bar{\nu}^l}}g^{ij}\bar{\nu}^j.$$

□

Proposition 7 (Approximate jumps and coordinates)**.** *Let $\Psi \colon \Omega \times \mathbb{R}^m \to E|_O$ be a local trivialization. Then, a section $u \in L^1_{\text{loc}}(E|_O)$ has an approximate jump at $x \in O$ with triplet $(a, b, \nu) \in E_x \times E_x \times T^*_xM$ if and only if Ψ^*u has an approximate jump at $\Phi^{-1}(x)$ in the usual Euclidean sense with triplet $(\Psi^*a, \Psi^*b, \bar{\nu})$, such that*

$$\nu^k = \frac{1}{\sqrt{g^{ij}\bar{\nu}^i\bar{\nu}^j}}g^{kl}\bar{\nu}^l \qquad \text{for } k \in \{1, \dots, n\}$$

and (g^{ij}) denotes the inverse of the metric tensor (g_{ij}).

Proof of Proposition 7. Let r_0 be chosen sufficiently small so that $B_{2r_0}(x) \subset O := \Phi(\Omega)$. As we will be interested in limits where $r \to 0$, without loss of generality, we assume that $r < r_0$. Furthermore, we will only prove the statement for the approximate upper limit as the arguments for the lower limit are the same.

Let us first assume that Ψ^*a is the approximate upper limit and Ψ^*b is the approximate lower limit of Ψ^*u at $\Phi^{-1}(x)$ in direction $\bar{\nu}$. Further, let $\nu = \nu^i\frac{\partial}{\partial x^i} \in T_xM$ be given by

$$\nu^i = \frac{1}{\sqrt{g^{kl}\bar{\nu}^k\bar{\nu}^l}}g^{ij}\bar{\nu}^j \qquad \text{for } i \in \{1, \dots, n\}.$$

Similarly to the proof of Proposition 6, to show that a is an approximate upper limit of u at x with respect to the unit normal ν it suffices to prove that

$$\lim_{r\to 0}\frac{1}{\mathcal{H}^n(\Phi^{-1}(B^+_r(x,\nu)))}\int_{\Phi^{-1}(B^+_r(x,\nu))}|\Psi^*u(y) - \Psi^*a| \,\mathrm{d}y = 0.$$

We define the $(n-1)$-dimensional geodesic disk centered at the point x with radius r orthogonal to ν as follows:

$$D_r(x,\nu) := \exp_x(\{X \in T_xM \colon |X| < r,\ \langle X, \nu\rangle = 0\}).$$

Note that $D_r(x,\nu)$ is a smooth $(n-1)$-dimensional submanifold of M and that the vector ν is orthogonal to $D_r(x,\nu)$ at x. By Lemma 24 and our choice of ν, the Euclidean unit normal onto $\Phi^{-1}(D_r(x,\nu))$ at $\Phi^{-1}(x)$ is given by $\bar{\nu}$. Therefore, there exists a $\delta > 0$ such that $\Phi^{-1}(D_r(x,\nu))$ is contained in the

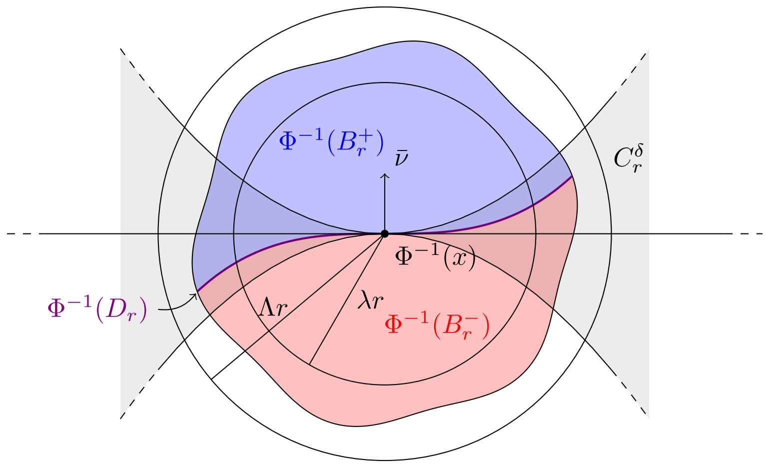


FIGURE 3. Geodesic half-balls in coordinates

parabolic cone

$$\{y \in \mathbb{R}^2 \colon |\Pi_{\bar{\nu}}(y-x)| \leq \delta |\Pi_{\bar{\nu}^\perp}(y-x)|^2\},$$

where $\Pi_{\bar{\nu}} y \coloneqq y^i \bar{\nu}^i$ and $\Pi_{\bar{\nu}^\perp} y \coloneqq y - \Pi_{\bar{\nu}} y$ (see Figure 3). Setting $H^+ \coloneqq \{y \in \mathbb{R}^n \colon (y^i - x^i)\bar{\nu}^i \geq 0\}$, we can estimate

$$|\Phi^{-1}(B_r^+(x,\nu)) \setminus H^+| + |\Phi^{-1}(B_r^-(x,\nu)) \cap H^+| \leq C_1 r^{n+1},$$

where $C_1 > 0$ is independent of r. As

$$\Phi^{-1}(B_r^+(x,\nu)) \supset B_{\lambda r}^+(\Phi^{-1}(x), \bar{\nu}) \setminus (\Phi^{-1}(B_r^-(x,\nu)) \cap H^+),$$

by possibly decreasing the value of r_0, the following bound holds:

$$\mathcal{H}^n(\Phi^{-1}(B_r^+(x,\nu))) \geq C_2 r^n$$

for some constant $C_2 > 0$ independent of r. As a consequence it follows for r small enough that

$$\begin{aligned}
&\frac{1}{\mathcal{H}^n(\Phi^{-1}(B_r^+(x,\nu)))} \int_{\Phi^{-1}(B_r^+(x,\nu))} |\Psi^* u(y) - \Psi^* a| \,\mathrm{d}y \\
&\quad \leq \frac{1}{C_2} \left(\omega_n \Lambda^n \fint_{B_{\Lambda r}^+(\Phi^{-1}(x),\bar{\nu})} |\Psi^* u(y) - \Psi^* a| \,\mathrm{d}y + \frac{1}{r^n} \int_{\Phi^{-1}(B_r^+(x,\nu)) \setminus H^+} |\Psi^* u(y) - \Psi^* b| \,\mathrm{d}y \right) + \frac{C_1 r^{n+1} |\Psi^* a - \Psi^* b|}{C_2 r^n} \\
&\quad \leq C \left(\fint_{B_{\Lambda r}^+(\Phi^{-1}(x),\bar{\nu})} |\Psi^* u(y) - \Psi^* a| \,\mathrm{d}y + \fint_{B_{\Lambda r}^-(\Phi^{-1}(x),\bar{\nu})} |\Psi^* u(y) - \Psi^* b| \,\mathrm{d}y + r \right),
\end{aligned}$$

where ω_n the volume of the unit ball in $\mathbb{R}^n$. Taking the limit $r \to 0$ leads to the desired result.

We omit the proof of the reverse implication as it follows by similar arguments. □

Lastly we prove the relation between approximate differentiability points on the manifold and in Euclidean space.

Proposition 8. *Let $\Psi \colon \Omega \times \mathbb{R}^m \to E|_O$ be a local trivialization with induced frame $\{\tau_1, \dots, \tau_m\}$. Then, any section $u \in L^1(E|_O)$ is approximately differentiable at $x \in O$ with approximate gradient $L \in E_x \otimes T_x^* M$*

if and only if $\Psi^* u$ is approximately differentiable at $\Phi^{-1}(x)$ in the usual Euclidean sense with approximate gradient $\bar{L} \in \mathbb{R}^{m\times n}$ and approximate limit $\bar{z} \in \mathbb{R}^m$ such that

$$L = (\bar{L}_i^\alpha + \Gamma_{i\beta}^\alpha \bar{z}^\beta)\, \tau_\alpha \otimes \mathrm{d}x^i, \tag{140}$$

where $(\Gamma_{i\beta}^\alpha)$ denotes the Christoffel symbols at x.

Proof of Proposition 8. Let r_0 be chosen sufficiently small so that $B_{2r_0}(x) \subset O := \Phi(\Omega)$. As we will be interested in limits where $r \to 0$, without loss of generality, we assume that $r < r_0$. Let us assume that $\Phi^{-1}(x)$ is an approximate differentiability point of $\Psi^* u$ with approximate gradient $\bar{L}$. By definition of approximate differentiability in the Euclidean setting, $\Psi^* u$ has an approximate limit $\Psi^* z$ at $\Phi^{-1}(x)$. With Proposition 6 this implies that u has the approximate limit z at x. Let L be defined as in the statement. Following the same lines of the proof of Proposition 6, in order to prove that u is approximately differentiable at x with approximate gradient L, it suffices to show that

$$\lim_{r\to 0} \fint_{B_r(\bar{x})} \frac{1}{r} |\Psi^* u(\bar{y}) - T(\bar{y})\Psi^* z - T(\bar{y})\Psi^* L(X)|\, \mathrm{d}\bar{y} = 0, \tag{141}$$

where $X := \exp_x^{-1}(y)$ and $\bar{x} := \Phi^{-1}(x)$. For each $\bar{y} \in B_r(\bar{x})$ let $y := \Phi(\bar{y})$ and $\gamma_y \colon [0, \mathrm{dist}(x, y)] \to M$ be the unique unit-speed geodesic such that $\gamma_y(0) = x$ and $\gamma_y(\mathrm{dist}(x, y)) = y$. By the smoothness of the map $(y, t) \mapsto \gamma_y(t)$ the following Taylor expansion holds true:

$$X^i := \mathrm{dist}(x, y)\dot{\gamma}_y^i(0) = y^i - x^i + O(r^2). \tag{142}$$

Thanks to (135) we have for any $\alpha \in \{1, \dots, m\}$ that

$$T(y)_\beta^\alpha L^\beta(X) = L^\alpha(X) - X^k \Gamma_{k\beta}^\alpha L^\beta(X) + O(r^2) = L^\alpha(X) + O(r^2),$$

where we have used $\sqrt{X^i X^i} = O(r)$. Hence using the choice of L and (142) we further derive for any $\alpha \in \{1, \dots, m\}$ that

$$T(y)_\beta^\alpha L^\beta(X) = \bar{L}_k^\alpha X^k + X^k \Gamma_{k\beta}^\alpha z^\beta + O(r^2) = \bar{L}_k^\alpha (y^k - x^k) + (\Gamma_0)_\beta^\alpha z^\beta + O(r^2),$$

where $(\Gamma_0)_\beta^\alpha := X^k \Gamma_{k\beta}^\alpha$. Consequently, the integrand in (141) can be written as

$$\begin{aligned}\frac{1}{r}|\Psi^* u(\bar{y}) - T(\bar{y})\Psi^* z - T(\bar{y})\Psi^* L(X)| &= \frac{1}{r}|\Psi^* u(y) - (\Psi^* z - \Gamma_0 \Psi^* z) - (\bar{L}(\bar{y} - \bar{x}) + \Gamma_0 \Psi^* z)| + O(r) \\ &= \frac{1}{r}|\Psi^* u(y) - \Psi^* z - \bar{L}(\bar{y} - \bar{x})| + O(r).\end{aligned}$$

The desired limit in (141) then follows by the approximate differentiability of $\Psi^* u$ at $\Phi^{-1}(x)$ with approximate limit $\Psi^* z$ and approximate gradient $\bar{L}$.

The reverse implication in the statement follows similarly. □

We are ready to investigate the decomposition of a section $u \in BV(E)$. We first provide several helpful results. The next lemma derives a formula for integration on submanifolds in coordinates. Without further mention we will assume that the metric tensor of an oriented $(n-1)$-dimensional C^1-submanifold $N \subset M$ is given by the restriction of g to TN. The corresponding volume form on N will be written as vol_N. Note that the orientation of M guarantees the existence of a unit-length C^1-vector field ν such that $\nu \llcorner \mathrm{vol} = \mathrm{vol}_N$, where $\nu \llcorner (\cdot)$ is the interior product with ν. We will call ν the *normal vector-field* of N.

Lemma 25 (Integration on a submanifold and coordinates)**.** *Let $N \subset M$ be an $(n-1)$-dimensional oriented C^1-submanifold of M. Furthermore, consider an open set $\Omega \subset \mathbb{R}^n$ and an orientation preserving chart $\Phi\colon \Omega \to O \subset M$. Then, for any $f \in C_c^\infty(O)$ we have*

$$\int_N f \operatorname{vol}_N = \int_{\bar N} f \circ \Phi \sqrt{g^{ij}\bar\nu^i\bar\nu^j}\sqrt{|g|}\operatorname{vol}_{\bar N},$$

$\bar N := \Phi^{-1}(N \cap O)$, $\bar\nu$ is the Euclidean unit normal onto $\bar N$ with orientation induced by the normal field ν on N, and $\operatorname{vol}_{\bar N}$ is the volume form on $\bar N$ induced by the restriction of the Euclidean metric to $\bar N$.

Remark 3. Note that by a standard approximation argument the result above holds for locally integrable functions on N.

Before coming to the proof of Lemma 25 we state the following classical result (see Proposition 4.1.54 in [32]). For any $l \in \{0, \dots, n\}$ we denote by $\Omega^l(M)$ the space of smooth l-forms on M.

Lemma 26. *For $k \in \{0, \dots, n\}$ let $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$. Furthermore, let N be an $(n-1)$-dimensional oriented C^1-submanifold of M with unit normal field ν. Then,*

$$(\alpha \wedge \star\beta)|_N = \alpha|_N \wedge \star_N(\nu \llcorner \beta)|_N, \tag{143}$$

where $(\cdot)|_N$ is the restriction to N and $\star_N$ denotes the Hodge star on N.

Proof of Lemma 25. Let $\star_N$ denote the Hodge star on N induced by the restriction of g to N. Note that $\star_N 1 = \operatorname{vol}_N$, the volume form on N. Using (143) with $\alpha = f$ and $\beta = \nu^\flat := \langle \nu, \cdot\rangle$, the linearity of $\star$, and a change of coordinates shows

$$\int_{N\cap O} f \operatorname{vol}_N = \int_{N\cap O} f \wedge \star_N(\nu^\flat(\nu)) = \int_{N\cap O} f \wedge \star_N(\nu \llcorner \nu^\flat) = \int_{N\cap O} \star(f\nu^\flat) = \int_{\bar N} \Phi^*(\star f\nu^\flat), \tag{144}$$

where Φ^* is the pull-back operator induced by Φ and $\bar N := \Phi^{-1}(N \cap O)$. By the definition of $\star$ we have for an arbitrary $\alpha \in \Omega^1(O)$ that

$$\alpha \wedge \star(f\nu^\flat) = f\langle \alpha, \nu^\flat\rangle \operatorname{vol}.$$

Taking the pull-back of both sides we have that

$$\Phi^*(\alpha) \wedge \Phi^*(f\nu^\flat) = fg^{ij}\alpha_i\nu_j^\flat\sqrt{|g|}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = fg^{ij}\alpha_i g_{jk}\nu^k\sqrt{|g|}\,\mathrm{d}x = f\alpha_i\nu^i\sqrt{|g|}\,\mathrm{d}x.$$

Let $\beta := \nu^i\,\mathrm{d}x^i$ and $\alpha = \alpha_i\,\mathrm{d}x^i$ be an arbitrary 1-form on Ω. By the definition of the Euclidean Hodge star $\star_{\mathbb{R}^n}$ it holds that

$$\alpha \wedge \star_{\mathbb{R}^n}(f\sqrt{|g|}\beta) = f\alpha_i\nu^i\sqrt{|g|}\,\mathrm{d}x.$$

By (144) and the arbitrariness of α we see that

$$\star_{\mathbb{R}^n}(f\sqrt{|g|}\beta) = \Phi^*(f\nu^\flat).$$

With this face and (143) applied for the Euclidean Hodge star we arrive that

$$\int_{N\cap U} f \operatorname{vol}_N = \int_{\bar N} f\sqrt{|g|} \wedge \star_{\mathbb{R}^n}\beta = \int_{\bar N} f\sqrt{|g|} \wedge \star_{\bar N}(\bar\nu \llcorner \beta) = \int_{\bar N} f\bar\nu^i\nu^i \operatorname{vol}_{\bar N},$$

where $\bar\nu$ is as in the statement. The desired result then ready follows from (139) as

$$\bar\nu^i\nu^i = \frac{\bar\nu^i g^{ij}\bar\nu^j}{\sqrt{g^{kl}\bar\nu^k\bar\nu^l}} = \sqrt{g^{ij}\bar\nu^i\bar\nu^j}.$$

□

In the next lemma we will relate the total variation of a section defined in (8) to the Euclidean total variation of its coordinate representations. We will assume that $\Psi\colon \Omega\times\mathbb{R}^m \to E|_O$ is a local trivialization such that its induced frame $\{\tau_1,\dots,\tau_m\}$ is orthonormal. As usual, we will denote by Φ the induced chart.

Lemma 27 (Coordinate representation of the total variation)**.** *Let $\Psi\colon \Omega\times\mathbb{R}^m \to E|_O$ be as above and let $u\in L^1(E|_O)$. Then, there exists a constant C independent of u such that*

$$\begin{aligned} \operatorname{var}(u,O) &\le C(\operatorname{var}(\Psi^*u,\Omega)+\|u\|_{L^1(E|_O)}),\\ \operatorname{var}(\Psi^*u,\Omega) &\le C(\operatorname{var}(u,O)+\|\Psi^*u\|_{L^1(\Omega;\mathbb{R}^d)}), \end{aligned} \tag{145}$$

where $\operatorname{var}(\Psi^*u,\Omega)$ stands for the Euclidean total variation of Ψ^*u in Ω.

Moreover, for any $v\in C_c^\infty(E|_O\otimes T^*O)$ it holds that

$$-\int_O \langle u,\nabla^*v\rangle \,\mathrm{d}\,\mathcal{H}^n_g = \int_\Omega u^\alpha \frac{\partial}{\partial x^k}(\sqrt{|g|}g^{ki}v_i^\alpha)\,\mathrm{d}x + \int_\Omega g^{ki}u^\alpha\Gamma^\alpha_{k\beta}v_i^\beta\sqrt{|g|}\,\mathrm{d}x. \tag{146}$$

Proof. We start by proving (146). In a coordinate domain with orthonormal frame, ∇^* has the following representation (see e.g. (10.1.8) in [32]):

$$\nabla^* = -\left[\sqrt{|g|}^{-1}\frac{\partial}{\partial x^k}(\sqrt{|g|}g^{ki}) + g^{ki}\nabla_{\frac{\partial}{\partial x^k}}\right]\frac{\partial}{\partial x^i}\llcorner,$$

where for any $w\in C_c^\infty(E|_O)$ and $\alpha\in\Omega^1(O)$ we set $\frac{\partial}{\partial x^i}\llcorner(w\otimes\alpha) := \alpha\left(\frac{\partial}{\partial x^i}\right)w$. Let $v=v_i^\alpha\tau_\alpha\otimes\mathrm{d}x^i\in C_c^\infty(E|_O\otimes T^*O)$ with $\|v\|_{L^\infty}\le 1$, passing to coordinates and using the representation of ∇^* from above we derive

$$\begin{aligned} -\int_M \langle u,\nabla^*v\rangle\,\mathrm{vol} &= \int_\Omega u^\alpha\frac{\partial}{\partial x^k}(\sqrt{|g|}g^{ki})v_i^\alpha\,\mathrm{d}x + \int_\Omega u^\alpha g^{ki}\sqrt{|g|}\frac{\partial v_i^\alpha}{\partial x^k}\,\mathrm{d}x + \int_\Omega g^{ki}u^\alpha\Gamma^\alpha_{k\beta}v_i^\beta\sqrt{|g|}\,\mathrm{d}x\\ &= \int_\Omega u^\alpha\frac{\partial}{\partial x^k}(\sqrt{|g|}g^{ki}v_i^\alpha)\,\mathrm{d}x + \int_\Omega g^{ki}u^\alpha\Gamma^\alpha_{k\beta}v_i^\beta\sqrt{|g|}\,\mathrm{d}x, \end{aligned}$$

which is (146).

We will now prove (145). Let $v=v_i^\alpha e_\alpha\otimes\mathrm{d}x^i\in C_c^\infty(E|_O\otimes T^*O)$. Using the smoothness of g, and the fact that $\|v\|_{L^\infty}\le 1$

$$\sum_{k,\alpha}(\sqrt{|g|}g^{ki}v_i^\alpha)^2 \le C\|v\|_{L^\infty}\le C.$$

Hence, by the definition of Euclidean total variation we have

$$\int_\Omega u^\alpha\frac{\partial}{\partial x^k}(\sqrt{|g|}g^{ki}v_i^\alpha)\,\mathrm{d}x \le C\operatorname{var}(\Psi^*u,\Omega).$$

Using the smoothness of the Christoffel symbols and the metric we can similarly estimate

$$\int_\Omega g^{ki}u^\alpha\Gamma^\alpha_{k\beta}v_i^\beta\sqrt{|g|}\,\mathrm{d}x \le C\int_O |u|\,\mathrm{d}\,\mathcal{H}^n_g.$$

for some constant C independent of u and v. This completes the proof of the first inequality in 145. The proof of the second inequality follows similarly. □

In the following we provide a definition for the push-forward of a vector-valued Radon measure in the Euclidean space to the manifold.

Definition 10 (Push-forward of vector measures)**.** Let F be a vector-bundle over M of rank m , $\Psi\colon \Omega\times\mathbb{R}^m \to F|_O$ a local trivialization, and $\bar\nu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ an $\mathbb{R}^m$-valued Radon measure on Ω. We denote by $\Psi\#\bar\nu \in \mathcal{M}(F|_O)$ the unique generalized vector-measure such that

$$\langle \Psi\#\bar\nu, v\rangle = \int_\Omega v^\alpha(\Phi(x))\bar\sigma^\alpha(x)\,\mathrm{d}|\bar\nu|(x) \tag{147}$$

for all $v \in C_c(F|_O)$, where $\bar\sigma$ and $|\bar\nu|$ are the polar density and total variation of $\bar\nu$, respectively.

Given $\nu \in \mathcal{M}(F|_O)$ and $A \in \mathcal{B}(O)$ we denote by $\nu \llcorner A$ the restriction of ν to A which is defined through

$$\sigma_{\nu\llcorner A} = \sigma_\nu, \quad |\nu \llcorner A| = |\nu| \llcorner A.$$

Furthermore, using the definition of $\Phi\#|\bar\nu|$ we see that Let F, Ψ, $\bar\nu$ be as in Definition 10. Using the definition of $\Phi\#\bar\nu$, where Φ is the chart associated to Ψ, we see that

$$\sigma^\alpha_{\Psi\#\bar\nu} = \frac{\tilde g^{\alpha\beta}\bar\sigma^\beta}{\sqrt{\tilde g^{\gamma\delta}\bar\sigma^\gamma\bar\sigma^\delta}}, \quad |\Psi\#\bar\nu| = \sqrt{\tilde g^{\alpha\beta}\bar\sigma^\alpha\bar\sigma^\beta}\,\Phi\#|\bar\nu|.$$

In the upcoming proof we will use the following relation

$$(\Psi\#\bar\nu)\llcorner A = \Psi\#(\bar\nu \llcorner \Phi^{-1}(A)). \tag{148}$$

We will now relate all components of the distributional derivative of a BV section u to the corresponding components of its coordinate representations.

Lemma 28. *Let $\Psi\colon \Omega\times\mathbb{R}^m \to E|_O$ be a local trivialization of E, $\tilde\Psi\colon \Omega\times\mathbb{R}^{m\times n} \to E|_O\otimes T^*O$ be a local trivialization of $E\otimes T^*M$ (both with the same induced coordinate chart), and $u \in BV_{\mathrm{loc}}(E|_O)$. Then, the following relations hold true:*

$$D^a u = \tilde\Psi\#(\sqrt{|g|}g^{-1}D^a(\Psi^*u)) + \Gamma u\,\mathcal{H}^n_g, \tag{149}$$

$$D^j u = \tilde\Psi\#(\sqrt{|g|}g^{-1}D^j(\Psi^*u)), \tag{150}$$

$$D^c u = \tilde\Psi\#(\sqrt{|g|}g^{-1}D^c(\Psi^*u)), \tag{151}$$

where $\Gamma u := \Gamma^\alpha_{i\beta}u^\beta e_\alpha \otimes \mathrm{d}x^i$ and $g^{-1} := (g^{ij})$.

Remark 4. Note that the right hand-sides of (149), (150), and (151) are well-defined since by Lemma 27 and $u \in BV_{\mathrm{loc}}(E|_O)$ implies that Ψ^*u has locally bounded variation.

Proof. Let $v \in C^\infty_c(E|_O \otimes T^*O)$, integrating by parts in (146), using $\Gamma^\beta_{i\alpha} = -\Gamma^\alpha_{i\beta}$ and the Euclidean Radon-Nikodym theorem, we have

$$\begin{aligned}
-\int_O \langle u, \nabla^* v\rangle\,\mathrm{vol} &= -\int_\Omega g^{ij}v^\alpha_j\bar\sigma^\alpha_i\sqrt{|g|}\,\mathrm{d}|D(\Psi^*u)| + \int_\Omega g^{ij}u^\alpha\Gamma^\alpha_{i\beta}v^\beta_j\sqrt{|g|}\,\mathrm{d}x \\
&= -\int_\Omega g^{ij}v^\alpha_j((\bar\sigma^a)^\alpha_i + \Gamma^\alpha_{i\beta}u^\beta)\sqrt{|g|}\,\mathrm{d}x - \int_\Omega g^{ij}v^\alpha_j(\bar\sigma^s)^\alpha_i\sqrt{|g|}\,\mathrm{d}|D^s(\Psi^*u)|,
\end{aligned}$$

where $\bar\sigma$, $\bar\sigma^a$, and $\bar\sigma^s$ are the polar densities of $D\Psi^*u$, $D^a\Psi^*u$, and $D^s\Psi^*u$, respectively. By the arbitrariness of v, the definition of the push-forward in (147), and the uniqueness of Radon-Nikodym decomposition of Du with respect to $\mathcal{H}^n_g$, we obtain equality (149) together with

$$D^s u = \Psi\#(\sqrt{|g|}g^{-1}D^s(\Psi^*u)). \tag{152}$$

Note that by Proposition 6 and Proposition 7 we have that $\mathcal{S}_u = \Phi(\mathcal{S}_{\Psi^*u})$ and $\mathcal{J}_u = \Phi(\mathcal{J}_{\Psi^*u})$. Consequently, (150) and (151) follow from (152) and (148). □

We are ready to prove the decomposition theorem for sections of bounded variation which generalizes Theorem 3.

Theorem 9 (Decomposition of sections of bounded variation)**.** *Let $u \in BV(E)$, then the discontinuity set $\mathcal{S}_u$ is $\mathcal{H}^{n-1}_g$-rectifiable, $\mathcal{H}^{n-1}_g(\mathcal{S}_u \setminus \mathcal{J}_u) = 0$, and the restriction $D^j u := D^s u \llcorner \mathcal{J}_u$ of the singular part of Du to $\mathcal{J}_u$ can be represented as*

$$D^j u = (u^+ - u^-) \otimes \nu^\flat \, \mathcal{H}^{n-1}_g \llcorner \mathcal{J}_u,$$

where the triplet (u^+, u^-, ν) is as in Definition 5 adapted to the setting of vector bundles and $\nu^\flat$ is the 1-form given by $\nu^\flat(X) = \langle \nu, X\rangle$ for any $X \in T_x M$.

Furthermore, u is approximately differentiable at a.e. point of M and the absolutely continuous part of Du can be written as

$$D^a u = \nabla u \; \mathcal{H}^n_g,$$

∇u being the approximate gradient of u.

Proof of Theorem 9. In what follows we will assume that $\Psi\colon \Omega \times \mathbb{R}^m \to E|_O$ is an arbitrary local trivialization with indued chart Φ. Note that by Lemma (27) $\Psi^* u \in BV(\Omega; \mathbb{R}^m)$.

Step 1 (Rectifiability of $\mathcal{S}_u$): By Proposition 6 and Proposition 7 we have that $\mathcal{S}_u \cap O = \Phi(\mathcal{S}_{\Psi^* u})$ and $\mathcal{J}_u \cap O = \Phi(\mathcal{J}_{\Psi^* u})$. By Theorem 3.78 in [8] $\mathcal{S}_{\Psi^* u}$ is $\mathcal{H}^{n-1}$-rectifiable and $\mathcal{H}^{n-1}(\mathcal{S}_{\Psi^* u} \setminus \mathcal{J}_{\Psi^* u}) = 0$. By the smoothness of Φ and (136) we derive that $S_u \cap O$ is $\mathcal{H}^{n-1}_g$-rectifiable and $\mathcal{H}^{n-1}_g((S_u \setminus \mathcal{J}_u) \cap O) = 0$. The result on M then follows by the arbitrariness of Ψ.

Step 2 (Characterization of the absolutely continuous part): Following the same argument in the first step but using Proposition 8 and Theorem 3.83 in [8] one has that u is approximately differentiable at $\mathcal{H}^n_g$-a.e. points in M. For any test-function $v \in C^\infty_c(E|_O \otimes T^* O)$ we derive from (149) and that (140)

$$\int_O \langle v, \sigma^a\rangle \, \mathrm{d}|D^a u| = \int_\Omega g^{ij} v^\alpha_j (\nabla(\Psi^* u))^\alpha_i \sqrt{|g|} \, \mathrm{d}x + \int_O \langle v, \Gamma u\rangle \, \mathrm{d}\,\mathcal{H}^n_g = \int_O \langle v, \nabla u\rangle \, \mathrm{d}\,\mathcal{H}^n_g,$$

where σ^a is the polar density of $D^a u$, and according to Theorem 3.83 in [8], $D^a(\Psi^* u) = \nabla(\Psi^* u)\, \mathcal{H}^n$. By the arbitrariness of v we have shown that $D^a u \llcorner O = \nabla u \, \mathcal{H}^n_g \llcorner O$. The result on M follows by a standard partition of unity argument.

Step 3 (Characterization of the jump set) As in the second step it is enough to prove the representation in O. With (150) and the representation of $D^j(\Psi^* u)$ in the Euclidean setting (see Theorem 3.78 in [8]) we have

$$\int_O \langle v, \sigma^j\rangle \, \mathrm{d}|D^j u| = \int_{\mathcal{J}_{\Psi^* u}} g^{ij} v^\alpha_j (((\Psi^* u)^+)^\alpha - ((\Psi^* u)^-)^\alpha) \bar{\nu}^i \sqrt{|g|} \, \mathrm{d}\,\mathcal{H}^{n-1},$$

where σ^a is the polar density of $D^j u$ and $\bar{\nu}$ is approximate normal to $\mathcal{J}_{\Psi^* u}$. Using Proposition (7) and $\nu^j = g^{jk}(\nu^\flat)_k$ it follows that

$$\begin{aligned}(((\Psi^* u)^+)^\alpha - ((\Psi^* u)^-)^\alpha) g^{ij} \bar{\nu}^i &= ((\Psi^* u^+)^\alpha - (\Psi^* u^-)^\alpha) \nu^j \sqrt{g^{ll'} \bar{\nu}^l \bar{\nu}^{l'}} \\ &= ((\Psi^* u^+)^\alpha - (\Psi^* u^-)^\alpha) g^{jk} (\nu^\flat)_k \sqrt{g^{ll'} \bar{\nu}^l \bar{\nu}^{l'}}.\end{aligned}$$

By the rectifiability of the jump set we can assume without loss of generality that $\mathcal{J}_{\Psi^* u}$ is contained in a C^1-submanifold $\bar{N} \subset \Omega$ such that $\bar{\nu}$ coincides with the normal to $\bar{N}$. Hence, by Lemma 25 it follows that

$$\begin{aligned}\int_O \langle v, \sigma^j\rangle \, \mathrm{d}|D^j u| &= \int_{\mathcal{J}_{\Psi^* u}} g^{jk} v^\alpha_j ((\Psi^* u^+)^\alpha - (\Psi^* u^-)^\alpha)(\nu^\flat)_k \sqrt{g^{ll'} \bar{\nu}^l \bar{\nu}^{l'}} \sqrt{|g|} \, \mathrm{d}\,\mathcal{H}^{n-1} \\ &= \int_{\mathcal{J}_u \cap O} \langle v, (u^+ - u^-) \otimes \nu^\flat\rangle \, \mathrm{d}\,\mathcal{H}^{n-1}_g \,.\end{aligned}$$

By the arbitrariness of v we derive that $D^j u \llcorner O = (u^+ - u^-) \otimes \nu^\flat \, \mathcal{H}_g^{n-1} \llcorner (\mathcal{J}_u \cap O)$, as desired. □

The next lemma is a extension of Lemma 1 to the case of a general vector bundle E.

Lemma 29. *A section $u \in L^1_{\mathrm{loc}}(E)$ is in $BV_{\mathrm{loc}}(E)$ ($SBV_{\mathrm{loc}}(E)$, $L^p_{\mathrm{loc}}(E) \cap SBV^p_{\mathrm{loc}}(E)$) if and only if for any local trivialization $\Psi\colon \Omega \times \mathbb{R}^m \to E|_O$ the pull-back $\Psi^* u$ is in $BV_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ ($SBV_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$, $L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^m) \cap SBV^p_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$) in the usual Euclidean sense.*

Proof of Lemma 29. Let $\Psi\colon \Omega \times \mathbb{R}^m \to E|_O$ be local trivialization of E for some $O \subset\subset M$ open. Suppose that $u \in BV(E|_O)$. Then, $\mathrm{var}(u, O) < \infty$ and $\|u\|_{L^1(E|_O)} < \infty$ by the definition of $BV(E|_O)$. Consequently, the second inequality in (145) implies $\mathrm{var}(\Psi^* u, \Omega) < \infty$. With (136) we also have that $\|\Psi^* u\|_{L^1(\Omega;\mathbb{R}^m)} < C\|u\|_{L^1(E|_O)} < \infty$ for some constant C independent of u. Hence, we have shown that $\Psi^* u \in BV(\Omega;\mathbb{R}^m)$.

Suppose now that $u \in SBV(E|_O)$; then, by the reasoning above we already know that $\Psi^* u \in BV(\Omega;\mathbb{R}^m)$. By (151) and (136) it holds that $D^c u \llcorner O = 0$ if and only if $D^c(\Psi^* u) \llcorner \Omega = 0$. Hence, $\Psi^* u \in SBV(\Omega;\mathbb{R}^m)$ follows.

We have shown the forward implication of the lemma due to the arbitrariness of O. The reverse implication can be shown in similar manner. □

In the next lemma we will investigate a similar relationship of SBV^p on the manifold and in Euclidean space.

Lemma 30. *Let $p \in (1,\infty)$, $q \in [p,\infty]$, and $O \subset\subset M$ be open set such that there exists a local trivialization $\Psi\colon \Omega \to E|_O$. Then, there exists a constant C such that for all $u \in SBV^p(E|_O) \cap L^q(E|_O)$ the following estimates hold true:*

$$\frac{1}{C}\|\Psi^* u\|_{L^q(\Omega;\mathbb{R}^m)} \le \|\Psi^* u\|_{L^q(E|_O)} \le C\|\Psi^* u\|_{L^q(\Omega;\mathbb{R}^m)}, \tag{153}$$

$$\frac{1}{C}\,\mathcal{H}^{n-1}(\mathcal{J}_{\Psi^* u} \cap \Omega) \le \mathcal{H}_g^{n-1}(\mathcal{J}_u \cap O) \le C\,\mathcal{H}^{n-1}(\mathcal{J}_{\Psi^* u} \cap \Omega), \tag{154}$$

$$\|\nabla u\|_{L^p(E|_O)} \le C\left(\|\nabla \Psi^* u\|_{L^p(\Omega;\mathbb{R}^{m\times n})} + \|\Psi^* u\|_{L^p(\Omega;\mathbb{R}^m)}\right) \tag{155}$$

$$\|\nabla \Psi^* u\|_{L^p(\Omega;\mathbb{R}^{m\times n})} \le C\left(\|\nabla u\|_{L^p(E|_O)} + \|u\|_{L^p(E|_O)}\right) \tag{156}$$

Proof. Both estimates in (153) directly follow from (136). Now, note that by Lemma 29 we have that $\Psi^* u \in SBV(E|_O)$. By (136) and the fact that $\mathcal{J}_u \cap O = \Phi(\mathcal{J}_{\Psi^* u} \cap \Omega)$ the inequalities in (154) follow.

Let us shortly write $\bar{u}$ for $\Psi^* u$. Using (140), a change of coordinates, (136), and the smoothness of the Christoffel symbols we derive that

$$\begin{aligned}
\int_O |\nabla u|^p \,\mathrm{vol} &= \int_\Omega \tilde{g}^{\alpha\beta} g_{ij} \left[\Big((\nabla\bar{u})^\alpha_i + \Gamma^\alpha_{i\gamma}\bar{u}^\gamma\Big)\Big((\nabla\bar{u})^\beta_j + \Gamma^\beta_{j\delta}\bar{u}^\delta\Big)\right]^{\frac{p}{2}} \sqrt{|g|}\,\mathrm{d}x \\
&\le C\int_\Omega \Big[\sum_{i,\alpha}\Big((\nabla\bar{u})^\alpha_i + \Gamma^\alpha_{i\beta}\bar{u}^\beta\Big)^2\Big]^{\frac{p}{2}}\,\mathrm{d}x \\
&\le C\|\nabla\Psi^* u\|_{L^p(\Omega;\mathbb{R}^{m\times n})}{}^p + C\int_\Omega \Big[\sum_\beta \Big(\sum_{i,\alpha}(\Gamma^\alpha_{i,\beta})^2\Big)(\bar{u}^\beta)^2\,\mathrm{d}x\Big]^{\frac{p}{2}}\,\mathrm{d}x \\
&\le C\left(\|\nabla\Psi^* u\|_{L^p(\Omega;\mathbb{R}^{m\times n})}{}^p + \|\Psi^* u\|_{L^p(\Omega;\mathbb{R}^m)}{}^p\right).
\end{aligned}$$

Taking the p-th root leads to (155). We can show (156) in similar fashion. □

Proof of Theorem 4. The result follows by a partition of unity argument as in the proof of the decomposition theorem, employing Lemma 30. □

Appendix B. Proof of the ball construction in open subsets

Let S be a closed, oriented, 2-dimensional Riemannian manifold. In this appendix, we will prove the localized version of the ball-construction stated in Theorem 7. As the argument closely follows the one presented in [27] we will only sketch the necessary modifications.

The first such modification is a localized version Lemma A.3 from [27] to an open subset of S.

Lemma 31. *Let $v \in C^\infty(TO)$ for an open subset $O \subset S$ with Lipschitz boundary such that the energy upper bound from (47) is satisfied for some $\varepsilon > 0$. Then, there exist $\varepsilon_0 > 0$ and a constant $C > 0$ independent of v or ε such that whenever $\varepsilon \in (0, \varepsilon_0)$ we can find a finite collection of closed, pairwise disjoint balls $\mathcal{B} = \{B_j\}$ whose union covers $\{x \in O \colon |v(x)| \leq \frac{1}{2}\}$, and such that*

$$\sum_j r_j \leq C\varepsilon|\log \varepsilon|,$$

where r_j is the radius of B_j.

Proof. As was done in the proof of Lemma A.3 in [27] using the coarea formula we can find a regular value $\alpha \in [\frac{1}{2}, \frac{3}{4}]$ of $|v|$ such that

$$\mathcal{H}^1_g(\{x \in O \colon |v(x)| = \alpha\}) \leq C\varepsilon|\log \varepsilon|,$$

for some constant $C > 0$ independent of v and ε. Furthermore, using the Lipschitz-regularity of ∂O, we can find a constant $\tilde{C} > 0$ only depending on ∂O such that

$$\mathcal{H}^1_g(\partial\{x \in O \colon |v(x)| \leq \alpha\}) \leq \tilde{C}\, \mathcal{H}^1_g(\{x \in O \colon |v(x)| = \alpha\}).$$

Combining both estimates we discover by the very definition of $\mathcal{H}^1_g$ that there exists a countable cover of $\partial\{x \in O \colon |v| \leq \alpha\}$ whose radii sum up to at most $C\varepsilon|\log \varepsilon|$. By compactness we can reduce ourselves to a finite cover and using a standard merging procedure to a disjoint cover without increasing the sum of all radii. □

We continue by showing that (123) from [27] is still satisfied if we replace S by an open subset of S.

Lemma 32. *There exists $r_0 = r_0(O) < r^*$ depending on the geometry of ∂O and $C > 0$ such that for any $v \in C^\infty(TO)$, $x \in O$, and $r \in (\varepsilon, r_0)$ we have*

$$\frac{1}{2}\int_{\partial B_r(x) \cap O} |\mathrm{d}|v||^2 + \frac{1}{2\varepsilon^2}(1 - |v|^2)^2 \,\mathrm{d}\mathcal{H}^1_g \geq \frac{C}{\varepsilon}\|1 - |v|\|_{L^\infty(\partial B_r(x))}. \tag{157}$$

Proof. By the Lipschitz regularity and compactness of ∂O we can find $\alpha \in (0, \pi)$ and $\tilde{r}_0 \in (0, r^*)$ such that for all $y \in \partial O$ there exist a unit-length vector $\nu \in T_y S$ satisfying

$$\partial O \cap B_r(y) \subset C_{\alpha,r}(y, \nu)$$

for all $r \in (0, \tilde{r}_0)$, where

$$C_{\alpha,r}(y,\nu) := \exp_y(\{X \in T_y S \colon |X| < r,\, |\langle X, \nu\rangle| \leq \cos(\tfrac{\alpha}{2})|\langle X, \nu^\perp\rangle|\}).$$

Let us set $r_0 := \frac{\tilde{r}_0}{2}$ and consider $x \in O$ and $r \in (0, r_0)$. Suppose that $B_r(x) \setminus O \neq \emptyset$. Then, we can find $y \in \partial O \cap B_r(x)$. Note that $B_r(x)$ is contained in $B_{2r}(y)$. As $2r < 2r_0 \leq \tilde{r}_0$ we can find by our choice of

$\tilde{r}_0$ a unit-length vector $\nu \in T_yS$ such that the set $A \coloneqq (\partial B_r(x) \cap O) \setminus C_{\alpha,2r}(y,\nu)$ is connected. Hence, by the compactness of S, we can find a constant $C > 0$ only depending on α and S such that

$$\mathcal{H}^1_g(A) \geq Cr. \tag{158}$$

Let us set $\zeta \coloneqq (1-|v|)^2$ on A. Using Young's inequality we derive that

$$|\zeta'| + \frac{1}{\varepsilon}|\zeta| \leq C\varepsilon \left(|\mathrm{d}|v||^2 + \frac{1}{2\varepsilon^2}(1-|v|^2)^2 \right).$$

Here, $(\cdot)'$ denotes the differential in tangential direction of $\partial B_r(x)$. Let us now select a point $z \in A$ such that $\zeta(z) = \fint_A \zeta \,\mathrm{d}\mathcal{H}^1_g$. Using the connectedness of A, the fundamental theorem of calculus, and (158) it then follows that

$$\begin{aligned}\|\zeta\|_{L^\infty} \leq |\zeta(z)| + \int_A |\zeta'| \,\mathrm{d}\mathcal{H}^1_g \leq \int_A \frac{1}{Cr}|\zeta| + |\zeta'| \,\mathrm{d}\mathcal{H}^1_g \leq C \int_A \frac{1}{\varepsilon}|\zeta| + |\zeta'| \,\mathrm{d}\mathcal{H}^1_g \\ \leq C\varepsilon \int_A |\mathrm{d}|v||^2 + \frac{1}{2\varepsilon^2}(1-|v|^2)^2 \,\mathrm{d}\mathcal{H}^1_g,\end{aligned}$$

which is (157).

The argument for the remaining case $B_r(x) \subset O$ follows as in ([27]). □

In the same way as was done in [27] we define $\Lambda_\varepsilon \colon [0,\infty) \to \mathbb{R}$ by

$$\Lambda_\varepsilon(\sigma) \coloneqq \int_0^\sigma \lambda_\varepsilon(r)\,\mathrm{d}r, \qquad \text{where } \lambda_\varepsilon(r) \coloneqq \min_{0<s\leq 1} \left[\frac{c_2}{4\varepsilon}(1-s)^2 + s^2\frac{\pi}{r}(1-c_2r^2) \right],$$

for constants c_2, c_3 as in (125) from Lemma A.1 in [27].

Sketch of the proof of Theorem 7. As our argument mostly coincides with the one provided in the proof of Proposition 8.2 in [27] we only briefly sketch the main differences. Employing Lemma 32 and the same Besicovitch covering argument as in [27] (see the proof of Proposition 8.2) we can find an initial cover of Z_E with a finite family $\mathcal{B} = \{B_k\}_k^K$ of pairwise disjoint, closed, geodesic balls each with radius denoted by r_k such that $\sum_{k=0}^K r_k \leq C\varepsilon|\log\varepsilon|$ for some universal constant $C > 0$ and

$$\frac{1}{2}\int_{B_k} |\nabla v|^2 + \frac{1}{2\varepsilon^2}F(|v|)\,\mathrm{vol} \geq \Lambda_\varepsilon(r_{k,0})$$

for any $B_k \subset O$.

Given $k = 1,\dots,K$ we set $d_k \coloneqq \mathrm{dg}(v, B_k; O)$ (see (46)) and define

$$\sigma^* \coloneqq \min_{d_k \neq 0} \frac{r_k}{|d_k|}.$$

Starting from the family $\mathcal{B}^{(\sigma^*)} \coloneqq \mathcal{B}$ we grow and merge every ball that does not intersect ∂O according to the standard ball-construction algorithm, while we leave unchanged all the remaining balls. For every $\sigma \in (\sigma^*, \sigma_0)$, where σ_0 is as in the statement of Lemma 32, this produces a finite family of pairwise disjoint, geodesic balls $\mathcal{B}^{(\sigma)} = \{B_k^{(\sigma)}\}$ each with radius $r_k^{(\sigma)}$ and degree $d_k^{(\sigma)} \coloneqq \mathrm{dg}(v, \partial B; O)$ such that $r_k^{(\sigma)} \geq \sigma|d_k^{(\sigma)}|$ for all k and

$$\frac{1}{2}\int_{B_k^{(\sigma)}} |\nabla v|^2 + \frac{1}{2\varepsilon^2}(1-|v|^2)^2\,\mathrm{vol} \geq \frac{r_k^{(\sigma)}}{\sigma}\Lambda_\varepsilon(\sigma),$$

as long as $B_k^{(\sigma)} \subset O$.

We conclude by following exactly the same lines of the proof of Proposition 8.2 in [27], since Lemma 31 provides the necessary extension of Lemma A.3 in [27]. □

Acknowledgements R. Badal acknowledges support by the DFG project FR 4083/5-1 and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 -390685587, Mathematics Münster: Dynamics–Geometry–Structure. The work of M. Cicalese was supported by the DFG Collaborative Research Center TRR 109, "Discretization in Geometry and Dynamics".

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