Asymptotic analysis of periodically-perforated nonlinear media at the critical exponent

Laura Sigalotti
Dipartimento di Matematica, Università di Roma 'La Sapienza'
p.le A.Moro 2, 00185 Roma, Italy
email: sigalott@mat.uniroma1.it

Abstract

We give a Γ-convergence result for vector-valued nonlinear energies defined on periodically perforated domains. We consider integrands with n-growth where n is the space dimension, showing that there exists a critical scale for the perforations such that the Γ-limit is non-trivial. We prove that the limit extra-term is given by a formula of homogenization type, which simplifies in the case of n-homogeneous energy densities.

Keywords: Γ-convergence; perforated domains; critical exponent.

1 Introduction

Variational problems on varying domains, and on perforated domains in particular, are a very much studied class of problems, with interesting applications in homogenization and shape optimization. In this paper we study the asymptotic behavior of a class of vector-valued nonlinear energies defined on periodically perforated domains.

A *perforated domain* is obtained from a fixed $\Omega \subset \mathbb{R}^n$ by removing some periodic set, the simplest of which is a periodic array of closed sets:

$$\Omega_\delta = \Omega \setminus \bigcup_{i \in \mathbb{Z}^n} (\delta i + \varepsilon K),$$

with $\varepsilon = \varepsilon(\delta)$ and K a bounded closed set with non-empty interior. If we consider Dirichlet boundary conditions on the boundary of Ω_δ (or on the boundary of Ω_δ interior to Ω), the asymptotic behavior of such problems is obtained by studying the Γ-convergence of the functionals

$$F_\delta(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\,dx & \text{if } u \in W_0^{1,p}(\Omega;\mathbb{R}^m) \text{ and } u = 0 \text{ on } \Omega \setminus \Omega_\delta \\ +\infty & \text{otherwise,} \end{cases} \tag{1}$$

where f is an energy density satisfying a growth condition of order $p > 1$. From early results by Marchenko and Khruslov [22] we know that if we set $f(Du) = |Du|^p$ in the functionals above, then there exists a critical scaling of the perforations such that the Γ-limit contains an additional *strange term* in place of the internal boundary conditions. The limit functional, indeed, is given by

$$F_0(u) = \int_\Omega |Du|^p\,dx + \kappa_p \int_\Omega |u|^p\,dx,$$

where κ_p is a positive constant, explicitly calculable. This result was recast in a rigorous variational setting by Cioranescu and Murat [13], who provided an explicit formula for the critical choice of ε according to the space dimension n:

$$\varepsilon = R\delta^{n/n-p} \quad \text{if } p < n, \quad \text{with } R > 0$$
$$\varepsilon = \exp(-a\delta^{\frac{-n}{n-1}}) \quad \text{if } p = n, \quad \text{with } a > 0.$$

In [2] Ansini and Braides performed a complete analysis by Γ-convergence for energies with a general integrand f with p-growth, and depending on vector-valued functions, in the case leading to the polynomial scaling; i.e., $p < n$. In that setting the additional term of the Γ-limit is given by $\int_\Omega \varphi(u)\,dx$, where the function φ is obtained through a nonlinear *capacitary formula.*

In this paper we study in details the critical case $p = n$ leading to the exponential scaling. We will see that in this setting the limit extra term is not defined by a capacitary formula, but rather by a formula of homogenization type. The additional extra term is given by $\int_\Omega \varphi(u)\,dx$, where $\varphi(z)$ is obtained as the limit of a family of minimum problems of the form

$$|\log\varepsilon_j|^{n-1} \inf\left\{ \int_{B_{c\delta_j/\varepsilon_j}} f(\varepsilon_j^{-1}Dv)\varepsilon_j^n :\ v = 0 \text{ on } K,\ v = z \text{ on } \partial B_{c\delta_j/\varepsilon_j} \right\}$$

(see (9) in Theorem 2.1), which always exists up to subsequences. In this critical case the energy does not concentrate at the same scale as the perforation radius, in a fashion similar to optimal sequences for Ginzburg-Landau functionals (see e.g. [1, 3, 23]). The proof of the Γ-convergence result is based on a careful use of a technical lemma by Ansini and Braides [2], which allows to separate the estimate of the energies near the perforations and far form them. The contribution of the energies close to the perforations leads to the formula defining φ.

In the last section we will detail the interesting case in which the energy density f is positively homogeneous of degree n. We will prove that under this assumption the whole family of functionals Γ-converges to the same limit functional. The function φ can be determined explicitly and is independent of the shape of the perforations. Our arguments highlight that the exponential radius of the perforations derives from the scaling invariance of the minimum problems and from the logarithmic behaviour of the minimizers.

Our results can be compared with previous ones obtained (for equations) in the framework of H-convergence and two-scale convergence (see e.g. Cioranescu and Murat [13], Casado-Diaz [11]). Furthermore, we note that although the paper is devoted to periodically perforated domains there exists a wide literature dealing with the asymptotic behaviour of Dirichlet monotone problems in varying domains without periodicity conditions (see e.g. [10, 12, 16, 17, 18, 19, 20, 21, 24]). An overview on this subject can be found in [14].

2 The main result

In all that follows $n > 1$ and $m \geq 1$ are fixed integers. With $\mathbb{M}^{m\times n}$ we denote the space of $m \times n$ matrices with real entries. If $E \subset \mathbb{R}^n$ is a Lebesgue-measurable set then $|E|$ is its Lebesgue measure. $B_r(x)$ is the open ball of centre x and radius r; if $x = 0$ we will write B_r in place of $B_r(x)$. The letter c denotes a generic strictly positive constant.

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega| = 0$. Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior. Let (δ_j) be a sequence of positive real numbers converging to zero. For all $j \in \mathbb{N}$ and $i \in \mathbb{Z}^n$ we denote by x_i^j the vector $i\delta_j \in \mathbb{R}^n$. Let $a > 0$; let

$$\varepsilon_j = \exp\big(- a\delta_j^{-n/n-1}\big). \tag{2}$$

For all $j \in \mathbb{N}$ and $i \in \mathbb{Z}^n$ we denote by K_i^j the perforation $K_i^j = x_i^j + \varepsilon_j K$. Let Ω_j be the periodically perforated domain

$$\Omega_j = \Omega \setminus \bigcup_{i\in\mathbb{Z}^n} K_i^j. \tag{3}$$

In the following theorem we state a general Γ-convergence result for vector-valued nonlinear energies defined on periodically perforated domains, in the case of integrands with n-growth. For an introduction to Γ-convergence see e.g. [4, 5, 14].

Theorem 2.1 *Let $f : \mathbb{M}^{m\times n} \to [0, +\infty)$ be a quasiconvex function satisfying $f(0) = 0$. We assume that there exist c_1, c_2, $k > 0$ such that*

$$c_1|A|^n \leq f(A) \leq c_2|A|^n \quad \textit{for all } A \in \mathbb{M}^{m\times n} \tag{4}$$

and

$$|f(A) - f(B)| \leq k|A - B| \big(|A|^{n-1} + |B|^{n-1}\big) \quad \textit{for all } A, B \in \mathbb{M}^{m\times n}. \tag{5}$$

Let (δ_j) be a positive sequence converging to zero and let ε_j be defined as in (2). For notational simplicity we also define

$$\begin{aligned} T_j &= \varepsilon_j^{-1} = \exp\big(a\delta_j^{-n/n-1}\big), &(6)\\ S_j &= \frac{\delta_j}{\varepsilon_j} = a^{(n-1)/n}\frac{T_j}{(\log T_j)^{(n-1)/n}}. &(7)\end{aligned}$$

For all $j \in \mathbb{N}$, $\alpha > 0$ and $z \in \mathbb{R}^m$ we set

$$\varphi_j^\alpha(z) = (\log T_j)^{n-1} \inf\Big\{ \int_{B_{\alpha S_j}} \frac{f(T_j Dv)}{T_j^n} \,:\, v \in z + W_0^{1,n}\big(B_{\alpha S_j};\mathbb{R}^m\big),\ v = 0 \textit{ on } K\Big\}. \tag{8}$$

Then, upon possibly passing to subsequences on j, there exists the limit

$$\varphi(z) = \lim_{\alpha\to 0^+}\lim_{j\to+\infty} \varphi_j^\alpha(z) = \sup_{\alpha>0}\lim_{j\to+\infty} \varphi_j^\alpha(z) \tag{9}$$

uniformly on the compact sets of $\mathbb{R}^m$.
Moreover, the functionals $F_j : W^{1,n}(\Omega;\mathbb{R}^m) \to [0, +\infty]$ defined by

$$F_j(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\,dx & \textit{if } u = 0 \textit{ a.e. on } \displaystyle\bigcup_{i\in\mathbb{Z}^n} K_i^j \cap \Omega, \\ +\infty & \textit{otherwise,} \end{cases} \tag{10}$$

Γ-converge, with respect to the strong convergence of $L^n(\Omega;\mathbb{R}^m)$, to the functional $F: W^{1,n}(\Omega;\mathbb{R}^m)\to[0,+\infty]$ defined by

$$F(u)=\int_\Omega f(Du)\,dx+a^{1-n}\int_\Omega \varphi(u)\,dx. \tag{11}$$

Finally, if in addition f is positively homogeneous of degree n, then the function φ is independent both of the subsequence of (T_j) and of the shape of the perforations. Indeed, for all $z\in\mathbb{R}^m$ there exists the limit

$$\psi(z)=\lim_{T\to+\infty}(\log T)^{n-1}\min\Big\{\int_{B_T} f(Du)\,dx:\ u\in z+W^{1,n}_0(B_T;\mathbb{R}^m),\ u=0 \text{ on } B_1\Big\}. \tag{12}$$

Moreover, $\varphi(z)=\psi(z)$ for all $z\in\mathbb{R}^m$ (in particular, the whole sequence (F_j) Γ-converges to F and the shape of K does not affect the result).

Remark 2.2 *Using the terminology introduced in [8] by Braides and Truskinovsky, our result can be summarized by saying that the functionals (F_j) in (10) are equivalent by Γ-convergence to the functionals G_j defined as*

$$G_j(u)=\int_\Omega f(Du_j)\,dx+\frac{|\log\varepsilon_j|^{n-1}}{\delta_j^n}\int_\Omega\varphi(u)\,dx\qquad u\in W^{1,n}_0(\Omega;\mathbb{R}^m),$$

meaning that both families have the same Γ-limits on all Γ-converging sequences as $j\to+\infty$.

Corollary 2.3 (Convergence of minimum problems) *Let (F_j) be a family of functionals of the form (10) satisfying the statement of Theorem 2.1. Then for all $\phi\in W^{-1,n}(\Omega;\mathbb{R}^m)$ the minimum values*

$$m_j=\inf\Big\{F_j(u)+\langle\phi,u\rangle : u\in W^{1,n}_0(\Omega;\mathbb{R}^m)\Big\}$$

converge to

$$m=\min\Big\{F(u)+\langle\phi,u\rangle : u\in W^{1,n}_0(\Omega;\mathbb{R}^m)\Big\}.$$

Moreover, if (u_j) is such that $F_j(u_j)+\langle\phi,u_j\rangle=m_j+o(1)$ as $j\to+\infty$, then it admits a subsequence weakly converging in $W^{1,n}_0(\Omega;\mathbb{R}^m)$ to a solution of the problem defining m.

Proof. By a cut-off argument near $\partial\Omega$ (see [6, Section 11.3]) we can prove that if $u\in W^{1,n}_0(\Omega;\mathbb{R}^m)$ then the recovery sequence for u can be taken in $W^{1,n}_0(\Omega;\mathbb{R}^m)$ as well. Since f satisfies a growth condition of order n, then $u_j\rightharpoonup u$ in $W^{1,n}_0(\Omega;\mathbb{R}^m)$. Note that $G(u)=\langle\phi,u\rangle$ is continuous with respect to the weak convergence of $W^{1,n}_0(\Omega;\mathbb{R}^m)$. There follows that the functionals

$$\Phi_j(u)=\begin{cases} F_j(u)+G(u) & \text{if } u\in W^{1,n}_0(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise}\end{cases}$$

Γ-converge to

$$\Phi_0(u)=\begin{cases} F(u)+G(u) & \text{if } u\in W^{1,n}_0(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise}\end{cases}$$

in $W^{1,n}_0(\Omega;\mathbb{R}^m)$. We can then apply the fundamental property of Γ-convergence (see, e.g. [5, Theorem 1.21]) and obtain the thesis. ∎

3 Preliminary results

3.1 Some auxiliary energy densities

In this section we study some properties of the energy densities φ_j^α and we prove the existence of the limit in (9). We show that the family (φ_j^α) is equi-bounded and equi-continuous (both with respect to α and j), hence we can apply Ascoli-Arzelà's Theorem.

We recall that for fixed $j \in \mathbb{N}$, $\alpha > 0$ and $z \in \mathbb{R}^m$ we set

$$\varphi_j^\alpha(z) = (\log T_j)^{n-1} \inf \Big\{ \int_{B_{\alpha S_j}} \frac{f(T_j Dv)}{T_j^n} :\ v \in z + W_0^{1,n}\big(B_{\alpha S_j};\mathbb{R}^m\big),\ v = 0 \text{ on } K \Big\},$$

where f is a function satisfying all the assumptions of Theorem 2.1. We preliminarily note that, for fixed $j \in \mathbb{N}$ and $z \in \mathbb{R}^m$, the function $\alpha \mapsto \varphi_j^\alpha(z)$ is decreasing in $(0, +\infty)$.

Up to translations we can assume that there exist $0 < r_1 \le r_2$ such that $B_{r_1} \subseteq K \subseteq B_{r_2}$. Then, for fixed α, j and z we have

$$\begin{aligned} &\inf \Big\{ \int_{B_{\alpha S_j}} \frac{f(T_j Dv)}{T_j^n}\, dx :\ v \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } B_{r_1} \Big\} \\ &\quad \le \inf \Big\{ \int_{B_{\alpha S_j}} \frac{f(T_j Dv)}{T_j^n}\, dx :\ v \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } K \Big\} \\ &\qquad \le \inf \Big\{ \int_{B_{\alpha S_j}} \frac{f(T_j Dv)}{T_j^n}\, dx :\ v \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } B_{r_2} \Big\}. \end{aligned}$$

By (4) and a scaling argument we get

$$\begin{aligned} &c_1 \inf \Big\{ \int_{B_{\frac{\alpha}{r_1} S_j}} |Dv|^n\, dx :\ v \in z + W_0^{1,n}(B_{\frac{\alpha}{r_1} S_j};\mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\} \\ &\quad \le \inf \Big\{ \int_{B_{\alpha S_j}} \frac{f(T_j Dv)}{T_j^n}\, dx :\ v \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } K \Big\} \\ &\qquad \le c_2 \inf \Big\{ \int_{B_{\frac{\alpha}{r_2} S_j}} |Dv|^n\, dx :\ v \in z + W_0^{1,n}(B_{\frac{\alpha}{r_2} S_j};\mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\}. \end{aligned} \tag{13}$$

The following remark shows how to get a growth condition for φ_j^α through this comparison argument.

Remark 3.1 *If we set $f(A) = |A|^n$ and $\overline{B}_1 = K$, then the expression of $\varphi_j^\alpha(z)$ becomes*

$$(\log T_j)^{n-1} \min \Big\{ \int_{B_{\alpha S_j}} |Dv|^n\, dx :\ v \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\}. \tag{14}$$

In this case the minimum can be computed explicitly. First of all we note that the minimum in (14) equals

$$|z|^n \min \Big\{ \int_{B_{\alpha S_j}} |Dv|^n\, dx :\ v \in \frac{z}{|z|} + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\}. \tag{15}$$

Up to rotations it is not restrictive to assume that $z/|z| = e_1$. It can be easily seen that

$$\min\Big\{\int_{B_{\alpha S_j}} |Dv|^n\,dx:\ v \in e_1 + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } B_1\Big\} \tag{16}$$

$$= \min\Big\{\int_{B_{\alpha S_j}} |Du|^n\,dx:\ u \in 1 + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}),\ u = 0 \text{ on } B_1\Big\}. \tag{17}$$

In fact, on one hand we can identify each test function u for the scalar minimum problem (17) with a vector-valued test function for (16), by setting $v = ue_1$, and we deduce that

$$\min\Big\{\int_{B_{\alpha S_j}} |Dv|^n\,dx:\ v \in e_1 + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } B_1\Big\}$$

$$\leq \min\Big\{\int_{B_{\alpha S_j}} |Du|^n\,dx:\ u \in 1 + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}),\ u = 0 \text{ on } B_1\Big\}.$$

The converse inequality can be obtained noticing that the minimum in (16) must be reached by a function of the form $v = (v_1, 0, \ldots, 0)$. Taking $u = v_1 \in 1 + W_0^{1,n}(B_{\alpha S_j};\mathbb{R})$ as a test function in (17) we get the desired inequality. Therefore we can restrict our attention to the scalar problem (17). For symmetry reasons the minimum is reached by a radial function $v(x) = w(|x|)$ satisfying $w(1) = 0$, $w(\alpha S_j) = 1$ and the Euler-Lagrange equation

$$\frac{\partial}{\partial\rho}(|w'(\rho)|^{n-2}w'(\rho)\rho^{n-1}) = 0, \qquad \text{where } \rho = |x|.$$

Hence we get $w(\rho) = \frac{\log\rho}{\log(\alpha S_j)} \vee 0$ and we can conclude that in this case

$$\varphi_j^\alpha(z) = (\log T_j)^{n-1}\frac{1}{(\log(\alpha S_j))^{n-1}}\omega_{n-1}|z|^n. \tag{18}$$

Since $S_j = a^{(n-1)/n}T_j(\log T_j)^{(1-n)/n}$ we have (for all $a, \alpha > 0$)

$$\lim_{j\to+\infty}(\log T_j)^{n-1}\frac{1}{(\log(\alpha S_j))^{n-1}} = 1.$$

In conclusion, in the case $f(A) = |A|^n$ and $K = \overline{B}_1$ there exists the limit

$$\varphi(z) = \lim_{\alpha\to0^+}\lim_{j\to+\infty}\varphi_j^\alpha(z) = \sup_{\alpha>0}\lim_{j\to+\infty}\varphi_j^\alpha(z) = \omega_{n-1}|z|^n,$$

uniformly on the compact sets of $\mathbb{R}^m$.

From Remark 3.1 we deduce that the functions φ_j^α satisfy a growth condition of order n: there exist two positive constants $\tilde{c}_1$, $\tilde{c}_2$ (independent of α, j and z) such that

$$\tilde{c}_1|z|^n \leq \varphi_j^\alpha(z) \leq \tilde{c}_2|z|^n. \tag{19}$$

In fact, let $\alpha > 0$, $j \in \mathbb{N}$ and $z \in \mathbb{R}^m$ be fixed. By (13) and the computations of Remark 3.1 ((18) in particular) we get

$$\omega_{n-1}c_1\Big(\log\Big(\frac{\alpha}{r_1}S_j\Big)\Big)^{1-n}|z|^n \leq \frac{\varphi_j^\alpha(z)}{(\log T_j)^{n-1}} \leq \omega_{n-1}c_2\Big(\log\Big(\frac{\alpha}{r_2}S_j\Big)\Big)^{1-n}|z|^n. \tag{20}$$

Finally we multiply (20) by $(\log T_j)^{n-1}$ and take into account that $\frac{\log T_j}{\log \alpha S_j}$ is bounded by two positive constants (independent of α and j). Thus we get (19) and conclude that the functions φ_j^α are equi-bounded on every bounded subset of $\mathbb{R}^m$.

It remains to show that φ_j^α are equi-continuous on the compact sets of $\mathbb{R}^m$. Let $\alpha > 0$, $j \in \mathbb{N}$ and z, $z' \in \mathbb{R}^m \setminus \{0\}$. For fixed $\eta > 0$ we consider a function $u^z_{\alpha,j} \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m)$ such that $u^z_{\alpha,j} = 0$ on K and

$$(\log T_j)^{n-1} \int_{B_{\alpha S_j}} \frac{f(T_j D u^z_{\alpha,j})}{T_j^n}\,dx < \varphi_j^\alpha(z) + \eta |z|^n. \tag{21}$$

A. We first consider the case in which $z' = \beta z$ for some constant $\beta \neq 0$. Note that $\beta u^z_{\alpha,j}$ is a good test function for $\varphi_j^\alpha(z')$. By (5) we get

$$\begin{aligned} |f(\beta T_j D u^z_{\alpha,j}) - f(T_j D u^z_{\alpha,j})| &\le k(|\beta T_j D u^z_{\alpha,j}|^{n-1} + |T_j D u^z_{\alpha,j}|^{n-1})|\beta T_j D u^z_{\alpha,j} - T_j D u^z_{\alpha,j}| \\ &= k(|\beta|^{n-1}+1)|\beta-1||T_j D u^z_{\alpha,j}|^n \end{aligned}$$

Therefore

$$f(\beta T_j D u^z_{\alpha,j}) \le f(T_j D u^z_{\alpha,j}) + k(|\beta|^{n-1}+1)|\beta - 1|\,|T_j D u^z_{\alpha,j}|^n. \tag{22}$$

Taking into account (21), (22) and the growth conditions on f and φ_j^α, we get

$$\begin{aligned} \varphi_j^\alpha(z') &\le (\log T_j)^{n-1} \int_{B_{\alpha S_j}} \frac{f(\beta T_j D u^z_{\alpha,j})}{T_j^n}\,dx \le (\log T_j)^{n-1} \int_{B_{\alpha S_j}} \frac{f(T_j D u^z_{\alpha,j})}{T_j^n}\,dx \\ &\qquad + k|\beta-1|(|\beta|^{n-1}+1)(\log T_j)^{n-1} \int_{B_{\alpha S_j}} |D u^z_{\alpha,j}|^n\,dx \\ &\le \varphi_j^\alpha(z) + \eta|z|^n + k|\beta-1|(|\beta|^{n-1}+1)(\log T_j)^{n-1} \int_{B_{\alpha S_j}} |D u^z_{\alpha,j}|^n\,dx \\ &\le \varphi_j^\alpha(z) + \eta|z|^n + \frac{k}{c_1}|\beta-1|(|\beta|^{n-1}+1)(\log T_j)^{n-1} \int_{B_{\alpha S_j}} \frac{f(T_j D u^z_{\alpha,j})}{T_j^n}\,dx \\ &\le \varphi_j^\alpha(z) + \eta|z|^n + \frac{k}{c_1}|\beta-1|(|\beta|^{n-1}+1)\big(\varphi_j^\alpha(z) + \eta|z|^n\big) \\ &\le \varphi_j^\alpha(z) + k|\beta-1|(|\beta|^{n-1}+1)\frac{\tilde{c}_2}{c_1}|z|^n + \eta|z|^n\Big(1 + \frac{k}{c_1}|\beta-1|(|\beta|^{n-1}+1)\Big) \\ &= \varphi_j^\alpha(z) + k\frac{\tilde{c}_2}{c_1}|z'-z|(|z'|^{n-1}+|z|^{n-1}) + \eta c|z|^n. \end{aligned}$$

By a symmetric argument and the arbitrariness of η we get

$$|\varphi_j^\alpha(z) - \varphi_j^\alpha(z')| \le k\frac{\tilde{c}_2}{c_1}|z'-z|(|z'|^{n-1}+|z|^{n-1}). \tag{23}$$

Thus, the functions φ_j^α are equi-locally Lipschitz continuous in the radial directions of $\mathbb{R}^m \setminus \{0\}$.

B. Now we consider the case in which $|z| = |z'|$; i.e., there exists $\mathcal{A} \in \mathrm{SO}(m)$ such that $\mathcal{A}z = z'$. Note that $\mathcal{A}u^z_{\alpha,j}$ is a test function for $\varphi^\alpha_j(z')$.
By (5) we get

$$|f(T_j \mathcal{A} Du^z_{\alpha,j}) - f(T_j Du^z_{\alpha,j})| \leq k|\mathcal{A} - Id|(|\mathcal{A}|^{n-1} + 1)|TDu^z_{\alpha,j}|^n.$$

Arguing as in **A** we get

$$\begin{aligned}
\varphi^\alpha_j(z') &\leq (\log T_j)^{n-1} \int_{B_{\alpha S_j}} \frac{f(T_j \mathcal{A} Du^z_{\alpha,j})}{T_j^n}\, dx \\
&\leq \varphi^\alpha_j(z) + k|\mathcal{A} - Id|(|\mathcal{A}|^{n-1} + 1)\frac{\tilde{c}_2}{c_1}|z|^n + \eta|z|^n\Big(1 + \frac{k}{c_1}|\mathcal{A} - Id|(|\mathcal{A}|^{n-1} + 1)\Big) \\
&= \varphi^\alpha_j(z) + k\frac{\tilde{c}_2}{c_1}|z' - z|(|z'|^{n-1} + |z|^{n-1}) + \eta c|z|^n.
\end{aligned}$$

By a symmetric argument and the arbitrariness of η we get

$$|\varphi^\alpha_j(z) - \varphi^\alpha_j(z')| \leq k\frac{\tilde{c}_2}{c_1}|z' - z|(|z'|^{n-1} + |z|^{n-1}). \tag{24}$$

We have proved that the functions φ^α_j are equi-locally Lipschitz continuous along the tangential directions of $\mathrm{R}^m \setminus \{0\}$.

C. Under general assumptions we can connect z and z' by the composition of a rotation and a homothety. By (23) and (24) we deduce that there exists a constant $c > 0$ (independent of α and j) such that

$$|\varphi^\alpha_j(z) - \varphi^\alpha_j(z')| \leq c|z - z'|(|z|^{n-1} + |z'|^{n-1}). \tag{25}$$

The equi-locally Lipschitz continuity on the whole $\mathbb{R}^m$ follows easily.

For fixed $\alpha > 0$, the sequence $(\varphi^\alpha_j)_{j\in\mathbb{N}}$ satisfies the assumptions of Ascoli-Arzelà's Theorem, hence there exists $\varphi^\alpha : \mathbb{R}^m \to [0, +\infty)$ such that up to subsequences

$$\varphi^\alpha(z) = \lim_{j\to+\infty} \varphi^\alpha_j(z), \qquad \text{uniformly on the compact sets of } \mathbb{R}^m.$$

The same argument holds for the family $(\varphi^\alpha_j)_{\alpha>0}$, with j fixed: there exists $\varphi_j : \mathbb{R}^m \to [0, +\infty)$ such that up to subsequences

$$\varphi_j(z) = \lim_{\alpha\to0^+} \varphi^\alpha_j(z), \qquad \text{uniformly on the compact sets of } \mathbb{R}^m.$$

We recall that $\alpha \mapsto \varphi^\alpha_j(z)$ is decreasing and note that $\alpha \mapsto \varphi^\alpha(z)$ is monotone as well.

In conclusion, there exists a function $\varphi : \mathbb{R}^m \to [0, +\infty)$ such that

$$\varphi(z) = \sup_{\alpha>0} \lim_{j\to+\infty} \varphi^\alpha_j(z) = \lim_{\alpha\to0^+} \lim_{j\to+\infty} \varphi^\alpha_j(z) = \lim_{j\to+\infty} \lim_{\alpha\to0^+} \varphi^\alpha_j(z),$$

uniformly on the compact sets of $\mathbb{R}^m$. This proves the first statement of Theorem 2.1.

3.2 A lemma for varying domains

This section deals with a technical result by Ansini and Braides which allows to modify sequences of functions near the perforations K_i^j. The following lemma is valid under general assumptions (see [2, Lemma 3.1]), but we will recall it in the form we need in this article.

Lemma 3.2 *Let (u_j) converge weakly to u in $W^{1,n}(\Omega;\mathbb{R}^m)$. Let (ρ_j) be a positive sequence of the form $\rho_j = \overline{c}\delta_j$, where $\overline{c} < 1/2$. For all $j \in \mathbb{N}$ we define*

$$Z_j = \{i \in \mathbb{Z}^n : \operatorname{dist}(x_i^j, \mathbb{R}^n \setminus \Omega) > \delta_j\}. \tag{26}$$

We fix $k \in \mathbb{N}$. Then, for all $i \in Z_j$ there exists $k_i \in \{0, 1, \ldots, k-1\}$ such that, having set

$$\begin{aligned} C_i^j &= \left\{x \in \Omega : \frac{1}{2^{k_i+1}}\rho_j < |x - x_i^j| < \frac{1}{2^{k_i}}\rho_j\right\}, \quad (27)\\ u_j^i &= |C_i^j|^{-1}\int_{C_i^j} u_j\,dx,\\ \rho_j^i &= \frac{3}{4}2^{-k_i}\rho_j, \end{aligned}$$

there exists a sequence (w_j), with $w_j \rightharpoonup u$ in $W^{1,n}(\Omega;\mathbb{R}^m)$, such that

$$w_j = u_j \text{ on } \Omega \setminus \bigcup_{i\in Z_j} C_i^j, \tag{28}$$

$$w_j(x) = u_j^i \text{ if } |x - x_i^j| = \rho_j^i, \tag{29}$$

$$\text{and} \qquad \int_\Omega |f(Dw_j) - f(Du_j)|\,dx \le \frac{c}{k}. \tag{30}$$

3.3 A discretization argument

In this section we prove that the extra-term of the Γ-limit can be obtained through a discretization argument. This technical result will play an important role in the proof of Theorem 2.1.

Proposition 3.3 *Let (u_j) be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$ converging to u weakly in $W^{1,n}(\Omega;\mathbb{R}^m)$. We fix $k \in \mathbb{N}$. Let (ρ_j) be a positive sequence of the form $\rho_j = \overline{c}\delta_j$, where $\overline{c} < 1/2$. For all $i \in Z_j$, defined as in (26), and for an arbitrary choice of $k_i \in \{0, 1, \ldots, k-1\}$, we consider an annuli C_i^j of the form (27). We denote by u_j^i the mean value of u_j on C_i^j and by Q_i^j the n-cube $Q_i^j = x_i^j + \left(-\frac{\delta_j}{2}, \frac{\delta_j}{2}\right)^n$. Let ψ_j^α be defined by*

$$\psi_j^\alpha = \sum_{i\in Z_j} \varphi_j^\alpha(u_j^i)\chi_{Q_i^j}, \tag{31}$$

where $\alpha > 0$ is fixed. Then, upon possibly passing to subsequences, we have

$$\lim_{j\to+\infty}\int_\Omega |\psi_j^\alpha - \varphi^\alpha(u)|\,dx = 0. \tag{32}$$

Proof. By the $W^{1,n}$-weak convergence of u_j to u and the uniform convergence of φ_j^α to φ^α on the compact sets of $\mathbb{R}^m$, we deduce that $\varphi_j^\alpha(u_j)$ tends to $\varphi^\alpha(u)$ in $L^1(\Omega)$. There follows that the limit in (32) equals the limits

$$\lim_j \int_\Omega |\psi_j^\alpha - \varphi_j^\alpha(u_j)|\, dx = \lim_j \sum_{i\in Z_j} \int_{Q_i^j} |\varphi_j^\alpha(u_j^i) - \varphi_j^\alpha(u_j)|\, dx.$$

In Section 3.1 we proved that the functions φ_j^α are equi-locally Lipschitz continuous, hence

$$\sum_{i\in Z_j} \int_{Q_i^j} |\varphi_j^\alpha(u_j^i) - \varphi_j^\alpha(u_j)|\, dx \le c\big(\sup_j \|u_j\|_\infty^{n-1}\big) \sum_{i\in Z_j} \int_{Q_i^j} |u_j^i - u_j|\, dx.$$

By Hölder's inequality we get

$$\int_{Q_i^j} |u_j^i - u_j|\, dx \le \delta_j^{n-1} \Big(\int_{Q_i^j} |u_j^i - u_j|^n\, dx \Big)^{\frac{1}{n}}.$$

We want to estimate the last integral with a quantity independent of the index i. Poincaré-Wirtinger's inequality states the existence of a constant $P > 0$ such that

$$\Big(\int_{Q_i^j} |u_j^i - u_j|^n\, dx \Big)^{\frac{1}{n}} \le P\delta_j \Big(\int_{Q_i^j} |Du_j|^n\, dx \Big)^{\frac{1}{n}}. \tag{33}$$

Note that P depends on C_i^j and hence on the choice of $k_i \in \{0, 1, \ldots, k-1\}$. However, the family of homothetic annuli $\{C_i^j : k = 0, \ldots, k-1\}$ is finite, hence it suffices to take P as the maximum among the finite family of corresponding Poincaré-Wirtinger's constants to get a constant independent of i. To sum up, we have:

$$\begin{aligned} \lim_j \int_\Omega |\psi_j^\alpha - \varphi^\alpha(u)|\, dx &\le c \limsup_j \sum_{i\in Z_j} \delta_j^{n-1} P \delta_j \Big(\int_{Q_i^j} |Du_j|^n\, dx \Big)^{\frac{1}{n}} \\ &= c \limsup_j \delta_j^n \sum_{i\in Z_j} \Big(\int_{Q_i^j} |Du_j|^n\, dx \Big)^{\frac{1}{n}}. \end{aligned} \tag{34}$$

Since the function $y \mapsto y^{\frac{1}{n}}$ is concave, we have

$$\begin{aligned} \sum_{i\in Z_j} \frac{1}{\#Z_j} \Big(\int_{Q_i^j} |Du_j|^n\, dx \Big)^{\frac{1}{n}} &\le \Big(\sum_{i\in Z_j} \frac{1}{\#Z_j} \int_{Q_i^j} |Du_j|^n\, dx \Big)^{\frac{1}{n}} \\ &\le \Big(\frac{1}{\#Z_j} \Big)^{\frac{1}{n}} \Big(\int_\Omega |Du_j|^n\, dx \Big)^{\frac{1}{n}}. \end{aligned} \tag{35}$$

Note that $\#Z_j \simeq |\Omega|/\delta_j^n$. By (34) and (35) we get

$$\begin{aligned} \lim_j \int_\Omega |\psi_j^\alpha - \varphi^\alpha(u)|\, dx &\le c \limsup_j \delta_j^n (\#Z_j) \sum_{i\in Z_j} \frac{1}{\#Z_j} \Big(\int_{Q_i^j} |Du_j|^n\, dx \Big)^{\frac{1}{n}} \\ &\le c \limsup_j \delta_j^n (\#Z_j)^{1-1/n} \Big(\int_\Omega |Du_j|^n\, dx \Big)^{\frac{1}{n}} \\ &\le c \limsup_j \delta_j = 0, \end{aligned}$$

as desired. ∎

4 Proof of the main result

We divide the proof of Theorem 2.1 into two steps: first we prove the Γ-liminf inequality and then we show how to build recovery sequences and get the Γ-limsup inequality.

4.1 Liminf inequality

Let $u \in W^{1,n}(\Omega;\mathbb{R}^m)$ and let $u_j \to u$ in $L^n(\Omega;\mathbb{R}^m)$ be such that $\sup_j F_j(u_j) < +\infty$. Since f satisfies a growth condition of order n, we deduce that $u_j \rightharpoonup u$ weakly in $W^{1,n}(\Omega;\mathbb{R}^m)$. In this section we will prove the following result:

Proposition 4.1 (Liminf inequality) *Let $u \in W^{1,n}(\Omega;\mathbb{R}^m)$ and let $u_j \to u$ in $L^n(\Omega;\mathbb{R}^m)$ be such that $\sup_j F_j(u_j) < +\infty$. Then*

$$\liminf_j \int_\Omega f(Du_j)\,dx \geq \int_\Omega f(Du)\,dx + a^{1-n}\int_\Omega \varphi(u)\,dx,$$

where φ is defined as in (9).

Proof. We divide the proof into two steps: first we assume that in addition (u_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$ and then we deal with the general case.

1 Let (u_j) be bounded in $L^\infty(\Omega;\mathbb{R}^m)$. We want to apply Lemma 3.2 to the sequence $u_j \to u$. To this end we fix $k \in \mathbb{N}$ and consider the infinitesimal sequence $\rho_j = \alpha\delta_j$, with $\alpha < 1/2$. Lemma 3.2 states the existence of $w_j \to u$ in $L^n(\Omega;\mathbb{R}^m)$ satisfying (28)-(30) with $\rho_j^i = \frac{3}{4}2^{-k_i}\rho_j$, where $k_i \in \{0,\dots,k-1\}$. We recall that $Z_j = \{i \in \mathbb{Z}^n : \operatorname{dist}(x_i^j, \mathbb{R}^n \setminus \Omega) > \delta_j\}$ and denote by E_j the set

$$E_j = \bigcup_{i\in Z_j} B_i^j, \text{ where } B_i^j = B_{\rho_j^i}(x_i^j). \tag{36}$$

We treat separately the contribution of $f(Du_j)$ on $\Omega \setminus E_j$ and on E_j (step **A** and **B** respectively).

1.A We first deal with the contribution of $f(Du_j)$ outside the set E_j. We will prove that

$$\liminf_j \int_{\Omega\setminus E_j} f(Du_j)\,dx \geq \int_\Omega f(Du)\,dx - \frac{c}{k}. \tag{37}$$

Let

$$v_j(x) = \begin{cases} u_j^i & \text{for } x \in B_i^j,\ i \in Z_j \\ w_j(x) & \text{for } x \in \Omega\setminus E_j. \end{cases}$$

Note that (v_j) is bounded in $W^{1,n}(\Omega;\mathbb{R}^m)$, hence there exists a function v such that $v_j \to v$ in $L^n(\Omega;\mathbb{R}^m)$ upon passing to subsequences. Let

$$\chi_j = \chi_{\Omega\setminus \bigcup_{i\in Z_j} B_{\rho_j}(x_i^j)}.$$

By construction there exists a constant $\beta \in \mathbb{R}^+$ such that χ_j converges weakly* to β in L^∞ (see e.g. [5, Example 2.4]). Hence $v_j\chi_j \rightharpoonup \beta v$ in L^n and $u_j\chi_j \rightharpoonup \beta u$ in L^n. Since

$v_j\chi_j = u_j\chi_j$ we can deduce that $u = v$. There follows that $v_j \rightharpoonup u$ in $W^{1,n}(\Omega;\mathbb{R}^m)$. From Lemma 3.2 and the quasiconvexity of f we obtain

$$\begin{aligned}\liminf_j \int_{\Omega\setminus E_j} f(Du_j)\,dx + \frac{c}{k} &\geq \liminf_j \int_{\Omega\setminus E_j} f(Dw_j)\,dx \\ &= \liminf_j \int_\Omega f(Dv_j)\,dx \geq \int_\Omega f(Du)\,dx\end{aligned}$$

as desired.

1.B We now turn to the estimate of the contribution of $f(Du_j)$ on E_j. We will prove that

$$\liminf_j \int_{E_j} f(Du_j)\,dx \geq a^{1-n}\int_\Omega \varphi^\alpha(u)\,dx - \frac{c}{k}. \tag{38}$$

Lemma 3.2 states that

$$\liminf_j \int_{E_j} f(Du_j)\,dx + \frac{c}{k} \geq \liminf_j \int_{E_j} f(Dw_j)\,dx = \liminf_j \sum_{i\in Z_j}\int_{B_i^j} f(Dw_j)\,dx.$$

We fix $j \in \mathbb{N}$ and $i \in Z_j$. Let

$$w_j^i(x) = \begin{cases} w_j(x + x_i^j) & \text{for } |x| \leq \rho_j^i, \\ u_j^i = |C_i^j|^{-1}\displaystyle\int_{C_i^j} u_j\,dx & \text{otherwise.}\end{cases}$$

We define a function $\zeta \in u_j^i + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m)$ by setting $\zeta(y) = w_j^i(\varepsilon_j y)$. Note that if $|y| = \alpha S_j$ then $|\varepsilon_j y| = \alpha\delta_j > \frac{3}{4}2^{-k_i}\alpha\delta_j$, hence $\zeta = u_j^i$ on $\partial B_{\alpha S_j}$. Moreover, $\zeta = 0$ on K. By a change of variables we obtain

$$\begin{aligned}\int_{B_i^j} f(Dw_j(x))\,dx &= \int_{B_{\rho_j}} f(Dw_j^i(x))\,dx = \int_{B_{\alpha S_j}} \frac{f(T_j D\zeta(y))}{T_j^n}\,dy \\ &\geq \inf\Big\{\int_{B_{\alpha S_j}} \frac{f(T_j Dv(y))}{T_j^n}\,dy:\ v \in u_j^i + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } K\Big\} \\ &= \varphi_j^\alpha(u_j^i)\frac{1}{(\log T_j)^{n-1}} = \varphi_j^\alpha(u_j^i)\frac{\delta_j^n}{a^{n-1}}.\end{aligned}$$

Now we apply Proposition 3.3 and get

$$\begin{aligned}\liminf_j \int_{E_j} f(Du_j)\,dx + \frac{c}{k} &\geq \liminf_j \sum_{i\in Z_j}\int_{B_i^j} f(Dw_j)\,dx \\ &\geq \liminf_j \sum_{i\in Z_j} a^{1-n}\varphi_j^\alpha(u_j^i)\delta_j^n = a^{1-n}\int_\Omega \varphi^\alpha(u)\,dx.\end{aligned}$$

To sum up, we have proved that for all $\alpha \in (0,1/2)$ the contribution of $f(Du_j)$ on E_j can be estimated as follows:

$$\liminf_j \int_{E_j} f(Du_j)\,dx \geq a^{1-n}\int_\Omega \varphi^\alpha(u)\,dx - \frac{c}{k}.$$

as desired.

Matching the results of steps **1.A** and **1.B** and taking into account the arbitrariness of k, we get

$$\liminf_j F_j(u_j) \geq \int_\Omega f(Du)\,dx + a^{1-n}\int_\Omega \varphi^\alpha(u)\,dx.$$

By Beppo Levi's Theorem we can take the supremum over $\alpha > 0$ and conclude that

$$\liminf_j F_j(u_j) \geq \int_\Omega f(Du)\,dx + a^{1-n}\int_\Omega \varphi(u)\,dx,$$

provided that (u_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$.

2 We now remove the boundedness assumption on (u_j). By [7, Lemma 3.5], upon passing to a subsequence, for all $M \in \mathbb{N}$ and $\eta > 0$ there exist $R_M > M$ and a Lipschitz function Φ_M of Lipschitz constant 1 such that

$$\begin{cases} \Phi_M(z) = z & \text{if } |z| < R_M, \\ \Phi_M(z) = 0 & \text{if } |z| > 2R_M, \\ \lim\limits_j F_j(u_j) \geq \liminf_j F_j(\Phi_M(u_j)) - \eta. \end{cases} \tag{39}$$

Let $\alpha \in (0, 1/2)$. We fix $M \in \mathbb{N}$ and $\eta > 0$ and we apply the arguments of steps **1.A** and **1.B** to the sequence $\Phi_M(u_j)$. Hence we get

$$\liminf_j F_j(\Phi_M(u_j)) \geq \int_\Omega f(D\Phi_M(u))\,dx + a^{1-n}\int_\Omega \varphi^\alpha(\Phi_M(u))\,dx.$$

By (39) we obtain

$$\lim_j F_j(u_j) + \eta \geq \liminf_j F_j(\Phi_M(u_j)) \geq \int_\Omega f(D\Phi_M(u))\,dx + a^{1-n}\int_\Omega \varphi^\alpha(\Phi_M(u))\,dx.$$

If $M \to +\infty$ we have $\Phi_M(u) \rightharpoonup u$ in $W^{1,n}(\Omega;\mathbb{R}^m)$, thus

$$\lim_j F_j(u_j) + \eta \geq \int_\Omega f(Du)\,dx + a^{1-n}\int_\Omega \varphi^\alpha(u)\,dx.$$

Taking the supremum over $\alpha > 0$ and letting $\eta \to 0$ and we get the thesis. ■

4.2 Limsup inequality

This section completes the proof of the Γ-convergence result in the general case. A careful use of Lemma 3.2 allows to build recovery sequences. For all $u \in W^{1,n}(\Omega;\mathbb{R}^m)$ we will prove an *approximate limsup inequality*, which is equivalent to the existence of a recovery sequence (see, e.g. [5]).

Proposition 4.2 (Limsup inequality) *For all $u \in W^{1,n}(\Omega;\mathbb{R}^m)$ and $k \in \mathbb{N}$ there exists a sequence $u_j \rightharpoonup u$ in $W^{1,n}(\Omega;\mathbb{R}^m)$ such that*

$$\limsup_j F_j(u_j) \leq \int_\Omega f(Du)\,dx + a^{1-n}\int_\Omega \varphi(u)\,dx + \frac{c}{k}. \tag{40}$$

Proof. Let $u \in W^{1,n}(\Omega;\mathbb{R}^m)$. We want to apply Lemma 3.2 to the constant sequence $u_j \equiv u$. To this end we fix $k \in \mathbb{N}$ and choose a positive sequence $\rho_j = 2^{k+1}\alpha\delta_j$, where α is such that $\alpha < \alpha 2^{k+1} < 1/2$. Following the notations of Lemma 3.2, we can state that there exist $k_i \in \{0,\ldots,k-1\}$ (for all $i \in Z_j$) and a sequence $(v_j) \subset W^{1,n}(\Omega;\mathbb{R}^m)$ such that

$$\begin{aligned}
&v_j \rightharpoonup u \text{ in } W^{1,n}(\Omega;\mathbb{R}^m), \qquad v_j = u \text{ on } \Omega \setminus \bigcup_{i\in Z_j} C_i^j,\\
&v_j(x) = u_j^i = |C_i^j|^{-1}\int_{C_i^j} u\,dx \text{ for } |x - x_i^j| = \rho_j^i = \frac{3}{4}2^{-k_i}\rho_j,\\
&\text{and} \qquad \int_\Omega |f(Dv_j) - f(Du)|\,dx \le \frac{c}{k}.
\end{aligned} \tag{41}$$

1 We first assume that in addition $u \in L^\infty(\Omega;\mathbb{R}^m)$. We want to modify the sequence (v_j) to obtain an approximate recovery sequence $u_j \in W^{1,n}(\Omega;\mathbb{R}^m)$; i.e., a sequence $u_j \to u$ in $L^n(\Omega;\mathbb{R}^m)$ possibly depending on k and such that (40) is satisfied.

In order to obtain the required sequence (u_j) we modify the functions v_j close to the perforations; i.e., on some convenient balls surrounding them. We recall that $Z_j = \{i \in \mathbb{Z}^n : \operatorname{dist}(x_i^j, \partial\Omega) > \delta_j\}$ and we define the set of indexes

$$Z'_j = \{i \in \mathbb{Z}^n : K_i^j \cap \Omega \ne \emptyset \text{ and } i \notin Z_j\}.$$

We deal separately with the case $i \in Z_j$ and $i \in Z'_j$. (step **1.A** and **1.B** respectively).

1.A We first consider the perforations such that $i \in Z_j$. We denote by G_j the union of the corresponding balls $B_{\rho_j^i}(x_i^j)$:

$$G_j = \bigcup_{i\in Z_j} B_{\rho_j^i}(x_i^j).$$

For fixed $i \in Z_j$ let $\zeta_{i,j}^\alpha \in u_j^i + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m)$ be a function such that $\zeta_{i,j}^\alpha = 0$ on K and

$$(\log T_j)^{n-1}\int_{B_{\alpha S_j}} \frac{f(T_j D\zeta_{i,j}^\alpha)}{T_j^n}\,dx < \varphi_j^\alpha(u_j^i) + \frac{1}{k}. \tag{42}$$

We define u_j on the ball $B_{\rho_j^i}(x_i^j)$ as follows:

$$u_j(x) = \begin{cases} \zeta_{i,j}^\alpha\Big(\dfrac{x - x_i^j}{\varepsilon_j}\Big) & \text{for } |x - x_i^j| \le \alpha\delta_j,\\ u_j^i & \text{for } \alpha\delta_j < |x - x_i^j| < \rho_j^i. \end{cases}$$

Note that u_j vanishes on K_i^j. By a change of variables we get

$$\begin{aligned}
\int_{B_{\rho_j^i}(x_i^j)} f(Du_j)\,dx &= \int_{B_{\rho_j^i}(x_i^j)\setminus B_{\alpha\delta_j}(x_i^j)} f(Du_j)\,dx + \int_{B_{\alpha\delta_j}(x_i^j)} f(Du_j)\,dx\\
&= \int_{B_{\alpha S_j}} \frac{f(T_j D\zeta_{i,j}^\alpha(y))}{T_j^n}\,dy \le \frac{1}{(\log T_j)^{n-1}}\varphi_j^\alpha(u_j^i) + \frac{1}{(\log T_j)^{n-1}}\frac{1}{k}\\
&= a^{1-n}\delta_j^n\varphi_j^\alpha(u_j^i) + a^{1-n}\delta_j^n\frac{1}{k}.
\end{aligned}$$

Summing up on $i \in Z_j$, we get

$$\begin{aligned}\limsup_j \int_{G_j} f(Du_j)\,dx &= \limsup_j \sum_{i\in Z_j} \int_{B_{\rho_j^i}(x_i^j)} f(Du_j)\,dx \\ &\le \limsup_j \sum_{i\in Z_j} a^{1-n}\delta_j^n \varphi_j^\alpha(u_j^i) + a^{1-n}|\Omega|\frac{1}{k}.\end{aligned}$$

Taking into account Proposition 3.3, we deduce that

$$\limsup_j \int_{G_j} f(Du_j)\,dx \le a^{1-n}\int_\Omega \varphi^\alpha(u)\,dx + \frac{c}{k} \le a^{1-n}\int_\Omega \varphi(u)\,dx + \frac{c}{k}. \tag{43}$$

1.B Now we deal with the case $i \in Z_j'$. We denote by G_j' the union of the corresponding balls $B_{\rho_j}(x_i^j)$; i.e.,

$$G_j' = \bigcup_{i\in Z_j'} B_{\rho_j}(x_i^j)\cap\Omega.$$

Moreover, we denote by Ω_j' the set given by the union of the corresponding cubes $Q_i^j = x_i^j + \Big(-\frac{\delta_j}{2}, \frac{\delta_j}{2}\Big)^n$:

$$\Omega_j' = \bigcup_{i\in Z_j'} Q_i^j.$$

We define u_j on G_j' so that

$$\int_{G_j'} f(Du_j)\,dx = o(1) \quad \text{as } j\to+\infty. \tag{44}$$

We recall that up to translations there exists a ball B_{r_2} containing K. Let $\zeta_j(x)$ be the radial minimizer for the (scalar) minimum problem

$$\begin{aligned}\mu_j &= \min\Big\{\int_{B_{\rho_j}} |Du|^n\,dx:\ u\in 1+W_0^{1,n}(B_{\rho_j};\mathbb{R}),\ u=0 \text{ on } B_{r_2\varepsilon_j}\Big\} \\ &= \min\Big\{\int_{B_{\frac{\rho_j}{r_2}}} |Du|^n\,dx:\ u\in 1+W_0^{1,n}(B_{\frac{\rho_j}{r_2}};\mathbb{R}),\ u=0 \text{ on } B_{\varepsilon_j}\Big\}\end{aligned}$$

Arguing as in Section 3.1 we get

$$\zeta_j(x) = \frac{\log\big(|x|T_j\big)}{\log\big(\frac{\rho_j}{r_2}T_j\big)}\vee 0 \qquad\text{and}\qquad \mu_j = \omega_{n-1}\Big(\log\Big(\frac{\rho_j}{r_2}T_j\Big)\Big)^{1-n}.$$

For all $x\in B_{\rho_j}(x_i^j)\cap\Omega$ we set $u_j(x) = u(x)\,\zeta_j(x-x_i^j)$. Note that u_j vanishes on $K_i^j\cap\Omega$. By the growth condition (4) we have

$$\begin{aligned}\int_{B_{\rho_j}(x_i^j)\cap\Omega} f(Du_j(x))\,dx &\le c_2\int_{B_{\rho_j}(x_i^j)\cap\Omega} |Du_j(x)|^n\,dx \\ &\le c\int_{B_{\rho_j}\cap\Omega} |Du(x)|^n\,dx + c\|u\|_\infty^n \int_{B_{\rho_j}} |D\zeta_j(x)|^n\,dx \\ &\le c\int_{B_{\rho_j}\cap\Omega} |Du(x)|^n\,dx + c\Big(\log\Big(\frac{\rho_j}{r_2}T_j\Big)\Big)^{1-n} \\ &= c\delta_j^n + o(1) \quad \text{as } j\to+\infty.\end{aligned}$$

There follows that

$$\begin{aligned}\limsup_j \int_{G'_j} f(Du_j)\,dx &= \limsup_j \sum_{i\in Z'_j} \int_{B_{\rho_j}(x_i^j)\cap\Omega} f(Du_j)\,dx \\ &\le c \limsup_j \sum_{i\in Z'_j} \delta_j^n \le c\limsup_j |\Omega'_j|.\end{aligned}$$

Since $\lim_j |\Omega'_j| = |\partial\Omega| = 0$, we get

$$\limsup_j \int_{G'_j} f(Du_j)\,dx = 0. \tag{45}$$

Finally, we set $u_j(x) = v_j(x)$ on $\Omega \setminus (G_j \cup G'_j)$. The sequence (u_j) we have defined is then an approximate recovery sequence. In fact, $u_j \to u$ in $L^n(\Omega;\mathbb{R}^m)$ since (u_j) is bounded in $W^{1,n}(\Omega;\mathbb{R}^m)$ and $\lim_j |\{u_j \neq v_j\}| = 0$ by construction. Moreover, conditions (41), (43) and (45) imply that

$$\begin{aligned}\limsup_j \int_\Omega f(Du_j)\,dx &\le \limsup_j \int_{G_j} f(Du_j)\,dx + \limsup_j \int_{G'_j} f(Dv_j)\,dx \\ &\quad + \limsup_j \int_{\Omega\setminus(G_j\cup G'_j)} f(Du_j)\,dx \\ &\le a^{1-n}\int_\Omega \varphi(u)\,dx + \frac{c}{k} + \limsup_j \int_\Omega f(Dv_j)\,dx \\ &\le a^{1-n}\int_\Omega \varphi(u)\,dx + \int_\Omega f(Du)\,dx + \frac{c}{k},\end{aligned}$$

as desired.

2 We now remove the boundedness assumption on u. Let $u \in W^{1,n}(\Omega;\mathbb{R}^m)$. Note that u can be approximated by a sequence $(u_k) \subset W^{1,n}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ with respect to the strong convergence of $W^{1,n}(\Omega;\mathbb{R}^m)$. By the lower semicontinuity of the functional $F''(u) = \Gamma\text{-}\limsup_j F_j(u)$ with respect to the $L^n(\Omega;\mathbb{R}^m)$-convergence, we then have

$$F''(u) \le \liminf_k F''(u_k) = \lim_k F(u_k) = F(u)$$

and the proof is complete. ■

5 The n-homogeneous case

In this section we want to focus our attention on the interesting case in which the integrand function f is positively homogeneous of degree n. We will prove the last part of Theorem 2.1, which states that in this case the Γ-limit is independent of the subsequence of (T_j) and that the limit extra-term can be determined through a formula of homogenization type. Moreover, from that formula we easily deduce that the result is independent of the shape of the perforations. Note that the independence of the shape of the perforations can be proved also without the homogeneity assumption (see Casado-Diaz [11] for a proof in the framework of two-scale convergence method for monotone Dirichlet problems, corresponding to f convex and smooth).

Our first step consists in proving the following proposition.

Proposition 5.1 *Let $f : \mathbb{R}^{m\times n} \to [0, +\infty)$ be a quasiconvex function satisfying conditions (4) and (5) and such that $f(0) = 0$. Suppose that f is positively homogeneous of degree n. Then, for all $\nu \in \mathbb{R}^m$ with $|\nu| = 1$, there exists the limit*

$$\psi(\nu) = \lim_{T\to+\infty} (\log T)^{n-1} \min \Big\{ \int_{B_T} f(Du)\, dx : \ u \in \nu + W_0^{1,n}(B_T; \mathbb{R}^m), \ u = 0 \ \text{on } B_1 \Big\}. \tag{46}$$

Proof. Let $\nu \in \mathbb{R}^m$ be such that $|\nu| = 1$. Let

$$g_\nu(T) = (\log T)^{n-1} \min \Big\{ \int_{B_T} f(Du)\, dx : \ u \in \nu + W_0^{1,n}(B_T; \mathbb{R}^m), \ u = 0 \text{ on } B_1 \Big\}. \tag{47}$$

We will prove that there exists a function $r_\nu(\cdot, \cdot)$ such that

$$\liminf_{T\to+\infty} \big(\limsup_{S\to+\infty} r_\nu(S, T) \big) = 0 \tag{48}$$

$$\text{and} \qquad g_\nu(S) \le g_\nu(T) + r_\nu(S, T) \text{ for all } S \ge T. \tag{49}$$

If we first take the $\limsup_{S\to+\infty}$ and then the $\liminf_{T\to+\infty}$ in (49), we get

$$\limsup_{S\to+\infty} g_\nu(S) \le \liminf_{T\to+\infty} g_\nu(T),$$

thus we obtain the thesis.

Let $S \ge T$ be fixed. Let $u_T \in \nu + W_0^{1,n}(B_T; \mathbb{R}^m)$ be such that $u = 0$ on B_1 and

$$(\log T)^{n-1} \int_{B_T} f(Du_T)\, dx = g_\nu(T).$$

We denote by k the positive integer such that $T^k \le S < T^{k+1}$; i.e., $k = \left[\frac{\log S}{\log T}\right]$. We will modify u_T in order to obtain a function $\tilde{u}_T$ (see Fig. 1), which will be re-scaled into a test function for $g_\nu(S)$. Let

$$\tilde{u}_T(x) = \begin{cases} u_T\Big(\dfrac{x}{T^j}\Big) + j\nu & \text{if } T^j \le |x| \le T^{j+1}, \text{ for } j \in \{0, 1, \dots, k-1\}, \\[2ex] \nu \log|x| (\log T)^{-1} & \text{if } T^k \le |x| \le S. \end{cases}$$

Note that $\tilde{u}_T \in \frac{\log S}{\log T}\nu + W_0^{1,n}(B_S; \mathbb{R}^m)$ and $\tilde{u}_T = 0$ on B_1; hence $v_S(x) = \frac{\log T}{\log S}\tilde{u}_T(x)$ is a test function for $g_\nu(S)$. By construction we have

$$Dv_S(x) = \frac{\log T}{\log S} D\tilde{u}_T(x) = \begin{cases} \dfrac{\log T}{\log S} Du_T\Big(\dfrac{x}{T^j}\Big)\dfrac{1}{T^j} & \text{if } T^j \le |x| \le T^{j+1}, \\[2ex] A(x)(\log S)^{-1} & \text{if } T^k \le |x| \le S, \end{cases}$$

where $A(x) = (a_{h,j}(x))$ is given by $a_{h,j}(x) = x_j\, \nu_h |x|^{-2}$, for all $h \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

There follows that

$$\begin{aligned}
\int_{B_S} f(Dv_S)\,dx &= \sum_{j=0}^{k-1} \Big(\frac{\log T}{\log S}\Big)^n \int_{B_{T^{j+1}}\setminus B_{T^j}} f\Big(Du_T\Big(\frac{x}{T^j}\Big)\Big)\frac{1}{(T^j)^n}\,dx \\
&\quad + \int_{B_S\setminus B_{T^k}} f\Big(\frac{A(x)}{\log S}\Big)\,dx \\
&= \Big(\frac{\log T}{\log S}\Big)^n k \int_{B_T} f(Du_T)\,dx + \frac{1}{(\log S)^n}\int_{B_S\setminus B_{T^k}} f(A(x))\,dx \\
&\le \Big(\frac{\log T}{\log S}\Big)^n k \int_{B_T} f(Du_T)\,dx + \frac{1}{(\log S)^n}\,c\int_{B_S\setminus B_{T^k}} |x|^{-n}\,dx \\
&= \Big(\frac{\log T}{\log S}\Big)^n k \int_{B_T} f(Du_T)\,dx + \frac{1}{(\log S)^n}\,c\int_{T^k}^{S} \frac{d\rho}{\rho}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(\log S)^{n-1}\int_{B_S} f(Dv_S)\,dx &\le \frac{\log T}{\log S}\,k\,(\log T)^{n-1}\int_{B_T} f(Du_T)\,dx \\
&\quad + \frac{c}{\log S}(\log S - \log T^k).
\end{aligned}$$

Therefore we get

$$g_\nu(S) \le (\log S)^{n-1}\int_{B_S} f(Dv_S)\,dx \le \Big(\frac{\log T^k}{\log S}\Big) g_\nu(T) + c\Big(1-\Big[\frac{\log S}{\log T}\Big]\frac{\log T}{\log S}\Big).$$

In conclusion, the function $r_\nu(S,T) := c\Big(1-\Big[\frac{\log S}{\log T}\Big]\frac{\log T}{\log S}\Big)$ satisfies conditions (48)-(49). In fact $g_\nu(S) \le g_\nu(T) + r_\nu(S,T)$ for $S \ge T$ and $\limsup_{S\to+\infty} r_\nu(S,T) = 0$. ■

Remark 5.2 *Once the existence of the limit in (46) is proved, the function ψ can be extended the whole space $\mathbb{R}^m$ by n-homogeneity; i.e.,*

$$\psi(z) = |z|^n \psi\Big(\frac{z}{|z|}\Big) \textit{ for all } z \in \mathbb{R}^m.$$

Remark 5.3 *Our argument shows that the exponential radius of the perforations derives form the scaling invariance of the problems in (46) and from the logarithmic behavior of the minimizers. This highlights that in the critical case the energy does not concentrate at the same scale as the perforation radius, in a fashion similar to optimal sequences for Ginzburg-Landau functionals (see e.g. [1, 3, 23]).*

Now we can prove the last statement of Theorem 2.1: in the n-homogeneous case $\varphi(z) = \psi(z)$ for all $z \in \mathbb{R}^m$. In particular, the whole sequence (F_j) Γ-converges to F and the shape of K does not affect the result.

We recall that the energy densities φ_j^α in (8) and φ in (9) were studied in Section 3.1. Firstly we consider $K = \overline{B}_r$ for fixed $r > 0$ and then we show that for any compact set K with non-empty interior the result is the same.

Let $K = \overline{B}_r$ for some $r > 0$. For fixed $z \in \mathbb{R}^m$, $\alpha > 0$ and $j \in \mathbb{N}$ we defined

$$\begin{aligned}\varphi_j^\alpha(z) &= (\log T_j)^{n-1} \inf\Big\{ \int_{B_{\alpha S_j}} f(Dv)\,dx:\ v \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m), v = 0 \text{ on } B_r \Big\}\\ &= (\log T_j)^{n-1} \inf\Big\{ \int_{B_{\alpha S_j/r}} f(Dv)\,dx:\ v \in z + W_0^{1,n}(B_{\alpha S_j/r};\mathbb{R}^m), v = 0 \text{ on } B_1 \Big\}.\end{aligned}$$

Since $S_j = a^{(n-1)/n} T_j (\log T_j)^{(1-n)/n}$ we have $\log T_j \backsimeq \log(\alpha S_j/r)$, and hence

$$\begin{aligned}\lim_{j\to+\infty} \varphi_j^\alpha(z) = \lim_{j\to+\infty} \Big(\log\Big(\frac{\alpha S_j}{r}\Big)\Big)^{n-1}&\\ \times \inf\Big\{ \int_{B_{\alpha S_j/r}} f(Dv)\,dx: v \in z + W_0^{1,n}(B_{\alpha S_j/r};\mathbb{R}^m), v = 0 \text{ on } B_1 \Big\}&. \quad (50)\end{aligned}$$

Now, in (50) we compute the limit of a subsequence of the quantity appearing in the definition of ψ (46). Since (50) is the definition of φ^α, we deduce that $\varphi^\alpha(z) = \psi(z)$ for all $\alpha > 0$ and $z \in \mathbb{R}^m$. Hence, $\varphi(z) = \psi(z)$ as desired. Moreover, we note that this result is independent of the radius r.

Let K be a generic compact set with non-empty interior. Up to translations, there exist two balls B_{r_1}, B_{r_2} such that $\overline{B}_{r_1} \subseteq K \subseteq \overline{B}_{r_2}$. By comparison,

$$\begin{aligned}&(\log T_j)^{n-1} \inf\Big\{ \int_{B_{\alpha S_j}} f(Dv)\,dx:\ v \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } B_{r_1} \Big\}\\ &\le (\log T_j)^{n-1} \inf\Big\{ \int_{B_{\alpha S_j}} f(Dv)\,dx:\ v \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } K \Big\}\\ &\le (\log T_j)^{n-1} \inf\Big\{ \int_{B_{\alpha S_j}} f(Dv)\,dx:\ v \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m),\ v = 0 \text{ on } B_{r_2} \Big\}.\end{aligned}$$

Having proved that the limit of the first and the third term (spherical perforations) is $\psi(z)$, independent of the radius of the balls, we deduce that $\varphi(z) = \psi(z)$ for any K.

Finally, we want to focus our attention on a family of more general energy densities f for which the asymptotic analysis of the functionals (F_j) can be easily reduced to the case of n-homogeneous integrands. We will prove the following proposition.

Proposition 5.4 *Let $f, g : \mathbb{M}^{m\times n} \to [0, +\infty)$ be two functions satisfying all the assumptions of Theorem 2.1. Assume that in addition g is positively homogeneous of degree n. Let F_j be defined as in (10). Assume that there exists a positive infinitesimal sequence (γ_j) such that*

$$\Big|\frac{f(T_j A)}{T_j^n} - g(A)\Big| \le \gamma_j |A|^n \quad \textit{for all } A \in \mathbb{M}^{m\times n},\ j \in \mathbb{N}. \quad (51)$$

Then, the whole sequence (F_j) Γ-converges to the functional

$$F(u) = \int_\Omega f(Du)\,dx + \int_\Omega \varphi(u)\,dx, \quad u \in W^{1,n}(\Omega;\mathbb{R}^m),$$

where for all $z \in \mathbb{R}^m$ the function φ is given by

$$\varphi(z) = \lim_{T\to+\infty} (\log T)^{n-1} \min\Big\{ g(Dv)\,dx : v \in z + W_0^{1,n}(B_T;\mathbb{R}^m),\ v = 0 \textit{ on } B_1 \Big\}. \quad (52)$$

In particular, the Γ-convergence result is independent of the shape of K.

Proof. Let $z \in \mathbb{R}^m$, $j \in \mathbb{N}$ and $\alpha > 0$. Let $v \in z + W_0^{1,n}(B_{\alpha S_j};\mathbb{R}^m)$ with $v = 0$ on B_1. By (51) we get

$$g(Dv) - \gamma_j|Dv|^n \leq \frac{f(T_jDv)}{T_j^n} \leq g(Dv) + \gamma_j|Dv|^n.$$

By assumption there exist two positive constants c_1, c_2 such that $c_1|A|^n \leq g(A) \leq c_2|A|^n$ for all $A \in \mathbb{M}^{m\times n}$. Hence,

$$\Big(1 - \frac{\gamma_j}{c_1}\Big)g(Dv) \leq \frac{f(T_jDv)}{T_j^n} \leq \Big(1 + \frac{\gamma_j}{c_1}\Big)g(Dv).$$

Now, we multiply by $(\log T_j)^{n-1}$, take the integral over $B_{\alpha S_j}$ and then minimize over the family of $v \in z + W_0^{1,n}(B_T;\mathbb{R}^m)$, $v = 0$ on B_1. We get:

$$\begin{aligned}
&\Big(1 - \frac{\gamma_j}{c_1}\Big)(\log T_j)^{n-1} \min\Big\{\int_{B_{\alpha S_j}} g(Dv)\,dx : v \in z + W_0^{1,n}(B_T;\mathbb{R}^m), v = 0 \text{ on } B_1\Big\}\\
&\leq (\log T_j)^{n-1} \min\Big\{\int_{B_{\alpha S_j}} \frac{f(T_jDv)}{T_j^n}\,dx : v \in z + W_0^{1,n}(B_T;\mathbb{R}^m), v = 0 \text{ on } B_1\Big\}\\
&\leq \Big(1 + \frac{\gamma_j}{c_1}\Big)(\log T_j)^{n-1} \min\Big\{\int_{B_{\alpha S_j}} g(Dv)\,dx : v \in z + W_0^{1,n}(B_T;\mathbb{R}^m), v = 0 \text{ on } B_1\Big\}.
\end{aligned}$$

In Proposition 5.1 we proved that there exist the limits (independent of α) of the first and the third member as $j \to +\infty$, and that it equals

$$\psi^g(z) := \lim_{T\to+\infty}(\log T)^{n-1} \min\Big\{g(Dv)\,dx : v \in z + W_0^{1,n}(B_T;\mathbb{R}^m), v = 0 \text{ on } B_1\Big\}.$$

By comparison we get that there exists the limit

$$\lim_{j\to+\infty}(\log T_j)^{n-1} \min\Big\{\int_{B_{\alpha S_j}} \frac{f(T_jDv)}{T_j^n}\,dx : v \in z + W_0^{1,n}(B_T;\mathbb{R}^m), v = 0 \text{ on } B_1\Big\} = \psi^g(z).$$

In conclusion, the whole family (F_j) Γ-converges and the function φ defined in (9) equals ψ^g. ■

References

[1] Alberti, G., Baldo, S. and Orlandi, G. Variational convergence for functionals of Ginzburg-Landau type. *Indiana Univ. Math. J.* **54** (2005), 1411-1472.

[2] Ansini, N. and Braides, A. Asymptotic analysis of periodically-perforated nonlinear media. *J. Math. Pures Appl.* **81** (2002), 439-451. *Erratum* in **84** (2005), 147-148.

[3] Bethuel, F., Brezis, H. and Hélein, F. *Ginzburg-Landau Vortices.* Birkhäuser, Boston, 1994.

[4] Braides, A. A handbook of Γ-convergence, in *Handbook of Differential Equations: Stationary Partial Differential Equations*, **3** (Chipot, M. and Quittner, P. eds). Elsevier, Dordrecht, 2006.

[5] Braides, A. *Γ-convergence for Beginners.* Oxford University Press, Oxford, 2002.

[6] Braides, A. and Defranceschi, A. *Homogenization of Multiple Integrals.* Oxford University Press, Oxford, 1999.

[7] Braides, A., Defranceschi, A. and Vitali, E. Homogenization of free discontinuity problems. *Arch. Ration. Mech. Anal.* **135** (1996), 297-356.

[8] Braides, A. and Truskinovsky, L. Asymptotic expansion by Γ-convergence. *Cont. Mech. Therm.* **20** (2008), 21-62.

[9] Bucur, D. and Buttazzo, G. *Variational Methods in Shape Optimization Problems.* Progr. in Nonlinear Diff. Eq. and their Appl. **65**. Birkhäuser, Boston, 2005.

[10] Buttazzo, G., Dal Maso, G. and Mosco, U. Asymptotic behaviour for Dirichlet problems in domains bounded by thin layers. *Partial Differential Equations and the Calculus of Variations, Essay in Honor of Ennio De Giorgi*, 193-249, Birkhäuser, Boston, 1989.

[11] Casado-Diaz, J. Asymptotic behaviour of nonlinear problems in periodically perforated domains. The case $n = p$. *Asymptot. Anal.* **30** (2002), 1-26.

[12] Casado-Diaz, J. and Garroni, A. Asymptotic behaviour of nonlinear elliptic systems on varying domains. *SIAM J. Math. Anal.* **31** (2000), 581-624.

[13] Cioranescu, D. and Murat, F. Un term étrange venu d'ailleurs, I and II. *Nonlinear Partial Differential Equations and Their Applications.* Collège de France Seminar. Vol. II, 98-138, and Vol. III, 154-178, Res. Notes in Math., 60 and 70, Pitman, London, 1982 and 1983, translated in: A strange term coming from nowhere. *Topics in the Mathematical Modelling of Composite Materials* (Cherkaev, A.V. and Kohn, R.V. eds.), Birkhäuser, Boston, 1994.

[14] Dal Maso, G. *An Introduction to Γ-convergence.* Birkhäuser, Boston, 1993.

[15] Dal Maso, G. Asymptotic behaviour of solutions of Dirichlet problems. *Boll. Unione Mat. Ital.* **11A** (1997), 253-277.

[16] Dal Maso, G. and Defranceschi, A. Limits of nonlinear Dirichlet problems in varying domains. *Manuscripta Math.* **61** (1988), 251-278.

[17] Dal Maso, G. and Garroni, A. New results on the asymptotic analysis of Dirichlet problems in perforated domains. *Math. Mod. Meth. Appl. Sci.* **4** (1994), 373-407.

[18] Dal Maso, G., Garroni, A. and Skrypnik, I.V. A capacitary method for the asymptotic analysis of Dirichlet problems for monotone operators.*J. Anal. Math.* **71** (1997), 263-313.

[19] Dal Maso, G. and Mosco, U. Wiener's criterion and Γ-convergence. *Appl. Math. Optim.* **15** (1987), 15-63.

[20] Dal Maso, G. and Murat, F. Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **24**(1997), 239-290.

[21] Defranceschi, A. and Vitali, E. Limits of minimum problems with convex obstacles for vector valued functions. *Appl. Anal.* **52** (1994), 1-33.

[22] Marchenko, A.V. and Khruslov, E.Ya. New results in the theory of boundary value problems for regions with closed-grained boundaries. *Uspekhi Mat. Nauk* **33** (1978), 127.

[23] Sandier, E. and Serfaty, S. *Vortices in the Magnetic Ginzburg-Landau Model.* Birkhäuser, Boston, 2007.

[24] Skrypnik, I.V. Asymptotic behaviour of solutions of nonlinear elliptic problems in perforated domains. *Math. Sb.* **184** (1993), 67-70.