

L^2 -GRADIENT FLOWS OF SPECTRAL FUNCTIONALS

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ABSTRACT. We study the L^2 -gradient flow of functionals \mathcal{F} depending on the eigenvalues of Schrödinger potentials V for a wide class of differential operators associated to closed, symmetric, and coercive bilinear forms, including the case of all the Dirichlet forms (as for second order elliptic operators in Euclidean domains or Riemannian manifolds).

We suppose that \mathcal{F} arises as the sum of a $-\theta$ -convex functional \mathcal{K} with proper domain $\mathbb{K} \subset L^2$ forcing the admissible potentials to stay above a constant V_{\min} and a term $\mathcal{H}(V) = \varphi(\lambda_1(V), \dots, \lambda_J(V))$ which depends on the first J eigenvalues associated to V through a C^1 function φ .

Even if \mathcal{H} is not a smooth perturbation of a convex functional (and it is in fact concave in simple important cases as the sum of the first J eigenvalues) and we do not assume any compactness of the sublevels of \mathcal{K} , we prove the convergence of the Minimizing Movement method to a solution $V \in H^1(0, T; L^2)$ of the differential inclusion $V'(t) \in -\partial_L^- \mathcal{F}(V(t))$, which under suitable compatibility conditions on φ can be written as

$$V'(t) + \sum_{i=1}^J \partial_i \varphi(\lambda_1(V(t)), \dots, \lambda_J(V(t))) u_i^2(t) \in -\partial_{\mathbb{F}}^- \mathcal{K}(V(t))$$

where $(u_1(t), \dots, u_J(t))$ is an orthonormal system of eigenfunctions associated to the eigenvalues $(\lambda_1(V(t)), \dots, \lambda_J(V(t)))$.

Dedicated to J.L. Vazquez in occasion of his 75th birthday

1. INTRODUCTION

Optimization problems for eigenvalues of elliptic operators have been a subject of great interest in the last few years, due both to the possible applications and to the challenging mathematical questions arising from these topics.

In particular, shape optimization problems for the eigenvalues of the Dirichlet Laplacian have been deeply investigated and many results concerning existence of optimal shapes in suitable admissible classes of domains together with regularity results have been proved, see [18, 19] for an overview.

A point of view that has not completely been understood yet for this class of problems is an evolutive approach through a gradient flow of shapes associated to a functional depending on the eigenvalues. One of the main issues is the choice of the natural metric driving the evolution and of a corresponding topology well adapted to shape optimization problems. In the case of stationary variational problems, the best approach in order to prove existence results (see [7]) is to relax the problem in the class of capacity measures, i.e. Borel measures that vanish on sets of zero capacity, where the γ -convergence provides a compact topology sufficiently strong to guarantee the continuity of the eigenvalues of the Dirichlet Laplacian. In the framework of capacity measures, a first gradient flow evolution for this problem was proposed by Bucur,

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Buttazzo and Stefanelli in [6]. They prove existence of (generalized) Minimizing Movements for a large class of functionals, but they do not characterize explicitly the gradient flow equation. A very interesting observation from their work is that, even if the evolution starts from a “nice” shape, then the relaxation in the capacitary measures can actually happen. We also quote the approach of [15] in shape optimization problems.

Eigenvalue problems associated to Schrödinger potentials. In the present paper we propose a different approach, and we focus on the evolution, driven by the L^2 metric, of a special class of capacitary measures, that is, those absolutely continuous with respect to a given reference measure (as the Lebesgue measure of \mathbb{R}^d). Even if in the strong L^2 -framework the driving functionals are not smooth nor convex and their sublevels are not compact, we are still able to prove that the Minimizing Movements solve a natural differential inclusion. Our approach is sufficiently strong to deal with eigenvalues of a wide class of operators, not only those of the Dirichlet Laplacian, avoiding the relaxation phenomenon.

In fact, we will address the problem in the general setting of a (weakly) coercive, symmetric bilinear form $\mathcal{E} : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$ on a Hilbert space \mathbb{V} densely and compactly embedded in $\mathbb{H} = L^2(\mathbb{D}, \mathbf{m})$ for a finite measure space (\mathbb{D}, \mathbf{m}) . Since \mathcal{E} is a nonnegative quadratic form we have

$$\alpha := \min_{u \in \mathbb{V} \setminus \{0\}} \frac{\mathcal{E}(u, u)}{\int_{\mathbb{D}} u^2 \, d\mathbf{m}} \geq 0. \quad (1.1)$$

We consider a convex subset \mathbb{K} of $L^2(\mathbb{D}, \mathbf{m})$ whose elements V satisfy the uniform lower bound $V(x) \geq V_{\min}$ for a fixed constant V_{\min} . Given a (Schrödinger) potential $V \in \mathbb{K}$ (that we can identify with the capacitary measure $\mu = V\mathbf{m}$ absolutely continuous with respect to the reference measure \mathbf{m}), we can introduce the symmetric bilinear form

$$\mathcal{E}_V(u, v) := \mathcal{E}(u, v) + \int_{\mathbb{D}} Vuv \, d\mathbf{m} \quad D(\mathcal{E}_V) := \left\{ u \in \mathbb{V} : \int_{\mathbb{D}} (V_+)u^2 \, d\mathbf{m} < +\infty \right\}, \quad (1.2)$$

and we say that λ is an eigenvalue associated to V if there exists a nonzero eigenfunction $u \in D(\mathcal{E}_V)$ solving

$$\mathcal{E}(u, w) + \int_{\mathbb{D}} Vuw \, d\mathbf{m} = \lambda \int_{\mathbb{D}} uw \, d\mathbf{m} \quad \text{for every } w \in D(\mathcal{E}_V). \quad (1.3)$$

When $D(\mathcal{E}_V) = \mathbb{V}$ (1.3) corresponds to the weak formulation of the equation

$$Lu + Vu = \lambda u \quad \text{in } \mathbb{D}, \quad (1.4)$$

where L is the linear selfadjoint operator associated to \mathcal{E} .

Since \mathbb{V} is compact in $L^2(\mathbb{D}, \mathbf{m})$ the standard spectral theory allows to prove that there exists a sequence $\boldsymbol{\lambda}(V) = (\lambda_1(V), \dots, \lambda_k(V), \dots)$ of eigenvalues satisfying

$$\lambda_{\min} \leq \lambda_1(V) \leq \dots \leq \lambda_k(V) \leq \dots, \quad \lambda_{\min} := \alpha + V_{\min}, \quad \lim_{k \rightarrow \infty} \lambda_k(V) = +\infty, \quad (1.5)$$

and a corresponding sequence $\mathbf{u} = (u_1, u_2, \dots, u_k, \dots) \in (D(\mathcal{E}_V))^{\mathbb{N}}$ of eigenfunctions which provides an L^2 -orthonormal basis for $\overline{D(\mathcal{E}_V)}^{L^2}$.

Optimization problems for eigenvalues of potentials have been recently treated in [8], and they can be used, somehow, to approximate shape optimization problems, as highlighted in [8, Example 5.8].

Our main structural assumption on $\mathbb{V}, \mathcal{E}, \mathbb{D}, \mathbf{m}$ is that for every choice of positive constants $C, \bar{\lambda} \in \mathbb{R}$ the set of eigenfunctions satisfying (1.3) for $\lambda \leq \bar{\lambda}$ and $V \in \mathbb{K}$ with $\|V\|_{L^2(\mathbb{D}, \mathbf{m})} \leq C$ is relatively compact in $L^4(\mathbb{D}, \mathbf{m})$. This property is always satisfied if, e.g., \mathbb{V} is compactly embedded in $L^4(\mathbb{D}, \mathbf{m})$ or \mathcal{E} is a Dirichlet form.

Apart from the (finite dimensional, but still interesting) case when \mathbb{D} is a finite set, simple relevant examples covered by our setting are provided by a bounded Lipschitz open set \mathbb{D} of \mathbb{R}^d

with the usual Lebesgue measure $\mathbf{m} = \mathcal{L}^d$ (or a compact smooth Riemannian manifold endowed with the Riemannian volume measure) and

1. The Dirichlet (resp. Neumann) Laplacian $Lu = -\Delta u$ (the Laplace-Beltrami operator in the Riemannian case) corresponding to $\mathbb{V} = H_0^1(\mathbf{D})$ (resp. $\mathbb{V} = H^1(\mathbf{D})$) and $\mathcal{E}(u, v) = \int_{\mathbf{D}} \nabla u \cdot \nabla v \, dx$.
2. The elliptic operator associated to the Dirichlet form $\mathcal{E}(u, v) = \int_{\mathbf{D}} A(x) \nabla u \cdot \nabla v \, dx$ in $H_0^1(\mathbf{D})$ or $H^1(\mathbf{D})$, where A satisfies the usual uniform ellipticity condition $\alpha|\xi|^2 \leq A(x)\xi \cdot \xi \leq \alpha^{-1}|\xi|^2$ for some $\alpha > 0$ and every $x \in \mathbf{D}$, $\xi \in \mathbb{R}^d$.
3. The fractional Laplacian, for $s \in (0, 1)$, with

$$\mathcal{E}(u, v) = \int_{\mathbf{D}} \int_{\mathbf{D}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+2s}} \, dx \, dy,$$

where the integral should be read in the principal value sense, and $\mathbb{V} = H^s(\mathbf{D})$ or $\mathbb{V} = H_0^s(\mathbf{D})$.

4. The Dirichlet (resp. Neumann) Bilaplacian corresponding to $\mathbb{V} = H_0^2(\mathbf{D})$ (or $H^2(\mathbf{D})$) and $\mathcal{E}(u, v) = \int_{\mathbf{D}} \nabla^2 u : \nabla^2 v \, d\mathbf{m}$ in dimension $d \leq 8$.
5. We can also consider the Dirichlet form induced by a nondegenerate Gaussian measure \mathbf{m} in a separable Hilbert space \mathbf{D} , see e.g. [12, Chap. 10].

L^2 -gradient flows. The aim of this paper is to study the L^2 -gradient flow

$$\frac{d}{dt} V(t) \in -\partial_L^- \mathcal{F}(V(t)) \quad t > 0, \quad V(0) = V_0, \quad (1.6)$$

of potentials driven by the limiting subdifferential (known also as Mordukhovich subdifferential [21, 24, 25]) of a functional $\mathcal{F} : L^2(\mathbf{D}, \mathbf{m}) \rightarrow \mathbb{R} \cup \{+\infty\}$ arising from the sum of two competing terms \mathcal{H} and \mathcal{K} :

- (1) a convex (or a quadratic perturbation of a convex) confining term \mathcal{K} (typically nonsmooth, as the indicator function of a closed convex set of $L^2(\mathbf{D}, \mathbf{m})$) such that $\mathcal{K}(V) = +\infty$ iff $V \notin \mathbb{K}$, which in particular forces the potential V to stay above the constant V_{\min} . \mathcal{K} will keep track of the class of admissible potentials (see for example (3.4) below).
- (2) A term

$$\mathcal{H}(V) := \varphi(\lambda_1(V), \dots, \lambda_J(V)) \quad (1.7)$$

which depends on the first J eigenvalues associated to V through a function $\varphi \in C^1(\Lambda^J)$ where Λ^J is the subset of $[\lambda_{\min}, +\infty)^J$ spanned by all the ordered vectors made of J real numbers.

At least when all the first $J + 1$ eigenvalues are distinct, the gradient flow equation (1.6) reads as

$$V'(t) + \sum_{i=1}^J \partial_i \varphi(\lambda_1(V(t)), \dots, \lambda_J(V(t))) u_i^2(t) \in -\partial^- \mathcal{K}(V(t)) \quad (1.8)$$

where $(u_1(t), u_2(t), \dots, u_J(t))$ is an orthonormal system of eigenfunctions associated to the potential $V(t)$ and to the eigenvalues $\lambda_i(V(t))$. When some of the eigenvalues are multiple the function \mathcal{H} loses its differentiability properties; however, we will still be able to recover (1.8) at least when φ satisfies a suitable compatibility condition at the boundary of Λ^J . Among the interesting examples that are covered we mention the monotonically increasing composition of symmetric functions of the first k -eigenvalues, $k \leq J$, as (here $\lambda_{\min} > 0$)

$$\sum_{j=1}^k \lambda_j, \quad \sum_{i \neq j, i, j \leq k} \lambda_i \lambda_j, \quad 2\lambda_1 + \sqrt{\lambda_1 \lambda_2 \lambda_3}, \quad (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1^2 + \lambda_2^2). \quad (1.9)$$

The special case when $\varphi(\lambda_1, \dots, \lambda_J) = \lambda_1 + \dots + \lambda_J$ (also with $J = 1$) is a quite interesting example of concave functional leading to the differential inclusion

$$V'(t) + \sum_{i=1}^J u_i^2(t) \in -\partial_F^- \mathcal{K}(V(t)). \quad (1.10)$$

From the viewpoint of gradient flows, the main difficulty and challenging feature arising from increasing functions of eigenvalues is that even the simplest map $V \mapsto \lambda_j(V)$ is not even a smooth perturbation of a convex function with respect to the potential V (when $j = 1$ it is in fact a concave function, which is not differentiable when λ_1 is a multiple eigenvalue). Therefore many standard results of gradient flow theory do not apply. We are thus lead to follow and adapt results for gradient flows of highly non-convex functionals proposed in [28]. We have also to circumvent a second important difficulty, related to the lack of compactness of the sublevels of \mathcal{F} . By analyzing the structure of the limiting subdifferential of (suitable regularizations of) \mathcal{H} and employing a sort of compensated-compactness argument, we are eventually able to prove the strong convergence of the Minimizing Movement scheme for \mathcal{F} and to show that all the limits satisfy (1.6) (and (1.8) for compatible φ).

Plan of the paper. Section 2 is devoted to clarify the structural assumptions we will refer in the paper. The discussion of the main examples and of some applications covered by the theory is carried on in Section 3.

Section 4 contains the precise statement of our main results.

The crucial tools concerning the regularity and the differentiability properties of the functionals \mathcal{H} and \mathcal{F} are developed in Sections 5 and 6 respectively.

The last Section 7 collects the main estimates concerning the Minimizing Movement scheme and is then devoted to the proof of its strong convergence.

The Appendix contains some basic material concerning convergence of eigenvalues and eigenfunctions and a useful result of convex analysis.

2. NOTATION AND ASSUMPTIONS

We briefly collect here the abstract setting in which we work for the whole paper and a few preliminary results. Let (D, \mathfrak{m}) be a finite measure space with $\mathbb{H} := L^2(D, \mathfrak{m})$ separable. We will denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the norm and the scalar product of \mathbb{H} . For the sake of simplicity, in the following we will assume that $\dim(\mathbb{H}) = +\infty$; it will be easy to adapt the various statements to the case when \mathbb{H} has finite dimension (e.g. when D is a finite set and we can identify $L^2(D, \mathfrak{m})$ with some \mathbb{R}^d).

2.A Closed, symmetric, and coercive bilinear forms. We will consider a Hilbert space \mathbb{V} satisfying

$$\mathbb{V} \hookrightarrow \mathbb{H} \text{ is densely and compactly imbedded in } \mathbb{H}, \quad (2.1)$$

and a

continuous and symmetric bilinear form $\mathcal{E}: \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$ satisfying

$$\mathcal{E}(u) := \mathcal{E}(u, u) \geq \alpha |u|^2, \quad M^{-1} \|u\|_{\mathbb{V}}^2 \leq \mathcal{E}(u, u) + |u|^2 \leq M \|u\|_{\mathbb{V}}^2 \quad \text{for every } u \in \mathbb{V} \quad (2.2)$$

for constants $\alpha \geq 0$ and $M > 0$.

The bilinear form $\tilde{\mathcal{E}}(u, v) = \mathcal{E}(u, v) + \langle u, v \rangle$ is a scalar product for \mathbb{V} inducing an equivalent norm.

2.B Admissible Shrödinger potentials. We will deal with a

$$\begin{aligned} &\text{lower-semicontinuous } -\theta\text{-convex functional } \mathcal{K}: \mathbb{H} \rightarrow (-\infty, +\infty] \\ &\text{with proper domain } \mathbb{K} = D(\mathcal{K}) := \{V \in \mathbb{H} : \mathcal{K}(V) < +\infty\} \end{aligned} \quad (2.3)$$

such that

$$V \in \mathbb{K} \quad \Rightarrow \quad V \geq V_{\min} \text{ m-a.e. in } D, \quad (2.4)$$

where $\theta \geq 0$, $V_{\min} \in \mathbb{R}$ are suitable constants that we will keep fixed throughout the paper. Notice that the domain \mathbb{K} of the functional \mathcal{K} characterizes the set of admissible potentials.

Recall that \mathcal{K} is $-\theta$ -convex if the function $\mathcal{K}_\theta : V \mapsto \mathcal{K}(V) + \frac{\theta}{2}|V|^2$ is convex; for later use, we will set

$$\mathbb{K}[c] := \left\{ V \in \mathbb{K} : |V| \leq c, \mathcal{K}_\theta(V) \leq c \right\} \quad c \geq 0, \quad (2.5)$$

so that $\mathbb{K}[c]$, $c \geq 0$, is an increasing family of closed and bounded convex subsets of \mathbb{H} whose union is \mathbb{K} . Since \mathbb{K} is not empty, it contains at least an element V_o , so that setting $c_o := \max(|V_o|, \mathcal{K}_\theta(V_o))$ $\mathbb{K}[c]$ is not empty for every $c \geq c_o$.

2.C Eigenvalues. For every $V \in \mathbb{K}$ we consider the symmetric bilinear form

$$\mathcal{E}_V(u, v) := \mathcal{E}(u, v) + \int_D V uv \, dm, \quad D(\mathcal{E}_V) := \left\{ u \in \mathbb{V} : \int_D V_+ u^2 \, dm < +\infty \right\}. \quad (2.6)$$

Denoting by \mathbb{H}_V the closure of $D(\mathcal{E}_V)$ in \mathbb{H} , it is clear that \mathcal{E}_V is a closed and symmetric bilinear form, whose domain $D(\mathcal{E}_V)$ is compactly imbedded in \mathbb{H}_V . It is also clear that

$$\mathbb{V}_4 := \mathbb{V} \cap L^4(D, \mathbf{m}) \subset D(\mathcal{E}_V) \quad \text{for every } V \in \mathbb{K}. \quad (2.7)$$

We say that $\lambda \in \mathbb{R}$ is an eigenvalue for (the linear operator induced by) \mathcal{E}_V if there exists $u \in D(\mathcal{E}_V) \setminus \{0\}$ such that

$$\mathcal{E}(u, v) + \int_D V uv \, dm = \lambda \int_D uv \, dm \quad \text{for every } v \in D(\mathcal{E}_V). \quad (2.8)$$

Any non trivial solution u to (2.8) is called a (V, λ) -eigenfunction (u is also normalized if $|u| = 1$).

The standard spectral theory applied to the bilinear form \mathcal{E}_V allows to prove that there exists a sequence $\boldsymbol{\lambda}(V) = (\lambda_1(V), \dots, \lambda_k(V), \dots)$ of eigenvalues satisfying

$$\lambda_{\min} \leq \lambda_1(V) \leq \dots \leq \lambda_k(V) \leq \dots, \quad \lambda_{\min} := \alpha + V_{\min}, \quad \lim_{k \rightarrow \infty} \lambda_k(V) = +\infty. \quad (2.9)$$

Notice that λ_{\min} is defined in terms of α and V_{\min} and it will remain fixed across the paper. Moreover the sequence of the eigenvalues can be characterized by means of a min-max principle,

$$\begin{aligned} \lambda_j(V) &= \min_{E_j \subset D(\mathcal{E}_V)} \max_{u \in E_j \setminus \{0\}} \frac{\mathcal{E}(u) + \int_D V u^2 \, dm}{\int_D u^2 \, dm} \\ &= \min_{E_j \subset D(\mathcal{E}_V)} \max_{u \in E_j \setminus \{0\}} \left\{ \mathcal{E}(u) + \int_D V u^2 \, dm : \int_D u^2 \, dm = 1 \right\}, \end{aligned} \quad (2.10)$$

where the minimum is taken over the subspaces $E_j \subset D(\mathcal{E}_V)$ of dimension j . For a given $J \in \mathbb{N}$ $\boldsymbol{\lambda}^J = \boldsymbol{\lambda}^J(V) \in \mathbb{R}^J$ will denote the vector of the first J eigenvalues; we will denote by $\mathbf{U}^J(V)$ the collection of all the orthonormal systems of eigenfunctions associated to $\boldsymbol{\lambda}^J(V)$, namely

$$\begin{aligned} \mathbf{U}^J(V) &:= \left\{ \mathbf{u} = (u_1, u_2, \dots, u_J) \in D(\mathcal{E}_V)^J : u_i \text{ is a normalized } (V, \lambda_i)\text{-eigenfunction,} \right. \\ &\quad \left. \int_D u_i u_j \, dm = 0 \text{ for every } 1 \leq i, j \leq J, i \neq j \right\}. \end{aligned} \quad (2.11)$$

We note that the eigenvalues satisfy the following monotonicity property with respect to the potential:

$$\text{if } V_1 \leq V_2 \text{ m-a.e. in } D \quad \text{then} \quad \lambda_k(V_1) \leq \lambda_k(V_2) \quad \text{for all } k \in \mathbb{N}. \quad (2.12)$$

2.D L^4 -summability of eigenfunctions. We will assume that every eigenfunction u solving (2.8) for some $V \in \mathbb{K}$ belongs to $L^4(\mathbf{D}, \mathbf{m})$ and for every constant $c \geq c_o$ and $\bar{\lambda} > \lambda_{\min}$ the set

$$\mathbf{U}[c, \bar{\lambda}] := \left\{ u \text{ is a normalized } (V, \lambda)\text{-eigenfunction with } V \in \mathbb{K}[c], \lambda \leq \bar{\lambda} \right\} \quad (2.13)$$

is relatively compact in $L^4(\mathbf{D}, \mathbf{m})$.

Remark 2.1. Since $\mathbb{K}[c]$ is weakly compact in \mathbb{H} and we will see that from every sequence u_n of (V_n, λ_n) -eigenfunctions, $n \in \mathbb{N}$, with $\lambda_n \rightarrow \lambda$ and $V_n \rightharpoonup V$ in \mathbb{H} it is possible to extract a subsequence $k \mapsto u_{n(k)}$ strongly converging to a (λ, V) -eigenfunction u in \mathbb{V} , (2.13) is in fact equivalent to the compactness of $\mathbf{U}[c, \bar{\lambda}]$ in $L^4(\mathbf{D}, \mathbf{m})$ and can also be formulated as a continuity property:

$$\text{for every sequence } (u_n)_{n \in \mathbb{N}} \text{ of normalized } (V_n, \lambda_n)\text{-eigenfunctions:} \quad (2.14)$$

$$V_n \rightharpoonup V \text{ in } \mathbb{K}[c], \lambda_n \rightarrow \lambda, u_n \rightarrow u \text{ strongly in } \mathbb{V} \quad \Rightarrow \quad u_n \rightarrow u \text{ strongly in } L^4(\mathbf{D}, \mathbf{m}).$$

Assumption (2.13) and (2.7) guarantee that every (V, λ) -eigenfunction u belongs to $\mathbb{V}_4 \subset D(\mathcal{E}_W)$ for every $W \in \mathbb{K}$ and

$$\mathbb{V}_4 = \mathbb{V} \cap L^4(\mathbf{D}, \mathbf{m}) \text{ is dense in } D(\mathcal{E}_V) \text{ for every } V \in \mathbb{K}, \quad (2.15)$$

$$\mathbb{H}_4 := \overline{\mathbb{V}_4}^{\mathbb{H}} = \overline{D(\mathcal{E}_V)}^{\mathbb{H}} \quad \text{for every } V \in \mathbb{K}, \quad (2.16)$$

$$\mathbb{V}_4 \text{ is dense in } \mathbb{V} \text{ and } \mathbb{H}_4 = \mathbb{H} \quad \text{if } \mathbb{K} \cap L^\infty(\mathbf{D}, \mathbf{m}) \neq \emptyset. \quad (2.17)$$

In fact, if $V \in \mathbb{K}$ and $(u_k)_{k \in \mathbb{N}}$ is an orthonormal system of the eigenfunctions of \mathcal{E}_V , the space $E := \text{Span}\{u_k : k \in \mathbb{N}\}$ is contained in \mathbb{V}_4 and it is dense in $D(\mathcal{E}_V)$. In particular $\overline{E}^{\mathbb{H}} = \overline{\mathbb{V}_4}^{\mathbb{H}} = \overline{D(\mathcal{E}_V)}^{\mathbb{H}}$. If $V \in \mathbb{K} \cap L^\infty(\mathbf{D}, \mathbf{m})$, then $D(\mathcal{E}_V) = \mathbb{V}$ which is dense in \mathbb{H} .

Moreover, Assumption (2.13) yields in particular that for every $c \geq c_o$, $\bar{\lambda} > \lambda_{\min}$ there exists a constant C such that

$$u \in \mathbf{U}[c, \bar{\lambda}] \quad \Rightarrow \quad \|u\|_{L^4(\mathbf{D}, \mathbf{m})} \leq C. \quad (2.18)$$

Conversely, if for every $c \geq c_o$, $\bar{\lambda} > \lambda_{\min}$ there exists $p > 4$ and $C > 0$ such that

$$u \in \mathbf{U}[c, \bar{\lambda}] \quad \Rightarrow \quad \|u\|_{L^p(\mathbf{D}, \mathbf{m})} \leq C, \quad (2.19)$$

then Assumption (2.13) is satisfied, since $\mathbf{U}[c, \bar{\lambda}]$ is clearly bounded in \mathbb{V} (thus relatively compact in $L^2(\mathbf{D}, \mathbf{m})$) and bounded in $L^p(\mathbf{D}, \mathbf{m})$ for $p > 4$, and therefore relatively compact in $L^4(\mathbf{D}, \mathbf{m})$. Here are a few simple examples where Assumption (2.13) is satisfied:

- (a) \mathbf{D} is a finite set.
- (b) \mathcal{E} is a Dirichlet form.
- (c) \mathbb{V} is continuously embedded in $L^4(\mathbf{D}, \mathbf{m})$.
- (d) For every $c \geq c_o$ $\mathbb{K}[c]$ is bounded in $L^\infty(\mathbf{D}, \mathbf{m})$ (so that $D(\mathcal{E}_V) = \mathbb{V}$ and $L_V = L + V$ for every $V \in \mathbb{K}$) and the resolvent operator $(I + L)^{-1}$ is bounded from $L^2(\mathbf{D}, \mathbf{m})$ to $L^p(\mathbf{D}, \mathbf{m})$ for some $p > 4$.

We will discuss these cases in the next Section 3

2.E Functionals depending on the first J eigenvalues. We denote by Λ^J the subset of \mathbb{R}^J spanned by all the ordered vectors made of J real numbers, namely,

$$\Lambda^J := \left\{ (\lambda_1, \dots, \lambda_J) \in \mathbb{R}^J : \lambda_{\min} \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_J \right\}. \quad (2.20)$$

This can be seen as the set of all the possible first J eigenvalues. Notice that Λ^J is a closed convex subset of \mathbb{R}^J whose interior in $[\lambda_{\min}, +\infty)^J$ is

$$\text{Int}(\Lambda^J) = \left\{ (\lambda_1, \dots, \lambda_J) \in \mathbb{R}^J : \lambda_{\min} \leq \lambda_1 < \lambda_2 < \dots < \lambda_J \right\}. \quad (2.21)$$

We consider a function $\varphi : \Lambda^J \rightarrow \mathbb{R}$ of class C^1 satisfying

$$\varphi(\boldsymbol{\lambda}) \geq -A(1 + \lambda_J \vee 0) \quad \text{for every } \boldsymbol{\lambda} \in \Lambda^J, \quad (2.22)$$

for some constant $A \geq 0$. We can then define the functionals

$$\mathcal{H} : \mathbb{K} \rightarrow \mathbb{R}, \quad \mathcal{H}(V) := \varphi(\boldsymbol{\lambda}^J(V)) \quad \text{for every } V \in \mathbb{K} \quad (2.23)$$

and

$$\mathcal{F} : \mathbb{H} \rightarrow (-\infty, +\infty], \quad \mathcal{F}(V) := \mathcal{H}(V) + \mathcal{K}(V) \text{ if } V \in \mathbb{K}, \quad \mathcal{F}(V) := +\infty \text{ if } V \notin \mathbb{K}. \quad (2.24)$$

2.F Additional structural compatibility. Our main existence result will only rely on the assumptions 2.A–2.E we have discussed above. To gain a more refined characterization of the gradient flows, we will sometimes assume a further structural property on φ in addition to C^1 regularity. More precisely, we will suppose that for every $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_J) \in \Lambda^J$

$$\begin{aligned} \partial_J \varphi(\boldsymbol{\lambda}) &\geq 0, \\ k \in \{2, \dots, J\}, \lambda_{k-1} = \lambda_k &\Rightarrow \partial_{k-1} \varphi(\boldsymbol{\lambda}) \geq \partial_k \varphi(\boldsymbol{\lambda}). \end{aligned} \quad (2.25)$$

Notice that if φ is considered as a function depending on the first K eigenvalues with $K > J$ (and therefore is trivially extended to Λ^K), then (2.25) also holds on Λ^K . Conversely, if φ just depends on the first I eigenvalues, $I < J$, then $\partial_k \varphi(\boldsymbol{\lambda}) = 0$ for $k > I$ and (2.25) yields $\partial_I \varphi(\boldsymbol{\lambda}) \geq 0$.

If (2.25) holds, then for every $k \in \{1, \dots, J\}$ and every $(\lambda_1, \dots, \lambda_{k-1}) \in \Lambda^{k-1}$ the map

$$z \mapsto \varphi(\lambda_1, \dots, \lambda_{k-1}, z, \dots, z) \quad \text{is increasing in } [\lambda_{k-1}, +\infty), \quad (2.26)$$

so that

$$\begin{aligned} \varphi(\lambda_{\min}, \lambda_{\min}, \dots, \lambda_{\min}) &\leq \varphi(\lambda_1, \lambda_1, \dots, \lambda_1) \leq \varphi(\lambda_1, \lambda_2, \dots, \lambda_2) \leq \varphi(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_3) \leq \dots \\ &\leq \varphi(\lambda_1, \lambda_2, \dots, \lambda_J); \end{aligned}$$

therefore φ is bounded from below and (2.22) holds as well with $A := (\varphi(\lambda_{\min}, \lambda_{\min}, \dots, \lambda_{\min}))_-$.

Clearly (2.25) characterizes a convex cone in $C^1(\Lambda^J)$. It is not difficult to check that if $\phi \in C^1([\lambda_{\min}, +\infty)^K)$, $K \leq J$, is a symmetric function satisfying

$$\begin{aligned} \phi(\sigma_*(\boldsymbol{\lambda})) &= \phi(\boldsymbol{\lambda}) \quad \text{for every } \boldsymbol{\lambda} \in [\lambda_{\min}, +\infty)^K, \quad \sigma \in \text{Sym}(K), \quad (\sigma_*(\boldsymbol{\lambda}))_k = \lambda_{\sigma(k)}, \\ \partial_K \phi(\boldsymbol{\lambda}) &\geq 0, \end{aligned} \quad (2.27)$$

then $\varphi(\lambda_1, \dots, \lambda_J) := \phi(\lambda_1, \dots, \lambda_K)$ satisfies (2.25).

So, if e.g. $\lambda_{\min} > 0$, monotone compositions of symmetric functions as

$$2\lambda_1 + \sqrt{\lambda_1 \lambda_2} + \ln(\lambda_1 \lambda_2 \lambda_3), \quad (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1^2 + \lambda_2^2)$$

satisfy (2.25). Other examples are provided by the functions

$$\boldsymbol{\lambda} \mapsto h(\lambda_j - \lambda_{j-1}), \quad 1 \leq j \leq J, \quad h \in C^1([0, +\infty)), \quad 0 = h'(0) \leq h'(r) \text{ for every } r \geq 0.$$

Remark 2.2. From now on we will always operate in the setting described in 2.A–2.E; in particular, the constants $\alpha, M, \theta, V_{\min}, \lambda_{\min}, c_o, J$ will be considered as fixed throughout the paper.

3. EXAMPLES AND APPLICATIONS

Let us briefly show a few examples where the assumptions of Section 2.A–2.D apply, considering in particular the cases (a, ..., d) of 2.D.

3.1. The finite dimensional case. Let \mathbb{D} be a finite set which we can identify with the set of indices $\{1, \dots, d\}$ so that $\mathbb{H} = L^2(\mathbb{D}, \mathbf{m})$ can be identified to \mathbb{R}^d for some $d \geq 1$. In this case the bilinear form \mathcal{E} can be identified with a $d \times d$ matrix $L = (a_{ij})$, symmetric and nonnegative definite, so that

$$\mathcal{E}(u, v) = \sum_{i,j=1}^d a_{ij} u_i v_j = \langle Lu, v \rangle \quad \text{for } u, v \in \mathbb{R}^d.$$

We can take as \mathcal{K} any convex and lower semicontinuous function in \mathbb{R}^d whose proper domain is a closed convex set $\mathbb{K} \subset [v_{\min}, +\infty)^d$, for some constant $v_{\min} \in \mathbb{R}$.

For every $V \in \mathbb{K}$ we have the symmetric bilinear form

$$\mathcal{E}_V(u, v) := \mathcal{E}(u, v) + \sum_{i=1}^d V_i u_i v_i = \langle Lu + Vu, v \rangle, \quad (Vu)_i := V_i u_i,$$

and $\lambda \in \mathbb{R}$ is a (λ, V) -eigenvalue if there exists $u \in \mathbb{R}^d \setminus \{0\}$ such that

$$Lu + Vu = \lambda u.$$

Since we are in a finite dimensional setting, Assumption (2.13) is trivially satisfied.

3.2. The case of a Dirichlet form. Let us now consider the case when \mathcal{E} is a Dirichlet form, thus satisfying the Markov condition (see [17])

$$\text{for all } u \in \mathbb{V}, \text{ then } v := \min\{\max\{u, 0\}, 1\} \in \mathbb{V} \text{ and } \mathcal{E}(v) \leq \mathcal{E}(u). \quad (3.1)$$

Since, for every $V \in \mathbb{K}$, $\mathbb{V} \cap L^\infty(\mathbb{D}, \mathbf{m}) \subset D(\mathcal{E}_V)$ and $\mathbb{V} \cap L^\infty(\mathbb{D}, \mathbf{m})$ is also dense in \mathbb{V} and thus in \mathbb{H} , we deduce that $D(\mathcal{E}_V)$ is dense in \mathbb{H} . If $\beta := 1 + (\lambda_{\min})_-$ the quadratic form $\mathcal{E}_V + \beta |\cdot|^2$ is a Dirichlet form associated to the selfadjoint operator by $L_V + \beta$ whose inverse $R_V^\beta := (L_V + \beta)^{-1} : \mathbb{H} \rightarrow \mathbb{H}$ is a sub-Markovian compact selfadjoint operator (and in particular a contraction) in \mathbb{H} ; it is well known that u is a (λ, V) eigenfunction if and only if u is a $(\lambda + \beta)^{-1}$ -eigenfunction of the operator R_V^β (see also Appendix A). The restriction of R_V^β to $L^p(\mathbb{D}, \mathbf{m})$ is also a contraction. By [14, Theorems 1.6.1-2-3] the spectrum and the eigenfunctions of R_V^β are independent of $p \in [2, +\infty)$ and in particular all the eigenfunctions associated to V belong to $L^p(\mathbb{D}, \mathbf{m})$ for every $p \in [2, +\infty)$.

Let now u_n be a sequence of normalized (V_n, λ_n) -eigenfunctions with $\lambda_n \leq \bar{\lambda}$ and $V_n \in \mathbb{K}[c]$. Up to extracting a suitable subsequence, it is not restrictive to assume that $\lambda_n \rightarrow \lambda$, $V_n \rightarrow V$ in \mathbb{H} , $u_n \rightarrow u$ strongly in \mathbb{V} for some $\lambda \leq \bar{\lambda}$, $V \in \mathbb{K}[c]$, $u \in D(\mathcal{E}_V)$.

By Lemma A.3 in the Appendix, $R_{V_n}^\beta$ converge uniformly to R_V^β in $\mathcal{L}(\mathbb{H})$: this implies that u is a normalized (V, λ) -eigenfunction. We want to show that $\|u_n - u\|_{L^4} \rightarrow 0$; we fix $p > 4$ and we show that the L^p -norm of u_n is bounded. We argue by contradiction, assuming that $\|u_n\|_{L^p} \rightarrow +\infty$ along a (not relabeled) subsequence.

Since R_V and R_{V_n} are contractions in $L^q(\mathbb{D}, \mathbf{m})$ for any $q > p > 4$, by Riesz-Thorin interpolation we also have that $\|R_V^\beta - R_{V_n}^\beta\|_{\mathcal{L}(L^p)} \rightarrow 0$ as $n \rightarrow +\infty$. Setting $\tilde{u}_n := \|u_n\|_{L^p}^{-1} u_n$ by [10, Thm. 7.4, p. 690] we find a (V, λ) -eigenfunction v_n such that $\|\tilde{u}_n - v_n\|_{L^p} \rightarrow 0$. We deduce that $\|v_n\|_{L^p}$ is bounded; since v_n belongs to a finite dimensional space, it admits a subsequence $v_{n(k)}$ strongly convergent to some limit v in $L^p(\mathbb{D}, \mathbf{m})$. Therefore $\tilde{u}_n \rightarrow v$ strongly in $L^p(\mathbb{D}, \mathbf{m})$ with $\|v\|_{L^p} = 1$; however $\tilde{u}_n \rightarrow 0$ in $L^2(\mathbb{D}, \mathbf{m})$, a contradiction.

Remark 3.1. *All the examples 1,2,3,5 considered in the Introduction fit in the framework of Dirichlet forms, with domain \mathbb{V} which is compact in $L^2(\mathbb{D}, \mathbf{m})$.*

3.3. The case when $\mathbb{V} \subset L^4(\mathbf{D}, \mathbf{m})$ or the resolvent has a regularizing effect. This case follows immediately from the equivalent characterization (2.14) of (2.13). Notice that the example 4 in the Introduction corresponds to this situation, thanks to the Sobolev imbedding of $H^2(\mathbf{D})$ in $L^4(\mathbf{D})$ when the dimension $d \leq 8$.

The last case (d) considered in Section 2.D can be easily discussed by observing that if $V \in L^\infty(\mathbf{D}, \mathbf{m})$ then a normalized (V, λ) -eigenfunction u satisfies the equation

$$Lu + u = f \quad \text{with } f := \lambda u - Vu;$$

since $\|f\|_{L^2} \leq |\lambda| + \|V\|_{L^\infty}$, we immediately recover a uniform estimate of u in $L^p(\mathbf{D}, \mathbf{m})$ as in (2.19).

We conclude this section by briefly discussing two possible applications of gradient flows of spectral functionals.

3.4. Optimal design problems arising from population dynamics and reaction diffusion equations. Let $\mathbf{D} \subset \mathbb{R}^d$ be an open and bounded set, let $V^- < v_0 < V^+ \in \mathbb{R}$ be real constants with $V^- < 0 < V^+$, and let

$$\mathbb{K} := \left\{ V \in L^\infty(\mathbf{D}; \mathbf{m}) : V^- \leq V \leq V^+, \int_{\mathbf{D}} V(x) dx \geq v_0 \right\}.$$

For $V \in \mathbb{K}$, the classical reaction-diffusion model in an heterogeneous environment proposed by Fisher and Kolmogorov can be generalized as:

$$\begin{cases} u_t = \Delta u - uV(x) - u^2, & \text{in } \mathbf{D} \times \mathbb{R}^+, \\ u = 0, \quad (\text{or } \partial_\nu u = 0), & \text{on } \partial\mathbf{D} \times \mathbb{R}^+, \\ u(x, 0) \geq 0, \quad u(x, 0) \not\equiv 0, & \text{in } \bar{\mathbf{D}}, \end{cases} \quad (3.2)$$

where $u(x, t)$ represents the population density at time t and position x , and $(-V(x))$ is the intrinsic grow rate of the species at the spatial point x . The condition proved in [3] for the survival of the species for large times (as $t \rightarrow \infty$) is that the first eigenvalue $\lambda_1(V)$ for the associated linearized problem (we stress that here we have the opposite sign in front of the potential, with respect to [3]) which is defined as

$$\begin{cases} -\Delta u + Vu = \lambda_1(V)u, & \text{in } \mathbf{D}, \\ u = 0, \quad (\text{or } \partial_\nu u = 0), & \text{on } \partial\mathbf{D}, \end{cases}$$

should be negative, so it is natural to try to minimize it under the constraint $V \in \mathbb{K}$. This problem has been widely studied (see for example [9, 22] and the references therein): it is known that an optimal potential V^* is of bang-bang type, i.e. $V^* = V^+$ on $\mathbf{D}^+ \subset \mathbf{D}$ and $V^* = V^-$ on $\mathbf{D}^- = \mathbf{D} \setminus \mathbf{D}^+$. On the other hand, there are still many open problems concerning the shape of the partition \mathbf{D}^\pm . The L^2 -gradient flow of the functional

$$\mathcal{F}(V) := \begin{cases} \lambda_1(V) & \text{if } V \in \mathbb{K}, \\ +\infty & \text{otherwise,} \end{cases}$$

can provide some useful new insights.

3.5. Optimization of eigenvalues of potentials. In the paper [8] some optimization problems for eigenvalues of potentials in the case of the Dirichlet Laplacian, i.e. when $\mathbf{D} \subset \mathbb{R}^d$ is an open and bounded set, $\mathbb{V} = H_0^1(\mathbf{D})$ and $\mathcal{E}(u, v) = \int_{\mathbf{D}} \nabla u \cdot \nabla v dx$, were considered. The authors studied the minimization problem

$$\min \left\{ \varphi(\lambda_1(V), \dots, \lambda_J(V)) : V \in \mathbb{K} \right\}, \quad (3.3)$$

for all $\varphi: \mathbb{R}^J \rightarrow \mathbb{R}$ regular and increasing in each variables, with the class of admissible potentials \mathbb{K} defined as follows:

$$\mathbb{K} = \left\{ V \in L^2(D, \mathbf{m}) : V \geq 0, \int_D \Psi(V) \leq c \right\}, \quad (3.4)$$

where $\Psi: [0, +\infty] \rightarrow [0, +\infty]$ denotes a strictly decreasing convex function and $c \in \mathbb{R}$ is such that

$$\lim_{r \rightarrow +\infty} \Psi(r) < c < \Psi(0).$$

It is clear that \mathbb{K} is convex and closed in $L^2(D, \mathbf{m})$. Some remarks about the choice of the class of potentials are in order. First of all, we note that examples of function satisfying the hypotheses above are $\Psi(s) = s^{-\beta}$ or $\Psi(s) = e^{-\beta s}$, for some $\beta > 0$. It is immediate to check that \mathbb{K} is not empty and that $0 \notin \mathbb{K}$, so that the trivial potential $V = 0$ is not allowed.

By choosing \mathcal{H} as the indicator function of \mathbb{K} (which is clearly convex and lower semicontinuous) and $\mathcal{H}(V) = \varphi(\lambda_1(V), \dots, \lambda_J(V))$, we provide a gradient flow evolution for the minimization problems studied in [8, Section 4]. We note that in our L^2 setting, the existence of minimizers for the problem

$$\min \left\{ \varphi(\lambda_1(V), \dots, \lambda_J(V)) : V \in \mathbb{K} \right\},$$

follows easily since the functional is weakly lower semicontinuous in $L^2(D, \mathbf{m})$.

When $\Psi(s) = e^{-\beta s}$ the interest for problem (3.3) lies also in the fact that it can be used as an approximation of a shape optimization problem (see [8, Example 5.8]), namely

$$\min \left\{ \varphi(\lambda_1(\Omega), \dots, \lambda_J(\Omega)) : \Omega \subset D, \mathbf{m}(\Omega) = c \leq \mathbf{m}(D) \right\}.$$

4. MAIN RESULTS

In order to make precise the notion of gradient flows we are going to study, let us first recall the main definitions of subdifferentials which are involved. We refer to [27, Chap. 8] for more details.

Definition 4.1 (Fréchet and limiting subdifferentials). *Let $\mathcal{G}: \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ and let $v \in D(\mathcal{G}) \subset \mathbb{H}$, $\xi \in \mathbb{H}$. We say that ξ belongs to the Fréchet subdifferential $\partial_F^- \mathcal{G}(v)$ if*

$$\liminf_{w \rightarrow v} \frac{\mathcal{G}(w) - \mathcal{G}(v) - \langle \xi, w - v \rangle}{|w - v|} \geq 0;$$

equivalently, by using the viscosity characterization [4, Remark 1.4], there exist $\varrho > 0$ and a function

$$\omega: \mathbb{H} \rightarrow [0, +\infty) \quad \text{of class } C^1, \text{ convex, and satisfying } \omega(0) = 0, \quad (4.1)$$

such that

$$\mathcal{G}(w) - \mathcal{G}(v) - \langle \xi, w - v \rangle \geq -\omega(w - v) \quad \text{for every } w \in B(v, \varrho). \quad (4.2)$$

ξ belongs to the limiting subdifferential (known also as Mordukhovich subdifferential [21, 24, 25]) $\partial_L^- \mathcal{G}(v)$ if there exist sequences $v_n, \xi_n \in \mathbb{H}$ such that

$$\xi_n \in \partial_F^- \mathcal{G}(v_n), \quad v_n \rightarrow v \text{ strongly in } \mathbb{H}, \quad \xi_n \rightharpoonup \xi \text{ weakly in } \mathbb{H}, \quad \mathcal{G}(v_n) \rightarrow \mathcal{G}(v). \quad (4.3)$$

We denote by $\partial_L^o \mathcal{G}(v)$ the element of minimal norm in $\partial_L^- \mathcal{G}(v)$.

Remark 4.2 (On the definition of Fréchet and limiting subdifferential). *If we restrict the functions ω to the class $\omega(\delta) := q|\delta|^2$ then (4.2) corresponds to the definition of proximal subdifferential. Notice that here we adopted a definition of limiting subdifferential which is stronger than the one considered in [28] (and denoted by $\partial_\ell^- \mathcal{G}$), since in (4.3) we require the convergence of the functionals $\mathcal{G}(v_n) \rightarrow \mathcal{G}(v)$ instead of their boundedness. This choice is justified by the better*

regularity properties of the functionals which we are considering, but in the case of \mathcal{F} the two definitions will lead to the same object.

It is well known that when \mathcal{G} is $-\eta$ -convex and lower semicontinuous, then $\partial_F^- \mathcal{G}$ and $\partial_L^- \mathcal{G}$ coincide [11] and can also be characterized by

$$\xi \in \partial_F^- \mathcal{G}(v) \iff \mathcal{G}(w) \geq \mathcal{G}(v) + \langle \xi, w - v \rangle - \frac{\eta}{2} |w - v|^2 \quad \text{for every } w \in \mathbb{H}. \quad (4.4)$$

In particular, when $\eta = 0$ and \mathcal{G} is convex we recover the usual subdifferential of convex analysis which we will simply denote by $\partial^- \mathcal{G}$.

For a given time interval $[0, T]$, $T > 0$, the gradient flow of a convex functional then reads as the solution $v : [0, T] \rightarrow \mathbb{H}$ of the differential inclusion

$$v'(t) \in -\partial_F^- \mathcal{G}(v(t)) \quad \text{for a.e. } t \in (0, T), \quad (4.5)$$

and for every given initial condition $v_0 \in D(\mathcal{G})$ there exists a unique solution $v \in H^1(0, T; \mathbb{H})$ satisfying (4.5) and $v(0) = v_0$ [5].

In our case the interesting functionals \mathcal{F} are typically neither convex nor $-\eta$ -convex for any choice of $\eta > 0$, since even simple examples as $\mathcal{H}(V) = \sum_{j=1}^J \lambda_j(V)$ are nonsmooth concave functionals. In this case the graph of the proximal (and also of the Fréchet) subdifferential is not closed and it is then natural to study the corresponding equation of (4.5) in terms of the limiting subdifferential (see e.g. the discussion in [28]). A further difficulty arises by the fact that we did not assume any compactness on the sublevel of \mathcal{F} .

In order to circumvent these difficulties, we adopt the variational approach of the Minimizing Movement method [28, 1], trying to obtain the gradient flow as a limit of a discrete approximation.

We introduce a uniform partition of the interval $[0, T]$:

$$0 = t_0 < t_1 < \dots < t_{N-1} < T \leq t_N, \quad t_n := n\tau, \quad n \in \{0, \dots, N\}, \quad N = N(\tau) := \lceil T/\tau \rceil$$

corresponding to a step size $\tau > 0$, the perturbed functionals

$$\Phi(\tau, V; W) := \mathcal{F}(W) + \frac{1}{2\tau} |W - V|^2, \quad V, W \in \mathbb{H}, \quad (4.6)$$

and we consider the discrete solutions $\{V_\tau^n\}_{n \in \mathbb{N}}$ in \mathbb{H} of the variational iterative scheme starting from a given initial datum $V_0 \in D(\mathcal{F})$:

$$V_\tau^n \in \arg \min_{V \in \mathbb{H}} \Phi(\tau, V_\tau^{n-1}; V), \quad n = 1, \dots, N(\tau), \quad V_\tau^0 := V_0 \in D(\mathcal{F}). \quad (4.7)$$

We will show (see Lemma 6.1) that a discrete solution always exists for every initial datum $V_0 \in D(\mathcal{F})$.

We then call \bar{V}_τ the piecewise constant interpolant and by V_τ the piecewise linear interpolant of the discrete values $\{V_\tau^n\}$:

$$\bar{V}_\tau(t) = V_\tau^n, \quad V_\tau(t) := \frac{t_n - t}{\tau} V_\tau^{n-1} + \frac{t - t_{n-1}}{\tau} V_\tau^n \quad \text{if } t \in (t_{n-1}, t_n]. \quad (4.8)$$

Definition 4.3 (Generalized Minimizing Movements). *We say that a curve $V : [0, T] \rightarrow \mathbb{H}$ is a (strong) Generalized Minimizing Movement for Φ starting from $V_0 \in \mathbb{H}$ in $[0, T]$ if there exists a decreasing vanishing sequence of step sizes $(\tau(k))_{k \in \mathbb{N}}$, $\tau(k) \downarrow 0$ as $k \rightarrow \infty$, and a corresponding sequence of discrete solutions $\bar{V}_{\tau(k)}$ such that*

$$V_{\tau(k)}(t) \rightarrow V(t) \quad \text{strongly in } \mathbb{H} \text{ for every } t \in [0, T]. \quad (4.9)$$

We denote by $\text{GMM}(\Phi, V_0, T)$ the collection of all the (strong) generalized minimizing movements for Φ starting from V_0 in the interval $[0, T]$.

Our first result reads as follows.

Theorem 4.4. *In the setting of Section 2 under the assumptions stated in 2.A–2.E, for every choice of $V_0 \in \mathbb{K}$ the set $\text{GMM}(\Phi, V_0, T)$ is not empty. Every $V \in \text{GMM}(\Phi, V_0, T)$ belongs to $H^1(0, T; \mathbb{H})$, it satisfies for almost every $t \in (0, T)$*

$$V'(t) \text{ is the projection of the origin on the affine hull } \text{aff}\left(-\partial_L^- \mathcal{F}(V(t))\right), \quad (4.10)$$

it solves the Cauchy problem,

$$V'(t) = -\partial_L^\circ \mathcal{F}(V(t)) \quad \text{for a.e. } t \in (0, T), \quad V(0) = V_0, \quad (4.11)$$

and satisfies the Energy Dissipation Identity

$$\mathcal{F}(V(t)) = \mathcal{F}(V_0) + \int_0^t |V'(s)|^2 ds \quad \text{for every } t \in [0, T] \quad (4.12)$$

Finally, if $k \mapsto \tau(k)$ is a vanishing sequence as in (4.9) we also have

$$V'_{\tau(k)} \rightarrow V' \quad \text{strongly in } L^2(0, T; \mathbb{H}), \quad \mathcal{F}(V_{\tau(k)}(t)) \rightarrow \mathcal{F}(V(t)) \quad \text{for every } t \in [0, T]. \quad (4.13)$$

Remark 4.5 (Affine projection and minimal selection). *We recall that the affine hull of a set $A \subset \mathbb{H}$ is defined as*

$$\text{aff}(A) = \left\{ \sum_i t_i a_i : t_i \in \mathbb{R}, \sum_i t_i = 1, a_i \in A \right\}.$$

Notice that we have retrieved the minimal section principle (4.11) (even in the stronger formulation (4.10)) in this non convex case: though in general $\partial_L^- \mathcal{F}(V(t))$ is not convex, $V'(t)$ is its unique element of minimal norm for a.e. $t \in (0, T)$.

Under the sole C^1 assumption on φ of Section 2.E, the precise characterization of $\partial_L^\circ \mathcal{F}(V(t))$ is not immediate. A first information is provided by the following proposition:

Proposition 4.6. *Let V be a solution to (4.12) and let us denote by $O \subset [0, T]$ the open set*

$$O := \left\{ t \in [0, T] : \lambda^{J+1}(V(t)) \in \text{Int}(\Lambda^{J+1}) \right\}. \quad (4.14)$$

For every $t \in O$ there exists a unique element $\mathbf{u}(t) \in \mathbf{U}^J(V(t))$ and

$$V'(t) + \sum_{j=1}^J \partial_i \varphi(\lambda^j(V(t))) u_i^2(t) \in -\partial_F^- \mathcal{H}(V(t)). \quad (4.15)$$

The refined structural condition (2.25) of Section 2.F guarantees that the decomposition (4.15) holds a.e. in $(0, T)$.

Theorem 4.7. *Under the same assumptions of Theorem 4.4, let us also assume that (2.25) of Section 2.F holds, let V be a solution of (4.11), and let D be the set (of full Lebesgue measure) where V is differentiable and the inclusion (4.11) holds. Then for every $t \in D$ there exists $\mathbf{u}(t) \in \mathbf{U}^J(V(t))$ satisfying (4.15).*

The proof of our main results will follow from the analysis carried out in the next Sections. First of all, in Section 5, we study the regularity and the differentiability properties of the functional \mathcal{H} . In Section 6 we use these results in order to prove the existence of discrete solutions to the Minimizing Movement scheme and to obtain crucial structural properties of the limiting subdifferential of \mathcal{F} . A crucial step will also be the Chain Rule formula in Proposition 6.4.

Section 7 contains the basic estimates on the Minimizing Movement solutions. It is not difficult to show that a *weak* Generalized Minimizing Movement exists (i.e. a curve V which is the pointwise *weak* limit of a subsequence $V_{\tau(k)}$). The main improvement is to show that such a curve is also a *strong* Generalized Minimizing Movement according to (4.9). This fact is not obvious, since we did not assume that \mathcal{F} has compact sublevels: it will be obtained by using the

compactness properties of the subdifferential of \mathcal{H} and a compensated compactness argument, see Proposition 7.3.

At that point we will have all the ingredients to apply the results of [28]: the final discussion will be carried out in Section 7.3.

5. REGULARITY AND DIFFERENTIABILITY PROPERTIES OF EIGENVALUES AND EIGENFUNCTIONS

In this Section we will always keep the structural assumptions 2.A–2.E of Section 2; we will explicitly mention the more refined property (2.25) of Section 2.F, in all the results where it will be involved.

We will study the regularity properties of \mathcal{H} with respect to V . We will still denote by $\mathcal{E}_V : \mathbb{H} \rightarrow (-\infty, +\infty]$ the (extended) quadratic form in \mathbb{H} induced by \mathcal{E}_V :

$$\mathcal{E}_V(u) := \begin{cases} \mathcal{E}_V(u, u) & \text{if } u \in D(\mathcal{E}_V), \\ +\infty & \text{if } u \in \mathbb{H} \setminus D(\mathcal{E}_V). \end{cases} \quad (5.1)$$

It is not difficult to check that $u \mapsto \mathcal{E}_V(u) + (\lambda_{\min})_- |u|^2$ is a convex and lower semicontinuous functional on \mathbb{H} .

5.1. Weak continuity and Lipschitzianity.

Lemma 5.1 (Weak continuity of eigenvalues and eigenfunctions). *Let $V_n \in \mathbb{K}$, $n \in \mathbb{N}$, be a sequence weakly converging in \mathbb{H} to $V \in \mathbb{K}$ as $n \rightarrow \infty$. For all $k, J \in \mathbb{N}$ we have*

$$\lambda_k(V_n) \rightarrow \lambda_k(V) \quad \text{as } n \rightarrow +\infty$$

and every sequence $\mathbf{u}_n \in \mathbf{U}^J(V_n)$ admits a subsequence $m \mapsto \mathbf{u}_{n(m)}$ and a limit $\mathbf{u} \in \mathbf{U}^J(V)$ such that

$$\mathbf{u}_{n(m),k} \rightarrow \mathbf{u}_k \quad \text{strongly in } \mathbb{V} \cap L^4(\mathbf{D}, \mathbf{m}) \quad \text{for every } k \in \{1, \dots, J\}. \quad (5.2)$$

The proof of Lemma 5.1 is well-known (see e.g. [8, Proposition 2.5] for the first part of the claim and [2, Proposition 3.69 and Theorem 3.71] for the second one in the case of elliptic operators such as the Dirichlet Laplacian). The $L^4(\mathbf{D}, \mathbf{m})$ -convergence of eigenfunctions in (5.2) is a consequence of (2.13) in Section 2.D. We provide a detailed proof of Lemma 5.1 in Appendix A.

Corollary 5.2. *For every $c \geq c_o$ the sets*

$$\begin{aligned} \mathbf{U}^J[c] &:= \bigcup \left\{ \mathbf{U}^J(V) : V \in \mathbb{K}[c] \right\} \\ \mathbf{U}_j[c] &:= \left\{ u \text{ is a normalized } (V, \lambda_j(V))\text{-eigenfunction with } V \in \mathbb{K}[c] \right\}, \end{aligned} \quad (5.3)$$

are nonempty and compact in $(\mathbb{V}_4)^J \subset (L^4(\mathbf{D}, \mathbf{m}))^J$ and in \mathbb{V}_4 respectively. In particular, for every $c \geq c_o$

$$A(c) := \sup \left\{ \|u\|_{L^4(\mathbf{D}, \mathbf{m})} : u \in \mathbf{U}_j[c], 1 \leq j \leq J \right\} < +\infty. \quad (5.4)$$

Proof. It is clearly sufficient to prove the statement for $\mathbf{U}^J[c]$. Thanks to Lemma 5.1, the map $\lambda_J : \mathbb{K} \rightarrow \mathbb{R}$ is continuous with respect to the weak topology of \mathbb{H} . Since $\mathbb{K}[c]$ is weakly compact (being a closed bounded convex set of \mathbb{H}),

$$\ell_J(c) := \sup_{V \in \mathbb{K}[c]} \lambda_J(V) < +\infty, \quad \mathbf{U}^J[c] \subset (\mathbf{U}(c, \ell_J(c)))^J. \quad (5.5)$$

If $\mathbf{u}_n \in \mathbf{U}^J(V_n)$, $n \in \mathbb{N}$, is a sequence with $V_n \in \mathbb{K}[c]$, we can find an increasing subsequence $k \mapsto n(k)$ and a limit $V \in \mathbb{K}[c]$ such that $V_{n(k)} \rightharpoonup V$ weakly in \mathbb{H} ; up to extracting a further (not relabeled) subsequence, Lemma 5.1 shows that $\mathbf{u}_{n(k)} \rightarrow \mathbf{u} \in \mathbf{U}^J(V) \subset \mathbf{U}^J[c]$ strongly in $(\mathbb{V}_4)^J$ as $k \rightarrow \infty$. \square

We now introduce the family of functions $\sigma_k : \mathbb{K} \rightarrow \mathbb{H}$

$$\sigma_k(V) := \sum_{h=1}^k \lambda_h(V) \quad \text{for every } V \in \mathbb{K}, \quad k \in \mathbb{N}, \quad (5.6)$$

which will play a crucial role in the following, since they have a nice representation formula, which involves orthonormal sets of cardinality k . We refer to [20, 26] for a more refined investigation in finite dimension.

If $E \subset \mathbb{H}$ is a subspace of \mathbb{H} , we denote by $\text{Ort}_k(E)$ the subset of orthonormal frames of E^k

$$\text{Ort}^k(E) := \left\{ \mathbf{w} = (w_1, \dots, w_k) \in E^k : \langle w_i, w_j \rangle = \delta_{ij} \right\}, \quad (5.7)$$

and we have

$$\sigma_k(V) = \min \left\{ \sum_{h=1}^k \mathcal{E}_V(w_h) : \mathbf{w} = (w_1, \dots, w_k) \in \text{Ort}^k(\mathbb{V}_4) \right\}, \quad (5.8)$$

where the minimum in (5.8) is attained precisely at the elements of $\mathbf{U}^k(V)$. An important property of the functions λ_k, σ_k is their Lipschitzianity in $\mathbb{K}[c]$.

Lemma 5.3. *Under the assumptions of Section 2.A–2.E, for every $k \in \{1, \dots, J\}$ and $c \geq c_o$ the functions $V \mapsto \lambda_k(V)$ and $V \mapsto \sigma_k(V)$ are weakly continuous in \mathbb{K} and Lipschitz in $\mathbb{K}[c]$. Moreover, σ_k is concave and the concave and globally Lipschitz function $\sigma_{k,c} : \mathbb{H} \rightarrow \mathbb{R}$*

$$\sigma_{k,c}(V) := \min \left\{ \sum_{h=1}^k \mathcal{E}(w_h) + \int_{\mathbb{D}} V w_h^2 \, \text{d}\mathbf{m} : \mathbf{w} \in \mathbf{U}^J[c] \right\} \quad (5.9)$$

satisfies $\sigma_{k,c} \geq \sigma_k$ on \mathbb{K} and coincides with σ_k on $\mathbb{K}[c]$.

Proof. The weak continuity is a consequence of Lemma 5.1; the regularity of λ_k clearly follows from the analogous property of σ_k since $\lambda_k = \sigma_k - \sigma_{k-1}$. We can thus focus on the case of σ_k .

The fact that σ_k is concave clearly follows from (5.8), which represents σ_k as a minimum of a family of bounded linear functionals on \mathbb{H} . It is also clear that $\sigma_k \leq \sigma_{k,c}$.

In order to prove that the local representation given by (5.9) coincides with σ_k if $V \in \mathbb{K}[c]$, it is sufficient to notice that the choice $\mathbf{w} \in \mathbf{U}^k(V)$ is admissible in the minimization (5.9) of $\sigma_{k,c}(V)$ (by the very definition (5.3)) so that

$$\sigma_k(V) = \sigma_{k,c}(V) \quad \text{for every } V \in \mathbb{K}[c]. \quad (5.10)$$

We now observe that for every $\mathbf{w} \in \mathbf{U}^k[c]$ the norm of the linear functionals

$$\ell_{w_h} : V \rightarrow \int_{\mathbb{D}} V w_h^2 \, \text{d}\mathbf{m}, \quad h = 1, \dots, k, \quad (5.11)$$

is uniformly bounded in $L^2(\mathbb{D}, \mathbf{m})$ by the constant $A^2(c)$ given by (5.4), since

$$\|w_h^2\|_{L^2(\mathbb{D}, \mathbf{m})} \leq \|w_h\|_{L^4(\mathbb{D}, \mathbf{m})}^2 \leq A^2(c),$$

so that $\sigma_{k,c}$ satisfies

$$\sigma_{k,c}(V) \geq \min \left\{ \sum_{h=1}^k \ell_{w_h}(V) : \mathbf{w} \in \mathbf{U}^k[c] \right\} \geq -kA^2(c)|V|$$

and it is finite everywhere. Moreover, $\sigma_{k,c}$ is the infimum of a family of $kA^2(c)$ -Lipschitz functions on \mathbb{H} so it is $kA^2(c)$ -Lipschitz as well. Thanks to (5.10) we deduce that σ_k is $kA^2(c)$ -Lipschitz in $\mathbb{K}[c]$. \square

5.2. Compactness properties of the limiting subdifferential of \mathcal{H} . Let us now compute the superdifferential of the concave functions $\sigma_{k,c}$ defined by (5.9); we recall that the Fréchet superdifferential $\partial_F^+ \mathcal{G}$ of a function $\mathcal{G} : \mathbb{H} \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined as $-\partial_F^-(-\mathcal{G})$. We will just write $\partial^+ \mathcal{G}$ if \mathcal{G} is concave.

For every $V \in \mathbb{H}$ and $c \geq c_o$ we set

$$\begin{aligned} \mathbf{U}^{k,c}(V) &:= \left\{ \mathbf{u} \in \mathbf{U}^k[c] : \mathbf{u} \text{ is a minimizer of (5.9)} \right\}, \\ \Sigma_{k,c}(V) &:= \left\{ \sum_{h=1}^k u_h^2 : \mathbf{u} = (u_1, \dots, u_k) \in \mathbf{U}^{k,c}(V) \right\}, \\ \Sigma_k(V) &:= \left\{ \sum_{h=1}^k u_h^2 : \mathbf{u} = (u_1, \dots, u_k) \in \mathbf{U}^k(V) \right\} \quad \text{if } V \in \mathbb{K}. \end{aligned} \quad (5.12)$$

Notice that

$$\mathbf{U}^{k,c}(V) = \mathbf{U}^k(V) \quad \text{and} \quad \Sigma_{k,c}(V) = \Sigma_k(V) \quad \text{if } V \in \mathbb{K}[c]. \quad (5.13)$$

Lemma 5.4. *For every $V \in \mathbb{H}$ and $c \geq c_o$ we have*

$$\partial^+ \sigma_{k,c}(V) = \overline{\text{co}}\left(\Sigma_{k,c}(V)\right); \quad (5.14)$$

in particular if $V \in \mathbb{K}[c]$

$$\partial^+ \sigma_{k,c}(V) = \overline{\text{co}}\left(\Sigma_k(V)\right). \quad (5.15)$$

For $V \in \mathbb{K}$ we also get

$$\xi \in \overline{\text{co}}\left(\Sigma_k(V)\right) \quad \Rightarrow \quad \sigma_k(W) - \sigma_k(V) \leq \langle \xi, W - V \rangle \quad \text{for every } W \in \mathbb{K}. \quad (5.16)$$

Finally, $\partial^+ \sigma_{k,c}$ takes compact values and it is upper semicontinuous w.r.t. the weak topology. $\sigma_{k,c}$ is also Fréchet differentiable at every $V \in \mathbb{K}[c]$ such that $\lambda_k(V) < \lambda_{k+1}(V)$.

Proof. We want to apply Lemma C.1 in the appendix and we observe that the functions $\sigma_{k,c}$ can be represented as in (C.1), where

$$C := \mathbf{U}^J[c] \subset (\mathbb{V}_4)^J, \quad f(\mathbf{w}) := \sum_{h=1}^k w_h^2, \quad g(\mathbf{w}) := \sum_{h=1}^k \mathcal{E}(w_h) \quad \text{for every } \mathbf{w} \in \mathbf{U}^J[c]. \quad (5.17)$$

We thus obtain all the properties stated for $\sigma_{k,c}$; notice that (5.15) just follows by (5.14) and (5.13). It is also worth noticing that

$$\text{if } V \in \mathbb{K}[c] \text{ and } \lambda_k(V) < \lambda_{k+1}(V) \text{ then } \Sigma_{k,c}(V) = \Sigma_k(V) \text{ is a singleton} \quad (5.18)$$

thanks to Corollary B.1. This implies that $\sigma_{k,c}$ is Fréchet differentiable at V .

Eventually, (5.16) follows from (5.15) by choosing c sufficiently large so that $V \in \mathbb{K}[c]$ and using the fact that $\sigma_k(V) = \sigma_{k,c}(V)$, $\sigma_k(W) \leq \sigma_{k,c}(W)$. \square

We now want to study the structure of the subdifferential of \mathcal{H} . We fix a constant $c \geq c_o$ and we denote by $\varphi_c : \mathbb{R}^J \rightarrow \mathbb{R}$ a C^1 and Lipschitz function whose restriction to $\Lambda^J \cap [\lambda_{\min}, 1 + \ell_J(c)]^J$ coincides with φ (recall (5.5)). We introduce the function $\psi_c \in C^1(\mathbb{R}^J)$

$$\psi_c(s_1, s_2, \dots, s_J) := \varphi_c(s_1, s_2 - s_1, s_3 - s_2, \dots, s_J - s_{J-1}) \quad (5.19)$$

which clearly satisfies

$$\partial_j \psi_c = \partial_j \varphi_c - \partial_{j+1} \varphi_c \quad \text{if } 1 \leq j < J, \quad \partial_J \psi_c = \partial_J \varphi_c, \quad (5.20)$$

and we set

$$\mathcal{H}_c(V) := \psi_c(\boldsymbol{\sigma}_c(V)), \quad \boldsymbol{\sigma}_c(V) := (\sigma_{1,c}(V), \sigma_{2,c}(V), \dots, \sigma_{J,c}(V)) \quad \text{for every } V \in \mathbb{H}. \quad (5.21)$$

\mathcal{H}_c is a weakly continuous and strongly Lipschitz function which coincides with \mathcal{H} on $\mathbb{K}[c]$. Calling the map

$$\mathbf{\Lambda}^J: \Lambda^J \rightarrow \mathbb{R}^J, \quad \mathbf{\Lambda}^J(\lambda_1, \dots, \lambda_J) = (\lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \dots + \lambda_J),$$

we define ψ as the restriction of ψ_c to $\mathbf{\Lambda}^J(\Lambda^J \cap [\lambda_{\min}, 1 + \ell_J(c)]^J)$. In particular,

$$\psi(s_1, s_2, \dots, s_J) = \varphi(s_1, s_2 - s_1, \dots, s_J - s_{J-1}), \quad \text{for } (s_1, s_2, \dots, s_J) \in \mathbf{\Lambda}^J(\Lambda^J \cap [\lambda_{\min}, 1 + \ell_J(c)]^J).$$

Let us now introduce the multivalued map $S_c: \mathbb{H} \rightarrow 2^{\mathbb{H}}$,

$$S_c(V) := \sum_{j=1}^J \partial_j \psi_c(\sigma_c(V)) \partial^+ \sigma_{j,c}(V) = \left\{ \sum_{j=1}^J \gamma_j \xi_j : \gamma_j = \partial_j \psi_c(\sigma_c(V)), \xi_j \in \partial^+ \sigma_{j,c}(V) \right\}. \quad (5.22)$$

When $V \in \mathbb{K}[c]$, $S_c(V)$ is independent of c and can be written as

$$S_c(V) = S(V) = \left\{ \sum_{j=1}^J \gamma_j \xi_j : \gamma_j = \partial_j \psi(\sigma(V)), \xi_j \in \overline{\text{co}}(\Sigma_j(V)) \right\}. \quad (5.23)$$

Proposition 5.5 (Compactness of the limiting subdifferential of \mathcal{H}_c). *Let $c \geq c_o$ be given.*

(1) *For every weakly compact set $B \subset \mathbb{H}$ (in particular for $B = \mathbb{K}[c]$) the set*

$$\bigcup_{V \in B} S_c(V) \quad \text{is strongly compact in } \mathbb{H} \quad (5.24)$$

and the graph of S_c is weakly closed in $\mathbb{H} \times \mathbb{H}$:

$$(V_n, \xi_n) \rightharpoonup (V, \xi) \quad \xi_n \in S_c(V_n) \quad \Rightarrow \quad \xi \in S_c(V). \quad (5.25)$$

(2) *For every $V \in \mathbb{H}$, $\partial_L^- \mathcal{H}_c(V)$ is not empty and $\partial_L^- \mathcal{H}_c(V) \subset S_c(V)$.*

Proof. Claim (1) is an easy consequence of Lemma 5.4, the representation (5.17) in terms of Lemma C.1, and the fact that ψ_c is of class C^1 .

Claim (2) follows by the application of the calculus properties of limiting subdifferentials of Lipschitz functions: the chain rule [11, Ch. 1, Thm. 10.4], the sum rule [11, Ch. 1, Prop. 10.1], and the fact that $\partial_L^- f \subset \partial^+ f \cup \partial^- f$ for convex or concave functions. \square

5.3. Superdifferentiability. We want now to show that if φ satisfies the further structural conditions stated in Section 2.F then \mathcal{H} has a nice superdifferentiability property in $\mathbb{K}[c]$.

Theorem 5.6 (Superdifferentiability of \mathcal{H}). *Let $c \geq c_o$ and let \mathcal{H} satisfy the structural assumptions (2.25) of Section 2.F. If $V \in \mathbb{K}[c]$, $\mathbf{u} = (u_1, \dots, u_J) \in \mathbf{U}^J(V)$ and $\xi = \sum_{i=1}^J \partial_i \varphi(\boldsymbol{\lambda}^J(V)) u_i^2$, then $\xi \in \partial_F^+ \mathcal{H}_c(V)$; in particular there exists a positive function $\omega: \mathbb{H} \rightarrow \mathbb{R}$ as in (4.1) and $\varrho > 0$ such that*

$$\mathcal{H}(W) - \mathcal{H}(V) - \langle \xi, W - V \rangle \leq \omega(W - V) \quad \text{for every } W \in B(V, \varrho) \cap \mathbb{K}[c]. \quad (5.26)$$

Proof. Let us recall that $\sigma_c(V) = \sigma(V)$ since $V \in \mathbb{K}[c]$; we set

$$\lambda_i := \lambda_i(V), \quad p_i := \partial_i \varphi(\boldsymbol{\lambda}^J(V)), \quad \gamma_i := \partial_i \psi(\sigma(V)) = p_i - p_{i+1}.$$

The differentiability of ψ_c and the fact that $W \mapsto \sigma_{i,c}(W)$ is Lipschitz entail

$$\psi_c(\sigma_c(W)) - \psi_c(\sigma_c(V)) = \sum_{i=1}^J \gamma_i (\sigma_{i,c}(W) - \sigma_{i,c}(V)) + o(|W - V|) \quad \text{as } W \rightarrow V. \quad (5.27)$$

Let us consider the set H of indices $\{j: 1 \leq j < J, \lambda_j < \lambda_{j+1}\}$ and observe that $\gamma_j \geq 0$ if $j \notin H$ thanks to (2.25).

By Lemma 5.4 $\sigma_{i,c}$ is Fréchet differentiable at V for every $i \in H$ and it is Fréchet superdifferentiable for every i . It follows that setting $\xi_i := \sum_{k=1}^i u_k^2$,

$$\gamma_i \xi_i \text{ belongs to the Fréchet superdifferential of } W \mapsto \gamma_i \sigma_{i,c}(W) \text{ at } V. \quad (5.28)$$

Using (5.27) we find a positive function $\omega : \mathbb{H} \rightarrow \mathbb{R}$ as in (4.1) and $\varrho > 0$ such that

$$\psi_c(\sigma_c(W)) - \psi_c(\sigma_c(V)) \leq \left\langle \sum_{i=1}^J \gamma_i \xi_i, W - V \right\rangle + \omega(W - V) \text{ for every } W \in B(V, \varrho). \quad (5.29)$$

On the other hand

$$\sum_{i=1}^J \gamma_i \xi_i = p_J \xi_J + \sum_{i=1}^{J-1} (p_i - p_{i+1}) \xi_i = \sum_{i=2}^J p_i (\xi_i - \xi_{i-1}) + p_1 \xi_1 = \sum_{j=1}^J p_j u_j^2 = \xi. \quad (5.30)$$

(5.26) then follows by (5.29) and the fact that $\mathcal{H}_c(V) = \mathcal{H}(V)$ and $\mathcal{H}_c(W) = \mathcal{H}(W)$ if $V, W \in \mathbb{K}[c]$. \square

5.4. The case when \mathcal{H} is concave. The result of the previous section can be further refined when φ satisfies the stronger condition,

$$\partial_i \varphi \geq \partial_{i+1} \varphi \text{ for every } 1 \leq i < J, \quad \partial_J \varphi \geq 0 \text{ in } \Lambda^J, \quad (5.31)$$

which is related to Schur-concavity [23]. Even if the superdifferentiability result, Theorem 5.6, covers a more general setting, let us briefly recap this different approach.

We consider here the situation when φ is the restriction to Λ^J of a C^1 symmetric function $\phi : [\lambda_{\min}, +\infty)^J \rightarrow \mathbb{R}$ (recall (2.27)). We consider the functions $S_k : \mathbb{R}^J \rightarrow \mathbb{R}$, $1 \leq k \leq J$, defined by

$$S_k(\boldsymbol{\mu}) = S_k(\boldsymbol{\mu}_\uparrow) := \sum_{h=1}^k \mu_{(h)} \quad (5.32)$$

where for every $\boldsymbol{\mu} = (\mu_1, \dots, \mu_J) \in \mathbb{R}^J$ we will denote by $\boldsymbol{\mu}_\uparrow = (\mu_{(1)}, \dots, \mu_{(J)}) \in \Lambda^J$ the vector obtained by increasing rearrangement of the component of $\boldsymbol{\mu}$.

The functions S_k induce a partial order on \mathbb{R}^J given by

$$\boldsymbol{\mu}' \prec \boldsymbol{\mu}'' \Rightarrow S_k(\boldsymbol{\mu}') \geq S_k(\boldsymbol{\mu}'') \text{ for every } k = 1, \dots, J; \quad (5.33)$$

if (5.33) holds we say that $\boldsymbol{\mu}'$ is weakly majorized by $\boldsymbol{\mu}''$.

If $E \subset \mathbb{H}$ is a subspace of dimension $d \geq J$ and $V \in \mathbb{H}$, we can consider the vector $\boldsymbol{\lambda}^J(V, E) = (\lambda_1(V, E), \dots, \lambda_J(V, E))$ of the eigenvalues of the restriction of \mathcal{E}_V to E . The variational characterization easily shows that

$$\lambda_k(V) \leq \lambda_k(V, E) \text{ for every } 1 \leq k \leq J, \quad (5.34)$$

so that in particular $\boldsymbol{\lambda}^J(V, E) \prec \boldsymbol{\lambda}^J(V)$. By a Theorem of Schur [23, Chap. 9, B1], if $\boldsymbol{w} = (w_1, \dots, w_J) \in \text{Ort}^J(E)$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_J)$ with $\mu_k = \mathcal{E}_V(w_k)$ we also have

$$\boldsymbol{\mu} = \mathcal{E}_V(\boldsymbol{w}) \prec \boldsymbol{\lambda}^J(V, E). \quad (5.35)$$

By selecting $E = \text{Span}(\boldsymbol{w})$ we conclude that

$$\mathcal{E}_V(\boldsymbol{w}) \prec \boldsymbol{\lambda}^J(V) \text{ for every } \boldsymbol{w} \in \text{Ort}^J(\mathbb{H}). \quad (5.36)$$

If $\phi \in C^1([\lambda_{\min}, +\infty)^J)$ is symmetric then it satisfies the monotonicity condition

$$\boldsymbol{\lambda}' \prec \boldsymbol{\lambda}'' \Rightarrow \phi(\boldsymbol{\lambda}') \geq \phi(\boldsymbol{\lambda}'') \quad (5.37)$$

if and only if ϕ is increasing and Schur-concave [23, Chap. 3, A8], i.e.

$$(\partial_1 \phi(\boldsymbol{\lambda}) - \partial_2 \phi(\boldsymbol{\lambda}))(\lambda_1 - \lambda_2) \leq 0 \text{ for every } \boldsymbol{\lambda} \in [\lambda_{\min}, +\infty)^J, \quad (5.38)$$

which implies (5.31) thanks to the symmetry of ϕ .

It is worth noticing that this class contains all the concave increasing functions, so that if φ is induced by a symmetric, increasing and concave function ϕ , then (5.37) holds. (5.39) In particular S_J satisfies (5.37). However, the class of symmetric Schur-concave functions is much wider and stable w.r.t. various kind of operations, see [23]. In particular

all the elementary symmetric polynomials are Schur concave and increasing if $\lambda_{\min} \geq 0$. We deduce the following result.

Proposition 5.7 (Concavity of \mathcal{H}). *Let $k \in \mathbb{N}$, $V \in \mathbb{K}$, $\phi \in C^1([\lambda_{\min}, +\infty)^k)$ be a symmetric, increasing and Schur-concave function. Then*

$$\mathcal{H}(V) = \min \left\{ \phi(\mathcal{E}_V(\mathbf{w})) : \mathbf{w} \in (\mathbb{V} \cap L^4(\mathbb{D}, \mathbf{m}))^k \cap \text{Ort}^k(\mathbb{H}) \right\}. \quad (5.40)$$

If moreover ϕ is concave, then the function \mathcal{H} is concave as well.

Proof. (5.36) and (5.37) yield

$$\phi(\boldsymbol{\lambda}^k(V)) \leq \phi(\mathcal{E}_V(\mathbf{w})) \quad \text{for every } \mathbf{w} \in (\mathbb{V} \cap L^4(\mathbb{D}, \mathbf{m}))^k \cap \text{Ort}^k(\mathbb{H}). \quad (5.41)$$

On the other hand, the equality is attained by selecting $\mathbf{w} \in \mathbf{U}^k(V)$.

When ϕ is concave the maps $V \mapsto \phi(\mathcal{E}_V(\mathbf{w}))$ are concave since they are the composition of a concave with a linear function w.r.t. V . It follows that $V \mapsto \phi(\boldsymbol{\lambda}^k(V))$ is concave as well, since it is the minimum of a family of concave functions. \square

6. REGULARITY AND SUBDIFFERENTIABILITY PROPERTIES OF \mathcal{F}

In this section we will collect the main properties of \mathcal{F} , according to the setting presented in Section 2.A–2.E. We will eventually discuss a further important consequence of (2.25) of Section 2.F.

Lemma 6.1 (Weak continuity and coerciveness of \mathcal{F}). *For every $\eta > \theta$ the function $\mathcal{F}_\eta := \mathcal{F} + \frac{\eta}{2} |\cdot|^2$ is weakly lower semicontinuous and there exists a constant $S(\eta) \geq 0$ such that*

$$\mathcal{F}_\eta(V) \geq \delta |V|^2 - S(\eta) \quad \text{for every } V \in \mathbb{H}, \quad \delta := (\eta - \theta)/6; \quad (6.1)$$

in particular the sublevels of \mathcal{F}_η are bounded (thus weakly compact) in \mathbb{H} .

For every $a \geq 0$ there exists $c = c(a)$ such that if $|V| \leq a$ and $\mathcal{F}(V) \leq a$ then $V \in \mathbb{K}[c]$.

In particular for every $\tau > 0$ such that $\tau\theta < 1$ and every $V \in \mathbb{H}$ the functional $\Phi(\tau, V; \cdot) : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\}$

$$\Phi(\tau, V; W) := \frac{1}{2\tau} |W - V|^2 + \mathcal{F}(W) \quad W \in \mathbb{H} \quad (6.2)$$

has a minimizer.

Proof. By (2.22) and the fact that

$$(\lambda_J)_+ \leq (\lambda_{\min})_+ + \sum_{j=1}^J (\lambda_j - \lambda_{\min}) \leq (\lambda_{\min})_+ + J(\lambda_{\min})_- + \sum_{j=1}^J \lambda_j, \quad \boldsymbol{\lambda} \in \Lambda^J,$$

we obtain

$$\mathcal{H}(V) \geq -A_1(1 + \sigma_J(V)) \geq -A_1(1 + \sigma_{J, c_0}(V)) \quad \text{for every } V \in \mathbb{K}$$

with $A_1 := A(1 + J|\lambda_{\min}|)$. Since the function $V \mapsto -A_1\sigma_{J, c_0}(V)$ is convex, finite, and continuous in \mathbb{K} thanks to the representation (5.9), it is bounded from below by an affine function, so that there exists a constant $A_2 > 0$ such that

$$\mathcal{H}(V) \geq -A_2(1 + |V|) \quad \text{for every } V \in \mathbb{H}. \quad (6.3)$$

Setting $\delta := (\eta - \theta)/6$ and $A_3 := A_2 + A_2^2/4\delta$ we get

$$\mathcal{F}_\eta(V) \geq -A_2(1 + |V|) + \mathcal{H}_\theta(V) + 3\delta|V|^2 \geq -A_3 + \mathcal{H}_\theta(V) + 2\delta|V|^2, \quad (6.4)$$

showing that every sublevel of \mathcal{F}_η is contained in a suitable sublevel of \mathcal{K}_θ . Since \mathcal{K}_θ is convex, we have for some $A_4 \geq 0$

$$\mathcal{K}_\theta(V) \geq -A_4(1 + |V|) \quad \text{for every } V \in \mathbb{H}, \quad (6.5)$$

so that (6.4) yields for $A_5 := A_3 + A_4$ and $A_6 := A_5 + A_5^2/4\delta$

$$\mathcal{F}_\eta(V) \geq -A_5(1 + |V|) + 2\delta|V|^2 \geq -A_6 + \delta|V|^2, \quad (6.6)$$

showing (6.1). In particular if $\mathcal{F}(V) \leq a$ and $|V| \leq a$ then (6.4) shows that $V \in \mathbb{K}[c]$ whenever $c \geq a + \frac{1}{2}\eta^2 a + A_3$.

Since the restriction of \mathcal{H} to $\mathbb{K}[c]$ is weakly continuous and \mathcal{K}_η is convex and weakly lower semicontinuous as well, we conclude that \mathcal{F}_η is also weakly lower semicontinuous. Since

$$\Phi(\tau, V; W) = \frac{1}{2\tau}|V|^2 - \frac{1}{\tau}\langle W, V \rangle + \mathcal{F}_{\tau^{-1}}(W)$$

if $\tau^{-1} > \theta$ we immediately get that $\Phi(\tau, V; \cdot)$ has a minimizer. \square

Let us now study the properties of the limiting subdifferential of \mathcal{F} . We will also consider a weaker notion of ℓ -subdifferential: we say that ξ belongs to $\partial_\ell^- \mathcal{F}(v)$ if there exist sequences $v_n, \xi_n \in \mathbb{H}$ such that

$$\xi_n \in \partial_F^- \mathcal{F}(v_n), \quad v_n \rightarrow v \text{ strongly in } \mathbb{H}, \quad \xi_n \rightharpoonup \xi \text{ weakly in } \mathbb{H}, \quad \sup_n \mathcal{F}(v_n) < \infty. \quad (6.7)$$

Lemma 6.2 (Decomposition of the limiting subdifferential of \mathcal{F} - I). *For every $V \in \mathbb{K}$ we have $\partial_\ell^- \mathcal{F}(V) = \partial_L^- \mathcal{F}(V)$.*

If $V \in \mathbb{K}[c]$, $\xi \in \partial_L^- \mathcal{F}(V)$, and $c_1 > c$ then there exist $\xi_H \in \partial_L^- \mathcal{H}_{c_1}(V)$ and $\xi_K \in \partial_F^- \mathcal{K}(V)$ such that $\xi = \xi_H + \xi_K$. In particular there exist $\xi_j \in \overline{\text{co}}(\Sigma_j(V))$ such that

$$\xi = \sum_{j=1}^J \gamma_j \xi_j + \xi_K, \quad \gamma_j = \partial_j \psi_{c_1}(\sigma(V)). \quad (6.8)$$

Proof. We set $a := c_1 - c$ and we first consider the case when $\xi \in \partial_F^- \mathcal{F}(V)$ is an element of the Fréchet subdifferential of \mathcal{F} .

In this case there exists $\varrho > 0$ and a positive function $\omega : \mathbb{H} \rightarrow \mathbb{R}$ as in (4.1) such that

$$\mathcal{H}(W) - \mathcal{H}(V) + \mathcal{K}(W) - \mathcal{K}(V) - \langle \xi, W - V \rangle \geq -\omega(W - V) \quad \text{for every } W \in B(V, \varrho). \quad (6.9)$$

If $\delta < a$, $W \in B(V, \delta)$, and $W \notin \mathbb{K}[c_1]$ then

$$\begin{aligned} \mathcal{H}(W) - \mathcal{K}(V) &= \mathcal{H}(W) + \frac{\theta}{2}|W|^2 - (\mathcal{H}(V) + \frac{\theta}{2}|V|^2) - \frac{\theta}{2}(|W|^2 - |V|^2) \\ &\geq a - \frac{\theta}{2}(|W| + |V|)|W - V| \geq a - \theta c_1 \delta, \end{aligned}$$

since $|W| \leq |V| + \delta \leq c_1$ and $|V| \leq c \leq c_1$. On the other hand, if L is the Lipschitz constant of \mathcal{H}_{c_1} we get

$$\mathcal{H}_{c_1}(W) - \mathcal{H}_{c_1}(V) + \mathcal{K}(W) - \mathcal{K}(V) - \langle \xi, W - V \rangle \geq a - (L + \theta c_1 + |\xi|)\delta \geq 0, \quad (6.10)$$

if we choose $\delta > 0$ so small that $(L + \theta c_1 + |\xi|)\delta < a$. Possibly replacing ϱ with δ , since \mathcal{H}_{c_1} coincides with \mathcal{H} on $\mathbb{K}[c_1]$ we deduce from (6.9) and (6.10) that $\xi \in \partial_F^- (\mathcal{H}_{c_1} + \mathcal{K})(V)$. We can then apply the sum rule for the limiting subdifferential [11] and we obtain the decomposition

$$\xi = \xi_H + \xi_K, \quad \xi_H \in \partial_L^- \mathcal{H}_{c_1}(V), \quad \xi_K \in \partial_F^- \mathcal{K}(V). \quad (6.11)$$

Let us now consider the general case when $\xi \in \partial_\ell^- \mathcal{F}(V)$. By (6.7), we can find $V_n \in \mathbb{K}$ and $\xi_n \in \partial_F^- \mathcal{F}(V_n)$ such that

$$V_n \rightarrow V, \quad \xi_n \rightharpoonup \xi, \quad \sup_n \mathcal{F}(V_n) \leq C < +\infty. \quad (6.12)$$

We thus find a suitably large constant c_2 such that $V_n \in \mathbb{K}[c_2 - 1]$ and therefore we can decompose ξ_n as

$$\xi_n = \xi_H^n + \xi_K^n, \quad \xi_H^n \in \partial_L^- \mathcal{H}_{c_2}(V_n), \quad \xi_K^n \in \partial_F^- \mathcal{H}(V_n). \quad (6.13)$$

It follows that ξ_H^n is uniformly bounded, so that also ξ_K^n is uniformly bounded. Since \mathcal{H} is $-\theta$ -convex, this implies that $\mathcal{H}(V_n) \rightarrow \mathcal{H}(V)$; on the other hand \mathcal{H} is continuous in \mathbb{K} so that $\mathcal{H}(V_n) \rightarrow \mathcal{H}(V)$ as well, showing that $\xi \in \partial_L^- \mathcal{F}(V)$.

Choosing $c' \in (c, c_1)$ we definitely have $V_n \in \mathbb{K}[c']$. We can thus refine the decomposition (6.13) and assume that $\xi_H^n \in \partial_L^- \mathcal{H}_{c_1}(V_n) \subset S_{c_1}(V_n)$. We can now extract an increasing subsequence $k \mapsto n(k)$ such that $\xi_H^{n(k)} \rightarrow \xi_H$ for some $\xi_H \in \partial_L^- \mathcal{H}_{c_1}(V_n)$ (here we use the closedness of the limiting subdifferential) and therefore $\xi_K^{n(k)} \rightarrow \xi - \xi_H \in \partial_F^- \mathcal{H}(V)$.

(6.8) then follows by Proposition 5.5 and (5.23). \square

Corollary 6.3. *Under the same assumption of Lemma 6.2, let us suppose that $\lambda^{J+1}(V) \in \text{Int}(\Lambda^{J+1})$. If $\xi \in \partial_L^- \mathcal{F}(V)$ and \mathbf{u} is the unique element in $\mathbf{U}^J(V)$ we have*

$$\xi - \sum_{i=1}^J \partial_i \varphi(\lambda^J(V)) u_i^2 \in \partial_F^- \mathcal{H}(V). \quad (6.14)$$

Proof. If \mathbf{u} is the unique element in $\mathbf{U}^J(V)$ we have $\Sigma_k(V) = \{\xi_k\}$ where $\xi_k = \sum_{j=1}^k u_j^2$ for every $k \in \{1, \dots, J\}$. Using (6.8), we can then argue as in (5.30) to obtain

$$\sum_{j=1}^J \gamma_j \xi_j = \sum_{i=1}^J \partial_i \varphi(\lambda^J(V)) u_i^2. \quad \square$$

As a further step, we will prove that the limiting subdifferential of \mathcal{F} contains sufficient information to get the following chain rule property (cf. condition (CHAIN₂) of [28, Thm. 3]).

Proposition 6.4 (Chain rule). *Let $V \in H^1(0, T; \mathbb{H})$, $\xi \in L^2(0, T; \mathbb{H})$ such that $\xi(t) \in \partial_L^- \mathcal{F}(V(t))$ for a.e. $t \in (0, T)$ and $\mathcal{F} \circ V$ is bounded. Then the map $\mathcal{F} \circ V$ is absolutely continuous in $[0, T]$ and*

$$\frac{d}{dt} \mathcal{F}(V(t)) = \langle \xi(t), V'(t) \rangle \quad \text{a.e. in } (0, T). \quad (6.15)$$

Proof. Since $\mathcal{F} \circ V$ is bounded and V is bounded as well in \mathbb{H} being $V \in H^1(0, T; \mathbb{H})$, by Lemma 6.1 there exists a constant $c \geq c_0 + 1$ such that $V(t) \in \mathbb{K}[c - 1]$ for every $t \in [0, T]$.

We deduce that $\mathcal{F} \circ V = \mathcal{F}_c \circ V$ where $\mathcal{F}_c = \mathcal{H} + \mathcal{H}_c$. Since \mathcal{H}_c is a Lipschitz function, the composition $t \mapsto \mathcal{H}_c \circ V(t)$ is absolutely continuous. Moreover by Lemma 6.2 we can decompose $\xi(t)$ as

$$\xi(t) = \xi_H(t) + \xi_K(t), \quad \xi_H \in \partial_L^- \mathcal{H}_c(V(t)), \quad \xi_K(t) \in \partial_F^- \mathcal{H}(V(t)) \quad \text{for a.e. } t \in (0, T). \quad (6.16)$$

Since \mathcal{H}_c is Lipschitz, ξ_H is uniformly bounded and therefore the minimal selection $t \mapsto \partial_F^- \mathcal{H}(V(t))$ is a function in $L^2(0, T; \mathbb{H})$. Being \mathcal{H} the difference between a convex function and a quadratic one, we conclude that $t \mapsto \mathcal{H}(V(t))$ is absolutely continuous as well.

We can then find a Borel set $D \subset (0, T)$ of full Lebesgue measure such that the functions V , $\mathcal{H}_c \circ V$, $\mathcal{H} \circ V$, $\sigma_{j,c} \circ V$ are differentiable at every $t \in D$, $j = 1, \dots, J$, and there exist $\xi_j(t) \in \partial^+ \sigma_{j,c}(V(t))$ and $\xi_K(t) \in \partial_F^- \mathcal{H}(V(t))$ such that

$$\xi(t) = \sum_{j=1}^J \gamma_j(t) \xi_j(t) + \xi_K(t), \quad \gamma_j(t) = \partial_j \psi_c(\sigma_{k,c}(V)) \quad \text{for every } t \in D, \quad (6.17)$$

thanks to (6.8). Since $\sigma_{j,c}$ are concave and \mathcal{H} is $-\theta$ -convex, we have

$$\frac{d}{dt} \sigma_{j,c}(V(t)) = \langle \xi_j(t), V'(t) \rangle, \quad \frac{d}{dt} \mathcal{H}(V(t)) = \langle \xi_K(t), V'(t) \rangle \quad \text{for every } t \in D. \quad (6.18)$$

Since ψ_c is of class C^1 we clearly have

$$\frac{d}{dt} \mathcal{H}_c(V(t)) = \frac{d}{dt} \psi_c(\sigma_c(V(t))) = \sum_{j=1}^J \gamma_j(t) \frac{d}{dt} \sigma_{j,c}(V(t)) \quad \text{for every } t \in D. \quad (6.19)$$

Combining (6.19), (6.18) and (6.17) we get (6.15). \square

We conclude this section by showing a more refined decomposition of $\partial_L^- \mathcal{F}$ in the case φ satisfies also the structural condition (2.25) of Section 2.F.

Lemma 6.5 (Decomposition of the subdifferential of \mathcal{F} - II). *Let us suppose that all the assumptions of Section 2 are satisfied, included (2.25).*

(1) *If $V \in \mathbb{K}$ and $\xi \in \partial_F^- \mathcal{F}(V)$ then for every $\mathbf{u} \in \mathbf{U}^J(V)$*

$$\xi - \sum_{i=1}^J \partial_i \varphi(\boldsymbol{\lambda}^J(V)) u_i^2 \in \partial_F^- \mathcal{H}(V). \quad (6.20)$$

(2) *If $V \in \mathbb{K}$ and $\xi \in \partial_L^- \mathcal{F}(V)$ then there exists $\mathbf{u} \in \mathbf{U}^J(V)$ and $\xi_K \in \partial_F^- \mathcal{H}(V)$ such that*

$$\xi = \sum_{i=1}^J \partial_i \varphi(\boldsymbol{\lambda}^J(V)) u_i^2 + \xi_K. \quad (6.21)$$

Proof. Let us first consider Claim (1). By Definition 4.1 we know that there exist $\varrho > 0$ and a positive function $\omega_{\mathcal{F}} : \mathbb{H} \rightarrow \mathbb{R}$ as in (4.1) such that

$$\mathcal{H}(W) - \mathcal{H}(V) + \mathcal{K}(W) - \mathcal{K}(V) - \langle \eta, W - V \rangle \geq -\omega_{\mathcal{F}}(W - V)$$

for every $W \in \mathbb{K} \cap \mathbf{B}(V, \varrho)$. Let us set $\xi_H := \sum_{i=1}^J \partial_i \varphi(\boldsymbol{\lambda}^J(V)) u_i^2$ for some $\mathbf{u} \in \mathbf{U}^J(V)$ and let us select $c \geq c_o$ such that

$$c \geq |V| + 1, \quad \text{and} \quad c \geq \mathcal{H}(V) + \frac{1}{2} \theta |V|^2 + 1$$

so that $V \in \mathbb{K}[c - 1]$.

We can now apply (5.26) and find $\varrho_1 \in (0, \varrho)$ and a positive function $\omega_{\mathcal{H}} : \mathbb{H} \rightarrow \mathbb{R}$ as in (4.1) such that

$$\mathcal{H}(W) - \mathcal{H}(V) \leq \langle \xi_H, W - V \rangle + \omega_{\mathcal{H}}(W - V) \quad \text{for every } W \in \mathbf{B}(V, \varrho_1) \cap \mathbb{K}[c] \quad (6.22)$$

so that

$$\mathcal{H}(W) - \mathcal{H}(V) - \langle \xi - \xi_H, W - V \rangle \geq -\left(\omega_{\mathcal{H}}(W - V) + \omega_{\mathcal{F}}(W - V)\right), \quad (6.23)$$

for every $W \in \mathbb{K}[c] \cap \mathbf{B}(V, \varrho_1)$.

On the other hand, if $W \notin \mathbb{K}[c]$ and $|W - V| \leq \delta$ we have

$$\begin{aligned} \mathcal{H}(W) - \mathcal{H}(V) &= \mathcal{H}(W) + \frac{\theta}{2} |W|^2 - \left(\mathcal{H}(V) + \frac{\theta}{2} |V|^2\right) - \frac{\theta}{2} (|W|^2 - |V|^2) \\ &\geq 1 - \frac{\theta}{2} (|W| + |V|) \delta, \end{aligned}$$

so that choosing $\delta < \varrho_1$ sufficiently small we obtain

$$\mathcal{H}(W) - \mathcal{H}(V) - \langle \xi - \xi_H, W - V \rangle \geq \delta/2 \quad \text{if } W \notin \mathbb{K}[c] \text{ and } |W - V| < \delta. \quad (6.24)$$

This implies that (6.23) holds for every $W \in \mathbb{K} \cap \mathbf{B}(V, \delta)$ and therefore $\xi - \xi_H \in \partial_F^- \mathcal{H}(V)$.

Claim (2) readily follows: by the definition of limiting subdifferential we can find a sequence $V_n \in \mathbb{K}$ strongly convergent to V and $\xi_n \in \partial_F^- \mathcal{F}(V_n)$ weakly convergent to ξ with $\mathcal{F}(V_n) \rightarrow \mathcal{F}(V)$ as $n \rightarrow \infty$. We can then select arbitrary $\mathbf{u}_n \in \mathbf{U}^J(V_n)$ setting $\xi_H^n := \sum_{i=1}^J \partial_i \varphi(\boldsymbol{\lambda}^J(V_n)) (u_i^n)^2$ and $\xi_K^n := \xi_n - \xi_H^n \in \partial_F^- \mathcal{H}(V_n)$.

Up to extracting a subsequence, we may assume that $\mathbf{u}^n \rightarrow \mathbf{u} \in \mathbf{U}^J(V)$ strongly in \mathbb{H}^J and $\lambda^J(V_n) \rightarrow \lambda^J(V)$ in Λ^J , so that $\xi_H^n \rightarrow \xi_H := \sum_{i=1}^J \partial_i \varphi(\lambda^J(V)) u_i^2$ strongly in \mathbb{H} thanks to the regularity of φ . Correspondingly we have $\xi_k^n \rightharpoonup \xi_K = \xi - \xi_H \in \partial_{F^-} \mathcal{K}(V)$ since $\partial_{F^-} \mathcal{K}$ is strongly-weakly closed. \square

7. CONVERGENCE OF THE MINIMIZING MOVEMENT SCHEME AND PROOF OF THE MAIN RESULTS

We now refer to the construction we introduced in Section 4 (see in particular (4.6), (4.7), (4.8) and Definition 4.3) and we briefly recap the main general properties and estimates from the abstract theory of minimizing movements, following [28, Section 4]. As usual, we operate in the setting of Section 2, 2.A–2.E.

7.1. Existence, stability estimates and weak convergence of generalized minimizing movements. We start by proving the existence of generalized minimizing movements in our setting.

Lemma 7.1. *Let $\tau_* > 0$ such that $\theta\tau_* < 1$.*

- (1) *For every $\tau \in (0, \tau_*)$ and $V_0 \in \mathbb{K}$ there exists a discrete solution $(V_\tau^n)_{0 \leq n \leq N(\tau)}$ to the Minimizing Movement scheme (4.7). The interpolating functions V_τ and \bar{V}_τ satisfy the discrete equation*

$$V_\tau'(t) \in -\partial_{F^-} \mathcal{F}(\bar{V}_\tau(t)) \quad \text{for a.e. } t \in (0, T). \quad (7.1)$$

- (2) *There exists a constant C independent of τ such that for every discrete solution and for every $\tau \in (0, \tau_*)$*

$$\sup_{t \in [0, T]} |V_\tau(t)| \leq \sup_{t \in [0, T]} |\bar{V}_\tau(t)| \leq C \quad (7.2)$$

$$\sup_{t \in [0, T]} \mathcal{F}(\bar{V}_\tau(t)) \leq \mathcal{F}(V_0) \leq C, \quad (7.3)$$

$$\|V_\tau'\|_{L^2(0, T; \mathbb{H})} \leq C, \quad (7.4)$$

$$\|\bar{V}_\tau - V_\tau\|_{L^\infty(0, T; \mathbb{H})} \leq C\tau^{1/2}. \quad (7.5)$$

- (3) *There exists a constant c independent of τ such that $V_\tau(t) \in \mathbb{K}[c]$ and $\bar{V}_\tau(t) \in \mathbb{K}[c]$ for every $t \in [0, T]$.*

Proof. (1) The existence of discrete solutions to the Minimizing Movement scheme follows directly from Lemma 6.1. Notice that in our case we did not assume that the sublevels of $\mathcal{F}(V) + \frac{1}{2\tau_*}|V|^2$ are strongly compact as in [28, Lemma 1.2]; however Lemma 6.1 guarantees the weak lower semicontinuity of \mathcal{F} and the weak compactness of the sublevels of $\mathcal{F}(V) + \frac{1}{2\tau_*}|V|^2$.

(7.1) is then a simple application of the definition of Fréchet subdifferential (see e.g. [28, (4.29)]). In fact the minimality of V_τ^n in (4.7) yields

$$\begin{aligned} \mathcal{F}(W) - \mathcal{F}(V_\tau^n) &\geq \frac{1}{2\tau} |V_\tau^n - V_\tau^{n-1}|^2 - \frac{1}{2\tau} |W - V_\tau^{n-1}|^2 \\ &= -\frac{1}{\tau} \langle V_\tau^n - V_\tau^{n-1}, W - V_\tau^n \rangle - \frac{1}{2\tau} |W - V_\tau^n|^2; \end{aligned}$$

using (4.2) with $\omega(Z) := \frac{1}{2\tau}|Z|^2$ we get

$$-\frac{V_\tau^n - V_\tau^{n-1}}{\tau} \in \partial_{F^-} \mathcal{F}(V_\tau^n). \quad (7.6)$$

(7.1) then follows since for every $1 \leq n \leq N(\tau)$

$$V_\tau'(t) = \frac{V_\tau^n - V_\tau^{n-1}}{\tau} \quad \text{in } (t_{n-1}, t_n). \quad (7.7)$$

(2) is a direct application of [28, Prop. 4.6].

(3) still follows by (7.2), (7.3), Lemma 6.1, and the convexity of $\mathbb{K}[c]$. \square

Lemma 7.2 (Weak convergence of the Minimizing Movement scheme). *Under the same assumptions of Lemma 7.1 from every vanishing sequence $k \mapsto \tau(k) \downarrow 0$ it is possible to extract a further subsequence (not relabeled) and to find a limit function*

$$V \in H^1(0, T; \mathbb{H}), \quad \sup_{t \in [0, T]} \mathcal{F}(V(t)) \leq \mathcal{F}(V_0) \quad (7.8)$$

such that

$$\bar{V}_{\tau(k)}(t) \rightharpoonup V(t), \quad V_{\tau(k)}(t) \rightharpoonup V(t) \quad \text{weakly in } \mathbb{H} \text{ for every } t \in [0, T], \quad (7.9)$$

$$V'_{\tau(k)} \rightharpoonup V' \quad \text{weakly in } L^2(0, T; \mathbb{H}). \quad (7.10)$$

Proof. The proof of the weak convergence is a simple application of the a priori estimates of Lemma 7.1 (see also [1, Prop. 2.2.3], by choosing as σ the weak topology of \mathbb{H}). \square

7.2. Strong convergence of the Minimizing Movements scheme. This section contains the crucial argument improving Lemma 7.2.

Proposition 7.3. *Let $V_k := V_{\tau(k)}$, $\bar{V}_k := \bar{V}_{\tau(k)}$ be sequences of discrete solutions of the Minimizing Movement scheme weakly converging to V along a decreasing sequence of step sizes $\tau(k) \downarrow 0$ as in Lemma 7.2. Then $V_k, \bar{V}_k \rightarrow V$ uniformly in \mathbb{H} so that V is a Generalized Minimizing Movement in $\text{GMM}(\Phi, V_0, T)$.*

Proof. First of all we note that V_k, \bar{V}_k satisfy the differential inclusion

$$V'_k(t) \in -\partial_{\bar{F}} \mathcal{F}(\bar{V}_k(t)) \quad \text{for a.e. } t \in (0, T) \quad (7.11)$$

as in (7.1), the a priori estimates of Lemma 7.1, and the weak convergences (7.9) and (7.10). By Lemma 6.2 we can decompose $-V'_k(t)$ as the sum of two piecewise constant terms

$$-V'_k(t) = A_k(t) + B_k(t) - \theta \bar{V}_k(t), \quad A_k(t) \in S(\bar{V}_k(t)), \quad B_k(t) \in \partial_{\bar{F}} \mathcal{H}_\theta(\bar{V}_k(t)). \quad (7.12)$$

For the sake of clarity, we now divide the proof in several steps.

Step 1: compactness of A_k . Thanks to Proposition 5.5 the image of A_k is contained in a compact set $\mathcal{C} \subset \mathbb{H}$ independent of k . For late use, we will introduce

$$\mathcal{C}_{0,t} := t \bar{\text{co}}(\mathcal{C} \cup \{0\}), \quad t \geq 0.$$

By [29, Theorem 3.25], we deduce that $\mathcal{C}_{0,t}$ is a family of compact sets in \mathbb{H} , which by definition are also a convex, contain the origin, and satisfies $\mathcal{C}_{0,t} \subset \mathcal{C}_{0,T}$ for every $t \in [0, T]$.

As a consequence, $\int_0^T |A_k(t)|^2 dt$ is uniformly bounded and, up to pass to subsequences,

$$A_k \rightharpoonup A, \quad \text{weakly in } L^2(0, T; \mathbb{H}) \text{ as } k \rightarrow +\infty.$$

Step 2: a limsup inequality. At this point we only have that, for all $t \in [0, T]$,

$$B_k \rightharpoonup B := -V' - A + \theta V \quad \text{weakly in } L^2(0, T; \mathbb{H}) \text{ as } k \rightarrow +\infty,$$

We want now to prove

$$\limsup_{k \rightarrow \infty} \int_0^T e^{-2\theta t} \langle B_k(t), V_k(t) \rangle dt \leq \int_0^T e^{-2\theta t} \langle B(t), V(t) \rangle dt. \quad (7.13)$$

We first introduce the perturbation $C_k := B_k + \theta(V_k - \bar{V}_k) = -V'_k - A_k + \theta V_k$; since $\sup_{t \in [0, T]} |\bar{V}_k(t) - V(t)| \rightarrow 0$ as $k \rightarrow \infty$, (7.13) is equivalent to

$$\limsup_{k \rightarrow \infty} \int_0^T e^{-2\theta t} \langle C_k(t), V_k(t) \rangle dt \leq \int_0^T e^{-2\theta t} \langle B(t), V(t) \rangle dt. \quad (7.14)$$

By definition of C_k we have

$$\begin{aligned} \int_0^T e^{-2\theta t} \langle C_k(t), V_k(t) \rangle dt &= - \int_0^T e^{-2\theta t} \langle V_k'(t) - \theta V_k, V_k(t) \rangle dt - \int_0^T e^{-2\theta t} \langle A_k(t), V_k(t) \rangle dt \\ &= - \int_0^T \frac{d}{dt} \left(\frac{1}{2} e^{-2\theta t} |V_k(t)|^2 \right) dt - \int_0^T e^{-2\theta t} \langle A_k(t), V_k(t) \rangle dt \\ &= \frac{1}{2} |V_0|^2 - \frac{1}{2} e^{-2\theta T} |V_k(T)|^2 - \int_0^T e^{-2\theta t} \langle A_k(t), V_k(t) \rangle dt, \end{aligned} \quad (7.15)$$

and a similar calculation holds for B :

$$\int_0^T e^{-2\theta t} \langle B(t), V(t) \rangle dt = \frac{1}{2} |V_0|^2 - \frac{1}{2} e^{-2\theta T} |V(T)|^2 - \int_0^T e^{-2\theta t} \langle A(t), V(t) \rangle dt. \quad (7.16)$$

The lower semicontinuity of the norm with respect to the weak convergence yields

$$\limsup_{k \rightarrow +\infty} -|V_k(T)|^2 \leq -|V(T)|^2.$$

Hence (7.13) will follow if we to prove the convergence

$$\lim_{k \rightarrow \infty} \int_0^T e^{-2\theta t} \langle A_k(t), V_k(t) \rangle dt = \int_0^T e^{-2\theta t} \langle A(t), V(t) \rangle dt. \quad (7.17)$$

We use a compensated-compactness argument and we introduce the integral function

$$\mathcal{A}_k(t) := \int_0^t e^{-2\theta s} A_k(s) ds \quad t \in [0, T].$$

Since for every $k \in \mathbb{N}$ the sequence $\mathcal{A}'_k(t) = e^{-2\theta t} A_k(t)$ takes values in the compact subset $\mathcal{C} \subset \mathbb{H}$ and thus is uniformly bounded, we deduce that \mathcal{A}_k is uniformly Lipschitz equicontinuous.

It is also easy to show that $\mathcal{A}_k(t) \in \mathcal{C}_{0,T}$ for every $k \in \mathbb{N}$ and every $t \in [0, T]$, since by Jensen inequality

$$t^{-1} \mathcal{A}_k(t) = \int_0^t e^{-2\theta t} A_k(t) dt \in \overline{\text{co}}(\mathcal{C} \cup \{0\}) \quad \text{for every } t \in (0, T].$$

All in all, by Ascoli-Arzelà Theorem, we deduce that $(\mathcal{A}_k)_k$ is relatively compact in $C^0([0, T]; \mathbb{H})$, and therefore

$$\mathcal{A}_k \rightarrow \mathcal{A} \text{ uniformly and in } L^2(0, T; \mathbb{H}) \text{ as } k \rightarrow +\infty, \quad (7.18)$$

where $\mathcal{A}(t) := \int_0^t e^{-2\theta s} A(s) ds$. An integration by parts then gives

$$\int_0^T e^{-2\theta t} \langle A_k(t), V_k(t) \rangle dt = - \int_0^T \langle \mathcal{A}_k(t), V_k'(t) \rangle dt + \langle \mathcal{A}_k(T), V_k(T) \rangle - \langle \mathcal{A}_k(0), V_k(0) \rangle \quad (7.19)$$

with a similar identity involving A, V , and \mathcal{A} . Then we combine (7.18), (7.9) and (7.10) to infer

$$\langle \mathcal{A}_k(T), V_k(T) \rangle \rightarrow \langle \mathcal{A}(T), V(T) \rangle, \quad \langle \mathcal{A}_k(0), V_k(0) \rangle \rightarrow \langle \mathcal{A}(0), V(0) \rangle$$

and

$$\lim_{k \rightarrow +\infty} \int_0^T \langle \mathcal{A}_k(t), V_k'(t) \rangle dt = \int_0^T \langle \mathcal{A}(t), V'(t) \rangle dt;$$

we can then pass to the limit in (7.19) and we get (7.17).

Step 3: for a.e. $t \in (0, T)$ we have $B(t) \in \partial^- \mathcal{K}_\theta(V(t))$. Introducing the integral functional

$$\widetilde{\mathcal{K}}_\theta(V) := \int_0^T e^{-2\theta t} \mathcal{K}_\theta(V) dt \quad (7.20)$$

in the Hilbert space $\widetilde{\mathbb{H}} := L^2((0, T), \mu_\theta, \mathbb{H})$ associated to the Borel measure $\mu_\theta := e^{-2\theta t} \mathcal{L}^1$ in $(0, T)$, since $V \in D(\widetilde{\mathcal{K}}_\theta)$ and $B \in \widetilde{\mathbb{H}}$, we can equivalently prove that for all $W \in D(\widetilde{\mathcal{K}}_\theta) \subset \widetilde{\mathbb{H}}$

$$\int_0^T e^{-2\theta t} \langle B(t), W(t) - V(t) \rangle dt \leq \int_0^T e^{-2\theta t} \mathcal{K}_\theta(W(t)) dt - \int_0^T e^{-2\theta t} \mathcal{K}_\theta(V(t)) dt. \quad (7.21)$$

Since $B_k(t) \in \partial^- \mathcal{K}_\theta(\bar{V}_k(t))$ we have

$$\int_0^T e^{-2\theta t} \langle B_k(t), W(t) - \bar{V}_k(t) \rangle dt \leq \int_0^T e^{-2\theta t} \mathcal{K}_\theta(W(t)) dt - \int_0^T \mathcal{K}_\theta(\bar{V}_k(t)) dt.$$

Then it is sufficient to use Step 2, the weak lower semicontinuity of $\widetilde{\mathcal{K}}_\theta$ in $\widetilde{\mathbb{H}}$ (since it is strongly lower semicontinuous and θ -convex), the weak convergence of B_k , and the strong convergence of $\bar{V}_k - V_k$ to 0 in $\widetilde{\mathbb{H}}$ to obtain

$$\begin{aligned} \int_0^T e^{-2\theta t} \langle B(t), W(t) - V(t) \rangle dt &\leq \liminf_{k \rightarrow \infty} \int_0^T e^{-2\theta t} \langle B_k(t), W(t) - V_k(t) \rangle dt \\ &= \liminf_{k \rightarrow \infty} \int_0^T e^{-2\theta t} \langle B_k(t), W(t) - \bar{V}_k(t) \rangle dt \\ &\leq \int_0^T e^{-2\theta t} \mathcal{K}_\theta(W(t)) dt + \liminf_{k \rightarrow \infty} \left(- \int_0^T e^{-2\theta t} \mathcal{K}_\theta(\bar{V}_k(t)) dt \right) \\ &\leq \int_0^T e^{-2\theta t} \mathcal{K}_\theta(W(t)) dt - \int_0^T e^{-2\theta t} \mathcal{K}_\theta(V(t)) dt, \end{aligned}$$

which means $B(t) \in \partial^- \mathcal{K}_\theta(V(t))$ for a.e. t .

Step 4: $V_k \rightarrow V$ **uniformly in** $C^0([0, T]; \mathbb{H})$. By the equicontinuity estimate and the weak convergence (7.10), it is sufficient to prove that for all $S \in (0, T]$

$$\limsup_{k \rightarrow +\infty} |V_k(S)|^2 = |V(S)|^2. \quad (7.22)$$

Using the identities (7.15) and (7.16) written in the interval $[0, S]$ and taking into account of (7.17), (7.22) is equivalent to

$$\limsup_{k \rightarrow \infty} - \int_0^S e^{-2\theta t} \langle C_k(t), V_k(t) \rangle dt \leq - \int_0^S \langle e^{-2\theta t} B(t), V(t) \rangle dt. \quad (7.23)$$

Recalling that $C_k = B_k + \theta(V_k - \bar{V}_k)$ and $V_k - \bar{V}_k \rightarrow 0$ uniformly in \mathbb{H} , (7.23) can be reduced to

$$\liminf_{k \rightarrow \infty} \int_0^T e^{-2\theta t} \langle B_k(t), \bar{V}_k(t) \rangle dt \geq \int_0^T e^{-2\theta t} \langle B(t), V(t) \rangle dt. \quad (7.24)$$

Since by step 3 we have $B(t) \in \mathcal{K}_\theta(V(t))$ a.e., the monotonicity property of the subdifferential of a convex function yields

$$\langle B_k(t) - B(t), \bar{V}_k(t) - V(t) \rangle \geq 0 \quad \text{for a.e. } t \in (0, T) \quad (7.25)$$

so that

$$\begin{aligned} \langle B_k(t), \bar{V}_k(t) \rangle - \langle B(t), V(t) \rangle &= \langle B_k(t) - B(t), \bar{V}_k(t) - V(t) \rangle + \langle B(t), \bar{V}_k(t) \rangle - \langle B_k(t), V(t) \rangle \\ &\geq \langle B(t), \bar{V}_k(t) \rangle - \langle B_k(t), V(t) \rangle \end{aligned}$$

and therefore

$$\begin{aligned} \liminf_{k \rightarrow \infty} \int_0^T e^{-2\theta t} \left(\langle B_k(t), V_k(t) \rangle - \langle B(t), V(t) \rangle \right) dt \\ \geq \liminf_{k \rightarrow \infty} \int_0^T e^{-2\theta t} \langle B(t), V_k(t) \rangle dt - \int_0^T e^{-2\theta t} \langle B_k(t), V(t) \rangle dt = 0 \end{aligned}$$

where we used the weak convergence of B_k to B and of V_k to V . \square

7.3. Proof of the main results of Section 4. We can now collect all the information on the convergence of the Minimizing Movement scheme to conclude the proofs of the main results of Section 4.

Proof of Theorem 4.4. The fact that $\text{GMM}(\Phi, V_0, T)$ is not empty just follows from Proposition 7.3. We can then apply [28, Theorem 3] which shows that every element $V \in \text{GMM}(\Phi, V_0, T)$ satisfies (4.10), (4.11), (4.12) and (4.13) if \mathcal{F} satisfies the Chain rule property we proved in Proposition 6.4. In fact the compactness assumption in [28, Theorem 3] was just needed to guarantee the existence of an element in $\text{GMM}(\Phi, V_0, T)$ but the proof of the characterization of the limit differential equation is independent of such an assumption.

Proof of Proposition 4.6. It is sufficient to combine Theorem 4.4 with Corollary 6.3.

Proof of Theorem 4.7. just follows by Theorem 4.4 and (6.21) of Lemma 6.5.

APPENDIX A. CONVERGENCE OF EIGENVALUES AND EIGENFUNCTIONS FOR SCHRÖDINGER POTENTIALS

In order to study the behaviour of the eigenvalues of \mathcal{E}_V with respect to V we will use Mosco convergence in \mathbb{H} . Recall that a sequence of functionals $\Phi_n : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ converges in the sense of Mosco to a limit functional $\Phi : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ if the following two conditions hold:

- (M1) for every sequence $(w_n)_{n \in \mathbb{N}} \subset \mathbb{H}$ weakly converging to $w \in \mathbb{H}$ we have $\liminf_{n \rightarrow \infty} \Phi_n(w_n) \geq \Phi(w)$;
- (M2) for every $v \in \mathbb{H}$ there exists a sequence $(w_n)_{n \in \mathbb{N}}$ strongly converging to w such that $\lim_{n \rightarrow \infty} \Phi_n(w_n) = \Phi(w)$.

Mosco convergence is equivalent to Γ -convergence with respect to the weak and strong \mathbb{H} -topology, see [13, Chapters 12,13]. Under equi-coercivity (guaranteed in our case by the compactness of the imbedding of \mathbb{V} in \mathbb{H}) weak and strong Γ -convergence are equivalent and are also related to uniform convergence of the resolvents.

We split the proof of Lemma 5.1 in two parts: first we prove that the weak convergence of potentials implies the Mosco convergence of the associated functionals, and then show that the Mosco convergence implies the convergence of eigenvalues and eigenfunctions.

Lemma A.1. *Let $V_n \in \mathbb{K}$, $n \in \mathbb{N}$, be a sequence weakly converging in \mathbb{H} to $V \in \mathbb{K}$ as $n \rightarrow +\infty$. Then the corresponding sequence of quadratic forms \mathcal{E}_{V_n} converges in the sense of Mosco to \mathcal{E}_V .*

Proof. We start from the condition (M1) and consider a sequence w_n weakly converging to w in \mathbb{H} such that $\mathcal{E}_{V_n}(w_n) \leq E$ definitely. In particular $\mathcal{E}(w_n)$ is uniformly bounded from above, so that w_n is converging strongly to w in \mathbb{H} and

$$\liminf_{n \rightarrow +\infty} \mathcal{E}(w_n) + V_{\min} \int_{\mathbb{D}} |w_n|^2 \, \mathbf{d}\mathbf{m} \geq \mathcal{E}(w) + V_{\min} \int_{\mathbb{D}} |w|^2 \, \mathbf{d}\mathbf{m}. \quad (\text{A.1})$$

On the other hand, for every $k > 0$, $w_n \wedge k$ converges strongly to $w \wedge k$ in $L^4(\mathbb{D}, \mathbf{m})$ so that

$$\liminf_{n \rightarrow +\infty} \int_{\mathbb{D}} (V_n - V_{\min}) |w_n|^2 \, \mathbf{d}\mathbf{m} \geq \liminf_{n \rightarrow +\infty} \int_{\mathbb{D}} (V_n - V_{\min}) |w_n \wedge k|^2 \, \mathbf{d}\mathbf{m} = \int_{\mathbb{D}} (V - V_{\min}) |w \wedge k|^2 \, \mathbf{d}\mathbf{m}.$$

Since $k > 0$ is arbitrary we conclude that

$$\liminf_{n \rightarrow +\infty} \int_{\mathbb{D}} (V_n - V_{\min}) |w_n|^2 \, \mathbf{d}\mathbf{m} \geq \int_{\mathbb{D}} (V - V_{\min}) |w|^2 \, \mathbf{d}\mathbf{m}. \quad (\text{A.2})$$

Combining (A.1) and (A.2) we obtain $\mathcal{E}_V(w) \leq E$ as well.

Concerning (M2), we first show that for every $w \in D(\mathcal{E}_V)$ there exists a sequence $(w_k)_{k \in \mathbb{N}}$ in $D(\mathcal{E}_V) \cap L^4(\mathbb{D}, \mathbf{m})$ converging strongly to w in \mathbb{H} such that $\mathcal{E}_V(w_k) \rightarrow \mathcal{E}_V(w)$ as $k \rightarrow +\infty$. It is sufficient to consider an orthonormal basis of eigenfunctions $(u_h)_{h \in \mathbb{N}}$ for \mathcal{E}_V and set

$$w_k := \sum_{h=1}^k \langle w, u_h \rangle u_h \quad (\text{A.3})$$

On the other hand, for every $k \in \mathbb{N}$ we have

$$\lim_{n \rightarrow +\infty} \mathcal{E}_{V_n}(w_k) = \mathcal{E}_V(w_k) \quad (\text{A.4})$$

so that a standard diagonal argument yields (M2). \square

Now we provide the proof of some well-known facts concerning the Mosco convergence and the convergence of eigenvalues.

Definition A.2. For all $\beta > \lambda_{\min}$ and $V \in \mathbb{K}$, the resolvent operator $R_V^\beta: \mathbb{H} \rightarrow \mathbb{H}$ maps every $f \in \mathbb{H}$ into the the unique solution u of the problem

$$\mathcal{E}(u, w) + \int_{\mathbb{D}} (V + \beta) u w \, \mathbf{d}\mathbf{m} = \int_{\mathbb{D}} f w \, \mathbf{d}\mathbf{m} \quad \text{for all } w \in D(\mathcal{E}_V).$$

$R_V^\beta f$ is the unique minimizer of the functional

$$v \mapsto \frac{1}{2} \mathcal{E}(v) + \frac{1}{2} \int_{\mathbb{D}} (V + \beta) v^2 \, \mathbf{d}\mathbf{m} - \int_{\mathbb{D}} f v \, \mathbf{d}\mathbf{m}$$

We list here some properties of the resolvent operator:

- The operator R_V^β is continuous.
- The operator R_V^β is compact, thanks to the compact embedding of \mathbb{V} into \mathbb{H} .
- The operator R_V^β is self-adjoint.
- The operator R_V^β is positive.

As a consequence, the spectrum of R_V^β is real, positive and discrete and it is made of eigenvalues ordered as

$$0 \leq \dots \leq \Lambda_k(\beta, V) \leq \dots \leq \Lambda_1(\beta, V) = \|R_V^\beta\|_{\mathcal{L}(\mathbb{H})},$$

which are related to the sequence $\lambda_k(V)$ by the formula

$$\Lambda_k(V) = (\lambda_k(V) + \beta)^{-1}, \quad u \text{ is a } (V, \lambda)\text{-eigenfunction if and only if } R_V^\beta u = (\lambda + \beta)^{-1} u. \quad (\text{A.5})$$

The next fundamental lemma relates Mosco convergence to the (uniform) norm convergence of the resolvent operators.

Lemma A.3. Let $V_n, V \in \mathbb{K}$ and let us assume that $V_n \rightharpoonup V$ in \mathbb{H} . Then for every $\beta > \lambda_{\min}$ the associated resolvent operators converge, namely

$$R_{V_n}^\beta \rightarrow R_V^\beta \quad \text{in } \mathcal{L}(\mathbb{H}) \quad \text{as } n \rightarrow +\infty.$$

Proof. We fix $\beta > \lambda_{\min}$. From the definition of operator convergence, for $t \geq 0$ fixed, we have

$$\|R_{V_n}^\beta - R_V^\beta\|_{\mathcal{L}(\mathbb{H})} = \sup_{|f| \leq 1} |R_{V_n}^\beta(f) - R_V^\beta(f)| \leq |R_{V_n}^\beta(f_n) - R_V^\beta(f_n)| + \frac{1}{n},$$

for a suitable sequence $(f_n) \subset \mathbb{H}$ with $|f_n| \leq 1$ and that we can assume to be weakly- \mathbb{H} converging to some $f \in \mathbb{H}$. We can then split

$$|R_{V_n}^\beta(f_n) - R_V^\beta(f_n)| \leq |R_{V_n}^\beta(f_n - f)| + |R_{V_n}^\beta(f) - R_V^\beta(f)| + |R_V^\beta(f_n) - R_V^\beta(f)|.$$

The last term is infinitesimal, as the resolvent operator is continuous and $\|R_V^\beta(f_n)\|_{\mathbb{V}}$ is uniformly bounded.

The second term is also infinitesimal thanks to [13, Theorem 13.12]. Concerning the first term, since $R_{V_n}^\beta(f_n - f)$ is uniformly bounded in \mathbb{V} , it is sufficient to prove its weak convergence in \mathbb{H} . For every $g \in \mathbb{H}$ we have

$$\langle R_{V_n}^\beta(f_n - f), g \rangle = \langle f_n - f, R_{V_n}^\beta g \rangle \rightarrow 0$$

since $R_{V_n}^\beta g \rightarrow R_V^\beta g$ strongly in \mathbb{H} and $f_n \rightarrow 0$. In conclusion, $w = u = R_V^\beta(f)$ and by the compact embedding of \mathbb{V} into \mathbb{H} , we conclude that

$$R_{V_n}^\beta(f_n) \rightarrow R_V^\beta(f), \quad \text{strongly in } \mathbb{H},$$

and the convergence holds for the whole sequence, since the limit is independent of the chosen subsequence. \square

Eventually, thanks to the classical theory of linear operators, the norm convergence of the operators implies the convergence of the spectrum, see for example [16, Lemma XI.9.5]. Passing to the limit in the equation

$$R_{V_n}^\beta u_n = \Lambda_n u_n$$

where u_n is normalized sequence of eigenvalues associated to a converging sequence Λ_n and using the uniform boundedness of $R_{V_n}^\beta$ and the compactness of the imbedding of \mathbb{V} in \mathbb{H} we can also prove the convergence (possibly up to subsequences) of the eigenfunctions: this concludes the proof of Lemma 5.1.

APPENDIX B. TRACE OF SYMMETRIC OPERATORS

Let $E \subset \mathbb{H}$ be a subspace. We denote by $\text{Ort}^k(E)$ the subset of orthonormal sets of E^k :

$$\text{Ort}^k(E) := \left\{ \mathbf{w} = (w_1, \dots, w_k) \in E^k : \langle w_i, w_j \rangle = \delta_{ij} \right\}. \quad (\text{B.1})$$

If E has finite dimension $\dim(E) = d$ then we set $\text{Ort}(E) := \text{Ort}^d(E)$. An orthogonal matrix $Q \in O(k)$ operates on E^k by

$$(\mathbf{Q}\mathbf{w})_j := \sum_{i=1}^k Q_{ij} w_i, \quad \mathbf{w} = (w_1, \dots, w_k) \in \mathbb{H}^k, \quad j = 1, \dots, k. \quad (\text{B.2})$$

It is clear that if $\mathbf{w} \in \text{Ort}^k(E)$ then also $\mathbf{Q}\mathbf{w} \in \text{Ort}^k(E)$ since

$$\langle (\mathbf{Q}\mathbf{w})_i, (\mathbf{Q}\mathbf{w})_j \rangle = \left\langle \sum_{h=1}^k Q_{hi} w_h, \sum_{l=1}^k Q_{lj} w_l \right\rangle = \sum_{h,l=1}^k Q_{hi} Q_{lj} \langle w_h, w_l \rangle = \sum_{h,l=1}^k Q_{hi} Q_{lj} \delta_{hl} = \sum_{h=1}^k Q_{hi} Q_{hj} = \delta_{ij}$$

Let now \mathcal{Q} be a symmetric bilinear form on E and let $Q \in O(k)$ be an orthogonal matrix. For every $\mathbf{w} \in \text{Ort}^k(E)$ with $\mathbf{w}' = \mathbf{Q}\mathbf{w}$ we have

$$\sum_{h=1}^k \mathcal{Q}(w_h, w_h) = \sum_{h=1}^k \mathcal{Q}(w'_h, w'_h). \quad (\text{B.3})$$

In fact

$$\begin{aligned} \sum_{h=1}^k \mathcal{Q}(w'_h, w'_h) &= \sum_h \mathcal{Q} \left(\sum_i Q_{ih} w_i, \sum_j Q_{jh} w_j \right) = \sum_h \sum_{i,j} Q_{ih} Q_{jh} \mathcal{Q}(w_i, w_j) = \\ &= \sum_{i,j} \mathcal{Q}(w_i, w_j) \left(\sum_h Q_{ih} Q_{jh} \right) = \sum_{i,j} \mathcal{Q}(w_i, w_j) \delta_{ij} = \sum_{h=1}^k \mathcal{Q}(w_h, w_h). \end{aligned}$$

In particular, if E has finite dimension $\dim(E) = d$ the quantity

$$\mathrm{tr}_E(\mathcal{Q}) := \sum_{h=1}^d \mathcal{Q}(w_h, w_h) \quad \mathbf{w} \in \mathrm{Ort}(E) \quad (\text{B.4})$$

is well defined and independent of the choice of $\mathbf{w} \in \mathrm{Ort}(E)$.

A first application concerns the function

$$|\mathbf{w}|^2(x) := \sum_{h=1}^d |w_h(x)|^2 \quad x \in D, \quad \mathbf{w} \in \mathrm{Ort}(E) \quad (\text{B.5})$$

which is defined \mathfrak{m} -a.e. in D and defines a quadratic form on E .

Corollary B.1. *If $E \subset \mathbb{H}$ is a finite dimensional space with $\dim(E) = d$ and $\mathbf{w}', \mathbf{w}'' \in \mathrm{Ort}(E)$ then $|\mathbf{w}'|^2(x) = |\mathbf{w}''|^2(x)$ for \mathfrak{m} -a.e. $x \in D$.*

Proof. Since $\mathbf{w}', \mathbf{w}''$ are orthonormal basis of E there exists an orthogonal matrix $\mathbf{Q} \in O(d)$ such that $\mathbf{w}'' = \mathbf{Q}\mathbf{w}'$. It is then sufficient to apply (B.3). \square

APPENDIX C. BASIC FACTS CONCERNING NON SMOOTH DIFFERENTIAL CALCULUS

Let C be a compact metrizable topological space, let $f : C \rightarrow \mathbb{H}$ be a continuous map with image $R := f(C)$, and let $g : C \rightarrow \mathbb{R}$ be a lower semicontinuous map. We denote by $\mathcal{K}(R)$ the space of compact subsets of R . We set

$$F(v) := \min \left\{ \langle v, f(u) \rangle + g(u), \quad u \in C \right\} \quad \text{for every } v \in \mathbb{H}, \quad (\text{C.1})$$

and we denote by $M(v)$ the set of $u \in C$ where the minimum in (C.1) is attained.

Lemma C.1. *F is a Lipschitz concave function whose superdifferential is given by*

$$\partial^+ F(v) = \overline{\mathrm{co}}(f(M(v))); \quad (\text{C.2})$$

in particular, for every $\xi \in \partial^+ F(v)$ there exists a Borel probability measure $\mu \in \mathcal{P}(C)$ such that

$$\mathrm{supp}(\mu) \subset M(v), \quad \xi = \int_C f(u) \, d\mu(u). \quad (\text{C.3})$$

The map $\partial^+ F : \mathbb{H} \rightarrow \mathcal{K}(R)$ is weak-strong upper semicontinuous and satisfies

$$v_n \rightarrow v, \quad \xi_n \in \partial^+ F(v_n) \quad \Rightarrow \quad \begin{cases} (\xi_n)_{n \in \mathbb{N}} \text{ is strongly relatively compact in } \mathbb{H}, \\ \text{every limit point } \xi \text{ of } (\xi_n)_{n \in \mathbb{N}} \text{ belongs to } \partial^+ F(v). \end{cases} \quad (\text{C.4})$$

F is Fréchet differentiable at v_0 if and only if $f(M(v_0))$ is a singleton.

Proof. If $\xi = f(u)$ for some $u \in M(v)$ we have

$$F(w) - F(v) \leq \langle w, f(u) \rangle - g(u) - (\langle v, f(u) \rangle - g(u)) = \langle w - v, f(u) \rangle = \langle w - v, \xi \rangle$$

showing that $\xi \in \partial^+ F(v)$. It follows that $f(M(v)) \subset \partial^+ F(v)$ and therefore also $\overline{\mathrm{co}}(f(M(v))) \subset \partial^+ F(v)$.

Let us now prove that if $\xi \notin \overline{\mathrm{co}}(f(M(v)))$ then $\xi \notin \partial^+ F(v)$. Since $\overline{\mathrm{co}}(f(M(v)))$ is a compact convex set, we can apply the second geometric form of Hahn-Banach theorem and find $\eta \in \mathbb{H}$ with $|\eta| = 1$ and $\alpha \in \mathbb{R}$ such that

$$\langle \eta, \xi \rangle < \alpha < \min_{u \in M(v)} \langle \eta, f(u) \rangle,$$

i.e. the compact set $f(M(v))$ is contained in the open set $H(\eta, \alpha) := \{x \in \mathbb{H} : \langle \eta, x \rangle > \alpha\}$. By a standard compactness argument, we can find $\varepsilon > 0$ such that for every $w \in B(v, 2\varepsilon)$ $f(M(w)) \subset H(\eta, \alpha)$. Choosing $w := v + \varepsilon\eta$ and $u \in M(w)$ so that $f(u) \in H(\eta, \alpha)$ we get

$$\begin{aligned} \langle w - v, \xi \rangle &= \varepsilon \langle \eta, \xi \rangle < \varepsilon \alpha < \varepsilon \langle \eta, f(u) \rangle = \langle w - v, f(u) \rangle \\ &= \langle w, f(u) \rangle - g(u) - (\langle v, f(u) \rangle - g(u)) = F(w) - (\langle v, f(u) \rangle - g(u)) \leq F(w) - F(v) \end{aligned}$$

which shows that $\xi \notin \partial^+ F(v)$.

The representation (C.3) is an immediate consequence of the continuity of f and Krein-Milman Theorem.

Let now suppose that $v_n \rightarrow v$ in \mathbb{H} and let $\xi_n \in \partial^+ F(v_n)$; we can find a Borel probability measure μ_n on C such that

$$\text{supp}(\mu_n) \subset M(v_n), \quad \xi_n = \int_C f(u) d\mu_n(u).$$

Since C is compact and metrizable, we can find a subsequence $k \mapsto n(k)$ and a limit measure μ such that $\mu_{n(k)} \rightarrow \mu$ weakly in $\mathcal{P}(C)$. For every point u of the support of μ there exists a sequence of points $u_n \in \text{supp}(\mu_n) \subset M(v_n)$ converging to u ; passing to the limit in the family of inequalities

$$\langle v_n, f(u_n) \rangle + g(u_n) \leq \langle v_n, f(w) \rangle + g(w) \quad \text{for every } w \in C,$$

we get

$$\langle v, f(u) \rangle + g(u) \leq \langle v, f(w) \rangle + g(w) \quad \text{for every } w \in C,$$

so that $u \in M(v)$. It follows that setting

$$\xi := \int_C f(u) d\mu(u) \in \partial^+ F(v)$$

we then conclude that $\xi_{n(k)} \rightarrow \xi$ strongly in \mathbb{H} as $k \rightarrow \infty$.

Concerning the Fréchet differential of F , it is obvious that if F is differentiable at v_0 then $\partial^+ F(v_0)$ reduces to a singleton. To prove the converse property, let ξ_0 be the unique element of $f(M(v_0))$: we have just to show that $\xi_0 \in \partial^- F(v_0)$. By (C.4), for every $\varepsilon > 0$ we can find $\delta > 0$ such that

$$f(M(w)) \subset B(\xi_0, \varepsilon) \quad \text{for every } w \in B(v_0, \delta).$$

For every $w \in B(v_0, \delta)$ and $\xi \in f(M(w))$ we thus have

$$F(w) - F(v) - \langle \xi_0, w - v \rangle \geq \langle \xi - \xi_0, w - v \rangle \geq -|\xi - \xi_0| \cdot |w - v| \geq -\varepsilon |w - v|$$

which shows that $\xi_0 \in \partial^- F(v_0)$. □

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