Strict convergence with equibounded area and minimal completely vertical liftings

Domenico Mucci

Abstract. Minimal lifting measures of vector-valued functions of bounded variation were introduced by Jerrard-Jung. They satisfy strong continuity properties with respect to the strict convergence in BV. Moreover, they can be described in terms of the action of the optimal Cartesian currents enclosing the graph of u. We deal with a good notion of completely vertical lifting for maps with values into the two dimensional Euclidean space. We then prove the lack of uniqueness in the high codimension case. The relationship with the relaxed area functional in the strict convergence is also discussed.

Keywords: lifting measures; functions of bounded variations; strict convergence; Cartesian currents; relaxed area functional.

AMS classification codes: 49Q20; 49J45; 26B30.

Introduction

R. L. Jerrard and N. Jung introduced in [16] an interesting notion of *minimal lifting* measures of vector-valued functions of bounded variation $u : \Omega \to \mathbb{R}^N$ defined on open sets $\Omega \subset \mathbb{R}^n$. Namely, they are able to define for each $u \in BV(\Omega, \mathbb{R}^N)$ an $\mathbb{R}^{N\times n}$-valued measure $\mu[u]$ in such a way that:

i) for Sobolev maps $u \in W^{1,1}(\Omega, \mathbb{R}^N)$

$$\mu_i^j[u] = (\mathrm{Id} \bowtie u)_\#(\nabla_i u^j \, \mathcal{L}^n \llcorner \Omega) \quad \forall\, i = 1, \dots, n, \; j = 1, \dots, N$$

where $(Id \bowtie u)(x) := (x, u(x))$ is the graph map;

ii) if $u_k \stackrel{BV}{\to} u$ in the *strict BV convergence*, i.e., $u_k \to u$ in $L^1(\Omega, \mathbb{R}^N)$ and $|Du_k|(\Omega) \to |Du|(\Omega)$, then $\mu[u_k] \rightharpoonup \mu[u]$ weakly as measures and $|\mu[u_k]|(\Omega \times \mathbb{R}^N) \to |\mu[u]|(\Omega \times \mathbb{R}^N)$.

They first observe that if an $\mathbb{R}^{N\times n}$-valued measure $\mu = (\mu)_j^j$ satisfies for each i

$$\int_\Omega \partial_{x_i} \phi(x, u)\, dx + \sum_{j=1}^N \int_\Omega \partial_{y_j} \phi(x, y)\, d\mu_i^j = 0 \qquad \forall \phi \in C_c^\infty(\Omega \times \mathbb{R}^N) \tag{0.1}$$

then the projection $\pi_\# \mu$ onto the domain Ω agrees with the distributional derivative Du of the given map $u \in BV(\Omega, \mathbb{R}^N)$, whence $\pi_\# |\mu| \geq |Du|$. *Minimality* is given by selecting among such measures μ the one, say $\mu = \mu[u]$, satisfying the total variation equality $\pi_\# |\mu|(\Omega) = |Du|(\Omega)$.

We recall that the existence of the minimal lifting is guaranteed by the density of smooth maps w.r.t. the topology of the strict convergence in BV, see [5]. Uniqueness, instead, is obtained through the validity for each i, j of the explicit formula

$$\int_{\Omega\times\mathbb{R}^N} \phi(x, y)\, d\mu_i^j[u] = \int_\Omega \Big(\int_0^1 \phi(x, u^\theta(x))\, d\theta \Big)\, d(Du)_i^j$$

u^θ being the jump interpolation function on the jump set J_u, and a precise representative outside J_u.

As an application, they prove a weak continuity property of the distributional Hessian matrix $u \mapsto \mathrm{Det}\, D^2 u$ of real-valued functions $u \in BV^2 \cap W^{1,\infty}(\Omega)$ defined on open sets $\Omega \subset \mathbb{R}^3$, see [16] for details.

Minimal lifting measures were also exploited by Rindler-Shaw in [26], where they analyzed the strict continuity property of wide classes of functionals with linear growth.

Completely vertical liftings. In this paper, we wish to extend the previous analysis to the measures corresponding, for smooth maps u, to the high order minors of the Jacobian matrix ∇u. We first consider the case when $n = N = 2$, and let for simplicity $\Omega = B^2$, the unit ball in $\mathbb{R}^2$. In this case, comparing with (0.1), we require that the action of a *completely vertical lifting* measure μ_v of a map $u \in BV(B^2, \mathbb{R}^2)$ on smooth functions $\phi \in C_c^\infty(B^2 \times \mathbb{R}^2)$ depends on the minimal lifting $\mu[u]$ in the sense of Jerrard-Jung by means of the formulas

$$\sum_{i=1}^{2} (-1)^{i+j} \int_{B^2 \times \mathbb{R}^2} \partial_{x_i} \phi(x,y)\, d\mu_i^{\bar{j}}[u] + \int_{B^2 \times \mathbb{R}^2} \partial_{y_j} \phi(x,y)\, d\mu_v = 0 \qquad j = 1,2 \tag{0.2}$$

where $\bar{1} := 2$ and $\bar{2} := 1$. Notice in fact that in the case of smooth maps $u : \overline{B^2} \to \mathbb{R}^2$, one can choose

$$\mu_v = \mu_v[u] = (Id \bowtie u)_\#(\det \nabla u\, \mathcal{L}^2 \llcorner B^2)\,.$$

The main idea, as already observed in [16], is that the lifting measures can be seen in an analytical-geometric approach through the theory of Cartesian currents by Giaquinta-Modica-Souček, see [14].

Roughly speaking, an element T in $\mathrm{cart}(B^2 \times \mathbb{R}^2)$ is an integer multiplicity (say i.m.) rectifiable 2-current in $B^2 \times \mathbb{R}^2$ that can be decomposed as $T = G_u + S_T$, where:

i) G_u is the 2-current carried by the "graph" of some map $u \in BV(B^2, \mathbb{R}^2)$;

ii) S_T is "vertical", in the sense that the action of S_T is zero on compactly supported smooth 2-forms of the type $\phi(x,y)\, dx$, where $dx := dx^1 \wedge dx^2$;

iii) the *mass* of T decomposes as $\mathbf{M}(T) = \mathbf{M}(G_u) + \mathbf{M}(S_T) < \infty$;

iv) the vertical component S_T "fills the holes" in the graph of u, i.e., the boundary current ∂T is zero inside the cylinder $B^2 \times \mathbb{R}^2$.

The underlying map u of a current T in $\mathrm{cart}(B^2 \times \mathbb{R}^2)$ is such that the determinant $\det \nabla u$ of the approximate gradient ∇u is a summable function, and the mass of the graph current G_u satisfies

$$\mathbf{M}(G_u) = \int_{B^2} \sqrt{1 + |\nabla u|^2 + (\det \nabla u)^2}\, dx < \infty \tag{0.3}$$

so that it agrees with the "area" of the "rectifiable graph" $\mathcal{G}_u$ of u. Moreover, property iv) yields

$$0 = \langle \partial T, (-1)^{i-1} \phi(x,y)\, dx^{\bar{i}} \rangle = \langle G_u, \partial_{x_i} \phi(x,y)\, dx \rangle + (-1)^{i-1} \sum_{j=1}^{2} \langle T, \partial_{y_j} \phi(x,y) dy^j \wedge dx^{\bar{i}} \rangle \qquad \forall\, i = 1,2$$

for any $\phi \in C_c^\infty(B^2 \times \mathbb{R}^2)$. Therefore, comparing the latter formula with (0.1), it turns out that in some sense the lifting measures μ are identified by the action of Cartesian currents $T = G_u + S_T$ on forms with one differential dy^j in the vertical directions.

If e.g. u is the vortex map $u(x) := x/|x|$, see Example 4.6 below, the graph current has a "hole" upon the origin O, and

$$(\partial G_u) \llcorner (B^2 \times \mathbb{R}^2) = -\delta_O \times [\![\mathbb{S}^1]\!]\,, \qquad [\![\mathbb{S}^1]\!] := \partial [\![D^2]\!]$$

where δ_O is the unit Dirac mass and

$$D^2 := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 < 1\} \tag{0.4}$$

is the oriented unit disk in the target space, compare [14, Sec. 3.2.2]. Roughly speaking, there are two qualitatively different ways to "fill the hole" in the graph of u, by inserting a disk or a cylinder:

$$S_{T_1} := \delta_O \times [\![D^2]\!]\,, \qquad S_{T_2} := [\![L]\!] \times [\![\mathbb{S}^1]\!]$$

where L is any oriented line segment connecting a point in the boundary ∂B^2 to the origin. On the other hand, letting $T_\ell := G_u + S_{T_\ell}$, where $\ell = 1, 2$, the minimal lifting measure $\mu[u]$ clearly corresponds to the action of the Cartesian current T_1.

From another point of view, it can be checked that if $\{u_k\} \subset C^1(B^2, \mathbb{R}^2)$ is a smooth sequence such that $u_k \stackrel{BV}{\to} u$ strictly in BV to the vortex map u, and in addition

$$\sup_k \int_{B^2} |\det \nabla u_k| \, dx < \infty \tag{0.5}$$

then, possibly passing to a (not relabeled) subsequence, it turns out that $G_{u_k} \stackrel{\mathcal{D}}{\rightharpoonup} T_1$ weakly in the sense of currents. In fact, the weak convergence $G_{u_k} \stackrel{\mathcal{D}}{\rightharpoonup} T_2$ is incompatible with the strict BV convergence $u_k \stackrel{BV}{\to} u$, as the total variation convergence $|Du_k|(B^2) \to |Du|(B^2)$ is violated.

In view of the validity of formula (0.2), we need to ensure the existence of a Cartesian current $T = G_u + S_T$ whose action identifies the minimal lifting measure $\mu[u]$ by Jerrard-Jung, as e.g. the current T_1 for the vortex map. By exploiting the closure-compactness theorem in $\mathrm{cart}(B^2 \times \mathbb{R}^2)$ from [14], that is based on Federer-Fleming's closure theorem [11], we are thus led to require the following hypothesis:

Definition 0.1 A map $u \in BV(B^2, \mathbb{R}^2)$ satisfies the area bounded strict density property, say *(ABS) density property*, if there exists a sequence of smooth maps $\{u_k\} \subset C^1(B^2, \mathbb{R}^2)$ such that $u_k \stackrel{BV}{\to} u$ strictly in BV and (0.5) holds.

MAIN RESULTS. If u satisfies the (ABS) density property, clearly $\det \nabla u \in L^1(B^2)$, and it turns out that the sequence of graphs $\{G_{u_k}\}$ may sub-converge to a *unique* Cartesian current T as above, that actually only depends on u and hence will be denoted by T_u. In that case, in fact, the null-boundary condition iv) yields that for any $\phi \in C_c^\infty(B^2 \times \mathbb{R}^2)$

$$\sum_{i=1}^2 (-1)^{i+j} \int_{B^2 \times \mathbb{R}^2} \partial_{x_i} \phi(x, y) \, d\mu_{\bar{i}}^{\bar{j}}[u] + \langle T_u, \partial_{y_j} \phi(x, y) \, dy \rangle = 0 \qquad j = 1, 2 \, .$$

Therefore, formulas (0.2) hold true provided that the completely vertical lifting measure $\mu_v[u]$ is defined through the action of the unique current T_u given as the weak limit point of sequences as in Definition 0.1.

In the case e.g. of the vortex map, using that $T_u = G_u + \delta_O \times [\![D^2]\!]$ we obtain:

$$\int_{B^2 \times \mathbb{R}^2} \phi(x, y) \, d\mu_v[u] = \int_{D^2} \phi(O, y) \, dy \qquad \forall \, \phi \in C_b(B^2 \times \mathbb{R}^2) \, .$$

Taking instead $u(x_1, x_2) = (x_1, x_2)$ if $x_1 < 0$ and $u(x_1, x_2) = (x_1 + 1, x_2)$ if $x_1 > 0$, see Example 4.7 below, we have $T_u = G_u + S_T$, where

$$S_T := -\psi_\#([\![-1, 1]\!] \times [\![0, 1]\!]) \, , \qquad \psi(\lambda, s) := (0, \lambda, s, \lambda) \, .$$

The vertical component S_T is given by the integration of 2-forms on the "wall" that encloses the fracture in the graph of u in correspondence to the discontinuity set of the SBV-map u, and this time we get:

$$\int_{B^2 \times \mathbb{R}^2} \phi(x, y) \, d\mu_v[u] = \int_{B^2} \phi(x, u(x)) \, dx + \int_{(-1,1) \times (0,1)} \phi(0, \lambda, s, \lambda) \, d\lambda \, ds \, .$$

We now wish describe the action of our completely vertical lifting $\mu_v[u]$, by first considering the image measure $\pi_\# \mu_v[u]$, where $\pi : B^2 \times \mathbb{R}^2 \to B^2$ is the orthogonal projection onto the domain.

For this purpose, we shall see that if u admits the completely vertical lifting measure $\mu_v[u]$, then $\overline{u}^1$ is locally summable w.r.t. Du^2 and $\overline{u}^2$ is locally summable w.r.t. Du^1, where $\overline{u}^j$ is the average of the j-th component u^j of u, see (1.1). As a consequence, the product $u^1 u^2$ is a function of bounded variation, even if $u \notin L^\infty(B^2, \mathbb{R}^2)$, see Remark 1.1.

We can thus introduce the $\mathbb{R}^2$-valued measure $\mathbf{m}_u = (\mathbf{m}_u^1, \mathbf{m}_u^2)$ with components

$$\mathbf{m}_u^1 := \frac{1}{2} \left(\overline{u}^1 \, (Du)_2^2 - \overline{u}^2 \, (Du)_2^1 \right) , \qquad \mathbf{m}_u^2 := \frac{1}{2} \left(\overline{u}^2 \, (Du)_1^1 - \overline{u}^1 \, (Du)_1^2 \right)$$

and its distributional divergence, given by:

$$\operatorname{Div} \mathbf{m}_u = \frac{1}{2} \sum_{i,j=1}^{2} (-1)^{i+j} D_i\big(\overline{u}^j\, (Du)^{\overline{j}}_{\overline{i}}\big) \, .$$

We shall in fact prove the following projection formula:

Theorem 0.2 *Let $u \in BV(B^2, \mathbb{R}^2)$, with $\det \nabla u \in L^1(B^2)$, and assume that the completely vertical lifting $\mu_v[u]$ of u does exist. Then $u^1 u^2 \in BV(B^2, \mathbb{R})$, the $\mathbb{R}^2$-valued measure $\mathbf{m}_u$ is well-defined, and*

$$\pi_{\#}\mu_v[u] = \operatorname{Div} \mathbf{m}_u \, .$$

Therefore, the distribution $\operatorname{Div} \mathbf{m}_u$ is a signed measure with finite total variation, and actually

$$|\operatorname{Div} \mathbf{m}_u|(B^2) \le |\mu_v[u]|(B^2 \times \mathbb{R}^2) < \infty \, . \tag{0.6}$$

Notice that if in addition $u \in W^{1,1}(\Omega, \mathbb{R}^2)$, then

$$\operatorname{Div} \mathbf{m}_u = \frac{1}{2} \operatorname{Div}\big(u^1 \nabla_2 u^2 - u^2 \nabla_2 u^1 \, , \, u^2 \nabla_1 u^1 - u^1 \nabla_1 u^2\big) \, .$$

We recall that the *distributional determinant* $\operatorname{Det} \nabla u$, first introduced in [19], see also [25, 8, 22], is well defined by the right-hand side of the latter formula, provided that $u \in W^{1,1}(\Omega, \mathbb{R}^2)$ is a bounded map. We refer to [2] for a detailed survey on the distributional Jacobian of Sobolev maps.

In our context, we have:

Corollary 0.3 *If $u \in W^{1,1}(B^2, \mathbb{R}^2)$ satisfies the hypotheses of Theorem 0.2, the distributional determinant of u is a signed measure with finite total variation, and*

$$\pi_{\#}\mu_v[u] = \operatorname{Det} \nabla u \, , \qquad |\operatorname{Det} \nabla u|(B^2) \le |\mu_v[u]|(B^2 \times \mathbb{R}^2) < \infty \, .$$

Now, if $u \in BV(B^2, \mathbb{R}^2)$ is a bounded map satisfying the hypotheses of Theorem 0.2, according to the decomposition of the derivative Du into its absolutely continuous, Cantor, and Jump components, we can write:

$$\langle \operatorname{Div} \mathbf{m}_u, g \rangle = \langle \operatorname{Det} \nabla u, g \rangle + \langle \operatorname{Div} F_u, g \rangle + \langle \mu_u^J, g \rangle \quad \forall\, g \in C_c^\infty(B^2) \, .$$

The second term is given by the distributional divergence of the $\mathbb{R}^2$-valued measure $F_u = (F_u^1, F_u^2)$ with components

$$F_u^1 := \frac{1}{2}\big(u^1\, (D^C u)^2_2 - u^2\, (D^C u)^1_2\big) \, , \qquad F_u^2 := \frac{1}{2}\big(u^2\, (D^C u)^1_1 - u^1\, (D^C u)^2_1\big)$$

u being a precise representative. Moreover, μ_u^J is the distribution

$$\langle \mu_u^J, g \rangle := \int_{J_u} \Delta_u^J(x)\, \partial_\tau g(x)\, d\mathcal{H}^1 \, , \quad g \in C_c^\infty(B^2)$$

where $\tau := *\nu = (-\nu_2, \nu_1)$ is the unit tangent vector to the Jump set, and for $\mathcal{H}^1$-a.e. $x \in J_u$

$$\Delta_u^J(x) := \frac{1}{2}\big(u^{1-}(x)\, u^{2+}(x) - u^{1+}(x)\, u^{2-}(x)\big) \, .$$

If u is the vortex map, we have $\operatorname{Det} \nabla u = \pi\, \delta_O$, $F_u = 0$, and $\mu_u^J = 0$. Instead, if u is the cited SBV-map from Example 4.7, we obtain that $\operatorname{Det} \nabla u = \det \nabla u\, \mathcal{L}^2 \llcorner B^2 + (\operatorname{Det} \nabla u)^s$, with $\det \nabla u = 1$ and singular part

$$(\operatorname{Det} \nabla u)^s = \frac{1}{2} \mathcal{H}^1 \llcorner J_u \, , \qquad J_u = \{0\} \times (-1, 1) \, .$$

Moreover, $F_u = 0$ and

$$\langle \mu_u^J, g \rangle = \frac{1}{2} \int_{-1}^{1} g(0, x_2)\, dx_2 = \langle (\operatorname{Det} \nabla u)^s, g \rangle \qquad \forall g \in C_c^\infty(B^2) \, .$$

Therefore, $\pi_\# \mu_v[u] = \mathcal{L}^2 \llcorner B^2 + \mathcal{H}^1 \llcorner J_u$, but $\operatorname{Det} \nabla u$ and μ_u^J are not mutually singular measures.

Notwithstanding, we expect that whenever Theorem 0.2 holds, at least for bounded maps, the three terms in the previous decomposition formula (4.6) are finite signed measures, and the total variation of the measure $\operatorname{Div} \mathbf{m}_u$ satisfies the additivity property:

$$|\operatorname{Div} \mathbf{m}_u| = |\operatorname{Det} \nabla u| + |\operatorname{Div} F_u| + |\mu_u^J| < \infty\,.$$

The action of the completely vertical lifting $\mu_v[u]$ is computed in Theorem 5.1 below for a dense class of test functions. This yields to an explicit total variation formula, see Corollary 5.3. In the case of Sobolev maps, it becomes:

$$|\mu_v[u]|(B^2 \times \mathbb{R}^2) = \sup\{|\operatorname{Det} \nabla[\Phi(u)]|(B^2) \,:\, \Phi \in \mathcal{F}\}$$

where

$$\mathcal{F} := \{\Phi \in C^\infty(\mathbb{R}^2, \mathbb{R}^2) \mid \Phi(y) = (\Phi_1(y_1), \Phi_2(y_2))\,,\ \Phi_j' \in C_c^\infty(\mathbb{R})\,,\ \|\Phi_j'\|_\infty \leq 1 \text{ for } j = 1, 2\}\,.$$

On the other hand, we expect that equality holds in the lower bound estimate (0.6), that in the Sobolev case reads as

$$|\mu_v[u]|(B^2 \times \mathbb{R}^2) = |\operatorname{Det} \nabla u|(B^2)\,. \tag{0.7}$$

In fact, we shall see that equation (0.7) holds true provided that in addition u is a continuous map in $W^{1,p}(B^2, \mathbb{R}^2)$ for some $p > 1$. For this purpose, we rely on arguments taken from the theory of *functions of bounded higher variation* due to Jerrard-Soner [15], and in particular on a result by De Lellis [9] concerning the validity of the *strong coarea formula* for the distributional Jacobian.

Failure in high codimension. All the previous results readily extend to the case of maps $u \in BV(B^n, \mathbb{R}^2)$, in any dimension $n \geq 3$, by introducing the expected modifications.

However, the situation is totally different if one considers maps taking values in high codimension Euclidean spaces. In fact, already in the case of maps $u \in W^{1,1}(B^2, \mathbb{R}^3)$, the strict convergence in the BV-sense of a smooth sequence of maps whose graphs have equibounded area, fails to contain sufficient information yielding to a good definition of the completely vertical lifting of the limit map u.

Following an example taken from [14, Sec. 3.2.3], this drawback can be seen by considering the 0-homogeneous extension of the Lipschitz-continuous map $\varphi : \partial B^2 \to \mathbb{R}^3$ given in polar coordinates by

$$\varphi(\cos\theta,\, \sin\theta) := \begin{cases} (\cos 4\theta,\, \sin 4\theta,\, 0) & \text{if } 0 \leq \theta \leq \pi/2 \\ (1,\, 0,\, \theta - \pi/2) & \text{if } \pi/2 \leq \theta \leq \pi \\ (\cos 4\theta,\, -\sin 4\theta,\, \pi/2) & \text{if } \pi \leq \theta \leq 3\pi/2 \\ (1,\, 0,\, 2\pi - \theta) & \text{if } 3\pi/2 \leq \theta < 2\pi\,. \end{cases}$$

In fact, recalling the notation (0.4), by Example 2 in [14, Sec. 3.2.2] we know that

$$(\partial G_u) \llcorner (B^2 \times \mathbb{R}^3) = -\delta_O \times \varphi_\# [\![\mathbb{S}^1]\!]\,, \qquad \varphi_\# [\![\mathbb{S}^1]\!] = \partial [\![D^2]\!] \times (\delta_0 - \delta_{\pi/2})\,.$$

Now, we can build up two Cartesian currents $T_\ell = G_u + S_{T_\ell} \in \operatorname{cart}(B^2 \times \mathbb{R}^3)$ with underlying function equal to u and such that the component S_{T_ℓ} is "completely vertical", namely

$$S_{T_\ell} = \delta_O \times S_\ell\,, \qquad S_\ell \in \mathcal{R}_2(\mathbb{R}^3)\,, \qquad \ell = 1, 2\,.$$

By considering the boundary of the cylinder $C = D^2 \times [0, \pi/2]$ in the target space, we in fact have:

$$\partial [\![C]\!] = S_1 - S_2\,, \quad S_1 := \partial [\![D^2]\!] \times [\![0, \pi/2]\!]\,, \quad S_2 := [\![D^2]\!] \times (\delta_0 - \delta_{\pi/2})$$

so that $S_1, S_2 \in \mathcal{R}_2(\mathbb{R}^3)$, whereas

$$\partial S_1 = \partial S_2 = \partial [\![D^2]\!] \times (\delta_0 - \delta_{\pi/2})\,.$$

Therefore, in codimension $N \geq 3$, uniqueness of the completely vertical lifting measure fails to hold, since both the currents T_1 and T_2 play the same role for the map u in the latter example.

Relaxed area. Coming back to Definition 0.1, we do not know if any map $u \in BV(B^2, \mathbb{R}^2)$ with $\det \nabla u \in L^1(B^2)$ satisfies the (ABS) density property, even assuming $u \in L^\infty$. On the other hand, the (ABS) density property holds true if and only if the map u has finite *relaxed area* w.r.t. the strict BV topology, i.e., $\overline{A}_{BV}(u) < \infty$, where

$$\overline{A}_{BV}(u) := \inf\Big\{ \liminf_{k\to\infty} A(u_k) \mid \{u_k\} \subset C^1(B^2, \mathbb{R}^2),\ u_k \overset{BV}{\to} u \text{ strictly in } BV \Big\}$$

$A(u_k)$ being the area of the graph of u_k, so that $A(u_k) = \mathbf{M}(G_{u_k})$, see (0.3).

For that reason, in the last section we discuss the relationship between our previous results and the relaxed area functional. Notice in particular that with our previous notation, if $\overline{A}_{BV}(u) < \infty$ by lower semicontinuity we clearly have:

$$\mathbf{M}(T_u) \le \overline{A}_{BV}(u)\,. \tag{0.8}$$

However, the "double eight" example by Giaquinta-Modica-Souček [13], that was inspired by Malý [17], shows the existence of 0-homogeneous Sobolev maps with finite relaxed area but for which the strict inequality holds in (0.8), whence a *gap phenomenon* occurs, see Example 9.4 below.

Finally, we point out that the previous example on the failure in high-codimension shows that even for Sobolev maps $u : B^2 \to \mathbb{R}^3$, the strict BV-convergence doesn't guarantee any control on the 2×2 minors of the gradient matrix along smooth approximating sequences with equibounded area, see Remark 9.2.

Plan of the paper. In Sec. 1, we collect some notation on vector-valued functions of bounded variation, and recall the notion of lifting measure due to Jerrard-Jung. In Sec. 2, we then briefly discuss the relevant class of Cartesian currents and how they can be used to re-write the lifting measure.

In Sec. 3, we introduce the completely vertical lifting measure $\mu_v[u]$ in the case $n = N = 2$, proving the uniqueness property. The formula on the projected measure $\pi_\# \mu_v[u]$ is then discussed in Sec. 4, where we also give some examples showing different features.

In Sec. 5, we compute the action of the completely vertical lifting on a dense class of test functions, thus obtaining an explicit formula for its total variation.

In Sec. 6, we prove the sufficient condition ensuring the validity of the total variation formula (0.7).

In Sec. 7, we briefly sketch how to extend our previous results on the completely vertical lifting to the case of maps in $BV(B^n, \mathbb{R}^2)$, in high dimension $n \ge 3$.

In Sec. 8, we show how our approach fails to give a good definition of completely vertical lifting of $\mathbb{R}^N$-valued maps, in high codimension $N \ge 3$.

In Sec. 9, we finally discuss the relationship with the relaxed area functional.

1 Liftings of BV maps

In this preliminary section, we collect some notation on vector-valued functions of bounded variation, referring to [14, Sec. 4.1] or to the treatise [4] for further details. We then recall the notion of lifting measure due to Jerrard-Jung [16].

Functions of bounded variation. Let $u : \Omega \to \mathbb{R}^N$ be a vector-valued summable function defined in a bounded domain $\Omega \subset \mathbb{R}^n$, where $n, N \ge 2$. We say that $u \in BV(\Omega, \mathbb{R}^N)$ if the distributional derivative Du is an $\mathbb{R}^{N\times n}$-valued measure of finite total variation, $|Du|(\Omega) < \infty$. In this case, denoting by $Du = \tilde{D}u + D^J u$ the decomposition into diffuse and Jump part, one has $\tilde{D}u = D^a u + D^C u$, where the absolutely continuous component $D^a u$ w.r.t. the Lebesgue measure $\mathcal{L}^n$ is equal to $\nabla u\, \mathcal{L}^n \llcorner \Omega$, with ∇u the approximate gradient map, and $D^C u$ denotes the Cantor component. Moreover, the jump component satisfies $D^J u = (u^+ - u^-) \otimes \nu\, \mathcal{H}^{n-1} \llcorner J_u$, where $\mathcal{H}^k$ is the Hausdorff measure and $u^\pm$ are the one-sided limits of u at $\mathcal{H}^{n-1}$-a.e. point of the jump set J_u w.r.t. a given unit normal ν. Therefore, the jump function $(u^+ - u^-) : J_u \to \mathbb{R}^N$ is $\mathcal{H}^{n-1} \llcorner J_u$-summable.

Let $u \in BV(\Omega, \mathbb{R}^N)$ and $f : \mathbb{R}^N \to \mathbb{R}$ a Lipschitz function of class C^1. It is proved in [27, 28] that the composition $f \circ u$ is in $BV(\Omega, \mathbb{R})$ and

$$\begin{aligned} D^a(f\circ u) &= \nabla f(u)\, \nabla u\, \mathcal{L}^n \llcorner \Omega \\ D^C(f\circ u) &= \nabla f(u)\, D^C u \\ D^J(f\circ u) &= (f(u^+) - f(u^-)) \otimes \nu\, \mathcal{H}^{n-1} \llcorner J_u \end{aligned}$$

where u is a precise representative (cf. [4, Thm. 3.96] for a proof). We also recall that the previous chain-rule formula involves the jump interpolation function, given for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$ by

$$u^\theta(x) := \theta\, u^+(x) + (1-\theta)\, u^-(x)\,, \qquad \theta \in [0,1]$$

and extended as equal to a precise representative $u(x)$ at $\mathcal{H}^{n-1}$-a.e. $x \in \Omega \setminus J_u$. In fact, one has:

$$D(f \circ u) = f_u\, Du\,, \qquad f_u(x) := \int_0^1 \nabla f(u^\theta(x))\, d\theta\,.$$

The chain-rule formula was extended in [3] to Lipschitz-continuous functions $f : \mathbb{R}^N \to \mathbb{R}^m$. When $N = 2$ and $f(y_1, y_2) = y_1 y_2$, given $u = (u^1, u^2) \in BV(\Omega, \mathbb{R}^2)$, and letting $\overline{u}(x) := (\overline{u}^1(x), \overline{u}^2(x))$, where

$$\overline{u}^j(x) := \int_0^1 u^{j\theta}(x)\, d\theta\,, \qquad j = 1, 2 \tag{1.1}$$

if in addition $u \in L^\infty(\Omega, \mathbb{R}^2)$ it turns out that the product $u^1 u^2$ is a BV-function with

$$D(u^1 u^2) = \overline{u}^1 Du^2 + \overline{u}^2 Du^1\,.$$

Remark 1.1 If u is not bounded, the same conclusion holds true provided that $\overline{u}^1$ is locally summable w.r.t. Du^2 and $\overline{u}^2$ is locally summable w.r.t. Du^1, compare e.g. [14, p. 487].

The weak-* convergence $u_k \overset{*}{\rightharpoonup} u$ in BV is defined by the strong L^1-convergence joined with the weak-* convergence $Du_k \rightharpoonup Du$ as $\mathbb{R}^{N\times n}$-valued measures in Ω. Finally, we say that $u_k \overset{BV}{\to} u$ *strictly* in BV if $u_k \to u$ in $L^1(\Omega, \mathbb{R}^N)$ and $|Du_k|(\Omega) \to |Du|(\Omega)$. Clearly, the strict convergence in BV is stronger than the weak-* convergence.

LIFTING MEASURES. Jerrard-Jung [16] analyzed for a given map $u \in BV(\Omega, \mathbb{R}^N)$ the measures μ_i^j in $\Omega \times \mathbb{R}^N$, where $i = 1, \ldots, n$ and $j = 1, \ldots, N$, defined through the formulas

$$\int_\Omega \partial_{x_i} \phi(x, u)\, dx + \sum_{j=1}^N \int_\Omega \partial_{y_j} \phi(x, y)\, d\mu_i^j = 0 \qquad \forall\, i = 1, \ldots, n \tag{1.2}$$

for any $\phi \in C_c^\infty(\Omega \times \mathbb{R}^N)$. Then, if u is smooth the formulas (1.2) are satisfied by taking

$$\mu_i^j = (\mathrm{Id} \bowtie u)_\#(\nabla_i u^j\, \mathcal{L}^n \llcorner \Omega) \tag{1.3}$$

where $(Id \bowtie u)(x) := (x, u(x))$ is the graph map, see Example 2.2 below.

In general, choosing $\phi(x, y) = g(x)\, y_j$, where $g \in C_c^\infty(\Omega)$, and using a cut-off argument when $u \notin L^\infty$, one has

$$\int_\Omega \partial_{x_i} g(x)\, u^j\, dx + \int_\Omega g(x)\, d\mu_i^j = 0 \qquad \forall\, i, j$$

and hence the image though the projection map $\pi : \Omega \times \mathbb{R}^N \to \Omega$ satisfies $\pi_\# \mu_i^j = (Du)_i^j$ for all i, j. For that reason, the $\mathbb{R}^{N\times n}$-valued measure $\mu = (\mu_i^j)$ is called in [16] a *lifting* of u. Of course, the lifting measure is not unique. However, using that $\pi_\# \mu = Du$, in general it satisfies $\pi_\# |\mu| \geq |Du|$.

Definition 1.2 A lifting measure μ is said to be *minimal* if $\pi_\# |\mu|(\Omega) = |Du|(\Omega)$.

Uniqueness of the minimal lifting, denoted $\mu = \mu[u]$, is obtained in [16] through the validity for each i, j of the explicit formula

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^N} \phi(x, y)\, d\mu_i^j[u] &= \int_\Omega \Big(\int_0^1 \phi(x, u^\theta(x))\, d\theta \Big)\, d(Du)_i^j \\ &= \int_{\Omega \setminus J_u} \phi(x, u(x))\, d(\bar{D}u)_i^j + \int_{J_u} \Big(\int_0^1 \phi(x, u^\theta(x))\, d\theta \Big)\, d(D^J u)_i^j \end{aligned} \tag{1.4}$$

for any $\phi \in C_c^\infty(\Omega \times \mathbb{R}^N)$ where, we recall,

$$(\bar{D}u)_i^j = \nabla_i u^j \, \mathcal{L}^n \llcorner \Omega + (D^C u)_i^j \,, \qquad (D^J u)_i^j = (u^{j+} - u^{j-}) \, \nu_i \, \mathcal{H}^{n-1} \llcorner J_u \,.$$

In particular, property (1.3) holds whenever u is a Sobolev map in $W^{1,1}(\Omega, \mathbb{R}^N)$.

Formula (1.4) is proved by exploiting the Disintegration theorem (cf. [4, Thm. 2.28]), and analyzing the $\mathbb{R}^{N\times n}$-valued measures μ_x on $\mathbb{R}^N$ this way obtained through the chain-rule formula.

Existence of the minimal lifting, instead, readily follows from the fact that if $u_k \overset{BV}{\to} u$ strictly in BV, then $\mu[u_k] \rightharpoonup \mu[u]$ as measures and $|\mu[u_k]|(\Omega \times \mathbb{R}^N) \to |\mu[u]|(\Omega \times \mathbb{R}^N)$, and that for any $u \in BV(\Omega, \mathbb{R}^N)$ there exists a smooth sequence $\{u_k\} \subset C^\infty(\Omega, \mathbb{R}^N)$ such that $u_k \overset{BV}{\to} u$ in the strict BV-sense, see [5].

Remark 1.3 Following [26], one defines for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ the $\mathbb{R}^{N\times n}$-valued measure ν_x on $\mathbb{R}^N$ as the functional acting on test functions $\varphi \in C_0(\mathbb{R}^N)$ by

$$\int_{\mathbb{R}^N} \varphi(y) \, d\nu_x(y) = \frac{dDu}{d|Du|}(x) \int_0^1 \varphi(u^\theta(x)) \, d\theta$$

where $dDu/d|Du|$ is the Radon-Nikodym derivative. Therefore, one has $\mu[u] = |Du| \otimes \nu$. Moreover, by the chain rule formula

$$0 = \int_\Omega dD\phi(\mathrm{Id} \bowtie u) = \int_\Omega \nabla_x \phi(x, u(x)) \, dx + \int_\Omega \Big(\int_0^1 \nabla_y \phi(x, u^\theta(x)) \, d\theta \Big) \, dDu(x)$$

for every $\phi \in C_c^1(\Omega \times \mathbb{R}^N)$, where according to (1.4)

$$\int_\Omega \Big(\int_0^1 \nabla_y \phi(x, u^\theta(x)) \, d\theta \Big) \, dDu(x) = \int_{\Omega \times \mathbb{R}^N} \nabla_y \phi(x, y) \, d\mu[u] \,.$$

Using this approach, the continuity of wide classes of functionals with linear growth w.r.t. the so called "area-strict convergence" is analyzed in [26].

2 Cartesian currents

In this section, we show how the above notation can be re-written by using tools from Geometric Measure Theory. Therefore, after introducing some general notation, we briefly discuss in our context the relevant class of Cartesian currents introduced by Giaquinta-Modica-Souček [14].

RECTIFIABLE CURRENTS. If $U \subset \mathbb{R}^m$ is an open set, $m \in \mathbb{N}^+$, and $n = 0, \ldots, m$, we denote by $\mathcal{D}_n(U)$ the strong dual of the space of compactly supported smooth n-forms $\mathcal{D}^n(U)$, whence $\mathcal{D}_0(U)$ is the vector space of distributions in U. For any $T \in \mathcal{D}_n(U)$, we define its *mass* $\mathbf{M}(T)$ as

$$\mathbf{M}(T) := \sup\{\langle T, \omega \rangle \mid \omega \in \mathcal{D}^n(U) \,, \ \|\omega\| \leq 1\}$$

and (for $n \geq 1$) its *boundary* as the $(n-1)$-current ∂T defined by the relation

$$\langle \partial T, \eta \rangle := \langle T, d\eta \rangle, \qquad \eta \in \mathcal{D}^{n-1}(U)$$

where $d\eta$ is the differential of η. The *weak convergence* $T_k \overset{\mathcal{D}}{\rightharpoonup} T$ in the sense of currents in $\mathcal{D}_n(U)$ is defined by duality as

$$\lim_{k\to\infty} \langle T_k, \omega \rangle = \langle T, \omega \rangle \qquad \forall \, \omega \in \mathcal{D}^n(U) \,.$$

If $T_k \overset{\mathcal{D}}{\rightharpoonup} T$, by lower semicontinuity we clearly have

$$\mathbf{M}(T) \leq \liminf_{k\to\infty} \mathbf{M}(T_k) \,.$$

For $n \geq 1$, an n-current T with finite mass is called *rectifiable* if

$$\langle T, \omega \rangle = \int_{\mathcal{M}} \theta \, \langle \omega, \xi \rangle \, d\mathcal{H}^n \qquad \forall \, \omega \in \mathcal{D}^n(U)$$

with $\mathcal{M}$ an n-rectifiable set in U, $\xi : \mathcal{M} \to \Lambda^n \mathbb{R}^m$ an $\mathcal{H}^n \llcorner \mathcal{M}$-measurable function such that $\xi(x)$ is a simple unit n-vector orienting the approximate tangent space to $\mathcal{M}$ at $\mathcal{H}^n$-a.e. $x \in \mathcal{M}$, and $\theta : \mathcal{M} \to [0, +\infty)$ an $\mathcal{H}^n \llcorner \mathcal{M}$-summable and non-negative function. Therefore, we get $\mathbf{M}(T) = \int_{\mathcal{M}} \theta \, d\mathcal{H}^n < \infty$.

In addition, if the multiplicity function θ is integer-valued, the current T is called *i.m. rectifiable* and the corresponding class is denoted by $\mathcal{R}_n(U)$.

Finally, if $f : U \to V$ is a smooth map with values in an open set $V \subset \mathbb{R}^d$, where $d \geq n$, and f is bounded on the support of a current $T \in \mathcal{R}_n(U)$, the image current $f_\# T$ in $\mathcal{R}_n(V)$ is defined by

$$\langle f_\# T, \omega \rangle := \langle T, f^\# \omega \rangle \qquad \forall \, \omega \in \mathcal{D}^n(V)$$

where $f^\# \omega$ denotes the pull-back of the n-form ω by f.

Remark 2.1 The relevance of the class $\mathcal{R}_n(U)$ in the Calculus of Variations relies of Federer-Fleming's closure-compactness theorem [11], stating that if a sequence $\{T_k\} \subset \mathcal{R}_n(U)$ satisfies $\sup_k \mathbf{M}(T_k) < \infty$ and $\sup_k \mathbf{M}((\partial T_k) \llcorner W) < \infty$ for each open set $W \subset\subset U$, then there exists $T \in \mathcal{R}_n(U)$ and a (not relabeled) subsequence of $\{T_k\}$ such that $T_k \stackrel{\mathcal{D}}{\rightharpoonup} T$.

Graph currents. Currents in $\mathcal{R}_n(U)$ generalize the action given by integration of n-forms on smooth oriented n-surfaces $\mathcal{M}$. If e.g. $U = \Omega \times \mathbb{R}^N$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $u : \overline{\Omega} \to \mathbb{R}^N$ is a smooth map, the graph current G_u, given by integration of n-forms on the naturally oriented graph $\mathcal{G}_u$, belongs to $\mathcal{R}_n(\Omega \times \mathbb{R}^N)$, and $\mathbf{M}(G_u) = \mathcal{H}^n(\mathcal{G}_u)$, see (2.3) for the case $n = N = 2$. In particular, Stokes theorem yields that G_u satisfies the *null-boundary condition* $(\partial G_u) \llcorner (\Omega \times \mathbb{R}^N) = 0$, see (2.5) below. Moreover, denoting by $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$ the graph map, by the area formula one has $G_u = (\mathrm{Id} \bowtie u)_\# [\![\Omega]\!]$, i.e.,

$$\langle G_u, \omega \rangle = \int_\Omega (\mathrm{Id} \bowtie u)^\# \omega \qquad \forall \, \omega \in \mathcal{D}^n(\Omega \times \mathbb{R}^N) \,. \tag{2.1}$$

Example 2.2 Property (1.3) is readily checked by means of an explicit computation in the integration by parts formulas that express the null-boundary condition of G_u. Precisely, setting $\widehat{dx^i}$ in such a way that $(-1)^{i-1} dx^i \wedge \widehat{dx^i} = dx$, where $dx := dx^1 \wedge \cdots \wedge dx^n$, for every $\phi \in C_c^\infty(\Omega \times \mathbb{R}^N)$ we have:

$$0 = \langle \partial G_u, (-1)^{i-1} \phi(x,y) \widehat{dx^i} \rangle = \langle G_u, \partial_{x_i} \phi(x,y) \, dx \rangle + (-1)^{i-1} \sum_{j=1}^N \langle G_u, \partial_{y_j} \phi(x,y) dy^j \wedge \widehat{dx^i} \rangle \qquad \forall \, i \,. \tag{2.2}$$

Therefore, using (2.1), definition (1.2) holds true with μ_i^j given by (1.3).

Cartesian currents. The class $\mathrm{cart}(\Omega \times \mathbb{R}^N)$ is defined in [14] by analyzing the properties of the currents T that are weak limits $G_{u_k} \stackrel{\mathcal{D}}{\rightharpoonup} T$ of smooth sequences $\{u_k\} \subset C^1(\Omega, \mathbb{R}^N)$ satisfying $\sup_k \mathbf{M}(G_{u_k}) < \infty$, on account of Federer-Fleming's theorem.

A current T in $\mathrm{cart}(\Omega \times \mathbb{R}^N)$ with underlying function u, decomposes as

$$T = G_u + S_T \,, \quad \mathbf{M}(T) = \mathbf{M}(G_u) + \mathbf{M}(S_T) < \infty$$

for some $u \in BV(\Omega, \mathbb{R}^N)$ such that all the minors of the $\mathbb{R}^{N \times n}$-valued Jacobian map $x \mapsto \nabla u(x)$ are summable in Ω. Therefore, when e.g. $n = N = 2$ we have $\det \nabla u \in L^1(\Omega)$. The graph current G_u is an i.m. rectifiable current in $\mathcal{R}_n(\Omega \times \mathbb{R}^N)$, and it can be defined by (2.1) through the action of the approximate gradient ∇u. As to the mass of G_u, when e.g. $n = N = 2$ by the area formula one has:

$$\mathbf{M}(G_u) = \int_\Omega \sqrt{1 + |\nabla u|^2 + (\det \nabla u)^2} \, dx < \infty \,. \tag{2.3}$$

Moreover, the i.m. rectifiable current $S_T \in \mathcal{R}_n(\Omega \times \mathbb{R}^N)$ has finite mass and is "vertical", i.e.,

$$\langle S_T, \phi(x,y) \, dx \rangle = 0 \tag{2.4}$$

for every $\phi \in C_c^\infty(\Omega \times \mathbb{R}^N)$. Finally, the current $T \in \mathcal{R}_n(\Omega \times \mathbb{R}^N)$ satisfies the null-boundary condition

$$\langle \partial T, \eta \rangle := \langle T, d\eta \rangle = 0 \qquad \forall \, \eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N) \tag{2.5}$$

that will be denoted by the equation $(\partial T) \llcorner (\Omega \times \mathbb{R}^N) = 0$.

Example 2.3 If $T \in \text{cart}(\Omega \times \mathbb{R}^N)$, by (2.4) and (2.5) one has

$$0 = \langle \partial T, (-1)^{i-1}\phi(x,y)\widehat{dx^i}\rangle = \langle G_u, \partial_{x_i}\phi(x,y)\,dx\rangle + (-1)^{i-1}\sum_{j=1}^N \langle T, \partial_{y_j}\phi(x,y)dy^j \wedge \widehat{dx^i}\rangle \qquad \forall\, i$$

for any $\phi \in C_c^\infty(\Omega \times \mathbb{R}^N)$. In particular, if $T = G_u$ for some smooth map u, the latter equation becomes the integration by parts formula in (2.2).

In the case $n = N$, for $j = 1, \ldots, n$ we define $\widehat{dy^j}$ so that $(-1)^{j-1}dy^j \wedge \widehat{dy^j} = dy := dy^1 \wedge \cdots \wedge dy^n$. In a similar way, by (2.5) we can write for each j and any $\phi \in C_c^\infty(\Omega \times \mathbb{R}^n)$

$$0 = \langle \partial T, \phi(x,y)(-1)^{j-1}\widehat{dy^j}\rangle = \sum_{i=1}^n \langle T, \partial_{x_i}\phi(x,y)(-1)^{j-1}dx^i \wedge \widehat{dy^j}\rangle + \langle T, \partial_{y_j}\phi(x,y)\,dy\rangle\,. \tag{2.6}$$

In particular, when $T = G_u$ for some smooth map u, equations (2.6) become the integration by parts formulas:

$$0 = \langle \partial G_u, \phi(x,y)(-1)^{j-1}\widehat{dy^j}\rangle = \sum_{i=1}^n \int_\Omega \partial_{x_i}\phi(x,u)\,(\text{adj}\,\nabla u)_i^j\,dx + \int_\Omega \partial_{y_j}\phi(x,u)\,\det \nabla u\,dx\,. \tag{2.7}$$

Remark 2.4 If $T = G_u + S_T \in \text{cart}(\Omega \times \mathbb{R}^N)$, since $\mathbf{M}(T) < \infty$, by dominated convergence we can define the lifting measure μ of u corresponding to T through the system:

$$\int_{\Omega\times\mathbb{R}^N} f(x,y)\,d\mu_i^j = (-1)^{i-1}\langle T, f(x,y)dy^j \wedge \widehat{dx^i}\rangle \quad \forall\, i,j \qquad \forall\, f \in C_b(\Omega \times \mathbb{R}^N)\,. \tag{2.8}$$

In fact, by the definition (1.2) we infer that

$$\int_{\Omega\times\mathbb{R}^N} g(x)\,d\mu_i^j = \int_\Omega g(x)\,\nabla_i u^j(x)\,dx + (-1)^{i-1}\langle S_T, g(x)dy^j \wedge \widehat{dx^i}\rangle \quad \forall\, i,j \qquad \forall\, g \in C_b(\Omega)\,.$$

Therefore, it turns out that in some sense the minimality assumption in Definition 1.2 involves the so called $(n-1,1)$-stratum of the vertical component S_T, see (3.1) below.

3 Completely vertical liftings

In this paper, we wish to extend the above minimality condition to the measures related to the higher strata of the vertical current S_T enclosing the graph current G_u. For this purpose, we shall again make use of the explicit formula of the null-boundary condition (2.5).

In this section, we deal with the case of dimensions $n = N = 2$, and let for simplicity $\Omega = B^2$, the unit ball in $\mathbb{R}^2$. We shall define the measure μ_v through the action of the completely vertical component of currents T in $\text{cart}(B^2 \times \mathbb{R}^2)$ with underlying BV-map equal to u. We recall that Jerrard-Jung were able to define the minimal lifting measure $\mu = \mu[u]$ by requiring that the total variation of the projection onto the domain satisfies $\pi_\# |\mu|(\Omega) = |Du|(\Omega)$, see Definition 1.2. Minimality of the completely vertical lifting, instead, is expressed by requiring that the $(1,1)$-*stratum* of T corresponds to the minimal lifting measure $\mu[u]$. With this notation, existence is guaranteed by Proposition 3.4, provided that u is the strict BV limit of a sequence of smooth maps whose gradient determinants are equibounded in L^1, see the (ABS) density property in Definition 3.2, whereas uniqueness is proved in Theorem 3.5. Finally, we discuss a class of (non-smooth) maps for which the explicit formula of μ_v is readily obtained.

COMPLETELY VERTICAL LIFTINGS. Any current $T \in \mathcal{R}_2(B^2 \times \mathbb{R}^2)$ is identified by the measures

$$\mu_h[T] := T \llcorner dx\,, \quad \mu_i^j[T] := T \llcorner (-1)^i dx^{\bar{i}} \wedge dy^j\,, \quad i,j = 1,2\,, \quad \mu_v[T] := T \llcorner dy \tag{3.1}$$

where $\bar{1} := 2$ and $\bar{2} := 1$. If $T = G_u + S_T$ is a current in $\text{cart}(B^2 \times \mathbb{R}^2)$, the horizontal component (i.e., the $(2,0)$-stratum of T) satisfies

$$\mu_h[T] = (Id \bowtie u)_\#(\mathcal{L}^2 \llcorner B^2)$$

and hence it only depends on the underlying map u. However, in general the intermediate components $\mu_i^j[T]$, i.e., the $(1,1)$-stratum of T, and the completely vertical component $\mu_v[T]$, the $(0,2)$-stratum, also depend on the component S_T.

On account of Remark 2.4, we shall restrict to the case when the intermediate components only depend on u through the formulas

$$\mu_i^j[T] = \mu_i^j[u] \quad \forall\, i,j \tag{3.2}$$

where $\mu_i^j[u]$ is for each $i,j=1,2$ the minimal lifting measure in the sense of Jerrard-Jung, see Definition 1.2. Notice in fact that equalities (3.2) make sense on all test functions in $C_b(B^2\times\mathbb{R}^2)$ by dominated convergence, see (2.8).

Definition 3.1 For any $u\in BV(B^2,\mathbb{R}^2)$ with $\det\nabla u\in L^1(B^2)$, we denote by $\mathcal{T}_u$ the class of currents $T=G_u+S_T$ in $\mathrm{cart}(B^2\times\mathbb{R}^2)$ such that (3.2) holds. A signed measure μ_v in $B^2\times\mathbb{R}^2$ is said to be a *completely vertical lifting* of u if

$$\mu_v=\mu_v[T] \tag{3.3}$$

where $\mu_v[T]:=T\llcorner dy$ for some $T\in\mathcal{T}_u$.

Now, if $T\in\mathcal{T}_u$, using (2.8) we have

$$\langle T,\partial_{x_i}\phi(x,y)(-1)^{j-1}dx^i\wedge\widehat{dy^j}\rangle=(-1)^{i+j}\int_{B^2\times\mathbb{R}^2}\partial_{x_i}\phi(x,y)\,d\mu_{\bar\imath}^{\bar\jmath}[u]$$

for every $\phi\in C_c^\infty(B^2\times\mathbb{R}^2)$ and hence we can equivalently write the null-boundary condition (2.6) as:

$$\sum_{i=1}^2(-1)^{i+j}\int_{B^2\times\mathbb{R}^2}\partial_{x_i}\phi(x,y)\,d\mu_{\bar\imath}^{\bar\jmath}[u]+\langle T,\partial_{y_j}\phi(x,y)\,dy\rangle=0 \qquad j=1,2\,. \tag{3.4}$$

Therefore, according to the notation (1.2), by (3.3) it turns out that the action of a completely vertical lifting measure of u on smooth functions $\phi\in C_c^\infty(B^2\times\mathbb{R}^2)$ satisfies the system

$$\sum_{i=1}^2(-1)^{i+j}\int_{B^2\times\mathbb{R}^2}\partial_{x_i}\phi(x,y)\,d\mu_{\bar\imath}^{\bar\jmath}[u]+\int_{B^2\times\mathbb{R}^2}\partial_{y_j}\phi(x,y)\,d\mu_v=0 \qquad j=1,2\,. \tag{3.5}$$

Notice that in the case of smooth maps $u:\overline{B^2}\to\mathbb{R}^2$, recalling that $(\mathrm{adj}\,\nabla u)_i^j=(-1)^{i+j}\nabla_{\bar\imath}u^{\bar\jmath}$ and taking $T=G_u$, on account of (2.7) we can choose

$$\mu_v=\mu_v[u]=(Id\bowtie u)_\#(\det\nabla u\,\mathcal{L}^2\llcorner B^2)\,. \tag{3.6}$$

Existence. The existence of a completely vertical lifting μ_v is guaranteed provided that the class $\mathcal{T}_u$ is non-empty. For this purpose, we give the following

Definition 3.2 Let $u\in BV(B^2,\mathbb{R}^2)$. We say that u satisfies the area bounded strict density property, say *(ABS) density property*, if there exists a sequence of smooth maps $\{u_k\}\subset C^1(B^2,\mathbb{R}^2)$ such that $u_k\overset{BV}{\to}u$ strictly in $BV(B^2,\mathbb{R}^2)$ and

$$\sup_k\int_{B^2}|\det\nabla u_k|\,dx<\infty\,. \tag{3.7}$$

Remark 3.3 The (ABS) density property implies that the area of the graphs of the smooth approximating sequence is equibounded, so that $\sup_k\mathbf{M}(G_{u_k})<\infty$, see (2.3). Therefore, it is equivalent to the boundedness of the relaxed area functional w.r.t. the strict BV convergence, see Sec. 9 below.

Our notation is motivated by the following

Proposition 3.4 *Let $u\in BV(B^2,\mathbb{R}^2)$ that satisfies the (ABS) density property, Definition 3.2. Then $\det\nabla u\in L^1(B^2)$ and the class $\mathcal{T}_u$ is non-empty, whence a completely vertical lifting is given according to Definition 3.1.*

PROOF: By Remark 3.3 we have $\sup_k \mathbf{M}(G_{u_k}) < \infty$. Since moreover u_k is smooth, Stokes theorem implies the null-boundary condition $(\partial G_{u_k}) \llcorner (B^2 \times \mathbb{R}^2) = 0$ for each k. Therefore, using the closure-compactness theorem in the class of Cartesian currents, see [14], it turns out that possibly passing to a (not relabeled) subsequence $G_{u_k} \stackrel{\mathcal{D}}{\rightharpoonup} T$ to some current T in $\mathrm{cart}(B^2 \times \mathbb{R}^2)$. By L^1-convergence, T has underlying map equal to u, whence we can write $T = G_u + S_T$. Since by lower semicontinuity $\mathbf{M}(T) \le \liminf_k \mathbf{M}(G_{u_k}) < \infty$, where $\mathbf{M}(T) = \mathbf{M}(G_u) + \mathbf{M}(S_T)$, by (2.3) we infer that $\det \nabla u \in L^1(B^2)$. Finally, the strict convergence $u_k \stackrel{BV}{\to} u$ implies that property (3.2) holds, whence $T \in \mathcal{T}_u$. □

UNIQUENESS. We now show that the completely vertical lifting of u is unique, when it does exist, whence we denote it by $\mu_v = \mu_v[u]$. In fact, we shall see in the next two sections that the completely vertical lifting only depends on u.

Theorem 3.5 *If for some u the class $\mathcal{T}_u$ in Definition 3.1 is non-empty, it contains only one element, that will be denoted by T_u.*

PROOF: If $T_1, T_2 \in \mathcal{T}_u$, the difference $\widehat{T} := T_1 - T_2$ is a current in $\mathcal{R}_2(B^2 \times \mathbb{R}^2)$ satisfying $\mu_h[\widehat{T}] = 0$, $\mu_i^j[\widehat{T}] = 0$ for $i, j = 1, 2$, see (3.1), and also $(\partial \widehat{T}) \llcorner (B^2 \times \mathbb{R}^2) = 0$. We now show that

$$\langle \widehat{T}, g(x)\phi_1(y_1)\phi_2(y_2)\, dy \rangle = 0 \qquad \forall\, g \in C_c^\infty(B^2)\,,\ \phi_j \in C_c^\infty(\mathbb{R})\,, \quad j = 1, 2\,. \tag{3.8}$$

Let $\Phi_j(y) := \int_{-\infty}^{y} \phi_j(t)\, dt$, so that Φ_j is a smooth bounded function, for $j = 1, 2$. We thus can write:

$$g(x)\phi_1(y_1)\phi_2(y_2)\, dy = \Psi^{\#}(g(x)\, dy)\,, \qquad \Psi(x, y) := (x, \Phi_1(y_1), \Phi_2(y_2))\,.$$

Denote now

$$\omega_2 := \frac{1}{2}\left(y_1\, dy^2 - y_2\, dy^2\right) \tag{3.9}$$

so that $d\omega_2 = dy := dy^1 \wedge dy^2$. Since the image current $\Psi_{\#}\widehat{T}$ satisfies the null-boundary condition $(\partial \Psi_{\#}\widehat{T}) \llcorner (B^2 \times \mathbb{R}^2) = 0$, by dominated convergence we have

$$0 = \langle \partial \Psi_{\#}\widehat{T}, g\, \omega_2 \rangle = \langle \Psi_{\#}\widehat{T}, d\,(g\, \omega_2) \rangle = \langle \Psi_{\#}\widehat{T}, dg \wedge \omega_2 \rangle + \langle \Psi_{\#}\widehat{T}, g\, dy \rangle\,.$$

Moreover, using that $\mu_i^j[\widehat{T}] = 0$ for $i, j = 1, 2$, we compute

$$\langle \Psi_{\#}\widehat{T}, dg \wedge \omega_2 \rangle = \langle \widehat{T}, \Psi^{\#}(dg \wedge \omega_2) \rangle = \frac{1}{2}\langle \widehat{T}, dg \wedge (\Phi_1 d\Phi_2 - \Phi_2 d\Phi_1) \rangle = 0$$

whereas

$$\langle \Psi_{\#}\widehat{T}, g\, dy \rangle = \langle \widehat{T}, \Psi^{\#}(g\, dy) \rangle = \langle \widehat{T}, g(x)\phi_1(y_1)\phi_2(y_2)\, dy \rangle$$

so that (3.8) holds true. Finally, since the vector space generated by linear combinations of products of functions $g(x)$, $\phi_1(y_1)$, and $\phi_2(y_2)$ as above is strongly dense in $\mathcal{D}^0(B^2 \times \mathbb{R}^2)$, by (3.8) we infer that $\mu_v[\widehat{T}] = 0$, see (3.1), whence $\widehat{T} = 0$ and $T_1 = T_2$, as required. □

Remark 3.6 If $u \in W^{1,1}(B^2, \mathbb{R}^2)$ satisfies the (ABS) density property, by equality (1.3) it turns out the component S_T of the Cartesian current T_u in $\mathcal{T}_u$ is "completely vertical", i.e., $\langle S_T, \omega \rangle = 0$ for each form $\omega \in \mathcal{D}^2(B^2 \times \mathbb{R}^2)$ with at most one differential dy^j.

CARTESIAN MAPS. If u is smooth, the completely vertical lifting $\mu_v[u]$ is given by (3.6). We now analyze a larger class of maps $u \in BV(\Omega, \mathbb{R}^2)$ for which formula (3.6) holds.

We recall from [14] that the class $\mathcal{A}^1(B^2, \mathbb{R}^2)$ is given by the summable maps u in $L^1(B^2, \mathbb{R}^2)$ that are approximately differentiable $\mathcal{L}^2$-a.e., with approximate gradient $\nabla u \in L^1(B^2, \mathbb{R}^{2\times 2})$ and with $\det \nabla u \in L^1(B^2)$. If $u \in \mathcal{A}^1(B^2, \mathbb{R}^2)$, the graph current G_u is well-defined by (2.1), with $\Omega = B^2$, through the action of the approximate gradient ∇u. It is again an i.m. rectifiable current in $\mathcal{R}_2(B^2 \times \mathbb{R}^2)$ with finite mass given by formula (2.3). However, in general the current $T = G_u$ fails to satisfy the null-boundary condition (2.5), where $n = N = 2$ and $\Omega = B^2$. In the case e.g. of the vortex map $u(x) = x/|x|$ discussed in Example 4.6 below, the graph current has a "hole" upon the origin O, see (4.8).

A map $u \in \mathcal{A}^1(B^2, \mathbb{R}^2)$ is said to be a *Cartesian map* in $\mathrm{cart}^1(B^2, \mathbb{R}^2)$ if in addition

$$(\partial G_u) \llcorner (B^2 \times \mathbb{R}^2) = 0\,.$$

Therefore, if $u \in \mathrm{cart}^1(B^2, \mathbb{R}^2)$, actually $u \in W^{1,1}(B^2, \mathbb{R}^2)$. Moreover, G_u is a current in $\mathrm{cart}(B^2 \times \mathbb{R}^2)$, and trivially $T_u = G_u$ satisfies (3.2), so that the completely vertical measure $\mu_v[u]$ is given by (3.6). Therefore, we conclude with the following

Corollary 3.7 *If $u \in \mathrm{cart}^1(B^2, \mathbb{R}^2)$, the completely vertical lifting of u is given by the formula* (3.6).

Remark 3.8 Notice however that it is not clear whether any map in $\mathrm{cart}^1(B^2, \mathbb{R}^2)$ satisfies the (ABS) density property. In fact, the strict BV density result from [5] is based on a convolution argument, but $|\det \nabla u|$ fails to be a convex function of the gradient, whence property (3.7) is not guaranteed. The same problem is open for maps $u \in BV(B^2, \mathbb{R}^2)$ with $\det \nabla u \in L^1(B^2)$, even in the bounded case.

4 Projection formula

In this section, we begin to analyze the completely vertical lifting $\mu_v[u]$, assuming it exists, obtaining the formula of the image measure $\pi_{\#}\mu_v[u]$, where $\pi : B^2 \times \mathbb{R}^2 \to B^2$ is the orthogonal projection onto the domain. This yields to a lower bound for the total variation of $\mu_v[u]$. We then focus on the case of bounded maps, and give some examples showing different features.

Projection formula. Let u and $\mu_v[u]$ as in Definition 3.1. Using (1.4), formula (3.5) reads as the system

$$\int_{B^2 \times \mathbb{R}^2} \partial_{y_j} \phi(x, y)\, d\mu_v[u] = -\sum_{i=1}^{2} (-1)^{i+j} \int_{B^2} \Big(\int_0^1 \partial_{x_i} \phi(x, u^\theta(x)) d\theta \Big)\, d(Du)^{\bar{j}}_{\bar{i}}\,, \quad j = 1, 2\,. \tag{4.1}$$

Taking in particular $\phi(x, y) = g(x)\, y_j$, by summating on $j = 1, 2$ and dividing by two we get:

$$\begin{aligned} \int_{B^2} g(x)\, d\mu_v[u] = & -\frac{1}{2} \sum_{i,j=1}^{2} (-1)^{i+j} \int_{B^2} \partial_{x_i} g(x) \Big(\int_0^1 u^{j\theta}(x) d\theta \Big)\, d(Du)^{\bar{j}}_{\bar{i}} \\ = & -\frac{1}{2} \sum_{i,j=1}^{2} (-1)^{i+j} \int_{B^2} \partial_{x_i} g(x)\, \overline{u}^j(x)\, d(Du)^{\bar{j}}_{\bar{i}} \end{aligned} \tag{4.2}$$

where the average functions are given by (1.1). As a consequence, it turns out that $\overline{u}^j$ is locally summable w.r.t. $Du^{\bar{j}}$, and hence $u^1 u^2 \in BV(B^2)$, see Remark 1.1.

We thus introduce the $\mathbb{R}^2$-valued measure $\mathbf{m}_u = (\mathbf{m}_u^1, \mathbf{m}_u^2)$ with components depending on the distributional derivative Du and on the average $\overline{u} = (\overline{u}^1, \overline{u}^2)$ of u as follows

$$\mathbf{m}_u^1 := \frac{1}{2} \big(\overline{u}^1\, (Du)^2_2 - \overline{u}^2\, (Du)^1_2 \big)\,, \qquad \mathbf{m}_u^2 := \frac{1}{2} \big(\overline{u}^2\, (Du)^1_1 - \overline{u}^1\, (Du)^2_1 \big)$$

and observe that in the distributional sense we can define:

$$\mathrm{Div}\, \mathbf{m}_u = \frac{1}{2} \sum_{i,j=1}^{2} (-1)^{i+j} D_i \big(\overline{u}^j\, (Du)^{\bar{j}}_{\bar{i}} \big)\,.$$

We have actually proved the following projection formula:

Theorem 4.1 *Let $u \in BV(B^2, \mathbb{R}^2)$, with $\det \nabla u \in L^1(B^2)$. Assume that the completely vertical lifting $\mu_v[u]$ of u does exist, see Definition* 3.1. *Then $u^1 u^2 \in BV(B^2, \mathbb{R})$, the $\mathbb{R}^2$-valued measure $\mathbf{m}_u$ is well-defined, and we have:*

$$\pi_{\#}\mu_v[u] = \mathrm{Div}\, \mathbf{m}_u\,. \tag{4.3}$$

Therefore, the distribution $\mathrm{Div}\, \mathbf{m}_u$ is a signed measure with finite total variation.

Assume now that in particular $u \in W^{1,1}(\Omega, \mathbb{R}^2)$. Then we have:

$$\operatorname{Div} \mathbf{m}_u = \frac{1}{2} \operatorname{Div}\big(u^1 \nabla_2 u^2 - u^2 \nabla_2 u^1 \, , \, u^2 \nabla_1 u^1 - u^1 \nabla_1 u^2\big) \, .$$

We recall that the *distributional determinant* $\operatorname{Det} \nabla u$ is well defined by the right-hand side of the latter formula, provided that $u \in W^{1,1}(\Omega, \mathbb{R}^2)$ is a bounded map. In our context, we have:

Corollary 4.2 *Let $u \in W^{1,1}(B^2, \mathbb{R}^2)$, with $\det \nabla u \in L^1(B^2)$, and assume that the completely vertical lifting $\mu_v[u]$ of u does exist. Then the distributional determinant of u is a signed measure with finite total variation, and we have*

$$\pi_\# \mu_v[u] = \operatorname{Det} \nabla u \, . \tag{4.4}$$

Total variation lower bound. Since $\mu_v[u]$ is a signed measure,

$$\langle \operatorname{Div} \mathbf{m}_u, B \rangle = \pi_\# \mu_v[u](B) = \mu_v[u](B \times \mathbb{R}^2)$$

for each Borel set $B \subset B^2$. Therefore, by Theorem 4.1 we readily obtain:

Corollary 4.3 *If u is a BV map satisfying the hypotheses of Theorem* 4.1*, then:*

$$|\operatorname{Div} \mathbf{m}_u|(B^2) \leq |\mu_v[u]|(B^2 \times \mathbb{R}^2) < \infty \, .$$

For Sobolev maps we know that (1.3) holds, with $\Omega = B^2$ and $n = 2$. Corollary 4.2 yields:

Corollary 4.4 *If u is a map in $W^{1,1}(B^2, \mathbb{R}^2)$ satisfying the hypotheses of Corollary* 4.2*, then:*

$$|\operatorname{Det} \nabla u|(B^2) \leq |\mu_v[u]|(B^2 \times \mathbb{R}^2) < \infty \, .$$

The bounded case. If $u \in BV(B^2, \mathbb{R}^2)$ is a bounded map satisfying the hypotheses of Theorem 4.1, the measure $\operatorname{Div} \mathbf{m}_u$ can be written as the sum of three distributions, involving the absolutely continuous, Cantor and Jump part of the distributional derivative of u, respectively.

The first term agrees with the distributional determinant $\operatorname{Det} \nabla u$, whereas the second term is given by the distributional divergence of the $\mathbb{R}^2$-valued measure $F_u = (F_u^1, F_u^2)$ with components

$$F_u^1 := \frac{1}{2} \big(u^1 \, (D^C u)_2^2 - u^2 \, (D^C u)_2^1\big) \, , \qquad F_u^2 := \frac{1}{2} \big(u^2 \, (D^C u)_1^1 - u^1 \, (D^C u)_1^2\big)$$

where u is a precise representative. By (4.2), the third term is given for any $g \in C_c^\infty(B^2)$ by:

$$\frac{1}{2} \sum_{i,j=1}^{2} (-1)^{i+j} \int_{J_u} \partial_{x_i} g(x) \Big(\int_0^1 u^{j\theta}(x) d\theta \Big) \, (u^{\bar{\jmath}+} - u^{\bar{\jmath}-})(x) \, \nu_{\bar{\imath}} \, d\mathcal{H}^1 \, .$$

Now, setting for $\mathcal{H}^1$-a.e. $x \in J_u$

$$\Delta_u^J(x) := \frac{1}{2} \big(u^{1-}(x) \, u^{2+}(x) - u^{1+}(x) \, u^{2-}(x)\big) \tag{4.5}$$

we compute

$$\Delta_u^J(x) = \frac{1}{2} \sum_{j=1}^{2} (-1)^{j+1} \Big(\int_0^1 u^{j\theta}(x) \, d\theta \Big) \, (u^{\bar{\jmath}+} - u^{\bar{\jmath}-})(x) \, .$$

Therefore, denoting by $\tau := *\nu = (-\nu_2, \nu_1)$ the unit tangent vector to the Jump set, we get:

$$\langle \operatorname{Div} \mathbf{m}_u, g \rangle = \langle \operatorname{Det} \nabla u, g \rangle + \langle \operatorname{Div} F_u, g \rangle + \langle \mu_u^J, g \rangle \quad \forall \, g \in C_c^\infty(B^2) \tag{4.6}$$

where μ_u^J is the distribution

$$\langle \mu_u^J, g \rangle := \int_{J_u} \Delta_u^J(x) \, \partial_\tau g(x) \, d\mathcal{H}^1 \, , \quad g \in C_c^\infty(B^2) \, .$$

Of course, each term in the decomposition formula (4.6) is equal to zero when the corresponding component of the distributional derivative of u is zero. We e.g. have $\operatorname{Div} F_u = 0$ if $D^C u = 0$, i.e., if u is a special function of bounded variation, say $u \in SBV$.

However, Example 4.7 shows that in general, even for an SBV map u such that both the distributions $\operatorname{Det}\nabla u$ and μ_u^J are measures, *they are not mutually singular*. Notwithstanding, we expect that whenever Theorem 4.1 holds, the three terms in the decomposition formula (4.6) are finite signed measures, and the total variation of the measure $\operatorname{Div}\mathbf{m}_u$ satisfies the additivity property:

$$|\operatorname{Div}\mathbf{m}_u| = |\operatorname{Det}\nabla u| + |\operatorname{Div} F_u| + |\mu_u^J| < \infty\,. \tag{4.7}$$

Remark 4.5 In definition (4.5), the number $|\Delta_u^J|$ is equal to the area of the triangle of the target space generated by the jump vectors u^- and u^+. Moreover, $\Delta_u^J = 0$ when u^- and u^+ are collinear, otherwise the sign of Δ_u^J is consistent with the orientation of the ordered couple of vectors (u^-, u^+). For future use, we also point out that for $\mathcal{H}^1$-a.e. $x \in J_u$

$$\Delta_u^J(x) = \int_{\gamma_{u,x}} \omega_2\,, \qquad \omega_2 := \frac{1}{2}\,(y_1\, dy^2 - y_2\, dy^2)$$

where $\gamma_{u,x}$ is the oriented segment $\gamma_{u,x} : [0,1] \to \mathbb{R}^2$ with end points u^- and u^+, given by

$$\gamma_{u,x}(\theta) := (u^{1\theta}(x), u^{2\theta}(x))\,, \quad u^{j\theta}(x) = \theta\, u^{j+}(x) + (1-\theta)\, u^{j-}(x)\,, \qquad \theta \in [0,1]\,.$$

Some examples. We now observe different behaviors of the projected measure $\pi_\# \mu_v[u]$. In all the following examples, the map u is in L^∞ and satisfies the hypotheses of Theorem 4.1, whence the decomposition formula (4.6) of the signed measure $\operatorname{Div}\mathbf{m}_u$ holds. Notice that according to equation (3.3), the value of the total variation of $\mu_v[u]$ can be computed in terms of the $(0,2)$-stratum of the unique Cartesian current T_u with underlying map u and with $(1,1)$-stratum satisfying (3.2). On the other hand, we do not know if equation (4.7) holds, in general.

Example 4.6 Taking the vortex map $u(x) = x/|x|$, we have $D^J u = D^C u = 0$, whence $\operatorname{Div} F_u = \mu_u^J = 0$, whereas $\det \nabla u = 0$ and $\operatorname{Det}\nabla u = (\operatorname{Det}\nabla u)^s = \pi\,\delta_O$, where δ_O is the unit Dirac mass at the origin. We thus get $\pi_\#\mu_v[u] = \pi\,\delta_O$. Notice that the latter formula can be checked by computing the action of the Cartesian current $T_u = G_u + \delta_O \times [\![\, D^2]\!]$ on 2-forms $g(x_1,x_2)\, dy$, where $dy := dy^1 \wedge dy^2$ and

$$D^2 := \{(y_1,y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 < 1\}$$

is the oriented unit disk in the target space. In fact, compare [14, Sec. 3.2.2], equation

$$(\partial G_u) \llcorner (B^2 \times \mathbb{R}^2) = -\delta_O \times \partial [\![\, D^2]\!] \tag{4.8}$$

yields that T_u satisfies the null-boundary condition (2.5), whereas (3.2) holds true. Since moreover the $(0,2)$-stratum of the current T_u is identified by the term $\delta_O \times [\![\, D^2]\!]$, we have:

$$\mathbf{M}(\delta_O \times [\![\, D^2]\!]) = |\mu_v[u]|(B^2 \times \mathbb{R}^2) = |\operatorname{Det}\nabla u|(B^2) = \pi$$

and hence equality holds in the lower bound from Corollary 4.4.

Example 4.7 Let $u(x_1,x_2) = (x_1,x_2)$ if $x_1 < 0$ and $u(x_1,x_2) = (x_1+1,x_2)$ if $x_1 > 0$. Since $D^C u = 0$ we have $\operatorname{Div} F_u = 0$, whereas

$$\operatorname{Det}\nabla u = \frac{1}{2}\,\operatorname{Div}(u_1,u_2)$$

whence $\operatorname{Det}\nabla u = \det\nabla u\,\mathcal{L}^2 \llcorner B^2 + (\operatorname{Det}\nabla u)^s$, with $\det\nabla u = 1$ and singular part

$$(\operatorname{Det}\nabla u)^s = \frac{1}{2}\mathcal{H}^1 \llcorner J_u\,, \qquad J_u = \{0\} \times (-1,1)\,.$$

Also, taking $\nu = (1,0)$ and $\tau = *\nu = (0,1)$, so that $\partial_\tau g = \partial_{x_2} g$, we have $u^-(x) = (0,x_2)$, $u^+(x) = (1,x_2)$ and $\Delta^J u(x) = -x_2/2$ for all $x \in J_u$. Therefore, integrating by parts we get for any $g \in C_c^1(B^2)$

$$\langle \mu_u^J, g\rangle := \int_{J_u} \Delta_u^J(x)\,\partial_\tau g(x)\, d\mathcal{H}^1 = -\frac{1}{2}\int_{-1}^1 x_2\,\partial_{x_2} g(0,x_2)\, dx_2 = \frac{1}{2}\int_{-1}^1 g(0,x_2)\, dx_2 = \langle (\operatorname{Det}\nabla u)^s, g\rangle\,.$$

We thus conclude that $\pi_\# \mu_v[u] = \mathcal{L}^2 \llcorner B^2 + \mathcal{H}^1 \llcorner J_u$, but the two terms $\operatorname{Det} \nabla u$ and μ_u^J are not mutually singular measures, even if (4.7) holds.

The latter formula can be checked by computing on forms $g(x)\, dy$ the action of the Cartesian current $T_u = G_u + S_T$, where $S_T := -\psi_\#([\![-1,1]\!] \times [\![0,1]\!])$ with $\psi(\lambda, s) := (0, \lambda, s, \lambda)$. Notice in fact that

$$(\partial G_u) \llcorner (B^2 \times \mathbb{R}^2) = (\partial \psi_\#([\![-1,1]\!] \times [\![0,1]\!])) \llcorner (B^2 \times \mathbb{R}^2) = \gamma_{0\#}[\![-1,1]\!] - \gamma_{1\#}[\![-1,1]\!]$$

where $\gamma_0(\lambda) := (0, \lambda, 0, \lambda)$ and $\gamma_1(\lambda) := (0, \lambda, 1, \lambda)$, for $\lambda \in (-1,1)$. Therefore, again T_u satisfies the null-boundary condition (2.5), whereas (3.2) holds true. In particular, this time the $(0,2)$-stratum of T_u satisfies

$$\langle \mu_v[T_u], \phi \rangle = \langle T_u, \phi\, dy \rangle = \int_{B^2} \phi(x, u(x))\, dx + \int_{(-1,1)\times(0,1)} \phi(0, \lambda, s, \lambda)\, d\lambda\, ds$$

for every $\phi \in C_c^\infty(B^2 \times \mathbb{R}^2)$, whence we obtain:

$$|\mu_v[u]|(B^2 \times \mathbb{R}^2) = |B^2| + \mathcal{L}^2((-1,1) \times (0,1)) = |\operatorname{Det} \nabla u|(B^2) + |\mu_u^J|(B^2)$$

and hence equality holds in the lower bound from Corollary 4.3.

Example 4.8 Taking instead $u(x_1, x_2) = (x_1 + a, x_2)$ if $x_1 < 0$ and $u(x_1, x_2) = (x_1 + b, x_2)$ if $x_1 > 0$, where $a < b$, we similarly obtain $\operatorname{Div} F_u = 0$,

$$\operatorname{Det} \nabla u = \mathcal{L}^2 \llcorner B^2 + \frac{b-a}{2}\, \mathcal{H}^1 \llcorner J_u \,, \qquad J_u = \{0\} \times (-1,1)$$

and $\Delta^J u(x) = (a-b)\, x_2/2$ for all $x \in J_u$, so that we again have

$$\langle \mu_u^J, g \rangle = \frac{b-a}{2} \int_{-1}^1 g(0, x_2)\, dx_2 = \langle (\operatorname{Det} \nabla u)^s, g \rangle \quad \forall\, g \in C_c^\infty(B^2)$$

and hence $\pi_\# \mu_v[u] = \mathcal{L}^2 \llcorner B^2 + (b-a)\, \mathcal{H}^1 \llcorner J_u$, so that similar conclusions to the one in the previuos example are readily checked.

Example 4.9 Take this time $\Omega = (0,1)^2$ and $u(x_1, x_2) = (v(x_1), v(x_2))$, where v is the classical Cantor-Vitali function. We have $\nabla u = 0$, $D^J u = 0$, and $(D^C u)_2^1 = (D^C u)_1^2 = 0$, whereas

$$(D^C u)_1^1 = D^C v \otimes (\mathcal{L}^1 \llcorner (0,1))\,, \qquad (D^C u)_2^2 = (\mathcal{L}^1 \llcorner (0,1)) \otimes D^C v$$

where $D^C v$ is concentrated on the classical middle thirds Cantor set in $[0,1]$. We thus get $\operatorname{Det} \nabla u = 0$ and $\mu_u^J = 0$, but

$$\begin{aligned} 2 \operatorname{Div} F_u = &\ \operatorname{Div}\big(u^1\, (D^C u)_2^2 - u^2\, (D^C u)_2^1\,,\ u^2\, (D^C u)_1^1 - u^1\, (D^C u)_1^2\big) \\ = &\ \operatorname{Div}\big(v(x_1)\, ((\mathcal{L}^1 \llcorner (0,1)) \otimes D^C v),\ v(x_2)\, (D^C v \otimes (\mathcal{L}^1 \llcorner (0,1))\big) \end{aligned}$$

so that integrating by parts we get:

$$\operatorname{Div} \mathbf{m}_u = \operatorname{Div} F_u = D^C v \otimes D^C v\,.$$

Now, taking $u_k(x) = (v_k(x_1), v_k(x_2))$, where $v_k : (0,1) \to (0,1)$ is the k-th step Lipschitz approximation of the Cantor-Vitali function, we clearly have $\int_\Omega |\det \nabla v_k|\, dx = 1$ for each k. Therefore, by an easy argument based on convolution and diagonalization, we can find a smooth sequence $\{u_k\} \subset C^1(\Omega, \mathbb{R}^2)$ such that $u_k \stackrel{BV}{\rightharpoonup} u$ and $\int_\Omega |\det \nabla u_k|\, dx \to 1$. Therefore, it turns out that $G_{u_k} \stackrel{\mathcal{D}}{\rightharpoonup} T_u$ to the unique current $T_u \in \operatorname{cart}(\Omega \times \mathbb{R}^2)$ with underlying map u and such that (3.2) holds true, whereas by lower semicontinuity

$$\mu_u[v](\Omega \times \mathbb{R}^2) \le \liminf_{k\to\infty} \int_\Omega |\det \nabla u_k|\, dx = 1$$

and hence, using that

$$\pi_\# \mu_v[u] = D^C v \otimes D^C v \quad \Longrightarrow \quad \pi_\# |\mu_v[u]|(\Omega) = |D^C v \otimes D^C v|(\Omega) = 1$$

we obtain

$$|\mu_v|(\Omega \times \mathbb{R}^2) = |\operatorname{Div} \mathbf{m}_u|(\Omega) = |\operatorname{Div} F_u|(\Omega) = 1\,.$$

Again, replacing B^2 with $(0,1)^2$, it turns out that equality holds in the lower bound from Corollary 4.3.

Example 4.10 S. Müller [23] showed the existence of a very interesting Sobolev map $u : \Omega \to \mathbb{R}^2$, where $\Omega = (0,1)^2$, such that $|\nabla u^1| \cdot |\nabla u^2| = 0$ a.e. in Ω, whence $\det \nabla u = 0$, but

$$\operatorname{Det} \nabla u = (\operatorname{Det} \nabla u)^s = D^C v \otimes D^C v$$

where v is a pure Cantor function, whence the singular part of the distributional determinant is concentrated on $C \times C$, where C is a Cantor-type set in $[0,1]$.

In [14, Sec. 4.2.5, Ex. 9], it is correspondingly shown the existence of the unique Cartesian current $T_u \in \operatorname{cart}(\Omega \times \mathbb{R}^2)$ with underlying map u and satisfying (3.2). Therefore, since $D^J u = D^C u = 0$, with our notation we have $\operatorname{Div} F_u = \mu_u^J = 0$ and

$$\pi_\# \mu_v[u] = \operatorname{Div} \mathbf{m}_u = \operatorname{Det} \nabla u \quad \Longrightarrow \quad \pi_\# |\mu_v[u]|(\Omega) = |D^C v \otimes D^C v|(\Omega)\,.$$

From the non-trivial construction in [14], we expect that in this example from [23] equality holds in the lower bound from Corollary 4.4, replacing B^2 with $(0,1)^2$, but we were not able to check all the details.

Notice that in [14] it is also proved that the boundary of the graph current G_u fails to have finite mass. This property can be readily checked by means of a contradiction argument as a direct consequence of a result taken from [21], where we showed that if $u \in W^{1,1}(\Omega, \mathbb{R}^2) \cap L^\infty$ satisfies $\det \nabla u \in L^1(\Omega)$ and condition $\mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^2) < \infty$, then the singular part $(\operatorname{Det} \nabla u)^s$ of the distributional determinant is concentrated on a countable set.

5 An explicit formula

In this section, we compute the action of the completely vertical lifting $\mu_v[u]$ on a dense class of test functions, giving an example. This yields that the completely vertical lifting only depends on the map u. As a consequence, we obtain an explicit formulation of the total variation of $\mu_v[u]$. Finally, for our purposes, the relationship between the distributional determinant of composition maps $\Phi(u)$ and the boundary current ∂G_u is discussed.

Explicit formulas. If $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth and bounded vector field, for any $u \in BV(B^2, \mathbb{R}^2)$ the composition $U := \Phi(u)$ is a map of bounded variation in $BV(B^2, \mathbb{R}^2) \cap L^\infty$. As a consequence, writing $U = (U^1, U^2)$, both the distributions

$$\operatorname{Det} \nabla[\Phi(u)] = \operatorname{Det} \nabla U := \frac{1}{2} \operatorname{Div}\Big(\big(U^1\, \nabla_2 U^2 - U^2\, \nabla_2 U^1\big)\,,\, \big(U^2\, \nabla_1 U^1 - U^1\, \nabla_1 U^2\big)\Big)$$

$$\operatorname{Div} F_{\Phi(u)}\,, \quad F_{\Phi(u)} := \frac{1}{2} \Big(\big(U^1\,(D^C U)^2_2 - U^2\,(D^C U)^1_2\big)\,,\, \big(U^2\,(D^C U)^1_1 - U^1\,(D^C U)^2_1\big)\Big)$$

are well defined. We also recall that $\gamma_{u,x}$ denotes the oriented segment $\gamma_{u,x} : [0,1] \to \mathbb{R}^2$ given for $\mathcal{H}^1$-a.e. $x \in J_u$ by $\gamma_{u,x}(\theta) := u^\theta(x)$, $\theta \in [0,1]$, see Remark 4.5, and that ω_2 is the 2-form given by (3.9).

Theorem 5.1 *Let $u \in BV(B^2, \mathbb{R}^2)$ with $\det \nabla u \in L^1(B^2)$. If $\mu_v[u]$ is the completely vertical lifting of u, see Definition 3.1, for every $g \in C_c^\infty(B^2)$ and $\phi_j \in C_c^\infty(\mathbb{R})$, $j = 1, 2$*

$$\int_{B^2} g(x) \phi_1(y_1) \phi_2(y_2)\, d\mu_v[u] = \langle \operatorname{Det} \nabla[\Phi(u)], g \rangle + \langle \operatorname{Div} F_{\Phi(u)}, g \rangle + \int_{J_u} \Big(\int_{\gamma_{u,x}} \Phi^\# \omega_2 \Big)\, \partial_\tau g(x)\, d\mathcal{H}^1$$

where $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is the smooth and bounded vector field

$$\Phi(y) := (\Phi_1(y_1), \Phi_2(y_2))\,, \qquad y = (y_1, y_2)$$

the function Φ_j being a primitive of ϕ_j, for $j = 1, 2$, so that

$$\Phi^\# \omega_2 = \frac{1}{2} \Big(\Phi_1(y_1)\, \phi_2(y_2)\, dy^2 - \Phi_2(y_2)\, \phi_1(y_1)\, dy^1 \Big)\,. \tag{5.1}$$

PROOF: Since $d(g\,\omega_2) = dg \wedge \omega_2 + g\,d\omega_2 = dg \wedge \omega_2 + g\,dy$, and $\Psi(x,y) := (x, \Phi(y))$ defines a smooth and bounded vector field from $B^2 \times \mathbb{R}^2$ into itself, we have:

$$\Psi^{\#}(g\,dy) = \Psi^{\#} d(g\,\omega_2) - \Psi^{\#}(dg \wedge \omega_2) = d\Psi^{\#}(g\,\omega_2) - dg \wedge \Phi^{\#}\omega_2$$

where for any $(x,y) \in B^2 \times \mathbb{R}^2$

$$\Psi^{\#}(g\,dy) = g\,\Phi^{\#} dy = g(x)\,\phi_1(y_1)\,\phi_2(y_2)\,dy\,.$$

Let $T_u = G_u + S_T$ the current in $\mathrm{cart}(B^2 \times \mathbb{R}^2)$ such that (3.2) holds true. By the null-boundary condition (2.5), we infer that

$$\langle T_u, d\Psi^{\#}(g\,\omega_2)\rangle = \langle \partial T_u, \Psi^{\#}(g\,\omega_2)\rangle = 0$$

and hence we get

$$\int_{B^2} g(x)\phi_1(y_1)\phi_2(y_2)\,d\mu_v[u] = \langle T_u, g(x)\phi_1(y_1)\phi_2(y_2)\,dy\rangle = \langle T_u, \Psi^{\#}(g\,dy)\rangle = -\langle T_u, dg \wedge \Phi^{\#}\omega_2\rangle\,.$$

Notice also that

$$\Psi_{\#}T_u = G_U + \Psi_{\#}S_T\,, \qquad U(x) := \Phi(u(x)) = (\Phi_1(u^1(x)), \Phi_2(u^2(x))\,.$$

On account of (5.1), we are thus led to apply formula (4.1) with $\phi(x,y) = g(x)\,F_j(y_j)\,\phi_{\bar{j}}(y_{\bar{j}})$. By using the decomposition $DU = \tilde{D}U + D^J U$ into diffuse and jump part, and recalling that $\tau = *\nu = (-\nu_2, \nu_1)$, on account of the chain-rule formula we obtain (after averaging between $j = 1, 2$):

$$\begin{aligned}
\int_{B^2} g(x)\phi_1(y_1)\phi_2(y_2)\,d\mu_v[u] = &-\frac{1}{2}\sum_{i=1}^{2}(-1)^{i+j}\int_{B^2}\partial_{x_i} g(x)\,U^j(x) d(\tilde{D}U)^{\bar{j}}_{\bar{i}} \\
&+\int_{J_u}\Big(\int_0^1 \frac{1}{2}\sum_{j=1}^{2}(-1)^{j-1}\Phi_j(u^{j\theta}(x))\,\phi_{\bar{j}}(u^{\bar{j}\theta}(x))\,(u^{\bar{j}+} - u^{\bar{j}-})(x)\,d\theta\Big)\,\partial_\tau g(x)\,d\mathcal{H}^1 \\
=: &\, I_1 + I_2
\end{aligned}$$

where the first term further decomposes as

$$I_1 = \langle \mathrm{Det}\,\nabla U, g\rangle + \langle \mathrm{Div}\,F_U, g\rangle\,, \qquad U = \Phi(u)\,.$$

As to the second term I_2, using that $\partial_\theta[\Phi_{\bar{j}}(u^{\bar{j}\theta}(x))] = \phi_{\bar{j}}(u^{\bar{j}\theta}(x))\,(u^{\bar{j}+} - u^{\bar{j}-})(x)$, by (5.1) we get:

$$\begin{aligned}
\int_{\gamma_{u,x}} \Phi^{\#}\omega_2 = &\int_0^1 \Big(\frac{1}{2}\sum_{j=1}^{2}(-1)^{j-1}\Phi_j((u^j)^\theta(x))\,\partial_\theta[\Phi_{\bar{j}}(u^{\bar{j}\theta}(x))]\Big)\,d\theta \\
= &\int_0^1 \Big(\frac{1}{2}\sum_{j=1}^{2}(-1)^{j-1}\Phi_j((u^j)^\theta(x))\,\phi_{\bar{j}}((u^{\bar{j}})^\theta(x))\,(u^{\bar{j}+} - u^{\bar{j}-})(x)\Big)\,d\theta
\end{aligned} \tag{5.2}$$

for $\mathcal{H}^1$-a.e. $x \in J_u$, as required. □

Notice that in general

$$\langle \mu^J_{\Phi(u)}, g\rangle \neq \int_{J_u}\Big(\int_{\gamma_{u,x}} \Phi^{\#}\omega_2\Big)\,\partial_\tau g(x)\,d\mathcal{H}^1\,.$$

In fact, with $U = \Phi(u)$, by Definition 4.5 we have

$$\langle \mu^J_{\Phi(u)}, g\rangle = \int_{J_u} \Delta^J_U(x)\,\partial_\tau g(x)\,d\mathcal{H}^1\,, \qquad \Delta^J_U(x) := \frac{1}{2}\big(U^{1-}(x)\,U^{2+}(x) - U^{1+}(x)\,U^{2-}(x)\big)$$

with $U^{j\pm}(x) = \Phi_j(u^{j\pm}(x))$, whereas the integral $\int_{\gamma_{u,x}} \Phi^{\#}\omega_2$ is given by (5.2).

Example 5.2 Coming back to the SBV-map u from Example 4.7, following the notation in Theorem 5.1 we readily compute $\mathrm{Div}\, F_{\Phi(u)} = 0$,

$$\mathrm{Det}\,\nabla[\Phi(u)] = \phi_1(u^1)\cdot\phi_2(u^2)\,\mathcal{L}^2 \llcorner B^2 + \frac{1}{2}\,[\Phi_1(1)-\Phi_1(0)] \int_{-1}^{1} g(0,\lambda)\,\phi_2(\lambda)\,d\lambda$$

and

$$\int_{J_u}\Big(\int_{\gamma_{u,x}} \Phi^{\#}\omega_2\Big)\,\partial_\tau g(x)\,d\mathcal{H}^1 = \frac{1}{2}\,[\Phi_1(1)-\Phi_1(0)] \int_{-1}^{1} g(0,\lambda)\,\phi_2(\lambda)\,d\lambda\,.$$

Finally, recalling that the vector space generated by linear combinations of products of functions $g(x)$, $\phi_1(y_1)$, and $\phi_2(y_2)$ as above is strongly dense in $\mathcal{D}^0(B^2\times\mathbb{R}^2)$, by Theorem 5.1 we immediately obtain:

Corollary 5.3 *Let $u\in BV(B^2,\mathbb{R}^2)$ with $\det\nabla u\in L^1(B^2)$ such that the completely vertical lifting $\mu_v[u]$ exists. Then*

$$|\mu_v[u]|(B^2\times\mathbb{R}^2) = \sup\Big\{\langle \mathrm{Det}\,\nabla[\Phi(u)],g\rangle + \langle \mathrm{Div}\,F_{\Phi(u)},g\rangle + \int_{J_u}\Big(\int_{\gamma_{u,x}}\Phi^{\#}\omega_2\Big)\,\partial_\tau g(x)\,d\mathcal{H}^1 \mid g\in\mathcal{G}\,,\ \Phi\in\mathcal{F}\Big\}$$

where $\mathcal{G} := \{g\in C_c^\infty(B^2) \,:\, \|g\|_\infty\le 1\}$ and

$$\mathcal{F} := \{\Phi\in C^\infty(\mathbb{R}^2,\mathbb{R}^2) \mid \Phi(y) = (\Phi_1(y_1),\Phi_2(y_2))\,,\ \Phi_j'\in C_c^\infty(\mathbb{R})\,,\ \|\Phi_j'\|_\infty\le 1 \text{ for } j=1,2\}\,. \tag{5.3}$$

In particular, if $u\in W^{1,1}(B^2,\mathbb{R}^2)$ we have:

$$|\mu_v[u]|(B^2\times\mathbb{R}^2) = \sup\{|\,\mathrm{Det}\,\nabla[\Phi(u)]|(B^2) \,:\, \Phi\in\mathcal{F}\}\,. \tag{5.4}$$

Distributional determinant of compositions. For future use, we finally point out the relationship between the distributional determinant of the composition maps $\Phi(u)$ and the boundary current ∂G_u.

Proposition 5.4 *Let $u\in W^{1,1}(B^2,\mathbb{R}^2)$ with $\det\nabla u\in L^1(B^2)$ such that the completely vertical lifting $\mu_v[u]$ exists. Then for every $\Phi\in\mathcal{F}$ and $g\in C_c^\infty(B^2)$*

$$\begin{aligned}\langle \mathrm{Det}\,\nabla[\Phi(u)],g\rangle &= \langle G_u,\Phi^{\#}\omega_2\wedge dg\rangle\\ &= -\langle\partial G_u, g\wedge\Phi^{\#}\omega_2\rangle + \langle G_u, g\wedge\Phi^{\#}dy\rangle\\ &= -\langle\partial G_u, g\wedge\Phi^{\#}\omega_2\rangle + \int_{B^2} g(x)\phi_1(u^1(x))\phi_2(u^2(x))\,\det\nabla u(x)\,dx\end{aligned} \tag{5.5}$$

where the 1-form ω_2 is given by (3.9).

Proof: For $\mathcal{L}^2$-a.e. $x\in B^2$ we compute

$$\begin{aligned}\nabla_i[g\,\Phi_j(u^j)\phi_{\bar{j}}(u^{\bar{j}})]\,(\mathrm{adj}\,\nabla u)_i^j &= \partial_{x_i} g\,\Phi_j(u^j)\phi_{\bar{j}}(u^{\bar{j}})\,(\mathrm{adj}\,\nabla u)_i^j\\ &+\ g\,\phi_j(u^j)\phi_{\bar{j}}(u^{\bar{j}})\nabla_i u^j\,(\mathrm{adj}\,\nabla u)_i^j\\ &+\ g\,\Phi_j(u^j)\phi_{\bar{j}}'(u^{\bar{j}})\nabla_i u^{\bar{j}}\,(\mathrm{adj}\,\nabla u)_i^j\end{aligned}$$

for $i,j=1,2$. Therefore, summing up on i, by the Laplace's formulas we obtain:

$$\sum_{i=1}^2 \nabla_i[g\,\Phi_j(u^j)\phi_{\bar{j}}(u^{\bar{j}})]\,(\mathrm{adj}\,\nabla u)_i^j = \sum_{i=1}^2 \partial_{x_i} g\,\Phi_j(u^j)\phi_{\bar{j}}(u^{\bar{j}})\,(\mathrm{adj}\,\nabla u)_i^j + g\,\phi_j(u^j)\phi_{\bar{j}}(u^{\bar{j}})\,\det\nabla u\,.$$

Taking $\phi(x,y) := (-1)^{j-1}\,g(x)\Phi_j(y_j)\phi(y_{\bar{j}})$ in the second equation in (2.7), with $n=N=2$ and $\Omega = B^2$, we thus have:

$$(-1)^{j-1}\langle\partial G_u, g(x)\Phi_j(y_j)\phi(y_{\bar{j}})\widehat{dy^j}\rangle = \int_{B^2}\Big(\sum_{i=1}^2 \partial_{x_i} g\,\Phi_j(u^j)\phi_{\bar{j}}(u^{\bar{j}})\,(\mathrm{adj}\,\nabla u)_i^j + g\,\phi_j(u^j)\phi_{\bar{j}}(u^{\bar{j}})\det\nabla u\Big)\,dx$$

so that averaging on j, by (5.1) we obtain

$$\langle \partial G_u, g\wedge \Phi^\# \omega_2\rangle = \frac{1}{2}\int_{B^2}\Big(\sum_{i,j=1}^{2}(-1)^{j-1}\partial_{x_i} g\,\Phi_j(u^j)\phi_{\bar{j}}(u^{\bar{j}})\,(\mathrm{adj}\,\nabla u)^j_i\Big)\,dx + \int_{B^2} g\,\phi_1(u^1)\phi_2(u^2)\det\nabla u\,dx\,.$$

Using equations

$$d\,(g\,\Phi^\#\omega_2) = dg\wedge\Phi^\#\omega_2 + g\,d\Phi^\#\omega_2\,,\quad d\Phi^\#\omega_2 = \Phi^\# dy = \phi_1(y_1)\,\phi_2(y_2)\,dy$$

we conclude with formula (5.5). □

6 Total variation formula

Our previous examples suggest that for any BV-map u satisfying the (ABS) density property, the equality sign holds in the total variation lower bound for the completely vertical lifting $\mu_v[u]$ obtained in Corollary 4.3. In particular, we expect that in the case of Sobolev maps u one has

$$|\mu_v[u]|(B^2\times\mathbb{R}^2) = |\operatorname{Det}\nabla u|(B^2)\,. \tag{6.1}$$

Notice that by the lower bound in Corollary 4.4 and the explicit formula (5.4), it turns out that equation (6.1) holds true provided that

$$\sup\{|\operatorname{Det}\nabla[\Phi(u)]|(B^2)\,:\,\Phi\in\mathcal{F}\}\le|\operatorname{Det}\nabla u|(B^2) \tag{6.2}$$

where, we recall, the sub-class $\mathcal{F}$ of smooth vector fields in $\mathbb{R}^2$ is given by (5.3).

In this section, we prove that formula (6.1) holds true when u is a continuous map in $W^{1,p}(B^2,\mathbb{R}^2)$ for some exponent $p>1$. To this purpose, we make use of arguments taken from the theory of *functions of bounded higher variation* due to Jerrard-Soner [15], and in particular of a result by De Lellis [9] concerning the validity of the *strong coarea formula* for the distributional Jacobian.

Strong coarea formula. If $u\in W^{1,1}(B^2,\mathbb{R}^2)$ satisfies the hypotheses of Corollary 4.2, by Corollary 4.4 we infer that u is a function of bounded higher variation, see [15]. Therefore, using the notation adopted here, the distributional determinant of u satisfies the following properties.

The *weak chain rule formula* holds: setting

$$u^y(x) := \frac{u(x)-y}{|u(x)-y|}$$

one has $u^y\in W^{1,1}(B^2,\mathbb{R}^2)\cap L^\infty$ for $\mathcal{L}^2$-a.e. $y\in\mathbb{R}^2$ and

$$\langle\operatorname{Det}\nabla u, g\rangle = \frac{1}{\pi}\int_{\mathbb{R}^2}\langle\operatorname{Det}\nabla u^y, g\rangle\,dy \qquad \forall\, g\in C_c^\infty(B^2)\,.$$

As a consequence, u satisfies the *weak coarea formula*:

$$|\operatorname{Det}\nabla u|(B^2)\le\frac{1}{\pi}\int_{\mathbb{R}^2}|\operatorname{Det}\nabla u^y|(B^2)\,dy\,. \tag{6.3}$$

When equality holds in (6.3), the function u is said to satisfy the *strong coarea formula.* Moreover, the *weak chain-rule* from [15] implies that for every bounded vector field $\Phi\in C^\infty(\mathbb{R}^2,\mathbb{R}^2)$

$$\langle\operatorname{Det}\nabla[\Phi(u)], g\rangle = \frac{1}{\pi}\int_{\mathbb{R}^2}\det\nabla\Phi(y)\,\langle\operatorname{Det}\nabla u^y, g\rangle\,dy \qquad \forall\, g\in C_c^\infty(B^2)\,. \tag{6.4}$$

We now recall that De Lellis [9, Thms. 13 and 14] proved that the strong coarea formula holds true if $u\in W^{1,p}(B^2,\mathbb{R}^2)$ is continuous, $p>1$, and

$$\int_{\mathbb{R}^2}|\operatorname{Det}\nabla u^y|(B^2)\,dy<\infty\,. \tag{6.5}$$

More precisely, he showed that with these hypotheses the Sobolev map u satisfies the *strong chain rule*, i.e., for every $\Phi \in C^\infty(\mathbb{R}^2,\mathbb{R}^2)$, the composition $\Phi(u)$ is a function of bounded higher variation and

$$\operatorname{Det}\nabla[\Phi(u)] = \det\nabla\Phi(u)\ \operatorname{Det}\nabla u$$

in the distributional sense, a condition ensuring the validity of the strong coarea formula.

In our framework, we have the following:

Theorem 6.1 *Let $p>1$ and $u \in W^{1,p}(B^2,\mathbb{R}^2)$ be a continuous map that satisfies the (ABS) density property in Definition* 3.2. *Then u satisfies the strong coarea formula:*

$$|\operatorname{Det}\nabla u|(B^2) = \frac{1}{\pi}\int_{\mathbb{R}^2} |\operatorname{Det}\nabla u^y|(B^2)\,dy\,. \tag{6.6}$$

PROOF: On account of the cited result by De Lellis [9], it suffices to prove inequality (6.5). For this purpose, we make use of an argument by Jerrard-Soner, cf. [15, Thm. 1.4]. Let $\{u_k\}$ the smooth sequence given by the property in Definition 3.2. Possibly passing to a not relabeled subsequence, we know that $G_{u_k} \overset{\mathcal{D}}{\rightharpoonup} T_u$ where $T_u = G_u + S_T \in \operatorname{cart}(B^2\times\mathbb{R}^2)$. This yields that for every $\eta \in \mathcal{D}^1(B^2)$

$$\lim_{k\to\infty}\langle G_{u_k},\omega_2\wedge\eta\rangle = \langle T_u,\omega_2\wedge\eta\rangle = \langle G_u,\omega_2\wedge\eta\rangle$$

where the 1-form ω_2 is given by (3.9), whence the second equation follows from Remark 3.6. With the notation from [15], since $\langle j(u),\eta\rangle := \langle G_u,\omega_2\wedge\eta\rangle$ we infer that $j(u_k)\rightharpoonup j(u)$ weakly in L^1_{loc}. Also, taking $\eta = dg$ for some $g\in C_c^\infty(B^2)$, since by (5.5) we have $\langle [Ju],g\rangle := \langle \operatorname{Det}\nabla u, g\rangle = \langle G_u,\omega_2\wedge\eta\rangle$, we infer that $[Ju_k]\rightharpoonup [J_u]$ weakly in $(C_c^1)^*$. Therefore, following [15, Lemma 4.9], it turns out that (possibily passing to a not relabeled subsequence)

$$\lim_{k\to\infty}\langle \operatorname{Det}\nabla u_k^y, g\rangle = \langle \operatorname{Det}\nabla u^y, g\rangle$$

for $\mathcal{L}^2$-a.e. $y\in\mathbb{R}^2$ and every $g\in C_c^\infty(B^2)$. By lower semicontinuity, this yields that for a.e. $y\in\mathbb{R}^2$

$$|\operatorname{Det}\nabla u^y|(B^2) \le \liminf_{k\to\infty} |\operatorname{Det}\nabla u_k^y|(B^2)$$

and hence, by Fatou's lemma,

$$\int_{\mathbb{R}^2} |\operatorname{Det}\nabla u^y|(B^2)\,dy \le \liminf_{k\to\infty}\int_{\mathbb{R}^2} |\operatorname{Det}\nabla u_k^y|(B^2)\,dy \le \sup_k \int_{\mathbb{R}^2} |\operatorname{Det}\nabla u_k^y|(B^2)\,dy\,. \tag{6.7}$$

Since moreover each map u_k is smooth, it satisfies the strong coarea formula and actually

$$\frac{1}{\pi}\int_{\mathbb{R}^2} |\operatorname{Det}\nabla u_k^y|(B^2)\,dy = \int_{\mathbb{R}^2} |\det\nabla u_k(x)|\,dx \qquad \forall\, k\,. \tag{6.8}$$

In conclusion, inequality (6.5) follows from (6.7), (6.8), and the L^1-equibondedness of the gradient determinants from (3.7). $\square$

TOTAL VARIATION. In conclusion, we obtain:

Theorem 6.2 *Let $p>1$ and $u\in W^{1,p}(B^2,\mathbb{R}^2)$ a continuous map satisfying the (ABS) density property, Definition* 3.2. *Then the completely vertical lifting $\mu_v[u]$ satisfies the total variation formula* (6.1).

PROOF: Let $\Phi\in\mathcal{F}$, see (5.3). Since $|\det\nabla\Phi(y)| = |\Phi_1'(y_1)|\cdot|\Phi_2'(y_2)| \le 1$ for each $y\in\mathbb{R}^2$, the weak chain rule (6.4) yields that

$$|\operatorname{Det}\nabla[\Phi(u)]|(B^2) \le \frac{1}{\pi}\int_{\mathbb{R}^2} |\operatorname{Det}\nabla u^y|(B^2)\,dy \qquad \forall\,\Phi\in\mathcal{F}\,.$$

We thus get the inequality

$$\sup\{|\operatorname{Det}\nabla[\Phi(u)]|(B^n)\,:\,\Phi\in\mathcal{F}\} \le \frac{1}{\pi}\int_{\mathbb{R}^2} |\operatorname{Det}\nabla u^y|(B^2)\,dy\,.$$

Since by Theorem 6.1 the map u satisfies the strong coarea formula (6.6), inequality (6.2) holds, whence equation (6.1) follows from the lower bound in Corollary 4.4 and the explicit formula (5.4). $\square$

7 The high dimension case

In this section, we show how to extend our previous results on the completely vertical lifting to the case of maps in $BV(B^n, \mathbb{R}^2)$, where B^n is the unit ball of dimension $n \geq 3$. We omit to write the proofs of almost all the results stated below, since they are an easy adaptation of the case when $n = 2$ already treated in the previous sections.

We first observe that if $n \geq 3$, a current $T \in \mathcal{R}_n(B^n \times \mathbb{R}^2)$ is identified by the measures

$$\mu_h[T] := T \llcorner dx\,, \quad \mu_i^j[T] := T \llcorner (-1)^{i-1} dy^j \wedge \widehat{dx^i}\,, \quad \mu_v^{\bar\alpha}[T] := T \llcorner \sigma(\alpha, \bar\alpha)\, dx^\alpha \wedge dy\,, \quad dy := dy^1 \wedge dy^2$$

for each $i = 1, \dots, n$, $j = 1, 2$, and each ordered multi-index α of length $n-2$ in $\{1, \dots, n\}$, where $\bar\alpha$ is the complementary ordered index of length two and the sign $\sigma(\alpha, \bar\alpha) = \pm 1$ is such that $dx^\alpha \wedge dx^{\bar\alpha} = \sigma(\alpha, \bar\alpha)\, dx$, see (3.1) when $n = 2$. In the sequel, we fix an order on the set of the $d(n) := n(n-1)/2$ multi-indexes $\bar\alpha$ of lenght two in $\{1, \dots, n\}$, and we correspondingly denote by $\mu_v[T]$ the $\mathbb{R}^{d(n)}$-valued measure in $B^n \times \mathbb{R}^2$ with components $\mu_v^{\bar\alpha}[T]$.

Notice that if $T = G_u$ for some smooth map $u \in C^1(\overline{B^n}, \mathbb{R}^2)$, then $\mu_h[G_u] = (\mathrm{Id} \bowtie u)_\#(\mathcal{L}^n \llcorner B^n)$, $\mu_i^j[G_u] = (\mathrm{Id} \bowtie u)_\#(\nabla_i u^j\, \mathcal{L}^n \llcorner B^n)$, and also

$$\mu_v^{\bar\alpha}[G_u] = (\mathrm{Id} \bowtie u)_\#(M_{\bar\alpha}(\nabla u)\, \mathcal{L}^n \llcorner B^n) \qquad \forall\, \alpha$$

where $M_{\bar\alpha}(\nabla u)$ is the 2×2 minor of the gradient matrix $\nabla u \in \mathbb{R}^{2\times n}$ with columns detected by $\bar\alpha$.

For any $u \in BV(B^n, \mathbb{R}^2)$, recalling that $D^a u = \nabla u\, \mathcal{L}^n \llcorner B^n$, we denote by $M_{(2)}(\nabla u)$ the $d(n)$-vector with ordered entries $M_{\bar\alpha}(\nabla u)$, for $\bar\alpha$ as above.

Let now $T \in \mathrm{cart}(B^n \times \mathbb{R}^2)$ be a Cartesian current with underlying function equal to u, so that we can write $T = G_u + S_T$. The horizontal component satisfying $\mu_h[T] = (Id \bowtie u)_\#(\mathcal{L}^n \llcorner B^n)$, we require again that the intermediate components only depend on u through the formulas (3.2), where $\mu_i^j[u]$ is the minimal lifting measure in the sense of Jerrard-Jung, see Definition 1.2. We thus give the following

Definition 7.1 For any $u \in BV(B^n, \mathbb{R}^2)$ with $M_{(2)}(\nabla u) \in L^1(B^n, \mathbb{R}^{d(n)})$, we denote by $\mathcal{T}_u$ the class of currents $T = G_u + S_T$ in $\mathrm{cart}(B^n \times \mathbb{R}^2)$ such that (3.2) holds. An $\mathbb{R}^{d(n)}$-valued measure μ_v in $B^n \times \mathbb{R}^2$ is said to be a completely vertical lifting of u if (3.3) holds for some $T \in \mathcal{T}_u$.

Extending Definition 3.2, we say that a map $u \in BV(B^n, \mathbb{R}^2)$ satisfies the (ABS) density property if there exists a sequence $\{u_k\} \subset C^1(B^n, \mathbb{R}^2)$ such that $u_k \overset{BV}{\to} u$ strictly in $BV(B^n, \mathbb{R}^2)$ and

$$\sup_k \int_{B^n} |M_{(2)}(\nabla u_k)|\, dx < \infty\,.$$

Arguing as in the proof of Proposition 3.4, it turns out that if u satisfies the (ABS) density property, then $M_{(2)}(\nabla u) \in L^1(B^n, \mathbb{R}^{d(n)})$ and the class $\mathcal{T}_u$ is non-empty, whence a completely vertical lifting exists.

As in Theorem 3.5, we now check the uniqueness of the completely vertical lifting $\mu_v = \mu_v[u]$, when it exists. We just have show again that the class $\mathcal{T}_u$ contains at most one element $T = T_u$.

Proof of uniqueness: If T_1, $T_2 \in \mathcal{T}_u$, the difference $\widehat{T} := T_1 - T_2$ is a current in $\mathcal{R}_n(B^n \times \mathbb{R}^2)$ satisfying $\mu_h[\widehat{T}] = 0$, $\mu_i^j[\widehat{T}] = 0$ for $i = 1, \dots, n$ and $j = 1, 2$, and $(\partial \widehat{T}) \llcorner (B^n \times \mathbb{R}^2) = 0$. Therefore, by a density argument it suffices to show that

$$\langle \widehat{T}, \eta(x) \wedge \phi_1(y_1)\phi_2(y_2)\, dy \rangle = 0 \qquad \forall\, \eta \in \mathcal{D}^{n-2}(B^n)\,,\ \phi_j \in C_c^\infty(\mathbb{R})\,, \quad j = 1, 2\,. \tag{7.1}$$

Recalling that ω_2 is given by (3.9), following the lines in the proof of Theorem 3.5 this time we get:

$$0 = \langle \partial \Psi_\# \widehat{T}, \eta \wedge \omega_2 \rangle = \langle \Psi_\# \widehat{T}, d\,(\eta \wedge \omega_2) \rangle = \langle \Psi_\# \widehat{T}, d\eta \wedge \omega_2 \rangle + (-1)^{n-2} \langle \Psi_\# \widehat{T}, \eta \wedge dy \rangle\,.$$

Moreover, using that $\mu_i^j[\widehat{T}] = 0$ for $i = 1, \dots, n$ and $j = 1, 2$, we compute

$$\langle \Psi_\# \widehat{T}, d\eta \wedge \omega_2 \rangle = \langle \widehat{T}, \Psi^\#(d\eta \wedge \omega_2) \rangle = \frac{1}{2}\, \langle \widehat{T}, d\eta \wedge (\Phi_1 d\Phi_2 - \Phi_2 d\Phi_1) \rangle = 0$$

whereas

$$\langle \Psi_{\#}\widehat{T}, \eta \wedge dy\rangle = \langle \widehat{T}, \Psi^{\#}(\eta \wedge dy)\rangle = \langle \widehat{T}, \eta(x) \wedge \phi_1(y_1)\phi_2(y_2)\, dy\rangle$$

so that (7.1) readily follows, as required. $\square$

Denote again by $\pi : B^n \times \mathbb{R}^2 \to B^n$ the orthogonal projection onto the domain. If u admits the completeley vertical lifting $\mu_v[u]$, similarly to Theorem 4.1 this time we get for each α as above

$$\int_{B^n} g(x)\, d\mu_v^{\bar\alpha}[u] = \langle \mathrm{Div}_{\bar\alpha}\, \mathbf{m}_u, g\rangle \qquad \forall\, g \in C_c^\infty(B^n)$$

where in the distributional sense (and with an obvious extension of the adjoint notation to the $\mathbb{R}^{2\times n}$-valued measure Du)

$$\mathrm{Div}_{\bar\alpha}\, \mathbf{m}_u := \frac{1}{2}\sum_{j=1}^{2}\sum_{i\in\overline{\alpha}} \frac{\partial}{\partial x_i}\big(u^j(x)\,((\mathrm{adj}\, Du)_{\overline{\alpha}})_i^j\big)\,.$$

Therefore, we get the lower bound:

$$|\,\mathrm{Div}_{\bar\alpha}\, \mathbf{m}_u|(B^n) \le |\mu_v^{\bar\alpha}[u]|(B^n \times \mathbb{R}^2) < \infty \qquad \forall\, \alpha\,.$$

Assume now in addition that $u \in W^{1,1}(B^n, \mathbb{R}^2)$. In this case, we have

$$\mathrm{Div}_{\bar\alpha}\, \mathbf{m}_u = \frac{1}{2}\sum_{j=1}^{2}\sum_{i\in\overline{\alpha}} \frac{\partial}{\partial x_i}\big(u^j(x)\,((\mathrm{adj}\, \nabla u)_{\overline{\alpha}})_i^j\big) =: J^{\bar\alpha}(u)$$

where the right-hand side agrees with the $\bar\alpha$-component of the *distributional Jacobian* $J(u)$. We thus get:

$$|J(u)|(B^n) \le |\mu_v[u]|(B^n \times \mathbb{R}^2) < \infty\,.$$

Following the proof of the explicit formulas in Theorem 5.1, for Sobolev maps we also obtain the total variation formula

$$|\mu_v[u]|(B^n \times \mathbb{R}^2) = \sup\{|J(\Phi(u))|(B^n) \mid \Phi \in \mathcal{F}\}\,.$$

Now, as in Theorem 6.1 we infer that a continuous map $u \in W^{1,p}(B^n, \mathbb{R}^2)$, where $p > 1$, that satisfies the (ABS) density property, also satisfies the strong coarea formula:

$$|J(u)|(B^n) = \frac{1}{\pi}\int_{\mathbb{R}^2} |J(u^y)|(B^n)\, dy\,.$$

As a consequence, as in Theorem 6.2 we conclude that the total variation of the completely vertical lifting of any such map u satisfies:

$$|\mu_v[u]|(B^n \times \mathbb{R}^2) = |J(u)|(B^n)\,.$$

Further details are omitted.

8 Failure of uniqueness in high codimension

In this section, we see how for $\mathbb{R}^N$-valued maps, in high codimension $N \ge 3$, even in the Sobolev case the previous approach fails to give a good definition of completely vertical lifting. This drawback is outlined by analyzing an example taken from [14, Sec. 3.2.3].

Denoting by $\mathbb{S}^1 := \{x \in \mathbb{R}^2 : |x| = 1\}$ the unit circle $\mathbb{S}^1 = \partial B^2$, where polar coordinates are used, we consider the Lipschitz-continuous map $\varphi : \mathbb{S}^1 \to \mathbb{R}^3$

$$\varphi(\cos\theta,\, \sin\theta) := \begin{cases} (\cos 4\theta,\, \sin 4\theta,\, 0) & \text{if } 0 \le \theta \le \pi/2 \\ (1,\, 0,\, \theta - \pi/2) & \text{if } \pi/2 \le \theta \le \pi \\ (\cos 4\theta,\, -\sin 4\theta,\, \pi/2) & \text{if } \pi \le \theta \le 3\pi/2 \\ (1,\, 0,\, 2\pi - \theta) & \text{if } 3\pi/2 \le \theta < 2\pi \end{cases} \tag{8.1}$$

and its 0-homogeneous extension $u : B^2 \to \mathbb{R}^3$, given by

$$u(x) := \varphi\left(\frac{x}{|x|}\right), \quad x \neq O \tag{8.2}$$

where O is the origin in $\mathbb{R}^2$. Then, u is a Sobolev map in $W^{1,p}(B^2, \mathbb{R}^3)$ for any $p < 2$, and all the 2×2 minors of the gradient matrix ∇u are equal to zero, by the area formula. Therefore, the graph current G_u is i.m. rectifiable in $\mathcal{R}_2(B^2 \times \mathbb{R}^3)$, with finite mass, $\mathbf{M}(G_u) < \infty$. Moreover, by Example 2 in [14, Sec. 3.2.2] we know that

$$(\partial G_u) \llcorner (B^2 \times \mathbb{R}^3) = -\delta_O \times \varphi_{\#} [\![\mathbb{S}^1]\!] .$$

Also, denoting by $[\![D^2]\!]$ the 2-current given by integration of 2-forms on the positively oriented disk

$$D^2 := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 < 1\}$$

it turns out that

$$\varphi_{\#} [\![\mathbb{S}^1]\!] = \partial [\![D^2]\!] \times (\delta_0 - \delta_{\pi/2})$$

whence the null-boundary condition $(\partial G_u) \llcorner (B^2 \times \mathbb{R}^3) = 0$ is violated, and u is not a Cartesian map.

Remark 8.1 Writing $u = (u^1, u^2, u^3)$, and denoting by $\widehat{u^j}$ the map in $W^{1,1}(B^2, \mathbb{R}^2)$ with components determined by the complementary ones to u^j, for each $j = 1, 2, 3$ it turns out that $\widehat{u^j}$ is a Cartesian map and in particular $\operatorname{Det} \nabla \widehat{u^j} = 0$, even if $(\partial G_u) \llcorner (B^2 \times \mathbb{R}^3) \neq 0$.

We now build up two Cartesian currents $T_\ell = G_u + S_{T_\ell} \in \operatorname{cart}(B^2 \times \mathbb{R}^3)$ with underlying function equal to u and such that the component S_{T_ℓ} is "completely vertical", namely

$$S_{T_\ell} = \delta_O \times S_\ell , \qquad S_\ell \in \mathcal{R}_2(\mathbb{R}^3) , \qquad \ell = 1, 2 .$$

Consider the cylinder $C = D^2 \times [0, \pi/2]$ in the target space, equipped with the natural orientation, and the i.m. rectifiable current $[\![C]\!] \in \mathcal{R}_3(\mathbb{R}^3)$ given by integration of 3-forms on C, i.e.,

$$[\![C]\!] = [\![D^2]\!] \times [\![0, \pi/2]\!] .$$

We have:

$$\partial [\![C]\!] = S_1 - S_2 , \quad S_1 := \partial [\![D^2]\!] \times [\![0, \pi/2]\!] , \quad S_2 := [\![D^2]\!] \times (\delta_0 - \delta_{\pi/2})$$

so that $S_1, S_2 \in \mathcal{R}_2(\mathbb{R}^3)$. Moreover, using that $\partial(\partial [\![C]\!]) = 0$, we get

$$\partial S_1 = \partial S_2 = \partial [\![D^2]\!] \times (\delta_0 - \delta_{\pi/2}) .$$

Therefore, both the i.m. rectifiable currents $T_\ell := G_u + \delta_O \times S_\ell \in \mathcal{R}_2(B^2 \times \mathbb{R}^3)$ satisfy the null-boundary condition $(\partial T_\ell) \llcorner (B^2 \times \mathbb{R}^3) = 0$, whence $T_\ell \in \operatorname{cart}(B^2 \times \mathbb{R}^3)$ for $\ell = 1, 2$.

Now, similarly to (3.1), any current $T \in \mathcal{R}_2(B^2 \times \mathbb{R}^3)$ is identified by the measures

$$\mu_h[T] := T \llcorner dx , \quad \mu_i^j[T] := T \llcorner (-1)^i dx^{\bar{\imath}} \wedge dy^j , \quad \mu_v^j[T] := T \llcorner (-1)^{j-1} \widehat{dy^j}$$

for $i = 1, 2$ and $j = 1, 2, 3$, where $\bar{1} := 2$, $\bar{2} := 1$, and $\widehat{dy^j}$ is such that $(-1)^{j-1} \widehat{dy^j} \wedge dy^j = dy^1 \wedge dy^2 \wedge dy^3$. Therefore, if $T = G_u + S_T$ is in $\operatorname{cart}(B^2 \times \mathbb{R}^3)$, since the underlying function u is in $BV(B^2, \mathbb{R}^3)$ and $\mu_h[T] = (Id \bowtie u)_{\#}(\mathcal{L}^2 \llcorner B^2)$, we may require again that the components $\mu_i^j[T]$ of the $(1,1)$-stratum of T only depend on u through the formulas (3.2), where $\mu_i^j[u]$ is the minimal lifting measure in the sense of Jerrard-Jung, see Definition 1.2.

However, this is the case of both the currents T_1 and T_2 of the previous example, since the vertical component S_ℓ has $(1,1)$-stratum equal to zero, for $\ell = 1, 2$. Therefore, using an approach as in Definition 3.2 in order to define a completely vertical lifting of u, it turns out that the uniqueness property fails to hold. Notice in fact that with the previous notation we have $\mu_v^j[T_1] \neq \mu_v^j[T_2]$ for $j = 1, 2, 3$ and

$$T_1 - T_2 = S_{T_1} - S_{T_2} = \delta_O \times (S_1 - S_2) = \delta_O \times \partial [\![C]\!] .$$

On the other hand, by modifying the map u in small disks around the origin, we can find two sequences of smooth maps $\{u_k^{(\ell)}\} \subset C^1(B^2, \mathbb{R}^3)$ such that $G_{u_k^{(\ell)}} \overset{\mathcal{D}}{\rightharpoonup} T_\ell$ and $\mathbf{M}(G_{u_k^{(\ell)}}) \to \mathbf{M}(T_\ell)$ as $k \to \infty$, for $\ell = 1, 2$. Roughly speaking, around the origin O, the image of $u_k^{(\ell)}$ covers the two bases of the cylinder C, for $\ell = 1$, and the lateral surface of C, for $\ell = 2$. In particular, we infer that $u_k^{(\ell)} \overset{BV}{\to} u$ strictly in $BV(B^2, \mathbb{R}^3)$ and

$$\sup_k \int_{B^2} |M_{(2)} \nabla u_k^{(\ell)}| \, dx < \infty$$

for $\ell = 1, 2$, where $|M_{(2)} \nabla u|^2$ is the sum of the square of the 2×2 minors of the matrix ∇u. Therefore, u satisfies a density property equivalent to the (ABS) one in Definition 3.2, but the two sequences $\{u_k^{(\ell)}\}$ have a qualitatively different behavior.

We thus conclude that for maps $u \in BV(B^n, \mathbb{R}^N)$ in high codimension $N \geq 3$, the strict convergence in the BV-sense of a smooth sequence of maps whose graphs have equibounded area, fails to contain sufficient information yielding to a good definition of completely vertical lifting of the limit map u. We shall return to this feature in the next section, see Remark 9.2.

9 The relaxed area functional

In this final section, we discuss the relationship between our previous results and the relaxed area functional. We first recall how the relaxation process w.r.t. a natural weak convergence gives rise to phenomena that are not linked to our notion of completely vertical lifting. We then deal with the relaxation in the strict BV topology, and finally with the relaxed Jacobian determinant.

Weak relaxed area functional. Let $n = N = 2$ and $u \in L^1(B^2, \mathbb{R}^2)$. Following the approach by Lebesgue-Serrin, we define:

$$\overline{A}_{L^1}(u) := \inf \Big\{ \liminf_{k \to \infty} A(u_k) \mid \{u_k\} \subset C^1(B^2, \mathbb{R}^2), \ u_k \to u \text{ strongly in } L^1 \Big\}$$

where for smooth maps $A(u)$ is the area of the graph $\mathcal{G}_u$

$$A(u) := \int_{B^2} \sqrt{1 + |\nabla u|^2 + (\det \nabla u)^2} \, dx = \mathcal{H}^2(\mathcal{G}_u)$$

so that $A(u) = \mathbf{M}(G_u)$, see (2.3). Now, if $\overline{A}_{L^1}(u) < \infty$, clearly $u \in BV(B^2, \mathbb{R}^2)$, and in this case the L^1-convergence can be replaced by the weak-* convergence $u_k \overset{*}{\rightharpoonup} u$ in BV. However, Acerbi-Dal Maso [1] proved that the weak relaxed area functional does not satisfy the locality property, in general. Namely, considering the localized functional, there exist maps u such that the set function $B \mapsto \overline{A}_{L^1}(u, B)$ fails to be sub-additive on open sets $B \subset B^2$. This is due to the following phenomenon observed in [1].

Consider e.g. the vortex map $u(x) := x/|x|$ in Example 4.6, so that (4.8) holds. Roughly speaking, there are two qualitatively different ways to fill the hole in the graph of u: inserting a disk $\delta_O \times [\![B^2]\!]$ or a cylinder $[\![L]\!] \times [\![\mathbb{S}^1]\!]$, where $[\![\mathbb{S}^1]\!] := \partial [\![D^2]\!]$ and L is any oriented line segment connecting a point in the boundary ∂B^2 of the domain to the origin O.

On the other hand, the Cartesian current $\overline{T} := G_u + [\![L]\!] \times [\![\mathbb{S}^1]\!]$ fails to satisfy equations (3.2), see Definition 3.1. In fact, for any smooth sequence $\{u_k\} \subset C^1(B^2, \mathbb{R}^2)$ such that $\sup_k \mathbf{M}(G_{u_k}) < \infty$ and $G_{u_k} \overset{\mathcal{D}}{\rightharpoonup} \overline{T}$, the strict convergence $u_k \overset{BV}{\to} u$ fails to hold. Notice in fact that the unique Cartesian current in the class $\mathcal{T}_u$ is $T_u := G_u + \delta_O \times [\![B^2]\!]$.

The explicit expression of the energy gap of the L^1-relaxed area of the vortex map in the ball B_r^2 of radius $r > 0$ has been computed in [6]. It turns out that for r large enough the gap is π, and the graphs of any optimal smooth approximating sequence weakly converge in $\mathcal{D}_2(B_r^2 \times \mathbb{R}^2)$ to the current $G_u + \delta_O \times [\![B^2]\!]$. More interestingly, for small radii r, they instead converge to a Cartesian current $T = G_u + S_T$, where the term S_T is essentially obtained by solving a suitable codimension one Plateau-type problem, and actually it looks like a catenoid that lives over a segment starting from the origin O.

STRICT RELAXED AREA FUNCTIONAL. For $u \in BV(B^2, \mathbb{R}^2)$, consider now the relaxed area functional in the strict BV topology, given by

$$\overline{A}_{BV}(u) := \inf\Big\{ \liminf_{k\to\infty} A(u_k) \mid \{u_k\} \subset C^1(B^2, \mathbb{R}^2),\ u_k \stackrel{BV}{\to} u \text{ strictly in } BV \Big\}$$

so that clearly $\overline{A}_{L^1}(u) \le \overline{A}_{BV}(u)$, where the strict inequality holds in general.

A part from the case of real valued functions, the strict BV convergence is not natural for the relaxation process of the area functional. In fact, if $\{u_k\} \subset C^1(B^2, \mathbb{R}^2)$ is such that $\sup_k A(u_k) < \infty$, then possibly passing to a (not relabeled) subsequence $u_k \stackrel{*}{\rightharpoonup} u$ to some $u \in BV(B^2, \mathbb{R}^2)$, but we cannot conclude that $u_k \stackrel{BV}{\to} u$. In fact, as we previously checked for the vortex map, it turns out that $G_{u_k} \stackrel{\mathcal{D}}{\rightharpoonup} T$ to some Cartesian current $T = G_u + S_T \in \text{cart}(B^2 \times \mathbb{R}^2)$ that in general fails to satisfy equations (3.2), whose validity is a necessary condition for occurrence of the strict convergence $u_k \stackrel{BV}{\to} u$.

On the other hand, a map $u \in BV(B^2, \mathbb{R}^2)$ satisfies the (ABS) density property in Definition 3.2 if and only if it has finite relaxed area w.r.t. the strict BV-convergence. More precisely, for any sequence $\{u_k\} \subset C^1(B^2, \mathbb{R}^2)$ such that $u_k \stackrel{BV}{\to} u$ and $\sup_k A(u_k) < \infty$, since $\sup_k \mathbf{M}(G_{u_k}) < \infty$, arguing as in the proof of Proposition 3.4, and on account of Theorem 3.5, possibly passing to a (not relabeled) subsequence $G_{u_k} \stackrel{\mathcal{D}}{\rightharpoonup} T_u$ to the unique Cartesian current in the class $\mathcal{T}_u$. In conclusion, if $\overline{\mathcal{A}}_{BV}(u) < \infty$, the completely vertical lifting $\mu_v[u]$ is well-defined.

Remark 9.1 The uniqueness of the weak limit current T_u in $\mathcal{T}_u$ leads us to expect that, differently to the weak relaxed functional previously discussed, this time the locality property holds, i.e., considering the localized functional, for any map $u \in BV(B^2, \mathbb{R}^2)$ with finite relaxed area, the set function $B \mapsto \overline{A}_{BV}(u, B)$ is sub-additive on open set, and hence it extends to a measure on Borel subsets of B^2.

This is in fact what happens e.g. in the case of vortex-type Sobolev maps with values into the unit circle, recently analyzed in [7]. More precisely, let $u : B^2 \to \mathbb{R}^2$ the 0-homogeneous extension (8.2) of some Lipschitz-continuous map $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$. Then $u \in W^{1,p}(B^2, \mathbb{R}^2)$ for each $p < 2$, and $\det \nabla u = 0$, by the area formula, whence the i.m. rectifiable graph current G_u has finite mass, whereas its boundary satisfies:

$$(\partial G_u) \llcorner (B^2 \times \mathbb{R}^2) = -\delta_O \times \varphi_{\#}[\![\mathbb{S}^1]\!] . \tag{9.1}$$

If the map φ takes values into the unit circle, i.e., if $|\varphi(\cos\theta, \sin\theta)| = 1$ for each θ, the degree of φ is well-defined as a map from $\mathbb{S}^1$ into itself, $\deg \varphi \in \mathbb{Z}$, we have $\varphi_{\#}[\![\mathbb{S}^1]\!] = (\deg \varphi)\, [\![\mathbb{S}^1]\!]$, and actually

$$\text{Det}\, \nabla u = (\deg \varphi)\, \pi\, \delta_O \quad \Longrightarrow \quad |\, \text{Det}\, \nabla u|(B^2) = \pi\, |\deg \varphi| .$$

In [7] it is proved that for any such vortex-type Sobolev map, the localized functional $B \mapsto \overline{A}_{BV}(u, B)$ is a measure, and

$$\overline{A}_{BV}(u) = \int_{B^2} \sqrt{1 + |\nabla u|^2}\, dx + |\, \text{Det}\, \nabla u|(B^2) . \tag{9.2}$$

With our notation, for any such vortex-type Sobolev map u, the unique Cartesian current in the class $\mathcal{T}_u$ from Definition 3.1 is $T_u := G_u + (\deg \varphi)\, \delta_O \times [\![B^2]\!]$, and there exists a sequence $\{u_k\} \subset C^1(B^2, \mathbb{R}^2)$ such that $G_{u_k} \stackrel{\mathcal{D}}{\rightharpoonup} T_u$ and $\mathbf{M}(G_{u_k}) \to \mathbf{M}(T_u)$ as $k \to \infty$.

Notice however that as for the weak relaxed functional, it is an open problem to characterize the subclass of maps $u \in BV(B^2, \mathbb{R}^2)$ such that $\overline{A}_{BV}(u) < \infty$, even in the case of bounded maps. By our previous results, we only know that if $\overline{A}_{BV}(u) < \infty$, then necessarily $\det \nabla u \in L^1(B^2)$, the product of the components $u^1 u^2 \in BV(B^2)$, and the distribution $\text{Div}\, \mathbf{m}_u$ is a finite measure in B^2. Moreover, by the lower semicontinuity of the mass w.r.t. the weak convergence as currents, one clearly has:

$$\mathbf{M}(T_u) \le \overline{A}_{BV}(u) < \infty .$$

On the other hand, there exist maps $u \in W^{1,1}(B^2, \mathbb{R}^2)$ such that $\overline{A}_{BV}(u) < \infty$, whence the unique Cartesian current T_u in the class $\mathcal{T}_u$ from Definition 3.1 does exist, but for any sequence $\{u_k\} \subset C^1(B^2, \mathbb{R}^2)$ such that $u_k \stackrel{BV}{\to} u$ and $G_{u_k} \stackrel{\mathcal{D}}{\rightharpoonup} T_u$, one has $\mathbf{M}(T_u) < \liminf_k \mathbf{M}(G_{u_k})$. This *gap phenomenon*, namely:

$$\mathbf{M}(T_u) < \overline{A}_{BV}(u) \tag{9.3}$$

is illustrated in Example 9.4 below.

Remark 9.2 Of course, the previous observations extend to the strict relaxed area functional of maps $u : B^n \to \mathbb{R}^2$ in high dimension $n \geq 3$, on account of the results sketched in Sec. 7. On the other hand, in the high codimension case, differently what happens for $\mathbb{R}^2$-valued maps, it turns out that even for Sobolev maps $u : B^2 \to \mathbb{R}^3$, the strict BV-convergence doesn't guarantee any control on the 2×2 minors of the matrix ∇u. This drawback is illustrated by the example from Sec. 8, where the two approximating sequences $\{u_k^{(\ell)}\}$ have a qualitatively different behavior, even if both of them strictly converge to the 0-homogeneous extension u of the map φ in (8.1).

Relaxed Jacobian determinant. Finally, if one considers on maps $u \in BV(B^2, \mathbb{R}^2)$ the relaxation problem:

$$\mathrm{TVJ}_{BV}(u) := \inf\Big\{\liminf_{k\to\infty} \int_{B^2} |\det \nabla u_k|\, dx \mid \{u_k\} \subset C^1(B^2, \mathbb{R}^2),\ u_k \overset{BV}{\to} u \text{ strictly in } BV\Big\}$$

by the lower semicontinuity of the total variation of the completely vertical lifting w.r.t. the weak convergence as measures, on account of Theorem 6.2 we readily obtain the following lower bound:

Proposition 9.3 *Let $u \in BV(B^2, \mathbb{R}^2)$ such that $\mathrm{TVJ}_{BV}(u) < \infty$. Then*

$$\mathrm{TVJ}_{BV}(u) \geq |\operatorname{Div} \mathbf{m}_u|(B^2)\,.$$

Therefore, in the particular case of Sobolev maps $u \in W^{1,1}(B^2, \mathbb{R}^2)$, the previous inequality becomes

$$\mathrm{TVJ}_{BV}(u) \geq |\operatorname{Det} \nabla u|(B^2)\,. \tag{9.4}$$

We address e.g. to [18, 12, 10] for the analysis of the relaxed Jacobian determinant w.r.t. the weak topologies of Sobolev spaces. As shown in [7], equality holds in formula (9.4) when considering the case of vortex-type maps with values into $\mathbb{S}^1$. However, there exist bounded Sobolev maps u for which the strict inequality holds in (9.4).

Example 9.4 Following an example by Malý [17], Giaquinta-Modica-Souček [13] considered the 0-homogeneous extension (8.2) of the map

$$\varphi(\cos\theta, \sin\theta) := \begin{cases} (-1 + \cos 4\theta, \sin 4\theta) & \text{if} \quad 0 \leq \theta < \pi/2 \\ (1 - \cos 4\theta, \sin 4\theta) & \text{if} \quad \pi/2 \leq \theta < \pi \\ (-1 + \cos 4\theta, -\sin 4\theta) & \text{if} \quad \pi \leq \theta < 3\pi/2 \\ (1 - \cos 4\theta, -\sin 4\theta) & \text{if} \quad 3\pi/2 \leq \theta < 2\pi \end{cases}$$

whose image covers an "eight" figure twice but with opposite orientation.

Since the loop φ is homotopically non-trivial, we have $A(u) < \overline{A}_{L^1}(u)$, see [14, Sec. 3.4.2]. However, the loop φ is homologically trivial, $\varphi_\# [\![\mathbb{S}^1]\!] = 0$, whence by (9.1) we infer that u is a Cartesian map, and actually $|\operatorname{Det} \nabla u|(B^2) = 0$. In particular, one clearly has $T_u = G_u$. For such map u the strict inequality holds in (9.4), the relaxed area functional fails to satisfy equation (9.2), and the energy gap (9.3) holds.

Finally, the value of the relaxed area functional and relaxed Jacobian determinant of the "double eight " example w.r.t. the weak topologies of Sobolev spaces was independently obtained in [20, 24].

Acknowledgments. I wish to thank P. Celada for some useful discussions. The author is a member of the GNAMPA of INDAM.

References

[1] E. Acerbi & G. Dal Maso: New lower semicontinuity results for polyconvex integrals. *Calc. Var. & PDE's* **2** (1994), 329–372.

[2] G. Alberti: Distributional Jacobian and singularities of Sobolev maps. *Ricerche Mat.* **54** (2005), no. 2, 375–394 (2006).

[3] L. Ambrosio & G. Dal Maso: A general chain rule for distributional derivatives. *Proc. Amer. Math. Soc.* **108** (1990), 691–702.

[4] L. Ambrosio, N. Fusco & D. Pallara: *Functions of bounded variation and free discontinuity problems.* Oxford Math. Monographs, Oxford, 2000.

[5] G. Anzellotti & M. Giaquinta: Funzioni BV e tracce. *Rend. Sem. Mat. Univ. Padova* **60** (1978), 1–22.

[6] G. Bellettini, A. Elshorbagy & R. Scala: The L^1-relaxed area of the vortex map. *Preprint* (2021), see https://arxiv.org/abs/2107.07236

[7] G. Bellettini, S. Scarano & R. Scala: The relaxed area of S^1-valued singular maps in the strict BV-convergence. *Preprint* (2022), see https://cvgmt.sns.it/paper/5440/

[8] J.M. Ball: Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.* **63** (1976), 337–403.

[9] C. De Lellis: Some remarks on the distributional Jacobian. *Nonlinear Anal.* **53** (2003), 1101–1114.

[10] G. De Philippis: Weak notions of Jacobian determinant and relaxation. *ESAIM Control Optim. Calc. Var.* **18** (2012), 181–207.

[11] H. Federer & W. Fleming: Normal and integral currents. *Ann. of Math.* **72** (1960), 458–520.

[12] I. Fonseca, N. Fusco & P. Marcellini: On the total variation of the Jacobian. *J. Funct. Anal.* **207** (2004), 1–32.

[13] M. Giaquinta, G. Modica & J. Souček: Graphs of finite mass which cannot be approximated in area by smooth graphs. *Manuscr. Math.* **78** (1993), 259–271.

[14] M. Giaquinta, G. Modica & J. Souček: *Cartesian currents in the calculus of variations. I. Cartesian currents.* Ergebnisse Math. Grenzgebiete (III Ser), vol. 37, Springer, Berlin, 1998.

[15] R.L. Jerrard & H. M. Soner: Functions of bounded higher variation. *Indiana Univ. Math. J.* **51** (2002), 645–677.

[16] R.L. Jerrard & N. Jung: Strict convergence and minimal liftings in BV. *Proc. Roy. Soc. Edinburgh Sect. A* **134** (2004), 1163–1176.

[17] J. Malý: L^p-approximation of Jacobians. *Comment. Math. Univ. Carolinae* **32** (1991), 659–666.

[18] P. Marcellini: Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals. *Manuscr. Math.* **51** (1985), 1–28.

[19] C.B. Morrey: *Multiple Integrals in the Calculus of Variations.* Springer, Berlin, 1966.

[20] D. Mucci: Remarks on the total variation of the Jacobian. *NoDEA Nonlinear Differential Equations Appl.* **13** (2006), 223–233.

[21] D. Mucci: A variational problem involving the distributional determinant. *Riv.Mat. Univ. Parma New Ser* **1** (2010), 321–345.

[22] S. Müller: Weak continuity of determinants and nonlinear elasticity. *C. R. Acad. Sci. Paris* **307** (1988), 501–506.

[23] S. Müller: On the singular support of the distributional determinant. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **10** (1993), 657–696.

[24] E. Paolini: On the relaxed total variation of singular maps. *Manuscr. Math.* **111** (2003) 499–512.

[25] Y. G. Reshetnyak: On the stability of conformal mappings in multidimensional spaces. *Sib. Math. Zhu.* **8** (1967), 91–114. English transl.: *Siberian Math. J.* **8** (1967), 69–85.

[26] F. Rindler & G. Shaw: Strictly continuous extension of functionals with linear growth to the space BV. *Q. J. Math.* **66** (2015), 953–978.

[27] A. I. Vol'pert: The spaces BV and quasi-linear equations. *Math. USSR-Sb.* **2** (1967), 225–267.

[28] A. I. Vol'pert & S. I. Hudjaev: *Analysis in classes of discontinuous functions and equations of mathematical physics.* Martinus Nijhoff Publisher, Dordrecht, 1985.

Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma.
Parco Area delle Scienze 53/A, I-43124 Parma, Italy
E-mail: domenico.mucci@unipr.it