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SINGULAR MULTIPLE INTEGRALS AND NONLINEAR POTENTIALS

CRISTIANA DE FILIPPIS AND BIANCA STROFFOLINI

ABSTRACT. We derive sharp partial regularity criteria of nonlinear potential theoretic nature for the Lebesgue-Serrin-Marcellini extension of nonhomogeneous singular multiple integrals featuring (p,q)-growth conditions.

1. INTRODUCTION

We provide optimal partial regularity criteria for relaxed minimizers of nonhomogeneous, singular multiple integrals of the form

$$W^{1,p}(\Omega,\mathbb{R}^N) \ni w \mapsto \mathcal{F}(w;\Omega) := \int_\Omega \left[F(Dw) - f\cdot w\right] \, \mathrm{d}x, \tag{1.1}$$

i.e., local minimizers of the Lebesgue-Serrin-Marcellini extension of $\mathcal{F}(\cdot)$:

$$\bar{\mathcal{F}}(w;\Omega) := \inf\left\{\liminf_{j\to\infty} \mathcal{F}(w_j;\Omega)\colon \{w_j\}_{j\in\mathbb{N}} \subset W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\colon w_j \rightharpoonup w \text{ in } W^{1,p}(\Omega,\mathbb{R}^N)\right\}, \tag{1.2}$$

using tools from Nonlinear Potential Theory, thus completing the analysis started in [26] for degenerate functionals. More precisely, we prove almost everywhere gradient continuity for local minimizers of (1.2) under sharp assumptions on the external datum f. Here, $\Omega\subset\mathbb{R}^n$ is an open, bounded set with Lipschitz boundary, $n\geq 2$, and $F\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ is a strictly quasiconvex integrand, verifying so-called (p,q)-growth conditions according to Marcellini's terminology [64]:

$$|z|^p \lesssim F(z) \lesssim 1+|z|^q, \qquad 1<p\leq q; \tag{1.3}$$

the singular behavior of $F(\cdot)$ around zero being encoded in the requirement $p\in(1,2)$. Let us recall that $F(\cdot)$ is quasiconvex when

$$\fint_{B_1(0)} F(z+D\varphi) \, \mathrm{d}x \geq F(z) \quad \text{holds for all } \ z\in\mathbb{R}^{N\times n}, \ \ \varphi\in C^\infty_{\mathrm{c}}(B_1(0),\mathbb{R}^N), \tag{1.4}$$

therefore the three main aspects of (1.1)-(1.2) we are interested in are the presence of a nontrivial forcing term f, the (p,q)-growth conditions in (1.3) and the quasiconvexity (1.4) of the integrand $F(\cdot)$. Let us briefly discuss some classical and recent results on these ingredients as each of them is currently object of intense investigation. The problem of determining the best conditions to impose on f in order to prove gradient continuity for minima is classical and received a considerable attention in the past decades. To better understand this issue, let us introduce the Lorentz space $L(n,1)$, defined by

$$w\in L(n,1) \iff \|w\|_{L(n,1)} := \int_0^\infty |\{x\in\mathbb{R}^n\colon |w(x)|>t\}|^{1/n} \, \mathrm{d}t < \infty.$$

A related deep result of Stein [80] states that

$$w\in W^{1,n} \ \text{ and } \ Dw\in L(n,1) \implies w \ \text{ is continuous}, \tag{1.5}$$

so (1.5) and the immersions $L^{n+\varepsilon}\hookrightarrow L(n,1)\hookrightarrow L^n$ for all $\varepsilon>0$, lead to the borderline characterization of $L(n,1)$ as the limiting space with respect to the Sobolev embedding theorem. A linear PDE interpretation of Stein's theorem relying on the combination of (1.5) with standard Calderón-Zygmund theory prescribes that

$$-\Delta u = f\in L(n,1) \implies Du \ \text{ is continuous},$$

which turns out to be sharp, in the light of Cianchi's counterexample [21]. Surprisingly enough, the same conclusion holds in a way more general setting than the linear one. It is indeed true for uniformly elliptic operators [3, 8, 22, 23, 32, 34, 57, 58, 60, 71, 72]; systems of differential forms [79]; fully nonlinear elliptic equations [7, 24], general nonuniformly elliptic functionals [9, 11, 27]; and it also holds at the level of partial regularity for

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systems of the p-Laplacian type without Uhlenbeck's structure [17,59]. The key point consists in the possibility of gaining local control on the gradient of solutions via the truncated Riesz potential of f, that is

$$\mathbf{I}_1^f(x_0,\varrho) := \int_0^\varrho \left(\fint_{B_\sigma(x_0)} |f| \ \mathrm{d}x \right) \ \mathrm{d}\sigma \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} \ \mathrm{d}y, \tag{1.6}$$

which is a standard aspect of linear equations and a remarkable feature of nonlinear ones, cf. Kuusi & Mingione's seminal works [58,60]. On the other hand, in [9,27] the gradient of minima is dominated via a nonlinear Wolff type potential, first introduced by Havin & Maz'ya [47], defined as:

$$\mathbf{I}_{1,m}^f(x_0,\varrho) := \int_0^\varrho \left(\sigma^m \fint_{B_\sigma(x_0)} |f|^m \ \mathrm{d}x \right)^{1/m} \frac{\mathrm{d}\sigma}{\sigma}, \qquad m>1,$$

sharing the same homogeneity - and therefore analogous mapping properties on function spaces - as the linear potential in (1.6). All the aforementioned results crucially rely on the strong ellipticity of the operators involved, while in (1.1) the integrand $F(\cdot)$ is only quasiconvex. This notion was first introduced by Morrey [69] and it turns out to be a natural condition in the multidimensional Calculus of Variations. Indeed, under polynomial growth conditions on the integrand $F(\cdot)$, quasiconvexity is a necessary and sufficient condition for sequential weak lower semicontinuity in $W^{1,p}$, [2,6,16,39,62,69]. A peculiar characteristic of quasiconvexity is that it is a purely nonlocal concept [52,69] in the sense that there is no condition involving only $F(\cdot)$ and a finite number of its derivatives, which is equivalent to quasiconvexity. Moreover, minimizers and critical points of quasiconvex functionals have a very different behavior. Precisely, a classical result of Evans [38] states that minima are regular outside a negligible "singular" set, while Müller & Šverák [70] proved that critical points, i.e. solutions to the associated Euler-Lagrange system, may have everywhere discontinuous gradients. This is coherent with the theory of elliptic systems: well-known counterexamples [68,82] show that solutions might develop singularities, therefore in the genuine vectorial setting the best one could hope for is partial regularity. The matter of almost everywhere regularity for minimizers of quasiconvex integrals was first treated by Evans [38] in the case of quadratic functionals, and, after that, it received lots of attention over the years. Subsequently, partial regularity for multiple integrals with standard p-growth was obtained in [1,18,54] exploiting Evans' blow up method, while in [35] a unified approach to the partial regularity for degenerate or singular quasiconvex integrals was proposed via the p-harmonic approximation and in [53] was derived an upper bound on the Hausdorff dimension of the singular set of minima of quasiconvex functionals. We refer to [10,14,26,30,33,43–46,49–51,61,75–77] and references therein for a non-exhaustive list of remarkable contributions in more general settings. The other main feature of the class of integrands considered in this paper is their (p,q)-growth conditions. This nomenclature was introduced by Marcellini in the fundamental papers [64,67] within the framework of nonlinear elasticity. In fact, a basic model describing the behavior of compressible materials subject to deformations is given by

$$W^{1,p}(\Omega,\mathbb{R}^n) \ni w \mapsto \mathscr{H}(w;\Omega) := \int_\Omega \left[|Dw|^p + \sqrt{1+|\det(Dw)|^2} - f\cdot w \right] \ \mathrm{d}x, \tag{1.7}$$

for some $f \in W^{1,p}(\Omega,\mathbb{R}^N)^*$, see [5,6,64,67]. A natural phenomenon in compressible elasticity is cavitation, i.e. the possible formation of cavities (holes) in elastic bodies after stretch, corresponding to the development of singularities in equilibrium solutions (minima) of $\mathscr{H}(\cdot)$. Functional $\mathscr{H}(\cdot)$ is quasiconvex in the sense of (1.4), [42, Chapter 5], however in general it is not $W^{1,p}$-quasiconvex[1] unless $p \ge n$, [6, Theorem 4.1], while its Lebesgue-Serrin-Marcellini extension $\bar{\mathscr{H}}(\cdot)$ is $W^{1,p}$-quasiconvex provided that $p > n-1$, [76, Lemma 7.6]. This means that the approach by relaxation based on the extension of the ambient space proposed in [64,67] fits the analysis of cavitation better than the pointwise one of [5,6], as it allows dealing with discontinuous maps thus describing the possible formations of cavities. We also point out that the integrand $H(z) := |z|^p + \sqrt{1+|\det(z)|^2}$ in (1.7) verifies

$$|z|^p \le H(z) \lesssim 1+|z|^n,$$

that is (1.3) with $q=n$. This was the first main reason behind the investigation of variational integrals with (p,q)-growth: starting with [64] for questions of semicontinuity and [65,66] about regularity, since then such class of functionals received lots of attention - with no pretence of completeness we mention the everywhere regularity results in [9,11–13,15,19,27,29,37,48,55,56], the partial regularity proven in [20,25,28,31,36,74] for general systems and manifold constrained problems with special structure and refer to [63] for a reasonable survey. The aforementioned results hold for strictly convex variational integrals. In the quasiconvex setting partial regularity has been obtained by Schmidt [75,77] for homogeneous - $f\equiv 0$ in (1.1) - functionals with (p,q)-growth and for their Lebesgue-Serrin-Marcellini extension [76], while [26] contains sharp partial regularity criteria in terms of a nontrivial forcing term f for relaxed minimizers of degenerate integrals of the form (1.1). The standard notion of relaxed local minimizers [76] reads as:

[1] I.e.: (1.4) holds for all $\varphi \in W_0^{1,p}(B_1(0),\mathbb{R}^N)$.

Definition 1. *Let $p \in (1,\infty)$. A function $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ is a local minimizer of (1.2) on Ω with $f \in W^{1,p}(\Omega,\mathbb{R}^N)^*$ if and only if every $x_0 \in \Omega$ admits a neighborhood $B \Subset \Omega$ so that $\bar{\mathcal{F}}(u;B) < \infty$ and $\bar{\mathcal{F}}(u;B) \le \bar{\mathcal{F}}(w;B)$ for all $w \in W^{1,p}(B,\mathbb{R}^N)$ so that $\operatorname{supp}(u-w) \Subset B$.*

Such definition can be immediately adapted to local minimizers of functional (1.1). Let us point out that when considering (1.1)-(1.2) we will assume with no loss of generality that f is defined on the whole $\mathbb{R}^n$, which is always possible if we set $f \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. For this reason, when stating that f belongs to a certain function space, we shall often avoid to specify the underlying domain. Further details about the notation employed can be found in Section 2 below. The main result of our paper is the following

Theorem 1. *Under assumptions (2.12)-(2.14), (2.16) and (2.23), let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of (1.2). Suppose that*

$$\lim_{\varrho\to 0} \mathbf{I}^f_{1,m}(x,\varrho) = 0 \qquad \text{locally uniformly in } x \in \Omega. \tag{1.8}$$

Then there exists an open "regular" set $\Omega_u \subset \Omega$ of full n-dimensional Lebesgue measure with $|\Omega\setminus\Omega_u| = 0$ such that $V_p(Du)$ and Du are continuous on Ω_u. In particular, the regular set Ω_u can be characterized as

$$\Omega_u := \Big\{ x_0 \in \Omega\colon \exists M \equiv M(x_0) \in (0,\infty),\ \tilde{\varepsilon} \equiv \tilde{\varepsilon}(\mathtt{data},M),\ \tilde{\varrho} \equiv \tilde{\varrho}(\mathtt{data},M,f(\cdot)) \in (0,\min\{d_{x_0},1\}) \ \text{such that } |(V_p(Du))_{B_\varrho(x_0)}| < M \text{ and } \mathfrak{F}(u;B_\varrho(x_0)) < \tilde{\varepsilon} \text{ for some } \varrho \in (0,\tilde{\varrho}] \Big\}.$$

Theorem 1 comes as a consequence of a fine connection established between the Lebesgue points of Du and $V_p(Du)$ and the pointwise behavior of the Wolff potential $\mathbf{I}^f_{1,m}(\cdot)$.

Theorem 2. *Under assumptions (2.12)-(2.14), (2.16) and (2.23), let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of (1.2), $x_0 \in \Omega$ be a point such that*

$$\mathbf{I}^f_{1,m}(x_0,1) < \infty \tag{1.9}$$

and $M \equiv M(x_0)$ be a positive constant. There exist $\tilde{\varepsilon} \equiv \tilde{\varepsilon}(\mathtt{data},M) \in (0,1)$ and $\tilde{\varrho} \equiv \tilde{\varrho}(\mathtt{data},M,f(\cdot)) \in (0,\min\{1,d_{x_0}\})$ such that if

$$\begin{cases} |(V_p(Du))_{B_\varrho(x_0)}| < M \\ \mathfrak{F}(u;B_\varrho(x_0)) + \mathbf{I}^f_{1,m}(x_0,\varrho)^{\frac{p}{2(p-1)}} + \mathbf{I}^f_{1,m}(x_0,\varrho)^{\frac{q}{2(p-1)}} + M^{(2-p)/p}\mathbf{I}^f_{1,m}(x_0,\varrho) < \tilde{\varepsilon}, \end{cases} \tag{1.10}$$

for some $\varrho \in (0,\tilde{\varrho}]$, then

$$\lim_{s\to 0}(V_p(Du))_{B_s(x_0)} = V_p(Du(x_0)), \qquad \lim_{s\to 0}(Du)_{B_s(x_0)} = Du(x_0) \tag{1.11}$$

and

$$\begin{cases} |V_p(Du(x_0)) - (V_p(Du))_{B_\sigma(x_0)}| \le c\mathfrak{N}(x_0;\sigma) \\ |Du(x_0) - (Du)_{B_\sigma(x_0)}| \le c\mathfrak{N}(x_0;\sigma)^{2/p} + c|(Du)_{B_\sigma(x_0)}|^{(2-p)/2}\mathfrak{N}(x_0;\sigma), \end{cases} \tag{1.12}$$

for all $\sigma \in (0,\varrho]$, where $c \equiv c(\mathtt{data},M)$ and

$$\begin{aligned} \mathfrak{N}(x_0;\sigma) \ &\approx\ \mathfrak{F}(u;B_\sigma(x_0)) + \boldsymbol{I}^f_{1,m}(x_0,\sigma)^{\frac{p}{2(p-1)}} + \boldsymbol{I}^f_{1,m}(x_0,\sigma)^{\frac{q}{2(p-1)}} \\ &\quad + |(V_p(Du))_{B_\sigma(x_0)}|^{(2-p)/p}\mathbf{I}^f_{1,m}(x_0,\sigma), \end{aligned}$$

up to constants depending on $(\mathtt{data},M)$. In particular, $x_0 \in \Omega$ satisfying (1.9) is a Lebesgue point of $V_p(Du)$ and of Du if and only if it verifies (1.10).

Conditions (1.8) or (1.9) can be guaranteed once prescribed the membership of f to a proper function space, as stated in the following optimal function space criterion.

Theorem 3. *Under assumptions (2.12)-(2.14), (2.16) and (2.23), let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of (1.2). There exists an open set $\Omega_u \subset \Omega$ of full n-dimensional Lebesgue measure such that $f \in L(n,1)$ yields that Du, $V_p(Du)$ are continuous on Ω_u, while if $f \in L^d$ for some $d > n$, then $Du, V_p(Du) \in C^{0,\tilde{\alpha}}_{\mathrm{loc}}(\Omega_u,\mathbb{R}^{N\times n})$ with $\tilde{\alpha} \equiv \tilde{\alpha}(n,N,p,d)$.*

Let us point out that Theorems 1-3 are new already for singular variational integrals with standard p-growth - the degenerate case coming as a straightforward consequence of the analysis carried out in [26]. Moreover, our results hold for general strictly $W^{1,p}$-quasiconvex functionals[2] as in (1.1), or for functionals that coincide with their Lebesgue-Serrin-Marcellini extension, and cover in particular relaxed local minimizer of the functional $\mathcal{H}(\cdot)$ in (1.7) with the choice $n = q = 2$ and $p \in (8/5,2)$, cf. [26, Section 2.4]. We refer to [77, 81] for

[2] I.e.: (2.14) below holds for all $\varphi \in W^{1,p}_0(B,\mathbb{R}^N)$.

further discussions and more examples. We finally remark that the nonlinear potential theory for singular nonhomogeneous equations or systems of the p-Laplacian type is a very recent achievement. In fact, after Duzaar & Mingione's breakthrough [34] on pointwise potential estimates with $p \in \big(2-1/n,\infty\big)$, lots of efforts have been devoted to the extension of such result to all $p \in (1,2)$: in [71] Nguyen & Phuc decreased the lower bound on p switching from $p > 2-1/n$ to $p > (3n-2)/(2n-1)$; later on Dong & Zhu [32] and Nguyen & Phuc [72] (singular equations with measure data) and Byun & Youn [17] (general subquadratic systems) eventually covered the full range $p \in (1,2)$. In this respect, our paper fits such line of research as we provide pointwise bounds on the gradient oscillation of minima of (1.2) that hold almost everywhere subject to the validity of a smallness condition on the excess functional that naturally involves also the potential $\mathbf{I}^{f}_{1,m}(\cdot)$. It is worth stressing that the strategy required to attack partial borderline regularity issues for singular quasiconvex integrals is deeply different in nature from the one presented in [26] for degenerate functionals. In fact, the iteration scheme designed here shares with [26] only the exit time (blocks and chains) technique introduced there, that compensates the destabilizing effect of a nontrivial right-hand side term within the nonsingular (resp. nondegenerate) regime, cf. [26, Section 1.2] for more details. All the rest of the proof requires a different strategy due to the problematic simultaneous presence of a rough forcing term f, featuring a very limited amount of regularity; a rather severe loss of ellipticity, that is a distinctive aspect of singular integrands; and (p,q)-growth conditions, leading to nonhomogeneous estimates. Since the full body of nonlinear potential theoretic techniques unavoidably breaks down as p approaches one [17,32,34,71,72], we first need an artificial, quadratic "upgrade" of the integrand's ellipticity fratures in terms of suitable vector fields encoding the scaling properties of the p-Laplacian. The price to pay for such a boost is a weakened control over gradient averages, that do not only have to remain bounded at successive scales to keep under control the rate of (p,q)-nonuniform ellipticity of the functional, but, in contrast with what happens in [26] but coherently with previous results on singular problems [17, 32, 72], also substantially impact the excess decay estimate. The combination of these issues forces us to develop a delicate double iteration scheme that on one hand preserves the boundedness of gradient averages, on the other guarantees a uniform control on the size of the excess functional in terms of hybrid terms involving both gradient average (at the initial scale) and Wolff potential related to the nonhomogeneity f, thus eventually leading to the almost pointwise gradient oscillation bounds in (1.12). We also mention that our gradient oscillation estimate improves that in [17, Theorem 1.1], valid for uniformly elliptic singular systems of the p-Laplacian type. In fact, setting $p = q$ in (1.12), a simple application of Jensen's inequality shows that the quantity on the right-hand side of (1.12) is smaller than its counterpart in [17, Theorem 1.1, (1.12)]. Let us finally point out that, since for basic homogeneity reasons functionals with (p,q)-growth do not admit higher integrability results of self-improving nature (see [35,77] for homogeneous problems), in antithesis with [17,59], our approach does not require any application of Gehring Lemma, so we believe it is flexible enough to find applications to possibly more general contexts [30,46,49,50,73,78,79].

2. Preliminaries

In this section we record the notation employed throughout the paper, describe the structural assumptions governing the ingredients appearing in (1.1) and collect certain basic results that will be helpful later on.

2.1. **Notation.** In this paper, $\Omega \subset \mathbb{R}^n$ will always be an open, bounded domain with Lipschitz-regular boundary, and $n \geq 2$. We denote by c a general constant larger than one depending on the main parameters governing the problem. We will still denote by c distinct occurrences of constant c from line to line. Specific occurrences will be marked with symbols $c_*, \tilde{c}$ or the like. Significant dependencies on certain parameters will be outlined by putting them in parentheses, i.e. $c \equiv c(n,p)$ means that c depends on n and p. By $B_r(x_0) := \{x \in \mathbb{R}^n : |x-x_0| < r\}$ we indicate the open ball with center in x_0 and radius $r > 0$; we shall avoid denoting the center when this is clear from the context, i.e., $B \equiv B_r \equiv B_r(x_0)$; this happens in particular with concentric balls. For $x_0 \in \Omega$, it is $d_{x_0} := \operatorname{dist}(x_0, \partial\Omega)$ and with $z_1, z_2 \in \mathbb{R}^{N\times n}$, $s \geq 0$ we set $\mathcal{D}_s(z_1,z_2) := (s^2+|z_1|^2+|z_2|^2)$. Given a measurable set $B \subset \mathbb{R}^n$ with bounded positive Lebesgue measure $|B| \in (0,\infty)$, and a measurable map $g\colon B \to \mathbb{R}^k$, $k \geq 1$, we set

$$(g)_B \equiv \fint_B g(x)\,\mathrm{d}x := \frac{1}{|B|}\int_B g(x)\,\mathrm{d}x.$$

A useful feature of the average is its almost minimality, i.e.:

$$\left(\fint_B |g-(g)_B|^t\ \mathrm{d}x\right)^{1/t} \leq 2\left(\fint_B |g-z|^t\,\mathrm{d}x\right)^{1/t} \qquad \text{for all}\ \ z \in \mathbb{R}^k, \quad t \geq 1. \tag{2.1}$$

For $t \geq 1$, $s \geq 0$, $q \geq p > 1$, we shorten:

$$\mathfrak{I}_t(g;\mathcal{B}) := \left(\fint_{\mathcal{B}} |g(x)|^t\,\mathrm{d}x\right)^{\frac{1}{t}}, \qquad\qquad \mathfrak{K}(s) := s+s^{q/p} \tag{2.2}$$

and define

$$\mathbb{1}_{\{q>p\}} := \begin{cases} 1 & \text{if } q>p \\ 0 & \text{if } q=p, \end{cases} \qquad \mathbb{1}_{\{q\geq 2\}} := \begin{cases} 1 & \text{if } q\geq 2 \\ 0 & \text{if } 1<q<2. \end{cases}$$

Finally, if $t>1$ is any number, its conjugate will be denoted by $t' := t/(t-1)$ and its Sobolev esponent as $t^* := nt/(n-t)$ when $t<n$ or any number larger than one for $t\geq n$. To streamline the notation, we gather together the main parameters governing our problem in the shorthand $\texttt{data} := \big(n,N,\lambda,\Lambda,p,q,\omega(\cdot),F(\cdot),m\big)$, we refer to Section 2.3 for more details on such quantities.

2.2. **Tools for p-Laplacian type problems.** The vector field $V_{s,p}\colon \mathbb{R}^{N\times n}\to\mathbb{R}^{N\times n}$, defined as

$$V_{s,p}(z) := (s^2+|z|^2)^{(p-2)/4}z, \qquad p\in(1,\infty) \ \text{ and } \ s\geq 0$$

for all $z\in\mathbb{R}^{N\times n}$, which encodes the scaling features of the p-Laplacian operator, is a useful tool for handling p-Laplacean type problems. If $s=0$, we simply write $V_{s,p}(\cdot)\equiv V_p(\cdot)$. Let us premise that although most of the properties of the vector field $V_{s,p}(\cdot)$ that we are going to list below hold for all $p\in(1,\infty)$, from now on, we shall always assume that $p\in(1,2)$. It is well-known that

$$\begin{cases} \dfrac{|V_{s,p}(z_1+z_2)|^2}{|z_1+z_2|} \lesssim \dfrac{|V_{s,p}(z_1)|^2}{|z_1|} + \dfrac{|V_{s,p}(z_2)|^2}{|z_2|} \\ |V_{s,p}(z_1)-V_{s,p}(z_2)| \approx (s^2+|z_1|^2+|z_2|^2)^{(p-2)/4}|z_1-z_2| \\ |V_{|z_1|,p}(z_2-z_1)| \approx |V_p(z_1)-V_p(z_2)| \\ \dfrac{|V_{s,p}(z_1)|^2|z_2|}{|z_1|} \lesssim |V_{s,p}(z_1)|^2+|V_{s,p}(z_2)|^2 \\ |V_{s,p}(kz)| \lesssim \max\{k,k^{p/2}\}|V_{s,p}(z)| \\ |V_{s,p}(z)| \approx \min\{|z|,|z|^{p/2}\} \\ |V_{s,p}(z_1+z_2)| \lesssim |V_{s,p}(z_1)|+|V_{s,p}(z_2)|, \end{cases} \tag{2.3}$$

for all $k>0$ cf. [18,35,77] - of course to avoid trivialities, above $|z_1+z_2|$, $|z_1|$ are supposed to be positive - and that whenever $t>-1$, $s\in[0,1]$ and $z_1,z_2\in\mathbb{R}^{N\times n}$ verify $s+|z_1|+|z_2|>0$, then

$$\int_0^1 \left[s^2+|z_1+y(z_2-z_1)|^2\right]^{\frac{t}{2}} \,\mathrm{d}y \approx (s^2+|z_1|^2+|z_2|^2)^{\frac{t}{2}}. \tag{2.4}$$

As useful consequences of $(2.3)_2$ we have

$$|z_1-z_2|^p \lesssim |V_{s,p}(z_1)-V_{s,p}(z_2)|^2 + |V_{s,p}(z_1)-V_{s,p}(z_2)|^p(|z_1|+s)^{p(2-p)/2}, \tag{2.5}$$

see [57, Lemma 2]. It is also worth recalling a Poincaré-type inequality involving the vector field $V_{s,p}(\cdot)$: with $p\in(1,2)$, $B_\varrho(x_0)\Subset\Omega$ and $w\in W^{1,p}(B_\varrho(x_0),\mathbb{R}^N)$ it is

$$\fint_{B_\varrho(x_0)} \left| V_{s,p}\left(\frac{w-(w)_{B_\varrho(x_0)}}{\varrho}\right)\right|^{p^\#} \,\mathrm{d}x \leq c\left(\fint_{B_\varrho(x_0)} |V_{s,p}(Dw)|^2 \,\mathrm{d}x\right)^{p^\#/2}, \tag{2.6}$$

with $p^\# := 2n/(n-p)$ and $c\equiv c(n,N,p)$, see [35, Lemma 8]. For $B_\varrho(x_0)\Subset\Omega$, $w\in W^{1,p}(B_\varrho(x_0),\mathbb{R}^N)$ and $z_0\in\mathbb{R}^N$, we define the excess functional by

$$\mathfrak{F}(w,z_0;B_\varrho(x_0)) := \left(\fint_{B_\varrho(x_0)} |V_p(Dw)-z_0|^2 \,\mathrm{d}x\right)^{1/2}$$

and further introduce the auxiliary integral

$$\tilde{\mathfrak{F}}(w,z_0;B_\varrho(x_0)) := \left(\fint_{B_\varrho(x_0)} |V_{|z_0|,p}(Dw-z_0)|^2 \,\mathrm{d}x\right)^{1/2}.$$

If $z_0 = (V_p(Dw))_{B_\varrho(x_0)}$ we shall abbreviate $\mathfrak{F}(w,(V_p(Dw))_{B_\varrho(x_0)};B_\varrho(x_0)) \equiv \mathfrak{F}(w;B_\varrho(x_0))$. Let us point out that combining (2.1) with [41, (2.6)] it holds that

$$\mathfrak{F}(w;B_\varrho(x_0)) \approx \mathfrak{F}(w,V_p((Dw)_{B_\varrho(x_0)});B_\varrho(x_0)), \tag{2.7}$$

while if $z_0 = (V_p)^{-1}((V_p(Dw))_{B_\varrho(x_0)})$ - recall that $V_p(\cdot)$ is an isomorphism of $\mathbb{R}^{N\times n}$ - via $(2.3)_3$ we have

$$\mathfrak{F}(w;B_\varrho(x_0)) \approx \tilde{\mathfrak{F}}(w,z_0;B_\varrho(x_0)). \tag{2.8}$$

In all the above displays, the constants implicit in "$\approx,\lesssim$" depend on (n,N,p,t) - the dependency on t accounts for the exponent appearing in (2.4). To the scopes of this paper, it is fundamental to record some well-known scaling features of the excess functional.

Lemma 2.1. *Let $1 < p < 2$ be a number, $B_\varrho(x_0) \subset \mathbb{R}^n$ be a ball, and $w \in W^{1,p}(B_\varrho(x_0), \mathbb{R}^N)$ be any function. With $\nu \in (0,1)$ it holds that*

$$
\begin{cases}
\mathfrak{F}(w; B_{\nu\varrho}(x_0)) \le \dfrac{2}{\nu^{n/2}}\mathfrak{F}(w; B_\varrho(x_0)) \\
|(V_p(Dw))_{B_{\nu\varrho}(x_0)}| \le \dfrac{1}{\nu^{n/2}}\mathfrak{F}(w; B_\varrho(x_0)) + |(V_p(Dw))_{B_\varrho(x_0)}| \\
|(V_p(Dw))_{B_{\nu\varrho}(x_0)} - (V_p(Dw))_{B_\varrho(x_0)}| \le \dfrac{1}{\nu^{n/2}}\mathfrak{F}(w; B_\varrho(x_0)).
\end{cases} \tag{2.9}
$$

Moreover, if for $\sigma \le \varrho$ there is $\kappa \in \mathbb{N} \cup \{0\}$ satisfying $\nu^{\kappa+1}\varrho < \sigma \le \nu^\kappa \varrho$, then

$$
\begin{cases}
\mathfrak{F}(w; B_{\nu^{\kappa+1}\varrho}(x_0)) \le \dfrac{2}{\nu^{n/2}}\mathfrak{F}(w; B_\sigma(x_0)) \le \dfrac{2^2}{\nu^n}\mathfrak{F}(w; B_{\nu^\kappa\varrho}(x_0)) \\
|(V_p(Dw))_{B_{\nu^{\kappa+1}\varrho}(x_0)}| \le |(V_p(Dw))_{B_\sigma(x_0)}| + \dfrac{1}{\nu^{n/2}}\mathfrak{F}(w; B_\sigma(x_0)) \le |(V_p(Dw))_{B_{\nu^\kappa\varrho}(x_0)}| + \dfrac{2^2}{\nu^n}\mathfrak{F}(w; B_{\nu^\kappa\varrho}(x_0)),
\end{cases} \tag{2.10}
$$

and, whenever $c_* \ge 1$ is an absolute constant it is

$$
\begin{aligned}
|(V_p(Dw))_{B_{\nu^{\kappa+1}\varrho}(x_0)}| + c_*\mathfrak{F}(w; B_{\nu^{\kappa+1}\varrho}(x_0)) &\le \frac{2^2}{\nu^{n/2}}\left[|(V_p(Dw))_{B_\sigma(x_0)}| + c_*\mathfrak{F}(w; B_\sigma(x_0))\right] \\
&\le \frac{2^4}{\nu^n}\left[|(V_p(Dw))_{B_{\nu^\kappa\varrho}(x_0)}| + c_*\mathfrak{F}(w; B_{\nu^\kappa\varrho}(x_0))\right].
\end{aligned} \tag{2.11}
$$

We conclude this section with a classical iteration lemma, [42, Lemma 6.1].

Lemma 2.2. *Let $h\colon [\varrho_0, \varrho_1] \to \mathbb{R}$ be a non-negative and bounded function, and let $\theta \in (0,1)$, $A, B, \gamma_1, \gamma_2 \ge 0$ be numbers. Assume that $h(t) \le \theta h(s) + A(s-t)^{-\gamma_1} + B(s-t)^{-\gamma_2}$ holds for all $\varrho_0 \le t < s \le \varrho_1$. Then the following inequality holds $h(\varrho_0) \le c(\theta, \gamma_1, \gamma_2)[A(\varrho_1 - \varrho_0)^{-\gamma_1} + B(\varrho_1 - \varrho_0)^{-\gamma_2}]$.*

2.3. **Structural assumptions.** We assume that $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is an integrand verifying:

$$
\begin{cases}
F \in C^1_{\operatorname{loc}}(\mathbb{R}^{N\times n}) \cap C^2_{\operatorname{loc}}(\mathbb{R}^{N\times n} \setminus \{0\}) \\
\Lambda^{-1}|z|^p \le F(z) \le \Lambda\left(1 + |z|^q\right)
\end{cases} \tag{2.12}
$$

for all $z \in \mathbb{R}^{N\times n}$, with $\Lambda \ge 1$ being a positive, absolute constant and exponents (p,q) satisfying:

$$
1 < p < 2 \qquad \text{and} \qquad \frac{q}{p} < 1 + \frac{1}{2n}. \tag{2.13}
$$

It is fundamental that $F(\cdot)$ is strictly degenerate quasiconvex, in the sense that whenever $B \Subset \Omega$ is a ball it holds that

$$
\int_B \left[F(z + D\varphi) - F(z)\right] \,\mathrm{d}x \ge \lambda \int_B (|z|^2 + |D\varphi|^2)^{\frac{p-2}{2}}|D\varphi|^2 \,\mathrm{d}x \qquad \text{for all } \ z \in \mathbb{R}^{N\times n}, \ \ \varphi \in C^\infty_c(B, \mathbb{R}^N), \tag{2.14}
$$

where λ is a positive, absolute constant. As a consequence, for all $z \in \mathbb{R}^{N\times n} \setminus \{0\}$, $\xi \in \mathbb{R}^N$, $\zeta \in \mathbb{R}^n$ it holds that

$$
\partial^2 F(z)\langle \xi \otimes \zeta, \xi \otimes \zeta\rangle \ge 2\lambda |z|^{p-2}|\xi|^2|\zeta|^2, \tag{2.15}
$$

see [42, Chapter 5] or [76, Lemma 7.14]. Moreover, as a minimal requirement on the second derivatives of $F(\cdot)$, we need to prescribe their behavior near the origin. Precisely, we need that

$$
\frac{|\partial^2 F(z) - \partial^2(|z|^p/p)|}{|z|^{p-2}} \to 0 \qquad \text{as } \ |z| \to 0. \tag{2.16}
$$

The by-product of (2.16) is summarized in the following lemma, that collects results from [1, 62, 77].

Lemma 2.3. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be an integrand verifying (2.12), $(2.13)_1$ and (2.16). Then,*

- *there exists a positive constant $c \equiv c(n, N, \Lambda, q)$ such that*

$$
|\partial F(z)| \le c(1 + |z|^{q-1}) \qquad \text{for all } \ z \in \mathbb{R}^{N\times n}; \tag{2.17}
$$

- *whenever $z_0 \in \mathbb{R}^{N\times n}$ verifies $|z_0| \le L + 1$ for some positive constant L, it is*

$$
\begin{cases}
|F(z_0 + z) - F(z_0) - \langle \partial F(z_0), z\rangle| \le c\left(|V_{|z_0|,p}(z)|^2 + |V_{|z_0|,q}(z)|^2\right) \\
|\partial F(z_0 + z) - \partial F(z_0)| \le c|z|^{-1}\left(|V_{|z_0|,p}(z)|^2 + |V_{|z_0|,q}(z)|^2\right),
\end{cases} \tag{2.18}
$$

  for all $z \in \mathbb{R}^{N\times n}$, with $c \equiv c(n, N, \Lambda, p, q, F(\cdot), L)$;
- *there exists a concave modulus of continuity $\omega\colon (0,\infty) \to (0,\infty)$ with $\lim_{s\to 0}\omega(s) = 0$ such that*

$$
|z| \le \omega(s) \implies |\partial F(z) - \partial F(0) - |z|^{p-2}z| \le s|z|^{p-1}; \tag{2.19}
$$

- *whenever* $L > 0$ *is a positive constant and* $z \in \left(\mathbb{R}^{N\times n} \setminus \{0\}\right) \cap \left\{|z| \le L\right\}$ *it is:*

$$|\partial^2 F(z)| \le c|z|^{p-2}, \tag{2.20}$$

for $c \equiv c(n, N, p, F(\cdot), L)$*;*

- *for all positive constants* L *and vectors* $z_1, z_2 \in \mathbb{R}^{N\times n}$ *so that* $0 < |z_1| \le L$, $0 < |z_2| \le 2L$ *it holds that*

$$|\partial^2 F(z_1) - \partial^2 F(z_2)| \le \left(\frac{|z_1|^2 + |z_2|^2}{|z_1|^2|z_2|^2}\right)^{(2-p)/2} \mu_L\left(\frac{|z_1 - z_2|^2}{|z_1|^2 + |z_2|^2}\right), \tag{2.21}$$

where $\mu_L \colon (0,\infty) \to (0,\infty)$ *is a nondecreasing modulus of continuity with* $s \mapsto \mu_L(s)^2$ *concave, depending on* $F(\cdot)$ *and on* L*.*

Proof. The proof of inequality (2.17) is contained in [62, Theorem 2.1, Step 2]; the bounds in (2.18) are the outcome of [77, Lemma 4.3], see also [1, Lemma II.3] for the nondegenerate case; while (2.20) is a direct consequence of (2.16) and of the uniform continuity of $\partial^2 F(\cdot)$ on compact sets away from zero prescribed by $(2.12)_1$. The implication in (2.19) comes by observing that (2.16) yields $(\partial F(z) - \partial F(0) - |z|^{p-2}z)/|z|^{p-1} \to 0$ as $|z| \to 0$, which in turn gives (2.19), cf. [77, Remark 5.8]. To prove (2.21), we first observe that if we denote by $g(z)$ either $\partial^2(|z|^p/p)$ or $|z|^{p-2}$, by homogeneity $g(\cdot)$ verifies the Hölder condition

$$|g(z_1) - g(z_2)| \lesssim \left(\frac{|z_1^2| + |z_2|^2}{|z_1|^2|z_2|^2}\right)^{\frac{2-p}{2}} \left(\frac{|z_2 - z_1|^2}{|z_1|^2 + |z_2|^2}\right)^{\frac{\gamma_0}{2}}, \tag{2.22}$$

for all $z_1, z_2 \in \mathbb{R}^{N\times n} \setminus \{0\}$ and some $\gamma_0 \in (0, 2-p)$, up to constants depending on (n, N, p), see [75, Sections 1-2]. Next, we fix $\varepsilon \in (0, L)$ and consider three cases: $0 < |z_1|, |z_2| < \varepsilon$, $0 < |z_1| < \varepsilon$ and $\varepsilon \le |z_2| \le 2L$, and $\varepsilon \le |z_1|, |z_2| \le 2L$. If $0 < |z_1|, |z_2| < \varepsilon$ or $0 < |z_1| < \varepsilon$ and $\varepsilon \le |z_2| \le 2L$, we recall that (2.16) and $(2.12)_1$ assure uniform continuity of $z \mapsto (\partial^2 F(z) - \partial^2(|z|^p/p))/|z|^{p-2}$ near the origin and of $z \mapsto \partial^2 F(z)$ on compact sets away from zero, with a concave modulus of continuity $\mu(\cdot)$ (that clearly can always be manipulated in such a way that $s \mapsto \mu(s)^2$ remains concave), so we bound via (2.22) and (2.20),

$$\begin{aligned}
|\partial^2 F(z_1) - \partial^2 F(z_2)| &\le |\partial^2(|z_1|^p/p) - \partial^2(|z_2|^p/p)| + |(\partial^2 F(z_1) - \partial^2(|z_1|^p/p)) - (\partial^2 F(z_2) - \partial^2(|z_2|^p/p))| \\
&\le c\left(\frac{|z_1|^2 + |z_2|^2}{|z_1|^2|z_2|^2}\right)^{\frac{2-p}{2}} \left(\frac{|z_2 - z_1|^2}{|z_1|^2 + |z_2|^2}\right)^{\frac{\gamma_0}{2}} \\
&\quad + |z_1|^{p-2}\left|\left(\frac{\partial^2 F(z_1) - \partial^2(|z_1|^p/p)}{|z_1|^{p-2}}\right) - \left(\frac{\partial^2 F(z_2) - \partial^2(|z_2|^p/p)}{|z_2|^{p-2}}\right)\right| \\
&\quad + \left||z_1|^{p-2} - |z_2|^{p-2}\right| \left(\frac{|\partial^2 F(z_2) - \partial^2(|z_2|^p/p)|}{|z_2|^{p-2}}\right) \\
&\le c\left(\frac{|z_1|^2 + |z_2|^2}{|z_1|^2|z_2|^2}\right)^{\frac{2-p}{2}} \left(\frac{|z_2 - z_1|^2}{|z_1|^2 + |z_2|^2}\right)^{\frac{\gamma_0}{2}} + |z_1|^{p-2}\mu(|z_1 - z_2|^2) \\
&\le c\left(\frac{|z_1|^2 + |z_2|^2}{|z_1|^2|z_2|^2}\right)^{\frac{2-p}{2}} \left(\frac{|z_2 - z_1|^2}{|z_1|^2 + |z_2|^2}\right)^{\frac{\gamma_0}{2}} + c\mu\left(\frac{|z_1 - z_2|^2}{|z_1|^2 + |z_2|^2}\right)\left(\frac{|z_1|^2 + |z_2|^2}{|z_1|^2|z_2|^2}\right)^{\frac{2-p}{2}},
\end{aligned}$$

where we also used that the composition of two concave, increasing functions is concave and increasing, that $|z_1|^{p-2}\left(|z_1|^2|z_2|^2(|z_1|^2 + |z_2|^2)^{-1}\right)^{(2-p)/2} \le 1$, and it is $c \equiv c(n, N, p, L)$. On the other hand, if $\varepsilon \le |z_1|, |z_2| \le 2L$, using $(2.12)_1$ and that $s \mapsto \mu(s)$ is increasing, we have

$$\begin{aligned}
|\partial^2 F(z_1) - \partial^2 F(z_1)| &\le \mu(|z_1 - z_2|^2) \le c\mu\left(\frac{|z_1 - z_2|^2}{|z_1|^2 + |z_2|^2}\right)\left(\frac{|z_1|^2 + |z_2|^2}{|z_1|^2|z_2|^2}\right)^{\pm\frac{2-p}{2}} \\
&\le c\mu\left(\frac{|z_1 - z_2|^2}{|z_1|^2 + |z_2|^2}\right)\left(\frac{|z_1|^2 + |z_2|^2}{|z_1|^2|z_2|^2}\right)^{\frac{2-p}{2}},
\end{aligned}$$

with $c \equiv c(n, N, p, L)$. The conclusion now follows by setting $\mu_L(s) := \max\left\{c\mu(s), s^{\frac{\gamma_0}{2}}\right\}$ and combining the two previous displays. □

Finally, the forcing term $f \colon \Omega \to \mathbb{R}^N$ displayed in (1.1) is such that

$$f \in L^m(\Omega, \mathbb{R}^N) \quad \text{with} \quad n > m > (p^*)' > 1 \tag{2.23}$$

which, together with $(2.13)_1$ yields:

$$1 < m' < p^* < \infty \qquad \text{and} \qquad f \in W^{1,p}(\Omega, \mathbb{R}^N)^*. \tag{2.24}$$

Assumption (2.23) should be interpreted as a minimal integrability requirement on the forcing term f, in the sense that it must at least belong to some intermediate Lebesgue space between $L^{(p^*)'}$ and L^n. The motivation behind this choice is twofold: the lower bound $m > (p^*)'$ assures that $f \in (W^{1,p})^*$, so that the linear functional $w \mapsto \int f \cdot w \, \mathrm{d}x$ is continuous on $W^{1,p}$; the upper bound $m < n$ reminds that in all the forthcoming estimates f should appear raised to a power strictly less than n - this will eventually contribute to the construction of the Wolff potential $\mathbf{I}^f_{1,m}(\cdot)$, which is well-behaved with respect to the embedding in Lorentz spaces exactly when $m < n$, cf. [57, Section 2.3].

Remark 2.1. When dealing with regularity issues for functionals with (p, q)-growth, it is in general necessary to control the size of the ratio q/p, that cannot be too far from one. In fact, a constraint of the type

$$\frac{q}{p} < 1 + \mathsf{o}(n) \tag{2.25}$$

turns out to be necessary and sufficient for guaranteeing regularity already in the case of (p, q)-nonuniformly elliptic equations [65, 66] - violations of such condition might cause rather striking irregularity phenomena, possibly affecting scalar problems [4, 37, 40]. Determining the optimal bound on the ratio q/p is an open question even for strictly convex, autonomous integrals with (p, q)-growth: the first results [65, 66] quantified the right-hand side of (2.25) as $\mathsf{o}(n) = 2/n$, that was later on updated in [11,12,19,74] to $\mathsf{o}(n) = 2/(n-1)$. So far, the only sharp results available cover zero-order regularity for minima of convex functionals with (p, q)-growth [48, 65], and gradient regularity for strictly convex, (p, q)-nonuniformly elliptic, nonautonomous functionals [4, 37, 40]. Concerning quasiconvex integrals, a constraint on the ratio q/p is needed already to secure basic semicontinuity properties [16, 39, 64]. More precisely, an autonomous quasiconvex functional with (p, q)-growth is $W^{1,p}$-sequentially weakly lower semicontinuous provided that $\mathsf{o}(n) = 1/(n-1)$ in (2.25). The very same constraint guarantees the applicability of extension lemmas based on the boundedness of certain trace preserving operators that improve the degree of integrability of a function and its gradient on suitable annuli, while conserving the function values elsewhere, [39, 46]. This lifting construction is fundamental to design comparison maps playing a crucial role when handling the delicate matter of partial regularity. In the setting of (p, q)-growing functionals, almost everywhere regularity for minima was first obtained in [75–77] (homogeneous case) assuming

$$\frac{q}{p} < 1 + \frac{1}{n \max\{p, 2\}}, \tag{2.26}$$

that is the same constraint imposed here, cf. (2.13). Notice that (2.26) yields two bounds that differ in the singular regime and in the degenerate one - in particular, when comparing the two cases, a natural deficit of $p/2$ occurs when passing from $p \geq 2$ to $1 < p < 2$. This typically happens also in the strictly convex setting [9, 11, 27, 29]. We further stress that, as in [75–77], condition (2.13) is used only when proving a Caccioppoli type inequality. In fact, due to the very rarefied structure of quasiconvex integrals and consequent lack of uniqueness for Dirichlet problems, we cannot apply any approximation procedure that artificially raises the integrand's growth and allows working within the (higher) energy class $W^{1,q}$. For this reason, to define appropriate test functions we need lifting theorems [39, 46], that force ratio q/p to be quite close to one, bound that is further reduced in order to derive various basic estimates, while remaining in $W^{1,p}$, the only available energy class. We further point out that constraint (2.26) has been recently relaxed in [46] to the essentially optimal range $q < \min\{np/(n-1), p+1\}$ [39] for nondegenerate, possibly signed quasiconvex integrals. In the light of the previous discussion, a restriction of type (2.25) plays a role only when proving the validity of Caccioppoli inequality, which in turn implies single-scale excess decay estimates, so, given that the nonhomogeneity is under control as soon as some ellipticity is available - and this is the case if a strict degenerate (resp. singular) or nondegenerate (resp. nonsingular) quasiconvexity condition is in force - we believe that our techniques can be adapted also to the class of variational integrals considered in [46]. We finally point out that Theorems 1-3 also hold in case of nondegenerate/nonsingular, strictly quasiconvex functionals and for their Lebesgue-Serrin-Marcellini extension with much easier proof due to the absence of the degenerate/singular alternative, see e.g. [38, 50, 75].

2.4. **Harmonic approximation lemmas.** This section is devoted to a quick overview of the main features of $\mathscr{A}$-harmonic maps and of p-harmonic maps. Let $\mathscr{A}$ be a constant bilinear form on $\mathbb{R}^{N\times n}$, elliptic in the sense of Legendre-Hadarmard i.e., satisfying

$$|\mathscr{A}| \leq H \qquad \text{and} \qquad \mathscr{A}\langle \xi \otimes \zeta, \xi \otimes \zeta \rangle \geq H^{-1}|\xi|^2|\zeta|^2, \tag{2.27}$$

for all $\zeta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^N$, with $H \geq 1$ being an absolute constant. An $\mathcal{A}$-harmonic map on an open set $\Omega \subset \mathbb{R}^n$ is a function $h \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega \mathcal{A}\langle Dh, D\varphi\rangle \, \mathrm{d}x = 0 \qquad \text{for all } \ \varphi \in C^\infty_c(\Omega, \mathbb{R}^N).$$

In [18, 33] we find that $\mathcal{A}$-harmonic maps have good regularity features; in fact for $B_\varrho(x_0) \Subset \Omega$ with $\varrho \in (0, 1]$ it holds that

$$\|Dh\|_{L^\infty(B_{\varrho/2}(x_0))} + \varrho\|D^2h\|_{L^\infty(B_{\varrho/2}(x_0))} \leq c \fint_{B_\varrho(x_0)} |Dh| \, \mathrm{d}x, \tag{2.28}$$

for $c \equiv c(n, N, H, d)$. We record an $\mathcal{A}$-harmonic approximation result from [35, Lemma 4], see also [33, Lemma 6].

Lemma 2.4. *Let $\mathcal{A}$ be a bilinear form on $\mathbb{R}^{N\times n}$ verifying (2.27), $B_\varrho(x_0) \Subset \Omega$ be a ball and $p > 1$ be a number. For any $\varepsilon > 0$ there exists $\delta \equiv \delta(n, N, H, p, \varepsilon) \in (0, 1]$ such that if $v \in W^{1,p}(B_\varrho(x_0), \mathbb{R}^N)$ with $\mathfrak{I}_2(V_{1,p}(Dv); B_\varrho(x_0)) \leq \sigma \leq 1$ is approximately $\mathcal{A}$-harmonic in the sense that*

$$\left| \fint_{B_\varrho(x_0)} \mathcal{A}\langle Dv, D\varphi\rangle \, \mathrm{d}x \right| \leq \sigma\delta\|D\varphi\|_{L^\infty(B_\varrho(x_0))} \qquad \text{for all } \ \varphi \in C^\infty_c(B_\varrho(x_0), \mathbb{R}^N),$$

then there exists an $\mathcal{A}$-harmonic map $h \in W^{1,p}(B_\varrho(x_0), \mathbb{R}^N)$ such that

$$\fint_{B_\varrho(x_0)} |V_{1,p}(Dh)|^2 \, \mathrm{d}x \leq c \qquad \text{and} \qquad \fint_{B_\varrho(x_0)} \left| V_{1,p}\left(\frac{v - \sigma h}{\varrho}\right) \right|^2 \, \mathrm{d}x \leq c\sigma^2\varepsilon,$$

for $c \equiv c(n, N, p)$.

We further recall the definition of p-harmonic map, i.e. a function $h \in W^{1,p}(\Omega, \mathbb{R}^N)$ satisfying

$$\int_\Omega \langle |Dh|^{p-2}Dh, D\varphi\rangle \, \mathrm{d}x = 0 \qquad \text{for all } \ \varphi \in C^\infty_c(\Omega, \mathbb{R}^N).$$

According to the regularity theory contained in [83, 84], whenever $B_\varrho(x_0) \Subset B_r(x_0) \Subset \Omega$ are concentric balls, it is

$$\|Dh\|_{L^\infty(B_{\varrho/2}(x_0))} \leq c' \left(\fint_{B_\varrho(x_0)} |Dh|^p \, \mathrm{d}x\right)^{1/p} \qquad \text{and} \qquad \mathfrak{F}(h; B_\varrho(x_0)) \leq c'' \left(\frac{\varrho}{r}\right)^\alpha \mathfrak{F}(h; B_r(x_0)), \tag{2.29}$$

with $c', c'' \equiv c', c''(n, N, p)$ and $\alpha \equiv \alpha(n, N, p) \in (0, 1)$. As a "singular" variant of Lemma 2.4, we have the following p-harmonic approximation lemma from [36, Lemma 1].

Lemma 2.5. *Let $p \in (1, \infty)$ be a number and $B_\varrho(x_0) \Subset \Omega$ be any ball. For all $\varepsilon > 0$, there exists $\delta \equiv \delta(n, N, p, \varepsilon) \in (0, 1]$ such that if $v \in W^{1,p}(B_\varrho(x_0), \mathbb{R}^N)$ with $\mathfrak{I}_p(Dv; B_\varrho(x_0)) \leq 1$ is approximately p-harmonic in the sense that*

$$\left| \fint_{B_\varrho(x_0)} \langle |Dv|^{p-2}Dv, D\varphi\rangle \, \mathrm{d}x \right| \leq \delta\|D\varphi\|_{L^\infty(B_\varrho(x_0))} \qquad \text{for all } \ \varphi \in C^\infty_c(B_\varrho(x_0), \mathbb{R}^N), \tag{2.30}$$

then there exits a p-harmonic map $h \in W^{1,p}(B_\varrho(x_0), \mathbb{R}^N)$ such that

$$\mathfrak{I}_p(Dh; B_\varrho(x_0)) \leq 1 \qquad \text{and} \qquad \fint_{B_\varrho(x_0)} \left| \frac{v - h}{\varrho} \right|^p \, \mathrm{d}x \leq c\varepsilon^p,$$

with $c \equiv c(n, N, p)$.

2.5. **On the Lebesgue-Serrin-Marcellini extension.** Let $\mathbb{B}_\Omega$ be the family of all open subsets of Ω and $B \in \mathbb{B}_\Omega$. For $1 < p \leq q < \infty$, an integrand $F \in C(\mathbb{R}^{N\times n})$, and maps $f \in W^{1,p}(\Omega, \mathbb{R}^N)^*$ and $w \in W^{1,p}(\Omega, \mathbb{R}^N)$, the Lebesgue-Serrin-Marcellini extension of a functional of type (1.1), i.e.:

$$\mathcal{F}(w; B) := \int_B [F(Dw) - f \cdot w] \, \mathrm{d}x =: \mathcal{F}_0(w; B) - \int_B f \cdot w \, \mathrm{d}x,$$

is defined as

$$\bar{\mathcal{F}}(w; B) := \inf_{\{w_j\}_{j\in\mathbb{N}} \in \mathcal{C}(w;B)} \liminf_{j\to\infty} \mathcal{F}(w_j; B),$$

with

$$\mathcal{C}(w; B) := \left\{ \{w_j\}_{j\in\mathbb{N}} \subset W^{1,q}_{\mathrm{loc}}(B, \mathbb{R}^N) \cap W^{1,p}(B, \mathbb{R}^N) \colon w_j \rightharpoonup w \text{ weakly in } W^{1,p}(B, \mathbb{R}^N) \right\}.$$

By density of smooth maps in $W^{1,p}(B, \mathbb{R}^N)$ it is $\mathcal{C}(w; B) \neq \emptyset$ and since in particular $f \in W^{1,p}(B, \mathbb{R}^N)^*$ we can rewrite

$$\bar{\mathcal{F}}(w; B) = \bar{\mathcal{F}}_0(w; B) - \int_B f \cdot w \, \mathrm{d}x, \tag{2.31}$$

therefore while describing the relevant features of relaxation we shall refer to the "bulk" component of $\bar{\mathcal{F}}(\cdot)$, i.e. $\bar{\mathcal{F}}_0(\cdot)$. A first crucial observation is that $\bar{\mathcal{F}}_0(\cdot)$ cannot be represented as an integral. In fact, a deep result from [16, 39] states that with $F(z) \lesssim (1+|z|^q)$ and $1<p\le q<np/(n-1)$, each $w\in W^{1,p}(\Omega,\mathbb{R}^N)$ such that $\bar{\mathcal{F}}_0(w;\Omega)<\infty$ uniquely determines a finite outer Radon measure μ_w verifying

$$\bar{\mathcal{F}}_0(w;\cdot)=\mu_w|_{\mathbb{B}_\Omega} \qquad \text{and} \qquad \frac{\mathrm{d}\mu_w}{\mathrm{d}\mathscr{L}^n}=QF(Dw). \tag{2.32}$$

Here $QF(\cdot)$ denotes the quasiconvex envelope of $F(\cdot)$, see [42, Section 5.3]. Moreover, a $W^{1,p}$-coercivity condition like $|z|^p \lesssim F(z)$ assures sequential lower semicontinuity[3] of $\bar{\mathcal{F}}_0(\cdot;\Omega)$ with respect to the weak topology of $W^{1,p}(\Omega,\mathbb{R}^N)$, cf. [39, 76]. In [6] it is proven that $W^{1,p}$-quasiconvexity is necessary for this semicontinuity property. However, the results of [6] hold for integral functionals, while, in the light of (2.32), $\bar{\mathcal{F}}_0(\cdot)$ cannot be represented as an integral. Despite the measure representation [39], the arguments developed in [6] can be adapted to prove that $\bar{\mathcal{F}}_0(\cdot)$ features the proper notion of $W^{1,p}$-quasiconvexity, i.e.: $\bar{\mathcal{F}}_0(\ell+\varphi;B)\ge\bar{\mathcal{F}}_0(\ell;B)$ holds for all $B\in\mathbb{B}_\Omega$, $\varphi\in W^{1,p}(B,\mathbb{R}^N)$ with $\operatorname{supp}(\varphi)\Subset B$ and any affine function $\ell(x):=v_0+\langle z_0,x-x_0\rangle$, cf. [76]. Other remarkable properties of $\bar{\mathcal{F}}_0(\cdot)$ such as additivity and extremality conditions can be found in [76,77]. Now, if $F(\cdot)$ is a continuous and $W^{1,p}$-coercive integrand and $f\in W^{1,p}(\Omega,\mathbb{R}^N)^*$, the weak sequential lower semicontinuity of $\bar{\mathcal{F}}_0(\cdot)$ in $W^{1,p}(\Omega,\mathbb{R}^N)$ and direct methods assure that once fixed a boundary datum $u_0\in W^{1,p}(\Omega,\mathbb{R}^N)$ such that $\bar{\mathcal{F}}_0(u_0;\Omega)<\infty$ - recall (2.31) - there exists a local minimizer $u\in u_0+W^{1,p}_0(\Omega,\mathbb{R}^N)$ of (1.2) in the sense of Definition 1. If in addition $F\in C^1_{\mathrm{loc}}(\mathbb{R}^{N\times n})$ with (1.4), $(2.12)_2$ and (2.23) in force and exponents (p,q) satisfying $1<p\le q<\min\{np/(n-1),p+1\}$, then any local minimizer $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ of (1.2) verifies by minimality the integral identity

$$0=\int_\Omega \left[\langle \partial F(Du),D\varphi\rangle - f\cdot\varphi\right]\,\mathrm{d}x \qquad \text{for all } \ \varphi\in C^\infty_c(\Omega,\mathbb{R}^N), \tag{2.33}$$

see [26, Section 2.7] and [76, Section 7.1].

3. Caccioppoli inequality

We start by recording a variation obtained in [75, Lemmas 6.3-6.5] and [76, Lemmas 4.4 and 4.6] of the extension result from [39], which will be crucial for constructing comparison maps for minima of (1.2).

Lemma 3.1. *Let $0<\tau_1<\tau_2$ be two numbers and $B_{\tau_2}\Subset\Omega$ be a ball. There exists a bounded, linear smoothing operator $\mathfrak{T}_{\tau_1,\tau_2}\colon W^{1,1}(\Omega,\mathbb{R}^N)\to W^{1,1}(\Omega,\mathbb{R}^N)$ defined as*

$$W^{1,1}(\Omega,\mathbb{R}^N)\ni w\mapsto \mathfrak{T}_{\tau_1,\tau_2}[w](x):=\fint_{B_1(0)} w(x+\vartheta(x)y)\ \mathrm{d}y,$$

where it is $\vartheta(x):=\frac{1}{2}\max\left\{\min\left\{|x|-\tau_1,\tau_2-|x|\right\},0\right\}$. If $w\in W^{1,p}(\Omega,\mathbb{R}^N)$ for some $p\ge1$, the map $\mathfrak{T}_{\tau_1,\tau_2}[w]$ has the following properties:

- *(i.) $\mathfrak{T}_{\tau_1,\tau_2}[w]\in W^{1,p}(\Omega,\mathbb{R}^N)$;*
- *(ii.) $w=\mathfrak{T}_{\tau_1,\tau_2}[w]$ almost everywhere on $(\Omega\setminus B_{\tau_2})\cup B_{\tau_1}$;*
- *(iii.) $\mathfrak{T}_{\tau_1,\tau_2}[w]\in w+W^{1,p}_0(B_{\tau_2}\setminus\bar{B}_{\tau_1},\mathbb{R}^N)$;*
- *(iv.) $|D\mathfrak{T}_{\tau_1,\tau_2}[w]|\le c(n)\mathfrak{T}_{\tau_1,\tau_2}[|Dw|]$ almost everywhere in Ω.*

Furthermore,

$$\begin{cases} \|\mathfrak{T}_{\tau_1,\tau_2}[w]\|_{L^p(B_{\tau_2}\setminus B_{\tau_1})}\le c\|w\|_{L^p(B_{\tau_2}\setminus B_{\tau_1})}\\ \|D\mathfrak{T}_{\tau_1,\tau_2}[w]\|_{L^p(B_{\tau_2}\setminus B_{\tau_1})}\le c\|Dw\|_{L^p(B_{\tau_2}\setminus B_{\tau_1})}\\ \|D\mathfrak{T}_{\tau_1,\tau_2}[w]\|_{L^p(B_\varsigma\setminus B_{\tau_1})}\le c\|Dw\|_{L^p(B_{2\varsigma-\tau_1}\setminus B_{\tau_1})} \quad \text{for } \ \tau_1\le\varsigma\le(\tau_1+\tau_2)/2\\ \|D\mathfrak{T}_{\tau_1,\tau_2}[w]\|_{L^p(B_{\tau_2}\setminus B_\varsigma)}\le c\|Dw\|_{L^p(B_{\tau_2}\setminus B_{2\varsigma-\tau_2})} \quad \text{for } \ (\tau_1+\tau_2)/2\le\varsigma\le\tau_2, \end{cases} \tag{3.1}$$

for $c\equiv c(n,p)$. Finally, let $\mathcal{N}\subset\mathbb{R}$ be a set with zero Lebesgue measure. There are

$$\tilde{\tau}_1\in\left(\tau_1,\frac{2\tau_1+\tau_2}{3}\right)\setminus\mathcal{N},\ \tilde{\tau}_2\in\left(\frac{\tau_1+2\tau_2}{3},\tau_2\right)\setminus\mathcal{N} \quad \text{verifying} \ \ (\tau_2-\tau_1)\approx(\tilde{\tau}_2-\tilde{\tau}_1) \tag{3.2}$$

up to absolute constants, such that if $1\le p\le 2$, $s\ge0$ and $\frac{2}{p}\le d<\frac{2n}{n-1}$, it is

$$\|V_{s,p}(D\mathfrak{T}_{\tilde{\tau}_1,\tilde{\tau}_2}[w])\|_{L^d(B_{\tilde{\tau}_2}\setminus B_{\tilde{\tau}_1})}\le\frac{c}{(\tau_2-\tau_1)^{n\left(\frac{1}{2}-\frac{1}{d}\right)}}\|V_{s,p}(Dw)\|_{L^2(B_{\tau_2}\setminus B_{\tau_1})}, \tag{3.3}$$

for $c\equiv c(n,p,d)$. Clearly, operator $\mathfrak{T}_{\tilde{\tau}_1,\tilde{\tau}_2}$ satisfies properties (i.)-(iv.) and (3.1) with $\tilde{\tau}_1$, $\tilde{\tau}_2$ substituting τ_1, τ_2.

In the next lemma we derive a preliminary version of Caccioppoli inequality that will be eventually adjusted depending on the singular/nonsingular behavior of $\bar{\mathcal{F}}(\cdot)$.

[3]Recall that we always work under the assumption that Ω is an open, bounded domain with Lipschitz boundary.

Lemma 3.2. *Assume (2.12)-(2.14), (2.16) and (2.23), let* $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ *be a local minimizer of (1.2),* $B_\varrho(x_0) \Subset \Omega$ *be any ball, and* $\ell(x) := v_0+\langle z_0, x-x_0\rangle$ *be an affine function with* $z_0 \in \left\{z \in \mathbb{R}^{N\times n} \colon |z| \le \left(80000(M+1)\right)^{2/p}\right\}$ *for some positive constant* M*, and* $v_0 \in \mathbb{R}^N$*. Then*

$$
\begin{aligned}
\tilde{\mathfrak{F}}(u,z_0;B_{\varrho/2}(x_0))^2 \ \le\ & c\mathfrak{K}\left(\fint_{B_\varrho(x_0)}\left|V_{|z_0|,p}\left(\frac{u-\ell}{\varrho}\right)\right|^2 \,\mathrm{d}x\right)+c\mathbb{1}_{\{q>p\}}\tilde{\mathfrak{F}}(u,z_0;B_\varrho(x_0))^{2q/p}\\
&+c\left[\left(\varrho^m\fint_{B_\varrho(x_0)}|f|^m\,\mathrm{d}x\right)^{\frac{p}{m(p-1)}}+|z_0|^{2-p}\left(\varrho^m\fint_{B_\varrho(x_0)}|f|^m\,\mathrm{d}x\right)^{2/m}\right]
\end{aligned}
\tag{3.4}
$$

holds with $c \equiv c(\mathtt{data}, M)$ *and* $\mathfrak{K}(\cdot)$ *being defined in (2.2).*

Proof. With $\varrho/2 \le \tau_1 < \tau_2 \le \varrho$ being parameters, $\tilde\tau_1,\tilde\tau_2$ as in (3.2)[4], let $\eta \in C^1_c(B_{\tilde\tau_2}(x_0))$ be a cut-off function satisfying

$$\mathbb{1}_{B_{\tilde\tau_1}(x_0)} \le \eta \le \mathbb{1}_{B_{\tilde\tau_2}(x_0)}, \qquad |D\eta| \lesssim \frac{1}{(\tilde\tau_2-\tilde\tau_1)},$$

set $S(x_0) := B_{\tau_2}(x_0)\setminus B_{\tau_1}(x_0)$, $\tilde S(x_0) := B_{\tilde\tau_2}(x_0)\setminus B_{\tilde\tau_1}(x_0)$, $\mathfrak{u}(x) := u(x)-\ell(x)$ and introduce the comparison maps

$$\varphi_1(x) := \mathfrak{T}_{\tilde\tau_1,\tilde\tau_2}[(1-\eta)\mathfrak{u}](x), \qquad \varphi_2(x) := \mathfrak{u}(x)-\varphi_1(x).$$

By Lemma 3.1 (*ii.*)-(*iii.*) it is

$$\varphi_1 \equiv 0 \ \text{ on } \ B_{\tilde\tau_1}(x_0), \quad \varphi_2 \in W^{1,p}_0(B_{\tilde\tau_2}(x_0),\mathbb{R}^N), \quad \varphi_2 \equiv \mathfrak{u} \ \text{ on } \ B_{\tilde\tau_1}(x_0), \quad D\mathfrak{u} = D\varphi_1+D\varphi_2. \tag{3.5}$$

Moreover, by (2.12)-(2.14), (2.23) and Lemma 3.1, we see that the construction developed in [26, Lemma 3.1], [76, Lemma 7.13] applies to our setting as well and renders:

$$
\begin{aligned}
c\int_{B_{\tilde\tau_2}(x_0)}|V_{|z_0|,p}(D\varphi_2)|^2\,\mathrm{d}x \ \le\ & \int_{\tilde S(x_0)}\left[F(Du-D\varphi_1)-F(Du)\right]\,\mathrm{d}x\\
&+\int_{\tilde S(x_0)}\left[F(z_0+D\varphi_1)-F(z_0)\right]\,\mathrm{d}x\\
&+\int_{B_{\tilde\tau_2}(x_0)} f\cdot\varphi_2\,\mathrm{d}x =: \left[(\mathrm{I})+(\mathrm{II})+(\mathrm{III})\right],
\end{aligned}
$$

with $c \equiv c(n,N,p,q,\lambda,\Lambda)$. Before proceeding further, let us notice that

$$\mathcal{D}_{|z_0|}(z_1,z_2)^{(q-p)/2} \lesssim \left(\mathcal{D}_{|z_0|}(z_1,z_2)^{(p-2)/2}\mathcal{D}_0(z_1,z_2)\right)^{(q-p)/p}+|z_0|^{q-p}, \tag{3.6}$$

for all $z_0,z_1,z_2 \in \mathbb{R}^{N\times n}$, where $\mathcal{D}_{|z_0|}(\cdot)$ has been defined in Section 2.1 and the constants implicit in "$\approx$, $\lesssim$" depend on (n,N,p,q), cf. [77, page 256]. We then rearrange:

$$
\begin{aligned}
(\mathrm{I})+(\mathrm{II}) \ =\ & \int_{\tilde S(x_0)}\left\langle\left(\int_0^1\left[\partial F(z_0)-\partial F(z_0+D\mathfrak{u}-sD\varphi_1)\right]\,\mathrm{d}s\right), D\varphi_1\right\rangle\,\mathrm{d}x\\
&+\int_{\tilde S(x_0)}\left\langle\left(\int_0^1\left[\partial F(z_0+sD\varphi_1)-\partial F(z_0)\right]\,\mathrm{d}s\right), D\varphi_1\right\rangle\,\mathrm{d}x =: (\mathrm{I'})+(\mathrm{II'}),
\end{aligned}
$$

and estimate (keep in mind the upper bound on $|z_0|$),

$$
\begin{aligned}
|(\mathrm{I'})| \ \overset{(2.18)_2}{\le}\ & c\int_{\tilde S(x_0)}\int_0^1\left[\frac{|V_{|z_0|,p}(D\mathfrak{u}-sD\varphi_1)|^2}{|D\mathfrak{u}-sD\varphi_1|}+\frac{|V_{|z_0|,q}(D\mathfrak{u}-sD\varphi_1)|^2}{|D\mathfrak{u}-sD\varphi_1|}\right]\mathrm{d}s|D\varphi_1|\,\mathrm{d}x\\
\overset{(2.3)_{1,5},(2.4)}{\le}\ & c\int_{\tilde S(x_0)}\left(\frac{|V_{|z_0|,p}(D\mathfrak{u})|^2|D\varphi_1|}{|D\mathfrak{u}|}+|V_{|z_0|,p}(D\varphi_1)|^2\right)\mathrm{d}x\\
&+c\int_{\tilde S(x_0)}\left(\int_0^1(|z_0|^2+|D\mathfrak{u}-sD\varphi_1|^2)^{(q-2)/2}\,\mathrm{d}s\right)(|D\mathfrak{u}|+|D\varphi_1|)|D\varphi_1|\,\mathrm{d}x\\
\overset{(2.3)_4,(2.4)}{\le}\ & c\int_{\tilde S(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2+|V_{|z_0|,p}(D\varphi_1)|^2\,\mathrm{d}x\\
&+c\int_{\tilde S(x_0)}(|z_0|^2+|D\mathfrak{u}|^2+|D\varphi_1|^2)^{(q-2)/2}(|D\mathfrak{u}|+|D\varphi_1|)|D\varphi_1|\,\mathrm{d}x
\end{aligned}
$$

---

[4]The negligible set $\mathcal{N}$ is the set of nondifferentiability points of $(\tau_1,\tau_2) \ni t \mapsto \|Du\|^p_{L^p(B_t(x_0))}$, cf. [26, Lemma 3.1] or in [76, Lemma 7.13].

$$
\begin{aligned}
&\overset{(3.6)}{\leq} && c\int_{\tilde{S}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2+|V_{|z_0|,p}(D\varphi_1)|^2 \ \mathrm{d}x\\
& && +c\int_{\tilde{S}(x_0)}|z_0|^{q-p}\mathcal{D}_{|z_0|}(D\mathfrak{u},D\varphi_1)^{(p-2)/2}(|D\mathfrak{u}|+|D\varphi_1|)|D\varphi_1| \ \mathrm{d}x\\
& && +c\int_{\tilde{S}(x_0)}\mathcal{D}_{|z_0|}(D\mathfrak{u},D\varphi_1)^{\frac{q(p-2)}{2p}}\mathcal{D}_0(D\mathfrak{u},D\varphi_1)^{(q-p)/p}(|D\mathfrak{u}|+|D\varphi_1|)|D\varphi_1| \ \mathrm{d}x\\
&\overset{(2.13)_1}{\leq} && c\int_{\tilde{S}(x_0)}(1+|z_0|^{q-p})\left(|V_{|z_0|,p}(D\mathfrak{u})|^2+|V_{|z_0|,p}(D\varphi_1)|^2\right) \ \mathrm{d}x\\
& && +c\int_{\tilde{S}(x_0)}\mathcal{D}_{|z_0|}(D\mathfrak{u},D\varphi_1)^{\frac{q(p-2)}{2p}}\left(|D\mathfrak{u}|^{\frac{2q}{p}-1}|D\varphi_1|+|D\varphi_1|^{2q/p}\right) \ \mathrm{d}x\\
&\leq && c\int_{\tilde{S}(x_0)}(1+|z_0|^{q-p})\left(|V_{|z_0|,p}(D\mathfrak{u})|^2+|V_{|z_0|,p}(D\varphi_1)|^2\right) \ \mathrm{d}x\\
& && +c\int_{\tilde{S}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^{\frac{2q}{p}-1}|V_{|z_0|,p}(D\varphi_1)|+|V_{|z_0|,p}(D\varphi_1)|^{2q/p} \ \mathrm{d}x\\
&\overset{(3.3)}{\leq} && c(1+|z_0|^{q-p})\int_{\tilde{S}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2+\left|V_{|z_0|,p}\left(\frac{\mathfrak{u}}{\tau_2-\tau_1}\right)\right|^2 \ \mathrm{d}x\\
& && +c\left(\int_{\tilde{S}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2 \ \mathrm{d}x\right)^{(2q-p)/2p}\left(\int_{\tilde{S}(x_0)}|V_{|z_0|,p}(D\varphi_1)|^{\frac{2p}{3p-2q}} \ \mathrm{d}x\right)^{\frac{3p-2q}{2p}}\\
& && +\frac{c\mathbb{1}_{\{q>p\}}}{(\tau_2-\tau_1)^{n\left(\frac{q}{p}-1\right)}}\left(\int_{S(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2+\left|V_{|z_0|,p}\left(\frac{\mathfrak{u}}{\tau_2-\tau_1}\right)\right|^2 \ \mathrm{d}x\right)^{q/p}\\
&\overset{(2.13)}{\leq} && c(1+|z_0|^{q-p})\int_{\tilde{S}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2+\left|V_{|z_0|,p}\left(\frac{\mathfrak{u}}{\tau_2-\tau_1}\right)\right|^2 \ \mathrm{d}x\\
& && +\frac{c\mathbb{1}_{\{q>p\}}}{(\tau_2-\tau_1)^{n\left(\frac{q}{p}-1\right)}}\left(\int_{S(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2+\left|V_{|z_0|,p}\left(\frac{\mathfrak{u}}{\tau_2-\tau_1}\right)\right|^2 \ \mathrm{d}x\right)^{q/p},
\end{aligned}
$$

where we also used $(2.18)_2$ with $L\equiv 80000(M+1)$, and exploited that by $(2.13)_2$ it is $\max\left\{2q/p,2p/(3p-2q)\right\}<2n/(n-1)$, $\min\left\{2q/p,2p/(3p-2q)\right\}\geq 2/p$ and $c\equiv c(n,N,\lambda,\Lambda,p,q,M,F(\cdot))$. In a totally similar way we bound:

$$
\begin{aligned}
|(\mathrm{II'})| &\overset{(2.18)_2,(2.3)_5}{\leq} && c\int_{\tilde{S}(x_0)}|V_{|z_0|,p}(D\varphi_1)|^2+(|z_0|^2+|D\varphi_1|^2)^{(q-2)/2}|D\varphi_1|^2 \ \mathrm{d}x\\
&\overset{(3.6),(3.3)}{\leq} && c(1+|z_0|^{q-p})\int_{\tilde{S}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2+\left|V_{|z_0|,p}\left(\frac{\mathfrak{u}}{\tau_2-\tau_1}\right)\right|^2 \ \mathrm{d}x\\
& && +c\int_{\tilde{S}(x_0)}|V_{|z_0|,p}(D\varphi_1)|^{2q/p} \ \mathrm{d}x\\
&\overset{(2.13),(3.3)}{\leq} && c(1+|z_0|^{q-p})\int_{\tilde{S}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2+\left|V_{|z_0|,p}\left(\frac{\mathfrak{u}}{\tau_2-\tau_1}\right)\right|^2 \ \mathrm{d}x\\
& && +\frac{c\mathbb{1}_{\{q>p\}}}{(\tau_2-\tau_1)^{n\left(\frac{q}{p}-1\right)}}\left(\int_{S(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2+\left|V_{|z_0|,p}\left(\frac{\mathfrak{u}}{\tau_2-\tau_1}\right)\right|^2 \ \mathrm{d}x\right)^{q/p}
\end{aligned}
$$

for $c\equiv c(n,N,\lambda,\Lambda,p,q,M,F(\cdot))$. Concerning term (III), we use (2.23), (2.24), $(3.5)_2$, Sobolev-Poincaré inequality, Young inequality and (2.5) with $s=|z_0|$, $z_1=0$, $z_2=D\varphi_2$ to estimate

$$
\begin{aligned}
|(\mathrm{III})| &\leq c|B_{\tilde{\tau}_2}(x_0)|\left(\tilde{\tau}_2^m\fint_{B_{\tilde{\tau}_2}(x_0)}|f|^m \ \mathrm{d}x\right)^{1/m}\left(\fint_{B_{\tilde{\tau}_2}(x_0)}|D\varphi_2|^p \ \mathrm{d}x\right)^{1/p}\\
&\leq c|B_{\tilde{\tau}_2}(x_0)|\left(\tilde{\tau}_2^m\fint_{B_{\tilde{\tau}_2}(x_0)}|f|^m \ \mathrm{d}x\right)^{1/m}\left(\fint_{B_{\tilde{\tau}_2}(x_0)}|V_{|z_0|,p}(D\varphi_2)|^2 \ \mathrm{d}x\right)^{1/p}\\
&\quad +|B_{\tilde{\tau}_2}(x_0)|\left(\tilde{\tau}_2^m\fint_{B_{\tilde{\tau}_2}(x_0)}|f|^m \ \mathrm{d}x\right)^{1/m}\left(\fint_{B_{\tilde{\tau}_2}(x_0)}|V_{|z_0|,p}(D\varphi_2)|^p|z_0|^{p(2-p)/2} \ \mathrm{d}x\right)^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{4}\int_{B_{\tilde{\tau}_2}(x_0)}|V_{|z_0|,p}(D\varphi_2)|^2\ \mathrm{d}x + c|B_{\tilde{\tau}_2}(x_0)|\left(\tilde{\tau}_2^m \fint_{B_{\tilde{\tau}_2}(x_0)}|f|^m\ \mathrm{d}x\right)^{\frac{p}{m(p-1)}}\\
&\quad +c|B_{\tilde{\tau}_2}(x_0)||z_0|^{2-p}\left(\tilde{\tau}_2^m \fint_{B_{\tilde{\tau}_2}(x_0)}|f|^m\ \mathrm{d}x\right)^{2/m},
\end{aligned}
$$

with $c \equiv c(n,N,p,m)$. Merging the content of the previous displays, reabsorbing terms and recalling $(3.5)_3$, (3.2) and the upper bound imposed on the size of $|z_0|$, we obtain

$$
\begin{aligned}
\int_{B_{\tau_1}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2\ \mathrm{d}x &\leq c\int_{B_{\tau_2}(x_0)\setminus B_{\tau_1}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2 + \left|V_{|z_0|,p}\left(\frac{\mathfrak{u}}{\tau_2-\tau_1}\right)\right|^2\ \mathrm{d}x\\
&\quad + \frac{c\mathbb{1}_{\{q>p\}}}{(\tau_2-\tau_1)^{n\left(\frac{q}{p}-1\right)}}\left(\int_{B_{\tau_2}(x_0)\setminus B_{\tau_1}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2 + \left|V_{|z_0|,p}\left(\frac{\mathfrak{u}}{\tau_2-\tau_1}\right)\right|^2\ \mathrm{d}x\right)^{q/p}\\
&\quad + c|B_{\tau_2}(x_0)|\left[\left(\tau_2^m\fint_{B_{\tau_2}(x_0)}|f|^m\ \mathrm{d}x\right)^{\frac{p}{m(p-1)}} + |z_0|^{2-p}\left(\tau_2^m\fint_{B_{\tau_2}(x_0)}|f|^m\ \mathrm{d}x\right)^{2/m}\right],
\end{aligned}
$$

for $c \equiv c(\mathtt{data}, M)$. We then sum to both sides of the above inequality the quantity $c\int_{B_{\tau_1}(x_0)}|V_{|z_0|,p}(D\mathfrak{u})|^2\ \mathrm{d}x$ and use Lemma 2.2 to conclude with (3.4). □

4. The nonsingular regime

Let us prove the approximate $\mathscr{A}$-harmonic character of minima of (1.2) within the nonsingular scenario.

Lemma 4.1. *Under assumptions* (2.12)-(2.14), (2.16) *and* (2.23), *let* $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ *be a local minimizer of* (1.2), $B_\varrho(x_0) \Subset \Omega$ *be a ball and* $z_0 \in \left(\mathbb{R}^{N\times n}\setminus\{0\}\right)\cap\left\{|z| \leq \left(80000(M+1)\right)^{2/p}\right\}$ *for some positive constant* M *be any matrix such that* $\tilde{\mathfrak{F}}(u,z_0;B_\varrho(x_0)) > 0$. *Then*

$$
\begin{aligned}
\left|\fint_{B_\varrho(x_0)}\frac{\partial^2F(z_0)}{|z_0|^{p-2}}\left\langle\frac{|z_0|^{(p-2)/2}(Du-z_0)}{\tilde{\mathfrak{F}}(u,z_0;B_\varrho(x_0))}, D\varphi\right\rangle\ \mathrm{d}x\right| &\leq \frac{c|z_0|^{(2-p)/2}\|D\varphi\|_{L^\infty(B_\varrho(x_0))}}{\tilde{\mathfrak{F}}(u,z_0;B_\varrho(x_0))}\left(\varrho^m\fint_{B_\varrho(x_0)}|f|^m\ \mathrm{d}x\right)^{1/m}\\
&\quad + c\left[\mu_M\left(\frac{\tilde{\mathfrak{F}}(u,z_0;B_\varrho(x_0))^2}{|z_0|^p}\right) + \left(\frac{\tilde{\mathfrak{F}}(u,z_0;B_\varrho(x_0))^2}{|z_0|^p}\right)^{(2\kappa-1)/2}\right]\|D\varphi\|_{L^\infty(B_\varrho(x_0))},
\end{aligned}
$$

where $\kappa := (q-1)/p$ *if* $q \geq 2$ *and* $\kappa := 1/p$ *when* $1 < q < 2$, *and* $c \equiv c(\mathtt{data}, M)$.

Proof. Let $\varphi \in C^\infty_c(B_\varrho(x_0),\mathbb{R}^N)$ be a map and, for the ease of reading, let us shorten $B_\varrho(x_0) \equiv B_\varrho$, $\tilde{\mathfrak{F}}(u,z_0,B_\varrho(x_0)) \equiv \tilde{\mathfrak{F}}_0(u)$, $\ell(x) := v_0 + \langle z_0, x-x_0\rangle$ with $v_0 \in \mathbb{R}^N$, $\mathfrak{u} := u-\ell$ and $\|D\varphi\|_{L^\infty(B_\varrho(x_0))} \equiv \|D\varphi\|_\infty$. Set

$$
B^- := B_\varrho \cap \left\{|D\mathfrak{u}| \leq |z_0|\right\}, \qquad B^+ := B_\varrho \cap \left\{|D\mathfrak{u}| > |z_0|\right\}
$$

and bound

$$
\begin{aligned}
\mathscr{I} &:= \left|\fint_{B_\varrho}\partial^2F(z_0)\langle Du-z_0, D\varphi\rangle\ \mathrm{d}x\right|\\
&\overset{(2.33)}{=} \left|\fint_{B_\varrho}\partial^2F(z_0)\langle Du-z_0, D\varphi\rangle - \langle\partial F(Du)-\partial F(z_0), D\varphi\rangle\ \mathrm{d}x + \fint_{B_\varrho(x_0)} f\cdot\varphi\ \mathrm{d}x\right|\\
&\leq \fint_{B_\varrho}|\partial^2F(z_0)\langle Du-z_0, D\varphi\rangle - \langle\partial F(Du)-\partial F(z_0), D\varphi\rangle|\ \mathrm{d}x + \fint_{B_\varrho(x_0)}|f\cdot\varphi|\ \mathrm{d}x\\
&= \frac{1}{|B_\varrho|}\int_{B^-}|\partial^2F(z_0)\langle Du-z_0, D\varphi\rangle - \langle\partial F(Du)-\partial F(z_0), D\varphi\rangle|\ \mathrm{d}x\\
&\quad + \frac{1}{|B_\varrho|}\int_{B^+}|\partial^2F(z_0)\langle Du-z_0, D\varphi\rangle - \langle\partial F(Du)-\partial F(z_0), D\varphi\rangle|\ \mathrm{d}x\\
&\quad + \fint_{B_\varrho(x_0)}|f\cdot\varphi|\ \mathrm{d}x =: \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3,
\end{aligned}
$$

where we also used that $\int_{B_\varrho}\langle\partial F(z_0), D\varphi\rangle\ \mathrm{d}x = 0$. We then estimate

$$
\mathscr{I}_1 \leq \frac{\|D\varphi\|_\infty}{|B_\varrho|}\int_{B^-}\left(\int_0^1|\partial^2F(z_0)-\partial^2F(z_0+sD\mathfrak{u})|\ \mathrm{d}s\right)|D\mathfrak{u}|\ \mathrm{d}x
$$

$$\begin{aligned}
&\overset{(2.21)}{\leq} \frac{c\|D\varphi\|_\infty}{|B_\varrho|}\int_{B^-}\mu_M\left(\frac{|V_{|z_0|,p}(D\mathfrak{u})|^2}{|z_0|^p}\right)\left(\int_0^1|z_0+sD\mathfrak{u}|^{p-2}\ \mathrm{d}s\right)|D\mathfrak{u}|\ \mathrm{d}x\\
&\overset{(2.4)}{\leq} c|z_0|^{(p-2)/2}\|D\varphi\|_\infty \fint_{B_\varrho}\mu_M\left(\frac{|V_{|z_0|,p}(D\mathfrak{u})|^2}{|z_0|^p}\right)|V_{|z_0|,p}(D\mathfrak{u})|\ \mathrm{d}x\\
&\leq c|z_0|^{(p-2)/2}\tilde{\mathfrak{F}}_0(u)\|D\varphi\|_\infty\left(\fint_{B_\varrho}\mu_M\left(\frac{|V_{|z_0|,p}(D\mathfrak{u})|^2}{|z_0|^p}\right)^2\ \mathrm{d}x\right)^{1/2}\\
&\leq c|z_0|^{(p-2)/2}\tilde{\mathfrak{F}}_0(u)\mu_M\left(\frac{\tilde{\mathfrak{F}}_0(u)^2}{|z_0|^p}\right)\|D\varphi\|_\infty,
\end{aligned}$$

for $c\equiv c(n,N,p,F(\cdot),M)$. We remark that the convergence of the singular integral in the previous display can be justified as in [35, Section 4]. Moreover, when applying (2.21) above we chose $L\equiv 80000(M+1)$ and consequently denote $\mu_L(\cdot)$ as $\mu_M(\cdot)$. Concerning term $\mathcal{I}_2$, notice that

$$|z|\overset{(2.13)_1}{\leq} 2|V_{|z_0|,p}(z)|^{2/p} \qquad \text{for all } z\in\mathbb{R}^{N\times n}\cap\left\{|z|\geq|z_0|\right\}, \tag{4.1}$$

so we have

$$\begin{aligned}
\mathcal{I}_2 &\overset{(2.18)_2,(2.20)}{\leq} \frac{c\|D\varphi\|_\infty}{|B_\varrho|}\int_{B^+}\left(|z_0|^{p-2}+\frac{|V_{|z_0|,p}(D\mathfrak{u})|^2}{|D\mathfrak{u}|^2}+\frac{|V_{|z_0|,q}(D\mathfrak{u})|^2}{|D\mathfrak{u}|^2}\right)|D\mathfrak{u}|\ \mathrm{d}x\\
&\overset{(4.1)}{\leq} \frac{c\|D\varphi\|_\infty}{|B_\varrho|}\int_{B^+}\left(1+\left(1-\mathbb{1}_{\{q\geq2\}}\right)|z_0|^{q-p}\right)|z_0|^{p-2}|V_{|z_0|,p}(D\mathfrak{u})|^{2/p}\ \mathrm{d}x\\
&\quad+\frac{c\|D\varphi\|_\infty}{|B_\varrho|}\int_{B^+}\mathbb{1}_{\{q\geq2\}}|V_{|z_0|,p}(D\mathfrak{u})|^{2(q-1)/p}\ \mathrm{d}x\\
&\leq \frac{c\|D\varphi\|_\infty(1+|z_0|^{q-p})}{|B_\varrho|}\int_{B^+}|z_0|^{p-2}|V_{|z_0|,p}(D\mathfrak{u})|^{2/p}\ \mathrm{d}x\\
&\quad+\frac{c\|D\varphi\|_\infty|z_0|^{q-p}}{|B_\varrho|}\int_{B^+}\frac{\mathbb{1}_{\{q\geq2\}}|V_{|z_0|,p}(D\mathfrak{u})|^{2(q-1)/p}}{|z_0|^{q-p}}\ \mathrm{d}x\\
&\overset{(2.13)}{\leq} c\|D\varphi\|_\infty|z_0|^{p-p\kappa-1}\tilde{\mathfrak{F}}_0(u)^{2\kappa},
\end{aligned}$$

where we used that $|z_0|\lesssim(M+1)$. In the above display, κ is defined as in the statement and $c\equiv c(\mathtt{data},M)$. Trivially, we also get

$$\mathcal{I}_3\leq 4\|D\varphi\|_\infty\left(\varrho^m\fint_{B_\varrho}|f|^m\ \mathrm{d}x\right)^{1/m}.$$

Merging the content of the previous displays, dividing both sides of the resulting inequality by $|z_0|^{(p-2)/2}\tilde{\mathfrak{F}}_0(u)$ and recalling that by $(2.13)_1$ it is $2\kappa>1$ we obtain (4.1) and the proof is complete. □

Now we are ready to prove a one-scale decay result valid in the nonsingular case. To this end, a fundamental observation is that under proper smallness conditions, local minimizers of (1.2) are approximately $\mathscr{A}$-harmonic in the sense of (2.27) for a suitable choice of the bilinear form $\mathscr{A}$.

Proposition 4.1. *Under hypotheses* (2.12)-(2.14), (2.16) *and* (2.23) *and let* $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ *be a local minimizer of* (1.2) *satisfying*

$$|(V_p(Du))_{B_\varrho(x_0)}|\leq 40000(M+1) \tag{4.2}$$

for some $M>0$ *on a ball* $B_\varrho(x_0)\Subset\Omega$. *Then, for any* $\beta\in(0,1)$ *there are* $\tau\equiv\tau,(\mathtt{data},M,\beta)\in(0,1/16)$ *and* $\varepsilon_0,\varepsilon_1\equiv\varepsilon_0,\varepsilon_1(\mathtt{data},M,\beta)\in(0,1]$ *such that if the smallness conditions*

$$\mathfrak{F}(u;B_\varrho(x_0))<\varepsilon_0|(V_p(Du))_{B_\varrho(x_0)}| \tag{4.3}$$

and

$$\left(\varrho^m\fint_{B_\varrho(x_0)}|f|^m\ \mathrm{d}x\right)^{1/m}\leq\varepsilon_1\mathfrak{F}(u;B_\varrho(x_0))|(V_p(Du))_{B_\varrho(x_0)}|^{(p-2)/p} \tag{4.4}$$

are verified on $B_\varrho(x_0)$, *it holds that*

$$\mathfrak{F}(u;B_{\tau\varrho}(x_0))\leq\tau^\beta\mathfrak{F}(u;B_\varrho(x_0)). \tag{4.5}$$

Proof. For the transparency of the exposition, let us introduce some abbreviations. As all balls considered here will be concentric to $B_\varrho(x_0)$, we shall omit denoting the center, given any ball $B_\varsigma(x_0) \subseteq B_\varrho(x_0)$ we shorten $(V_p(Du))_{B_\varsigma(x_0)} \equiv (V_p(Du))_\varsigma$ and for all $\varphi \in C_c^\infty(B_\varrho, \mathbb{R}^N)$ we denote $\|D\varphi\|_{L^\infty(B_\varrho)} \equiv \|D\varphi\|_\infty$. In the light of (4.3), there is no loss of generality in assuming that

$$|(V_p(Du))_\varrho| > 0 \qquad \text{and} \qquad \mathfrak{F}(u; B_\varrho) > 0, \tag{4.6}$$

otherwise (4.5) would be trivially true because of $(2.9)_1$. Being $V_p(\cdot)$ an isomorphism of $\mathbb{R}^{N\times n}$, we can find $\bar{z} \in \mathbb{R}^{N\times n} \setminus \{0\}$ such that $V_p(\bar{z}) = (V_p(Du))_\varrho$ and

$$|\bar{z}| \stackrel{(4.6)_1}{>} 0, \qquad \tilde{\mathfrak{F}}(u, \bar{z}; B_\varrho) \stackrel{(2.8)}{\approx} \mathfrak{F}(u; B_\varrho) \stackrel{(4.6)_2}{>} 0, \qquad |\bar{z}| = |(V_p(Du))_\varrho|^{2/p}. \tag{4.7}$$

We then define $u_0(x) := |\bar{z}|^{-1}\left(u(x) - (u)_\varrho - \langle \bar{z}, x - x_0\rangle\right)$ and, motivated by (4.7), we apply Lemma 4.1 to get

$$\begin{aligned}
\left| \fint_{B_\varrho} \frac{\partial^2 F(\bar{z})}{|\bar{z}|^{p-2}} \langle Du_0, D\varphi\rangle \, \mathrm{d}x \right| \quad &\le \quad \frac{c\tilde{\mathfrak{F}}(u, \bar{z}; B_\varrho)}{|\bar{z}|^{p/2}} \left[\mu_M\left(\frac{\tilde{\mathfrak{F}}(u, \bar{z}; B_\varrho)^2}{|\bar{z}|^p}\right) + \left(\frac{\tilde{\mathfrak{F}}(u, \bar{z}; B_\varrho)^2}{|\bar{z}|^p}\right)^{\frac{2\kappa-1}{2}} \right] \|D\varphi\|_\infty \\
&\quad + c|\bar{z}|^{1-p}\left(\varrho^m \fint_{B_\varrho} |f|^m \, \mathrm{d}x\right)^{1/m} \|D\varphi\|_\infty \\
&\stackrel{(4.7)_{2,3}}{\le} \frac{c\mathfrak{F}(u; B_\varrho)}{|(V_p(Du))_\varrho|} \left[\mu_M\left(\frac{\mathfrak{F}(u; B_\varrho)^2}{|(V_p(Du))_\varrho|^2}\right) + \left(\frac{\mathfrak{F}(u; B_\varrho)^2}{|(V_p(Du))_\varrho|^2}\right)^{\frac{2\kappa-1}{2}} \right] \|D\varphi\|_\infty \\
&\quad + c|(V_p(Du))_\varrho|^{2(1-p)/p}\left(\varrho^m \fint_{B_\varrho} |f|^m \, \mathrm{d}x\right)^{1/m} \|D\varphi\|_\infty \\
&\stackrel{(4.3),(4.4)}{\le} \tilde{c}\varepsilon_0\left[\mu_M(\varepsilon_0^2) + \varepsilon_0^{2\kappa-1} + \varepsilon_1\right] \|D\varphi\|_\infty,
\end{aligned}$$

with $\tilde{c} \equiv \tilde{c}(\mathtt{data}, M)$. Moreover, it holds that

$$\left(\fint_{B_\varrho} |V_{1,p}(Du_0)|^2 \, \mathrm{d}x\right)^{1/2} = \frac{\tilde{\mathfrak{F}}(u, \bar{z}; B_\varrho)}{|\bar{z}|^{p/2}} \stackrel{(4.7)_{2,3}}{\le} \frac{c_*\mathfrak{F}(u; B_\varrho)}{|(V_p(Du))_\varrho|} \stackrel{(4.3)}{\le} c_*\varepsilon_0,$$

where $c_* \equiv c_*(n, N, p)$ is the constant from the upper bound in (2.8). Now notice that by (4.2), $(4.7)_3$, (2.20) with $L = (40000(M+1))^{2/p}$ and (2.15) we see that the bilinear form $\mathscr{A} := \partial^2 F(\bar{z})|\bar{z}|^{2-p}$ satisfies (2.27) for some $H \equiv H(n, N, \lambda, p, F(\cdot), M) \ge 1$. We then set $\sigma := c_*\mathfrak{F}(u; B_\varrho)/|(V_p(Du))_\varrho|$, let $\varepsilon \in (0, 1]$ be any number to be determined later on and, recalling that by $(2.13)_1$ it is $2\kappa > 1$, we assume the following smallness conditions:

$$\max\{\tilde{c}, c_*\}\varepsilon_0 < \frac{\delta}{2^{10}} \qquad \text{and} \qquad \mu_M(\varepsilon_0^2) + \varepsilon_0^{2\kappa-1} + \varepsilon_1 \le \frac{1}{2^{10}}, \tag{4.8}$$

where $\delta \equiv \delta(n, N, p, \varepsilon) \in (0, 1]$ is the small parameter given by Lemma 2.4, further restrictions on the size of the various parameters appearing in (4.8) will be imposed later on. The choice in (4.8) requires in particular that $\varepsilon_1 \in (0, 1/3]$, fixes dependency $\varepsilon_0 \equiv \varepsilon_0(\mathtt{data}, M, \varepsilon)$ and ultimately gives that

$$\fint_{B_\varrho} |V_{1,p}(Du_0)|^2 \, \mathrm{d}x \le \sigma^2 \le 1 \quad \text{and} \quad \left| \fint_{B_\varrho} \mathscr{A}\langle Du_0, D\varphi\rangle \, \mathrm{d}x \right| \le \delta\|D\varphi\|_\infty \quad \text{for all} \quad \varphi \in C_c^\infty(B_\varrho, \mathbb{R}^N),$$

so Lemma 2.4 applies: there exists a $\mathscr{A}$-harmonic map $h \in W^{1,p}(B_\varrho, \mathbb{R}^N)$ such that

$$\fint_{B_\varrho(x_0)} |V_{1,p}(Dh)|^2 \, \mathrm{d}x \le c \qquad \text{and} \qquad \fint_{B_\varrho(x_0)} \left| V_{1,p}\left(\frac{u_0 - \sigma h}{\varrho}\right) \right|^2 \, \mathrm{d}x \le c\sigma^2\varepsilon, \tag{4.9}$$

for $c \equiv c(n, N, p)$. Let $\tau \in (0, 2^{-10})$ be a small number whose size will be determined later on and estimate by $(2.3)_{5,7}$, (4.9), (2.28), (4.1) with $|z_0| = 1$ and the mean value theorem:

$$\begin{aligned}
&\fint_{B_{2\tau\varrho}} \left| V_{1,p}\left(\frac{u_0 - \sigma(h(x_0) + \langle Dh(x_0), x - x_0\rangle)}{2\tau\varrho}\right) \right|^2 \, \mathrm{d}x \\
&\qquad \le c\fint_{B_{2\tau\varrho}} \left| V_{1,p}\left(\frac{\sigma(h - h(x_0) - \langle Dh(x_0), x - x_0\rangle)}{2\tau\varrho}\right) \right|^2 \, \mathrm{d}x + c\fint_{B_{2\tau\varrho}} \left| V_{1,p}\left(\frac{u_0 - \sigma h}{2\tau\varrho}\right) \right|^2 \, \mathrm{d}x \\
&\qquad \le \frac{c\varepsilon\sigma^2}{\tau^{n+2}} + c\sigma^2\fint_{B_{2\tau\varrho}} \left| \frac{h - h(x_0) - \langle Dh(x_0), x - x_0\rangle}{2\tau\varrho} \right|^2 \, \mathrm{d}x \le \frac{c\varepsilon\sigma^2}{\tau^{n+2}} + c\sigma^2\tau^2\varrho^2\|D^2 h\|^2_{L^\infty(B_{4\tau\varrho})} \\
&\qquad \le \frac{c\varepsilon\sigma^2}{\tau^{n+2}} + c\sigma^2\tau^2\mathfrak{I}_1(Dh; B_\varrho)^2 \le \frac{c\varepsilon\sigma^2}{\tau^{n+2}} + c\sigma^2\tau^2,
\end{aligned}$$

with $c \equiv c(n,N,p)$. In the above display, we fix $\varepsilon := \tau^{n+4}$ thus getting

$$\fint_{B_{2\tau\varrho}} \left| V_{1,p}\left(\frac{u_0-\sigma(h(x_0)+\langle Dh(x_0),x-x_0\rangle)}{2\tau\varrho}\right)\right|^2 \ \mathrm{d}x \le c\sigma^2\tau^2, \tag{4.10}$$

for $c \equiv c(n,N,p)$, which yields that

$$\begin{aligned}
\fint_{B_{2\tau\varrho}} |V_{|\bar z|,p}(\mathcal{S}(x))|^2 \ \mathrm{d}x \quad &:= \quad \fint_{B_{2\tau\varrho}} \left| V_{|\bar z|,p}\left(\frac{u-(u)_\varrho-\langle \bar z,x-x_0\rangle-\sigma|\bar z|(h(x_0)+\langle Dh(x_0),x-x_0\rangle)}{2\tau\varrho}\right)\right|^2 \ \mathrm{d}x\\
&= \quad |\bar z|^p \fint_{B_{2\tau\varrho}} \left| V_{1,p}\left(\frac{u_0-\sigma(h(x_0)-\langle Dh(x_0),x-x_0\rangle)}{2\tau\varrho}\right)\right|^2 \ \mathrm{d}x\\
&\overset{(4.10)}{\le} \quad c\tau^2|\bar z|^p\left(\frac{\mathfrak{F}(u;B_\varrho)}{|(V_p(Du))_\varrho|}\right)^2 \overset{(4.7)_3}{\le} c\tau^2\mathfrak{F}(u;B_\varrho)^2, \qquad (4.11)
\end{aligned}$$

with $c \equiv c(n,N,p)$. Now, notice that by (2.28) and $(4.9)_1$ it is $|Dh(x_0)| \le c(n,N,p)$, so recalling (4.3) and reducing further (with respect to (4.8)) the size of ε_0 in such a way that $c\sigma \le c\varepsilon_0 \le \min\{2^{-10},\tau^n\}$ we obtain

$$|\bar z|(1-c\sigma) \le |\bar z+\sigma|\bar z|Dh(x_0)| \le |\bar z|(1+c\sigma) \implies \frac{1}{2}|\bar z| \le |\bar z+\sigma|\bar z|Dh(x_0)| \le \frac{3}{2}|\bar z|. \tag{4.12}$$

We can then estimate

$$\begin{aligned}
&\left| \, |V_{|\bar z|,p}(\mathcal{S}(x))|^2 - |V_{|\bar z+\sigma|\bar z|Dh(x_0)|,p}(\mathcal{S}(x))|^2 \, \right|\\
&\quad \le c|\mathcal{S}(x)|^2 \sup_{t\in[|\bar z|(1-c\sigma),|\bar z|(1+c\sigma)]} \left(t^2+|\mathcal{S}(x)|^2\right)^{(p-3)/2} \big||\bar z|-|\bar z+\sigma|\bar z|Dh(x_0)|\big|\\
&\quad \le c\sigma|\bar z||Dh(x_0)||\mathcal{S}(x)|^2\left(|\bar z|(1-c\sigma)^2+|\mathcal{S}(x)|^2\right)^{(p-3)/2}\\
&\quad \le \frac{c\sigma|\bar z||V_{|\bar z|,p}(\mathcal{S}(x))|^2}{(1-c\sigma)^{3-p}(|\mathcal{S}(x)|^2+|\bar z|^2)^{1/2}} \le c|V_{|\bar z|,p}(\mathcal{S}(x))|^2,
\end{aligned}$$

for $c \equiv c(n,N,p)$. This and (4.11) imply that

$$\fint_{B_{2\tau\varrho}} |V_{|\bar z+\sigma|\bar z|Dh(x_0)|,p}(\mathcal{S}(x))|^2 \ \mathrm{d}x \le c\fint_{B_{2\tau\varrho}} |V_{|\bar z|,p}(\mathcal{S}(x))|^2 \ \mathrm{d}x \le c\tau^2\mathfrak{F}(u;B_\varrho)^2, \tag{4.13}$$

with $c \equiv c(n,N,p)$. Next, notice that

$$\begin{aligned}
\tilde{\mathfrak{F}}(u,\bar z+\sigma|\bar z|Dh(x_0);B_{2\tau\varrho})^2 \quad &\overset{(4.12)}{\le} \quad c\fint_{B_{2\tau\varrho}} |V_{|\bar z|,p}(Du-\bar z-\sigma|\bar z|Dh(x_0))|^2 \ \mathrm{d}x\\
&\overset{(2.3)_7}{\le} \quad c\fint_{B_{2\tau\varrho}} |V_{|\bar z|,p}(Du-\bar z)|^2 \ \mathrm{d}x + c\fint_{B_{2\tau\varrho}} |V_{|\bar z|,p}(\sigma\bar z Dh(x_0))|^2 \ \mathrm{d}x\\
&\overset{(4.7)_2}{\le} \quad c\tau^{-n}\mathfrak{F}(u;B_\varrho)^2 + c\sigma^2|\bar z|^p \overset{(4.7)_3}{\le} c\tau^{-n}\mathfrak{F}(u;B_\varrho)^2, \qquad (4.14)
\end{aligned}$$

for $c \equiv c(n,N,p)$. At this stage, keeping in mind (4.12) we apply Caccioppoli inequality (3.4) to bound

$$\begin{aligned}
\tilde{\mathfrak{F}}(u,\bar z+\sigma|\bar z|Dh(x_0);B_{\tau\varrho})^2 \quad &\le \quad c\mathfrak{K}\left(\fint_{B_{2\tau\varrho}} \left| V_{|\bar z+\sigma|\bar z|Dh(x_0)|,p}(\mathcal{S}(x)) \right|^2 \ \mathrm{d}x\right)\\
&\quad +c\mathbb{1}_{\{q>p\}}\tilde{\mathfrak{F}}(u,\bar z+\sigma|\bar z|Dh(x_0);B_{2\tau\varrho})^{2q/p} + c\left((\tau\varrho)^m\fint_{B_{2\tau\varrho}} |f|^m \ \mathrm{d}x\right)^{\frac{p}{m(p-1)}}\\
&\quad +c|\bar z+\sigma|\bar z|Dh(x_0)|^{2-p}\left((\tau\varrho)^m\fint_{B_{2\tau\varrho}} |f|^m \ \mathrm{d}x\right)^{2/m}\\
&= \quad c\left(\mathrm{(I)}+\mathrm{(II)}+\mathrm{(III)}\right),
\end{aligned}$$

with $c \equiv c(n,N,p,q,M)$. We continue estimating:

$$\begin{aligned}
\mathrm{(I)} \quad &\overset{(4.13)}{\le} \quad c\mathfrak{K}\left(\tau^2\mathfrak{F}(u;B_\varrho)^2\right) \overset{(2.2)}{=} c\tau^2\mathfrak{F}(u;B_\varrho)^2 + c\tau^{2q/p}\mathfrak{F}(u;B_\varrho)^{2q/p}\\
&\overset{(4.2)}{\le} \quad c\tau^2\mathfrak{F}(u;B_\varrho)^2 + cM^{2(q-p)/p}\tau^{2q/p}\left(\frac{\mathfrak{F}(u;B_\varrho)}{|(V_p(Du))_\varrho|}\right)^{2(q-p)/p}\mathfrak{F}(u;B_\varrho)^2\\
&\overset{(4.3)}{\le} \quad c\tau^2\mathfrak{F}(u;B_\varrho)^2\left(1+M^{2(q-p)/p}\tau^{2(q-p)/p}\varepsilon_0^{2(q-p)/p}\right) \le c\tau^2\mathfrak{F}(u;B_\varrho)^2,
\end{aligned}$$

for $c \equiv c(\texttt{data}, M)$,

$$
\begin{aligned}
\text{(II)} &\overset{(4.14)}{\leq} c\mathbb{1}_{\{q>p\}}\tau^{-nq/p}\mathfrak{F}(u;B_\varrho)^{2q/p} \overset{(4.2)}{\leq} \frac{c\mathbb{1}_{\{q>p\}}M^{2(q-p)/p}}{\tau^{nq/p}}\left(\frac{\mathfrak{F}(u;B_\varrho)}{|(V_p(Du))_\varrho|}\right)^{2(q-p)/p}\mathfrak{F}(u;B_\varrho)^2\\
&\overset{(4.3)}{\leq} c\mathbb{1}_{\{q>p\}}\tau^{-nq/p}\varepsilon_0^{2(q-p)/p}\mathfrak{F}(u;B_\varrho)^2,
\end{aligned}
$$

with $c \equiv c(n, N, p, q, M)$ and

$$
\begin{aligned}
\text{(III)} &\overset{(4.12)}{\leq} c\tau^{\frac{(m-n)p}{m(p-1)}}\left(\varrho^m \fint_{B_\varrho}|f|^m \ \mathrm{d}x\right)^{\frac{p}{m(p-1)}} + c\tau^{\frac{2(m-n)}{m}}|\bar z|^{2-p}\left(\varrho^m \fint_{B_\varrho}|f|^m \ \mathrm{d}x\right)^{2/m}\\
&\overset{(4.4)}{\leq} c\varepsilon_1^{p/(p-1)}\tau^{\frac{(m-n)p}{m(p-1)}}\mathfrak{F}(u;B_\varrho)^{p/(p-1)}|(V_p(Du))_\varrho|^{\frac{p-2}{p-1}}\\
&\quad +c\varepsilon_1^2\tau^{\frac{2(m-n)}{m}}|\bar z|^{2-p}\mathfrak{F}(u;B_\varrho)^2|(V_p(Du))_\varrho|^{2(p-2)/p}\\
&\overset{(4.7)_3}{\leq} c\varepsilon_1^{p/(p-1)}\tau^{\frac{(m-n)p}{m(p-1)}}\left(\frac{\mathfrak{F}(u;B_\varrho)}{|(V_p(Du))_\varrho|}\right)^{\frac{2-p}{p-1}}\mathfrak{F}(u;B_\varrho)^2 + c\varepsilon_1^2\tau^{\frac{2(m-n)}{m}}\mathfrak{F}(u;B_\varrho)^2\\
&\overset{(4.3)}{\leq} c\left(\varepsilon_1^{p/(p-1)}\tau^{\frac{(m-n)p}{m(p-1)}}\varepsilon_0^{\frac{2-p}{p-1}} + \varepsilon_1^2\tau^{\frac{2(m-n)}{m}}\right)\mathfrak{F}(u;B_\varrho)^2,
\end{aligned}
$$

for $c \equiv c(n, p, m)$. Merging the previous four displays we obtain

$$
\tilde{\mathfrak{F}}(u, \bar z + \sigma|\bar z|Dh(x_0); B_{\tau\varrho})^2 \quad \leq \quad c\mathscr{T}\mathfrak{F}(u;B_\varrho)^2, \tag{4.15}
$$

where $c \equiv c(\texttt{data}, M)$ and we set

$$
\mathscr{T} := \tau^2 + \mathbb{1}_{\{q>p\}}\tau^{-nq/p}\varepsilon_0^{2(q-p)/p} + \varepsilon_1^{p/(p-1)}\varepsilon_0^{(2-p)/(p-1)}\tau^{\frac{p(m-n)}{m(p-1)}} + \varepsilon_1^2\tau^{2(m-n)/m}.
$$

Finally, let us observe that by triangular inequality it is

$$
\begin{aligned}
|V_p(Du) - V_p(\bar z + \sigma|\bar z|Dh(x_0))|^2 &\overset{(2.3)_2}{\leq} c\left(|Du|^2 + |\bar z + \sigma|\bar z|Dh(x_0)|^2\right)^{(p-2)/2}|Du - (\bar z + \sigma|\bar z|Dh(x_0))|^2\\
&\leq c|V_{|\bar z + \sigma|\bar z|Dh(x_0)|,p}(Du - \bar z - \sigma|\bar z|Dh(x_0))|^2,
\end{aligned} \tag{4.16}
$$

for $c \equiv c(n, N, p)$, therefore

$$
\mathfrak{F}(u;B_{\tau\varrho})^2 \overset{(2.1)}{\leq} c\mathfrak{F}(u, V_p(\bar z + \sigma|\bar z|Dh(x_0)); B_{\tau\varrho})^2 \overset{(4.16)}{\leq} c\tilde{\mathfrak{F}}(u, \bar z + \sigma|\bar z|Dh(x_0); B_{\tau\varrho})^2 \overset{(4.15)}{\leq} c\mathscr{T}\mathfrak{F}(u;B_\varrho)^2, \tag{4.17}
$$

with $c \equiv c(\texttt{data}, M)$. Looking at the explicit expression of $\mathscr{T}$, we let $\beta \in (0, 1)$ be any number, first fix $\tau \in (0, 2^{-10})$, then reduce further the size of $\varepsilon_0 \in (0, 1)$ and finally restrict ε_1 in such a way that

$$
\begin{cases}
c\max\left\{\tau^{2(1-\beta)}, \tau^{\alpha/4}\right\} \leq \dfrac{1}{2^{10}}\\[2mm]
\dfrac{c2^2\varepsilon_0}{\tau^n} + c\mathbb{1}_{\{q>p\}}\tau^{-nq/p}\varepsilon_0^{2(q-p)/p} < \dfrac{\tau^{2\beta}}{2^{10}}\\[2mm]
c\max\left\{\varepsilon_1^2\tau^{\frac{2(m-n)}{m}}, \varepsilon_1^{p/(p-1)}\tau^{\frac{p(m-n)}{m(p-1)}}\right\} < \dfrac{\tau^{2\beta_0}}{2^{10}},
\end{cases} \tag{4.18}
$$

where $\alpha \equiv \alpha(n, N, p)$ is the same exponent appearing in $(2.29)_2$. This way, we determine dependencies: $\tau, \varepsilon_0, \varepsilon_1 \equiv \tau, \varepsilon_0, \varepsilon_1(\texttt{data}, M, \beta)$. Plugging the above restrictions in (4.17) we get (4.5) and the proof is complete. □

Let us look at what happens when the complementary condition to (4.4) is in force.

Proposition 4.2. *In the setting of Proposition 4.1, assume*

$$
\varepsilon_1\mathfrak{F}(u;B_\varrho(x_0))|(V_p(Du))_{B_\varrho(x_0)}|^{(p-2)/p} \leq \left(\varrho^m \fint_{B_\varrho(x_0)}|f|^m \ \mathrm{d}x\right)^{1/m} \tag{4.19}
$$

instead of (4.4). *Then, for all* $\tau \in (0, 1)$ *it is*

$$
\mathfrak{F}(u;B_{\tau\varrho}(x_0)) \leq c_0|(V_p(Du))_{B_\varrho(x_0)}|^{(2-p)/p}\left(\varrho^m \fint_{B_\varrho(x_0)}|f|^m \ \mathrm{d}x\right)^{1/m}, \tag{4.20}
$$

where $c_0 := 2\varepsilon_1^{-1}\tau^{-n/2}$ *and* $\varepsilon_1 \equiv \varepsilon_1(\texttt{data}, M) \in (0, 1]$ *is the same determined in* (4.18).

Proof. Inequality (4.20) is a direct consequence of $(2.9)_1$ and (4.19). □

5. The singular regime

We start by proving that local minimizers of (1.2) are approximately p-harmonic within the singular scenario.

Lemma 5.1. *Under assumptions (2.12)-(2.14), (2.16) and (2.23) let $B_\varrho(x_0) \Subset \Omega$ be any ball and $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of (1.2). Then*

$$\begin{aligned}\left| \fint_{B_\varrho(x_0)} |Du|^{p-2}\langle Du, D\varphi\rangle \ \mathrm{d}x \right| \quad &\le \quad 4\|D\varphi\|_\infty \left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \ \mathrm{d}x\right)^{1/m} + s\|D\varphi\|_\infty \mathfrak{I}_p(Du; B_\varrho)^{p-1} \\ &\quad + c\|D\varphi\|_\infty \left(\omega(s)^{-1} + \omega(s)^{-p} + \omega(s)^{q-p-1}\right) \mathfrak{I}_p(Du; B_\varrho)^p, \end{aligned} \tag{5.1}$$

for all $\varphi \in C^\infty_c(B_\varrho(x_0), \mathbb{R}^N)$ and any $s \in (0, \infty)$, with $c \equiv c(n, N, \Lambda, q)$.

Proof. With the same abbreviations used in Lemma 4.1, let $\varphi \in C^\infty_c(B_\varrho, \mathbb{R}^N)$ be any smooth map. We use (2.33) to control

$$\begin{aligned}\left| \fint_{B_\varrho} |Du|^{p-2}\langle Du, D\varphi\rangle \ \mathrm{d}x \right| \quad &= \quad \left| \fint_{B_\varrho} \langle \partial F(Du) - \partial F(0) - |Du|^{p-2}Du, D\varphi\rangle - f\cdot\varphi \ \mathrm{d}x \right| \\ &\le \quad \left| \fint_{B_\varrho} \langle \partial F(Du) - \partial F(0) - |Du|^{p-2}Du, D\varphi\rangle \ \mathrm{d}x \right| \\ &\quad + \fint_{B_\varrho} |f||\varphi| \ \mathrm{d}x =: \mathcal{G}_1 + \mathcal{G}_2. \end{aligned}$$

We fix $s \in (0, \infty)$, notice that

$$\frac{|B_\varrho \cap \{|Du| > \omega(s)\}|}{|B_\varrho|} \le \left(\frac{\mathfrak{I}_p(Du; B_\varrho)}{\omega(s)}\right)^p \tag{5.2}$$

and then bound

$$\begin{aligned} \mathcal{G}_1 \quad &\le \quad \frac{1}{|B_\varrho|}\int_{B_\varrho\cap\{|Du|<\omega(s)\}} |\langle \partial F(Du) - \partial F(0) - |Du|^{p-2}Du, D\varphi\rangle| \ \mathrm{d}x \\ &\quad + \frac{1}{|B_\varrho|}\int_{B_\varrho\cap\{|Du|\ge\omega(s)\}} |\langle \partial F(Du) - \partial F(0) - |Du|^{p-2}Du, D\varphi\rangle| \ \mathrm{d}x \\ &\overset{(2.19),(2.17)}{\le} \|D\varphi\|_\infty \left(s\mathfrak{I}_p(Du; B_\varrho)^{p-1} + \frac{c}{|B_\varrho|}\int_{B_\varrho\cap\{|Du|\ge\omega(s)\}} 1 + |Du|^{p-1} + |Du|^{q-1} \ \mathrm{d}x\right) \\ &\overset{(2.13)_2}{\le} s\|D\varphi\|_\infty \mathfrak{I}_p(Du; B_\varrho)^{p-1} + c\|D\varphi\|_\infty \frac{|B_\varrho \cap \{|Du| > \omega(s)\}|}{|B_\varrho|} \\ &\quad + c\|D\varphi\|_\infty \left(\frac{|B_\varrho \cap \{|Du| > \omega(s)\}|}{|B_\varrho|}\right)^{1/p} \mathfrak{I}_p(Du; B_\varrho)^{p-1} \\ &\quad + c\|D\varphi\|_\infty \left(\frac{|B_\varrho \cap \{|Du| > \omega(s)\}|}{|B_\varrho|}\right)^{(p-q+1)/p} \mathfrak{I}_p(Du; B_\varrho)^{q-1} \\ &\overset{(5.2)}{\le} s\|D\varphi\|_\infty \mathfrak{I}_p(Du; B_\varrho)^{p-1} + c\|D\varphi\|_\infty \left(\omega(s)^{-1} + \omega(s)^{-p} + \omega(s)^{q-p-1}\right) \mathfrak{I}_p(Du; B_\varrho)^p, \end{aligned}$$

for $c \equiv c(n, N, \Lambda, q)$ and

$$\mathcal{G}_2 \le 4\|D\varphi\|_\infty \left(\varrho^m \fint_{B_\varrho} |f|^m \ \mathrm{d}x\right)^{1/m}.$$

Merging the content of the two previous display we obtain (5.1) and the proof is complete. □

Next, a one-scale decay estimate for the excess functional valid in the singular regime.

Proposition 5.1. *Under hypotheses (2.12)-(2.14), (2.16) and (2.23), let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of (1.2) satisfying (4.2) on a ball $B_\varrho(x_0) \Subset \Omega$ for some positive constant M. Then, for any $\gamma \in (0, \alpha)$*[5]*, $\chi \in (0, 1]$ there are $\theta \equiv \theta(\texttt{data}, \chi, \gamma, M) \in (0, 2^{-10})$, $\varepsilon_i \equiv \varepsilon_i(\texttt{data}, \gamma, M)$, $i \in \{2, 3\}$, such that if the smallness conditions*

$$\chi|(V_p(Du))_{B_\varrho(x_0)}| \le \mathfrak{F}(u; B_\varrho(x_0)), \qquad \mathfrak{F}(u; B_\varrho(x_0)) < \varepsilon_2, \qquad \left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \ \mathrm{d}x\right)^{1/m} < \varepsilon_3 \tag{5.3}$$

[5] Here, $\alpha \equiv \alpha(n, N, p)$ is the exponent appearing in $(2.29)_2$.

hold on $B_\varrho(x_0)$, *then*

$$\mathfrak{F}(u;B_{\theta\varrho}(x_0)) \le \theta^{\gamma}\mathfrak{F}(u;B_\varrho(x_0)) + c_1\mathfrak{K}\left[\left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \ \mathrm{d}x\right)^{1/m}\right]^{\frac{p}{2(p-1)}}, \tag{5.4}$$

with $c_1 \equiv c_1(\texttt{data},\chi,\gamma,M)$, *and* $\mathfrak{K}(\cdot)$ *defined in* (2.2).

Proof. Let us premise that the same abbreviations appearing in Proposition 4.1 will be adopted also here. By triangular inequality and $(5.3)_1$ we have that

$$\mathfrak{I}_p(Du;B_\varrho)^p \le 2\mathfrak{F}(u;B_\varrho)^2 + 2|(V_p(Du))_\varrho|^2 \le 2\left(1+\frac{1}{\chi^2}\right)\mathfrak{F}(u;B_\varrho)^2 =: c_\chi\mathfrak{F}(u;B_\varrho)^2. \tag{5.5}$$

With this estimate at hand, (5.1) becomes

$$\begin{aligned}\left|\fint_{B_\varrho}\langle |Du|^{p-2}Du, D\varphi\rangle \ \mathrm{d}x\right| \ &\le \ 4\|D\varphi\|_\infty\left(\varrho^m\fint_{B_\varrho}|f|^m \ \mathrm{d}x\right)^{1/m} + sc_\chi^{(p-1)/p}\|D\varphi\|_\infty\mathfrak{F}(u;B_\varrho)^{2(p-1)/p}\\ &\quad + cc_\chi\|D\varphi\|_\infty\left(\omega(s)^{-1}+\omega(s)^{-p}+\omega(s)^{q-p-1}\right)\mathfrak{F}(u;B_\varrho)^2,\end{aligned} \tag{5.6}$$

for all $\varphi \in C^\infty_c(B_\varrho,\mathbb{R}^N)$, $s\in(0,\infty)$, with $c\equiv c(n,N,\Lambda,q)$. Notice that there is no loss of generality in assuming $\mathfrak{F}(u;B_\varrho)>0$, otherwise (5.4) would trivially be true by means of $(2.9)_1$. We then let $\varepsilon_4\in(0,1)$ to be fixed in a few lines, and set

$$\psi := c_\chi^{1/p}\mathfrak{F}(u;B_\varrho)^{2/p} + \left(\frac{4}{\varepsilon_4}\right)^{1/(p-1)}\left(\varrho^m\fint_{B_\varrho}|f|^m \ \mathrm{d}x\right)^{\frac{1}{m(p-1)}}, \qquad u_0 := \frac{u}{\psi}$$

and divide both sides of (5.6) by ψ^{p-1} to get

$$\begin{aligned}\left|\fint_{B_\varrho}\langle |Du_0|^{p-2}Du_0, D\varphi\rangle \ \mathrm{d}x\right| \ &\le \ (\varepsilon_3+s)\,\|D\varphi\|_\infty\\ &\quad + cc_\chi^{1/p}\|D\varphi\|_\infty\left(\omega(s)^{-1}+\omega(s)^{-p}+\omega(s)^{q-p-1}\right)\mathfrak{F}(u;B_\varrho)^{2/p}\\ &\overset{(5.3)_2}{\le} \left[\varepsilon_3+s+cc_\chi^{1/p}\left(\omega(s)^{-1}+\omega(s)^{-p}+\omega(s)^{q-p-1}\right)\varepsilon_2^{2/p}\right]\|D\varphi\|_\infty,\end{aligned} \tag{5.7}$$

for $c\equiv c(n,N,\Lambda,q)$. Now as a direct consequence of (5.5) we obtain

$$\mathfrak{I}_p(Du_0;B_\varrho) \le c_\chi^{1/p}\psi^{-1}\mathfrak{F}(u;B_\varrho)^{2/p} \le 1, \tag{5.8}$$

so fixed $\varepsilon\in(0,1)$ to be determined later on and, with $\delta\equiv\delta(n,N,p,\varepsilon)\in(0,1]$ being the small parameter given by Lemma 2.5, we reduce the size of parameters ε_4, s first and then that of $\varepsilon_2,\varepsilon_3$ in such a way that

$$\varepsilon_4+s<\frac{\delta}{4}, \qquad cc_\chi^{1/p}\left(\omega(s)^{-1}+\omega(s)^{-p}+\omega(s)^{q-p-1}+c'\right)\varepsilon_2^{2/p}<\frac{\delta}{4}, \qquad \max\{c',1\}\left(\frac{4\varepsilon_3}{\varepsilon_4}\right)^{1/(p-1)}<\frac{1}{4} \tag{5.9}$$

thus (5.7) is turned into (2.30) and together with (5.8) legalizes the application of Lemma 2.5 which renders a p-harmonic map $h\in W^{1,p}(B_\varrho,\mathbb{R}^N)$ such that

$$\mathfrak{I}_p(Dh;B_\varrho)\le 1 \qquad \text{and} \qquad \fint_{B_\varrho}\left|\frac{u_0-h}{\varrho}\right|^p \le c\varepsilon^p, \tag{5.10}$$

for $c\equiv c(n,N,p)$. In $(5.9)_3$, c' is the constant appearing in the Lipschitz bound $(2.29)_1$. We point out that the choices in (5.9) fix dependencies $\varepsilon_3,\varepsilon_4,s\equiv\varepsilon_3,\varepsilon_4,s(n,N,p,\varepsilon)$ and $\varepsilon_2\equiv\varepsilon_2(n,N,p,\omega(\cdot),\varepsilon,\chi)$. Further restrictions on the size of these parameters will be imposed in a few lines. Next, for $\theta\in(0,2^{-10})$, we exploit the isomorphism properties of $V_p(\cdot)$ to determine $z_{2\theta\varrho}\in\mathbb{R}^{N\times n}$ such that $V_p(z_{2\theta\varrho})=(V_p(Dh))_{2\theta\varrho}$ and estimate via $(2.3)_{3,7}$, (2.6), (5.10) and $(2.29)_2$,

$$\begin{aligned}\fint_{B_{2\theta\varrho}}\left|V_{|z_{2\theta\varrho}|,p}\left(\frac{u_0-(h)_{2\theta\varrho}-\langle z_{2\theta\varrho},x-x_0\rangle}{2\theta\varrho}\right)\right|^2 \ \mathrm{d}x &\le c\fint_{B_{2\theta\varrho}}\left|V_{|z_{2\theta\varrho}|,p}\left(\frac{u_0-h}{2\theta\varrho}\right)\right|^2 \ \mathrm{d}x\\ &\quad + c\fint_{B_{2\theta\varrho}}\left|V_{|z_{2\theta\varrho}|,p}\left(\frac{h-(h)_{2\theta\varrho}-\langle z_{2\theta\varrho},x-x_0\rangle}{2\theta\varrho}\right)\right|^2 \ \mathrm{d}x\\ &\le c\theta^{-n-p}\fint_{B_\varrho}\left|\frac{u_0-h}{\varrho}\right|^p \ \mathrm{d}x + c\fint_{B_{2\theta\varrho}}|V_{|z_{2\theta\varrho}|,p}(Dh-z_{2\theta\varrho})|^2 \ \mathrm{d}x\\ &\le c\theta^{-n-p}\varepsilon^p + c\fint_{B_{2\theta\varrho}}|V_p(Dh)-(V_p(Dh))_{2\theta\varrho}|^2 \ \mathrm{d}x \le c\theta^{-n-p}\varepsilon^p + c\theta^{2\alpha},\end{aligned}$$

with $c \equiv c(n,N,p)$. Scaling back to u in the previous display we obtain

$$\tag{5.11} \fint_{B_{2\theta\varrho}} \left| V_{\psi|z_{2\theta\varrho}|,p}\left(\frac{u-\psi((h)_{2\theta\varrho}+\langle z_{2\theta\varrho},x-x_0\rangle)}{2\theta\varrho}\right)\right|^2 \,\mathrm{d}x \le c\psi^p\left(\theta^{-n-p}\varepsilon^p+\theta^{2\alpha}\right),$$

for $c \equiv c(n,N,p)$. Notice that

$$\tag{5.12} \begin{cases} |z_{2\theta\varrho}| = |(V_p(Du))_\varrho|^{2/p} \le \mathfrak{I}_p(Dh;B_{2\theta\varrho}) \overset{(2.29)_1}{\le} c'\mathfrak{I}_p(Dh;B_\varrho) \overset{(5.10)_1}{\le} c' \\ \psi \overset{(5.3)_{2,3}}{<} c_\chi^{1/p}\varepsilon_2^{2/p} + \left(\frac{4\varepsilon_3}{\varepsilon_4}\right)^{1/(p-1)} \overset{(5.9)}{\le} 1, \end{cases}$$

so we can bound

$$\begin{aligned} \mathfrak{F}(u;B_{\theta\varrho})^2 \quad &\overset{(2.1)}{\le} \quad 4\fint_{B_{\theta\varrho}} |V_p(Du)-V_p(\psi z_{2\theta\varrho})|^2 \,\mathrm{d}x \overset{(2.3)_3}{\le} c\tilde{\mathfrak{F}}(u,\psi z_{2\theta\varrho};B_{\theta\varrho})^2 \\ &\overset{(5.12),(3.4)}{\le} \quad c\mathfrak{K}\left(\fint_{B_{2\theta\varrho}}\left|V_{\psi|z_{2\theta\varrho}|,p}\left(\frac{u-\psi((h)_{2\theta\varrho}-\langle z_{2\theta\varrho},x-x_0\rangle)}{2\theta\varrho}\right)\right|^2 \,\mathrm{d}x\right) \\ &\qquad + c\psi^{2-p}|z_{2\theta\varrho}|^{2-p}\left((\theta\varrho)^m\fint_{B_{2\theta\varrho}}|f|^m \,\mathrm{d}x\right)^{2/m} \\ &\qquad + c\left((\theta\varrho)^m\fint_{B_{2\theta\varrho}}|f|^m \,\mathrm{d}x\right)^{\frac{p}{m(p-1)}} + c\mathbb{1}_{\{q>p\}}\tilde{\mathfrak{F}}(u,\psi z_{2\theta\varrho};B_{2\theta\varrho})^{2q/p} \\ &\overset{(5.11)}{\le} \quad c\mathfrak{K}\left(\psi^p\left(\theta^{-n-p}\varepsilon^p+\theta^{2\alpha}\right)\right) + c\psi^{2-p}|z_{2\theta\varrho}|^{2-p}\left((\theta\varrho)^m\fint_{B_{2\theta\varrho}}|f|^m \,\mathrm{d}x\right)^{2/m} \\ &\qquad + c\left((\theta\varrho)^m\fint_{B_{2\theta\varrho}}|f|^m \,\mathrm{d}x\right)^{\frac{p}{m(p-1)}} + c\mathbb{1}_{\{q>p\}}\tilde{\mathfrak{F}}(u,\psi z_{2\theta\varrho};B_{2\theta\varrho})^{2q/p} \\ &=: \quad \mathrm{(I)}+\mathrm{(II)}+\mathrm{(III)}+\mathrm{(IV)}, \end{aligned}$$

with $c \equiv c(\mathtt{data}, M)$. We continue estimating

$$\begin{aligned} \mathrm{(I)} \quad &\overset{(2.2)_2}{\le} \quad c\psi^p\left(\theta^{-n-p}\varepsilon^p+\theta^{2\alpha}\right) + c\psi^q\left(\theta^{-n-p}\varepsilon^p+\theta^{2\alpha}\right)^{q/p} \\ &\overset{(5.3)_2}{\le} \quad cc_\chi^{q/p}\left(1+\varepsilon_2^{2(q-p)/p}\right)\left(\theta^{-\frac{(n+p)q}{p}}\varepsilon^p+\theta^{2\alpha}\right)\mathfrak{F}(u;B_\varrho)^2 \\ &\qquad + \frac{c}{\varepsilon_4^{q/(p-1)}}\mathfrak{K}\left[\left(\varrho^m\fint_{B_\varrho}|f|^m \,\mathrm{d}x\right)^{1/m}\right]^{p/(p-1)}, \end{aligned}$$

for $c \equiv c(\mathtt{data}, M)$. Moreover, by Young inequality with conjugate exponents $\left(\frac{p}{2-p},\frac{p}{2(p-1)}\right)$ we have

$$\begin{aligned} \mathrm{(II)}+\mathrm{(III)} \quad &\overset{(5.12)_1}{\le} \quad cc_\chi^{(2-p)/p}\mathfrak{F}(u;B_\varrho)^{\frac{2(2-p)}{p}}\theta^{\frac{2(m-n)}{m}}\left(\varrho^m\fint_{B_\varrho}|f|^m \,\mathrm{d}x\right)^{2/m} \\ &\qquad + c\left(\theta^{\frac{2(m-n)}{m}}\varepsilon_4^{\frac{p-2}{p-1}}+\theta^{\frac{(m-n)p}{m(p-1)}}\right)\left(\varrho^m\fint_{B_\varrho}|f|^m \,\mathrm{d}x\right)^{\frac{p}{m(p-1)}} \\ &\le \quad cc_\chi\varepsilon\mathfrak{F}(u;B_\varrho)^2 + c\left(\varepsilon^{\frac{p-2}{2(p-1)}}\theta^{\frac{p(m-n)}{m(p-1)}}+\theta^{\frac{2(m-n)}{m}}\varepsilon_4^{\frac{p-2}{p-1}}\right)\left(\varrho^m\fint_{B_\varrho}|f|^m \,\mathrm{d}x\right)^{\frac{p}{m(p-1)}}, \end{aligned}$$

with $c \equiv c(\mathtt{data}, M)$. Finally, we control

$$\begin{aligned} \mathrm{(IV)} \quad &\le \quad c\psi^q\mathbb{1}_{\{q>p\}}\tilde{\mathfrak{F}}(u_0,z_{2\theta\varrho};B_{2\theta\varrho})^{2q/p} \\ &\overset{(2.13)_1}{\le} \quad c\psi^q\theta^{-nq/p}\mathbb{1}_{\{q>p\}}\mathfrak{I}_p(Du_0-z_{2\theta\varrho};B_\varrho)^q \overset{(5.8),(5.12)_1}{\le} c\mathbb{1}_{\{q>p\}}\psi^q\theta^{-nq/p} \\ &\overset{(5.3)_2}{\le} \quad cc_\chi^{q/p}\mathbb{1}_{\{q>p\}}\theta^{-nq/p}\varepsilon_2^{2(q-p)/p}\mathfrak{F}(u;B_\varrho)^2 + \frac{c\mathbb{1}_{\{q>p\}}}{\varepsilon_4^{q/(p-1)}\theta^{nq/p}}\left(\varrho^m\fint_{B_\varrho}|f|^m \,\mathrm{d}x\right)^{\frac{q}{m(p-1)}}, \end{aligned}$$

for $c \equiv c(\texttt{data}, M)$. Setting

$$\mathcal{J}_1 := \theta^{-\frac{(n+p)q}{p}}\varepsilon^p + \theta^{2\alpha} + \varepsilon + \theta^{-nq/p}\mathbb{1}_{\{q>p\}}\varepsilon_2^{2(q-p)/p};$$
$$\mathcal{J}_2 := \varepsilon_4^{-q/(p-1)}\theta^{-nq/p} + \varepsilon^{\frac{p-2}{2(p-1)}}\theta^{\frac{p(m-n)}{m(p-1)}} + \theta^{\frac{2(m-n)}{m}}\varepsilon_4^{\frac{p-2}{p-1}}$$

and merging the content of all the previous displays we end up with

$$\mathfrak{F}(u;B_{\theta\varrho})^2 \le cc_\chi^{q/p}\mathcal{J}_1\mathfrak{F}(u;B_\varrho)^2 + c\mathcal{J}_2\mathfrak{K}\left[\left(\varrho^m \fint_{B_\varrho} |f|^m \,\mathrm{d}x\right)^{1/m}\right]^{p/(p-1)}, \tag{5.13}$$

for $c \equiv c(\texttt{data}, M)$. We then reduce the size of the various parameter appearing in the definition of $\mathcal{J}_1$ to get

$$\theta^{-\frac{(n+p)q}{p}}(\varepsilon^p + \varepsilon) < \frac{\theta^{2\alpha}}{2^{10}}, \qquad 4\theta^{-(n+2)}\varepsilon_2 + \theta^{-nq/p}\mathbb{1}_{\{q>p\}}\varepsilon_2^{2(q-p)/p} < \frac{\theta^{2\alpha}}{2^{10}}, \tag{5.14}$$

thus fixing dependencies $\varepsilon, \varepsilon_2 \equiv \varepsilon, \varepsilon_2(n, N, p, q, m, \theta)$, and (5.13) becomes

$$\mathfrak{F}(u;B_{\theta\varrho})^2 \le cc_\chi^{q/p}\theta^{2\alpha}\mathfrak{F}(u;B_\varrho)^2 + c\mathfrak{K}\left[\left(\varrho^m \fint_{B_\varrho} |f|^m \,\mathrm{d}x\right)^{1/m}\right]^{p/(p-1)}, \tag{5.15}$$

with $c \equiv c(\texttt{data}, \theta, M)$. Finally, we pick any $\gamma \in (0, \alpha)$, with $\alpha \equiv \alpha(n, N, p)$ being the exponent in $(2.29)_2$ and select $\theta \in (0, 2^{-10})$ so small that

$$cc_\chi^{q/p}\theta^{2(\alpha-\gamma)} \le \frac{1}{2^{10}} \implies \theta \equiv \theta(\texttt{data}, \chi, \gamma, M), \tag{5.16}$$

so with this choice and (5.15) we obtain (5.4) and the proof is complete. □

6. Excess decay and the Regular set

In this section we prove that the excess functional $\mathfrak{F}(\cdot)$ decays on a certain subset of Ω provided the boundedness of the potential $\mathbf{I}^f_{1,m}(\cdot)$. Precisely, with $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ being a local minimizer of (1.2), we set

$$\mathcal{R}_u := \Bigg\{ x_0 \in \Omega \colon \exists\ M \equiv M(x_0) \in (0,\infty),\ \bar{\varrho} \equiv \bar{\varrho}(\texttt{data}, M, f(\cdot)) < \min\{d_{x_0}, 1\},\ \bar{\varepsilon} \equiv \bar{\varepsilon}(\texttt{data}, M)$$
$$\text{such that } |(V_p(Du))_{B_\varrho(x_0)}| < M \text{ and } \mathfrak{F}(u; B_\varrho(x_0)) < \bar{\varepsilon} \text{ for some } \varrho \in (0, \bar{\varrho}] \Bigg\}. \tag{6.1}$$

According to the discussion in [26, Section 5.1], the set $\mathcal{R}_u$ is well defined and open, with full n-dimensional Lebesgue measure. In particular, given any point $x_0 \in \mathcal{R}_u$, there exists an open neighborhood $B(x_0) \subset \mathcal{R}_u$ of x_0 and a radius $\varrho_{x_0} \in (0, \bar{\varrho}]$ such that

$$|(V_p(Du))_{B_{\varrho_{x_0}}(x)}| < M \quad \text{and} \quad \mathfrak{F}(u; B_{\varrho_{x_0}}(x)) < \bar{\varepsilon} \qquad \text{for all } \ x \in B(x_0). \tag{6.2}$$

We stress that for a given point $x_0 \in \mathcal{R}_u$, all the radii considered from now on will be implicitly assumed to be less than $\min\{d_{x_0}, 1\}$. Next, for $x_0 \in \mathcal{R}_u$ verifying conditions (6.1) for some $M \equiv M(x_0) > 0$, and parameters $\bar{\varepsilon}, \bar{\varrho}$ still to be fixed, we set $\nu := 2^{-2}$, choose $\gamma = \alpha/2$ in (5.4), $\beta = \gamma$ in (4.5), define $\alpha_0 := \gamma$ and let $\chi := \varepsilon_0$ in $(5.3)_1$. This eventually fixes the dependency of all the parameters appearing in Propositions 4.1, 4.2 and 5.1 on $(\texttt{data}, M)$. We then define parameters

$$\hat{\varepsilon} := \frac{\varepsilon_2\varepsilon_0^2\mathfrak{m}(\tau\theta)^{32npq}}{2^{40npq+10}c_3c_3'H}, \qquad H := 2^{8(n+10)}c_3\max\{\tau^{-n}, \varepsilon_0^{-1}\}, \tag{6.3}$$

constants

$$c_2 := 4(c_0 + c_1), \qquad c_3 := c_2 \max_{\delta \in \{\nu,\tau,\theta\}}(1-\delta^{\alpha_0})^{-1}$$
$$c_3' := \left(\frac{c_3 2^{28nq}H}{\varepsilon_0\mathfrak{m}(\tau\theta)^{16nq}}\right)^{\frac{p}{2(p-1)}}, \qquad \mathfrak{m} := \min_{\delta \in \{\nu,\tau,\theta\}}(1-\delta^{\alpha_0}), \tag{6.4}$$

introduce the balanced composite excess functional

$$(0, \varrho] \ni s \mapsto \mathfrak{C}(x_0; s) := \mathfrak{F}(u; B_s(x_0)) + |(V_p(Du))_{B_s(x_0)}| \tag{6.5}$$

and assume that

$$\mathbf{I}^f_{1,m}(x_0, 1) < \infty. \tag{6.6}$$

Notice that, up to extending $f \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, the above position makes sense. Moreover, in (6.6) the finiteness of $\mathbf{I}^f_{1,m}(x_0, \cdot)$ is assumed to hold at radius one, but of course we can suppose that it holds at any positive radius.

By (6.6) and the absolute continuity of Lebesgue integral, we can find $\hat{\varrho} \equiv \hat{\varrho}(\texttt{data}, f(\cdot), M) \in (0, \min\{1, d_{x_0}\})$ such that

$$c_4\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,s)\right)^{\frac{p}{2(p-1)}} + c_4M^{(2-p)/p}\mathbf{I}^f_{1,m}(x_0,s) < \hat{\varepsilon}, \qquad c_4 := \left(\frac{2^{80npq}c_3^2c_3'H^2}{\varepsilon_0^4\varepsilon_3(\tau\theta)^{64npq}\mathfrak{m}}\right)^{\frac{p^2}{4(p-1)^2}} \tag{6.7}$$

for all $s \in (0, \hat{\varrho}]$. Moreover, since potentials can be discretized via dyadic type sums, if $\delta \in \{\nu, \tau, \theta\}$ it is

$$\max_{\delta\in\{\nu,\tau,\theta\}}\left\{\sum_{j=0}^{\infty}\left((\delta^{j+1}s)^m \fint_{B_{\delta^{j+1}s}(x_0)}|f|^m\,\mathrm{d}x\right)^{1/m}\right\} \le \frac{2^{4n}\mathbf{I}^f_{1,m}(x_0,s)}{(\tau\theta)^{2n}}, \tag{6.8}$$

for all $s \in (0, \hat{\varrho}]$. Recalling that $\nu > \max\{\tau, \theta\}$, by (6.8) and routine interpolation arguments we obtain that

$$\left(\sigma^m \fint_{B_\sigma(x_0)}|f|^m\,\mathrm{d}x\right)^{1/m} \le \frac{2^{8n}\mathbf{I}^f_{1,m}(x_0,s)}{(\tau\theta)^{4n}} \qquad \text{for all } \ 0 < \sigma \le s/4, \tag{6.9}$$

which, together with (6.7) yields:

$$\sup_{\sigma\le s/4}\mathfrak{K}\left[\left(\sigma^m \fint_{B_\sigma(x_0)}|f|^m\,\mathrm{d}x\right)^{1/m}\right]^{\frac{p}{2(p-1)}} + M^{(2-p)/p}\sup_{\sigma\le s/4}\left(\sigma^m \fint_{B_\sigma(x_0)}|f|^m\,\mathrm{d}x\right)^{1/m} < \hat{\varepsilon}\left(\frac{\varepsilon_0^4\varepsilon_3\mathfrak{m}(\tau\theta)^{56npq}}{2^{64npq}c_3^2c_3'H^2}\right)^{\frac{p^2}{4(p-1)^2}} \tag{6.10}$$

for all $s \in (0, \hat{\varrho}]$ and

$$\lim_{\sigma\to 0}\left(\sigma^m \fint_{B_\sigma(x_0)}|f|^m\,\mathrm{d}x\right)^{1/m} = 0. \tag{6.11}$$

We refer to [26, Section 5.2] for more details on this matter. In (6.1), we pick $\bar{\varepsilon} = \hat{\varepsilon}$, $\bar{\varrho} = \hat{\varrho}$, thus determining a ball $B_\varrho(x_0) \Subset \Omega$ with $\varrho \in (0, \hat{\varrho}]$ on which

$$|(V_p(Du))_{B_\varrho(x_0)}| < M \qquad \text{and} \qquad \mathfrak{F}(u; B_\varrho(x_0)) < \hat{\varepsilon} \tag{6.12}$$

hold true. Now we are ready to prove the main result of this section.

Theorem 4. *Under assumptions* (2.12)-(2.14), (2.16), (2.23) *and* (6.6), *let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of* (1.2), $x_0 \in \mathcal{R}_u$ *be a point and* $M \equiv M(x_0) > 0$ *be the corresponding constant in* (6.1). *There are* $\hat{\varepsilon} \equiv \hat{\varepsilon}(\texttt{data}, M) \in (0,1)$ *and* $\hat{\varrho} \equiv \hat{\varrho}(\texttt{data}, M, f(\cdot)) < d_{x_0}$ *as in* $(6.3)_1$ *and* (6.7) *respectively, such that if* $\bar{\varepsilon} \equiv \hat{\varepsilon}$ *and* $\bar{\varrho} \equiv \hat{\varrho}$ *in* (6.1), *then for all balls* $B_\varsigma(x_0) \subset B_\varrho(x_0)$ *it holds*

$$\begin{cases} |(V_p(Du))_{B_\varsigma(x_0)}| < 8(1+M) \\ |(V_p(Du))_{B_\varsigma(x_0)}| \le c_6\left(\mathfrak{C}(x_0;\varrho) + \mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varrho)\right)^{\frac{p}{2(p-1)}}\right), \end{cases} \tag{6.13}$$

and

$$\begin{aligned} \mathfrak{F}(u; B_\varsigma(x_0)) \quad &\le \quad c_5\left(\frac{\varsigma}{\varrho}\right)^{\alpha_0}\mathfrak{F}(u; B_\varrho(x_0)) + c_6\sup_{\sigma\le\varrho/4}\mathfrak{K}\left[\left(\sigma^m \fint_{B_\sigma(x_0)}|f|^m\ \mathrm{d}x\right)^{1/m}\right]^{\frac{p}{2(p-1)}} \\ &\quad + c_6\left(\mathfrak{C}(x_0;\varrho) + \mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varrho)\right)^{\frac{p}{2(p-1)}}\right)^{\frac{2-p}{p}}\sup_{\sigma\le\varrho/4}\left(\sigma^m \fint_{B_\sigma(x_0)}|f|^m\ \mathrm{d}x\right)^{1/m}, \end{aligned} \tag{6.14}$$

with $c_5, c_6 \equiv c_5, c_6(\texttt{data}, M)$ *and* $\alpha_0 \equiv \alpha_0(n, N, p) \in (0,1)$.

Proof. For the ease of reading, we split the proof into five steps.

Step 1: decay estimates at the first scale. For $j \in \mathbb{N} \cup \{0\}$, $\nu = 1/4$, and τ, θ from Propositions 4.1, 5.1 respectively, we introduce the following notation: $\tau_j := \tau^j$, $\theta_j := \theta^j$, $\nu_j := \nu^j$ with $\tau_0 \equiv \theta_0 \equiv \nu_0 = 1$, $r_1 := \nu_1\varrho$ and, for $s > 0$ we set:

$$\mathfrak{F}(s) := \mathfrak{F}(u; B_s(x_0)), \qquad V(s) := |(V_p(Du))_{B_s(x_0)}|,$$

$$\mathfrak{C}(s) := \mathfrak{C}(x_0; s), \qquad \mathfrak{S}(s) := \left(s^m \fint_{B_s(x_0)}|f|^m\ \mathrm{d}x\right)^{1/m}$$

$$\mathfrak{H}_{\mathfrak{s}} := \sup_{s\le\varrho/4}\mathfrak{S}(s), \qquad \mathfrak{K}_{\mathfrak{s}} := \sup_{s\le\varrho/4}\mathfrak{K}\left(\mathfrak{S}(s)\right).$$

We then estimate

$$
(6.15)\qquad \begin{cases} \mathfrak{F}(r_1) \stackrel{(2.9)_1}{\le} 2^{1+n}\mathfrak{F}(\varrho) \stackrel{(6.12)_2}{<} 2^{n+1}\hat{\varepsilon} \stackrel{(6.3)_1}{<} \dfrac{\varepsilon_2(\tau\theta)^{4npq}}{2^{8npq}} < \varepsilon_2 \\ V(r_1) \stackrel{(2.9)_2}{\le} 2^n\mathfrak{F}(\varrho)+V(\varrho) \stackrel{(6.12)}{<} 2^n\hat{\varepsilon}+M \stackrel{(6.3)_1}{\le} \dfrac{1}{2}+M. \end{cases}
$$

With the content of display (6.15) at hand, we can start iterations.

Step 2: maximal iteration chains. Let us recall from [58, Section 12.4] the definition of maximal iteration chains. Given any nonempty set of indices $\mathcal{J}_0 \subset \mathbb{N}\cup\{0\}$, for $\kappa\in\mathbb{N}$ the maximal iteration chain of length κ starting at ι is defined as:

$$
\mathcal{C}^{\kappa}_{\iota} := \left\{ j\in\mathbb{N}\cup\{0\} \colon \iota\le j\le \iota+\kappa,\ \iota\in\mathcal{J}_0,\ \iota+\kappa+1\in\mathcal{J}_0,\ j\notin\mathcal{J}_0 \text{ if } j>\iota \right\},
$$

i.e., $\mathcal{C}^{\kappa}_{\iota}=\{\iota,\iota+1,\cdots,\iota+\kappa\}$ and all its elements lie outside $\mathcal{J}_0$ except ι, which belongs to $\mathcal{J}_0$. Furthermore, $\mathcal{C}^{\kappa}_{\iota}$ is maximal, in the sense that it cannot be properly contained in any other set of the same kind. Similarly, the infinite maximal chain starting at ι is given by

$$
\mathcal{C}^{\infty}_{\iota} := \left\{ j\in\mathbb{N}\cup\{0\} \colon \iota\le j<\infty,\ \iota\in\mathcal{J}_0,\ j\notin\mathcal{J}_0 \text{ if } j>\iota \right\}.
$$

We then look at two different alternatives:

$$
(6.16)\qquad \mathfrak{C}(r_1) > \left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} \qquad \text{or} \qquad \mathfrak{C}(r_1) \le \left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}},
$$

with $\mathfrak{H}_{\mathfrak{s}}$ and r_1 defined at the beginning of **Step 1**, H is the constant in $(6.3)_2$ and ε_0 is the same parameter appearing in (4.3).

Step 3: large composite excess at the first scale. In order to repeatedly apply (4.5), (4.20) and (5.4) while keeping under control the various parameters involved and avoiding any blow-up of the bounding constants, let us prepare the set-up for the Blocks and Chains technique introduced in [26, Section 5.2]. We assume that $(6.16)_1$ holds and, with $\nu=1/4$ as in **Step 1**, we consider the set of indices

$$
\mathcal{J}_0 := \left\{ j\in\mathbb{N}\cup\{0\} \colon \mathfrak{C}(\nu_j r_1) > \left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} \right\}.
$$

Notice that $\mathcal{J}_0\neq\emptyset$ by $(6.16)_1$. We then look at two possibilities:

i. there is at least one maximal iteration chain $\mathcal{C}^{\kappa}_{\iota}$ starting at $\iota\in\mathcal{J}_0$ for some $\kappa\le\infty$;
ii. $\mathcal{J}_0\equiv\mathbb{N}\cup\{0\}$.

We first examine occurrence (*i.*) at its worst: we assume that there are (countably) infinitely many finite maximal iteration chains $\{\mathcal{C}^{\kappa_d}_{\iota_d}\}_{d\in\mathbb{N}}$ corresponding to the discrete sequences $\{\iota_d\}_{d\in\mathbb{N}},\{\kappa_d\}_{d\in\mathbb{N}}\subset\mathbb{N}$. By maximality it is easy to see that $\mathcal{C}^{\kappa_{d_1}}_{\iota_{d_1}}\cap\mathcal{C}^{\kappa_{d_2}}_{\iota_{d_2}}=\emptyset$ for $d_1\neq d_2$ and

$$
(6.17)\qquad \iota_{d+1}\ge\iota_d+\kappa_d+1 \implies \{\iota_d\}_{d\in\mathbb{N}} \text{ is increasing and } \iota_d\to\infty.
$$

By (6.17) we can split the reference interval $(0,r_1]$ into the union of disjoint blocks as $(0,r_1]=\bigcup_{d\in\mathbb{N}\cup\{0\}}\mathtt{B}_d$, where it is $\mathtt{B}_0:=\mathtt{I}_0\cup\mathtt{I}^1_1\cup\mathtt{K}_1$, $\mathtt{B}_d:=\mathtt{I}^2_d\cup\mathtt{I}^1_{d+1}\cup\mathtt{K}_{d+1}$ for $d\in\mathbb{N}$, with

$$
\begin{aligned}
&\mathtt{I}_0 := (\nu_{\iota_1}r_1, r_1], && \mathtt{K}_d := (\nu_{\iota_d+\kappa_d+1}r_1, \nu_{\iota_d+1}r_1]\\
&\mathtt{I}^1_d := (\nu_{\iota_d+1}r_1, \nu_{\iota_d}r_1], && \mathtt{I}^2_d := (\nu_{\iota_{d+1}}r_1, \nu_{\iota_d+\kappa_d+1}r_1],
\end{aligned}
$$

and we shall implicitly identify $\mathtt{I}_0\equiv\mathtt{I}^2_0$. By construction, the intervals described in the above display are disjoint and only $\mathtt{I}^2_d$ may be empty. The very definition of maximal iteration chains for the choice of $\mathcal{J}_0$ made above yields that

$$
(6.18)\qquad \begin{cases} \mathfrak{C}(\nu_{\iota_d}r_1) > \left(\dfrac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} & \text{for all } d\in\mathbb{N} \\ \mathfrak{C}(\nu_j r_1) \le \left(\dfrac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} & \text{for all } j\in\{\iota_d+1,\cdots,\iota_d+\kappa_d\},\quad d\in\mathbb{N}, \end{cases}
$$

so if $\varsigma\in\mathtt{K}_d$ we can find $j_\varsigma\in\{\iota_d+1,\cdots,\iota_d+\kappa_d\}$ such that $\nu_{j_\varsigma+1}r_1<\varsigma\le\nu_{j_\varsigma}r_1$ and

$$
(6.19)\qquad \mathfrak{C}(\varsigma) \stackrel{(2.11)}{\le} 2^{2+n}\mathfrak{C}(\nu_{j_\varsigma}r_1) \stackrel{(6.18)_2}{\le} 2^{n+2}\left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}.
$$

On the other hand, if $\varsigma\in\mathtt{I}_0$ or $\varsigma\in\mathtt{I}^2_d$, $d\in\mathbb{N}$, it is possible to determine $j_\varsigma\in\{0,\cdots,\iota_1-1\}$ or $j_\varsigma\in\{\iota_d+\kappa_d+1,\cdots,\iota_{d+1}-1\}$ verifying $\nu_{j_\varsigma+1}r_1<\varsigma\le\nu_{j_\varsigma}r_1$ and

$$
(6.20)\qquad \mathfrak{C}(\varsigma) \stackrel{(2.11)}{\ge} \frac{1}{2^{2+n}}\mathfrak{C}(\nu_{j_\varsigma+1}r_1) \stackrel{(6.18)_1}{>} \frac{1}{2^{2+n}}\left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}.
$$

Next, if $\mathtt{I}_d^2 = \emptyset$ so $\mathtt{B}_d = \mathtt{I}_{d+1}^1 \cup \mathtt{K}_{d+1}$, it turns out that the adjacent blocks $\mathtt{B}_{d-1}$-$\mathtt{B}_d$ contain two consecutive chains. In fact, in this case it is $\iota_{d+1} = \iota_d + \kappa_d + 1$, therefore $\mathtt{B}_{d-1} \cup \mathtt{B}_d = \mathtt{I}_{d-1}^2 \cup \mathtt{I}_d^1 \cup \mathtt{K}_d \cup \mathtt{I}_{d+1}^1 \cup \mathtt{K}_{d+1}$ and if in particular $\varsigma \in \mathtt{K}_d \cup \mathtt{I}_{d+1}^1 \cup \mathtt{K}_{d+1}$, there is $j_\varsigma \in \{\iota_d + 1, \cdots, \iota_{d+1} + \kappa_{d+1}\}$ such that $\nu_{j_\varsigma+1} r_1 < \varsigma \leq \nu_{j_\varsigma} r_1$ and

$$\begin{cases} \mathfrak{C}(\varsigma) \overset{(2.11)}{\leq} 2^{n+2}\mathfrak{C}(\nu_{j_\varsigma} r_1) \overset{(6.18)_2}{<} 2^{n+2} \left(\dfrac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} & \text{if } \ j_\varsigma \neq \iota_{d+1} \\ \mathfrak{C}(\varsigma) \overset{(2.11)}{\leq} 2^{n+2}\mathfrak{C}(\nu_{\iota_{d+1}} r_1) \overset{(2.11)}{\leq} 2^{2n+4}\mathfrak{C}(\nu_{\iota_d+\kappa_d} r_1) \overset{(6.18)_2}{<} 2^{2n+4} \left(\dfrac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} & \text{if } \ j_\varsigma = \iota_{d+1}, \end{cases}$$

so in any case we have that

$$\mathfrak{C}(\varsigma) < 2^{2n+4} \left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} \qquad \text{for all } \ \varsigma \in \mathtt{K}_d \cup \mathtt{B}_d. \tag{6.21}$$

We then consider two occurrences:

$$\varepsilon_0 V(r_1) \leq \mathfrak{F}(r_1) \qquad \text{or} \qquad \varepsilon_0 V(r_1) > \mathfrak{F}(r_1) \tag{6.22}$$

assume that $(6.22)_1$ holds and introduce a second set of indices defined as

$$\mathcal{J}_1 := \left\{ j \in \mathbb{N} \cup \{0\} \colon \varepsilon_0 V(\theta_j r_1) \leq \mathfrak{F}(\theta_j r_1) \right\},$$

which is nonempty given that $0 \in \mathcal{J}_1$ by $(6.22)_1$.

Step 3.1: the singular regime is stable. In this case

$$\mathcal{J}_1 \equiv \mathbb{N} \cup \{0\}, \tag{6.23}$$

so we can ignore the presence of blocks $\{\mathtt{B}_d\}_{d\in\mathbb{N}\cup\{0\}}$ and proceed in a more standard way, cf. [35,77]. By (6.15), (6.10) and $(6.22)_1$ we see that Proposition 5.1 applies and gives

$$\begin{cases} \mathfrak{F}(\theta_1 r_1) \overset{(5.4)}{\leq} \theta^{\alpha_0}\mathfrak{F}(r_1) + c_1\mathfrak{K}\left(\mathfrak{S}(r_1)\right)^{\frac{p}{2(p-1)}} \overset{(6.15)_1,(6.10)}{<} \varepsilon_2 \\ V(\theta_1 r_1) \overset{(6.23)}{\leq} \dfrac{\mathfrak{F}(\theta_1 r_1)}{\varepsilon_0} \overset{(6.24)_1}{\leq} \dfrac{1}{\varepsilon_0}\left(\theta^{\alpha_0}\mathfrak{F}(r_1) + c_1\mathfrak{K}(\mathfrak{S}(r_1))^{\frac{p}{2(p-1)}}\right) \overset{(2.9)_1}{\leq} \dfrac{1}{\varepsilon_0}\left(2^{n+1}\mathfrak{F}(\varrho) + c_1\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right) \\ V(\theta_1 r_1) \overset{(6.24)_2,(6.15)_1,(6.10)}{\leq} 1, \end{cases} \tag{6.24}$$

where we also used that by $(2.13)_1$ it is $\frac{p}{2(p-1)} > 1$. Let us fix $j \in \mathbb{N}$ and assume that

$$\mathfrak{F}(\theta_i r_1) < \varepsilon_2 \qquad \text{for all } \ i \in \{0, \cdots, j\}. \tag{6.25}$$

As a consequence of (6.23) and (6.25), we have

$$V(\theta_i r_1) \ \leq \ \frac{\mathfrak{F}(\theta_i r_1)}{\varepsilon_0} \overset{(6.25)}{<} \frac{\varepsilon_2}{\varepsilon_0} \overset{(5.14),(5.16)}{\leq} 1. \tag{6.26}$$

Thanks to (6.25)-(6.26) we can apply (5.4) at the $\theta_j r_1$-scale to get

$$\begin{aligned} \mathfrak{F}(\theta_{j+1} r_1) &\overset{(5.4)}{\leq} \theta^{\alpha_0}\mathfrak{F}(\theta_j r_1) + c_1\mathfrak{K}\left(\mathfrak{S}(\theta_j r_1)\right)^{\frac{p}{2(p-1)}} \\ &\leq \theta^{\alpha_0(j+1)}\mathfrak{F}(r_1) + c_1\sum_{i=0}^{j}\theta^{\alpha_0(j-i)}\mathfrak{K}\left(\mathfrak{S}(\theta_i r_1)\right)^{\frac{p}{2(p-1)}} \\ &\leq \theta^{\alpha_0(j+1)}\mathfrak{F}(r_1) + c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}} \overset{(6.10),(6.15)_1}{<} \varepsilon_2, \end{aligned} \tag{6.27}$$

where $\alpha_0 = \alpha/2$ has been introduced at the beginning of Section 6, and, via (6.23), (6.27), (5.14), (5.16), the choice of $\chi(=\varepsilon_0)$ made at the beginning of Section 6 and $(5.9)_2$, we obtain

$$V(\theta_{j+1} r_1) \leq \frac{\mathfrak{F}(\theta_{j+1} r_1)}{\varepsilon_0} \leq \frac{\varepsilon_2}{\varepsilon_0} \leq 1. \tag{6.28}$$

The arbitrariety of $j \in \mathbb{N}$ and (6.23) allow concluding that (6.27)-(6.28) hold for all $j \in \mathbb{N} \cup \{0\}$; in particular it is

$$V(\theta_{j+1} r_1) \overset{(6.23)}{\leq} \frac{\mathfrak{F}(\theta_{j+1} r_1)}{\varepsilon_0} \overset{(6.27)}{\leq} \frac{1}{\varepsilon_0}\left(\theta^{\alpha_0(j+1)}\mathfrak{F}(r_1) + c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right) \overset{(2.9)}{\leq} \frac{1}{\varepsilon_0}\left(2^{n+1}\mathfrak{F}(\varrho) + c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right), \tag{6.29}$$

for all $j \in \mathbb{N} \cup \{0\}$. Standard interpolative arguments, (6.27), (6.29), and (6.15) then yield that whenever $\varsigma \in (0, r_1]$ there is $j_\varsigma \in \mathcal{J}_1$ such that $\theta_{j_\varsigma+1} r_1 < \varsigma \leq \theta_{j_\varsigma} r_1$,

$$\mathfrak{F}(\varsigma) \leq \frac{2^{4+n}}{\theta^{1+n/2}}\left(\frac{\varsigma}{\varrho}\right)^{\alpha_0}\mathfrak{F}(\varrho) + \frac{2c_3}{\theta^{n/2}}\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}, \qquad V(\varsigma) \leq \frac{3}{2} \tag{6.30}$$

and

$$V(\varsigma) \le |V(\varsigma)-V(\theta_{j_\varsigma}r_1)|+V(\theta_{j_\varsigma}r_1) \stackrel{(2.10)_2,(6.29)}{\le} \frac{\mathfrak{F}(\theta_{j_\varsigma}r_1)}{\theta^{n/2}}+\frac{1}{\varepsilon_0}\left(2^{n+1}\mathfrak{F}(\varrho)+c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right)$$

$$\stackrel{(6.27)}{\le} \frac{2^{n+4}c_3}{\theta^{n/2}\varepsilon_0}\left(\mathfrak{F}(\varrho)+V(\varrho)+\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right)=:\mathfrak{V}_1. \tag{6.31}$$

Finally, if $\varsigma\in(r_1,\varrho]$ via (6.15) and (2.10) it is

$$\mathfrak{F}(\varsigma)\le 2^{4+n}\left(\frac{\varsigma}{\varrho}\right)^{\alpha_0}\mathfrak{F}(\varrho) \qquad \text{and} \qquad V(\varsigma)\le 2^n\mathfrak{F}(\varrho)+V(\varrho)<1+M \tag{6.32}$$

Merging the content of the three previous displays we obtain

$$\begin{cases} \mathfrak{F}(\varsigma)\le \dfrac{2^{4+n}}{\theta^{1+n/2}}\left(\dfrac{\varsigma}{\varrho}\right)^{\alpha_0}\mathfrak{F}(\varrho)+\dfrac{2c_3}{\theta^{n/2}}\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}} \\ V(\varsigma)\le 2+M, \qquad V(\varsigma)\le\mathfrak{V}_1 \end{cases} \quad \text{for all} \ \ \varsigma\in(0,\varrho]. \tag{6.33}$$

Step 3.2: first change of scale. If (6.23) does not hold, there exists $j_1\in\mathbb{N}$ such that

$$j_1:=\min\{j\in\mathbb{N}\colon \varepsilon_0V(\theta_jr_1)>\mathfrak{F}(\theta_jr_1)\},$$

and by $(6.22)_1$ it is $j_1\ge 1$. We can rephrase the minimality character of j_1 as

$$\varepsilon_0V(\theta_jr_1)\le\mathfrak{F}(\theta_jr_1) \ \text{ for all } \ j\in\{0,\cdots,j_1-1\} \qquad \text{and} \qquad \varepsilon_0V(\theta_{j_1}r_1)>\mathfrak{F}(\theta_{j_1}r_1), \tag{6.34}$$

therefore by $(6.34)_1$ and (6.10) we deduce that (6.27)-(6.29) hold for all $j\in\{0,\cdots,j_1-1\}$. In particular, it is

$$\mathfrak{F}(\theta_{j_1}r_1)\le\theta^{\alpha_0j_1}\mathfrak{F}(r_1)+c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}, \qquad V(\theta_{j_1}r_1)\le\frac{3}{2}, \qquad V(\theta_{j_1}r_1)\le\mathfrak{V}_1, \tag{6.35}$$

with $\mathfrak{V}_1$ being defined in (6.31). We set $r_2:=\theta_{j_1}r_1$ and, keeping in mind the shorthand described at the beginning of **Step 1**, we introduce a new set of indices

$$\mathcal{J}_2:=\left\{j\in\mathbb{N}\cup\{0\}\colon \varepsilon_0V(\tau_jr_2)>\mathfrak{F}(\tau_jr_2)\right\},$$

which is nonempty given that $0\in\mathcal{J}_2$ because of $(6.34)_2$ and the very definition of r_2.

Step 3.3: the nonsingular regime is stable. Let us assume that

$$\mathcal{J}_2\equiv\mathbb{N}\cup\{0\} \ \text{ that is } \ \varepsilon_0V(\tau_jr_2)>\mathfrak{F}(\tau_jr_2) \ \text{ for all } \ j\in\mathbb{N}\cup\{0\}. \tag{6.36}$$

By induction, we want to show that

$$\begin{cases} V(\tau_jr_2)\le 2, \qquad V(\tau_jr_2)\le\mathfrak{V}_2 \\ \mathfrak{F}(\tau_{j+1}r_2)\le\tau^{(j+1)\alpha_0}\mathfrak{F}(r_2)+c_0\displaystyle\sum_{i=0}^{j}\tau^{\alpha_0(j-i)}V(\tau_ir_2)^{(2-p)/p}\mathfrak{S}(\tau_ir_2) \end{cases} \tag{6.37}$$

for all $j\in\mathbb{N}\cup\{0\}$, where we set

$$\mathfrak{V}_2:=c_3'\left(\mathfrak{F}(\varrho)+V(\varrho)+\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varrho)\right)^{\frac{p}{2(p-1)}}\right) \tag{6.38}$$

and $c_3'\equiv c_3'(\texttt{data},M)$ has been defined at the beginning of Section 6. For $j=0$, by $(6.35)_{2,3}$, $(6.34)_2$ and Propositions 4.1-4.2 we obtain

$$\begin{cases} \mathfrak{F}(\tau_1r_2)\stackrel{(4.5),(4.20)}{\le}\tau^{\alpha_0}\mathfrak{F}(r_2)+c_0V(r_2)^{(2-p)/p}\mathfrak{S}(r_2) \\ V(r_2)\le\dfrac{3}{2}, \qquad V(r_2)\le\mathfrak{V}_1, \end{cases} \tag{6.39}$$

thus, recalling that

$$\mathfrak{V}_1<2^{-4}\mathfrak{V}_2 \tag{6.40}$$

by definition, (6.37) is proven for $j=0$. Next, let us fix $j\in\mathbb{N}$ and assume the validity of (6.37) for all $i\in\{0,\cdots,j\}$. In particular it holds that

$$\begin{cases} \mathfrak{F}(\tau_{i+1}r_2)\le\tau^{\alpha_0(i+1)}\mathfrak{F}(r_2)+c_0\displaystyle\sum_{k=0}^{i}\tau^{\alpha_0(i-k)}V(\tau_kr_2)^{(2-p)/p}\mathfrak{S}(\tau_kr_2) \\ V(\tau_ir_2)\le 2, \qquad V(\tau_ir_2)\le\mathfrak{V}_2, \end{cases} \tag{6.41}$$

for all $i \in \{0, \cdots, j\}$, therefore we estimate using the discrete Fubini theorem and Young inequality with conjugate exponents $\left(\frac{p}{2-p}, \frac{p}{2(p-1)}\right)$,

$$
\begin{aligned}
V(\tau_{j+1}r_2) \quad & \leq \quad V(r_2)+\sum_{i=0}^{j}|V(\tau_{i+1}r_2)-V(\tau_i r_2)| \overset{(2.9)_3,(6.39)_2}{\leq} \mathfrak{V}_1+\frac{1}{\tau^{n/2}}\sum_{i=0}^{j}\mathfrak{F}(\tau_i r_2)\\
& \overset{(6.41)_1}{\leq} \quad \mathfrak{V}_1+\frac{\mathfrak{F}(r_2)}{\tau^{n/2}}+\frac{\mathfrak{F}(r_2)}{\tau^{n/2}}\sum_{i=0}^{j-1}\tau^{\alpha_0(i+1)}+\frac{c_0}{\tau^{n/2}}\sum_{i=0}^{j-1}\sum_{k=0}^{i}\tau^{\alpha_0(i-k)}V(\tau_k r_2)^{\frac{2-p}{p}}\mathfrak{S}(\tau_k r_2)\\
& \overset{(6.35)}{\leq} \quad \mathfrak{V}_1+\frac{2}{\tau^{n/2}\mathfrak{m}}\left(\theta^{\alpha_0 j_1}\mathfrak{F}(r_1)+c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right)+\frac{c_0}{\tau^{n/2}}\sum_{i=0}^{j}\sum_{k=0}^{i}\tau^{\alpha_0(i-k)}V(\tau_k r_2)^{\frac{2-p}{p}}\mathfrak{S}(\tau_k r_2)\\
& \overset{(2.9)_1}{\leq} \quad \mathfrak{V}_1+\frac{2}{\tau^{n/2}\mathfrak{m}}\left(2^{n+1}\mathfrak{F}(\varrho)+c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right)+\frac{c_0}{\tau^{n/2}}\sum_{k=0}^{j}V(\tau_k r_2)^{\frac{2-p}{p}}\mathfrak{S}(\tau_k r_2)\left(\sum_{i=k}^{j}\tau^{\alpha_0(i-k)}\right)\\
& \overset{(6.9),(6.41)_2}{\leq} \quad \mathfrak{V}_1+\frac{2}{\tau^{n/2}\mathfrak{m}}\left(2^{n+1}\mathfrak{F}(\varrho)+\frac{c_3 2^{\frac{4npq}{p-1}}}{(\tau\theta)^{\frac{2npq}{p-1}}}\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varrho)\right)^{\frac{p}{2(p-1)}}\right)\\
& \qquad +\frac{c_0\mathfrak{V}_2^{\frac{2-p}{p}}}{\tau^{n/2}\mathfrak{m}}\left(\mathfrak{S}(r_2)+\sum_{k=0}^{j-1}\mathfrak{S}(\tau_{k+1}r_2)\right)\\
& \overset{(6.40),(6.8)}{\leq} \quad \left(\frac{1}{2^4}+\frac{1}{2^2}\right)\mathfrak{V}_2+\frac{2^{n+2}\mathfrak{F}(\varrho)}{\tau^{n/2}\mathfrak{m}}+\frac{2^{4n+1}c_0}{(\tau\theta)^{3n}\mathfrak{m}}\mathfrak{V}_2^{\frac{2-p}{p}}\mathbf{I}^f_{1,m}(x_0,\varrho)\\
& \qquad +\frac{c_3 2^{\frac{4npq}{p-1}+1}}{\mathfrak{m}(\tau\theta)^{\frac{3npq}{p-1}}}\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varrho)\right)^{\frac{p}{2(p-1)}}\\
& \overset{(6.3),(6.12)_2}{\leq} \quad \left(\frac{1}{2^4}+\frac{1}{2^2}+\frac{1}{2^{10}}\right)\mathfrak{V}_2+\frac{c_3 2^{\frac{8npq}{p-1}}}{\mathfrak{m}(\tau\theta)^{\frac{4npq}{p-1}}}\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varrho)\right)^{\frac{p}{2(p-1)}}\\
& \overset{(6.7)}{\leq} \quad \left(\frac{1}{2^4}+\frac{1}{2^2}+\frac{1}{2^{10}}+\frac{1}{2^{20}}\right)\mathfrak{V}_2 \leq \mathfrak{V}_2
\end{aligned}
\tag{6.42}
$$

where $\mathfrak{m}$ has been defined in $(6.4)_4$, and, estimating $V(\tau_{j+1}r_2)$ in a slightly different way than (6.42) we also get

$$
\begin{aligned}
V(\tau_{j+1}r_2) \quad & \leq \quad V(r_2)+\frac{2\mathfrak{F}(r_2)}{\tau^{n/2}\mathfrak{m}}+\frac{c_0}{\tau^{n/2}}\sum_{i=0}^{j-1}\sum_{k=0}^{i}\tau^{\alpha_0(i-k)}V(\tau_k r_2)^{(2-p)/p}\mathfrak{S}(\tau_k r_2)\\
& \overset{(6.41)_2}{\leq} \quad V(r_2)+\frac{2\mathfrak{F}(r_2)}{\tau^{n/2}\mathfrak{m}}+\frac{c_0 2^{(2-p)/p}}{\tau^{n/2}\mathfrak{m}}\sum_{k=0}^{j}\mathfrak{S}(\tau_k r_2)\\
& \overset{(6.35)}{\leq} \quad \frac{3}{2}+\frac{2}{\tau^{n/2}\mathfrak{m}}\left(\theta^{j_1\alpha_0}\mathfrak{F}(r_1)+c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right)+\frac{c_0 2^{(2-p)/p}}{\tau^{n/2}\mathfrak{m}}\left(\mathfrak{S}(r_2)+\sum_{k=0}^{j}\mathfrak{S}(\tau_{k+1}r_2)\right)\\
& \overset{(2.9)_1,(6.8)}{\leq} \quad \frac{3}{2}+\frac{2}{\tau^{n/2}\mathfrak{m}}\left(2^{n+1}\mathfrak{F}(\varrho)+c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right)+\frac{c_0 2^{4n+2}\mathbf{I}^f_{1,m}(x_0,\varrho)}{(\tau\theta)^{3n}\mathfrak{m}}\\
& \overset{(6.3)_1,(6.10)}{\leq} \quad \frac{3}{2}+\frac{1}{2^{10}}+\frac{1}{2^{20}}+\frac{c_0 2^{4n+2}\mathbf{I}^f_{1,m}(x_0,\varrho)}{(\tau\theta)^{3n}\mathfrak{m}} \overset{(6.7)}{\leq} \frac{3}{2}+\frac{1}{2^{10}}+\frac{1}{2^{20}}+\frac{1}{2^{20}} \leq 2.
\end{aligned}
\tag{6.43}
$$

We can then combine (6.36), (6.42)-(6.43) and Proposition 4.1-4.2 to get

$$
\begin{aligned}
\mathfrak{F}(\tau_{j+2}r_2) \quad & \leq \quad \tau^{\alpha_0}\mathfrak{F}(\tau_{j+1}r_2)+c_0V(\tau_{j+1}r_2)^{(2-p)/p}\mathfrak{S}(\tau_{j+1}r_2)\\
& \overset{(6.37)_2}{\leq} \quad \tau^{\alpha_0(j+2)}\mathfrak{F}(r_2)+c_0\sum_{k=0}^{j+1}\tau^{\alpha_0(j+1-k)}V(\tau_k r_2)^{(2-p)/p}\mathfrak{S}(\tau_k r_2).
\end{aligned}
\tag{6.44}
$$

Inequalities (6.42)-(6.44) prove the validity of the induction step, so by the arbitrariety of $j \in \mathbb{N}$ we can conclude that (6.37) holds for all $j \in \mathbb{N} \cup \{0\}$ and, once established this, we can refine $(6.37)_2$ as

$$
\mathfrak{F}(\tau_{j+1}r_2) \leq \tau^{(j+1)\alpha_0}\mathfrak{F}(r_2)+c_3\mathfrak{V}_2^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}. \tag{6.45}
$$

Next, if $\varsigma \in (0,r_2]$ there is $j_\varsigma \in \mathbb{N}\cup\{0\}$ such that $\tau_{j_\varsigma+1}r_2<\varsigma\le \tau_{j_\varsigma}r_2$ and, via (2.9), (2.10), (6.7), (6.10), (6.35), (6.37) and (6.42)-(6.44) we have

$$\begin{cases} \mathfrak{F}(\varsigma)\le \frac{2^{n+4}}{\tau^{1+n/2}}\left(\frac{\varsigma}{\varrho}\right)^{\alpha_0}\mathfrak{F}(\varrho)+\frac{2c_3}{\tau^{n/2}}\left(\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\mathfrak{V}_2^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\right)\\ V(\varsigma)\le 3,\qquad V(\varsigma)\le 2\mathfrak{V}_2 \end{cases}$$

while if $\varsigma\in(r_2,r_1]$ we can find $j_\varsigma\in\{0,\cdots,j_1-1\}$ verifying $\theta_{j_\varsigma+1}r_1<\varsigma\le\theta_{j_\varsigma}r_1$, so as done before we can confirm the validity of estimates (6.30)-(6.31), and when $\varsigma\in(r_1,\varrho]$ the bounds in (6.32) trivially hold true. All in all, we can conclude with

$$\begin{cases} \mathfrak{F}(\varsigma)\le \frac{2^{n+4}}{(\tau\theta)^{1+n/2}}\left(\frac{\varsigma}{\varrho}\right)^{\alpha_0}\mathfrak{F}(\varrho)+\frac{2c_3}{(\tau\theta)^{n/2}}\left(\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\mathfrak{V}_2^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\right)\\ V(\varsigma)\le 3(1+M),\qquad V(\varsigma)\le 2\mathfrak{V}_2 \end{cases} \tag{6.46}$$

for all $\varsigma\in(0,\varrho]$, where we accounted also for the case in which we directly started from $(6.22)_2$ in case of stable nonsingular regime - just set $j_1=0$, replace r_2 with r_1 above and recall $(6.15)_2$.

Step 3.4: second change of scale and block $\mathtt{B}_0$. We now examine the case in which $\mathcal{J}_2\not\equiv\mathbb{N}\cup\{0\}$, i.e., there exists $j_2\in\mathbb{N}$ such that

$$j_2:=\min\left\{j\in\mathbb{N}\colon \varepsilon_0V(\tau_jr_2)\le\mathfrak{F}(\tau_jr_2)\right\}, \tag{6.47}$$

and $(6.34)_2$ assures that $j_2\ge1$. The minimality of j_2 renders that

$$\varepsilon_0V(\tau_jr_2)>\mathfrak{F}(\tau_jr_2)\ \text{ for all }\ j\in\{0,\cdots,j_2-1\}\qquad\text{and}\qquad \varepsilon_0V(\tau_{j_2}r_2)\le\mathfrak{F}(\tau_{j_2}r_2). \tag{6.48}$$

Set $r_3:=\tau_{j_2}r_2$ and notice that we can repeat the same procedure described in *Step 3.3* a finite number of times for getting

$$\mathfrak{F}(\tau_{j+1}r_2)\le\tau^{\alpha_0(j+1)}\mathfrak{F}(r_2)+c_3\mathfrak{V}_2^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}},\qquad V(\tau_jr_2)\le 2(M+1),\qquad V(\tau_jr_2)\le\mathfrak{V}_2, \tag{6.49}$$

for all $j\in\{0,\cdots,j_2-1\}$[6]. Next we prove that

$$r_3 \text{ cannot belong to } \mathtt{I}_0. \tag{6.50}$$

By contradiction, assume that (6.50) does not hold. We would then have

$$\begin{aligned} V(\tau_{j_2-1}r_2) &\overset{(2.9)_3}{\le} V(r_3)+\frac{1}{\tau^{n/2}}\mathfrak{F}(\tau_{j_2-1}r_2)\\ &\overset{(6.48)_1}{\le} V(r_3)+\frac{\varepsilon_0}{\tau^{n/2}}V(\tau_{j_2-1}r_2)\ \overset{(4.18)_2}{\Longrightarrow}\ 2V(r_3)\ge V(\tau_{j_2-1}r_2), \end{aligned} \tag{6.51}$$

so recalling that $(6.48)_1$-$(6.49)_2$ legalize the application of Propositions 4.1-4.2, we obtain

$$\begin{aligned} \mathfrak{F}(r_3) &\overset{(4.5),(4.20)}{\le} \tau^{\alpha_0}\mathfrak{F}(\tau_{j_2-1}r_2)+c_0V(\tau_{j_2-1}r_2)^{(2-p)/p}\mathfrak{S}(\tau_{j_2-1}r_2)\\ &\overset{(6.48)_1,(6.51)}{\le} 2\tau^{\alpha_0}\varepsilon_0V(r_3)+c_02^{(2-p)/p}V(r_3)^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\\ &\overset{(6.20),(6.5)}{\le} 2\tau^{\alpha_0}\varepsilon_0V(r_3)+\frac{2^{2n+6}\varepsilon_0c_0\mathfrak{C}(r_3)}{H}\\ &\overset{(2.9)_1}{\le} \varepsilon_0V(r_3)\left(2\tau^{\alpha_0}+\frac{2^{2n+6}c_0}{H}\right)+\frac{2^{2n+8}\varepsilon_0c_0\mathfrak{F}(\tau_{j_2-1}r_2)}{\tau^{n/2}H}\\ &\overset{(6.48)_1,(6.51)}{\le} \varepsilon_0V(r_3)\left(2\tau^{\alpha_0}+\frac{2^{2n+6}c_0}{H}+\frac{2^{2n+10}\varepsilon_0c_0}{\tau^{n/2}H}\right)\overset{(4.18)_1,(6.3)_2}{<}\varepsilon_0V(r_3), \end{aligned} \tag{6.52}$$

thus contradicting $(6.48)_2$[7]. Therefore (6.50) is true and in particular it holds that

$$\mathtt{I}_0\subseteq(r_3,r_2]\cup(r_2,r_1]\qquad\text{or}\qquad\mathtt{I}_0\subseteq(r_3,r_1], \tag{6.53}$$

depending on whether we started from (6.22) with a change of scale or from $(6.22)_2$. This means that if $\varsigma\in\mathtt{I}_0$ we can find $j_\varsigma\in\{0,\cdots,j_1-1\}$ or $j_\varsigma\in\{0,\cdots,j_2-1\}$ such that either $\theta_{j_\varsigma+1}r_1<\varsigma\le\theta_{j_\varsigma}r_1$ or $\tau_{j_\varsigma+1}r_2<\varsigma\le\tau_{j_\varsigma}r_2$ but in any case the estimates in (6.46) are valid. Next, we observe that

$$\begin{aligned} \mathfrak{F}(\nu_{\iota_1}r_1) &\overset{(2.9)_1}{\le} 2^{1+n}\mathfrak{F}(\nu_{\iota_1-1}r_1)\\ &\overset{(6.46)_1}{\le} \frac{2^{2n+5}}{(\tau\theta)^{1+n/2}}\left(\frac{\nu_{\iota_1-1}r_1}{\varrho}\right)^{\alpha_0}\mathfrak{F}(\varrho)+\frac{2^{n+2}c_3}{(\tau\theta)^{n/2}}\left(\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\mathfrak{V}_2^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\right) \end{aligned}$$

[6]In comparison to (6.37), here we included also the case in which we directly start from $(6.22)_2$. In fact, by $(6.15)_2$ the bound on averages increases by $2M$.

[7]Notice that in (6.51) we have also used that the constant c appearing in $(4.18)_2$ is larger than one.

$$\leq \frac{2^{2n+7}}{(\tau\theta)^{1+n/2}}\left(\frac{\nu_{\iota_1}r_1}{\varrho}\right)^{\alpha_0}\mathfrak{F}(\varrho)+\frac{2^{n+2}c_3}{(\tau\theta)^{n/2}}\left(\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\mathfrak{V}_2^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\right) \tag{6.54}$$

and, concerning the average via Young inequality we have

$$\begin{aligned}
V(\nu_{\iota_1}r_1) &\overset{(2.9)_2}{\leq} 2^n\mathfrak{F}(\nu_{\iota_1-1}r_1)+V(\nu_{\iota_1-1}r_1)\\
&\overset{(6.46)}{\leq} \frac{2^{2n+4}}{(\tau\theta)^{1+n/2}}\left(\frac{\nu_{\iota_1-1}r_1}{\varrho}\right)^{\alpha_0}\mathfrak{F}(\varrho)+\frac{2^{n+1}c_3}{(\tau\theta)^{n/2}}\left(\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\mathfrak{V}_2^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\right)+3(1+M)\\
&\overset{(6.12)_1,(6.38)}{\leq} 3(M+1)+\left(\frac{2^{2n+4}}{(\tau\theta)^{1+n/2}}+\frac{c_3'}{4}\right)\mathfrak{F}(\varrho)+\left(\frac{c_3 2^{n+1}}{(\tau\theta)^{n/2}}+2^{\frac{2-p}{p-1}}\left(\frac{2^{2n+2}c_3}{(\tau\theta)^{n/2}}\right)^{\frac{p}{2(p-1)}}\right)\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\\
&\qquad+\frac{c_3'}{4}\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varrho)\right)^{\frac{p}{2(p-1)}}+\frac{2^{n+1}c_3(c_3')^{(2-p)/p}}{(\tau\theta)^{n/2}}M^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\\
&\overset{(6.7),(6.10)}{\leq} \frac{1}{2}+3(M+1)+\mathfrak{F}(\varrho)\left(\frac{2^{2n+4}}{(\tau\theta)^{1+n/2}}+\frac{c_3'}{4}\right)\overset{(6.12)_2,(6.3)}{\leq} 4(M+1),
\end{aligned} \tag{6.55}$$

and, using also the definition of c_3' we get

$$\begin{aligned}
V(\nu_{\iota_1}r_1) &\overset{(2.9)_2,(6.46)_2}{\leq} 2^n\mathfrak{F}(\nu_{\iota_1-1}r_1)+2\mathfrak{V}_2\\
&\overset{(6.46)_1}{\leq} \frac{2^{2n+4}}{(\tau\theta)^{1+n/2}}\mathfrak{F}(\varrho)+\frac{2^{n+1}c_3}{(\tau\theta)^{n/2}}\left(\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\mathfrak{V}_2^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\right)+2\mathfrak{V}_2\\
&\overset{(6.9)}{\leq} \frac{9}{4}\mathfrak{V}_2+\frac{2^{2n+4}}{(\tau\theta)^{1+n/2}}\mathfrak{F}(\varrho)+\left(\frac{2^{16nq}}{(\tau\theta)^{8n}}\right)^{\frac{p}{2(p-1)}}\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varrho)\right)^{\frac{p}{2(p-1)}}\leq 3\mathfrak{V}_2.
\end{aligned} \tag{6.56}$$

Once (6.54)-(6.56) are available, given any $\varsigma\in\mathtt{I}_1^1$ by (2.9) we obtain

$$\begin{cases}\mathfrak{F}(\varsigma)\leq \frac{2^{3n+10}}{(\tau\theta)^{1+n/2}}\left(\frac{\varsigma}{\varrho}\right)^{\alpha_0}\mathfrak{F}(\varrho)+\frac{2^{2n+3}c_3}{(\tau\theta)^{n/2}}\left(\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\mathfrak{V}_2^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\right)\\ V(\varsigma)\leq 5(1+M),\qquad V(\varsigma)\leq 4\mathfrak{V}_2.\end{cases} \tag{6.57}$$

Finally, if $\varsigma\in\mathtt{K}_1$, by (6.19), (6.10), (6.9), and (6.7) we can conclude that

$$\begin{cases}\mathfrak{F}(\varsigma)\leq 2^{n+2}\left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}\\ V(\varsigma)\leq 1,\qquad V(\varsigma)\leq\mathfrak{V}_2.\end{cases} \tag{6.58}$$

Combining estimates (6.46), (6.57), and (6.58) we can conclude that for all $\varsigma\in\mathtt{B}_0$ we obtain

$$\begin{cases}\mathfrak{F}(\varsigma)\leq \frac{2^{4n+12}}{(\tau\theta)^{1+n/2}}\left(\frac{\varsigma}{\varrho}\right)^{\alpha_0}\mathfrak{F}(\varrho)+\frac{2^{2n+4}c_3}{(\tau\theta)^{n/2}}\left(\frac{H}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}\left(\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\mathfrak{V}_2^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\right)\\ V(\varsigma)\leq 5(1+M),\qquad V(\varsigma)\leq 4\mathfrak{V}_2.\end{cases} \tag{6.59}$$

Step 3.5: the general block $\mathtt{B}_d$. Here we prove that in the regularity perspective, each block $\mathtt{B}_d$ acts independently for all $d\in\mathbb{N}$ (the case $d=0$ is contained in *Step 3.4*). Recalling the definition of $\mathtt{B}_d$ given in **Step 3**, we immediately notice that if $\mathtt{I}_d^2=\emptyset$, we can conclude with (6.21) and, recalling the definitions given in (6.5) and (6.38), estimate (6.9), and the smallness condition in $(6.7)_1$, we also secure $(6.58)_2$. Next, define quantities

$$\mathfrak{V}_{1,d}:=c_{3,d}'\left(\mathbf{I}^f_{1,m}(x_0,\varrho)\right)^{\frac{p}{2(p-1)}},\qquad c_{3,d}':=\left(\frac{2^{10n}H}{(\tau\theta)^{4n}\varepsilon_0}\right)^{\frac{p}{2(p-1)}}$$

$$\mathfrak{V}_{2,d}:=c_{3,d}''\left(\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varrho)\right)\right)^{\frac{p}{2(p-1)}},\qquad c_{3,d}'':=\left(\frac{2^{20nq}c_3H}{\mathfrak{m}(\tau\theta)^{16nq}\varepsilon_0^2}\right)^{\frac{p}{2(p-1)}},$$

assume $\mathtt{I}_d^2\neq\emptyset$ and observe that since $d\geq 1$, the construction carried out in **Step 2**-**Step 3** gives a chain $\mathtt{K}_d$ preceding $\mathtt{I}_d$, thus

$$\mathfrak{C}(\nu_{\iota_d+\kappa_d+1}r_1)\overset{(2.11)}{\leq} 2^{n+2}\mathfrak{C}(\nu_{\iota_d+\kappa_d}r_1)\overset{(6.18)_2}{\leq} 2^{n+2}\left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}, \tag{6.60}$$

which, by means of (6.5), (6.10), (6.9) and $(6.3)_1$ yields:

$$\begin{cases} \mathfrak{F}(\nu_{\iota_d+\kappa_d+1}r_1)\le 2^{n+2}\left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}<\frac{\varepsilon_2(\tau\theta)^{4npq}}{2^{8npq}}<\varepsilon_2\\ V(\nu_{\iota_d+\kappa_d+1}r_1)\le 1,\qquad V(\nu_{\iota_d+\kappa_d+1}r_1)\le \mathfrak{V}_{1,d}. \end{cases} \tag{6.61}$$

Therefore, up to make the following substitutions:

$$\begin{aligned} &r_1\rightsquigarrow r_d:=\nu_{\iota_d+\kappa_d+1}r_1,\qquad r_2\rightsquigarrow r_{2;d}:=\theta_{j_1}r_d,\qquad r_3\rightsquigarrow r_{3;d}:=\tau_{j_2}r_{2;d},\\ &\mathfrak{V}_1\rightsquigarrow\mathfrak{V}_{1,d},\qquad \mathfrak{V}_2\rightsquigarrow\mathfrak{V}_{2,d}, \end{aligned} \tag{6.62}$$

we can replicate the whole procedure developed in *Step 3.1-Step 3.4*[8] to obtain

$$\begin{cases} \mathfrak{F}(\theta_{j+1}r_d)\overset{(6.30)}{\le}\theta^{\alpha_0(j+1)}\mathfrak{F}(r_d)+c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\overset{(6.60)}{\le}\left(\frac{2^{4n}Hc_3}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\\ V(\theta_{j+1}r_d)\overset{(6.29)}{\le}\frac{1}{\varepsilon_0}\left(\theta^{\alpha_0(j+1)}\mathfrak{F}(r_d)+c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right)\overset{(6.60)}{\le}2^{n+2}\left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0^2}\right)^{\frac{p}{2(p-1)}}+\frac{c_3}{\varepsilon_0}\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\overset{(6.10)}{\le}1\\ V(\theta_{j+1}r_d)\overset{(6.29)}{\le}\frac{1}{\varepsilon_0}\left(\theta^{\alpha_0(j+1)}\mathfrak{F}(r_d)+c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\right)\overset{(6.60)}{\le}2^{n+2}\left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0^2}\right)^{\frac{p}{2(p-1)}}+\frac{c_3}{\varepsilon_0}\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}\overset{(6.9)}{\le}\mathfrak{V}_{2,d}, \end{cases}$$

for all $j\in\mathbb{N}\cup\{0\}$ in case of stability of the singular regime or, in case of a change of scale,

$$\begin{aligned} \mathfrak{F}(\tau_{j+1}r_{2;d})&\overset{(6.45)}{\le}\tau^{(j+1)\alpha_0}\mathfrak{F}(r_{2;d})+c_3\mathfrak{V}_{2,d}^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\\ &\overset{(6.35)_1}{\le}\tau^{(j+1)\alpha_0}\theta^{j_1\alpha_0}\mathfrak{F}(r_d)+\tau^{(j+1)\alpha_0}c_3\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+c_3\mathfrak{V}_{2,d}^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\\ &\overset{(6.60)}{\le}\left(\frac{2^{4n}Hc_3}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+2^4c_3\mathfrak{V}_{2,d}^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}, \end{aligned} \tag{6.63}$$

and, concerning averages,

$$V(\tau_jr_{2;d})\overset{(6.43)}{\le}2,\qquad V(\tau_jr_{2;d})\overset{(6.42)}{\le}\mathfrak{V}_{2,d}, \tag{6.64}$$

for any $j\in\mathbb{N}\cup\{0\}$ if the nonsingular regime is stable. Keeping in mind (6.61) and following the same procedure in *Step 3.3*, we deduce that the same bounds in (6.63)-(6.64) hold also if we started within the nonsingular, stable scenario, modulo replacing $r_{2;d}$ with r_d and letting $j_1=0$. In any case, for all $\varsigma\in(0,r_d]$ it is

$$\begin{cases} \mathfrak{F}(\varsigma)\le\left(\frac{2^{6n}Hc_3}{\varepsilon_0(\tau\theta)^{n/2}}\right)^{\frac{p}{2(p-1)}}\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\frac{2^4c_3}{(\tau\theta)^{n/2}}\mathfrak{V}_{2,d}^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\\ V(\varsigma)\le 5,\qquad V(\varsigma)\le 5\mathfrak{V}_{2,d}. \end{cases} \tag{6.65}$$

Next, let us generalize (6.50) to arbitrary $d\in\mathbb{N}$ by showing that the nonsingular regime remains stable in $\mathtt{I}_d^2$, i.e.:

$$r_{3;d}\text{ cannot belong to }\mathtt{I}_d^2\text{ for all }d\in\mathbb{N}. \tag{6.66}$$

In fact, if (6.66) were false, we would be in the same situation as in (6.51), that, together with the minimality of j_2, cf. (6.47), and $(6.64)_1$, allows reproducing the same computations displayed in (6.52) to contradict the very definition of j_2. As a consequence, we obtain that $\mathtt{I}_d^2\subseteq(r_{3;d},r_{2;d}]\cup(r_{2;d},r_d]$ or $\mathtt{I}_d^2\subseteq(r_{3;d},r_d]$, so whenever $\varsigma\in\mathtt{I}_d^2$, as in *Step 3.4* we assure the validity of (6.65). Since by $(2.9)_1$, (6.65), (6.7), (6.10) and $(6.3)_1$ it holds:

$$\begin{cases} \mathfrak{F}(\nu_{\iota_{d+1}}r_1)\le\left(\frac{2^{8n}Hc_3}{\varepsilon_0(\tau\theta)^{n/2}}\right)^{\frac{p}{2(p-1)}}\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\frac{2^{n+6}c_3}{(\tau\theta)^{n/2}}\mathfrak{V}_{2;d}^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\\ V(\nu_{\iota_{d+1}}r_1)\le 6,\qquad V(\nu_{\iota_{d+1}}r_1)\le 6\mathfrak{V}_{2,d} \end{cases} \tag{6.67}$$

for $\varsigma\in\mathtt{I}_{d+1}^1$ we obtain thanks to (6.67):

$$\begin{cases} \mathfrak{F}(\varsigma)\le\left(\frac{2^{10n}Hc_3}{\varepsilon_0(\tau\theta)^{n/2}}\right)^{\frac{p}{2(p-1)}}\mathfrak{K}_{\mathfrak{s}}^{\frac{p}{2(p-1)}}+\frac{2^{2n+8}c_3}{(\tau\theta)^{n/2}}\mathfrak{V}_{2;d}^{(2-p)/p}\mathfrak{H}_{\mathfrak{s}}\\ V(\varsigma)\le 7,\qquad V(\varsigma)\le 7\mathfrak{V}_{2,d} \end{cases} \tag{6.68}$$

[8]Of course parameters j_1, j_2 appearing in the previous display are defined as in *Step 3.2* and *Step 3.4*, but do not necessarily numerically coincide with those in *Step 3.2* and in *Step 3.4* respectively.

Finally, if $\varsigma \in \mathtt{K}_{d+1}$ we directly have (6.19) and $(6.58)_2$. To summarize, we have just proven the validity of estimates (6.68) for all $\varsigma \in \mathtt{B}_d$, $d \in \mathbb{N}$, thus completing the analysis of the occurrence of an infinite number of finite iteration chains.

Step 3.6: a finite number of finite iteration chains. Assume now that there is only a finite number, say $e_* \in \mathbb{N}$, of finite iteration chains, $\{\mathcal{C}^{\kappa_d}_{\iota_d}\}_{d\in\{1,\cdots,e_*\}}$. Such chains determine blocks $\{\mathtt{B}_d\}_{d\in\{0,\cdots,e_*-1\}}$, on which estimates (6.59) and (6.68) apply and, being only e_* chains, by definition it follows that $\{j \in \mathbb{N}\colon j \ge \iota_{e_*} + \kappa_{e_*} + 1\} \subset \mathcal{J}_0$. Then, for all $\varsigma \in (0, r_1] \setminus \bigcup_{d\in\{0,\cdots,e_*-1\}} \mathtt{B}_d \equiv (0, \nu_{\iota_{e_*}+\kappa_{e_*}+1} r_1]$ there is $j \ge \iota_{e_*} + \kappa_{e_*} + 1$ such that $\nu_{j_\varsigma+1} r_1 < \varsigma \le \nu_{j_\varsigma} r_1$ and

$$\mathfrak{C}(\varsigma) \overset{(6.20)}{\ge} \frac{1}{2^{2+n}} \left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} \qquad \text{for all } \varsigma \in (0, \nu_{\iota_{e_*}+\kappa_{e_*}+1} r_1]. \tag{6.69}$$

Furthermore, we also have that

$$\mathfrak{C}(\nu_{\iota_{e_*}+\kappa_{e_*}+1} r_1) \overset{(2.11)}{\le} 2^{2+n} \mathfrak{C}(\nu_{\iota_{e_*}+\kappa_{e_*}} r_1) \overset{(6.18)_2}{\le} 2^{2+n} \left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} \overset{(6.10),(6.3)_1}{<} \frac{\varepsilon_2 (\tau\theta)^{4npq}}{2^{8npq}},$$

which means that (6.61) holds and, via (6.69) we also see that the same argument leading to (6.50)-(6.66) works in this case as well and renders that the nonsingular regime is stable over the whole $(0, \nu_{\iota_{e_*}+\kappa_{e_*}+1} r_1]$. With these last informations at hand we gain that (6.63) (with $\nu_{\iota_{e_*}+\kappa_{e_*}+1} r_1$ instead of $r_{2;d}$) holds, and as a consequence (6.65) is satisfied for all $\varsigma \in (0, \nu_{\iota_{e_*}+\kappa_{e_*}+1} r_1]$. To summarize, we have just proven that (6.59) or (6.68) hold for all $\varsigma \in (0, r_1]$.

Step 3.7: an infinite iteration chain. We describe the presence of an infinite iteration chains by introducing a number $e_* \in \mathbb{N}$ - assume $e_* \ge 2$ for the moment - and corresponding sets of integers $\{\iota_1, \cdots, \iota_{e_*}\} \subset \mathbb{N}$, $\{\kappa_1, \cdots, \kappa_{e_*-1}\} \subset \mathbb{N}$ and $\kappa_{e_*} = \infty$, determining $e_* - 1$ finite iteration chains $\{\mathcal{C}^{\kappa_d}_{\iota_d}\}_{d\in\{1,\cdots,e_*-1\}}$ and one infinite iteration chains $\mathcal{C}^{\infty}_{\iota_{e_*}}$ that must be unique by maximality. On each of blocks $\{\mathtt{B}_d\}_{d\in\{0,\cdots,e_*-2\}}$ determined by chains $\{\mathcal{C}^{\kappa_d}_{\iota_d}\}_{d\in\{1,\cdots,e_*-1\}}$ estimates (6.59) or (6.68) hold true. Concerning the last chain $\mathcal{C}^{\infty}_{\iota_{e_*}}$, it generates the last block $\mathtt{B}_{e_*-1} = \mathtt{I}^2_{e_*-1} \cup \mathtt{I}^1_{e_*} \cup \mathtt{K}_{e_*}$ with $\mathtt{K}_{e_*} = (0, \nu_{\iota_{e_*}+1} r_1]$. On intervals $\mathtt{I}^2_{e_*-1}$-$\mathtt{I}^1_{e_*}$ (6.67)-(6.68) are verified, while on $\mathtt{K}_{e_*}$ we can simply conclude by means of (6.58). On the other hand, if $e_* = 1$, there is only one block $\mathtt{B}_0 = \mathtt{I}_0 \cup \mathtt{I}^1_1 \cup \mathtt{K}_1 \equiv (0, r_1]$, on which (6.46) and (6.57)-(6.58) holds, therefore we can conclude with (6.59) also in this case.

Step 3.8: occurrence (ii.) Since $\mathcal{J}_0 \equiv \mathbb{N} \cup \{0\}$, inequality (6.20) is satisfied by all $\varsigma \in (0, r_1]$, so the validity of (6.50)-(6.66) is now extended to the full interval $(0, r_1]$ and this guarantees the stability of the nonsingular regime. Therefore we can proceed as done in *Step 3.1*-*Step 3.3* to get (6.46).

Step 4: small composite excess at the first scale. This time, the set $\mathcal{J}_0 \subseteq \mathbb{N} \cup \{0\}$ is defined as

$$\mathcal{J}_0 := \left\{ j \in \mathbb{N} \cup \{0\} \colon \mathfrak{C}(\nu_j r_1) \le \left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} \right\} \overset{(6.16)_2}{\ne} \emptyset.$$

We immediately notice that if $\mathcal{J}_0 \equiv \mathbb{N} \cup \{0\}$, then (6.19) holds for all $\varsigma \in (0, r_1]$ so this, (6.58) and (6.32) give the result. We then look at the case in which there exist infinitely many finite iteration chains $\{\mathcal{C}^{\kappa_d}_{\iota_d}\}_{d\in\mathbb{N}}$ with $\{\iota_d\}_{d\in\mathbb{N}}$, $\{\kappa_d\}_{d\in\mathbb{N}}$ as in (6.17), determining intervals

$$\mathtt{I}_0 := (\nu_{\iota_1+1} r_1, r_1], \qquad \mathtt{K}^1_d := (\nu_{\iota_d+\kappa_d} r_1, \nu_{\iota_d+1} r_1]$$
$$\mathtt{K}^2_d := (\nu_{\iota_d+\kappa_d+1} r_1, \nu_{\iota_d+\kappa_d} r_1] \qquad \mathtt{I}^2_d := (\nu_{\iota_{d+1}+1} r_1, \nu_{\iota_d+\kappa_d+1} r_1]$$

and blocks $\mathtt{B}_0 := \mathtt{I}_0 \cup \mathtt{K}^1_1 \cup \mathtt{K}^2_1$, $\mathtt{B}_d := \mathtt{I}^2_d \cup \mathtt{K}^1_{d+1} \cup \mathtt{K}^2_{d+1}$ such that $(0, r_1] \equiv \bigcup_{d\in\mathbb{N}\cup\{0\}} \mathtt{B}_d$ by (6.17). As done in **Step 3**, we readily observe that

$$\{0, \cdots, \iota_1\},\ \{\iota_d + \kappa_d + 1, \cdots, \iota_{d+1}\} \subset \mathcal{J}_0 \qquad \text{and} \qquad \{\iota_d + 1, \cdots, \iota_d + \kappa_d\} \subset \mathcal{C}^{\kappa_d}_{\iota_d}, \tag{6.70}$$

therefore we have

$$\begin{cases} \varsigma \in \mathtt{I}_0 \ \text{ or } \ \varsigma \in \mathtt{I}^2_d \implies \mathfrak{C}(\varsigma) \le 2^{2+n} \left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}} \\ \varsigma \in \mathtt{K}^1_d \implies \mathfrak{C}(\varsigma) > \frac{1}{2^{2+n}} \left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}. \end{cases} \tag{6.71}$$

Notice that $\mathtt{I}^2_d$ cannot be empty otherwise $\iota_{d+1} = \kappa_d + \iota_d$ and this is not possible by means of (6.70), while if $\mathtt{K}^1_{d+1} = \emptyset$ (i.e. if $\kappa_{d+1} = 1$), we can exploit $(6.71)_1$, (2.11) and that $\iota_d \in \mathcal{J}_0$ for all $d \in \mathbb{N}$ to derive

$$\begin{matrix} \varsigma \in \mathtt{B}_0 \ \text{ with } \ \mathtt{K}^1_1 = \emptyset \\ \varsigma \in \mathtt{B}_d \ \text{ with } \ \mathtt{K}^1_{d+1} = \emptyset, \ \ d \in \mathbb{N} \end{matrix} \implies \mathfrak{C}(\varsigma) \le 2^{2n+4} \left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}; \tag{6.72}$$

in other words, whenever $\mathtt{K}^1_{d+1}=\emptyset$ there is nothing to prove on the related block $\mathtt{B}_d$ and (6.72) and $(6.58)_2$ immediately follow. Now, given a general block $\mathtt{B}_d$ with $\mathtt{K}^1_{d+1}\neq\emptyset$ and $d\in\mathbb{N}\cup\{0\}$, if $\varsigma\in\mathtt{I}_0$ or $\varsigma\in\mathtt{I}^2_d$ estimate (6.71) holds. Next, observe that

$$
\begin{aligned}
\mathfrak{C}(\nu_{\iota_{d+1}+1}r_1) &\overset{(2.11)}{\le} 2^{2+n}\mathfrak{C}(\nu_{\iota_{d+1}}r_1)\overset{(6.70)}{\le} 2^{2+n}\left(\frac{H\mathfrak{H}_{\mathfrak{s}}}{\varepsilon_0}\right)^{\frac{p}{2(p-1)}}\\
&\overset{(6.10),(6.3)_1}{\Longrightarrow} \mathfrak{F}(\nu_{\iota_{d+1}+1}r_1)<\frac{\varepsilon_2(\tau\theta)^{4npq}}{2^{8npq}},\qquad V(\nu_{\iota_{d+1}+1}r_1)\le 1,\qquad V(\nu_{\iota_{d+1}+1}r_1)\le \mathfrak{V}_{1,d},
\end{aligned}
\tag{6.73}
$$

therefore, setting this time $r_d:=\nu_{\iota_{d+1}+1}r_1$ we can plug in the substitutions in (6.62) and apply the content of *Step 3.5* (with $\mathtt{I}^2_d$ replaced by $\mathtt{K}^1_{d+1}$) to get (6.65) and, recalling that $\mathtt{K}^2_{d+1}$ differs from $\mathtt{K}^1_{d+1}$ only by one scale, we recover also (6.68). Merging (6.71), (6.72), (6.65) and (6.68) we can conclude that (6.68) holds for all $\varsigma\in\mathtt{B}_d$, $d\in\mathbb{N}\cup\{0\}$.

Step 4.1: a finite number of finite iteration chains. Let us assume now that there is a finite number, say $e_*\in\mathbb{N}$ of finite iteration chains $\{\mathcal{C}^{\kappa_d}_{\iota_d}\}_{d\in\{1,\cdots,e_*\}}$ and corresponding blocks $\{\mathtt{B}_d\}_{d\in\{0,\cdots,e_*-1\}}$. On every block $\mathtt{B}_d$, $d\in\mathbb{N}\cup\{0\}$, estimates (6.68) apply. Notice that $(0,r_1]\setminus\bigcup_{d=0}^{e_*-1}\mathtt{B}_d=(0,\nu_{\iota_{e_*}+\kappa_{e_*}+1}r_1]$ and, since the last finite iteration chain is $\mathcal{C}^{\kappa_{e_*}}_{\iota_{e_*}}$, it follows that $\{j\in\mathbb{N}\colon j\ge\iota_{e_*}+\kappa_{e_*}+1\}\subset\mathcal{J}_0$, therefore for all $\varsigma\in(0,\nu_{\iota_{e_*}+\kappa_{e_*}+1}r_1]$ the bound in (6.71) is verified, so we confirm again the validity of (6.68).

Step 4.2: an infinite iteration chain. In this case, for $e_*\in\mathbb{N}$ (assume for the moment that $e_*\ge 2$) we can find finite set of integers $\{\iota_1,\cdots,\iota_{e_*}\}\subset\mathbb{N}$, $\{\kappa_1,\cdots,\kappa_{e_*-1}\}\subset\mathbb{N}$ and $\kappa_{e_*}=\infty$, thus determining e_*-1 finite iteration chains $\{\mathcal{C}^{\kappa_d}_{\iota_d}\}_{d\in\{1,\cdots,e_*-1\}}$ and one infinite iteration chain $\mathcal{C}^{\infty}_{\iota_{e_*}}$, that is unique by maximality. Chains $\{\mathcal{C}^{\kappa_d}_{\iota_d}\}_{d\in\{1,\cdots,e_*-1\}}$ determine blocks $\{\mathtt{B}_d\}_{d\in\{1,\cdots,e_*-2\}}$ on which the content of **Step 4** applies and (6.68) holds true, while the presence of $\mathcal{C}^{\infty}_{\iota_{e_*}}$ results into $\mathtt{B}_{e_*-1}=\mathtt{I}^2_{e_*-1}\cup(0,\nu_{\iota_{e_*}+1}r_1]$. If $\varsigma\in\mathtt{I}^2_{e_*-1}$ we directly have (6.71) which in particular implies the validity of (6.73) with $d=e_*-1$, so we can reproduce the content of *Step 3.5* with $r_d=\nu_{\iota_{e_*}+1}r_1$ and eventually arrive at (6.68). Finally if $e_*=1$, there is only the infinite iteration chain, thus $(0,r_1]=\mathtt{I}_0\cup(0,\nu_{\iota_{e_*}+1}r_1]$. On $\mathtt{I}_0$ the bound in (6.71) is in force, this in turn yields (6.73) so, proceeding as in *Step 3.5* we obtain (6.68).

Step 5: conclusions. Collecting estimates (6.33), (6.46), (6.59), and (6.68), and setting

$$
c_5:=\frac{2^{4n+12}}{(\tau\theta)^{1+n/2}},\qquad\qquad c_6:=\left(\frac{2^{40nq}H^2c_3^2c_3'}{\varepsilon_0^4\mathfrak{m}(\tau\theta)^{32nq}}\right)^{\frac{p}{2(p-1)}}
$$

we obtain (6.13)-(6.14) and the proof is complete. □

For later use, let us record a couple of consequences of Theorem 4, that come along the lines of [26, Proposition 5.1 and Corollary 5.1].

Corollary 6.1. *Assume* (2.12)-(2.14), (2.23), *let* $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ *be a local minimizer of* (1.2), $x_0\in\mathcal{R}_u$ *be a point,* $M\equiv M(x_0)>0$ *be the constant in* (6.1), $\hat{\varepsilon}\equiv\hat{\varepsilon}(\mathtt{data},M)$ *and* $\hat{\varrho}\equiv\hat{\varrho}(\mathtt{data},M,f(\cdot))$ *be as in* $(6.3)_1$ *and* (6.7) *respectively.*

- *If*

$$
\mathbf{I}^f_{1,m}(x,\sigma)\to 0\quad\textit{locally uniformly in}\quad x\in\Omega,
\tag{6.74}
$$

then if $\bar{\varepsilon}\equiv\hat{\varepsilon}$ *and* $\bar{\varrho}\equiv\hat{\varrho}$ *in* (6.1), *there is an open neighborhood* $B(x_0)\subset\mathcal{R}_u$ *and a positive radius* $\varrho_{x_0}\equiv\varrho_{x_0}(\mathtt{data},M,f(\cdot))\in(0,\hat{\varrho}]$ *such that*

$$
\begin{cases}
|(V_p(Du))_{B_\varsigma(x)}|<8(1+M)\\[2mm]
|(V_p(Du))_{B_\sigma(x)}|\le c_8\left(\mathfrak{C}(x;\varsigma)+\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x,\varsigma)\right)^{\frac{p}{2(p-1)}}\right)
\end{cases}
\tag{6.75}
$$

and

$$
\begin{aligned}
\mathfrak{F}(u;B_\sigma(x)) \le{}& c_7\left(\frac{\sigma}{\varsigma}\right)^{\alpha_0}\mathfrak{F}(u;B_\varsigma(x))+c_8\sup_{s\le\varsigma/4}\mathfrak{K}\left[\left(s^m\fint_{B_s(x)}|f|^m\,\mathrm{d}x\right)^{1/m}\right]^{\frac{p}{2(p-1)}}\\
&+c_8\left(\mathfrak{C}(x;\varsigma)+\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x,\varsigma)\right)^{\frac{p}{2(p-1)}}\right)^{(2-p)/p}\sup_{s\le\varsigma/4}\left(s^m\fint_{B_s(x)}|f|^m\,\mathrm{d}x\right)^{1/m},
\end{aligned}
\tag{6.76}
$$

hold for all $x\in B(x_0)$, $0<\sigma\le\varsigma\le\varrho_{x_0}$, *where* $c_7:=c_5(2^{12}c_5)^{1+\frac{n}{2\alpha_0}}$ *and* $c_8:=c_6(2^{16}c_5)^{2+\frac{n}{2\alpha_0}}$, $c_7,c_8\equiv c_7,c_8(\mathtt{data},M)$.

- *If* (6.6) *is in force instead of* (6.74)*, then the following "restricted" versions of* (6.75)-(6.76) *hold:*

$$\begin{cases} |(V_p(Du))_{B_\varsigma(x_0)}| < 8(1+M) \\ |(V_p(Du))_{B_\sigma(x_0)}| \le c_8\left(\mathfrak{C}(x_0;\varsigma)+\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varsigma)\right)^{\frac{p}{2(p-1)}}\right) \end{cases} \tag{6.77}$$

and

$$\begin{aligned} \mathfrak{F}(u;B_\sigma(x_0)) \quad \le \quad & c_7\left(\frac{\sigma}{\varsigma}\right)^{\alpha_0}\mathfrak{F}(u;B_\varsigma(x_0))+c_8\sup_{s\le\varsigma/4}\mathfrak{K}\left[\left(s^m\fint_{B_s(x_0)}|f|^m\ \mathrm{d}x\right)^{1/m}\right]^{\frac{p}{2(p-1)}} \\ & +c_8\left(\mathfrak{C}(x_0;\varsigma)+\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0,\varsigma)\right)^{\frac{p}{2(p-1)}}\right)^{(2-p)/p}\sup_{s\le\varsigma/4}\left(s^m\fint_{B_s(x_0)}|f|^m\ \mathrm{d}x\right)^{1/m}, \end{aligned} \tag{6.78}$$

for all balls $B_\sigma(x_0)\subseteq B_\varsigma(x_0)\subseteq B_{\varrho_{x_0}}(x_0)$*.*

- *With* (6.6) *in force, if in* (6.1) *it is* $\bar\varepsilon\equiv\hat\varepsilon$*,* $\bar\varrho\equiv\hat\varrho$*, then*

$$\sup_{\sigma\le\varrho_{x_0}}\mathfrak{F}(u;B_\varrho(x_0))\le c_9\hat\varepsilon, \tag{6.79}$$

with $c_9:=2^8(c_7+c_8)$*,* $c_9\equiv c_9(\mathtt{data},M)$*.*

We conclude this section with an almost everywhere VMO result. To do so, we need some preliminaries. Assume (6.74), let $x_0\in\mathscr{R}_u$ be any point, $M\equiv M(x_0)$ be the positive constant in (6.1). With $\bar\varepsilon$, $\bar\varrho$ still to be determined, we introduce constants:

$$H_1:=\max\left\{2^{24n}Hc_9,\frac{2^{24n}H}{\varepsilon_1(\tau\theta)^{6n}}\right\},\qquad H_2:=\left(\frac{2^{36nq}c_8H}{\varepsilon_1(\tau\theta)^{20nq}}\right)^{\frac{p}{2(p-1)}}, \tag{6.80}$$

and fix

$$\varepsilon_*:=\frac{\hat\varepsilon}{2^{20}c_9}, \tag{6.81}$$

where $\hat\varepsilon\equiv\hat\varepsilon(\mathtt{data},M)$, $H\equiv H(\mathtt{data},M)$ are defined in (6.3). Notice that (6.74) implies that

$$\mathbf{I}^f_{1,m}(\cdot,1)\in L^\infty_{\mathrm{loc}}(\Omega)\qquad\text{and}\qquad\left(s^m\fint_{B_s(x)}|f|^m\ \mathrm{d}x\right)^{1/m}\to 0\quad\text{locally uniformly in}\quad x\in\Omega. \tag{6.82}$$

By means of (6.74), we determine a threshold radius $\varrho_*\equiv\varrho_*(\mathtt{data},M,f(\cdot))\in(0,\hat\varrho]$ such that

$$c_{10}\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x,s)\right)^{\frac{p}{2(p-1)}}+c_{10}M^{(2-p)/p}\mathbf{I}^f_{1,m}(x,s)<\varepsilon_*,\qquad c_{10}:=\left(\frac{2^{24npq}c_9H_2}{\varepsilon_0(\tau\theta)^{20npq}\mathfrak{m}}\right)^{\frac{p}{2(p-1)}} \tag{6.83}$$

for all $s\le\varrho_*$, $x\in B_{d_{x_0}}(x_0)$, which implies via (6.9) that

$$\begin{cases} c_8\mathfrak{K}\left[\sup_{s\le\varrho_*/4}\left(s^m\fint_{B_s(x)}|f|^m\ \mathrm{d}x\right)^{1/m}\right]^{\frac{p}{2(p-1)}}<\frac{1}{2^{10}} \\ c_8\,(2+M)^{\frac{2-p}{p}}\left[\sup_{s\le\varrho_*/4}\left(s^m\fint_{B_s(x)}|f|^m\ \mathrm{d}x\right)^{1/m}\right]<\frac{1}{2^{10}}, \end{cases} \tag{6.84}$$

for all $x\in B_{d_{x_0}}(x_0)$, and recalling the definition of c_9 given in Corollary 6.1, we have $c_{10}>c_4$ and by (6.80) that it is $H_2>H$, so the choice made in (6.83) immediately implies the validity of (6.7) on $B_{d_{x_0}}(x_0)$. Now we are ready to prove:

Proposition 6.1. *Under assumptions* (2.12)-(2.14)*,* (2.23) *and* (6.74)*, let* $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ *be a local minimizer of* (1.2)*. There exists an open set* $\Omega_u\subset\Omega$ *of full n-dimensional Lebesgue measure such that* $Du\in VMO_{\mathrm{loc}}(\Omega_u,\mathbb{R}^{N\times n})$ *which can be characterized as*

$$\Omega_u:=\left\{x_0\in\Omega\colon\exists M\equiv M(x_0)>0\colon|(V_p(Du))_{B_\varrho(x_0)}|<M\ \ \text{and}\ \ \mathfrak{F}(u;B_\varrho(x_0))<\varepsilon_*\ \ \text{for some}\ \ \varrho\in(0,\varrho_*]\right\},$$

with $\varepsilon_*\equiv\varepsilon_*(\mathtt{data},M)$ *as in* (6.83) *and* $\varrho_*\equiv\varrho_*(\mathtt{data},M,f(\cdot))$ *defined by* (6.83)-(6.84)*. In particular, for all* $x_0\in\Omega_u$ *there is an open neighborhood* $B(x_0)\subset\Omega_u$ *such that*

$$\lim_{\varrho\to 0}\mathfrak{F}(u;B_\varrho(x))=0\qquad\text{uniformly for all}\ \ x\in B(x_0). \tag{6.85}$$

Proof. In the light of the discussion at the beginning of Section 6, the ideal candidate for Ω_u is set $\mathcal{R}_u$ as in (6.1) with $\bar{\varepsilon} \equiv \varepsilon_*$ and $\bar{\varrho} \equiv \varrho_*$: in fact, we already know that it is an open set of full n-dimensional Lebesgue measure so we only need to prove the VMO-result. We take $x_0 \in \mathcal{R}_u$ with $\bar{\varepsilon} \equiv \varepsilon_*$, $\bar{\varrho} \equiv \varrho_*$ in (6.1) and observe that (6.74) allows applying the first part of Corollary 6.1, so there exists an open neighborhood $B(x_0) \subset \mathcal{R}_u$ and a positive radius $\varrho_{x_0} \equiv \varrho_{x_0}(\texttt{data}, M, f(\cdot))$ such that (6.75)-(6.76) are verified for all $x \in B(x_0)$ and any $0 < \sigma \le \varsigma \le \varrho_{x_0}$. Of course, we can always assume that $B(x_0) \subset B_{d_{x_0}}(x_0)$. Fixed an arbitrary $r \in (0,1)$, by $(6.82)_2$ we can find a radius $\varrho'' \equiv \varrho''(\texttt{data}, M, f(\cdot)) \in (0, \varrho_{x_0}]$ satisfying

$$c_8 \sup_{s \le \varrho''} \mathfrak{K}\left[\left(\left(\fint_{B_s(x)} |f|^m \, \mathrm{d}x\right)^{1/m}\right]^{\frac{p}{2(p-1)}} + c_8\,(2+M)^{(2-p)/p} \sup_{s \le \varrho''}\left(\fint_{B_s(x)} |f|^m \, \mathrm{d}x\right)^{1/m} \le \frac{r}{2^4}. \tag{6.86}$$

Moreover, via (6.76) with $\sigma \equiv \varrho''$ and $\varsigma \equiv \varrho_{x_0}$, (6.81), (6.83), (6.84) and (6.2) with $\bar{\varepsilon} \equiv \varepsilon_*$, $\bar{\varrho} \equiv \varrho_*$ we obtain

$$\begin{aligned}
\mathfrak{F}(u; B_{\varrho''}(x)) \quad &\le \quad c_7 \left(\frac{\varrho''}{\varrho_{x_0}}\right)^{\alpha_0} \mathfrak{F}(u; B_{\varrho_{x_0}}(x)) + c_8 \sup_{s \le \varrho_{x_0}/4} \mathfrak{K}\left[\left(s^m \fint_{B_s(x)} |f|^m \, \mathrm{d}x\right)^{1/m}\right]^{\frac{p}{2(p-1)}} \\
&\quad + c_8\left(\mathfrak{C}(x; \varrho_{x_0}) + \mathfrak{K}\left(\mathbf{I}^f_{1,m}(x, \varrho_{x_0})\right)^{\frac{p}{2(p-1)}}\right)^{(2-p)/p} \sup_{s \le \varrho_{x_0}/4}\left(s^m \fint_{B_s(x)} |f|^m \, \mathrm{d}x\right)^{1/m} \\
&\le \quad c_7 \varepsilon_* + \frac{1}{2^{10}} \le 1
\end{aligned} \tag{6.87}$$

Finally we pick $\sigma_r \equiv \sigma_r(\texttt{data}, M, f(\cdot)) \in (0, \varrho'']$ small enough that

$$c_9 (\sigma_r/\varrho'')^{\alpha_0} \le r/2. \tag{6.88}$$

Plugging (6.86)-(6.88) in (6.76) with $\sigma \equiv \sigma_r$ and $\varsigma \equiv \varrho''$ we obtain that

$$\sigma \le \sigma_r \implies \mathfrak{F}(u; B_\sigma(x)) \le r \qquad \text{for all} \quad x \in B(x_0).$$

The arbitrariety of r, (6.75) and a standard covering argument eventually lead to (6.85) and the proof is complete. □

Remark 6.1. Let us list some relevant observations.

- Replacing (6.74) with (6.6) in Proposition 6.1, we obtain that whenever $x_0 \in \Omega$ verifies the conditions in (6.1) with $\bar{\varepsilon} \equiv \varepsilon_*$ and $\bar{\varrho} \equiv \varrho_*$ it holds that

$$\lim_{\varrho \to 0} \mathfrak{F}(u; B_\varrho(x_0)) = 0. \tag{6.89}$$

- Corollary 6.1 and Proposition 6.1 remain valid if M is replaced by $8(1+M)$, without affecting the magnitude of the bounding constants appearing in the various estimates as they are all derived in correspondence of larger values than M.
- Corollary 6.1 guarantees in particular that once (6.12) - with $\hat{\varepsilon}$, $\hat{\varrho}$ as in $(6.3)_1$ and (6.7) respectively - is verified for a certain $\varrho \in (0, \hat{\varrho}]$, then the Morrey type decay estimates (6.76) and (6.78) for the excess functional $\mathfrak{F}(\cdot)$ hold at all scales smaller than ϱ. This will allow us to work on all scales smaller that ϱ.

7. Borderline gradient continuity

This final section is devoted to the proof of the partial gradient continuity for minima of (1.2). Let $x_0 \in \mathcal{R}_u$ be any point, $M \equiv M(x_0) > 0$, $\bar{\varepsilon}$, $\bar{\varrho}$ be the parameters appearing in (6.1), still to be fixed as functions of $(\texttt{data}, M)$ and $(\texttt{data}, M, f(\cdot))$ respectively. We assume the validity of (6.6) at x_0, define the smallness threshold

$$\varepsilon' := \frac{\varepsilon_*}{2^8 c_9 \max\{H_1, H_2\}} \implies \varepsilon' \equiv \varepsilon'(\texttt{data}, M) \tag{7.1}$$

and determine the radius $\varrho' \equiv \varrho'(\texttt{data}, M, f(\cdot)) \in (0, \varrho_*]$ so small that

$$c_{11}\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0, s)\right)^{\frac{p}{2(p-1)}} + c_{11} M^{(2-p)/p}\mathbf{I}^f_{1,m}(x_0, s) < \varepsilon', \qquad c_{11} := \left(\frac{c_{10} 2^{32npq} \max\{H_1, H_2\}}{(\tau\theta)^{16npq}}\right)^{\frac{p}{2(p-1)}}. \tag{7.2}$$

for all $s \in (0, \varrho']$, which yields

$$\begin{aligned}
&\sup_{\sigma \le s/4} \mathfrak{K}\left[\left(\sigma^m \fint_{B_\sigma(x_0)} |f|^m \, \mathrm{d}x\right)^{1/m}\right]^{\frac{p}{2(p-1)}} \\
&\quad + M^{(2-p)/p} \sup_{\sigma \le s/4}\left(\sigma^m \fint_{B_\sigma(x_0)} |f|^m \, \mathrm{d}x\right)^{1/m} < \varepsilon' \left(\frac{(\tau\theta)^{12npq}}{2^{24npq} c_{10} \max\{H_1, H_2\}}\right)^{\frac{p}{2(p-1)}},
\end{aligned} \tag{7.3}$$

cf. Section 6. Of course there is no loss of generality in assuming that $0 < \varrho' \le \varrho_* \le \hat{\varrho}$, so setting $\bar{\varepsilon} \equiv \varepsilon'$ and $\bar{\varrho} \equiv \varrho'$ in (6.1) we can find $\varrho \in (0, \varrho']$ such that

$$\mathfrak{F}(u; B_\varrho(x_0)) < \varepsilon' \qquad \text{and} \qquad |(V_p(Du))_{B_\varrho(x_0)}| < M. \tag{7.4}$$

Thanks to the choices made in (7.1)-(7.2) we see that (6.77)-(6.78) and (6.89) are available; later on, we shall strengthen (6.6) by assuming (6.74) thus (6.75)-(6.76) and Proposition 6.1 will be at hand. Now, with $H \equiv H(\texttt{data}, M)$ as in $(6.3)_2$ and $H_1, H_2 \equiv H_1, H_2(\texttt{data}, M)$ being defined in (6.80), we slightly modify the definition of the composite excess functional given in (6.5) and consider its "unbalanced" version:

$$(0, \varrho] \ni s \mapsto \mathfrak{C}_H(x_0; s) := H\mathfrak{F}(u; B_s(x_0)) + |(V_p(Du))_{B_s(x_0)}|$$

and, for $s \in (0, \varrho]$, introduce the nonhomogeneous excess functional:

$$\mathfrak{N}(x_0; s) := H_1\mathfrak{F}(u; B_s(x_0)) + c_{12}H_2\left(\mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0, s)\right)^{\frac{p}{2(p-1)}} + |(V_p(Du))_{B_s(x_0)}|^{(2-p)/p}\mathbf{I}^f_{1,m}(x_0, s)\right),$$

where $\mathfrak{K}(\cdot)$ is defined in (2.2) and $c_{12} := \left(2^{16npq}(\tau\theta)^{-8npq}\right)^{\frac{p}{2(p-1)}}$. Notice that by (6.89), $(6.77)_1$ and (6.6) we have

$$\lim_{\varrho \to 0} \mathfrak{F}(u; B_\varrho(x_0)) = 0 \implies \lim_{\varrho \to 0} \mathfrak{N}(x_0; \varrho) = 0. \tag{7.5}$$

For the ease of exposition, we shall adopt some abbreviations. With $\tau \equiv \tau(\texttt{data}, M)$ being the parameter determined in Propositions 4.1-4.2, $j \in \mathbb{N} \cup \{-1, 0\}$ and $\sigma \in (0, \varrho]$ set $\sigma_j := \tau^{j+1}\sigma$, $\sigma_{-1} := \sigma$ and $B_j := B_{\sigma_j}(x_0)$. From now on we will mostly employ the shorthands described in **Step 1** of the proof of Theorem 4 and, with Remark 6.1 in mind, unless otherwise specified we shall work within the setting designed at the beginning of Section 7.

7.1. **An inductive lemma.** The key tool for proving our sharp partial continuity result is an inductive technical lemma that is the subquadratic counterpart of [26, Lemma 6.1], both inspired by [59, Lemma 6.1].

Lemma 7.1. *Let $x_0 \in \mathscr{R}_u$ be a point with $M \equiv M(x_0)$ being the positive constant in* (6.1)*, γ be a positive number and assume that $\bar{\varepsilon} \equiv \varepsilon'$, $\bar{\varrho} \equiv \varrho'$ in* (6.1) *with $\varepsilon' \equiv \varepsilon'(\texttt{data}, M)$, $\varrho' \equiv \varrho'(\texttt{data}, M, f(\cdot))$ defined in* (7.1)-(7.2)*; that*

$$\mathfrak{N}(x_0; \sigma) \le 2\gamma \qquad \text{for some} \quad \sigma \in (0, \varrho] \tag{7.6}$$

and that, for integers $k \ge i \ge 0$ inequalities

$$\mathfrak{C}_H(\sigma_j) \le \gamma, \qquad \mathfrak{C}_H(\sigma_{j+1}) \ge \frac{\gamma}{16} \qquad \text{for all} \quad j \in \{i, \cdots, k\}, \qquad \mathfrak{C}_H(\sigma_i) \le \frac{\gamma}{4} \tag{7.7}$$

are verified. Then the following holds:

$$\mathfrak{C}_H(\sigma_{k+1}) \le \gamma, \qquad \sum_{j=i}^{k+1} \mathfrak{F}(\sigma_j) \le \frac{\gamma}{2H} \tag{7.8}$$

and

$$\sum_{j=i}^{k+1} \mathfrak{F}(\sigma_j) \le \frac{4\mathfrak{F}(\sigma_i)}{3} + \frac{2^3\gamma^{(2-p)/p}}{3\varepsilon_1\tau^{n/2}}\sum_{j=i}^{k} \mathfrak{G}(\sigma_j), \tag{7.9}$$

where $H, \varepsilon_1 \equiv H, \varepsilon_1(\texttt{data}, M)$ defined in $(6.3)_2$ and in Propositions 4.1-4.2 respectively.

Proof. Our preliminary observation is that $x_0 \in \mathscr{R}_u$ with $\bar{\varepsilon} \equiv \varepsilon'$ and $\bar{\varrho} \equiv \varrho'$ guarantees the validity of (7.4) and of (6.77)-(6.78). A straightforward computation shows that

$$|V(\sigma_j) - V(\sigma_{j+1})| \overset{(2.9)_3}{\le} \frac{\mathfrak{F}(\sigma_j)}{\tau^{n/2}} \le \frac{\mathfrak{C}_H(\sigma_j)}{\tau^{n/2}H} \overset{(6.3)_2,(7.7)_1}{\le} \frac{\gamma}{2^6}. \tag{7.10}$$

Next, let us prove that under (7.6)-(7.7) the singular regime cannot be in force, i.e.:

$$\varepsilon_0 V(\sigma_j) \le \mathfrak{F}(\sigma_j) \quad \text{cannot hold for all} \quad j \in \{i, \cdots, k\}. \tag{7.11}$$

By contradiction, we assume that

$$\text{there is} \quad j \in \{i, \cdots, k\} \quad \text{such that} \quad \varepsilon_0 V(\sigma_j) \le \mathfrak{F}(\sigma_j) \quad \text{holds true} \tag{7.12}$$

and estimate via Young inequality with conjugate exponents $\left(\frac{p}{2-p}, \frac{p}{2(p-1)}\right)$,

$$\begin{aligned} H\mathfrak{F}(\sigma_{j+1}) \quad &\overset{(6.78)}{\le} \quad c_7H\tau^{\alpha_0(j+2)}\mathfrak{F}(\sigma) + c_8H\sup_{s \le \sigma/4}\mathfrak{K}\left(\mathfrak{G}(s)\right)^{\frac{p}{2(p-1)}} \\ &\quad + c_8H\left(\mathfrak{C}(\sigma) + \mathfrak{K}\left(\mathbf{I}^f_{1,m}(x_0, \sigma)\right)^{\frac{p}{2(p-1)}}\right)^{(2-p)/p}\sup_{s \le \sigma/4}\mathfrak{G}(s) \end{aligned}$$

$$
\begin{aligned}
&\leq H(c_7+2c_8)\mathfrak{F}(\sigma)+c_8H\sup_{s\leq\sigma/4}\mathfrak{K}\left(\mathfrak{S}(s)\right)^{\frac{p}{2(p-1)}}\\
&\quad +c_8H\mathfrak{K}(\mathbf{I}^f_{1,m}(x_0,\sigma))^{\frac{2-p}{2(p-1)}}\sup_{s\leq\sigma/4}\mathfrak{S}(s)+c_8HV(\sigma)^{(2-p)/p}\sup_{s\leq\sigma/4}\mathfrak{S}(\sigma)\\
&\overset{(6.9)}{\leq} \mathfrak{N}(x_0;\sigma)\left(\frac{H(c_7+c_8)}{H_1}+\frac{Hc_8}{c_{12}H_2}\left(\frac{2^{8nq}}{(\tau\theta)^{4nq}}\right)^{\frac{p}{2(p-1)}}+\frac{2Hc_8}{c_{12}H_2}\left(\frac{2^{8n}}{(\tau\theta)^{4n}}\right)\right)\\
&\overset{(7.6)}{\leq} \gamma\left(\frac{2H(c_7+c_8)}{H_1}+\frac{2Hc_8}{c_{12}H_2}\left(\frac{2^{8nq}}{(\tau\theta)^{4nq}}\right)^{\frac{p}{2(p-1)}}+\frac{2^2Hc_8}{c_{12}H_2}\left(\frac{2^{8n}}{(\tau\theta)^{4n}}\right)\right)\overset{(6.80)}{\leq}\frac{\gamma}{2^6}. \qquad (7.13)
\end{aligned}
$$

Furthermore, we have

$$
V(\sigma_j)\overset{(7.11)}{\leq}\frac{\mathfrak{F}(\sigma_j)}{\varepsilon_0}\leq\frac{\mathfrak{C}_H(\sigma_j)}{\varepsilon_0H}\overset{(7.7)_1}{\leq}\frac{\gamma}{\varepsilon_0H}\overset{(6.3)_2}{\leq}\frac{\gamma}{2^6} \qquad (7.14)
$$

and so

$$
\mathfrak{C}_H(\sigma_{j+1})\leq|V(\sigma_{j+1})-V(\sigma_j)|+V(\sigma_j)+H\mathfrak{F}(\sigma_{j+1})\overset{(7.10),(7.13),(7.14)}{\leq}\frac{3\gamma}{2^6}<\frac{\gamma}{16},
$$

in contradiction with $(7.7)_2$ and (7.11) is verified. Next, we prove the validity of

$$
\mathfrak{F}(\sigma_{j+1})\leq\frac{\mathfrak{F}(\sigma_j)}{4}+\frac{2\gamma^{(2-p)/p}\mathfrak{S}(\sigma_j)}{\varepsilon_1\tau^{n/2}}. \qquad (7.15)
$$

In the light of (7.11), we have to consider only two possibilities: either (4.4) holds and, given $(7.7)_1$ and the bound imposed on the size of γ, via (4.5) and (4.18) we directly have (7.15); or (4.19) is satisfied and

$$
\mathfrak{F}(\sigma_{j+1})\overset{(2.9)_1}{\leq}\frac{2}{\tau^{n/2}}\mathfrak{F}(\sigma_j)\overset{(4.19)}{\leq}\frac{2V(\sigma_j)^{(2-p)/p}\mathfrak{S}(\sigma_j)}{\varepsilon_1\tau^{n/2}}\overset{(2.13)_1,(7.7)_1}{\leq}\frac{2\gamma^{(2-p)/p}\mathfrak{S}(\sigma_j)}{\varepsilon_1\tau^{n/2}},
$$

and (7.15) follows in any case. Before proceeding further, notice that

$$
\left(\sum_{j=i}^{k}\mathfrak{S}(\sigma_j)\right)^{\frac{p}{2(p-1)}}\overset{(6.8)}{\leq}\frac{2^{\frac{4np}{p-1}}\left(\mathbf{I}^f_{1,m}(x_0,\sigma)\right)^{\frac{p}{2(p-1)}}}{(\tau\theta)^{\frac{2np}{p-1}}}\leq\frac{\mathfrak{N}(x_0;\sigma)}{H_2}\overset{(7.6)}{\leq}\frac{2\gamma}{H_2}. \qquad (7.16)
$$

Summing (7.15) for $j\in\{i,\cdots,k\}$ we obtain

$$
\sum_{j=i+1}^{k+1}\mathfrak{F}(\sigma_j)\leq\frac{1}{4}\sum_{j=i}^{k}\mathfrak{F}(\sigma_j)+\frac{2\gamma^{(2-p)/p}}{\varepsilon_1\tau^{n/2}}\sum_{j=i}^{k}\mathfrak{S}(\sigma_j)
$$

Adding on both sides of the previous inequality $\mathfrak{F}(\sigma_i)$ and reabsorbing terms, we get (7.9). We continue estimating in (7.9):

$$
\sum_{j=i}^{k+1}\mathfrak{F}(\sigma_j)\overset{(7.16)}{\leq}\frac{4\mathfrak{C}_H(\sigma_i)}{3H}+\frac{2^5\gamma}{3\varepsilon_1\tau^{n/2}H_2^{\frac{2(p-1)}{p}}}\overset{(7.7)_3}{\leq}\left(\frac{1}{3H}+\frac{2^5}{3\varepsilon_1\tau^{n/2}H_2^{\frac{2(p-1)}{p}}}\right)\gamma\overset{(6.80)}{\leq}\frac{5\gamma}{12H},
$$

which implies $(7.8)_2$. Finally, we estimate

$$
\begin{aligned}
V(\sigma_{k+1}) &\leq |V(\sigma_{k+1})-V(\sigma_i)|+V(\sigma_i)\overset{(7.7)_3}{\leq}\frac{\gamma}{4}+\sum_{j=i}^{k}|V(\sigma_{j+1})-V(\sigma_j)|\\
&\leq \frac{\gamma}{4}+\frac{1}{\tau^{n/2}}\sum_{j=i}^{k}\mathfrak{F}(\sigma_j)\overset{(7.8)_2}{\leq}\frac{\gamma}{4}+\frac{\gamma}{2H\tau^{n/2}}\overset{(6.3)_2}{\leq}\frac{\gamma}{2}
\end{aligned}
$$

and, combining the content of the above display with $(7.8)_2$ we obtain $(7.8)_1$ and the proof is complete. □

7.2. **Oscillation estimates for large gradients.** For some $\sigma\in(0,\varrho]$ we consider the case in which

$$
\frac{\gamma}{8}:=V(\sigma_0)>\frac{\mathfrak{N}(x_0;\sigma)}{16}\implies\mathfrak{N}(x_0;\sigma)\leq 2\gamma. \qquad (7.17)
$$

Before proceeding further, let us recall that by (6.6), (7.1), (7.2) and (7.4), estimates (6.77)-(6.78) of Corollary 6.1 are available, keep also in mind Remark 6.1. We then prove two technical lemmas, eventually leading to quantitative oscillation estimates for nonzero gradients.

Lemma 7.2. *Assume* (7.17). *Then*

$$\sum_{j=0}^{\infty}\mathfrak{F}(\sigma_j)\le\frac{\mathfrak{N}(x_0;\sigma)}{H}\qquad and\qquad \frac{\gamma}{16}\le V(\sigma_j)\le\gamma\ \ for\ all\ \ j\in\mathbb{N}\cup\{0\},\tag{7.18}$$

for any $\sigma\in(0,\varrho]$ *with* $H\equiv H(\mathtt{data},M)$ *defined in* $(6.3)_2$.

Proof. Let us first prove that

$$V(\sigma_j)\ge\frac{V(\sigma_0)}{2}\qquad\text{for all }\ j\in\mathbb{N}\cup\{0\}.\tag{7.19}$$

Notice that

$$\begin{aligned}V(\sigma_1)\ &\ge\ V(\sigma_0)-|V(\sigma_1)-V(\sigma_0)|\overset{(2.9)_3}{\ge}V(\sigma_0)-\frac{\mathfrak{F}(\sigma_0)}{\tau^{n/2}}\\ &\overset{(2.9)_1}{\ge}\ V(\sigma_0)-\frac{2\mathfrak{F}(\sigma_{-1})}{\tau^n}\ge V(\sigma_0)-\frac{2\mathfrak{N}(x_0;\sigma)}{\tau^nH_1}\\ &\overset{(6.80)}{\ge}\ V(\sigma_0)-\frac{\mathfrak{N}(x_0;\sigma)}{2^7}\overset{(7.17)}{\ge}\frac{V(\sigma_0)}{2}.\end{aligned}$$

By contradiction, we assume that there is a finite exit time index $J\ge2$ such that

$$V(\sigma_J)<\frac{V(\sigma_0)}{2}\qquad\text{and}\qquad V(\sigma_j)\ge\frac{V(\sigma_0)}{2}\quad\text{for all }\ j\in\{0,\cdots,J-1\}.\tag{7.20}$$

Let us preliminary observe that

$$V(\sigma_j)\ge\frac{V(\sigma_0)}{2}\ \text{ for all }\ j\in\{0,\cdots,J-1\}\implies\mathfrak{C}_H(\sigma_j)\le\gamma\ \text{ for all }\ j\in\{0,\cdots,J-1\}.\tag{7.21}$$

To show the validity of implication (7.21), we proceed by induction. By direct calculation, we see that

$$\mathfrak{C}_H(\sigma_0)\overset{(2.9)_1}{\le}V(\sigma_0)+\frac{2H\mathfrak{F}(\sigma_{-1})}{\tau^{n/2}}\le V(\sigma_0)+\frac{2H\mathfrak{N}(x_0;\sigma)}{\tau^{n/2}H_1}\overset{(7.17)}{\le}\gamma\left(\frac{1}{8}+\frac{2^2H}{\tau^{n/2}H_1}\right)\overset{(6.80)}{\le}\frac{\gamma}{4}.\tag{7.22}$$

We then fix an arbitrary $k\in\{0,\cdots,J-2\}$, assume that $\mathfrak{C}_H(\sigma_j)\le\gamma$ holds for all $j\in\{0,\cdots,k\}$ and notice that $(7.20)_2$ and (7.17) yield that $\mathfrak{C}_H(\sigma_{j+1})\ge\gamma/16$ for all $j\in\{0,\cdots,k\}$, therefore keeping (7.17) in mind, we deduce that the assumptions of Lemma 7.1 are verified with $i=0$ and k being the number used here so $\mathfrak{C}_H(\sigma_{k+1})\le\gamma$. Implication (7.21) then follows by the arbitrariety of $k\in\{0,\cdots,J-2\}$. By (7.20)-(7.21) now we know that $\mathfrak{C}_H(\sigma_j)\le\gamma$ for all $j\in\{0,\cdots,J-1\}$ and $\mathfrak{C}_H(\sigma_{j+1})\ge\gamma/16$ for all $j\in\{0,\cdots,J-2\}$, thus via (7.18) we can apply again Lemma 7.1 with $i=0$ and $k=J-2$ to get

$$\sum_{j=0}^{J-1}\mathfrak{F}(\sigma_j)\le\frac{\gamma}{2H}\overset{(7.17)}{\le}\frac{4V(\sigma_0)}{H},\tag{7.23}$$

so we can bound

$$|V(\sigma_J)-V(\sigma_0)|\le\sum_{j=0}^{J-1}|V(\sigma_{j+1})-V(\sigma_j)|\overset{(2.9)_1}{\le}\frac{1}{\tau^{n/2}}\sum_{j=0}^{J-1}\mathfrak{F}(\sigma_j)\overset{(7.23)}{\le}\frac{4V(\sigma_0)}{\tau^{n/2}H}\overset{(6.3)_2}{\le}\frac{V(\sigma_0)}{4}\tag{7.24}$$

for concluding:

$$V(\sigma_J)\ge V(\sigma_0)-|V(\sigma_0)-V(\sigma_J)|\overset{(7.24)}{\ge}\frac{3V(\sigma_0)}{4},$$

in contradiction with $(7.20)_1$. This and the arbitrariety of $J\ge2$ yield validity of (7.19), which in turn implies the left-hand side of inequality $(7.18)_2$ and, applying (7.21) for all $j\in\mathbb{N}\cup\{0\}$ we derive the full chain of inequalities in $(7.18)_2$. We only need to verify $(7.18)_1$. Using (7.17), $(7.18)_2$ and (7.22) we apply Lemma 7.1 with $i=0$ and for every integer k to have

$$\begin{aligned}\sum_{j=0}^{\infty}\mathfrak{F}(\sigma_j)\ &\overset{(7.9)}{\le}\ \frac{4\mathfrak{F}(\sigma_0)}{3}+\frac{2^3\gamma^{(2-p)/p}}{3\varepsilon_1\tau^{n/2}}\sum_{j=0}^{\infty}\mathfrak{S}(\sigma_j)\\ &\overset{(2.9)_1,(6.8)}{\le}\ \frac{8\mathfrak{F}(\sigma_{-1})}{3\tau^{n/2}}+\frac{2^{3+4n}\gamma^{(2-p)/p}\mathbf{I}^f_{1,m}(x_0,\sigma)}{3\varepsilon_1(\tau\theta)^{4n}}\\ &\overset{(7.17)}{\le}\ \frac{8\mathfrak{N}(x_0;\sigma)}{3\tau^{n/2}H_1}+\frac{2^{10n}V(\sigma_0)^{(2-p)/p}\mathbf{I}^f_{1,m}(x_0,\sigma)}{3\varepsilon_1(\tau\theta)^{4n}}\\ &\overset{(2.9)_2}{\le}\ \frac{8\mathfrak{N}(x_0;\sigma)}{3\tau^{n/2}H_1}+\frac{2^{10n}V(\sigma_{-1})^{(2-p)/p}\mathbf{I}^f_{1,m}(x_0,\sigma)}{3\varepsilon_1(\tau\theta)^{4n}}+\frac{2^{12n}\mathfrak{F}(\sigma_{-1})^{(2-p)/p}\mathbf{I}^f_{1,m}(x_0,\sigma)}{3\varepsilon_1(\tau\theta)^{6n}}\end{aligned}$$

$$\leq \mathfrak{N}(x_0;\sigma)\left(\frac{8}{3\tau^{n/2}H_1}+\frac{2^{10n}}{3\varepsilon_1(\tau\theta)^{4n}c_{12}H_2}+\frac{2^{12n}}{3\varepsilon_1(\tau\theta)^{6n}H_1}+\frac{2^{12n}}{3\varepsilon_1c_{12}H_2(\tau\theta)^{6n}}\right)$$

$$\overset{(6.80)}{\leq} \frac{\mathfrak{N}(x_0;\sigma)}{H},$$

where we also used Young inequality with conjugate exponents $\left(\frac{p}{2(p-1)},\frac{p}{2-p}\right)$ and the proof is complete. □

Lemma 7.3. *Whenever $\sigma\in(0,\varrho]$ is such that* (7.17) *holds, the limits in* (1.11) *exist and inequalities*

$$\begin{cases} |V_p(Du(x_0))-(V_p(Du))_{B_\sigma(x_0)}|\leq c\mathfrak{N}(x_0;\sigma) \\ |Du(x_0)-(Du)_{B_\sigma(x_0)}|\leq c\mathfrak{N}(x_0;\sigma)^{2/p}+c|(Du)_{B_\sigma(x_0)}|^{(2-p)/2}\mathfrak{N}(x_0;\sigma)\end{cases} \tag{7.25}$$

hold true for a constant $c\equiv c(\mathtt{data},M)$.

Proof. We start by showing that $\{(V_p(Du))_{B_j}\}_{j\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence. In fact, fixed integers $0\leq i\leq k-1$ we bound

$$\begin{aligned}|(V_p(Du))_{B_k}-(V_p(Du))_{B_i}| &\leq \sum_{j=i}^{k-1}|(V_p(Du))_{B_{j+1}}-(V_p(Du))_{B_j}|\\ &\overset{(2.9)_3}{\leq} \frac{1}{\tau^{n/2}}\sum_{j=i}^{k-1}\mathfrak{F}(\sigma_j)\leq\frac{1}{\tau^{n/2}}\sum_{j=i}^{\infty}\mathfrak{F}(\sigma_j)\overset{(7.18)}{\leq}c\mathfrak{N}(x_0;\sigma),\end{aligned} \tag{7.26}$$

and

$$|(V_p(Du))_{B_0}-(V_p(Du))_{B_{-1}}|\overset{(2.9)_3}{\leq}\frac{\mathfrak{F}(\sigma)}{\tau^{n/2}}\leq c\mathfrak{N}(x_0;\sigma) \tag{7.27}$$

with $c\equiv c(\mathtt{data},M)$, therefore there exists $\ell_V\in\mathbb{R}^{N\times n}$ such that

$$\lim_{j\to\infty}(V_p(Du))_{B_j}=\ell_V. \tag{7.28}$$

Sending $k\to\infty$ in (7.26) we obtain

$$|\ell_V-(V_p(Du))_{B_i}|\leq c\mathfrak{N}(x_0;\sigma)\qquad\text{for all}\ \ i\in\mathbb{N}\cup\{0\}.$$

Now, given any $s\in(0,\sigma]$ - and since we are interested in $s\to0$ we can assume $s\leq\sigma_0$ - there is $j_s\in\mathbb{N}\cup\{0\}$ such that $\sigma_{j_s+1}<s\leq\sigma_{j_s}$ and

$$\begin{aligned}\lim_{s\to0}|\ell_V-(V_p(Du))_{B_s(x_0)}| &\leq \lim_{j_s\to\infty}|\ell_V-(V_p(Du))_{B_{j_s}}|+\lim_{j_s\to\infty}|(V_p(Du))_{B_s(x_0)}-(V_p(Du))_{B_{j_s}}|\\ &\overset{(2.9)_3}{\leq}\lim_{j_s\to\infty}|\ell_V-(V_p(Du))_{B_{j_s}}|+\frac{1}{\tau^{n/2}}\lim_{j_s\to\infty}\mathfrak{F}(\sigma_{j_s})\overset{(7.28),(6.89)}{=}0,\end{aligned} \tag{7.29}$$

and the first limit in (1.11) equals ℓ_V, which defines the precise representative of $V_p(Du)$ at x_0, i.e.: $\ell_V=(V_p(Du))(x_0)$. Next, notice that whenever $B\Subset\Omega$ is a ball, by (2.1) and [41, (2.6)] it is

$$\mathfrak{F}(u;B)\approx\left(\fint_B|V_p(Du)-V_p((Du)_B)|^2\ \mathrm{d}x\right)^{1/2}, \tag{7.30}$$

with constants implicit in "$\approx$" depending only on p, so for any given $j\in\mathbb{N}\cup\{0\}$ it is

$$|(Du)_{B_j}|\ \leq\ \mathfrak{I}_2(V_p(Du);B_j)^{2/p}\leq c\mathfrak{F}(\sigma_j)^{2/p}+cV(\sigma_j)^{2/p} \tag{7.31}$$

with $c\equiv c(p)$, while for $j=-1$ via Hölder and Young inequalities with conjugate exponents $\left(\frac{2-p}{2},\frac{2}{p}\right)$ we have

$$\begin{aligned}|(Du)_{B_{-1}}| &\leq \mathfrak{I}_2(V_p(Du);B_0)^{2/p}+\left(\fint_{B_0}|Du-(Du)_{B_{-1}}|^p\ \mathrm{d}x\right)^{1/p}\\ &\overset{(2.9)_1,(2.5)}{\leq}\frac{c\mathfrak{F}(\sigma_{-1})^{2/p}}{\tau^{n/p}}+cV(\sigma_0)^{2/p}+c\left(\fint_{B_0}|V_p(Du)-V_p((Du)_{B_{-1}})|^2\ \mathrm{d}x\right)^{1/p}\\ &\quad+c|(Du)_{B_{-1}}|^{(2-p)/2}\left(\fint_{B_0}|V_p(Du)-V_p((Du)_{B_{-1}})|^p\ \mathrm{d}x\right)^{1/p}\\ &\overset{(7.30),(2.9)_1}{\leq}\frac{1}{2}|(Du)_{B_{-1}}|+c\mathfrak{F}(\sigma_{-1})^{2/p}+cV(\sigma_0)^{2/p},\end{aligned} \tag{7.32}$$

for $c \equiv c(\texttt{data}, M)$ and using this time Young inequality with conjugate exponents $\left(\frac{2}{2-p}, \frac{2}{p}\right)$ we get

$$\begin{aligned}
V(\sigma_0) &\le \mathfrak{I}_p(Du; B_0)^{p/2} \le c\left(\fint_{B_0} |Du - (Du)_{B_{-1}}|^p \,\mathrm{d}x\right)^{1/2} + c|(Du)_{B_{-1}}|^{p/2}\\
&\stackrel{(2.5)}{\le} c|(Du)_{B_{-1}}|^{p/2} + c\left(\fint_{B_0} |V_p(Du) - V_p((Du)_{B_{-1}})|^2 \,\mathrm{d}x\right)^{1/2}\\
&\quad +c|(Du)_{B_{-1}}|^{p(2-p)/4}\left(\fint_{B_0} |V_p(Du) - V_p((Du)_{B_{-1}})|^p \,\mathrm{d}x\right)^{1/2}\\
&\stackrel{(7.30),(2.9)_1}{\le} c|(Du)_{B_{-1}}|^{p/2} + \frac{c}{\tau^{n/2}}\mathfrak{F}(\sigma_{-1}) + \frac{c}{\tau^{\frac{np}{4}}}|(Du)_{B_{-1}}|^{p(2-p)/4}\mathfrak{F}(\sigma_{-1})^{p/2}\\
&\le c|(Du)_{B_{-1}}|^{p/2} + c\mathfrak{F}(\sigma_{-1}), \qquad (7.33)
\end{aligned}$$

for $c \equiv c(\texttt{data}, M)$, therefore in any case it is

$$|(Du)_{B_j}| \stackrel{(7.18)}{\le} c\mathfrak{N}(x_0;\sigma)^{2/p} + c\gamma^{2/p} \qquad \text{for all } \ j \in \mathbb{N}\cup\{0,-1\}, \tag{7.34}$$

with $c \equiv c(\texttt{data}, M)$. Moreover, given any ball $B \Subset \Omega$, by triangular and Hölder inequalities we bound

$$|V_p((Du)_B) - (V_p(Du))_B| \le \left(\fint_B |V_p(Du) - V_p((Du)_B)|^2 \,\mathrm{d}x\right)^{1/2} \stackrel{(7.30)}{\le} c(p)\mathfrak{F}(u; B) \tag{7.35}$$

and then estimate for integers $0 \le i \le k-1$:

$$\begin{aligned}
|(Du)_{B_i} - (Du)_{B_k}| &\le \sum_{j=i}^{k-1}|(Du)_{B_{j+1}} - (Du)_{B_i}|\\
&\stackrel{(2.3)_2}{\le} c\sum_{j=i}^{k-1}|V_p((Du)_{B_{j+1}}) - V_p((Du)_{B_j})|(|(Du)_{B_{j+1}}| + |(Du)_{B_j}|)^{(2-p)/2}\\
&\stackrel{(2.13)_1,(7.34)}{\le} c\left(\mathfrak{N}(x_0;\sigma)^{(2-p)/p} + \gamma^{(2-p)/p}\right)\sum_{j=i}^{k-1}|V_p((Du)_{B_{j+1}}) - V_p((Du)_{B_j})|\\
&\le c\left(\mathfrak{N}(x_0;\sigma)^{(2-p)/p} + \gamma^{(2-p)/p}\right)\sum_{j=i}^{k}|V_p((Du)_{B_j}) - (V_p(Du))_{B_j}|\\
&\quad +c\left(\mathfrak{N}(x_0;\sigma)^{(2-p)/p} + \gamma^{(2-p)/p}\right)\sum_{j=i}^{k-1}|(V_p(Du))_{B_{j+1}} - (V_p(Du))_{B_j}|\\
&\stackrel{(2.9)_3,(7.35)}{\le} \frac{c}{\tau^{n/2}}\left(\mathfrak{N}(x_0;\sigma)^{(2-p)/p} + \gamma^{(2-p)/p}\right)\sum_{j=i}^{k}\mathfrak{F}(\sigma_j)\\
&\stackrel{(7.18)_1}{\le} c\mathfrak{N}(x_0,\sigma)^{2/p} + c\gamma^{(2-p)/p}\mathfrak{N}(x_0;\sigma)\\
&\stackrel{(7.17),(7.33)}{\le} c\mathfrak{N}(x_0;\sigma)^{2/p} + c|(Du)_{B_{-1}}|^{(2-p)/2}\mathfrak{N}(x_0;\sigma)
\end{aligned}$$

for $c \equiv c(\texttt{data}, M)$. Recalling $(6.75)_1$, we get that $\{(Du)_{B_j}\}_{j\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence and there exists $\ell \in \mathbb{R}^{N\times n}$ such that $\lim_{j\to\infty}(Du)_{B_j} = \ell$. A standard interpolative argument analogous to that leading to (7.29) allows concluding that ℓ defines the precise representative of Du at x_0, i.e. $Du(x_0) = \ell$ and this assures the validity of the second limit in (1.11). Combining this last information with (7.35) and (7.29) we get that $\ell_V = (V_p(Du))(x_0) = V_p(Du(x_0))$ so via (7.28) we eventually recover the first limit in (1.11). Finally, merging (7.27)-(7.28) and recalling that $V_p(Du(x_0)) = (V_p(Du))(x_0)$ we obtain (7.25). The proof is complete. □

7.3. **Oscillation estimates for small gradients.** In this section we look at what happens when the complementary condition to (7.17) holds, i.e. when for $\sigma \in (0,\varrho]$ it is

$$\frac{\gamma}{8} =: \frac{\mathfrak{N}(x_0;\sigma)}{16} \ge V(\sigma_0) \implies \mathfrak{N}(x_0;\sigma) = 2\gamma. \tag{7.36}$$

Let us first observe that to avoid trivialities, we can suppose $\gamma > 0$, and that there is no loss of generality in assuming that (7.36) actually holds for all $s \in (0,\sigma]$. In fact, if for some $s \in (0,\sigma]$ the opposite inequality to (7.36), i.e. (7.17) holds, then Lemmas 7.2-7.3 apply and we can directly conclude with (1.11) and (7.25). The validity of (7.36) for all $s \in (0,\sigma]$, $(6.77)_1$ and (6.89) yield that $\lim_{s\to 0}(V_p(Du))_{B_s(x_0)} = 0$, therefore keeping in mind that also $\lim_{s\to 0} V(s) = 0$ and that

$$|(Du)_{B_s(x_0)}| \le c(p)\left(\mathfrak{F}(u;B_s(x_0))^{2/p} + V(s)^{2/p}\right) \tag{7.37}$$

we can conclude that $\lim_{s\to 0}(Du)_{B_s(x_0)} = 0$ and the existence of the two limits in (1.11) is proven. Next, we show the validity of (7.25) also in the case in which (7.36) is in force. Let us prove by induction that

$$\mathfrak{C}_H(\sigma_j) \le \gamma \qquad \text{for all } \ j \in \mathbb{N}\cup\{0\}. \tag{7.38}$$

A direct computation renders:

$$\mathfrak{C}_H(\sigma_0) \overset{(7.36)}{\le} \frac{\mathfrak{N}(x_0;\sigma)}{16} + \frac{2H\mathfrak{F}(\sigma_{-1})}{\tau^{n/2}} \le \mathfrak{N}(x_0;\sigma)\left(\frac{1}{16}+\frac{2H}{\tau^{n/2}H_1}\right) \overset{(6.80),(7.36)}{\le} \frac{\gamma}{4}. \tag{7.39}$$

Then, we assume by contradiction that $\{j \in \mathbb{N}\cup\{0\}\colon \mathfrak{C}_H(\sigma_{j+1}) > \gamma\} \neq \emptyset$, define $\mathfrak{l} := \min\{j \in \mathbb{N}\cup\{0\}\colon \mathfrak{C}_H(\sigma_{j+1}) > \gamma\}$, i.e. the smallest integer minus one for which (7.38) fails, introduce the set $\mathcal{I}_{\mathfrak{l}} := \{j \in \mathbb{N}\cup\{0\}\colon \mathfrak{C}_H(\sigma_j) \le \gamma/4,\ j < \mathfrak{l}+1\}$ and set $\chi := \max \mathcal{I}_{\mathfrak{l}}$. Notice that by (7.39) it is $\mathcal{I}_{\mathfrak{l}} \neq \emptyset$, by definition $\mathfrak{C}_H(\sigma_\chi) \le \gamma/4$ and for $j \in \{\chi,\cdots,\mathfrak{l}\}$ we have $\gamma \ge \mathfrak{C}_H(\sigma_{j+1}) \ge \gamma/4 > \gamma/16$, therefore, recalling also (7.36) we can apply Lemma 7.1 with $i \equiv \chi$ and $k \equiv \mathfrak{l}$ to conclude that $\mathfrak{C}_H(\sigma_{\mathfrak{l}+1}) \le \gamma$ in contradiction with the definition of $\mathfrak{l}$. This means that $\{j \in \mathbb{N}\cup\{0\}\colon \mathfrak{C}_H(\sigma_{j+1}) > \gamma\} = \emptyset$ and (7.38) holds true. Next, we take any $s \in (0,\sigma_0]$, determine $j_s \in \mathbb{N}\cup\{0\}$ such that $\sigma_{j_s+1} < s \le \sigma_{j_s}$ and estimate

$$V(s) \overset{(2.10)_2}{\le} V(\sigma_{j_s}) + \frac{\mathfrak{F}(\sigma_{j_s})}{\tau^{n/2}} \overset{(6.3)_2}{\le} \mathfrak{C}_H(\sigma_{j_s}) \le \gamma \le \mathfrak{N}(x_0;\sigma).$$

Moreover, if $s \in (\sigma_0,\sigma_{-1}]$ we directly obtain

$$V(s) \le V(\sigma_0) + |V(\sigma_0)-V(s)| \overset{(2.10)_1}{\le} V(\sigma_0) + \frac{2\mathfrak{F}(\sigma_{-1})}{\tau^n} \overset{(7.38)}{\le} \gamma + \frac{2\mathfrak{N}(x_0;\sigma)}{\tau^n H_1} \overset{(6.80)}{\le} 2\gamma = \mathfrak{N}(x_0;\sigma),$$

so in any case it is $\sup_{s\le\sigma} V(s) \le \mathfrak{N}(x_0;\sigma)$, which in turn implies that $|V(\sigma_{-1})-V(s)| \le 2\mathfrak{N}(x_0;\sigma)$ and $(7.25)_1$ can now be derived by sending $s \to 0$ in the previous inequality and recalling (1.11). Concerning $(7.25)_2$, we use (7.31)-(7.32) and (7.38) to deduce that $|(Du)_{B_j}| \le c\gamma^{2/p}$ for all $j \in \mathbb{N}\cup\{-1,0\}$. This, the same interpolation argument exploited before and standard manipulations eventually render that $|(Du)_{B_{-1}} - (Du)_{B_s(x_0)}| \le c\gamma^{2/p}$, which, together with (7.36) and (1.11) yield $(7.25)_2$ by sending $s \to 0$. In conclusion, we have just proven the following lemma.

Lemma 7.4. *Assume that* (7.36) *holds for some* $\sigma \in (0,\varrho]$. *Then the limits in* (1.11) *exist and the bounds in* (7.25) *are verified.*

7.4. **Sharp partial gradient continuity and proof of Theorems 1-2.** Let us complete the proof of Theorem 2, started in Sections 7.2-7.3.

Proof of Theorem 2. Let $x_0 \in \mathcal{R}_u$ be a point satisfying (1.9), $M \equiv M(x_0) > 0$, $\bar{\varepsilon} \in (0,1)$, $\bar{\varrho} \in (0,\min\{1,d_{x_0}\})$ be the corresponding parameters in (6.1) with $\bar{\varepsilon},\bar{\varrho}$ to be determined. We define $\tilde{\varepsilon} := 2^{-10}\varepsilon'$ and suitably reduce the threshold radius to determine $\tilde{\varrho} \in (0,\varrho']$ in such a way that inequality (7.2) holds with $\tilde{\varepsilon}2^{-2}$ replacing ε' for all $s \in (0,\tilde{\varrho}]$. Setting $\bar{\varepsilon} \equiv \tilde{\varepsilon}/2$ and $\bar{\varrho} \equiv \tilde{\varrho}$ in (6.1) we see that both (1.10) and the assumptions in force in Sections 7.2-7.3 are satisfied, therefore the existence of the limits in (1.11) follows from Lemmas 7.3-7.4, while the (almost) pointwise oscillation estimates in (1.12) are exactly those appearing in (7.25). We are only left with the proof of the assertion on the Lebesgue points of $V_p(Du)$ and of Du. Let us first assume that x_0 verifies both (1.9) and (1.10) with the just fixed parameters $\tilde{\varepsilon} \equiv \tilde{\varepsilon}(\mathtt{data},M)$ and $\tilde{\varrho} \equiv \tilde{\varrho}(\mathtt{data},M,f(\cdot))$. This choice assures that (1.11), (6.77) and (6.89) are available, and this in particular assures that x_0 is a Lebesgue point of $V_p(Du)$. Moreover, with $\sigma \in (0,\varrho]$, recalling $(1.11)_2$, we bound by means of (2.5), (7.30), (7.37), (6.77) and (6.89),

$$\begin{aligned}
\left(\fint_{B_\sigma(x_0)} |Du-(Du)_{B_\sigma(x_0)}|^p \,\mathrm{d}x\right)^{1/p} &\le c\left(\fint_{B_\sigma(x_0)} |V_p(Du)-V_p((Du)_{B_\sigma(x_0)})|^p |(Du)_{B_\sigma(x_0)}|^{p(2-p)/2}\,\mathrm{d}x\right)^{1/p}\\
&\quad + c\left(\fint_{B_\sigma(x_0)} |V_p(Du)-V_p((Du)_{B_\sigma(x_0)})|^2\,\mathrm{d}x\right)^{1/p}\\
&\le c\mathfrak{F}(u;B_\sigma(x_0))^{2/p} + c|(Du)_{B_\sigma(x_0)}|^{(2-p)/2}\mathfrak{F}(u;B_\sigma(x_0))\\
&\le c\mathfrak{F}(u;B_\sigma(x_0))^{2/p} + c(1+M)^{(2-p)/p}\mathfrak{F}(u;B_\sigma(x_0)) \to 0
\end{aligned}$$

with $c \equiv c(n,N,p)$ and x_0 is a Lebesgue point of Du as well. On the other hand, if x_0 is a Lebesgue point of $V_p(Du)$ we know that $\mathfrak{F}(u;B_\sigma(x_0)) \to 0$ and that $(1.11)_1$ exists, therefore, recalling that (1.9) is in force, we can fix ϱ so small that $(1.10)_2$ holds and set $M := 2\limsup_{\sigma\to 0}|(V_p(Du))_{B_\sigma(x_0)}|+1$ to verify also $(1.10)_1$. Finally, if x_0 is a Lebesgue point of Du, then

$$\left(\fint_{B_\sigma(x_0)}|Du-(Du)_{B_\sigma(x_0)}|^p \ \mathrm{d}x\right)^{1/p}\to 0, \qquad \limsup_{\sigma\to 0}|(Du)_{B_\sigma(x_0)}|<\infty \tag{7.40}$$

and $(1.11)_2$ exists. Since

$$|(V_p(Du))_{B_\sigma(x_0)}|\le \mathfrak{I}_p(Du;B_\sigma(x_0))^{p/2}\le c\left(\fint_{B_\sigma(x_0)}|Du-(Du)_{B_\sigma(x_0)}|^p \ \mathrm{d}x\right)^{1/2}+c|(Du)_{B_\sigma(x_0)}|^{p/2}, \tag{7.41}$$

with $c\equiv c(p)$ and, via triangular inequality,

$$\begin{aligned}\mathfrak{F}(u;B_\sigma(x_0)) &\overset{(7.30)}{\le} c\left(\fint_{B_\sigma(x_0)}|V_p(Du)-V_p((Du)_{B_\sigma(x_0)})|^2 \ \mathrm{d}x\right)^{1/2}\\ &\overset{(2.3)_2}{\le} c\left(\fint_{B_\sigma(x_0)}(|Du|^2+|(Du)_{B_\sigma(x_0)}|^2)^{(p-2)/2}|Du-(Du)_{B_\sigma(x_0)}|^2 \ \mathrm{d}x\right)^{1/2}\\ &\le c\left(\fint_{B_\sigma(x_0)}|Du-(Du)_{B_\sigma(x_0)}|^p \ \mathrm{d}x\right)^{1/2}\overset{(7.40)}{\to} 0,\end{aligned} \tag{7.42}$$

for $c\equiv c(n,N,p)$, keeping (1.9) and (7.42) in mind we can choose ϱ so small that $(1.10)_2$ holds true, and setting $M := c+2c\limsup_{\sigma\to 0}|(Du)_{B_\sigma(x_0)}|^{p/2}$ where $c\equiv c(p)$ is the constant appearing in (7.41), we obtain also $(1.10)_1$ and the proof is complete. □

Next, we prove Theorem 1.

Proof of Theorem 1. Since our results are local in nature, we can assume that (1.8) holds globally in Ω - notice that being (1.8) in force, we can always assume the validity of (6.82). Let $\mathcal{R}_u$ be the set defined in (6.1) with $\bar\varepsilon\equiv\tilde\varepsilon$, $\bar\varrho\equiv\tilde\varrho$ and $\tilde\varepsilon$, $\tilde\varrho$ defined in the proof of Theorem 2. The discussion at the beginning of Section 6, see also [26, Section 5.1], yields that $\mathcal{R}_u$ is an open set of full n-dimensional Lebesgue measure and $|\Omega\setminus\mathcal{R}_u|=0$ therefore given any $x_0\in\mathcal{R}_u$ with the specifics described before there is an open neighborhood $B(x_0)$ of x_0 and a positive radius $\varrho_{x_0}\in(0,\tilde\varrho]$ such that $|(V_p(Du))_{B_{\varrho_{x_0}}(x)}|<M$ and $\mathfrak{F}(u;B_{\varrho_{x_0}}(x))<\tilde\varepsilon$. Given (1.8) and our choice of $\bar\varrho$, $\bar\varepsilon$ we see that $(1.10)_2$ holds on $B(x_0)$, Corollary 6.1, Theorem 2 and Proposition 6.1 apply, the limits in (1.11) exist and define the precise representative of $V_p(Du)$ and of Du at all $x\in B(x_0)$. With these informations at hand, we aim to prove that the limits in (1.11) are uniform in the sense that the continuous maps $B(x_0)\ni x\mapsto (V_p(Du))_{B_\sigma(x)}$, $B(x_0)\ni x\mapsto (Du)_{B_\sigma(x)}$ with $\sigma\in(0,\varrho_{x_0}]$ uniformly converge to $V_p(Du(x))$ and to $Du(x)$ respectively as $\sigma\to 0$ thus yielding that $V_p(Du)$ and Du are continuous on $B(x_0)$. This is a consequence of the two inequalities in (1.12) as their right-hand side uniformly converges to zero by means of (1.8), (6.85), (6.75) and (7.37). The proof is complete. □

7.5. **Optimal function space criteria and proof of Theorem 3.** This final section is devoted to the proof of Theorem 3. Once noticed that

$$\begin{cases} f\in L(n,1) \implies \mathbf{I}^f_{1,m}(x,s)\to_{s\to 0} \quad \text{uniformly in } \ x\in\Omega\\[2mm] f\in L^d, \ \ d>n \implies \mathbf{I}^f_{1,m}(x,s)\le \dfrac{ds^{1-n/d}\|f\|_{L^d}}{(d-n)\omega_n^{1/d}},\end{cases}$$

cf. [57, Section 2.3] and [26, Section 6.5] respectively, keeping in mind (6.75), the proof goes exactly as in [26, Proof of Theorem 2].

References

[1] E. Acerbi, N. Fusco, A regularity theorem for minimizers of quasiconvex integrals. *Arch. Ration. Mech. Anal.* 99, 261-281, (1987).
[2] E. Acerbi, N. Fusco, Semicontinuity problems in the calculus of variations. *Arch. Ration. Mech. Anal.* 86, 125-145, (1984).
[3] B. Avelin, T. Kuusi, G. Mingione, Nonlinear Calderón–Zygmund theory in the limiting case. *Arch. Ration. Mech. Anal.* 227, 663-714, (2018).
[4] A. Kh. Balci, L. Diening, M. Surnachev, New examples on Lavrentiev gap using fractals. *Calc. Var. & PDE* 59:180, (2020).
[5] J. M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. *Phil. Trans. R. Soc. Lond.* 306, 557-611, (1982).
[6] J. M. Ball, F. Murat, $W^{1,p}$-quasiconvexity and variational problems for multiple integrals. *J. Funct. Anal.* 58, 225-253, (1984).
[7] A. Banerjee, I. Munive, Gradient continuity estimates for the normalized p-Poisson equation. *Comm. Cont. Math.* 22, no. 08, 1950069, (2020).
[8] P. Baroni, Riesz potential estimates for a general class of quasilinear equations, *Calc. Var. & PDE* 53, 803-846, (2015).
[9] L. Beck, G. Mingione, Lipschitz bounds and nonuniform ellipticity, *Comm. Pure Appl. Math.* 73, 944-1034, (2020).

[10] L. Beck, B. Stroffolini, Regularity results for differential forms solving degenerate elliptic systems. *Calc. Var. & PDE* 46(3-4), 769-808, (2013).

[11] P. Bella, M. Schäffner, Lipschitz bounds for integral functionals with (p,q)-growth conditions. *Adv. Calc. Var*, to appear. 10.1515/acv-2022-0016

[12] P. Bella, M. Schäffner, On the regularity of minimizers for scalar integral functionals with (p,q)-growth. *Anal. & PDE* 13, 2241-2257, (2020).

[13] M. Bildhauer, M. Fuchs, $C^{1,\alpha}$-solutions to non-autonomous anisotropic variational problems. *Calc. Var. & PDE* 24, 309-340, (2005).

[14] V. Bögelein, B. Dacorogna, F. Duzaar, P. Marcellini, C. Scheven, Integral Convexity and Parabolic Systems. *SIAM J. Math. Anal.* 52(2), 1489-1525, (2020).

[15] V. Bögelein, F. Duzaar, P. Marcellini, C. Scheven, Boundary regularity for elliptic systems with p,q-growth *J. Math. Pures Appl.* 159, 250-293, (2022).

[16] G. Bouchitté, I. Fonseca, J. Maly', The effective bulk energy of the relaxed energy of multiple integrals below the growth exponent. *Proc. R. Soc. Edinb. Sect. A Math.* 128, 463-479, (1998).

[17] S.-S. Byun, Y. Youn, Potential estimates for elliptic systems with subquadratic growth. *J. Math. Pures Appl.* 131, 193-224, (2019).

[18] M. Carozza, N. Fusco, G. Mingione, Partial regularity of minimizers of quasiconvex integrals with subquadratic growth. *Ann. Mat. Pura App.* IV, 175, 141-164, (1998).

[19] M. Carozza, J. Kristensen, A. Passarelli di Napoli, Regularity of minimizers of autonomous convex variational integrals. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5), XIII, 1065-1089, (2014).

[20] I. Chlebicka, C. De Filippis, L. Koch, Boundary regularity for manifold constrained $p(x)$-harmonic maps. *J. London Math. Soc.* (2) 104, 2335-2375, (2021).

[21] A. Cianchi, Maximizing the L^∞-norm of the gradient of solutions to the Poisson equation. *J. Geom. Anal.* 2, 499-515, (1992).

[22] A. Cianchi, V. G. Maz'ya, Global Lipschitz regularity for a class of quasilinear elliptic equations. *Comm. Partial Differential Equations* 36, 1, 100-133, (2011).

[23] A. Cianchi, V. G. Maz'ya, Global boundedness of the gradient for a class of nonlinear elliptic systems. *Arch. Ration. Mech. Anal.* 212, 1, 129-177, (2014).

[24] P. Daskalopoulos, T. Kuusi, G. Mingione, Borderline estimates for fully nonlinear elliptic equations. *Comm. PDE* 39, 574-590, (2014).

[25] C. De Filippis, Partial regularity for manifold constrained $p(x)$-harmonic maps. *Calc. Var. & PDE* 58:47, (2019).

[26] C. De Filippis, Quasiconvexity and partial regularity via nonlinear potentials. *J. Math. Pures Appl.* 163, 11-82, (2022).

[27] C. De Filippis, G. Mingione, Lipschitz bounds and nonautonomous integrals. *Arch. Ration. Mech. Anal.* 242, 973-1057, (2021).

[28] C. De Filippis, G. Mingione, Manifold constrained non-uniformly elliptic problems. *J. Geom. Anal.* 30:1661-1723, (2020).

[29] C. De Filippis, G. Mingione, On the regularity of minima of non-autonomous functionals. *J. Geom. Anal.* 30:1584-1626, (2020).

[30] L. Diening, D. Lengeler, B. Stroffolini, A. Verde, Partial regularity for minimizers of quasi-convex functionals with general growth. *SIAM J. Math. Anal.* 44, 5, 3594-3616, (2012).

[31] L. Diening, B. Stroffolini, A. Verde, The φ-harmonic approximation and the regularity of φ-harmonic maps. *J. Differential Equations* 253, 1943-1958, (2012).

[32] H. Dong, H. Zhu, Gradient estimates for singular p-Laplace type equations with measure data. *Preprint* (2021). arXiv:2102.08584

[33] F. Duzaar, J. Grotowski, M. Kronz, Regularity of almost minimizers of quasiconvex integrands with subquadratic growth. *Ann. Mat. Pura Appl.* 184, 421-448, (2005).

[34] F. Duzaar, G. Mingione, Gradient estimates via linear and nonlinear potentials. *J. Funct. Anal.* 259, 11, 2961-2998, (2010).

[35] F. Duzaar, G. Mingione, Regularity for degenerate elliptic problems via p-harmonic approximation. *Ann. I. H. Poincaré - AN* 21, 735-766, (2004).

[36] F. Duzaar, G. Mingione, The p-harmonic approximation and the regularity of p-harmonic maps. *Calc. Var. & PDE* 20, 235-256, (2004).

[37] L. Esposito, F. Leonetti, G. Mingione, Sharp regularity for functionals with (p,q) growth. *J. Diff. Equ.* 204, 5-55, (2004).

[38] L. C. Evans, Quasiconvexity and partial regularity in the calculus of variations. *Arch. Ration. Mech. Anal.* 95, 227-252, (1986).

[39] I. Fonseca, J. Maly', Relaxation of multiple integrals below the growth exponent. *Ann. I. H. Poincaré - AN* 14, 309-338, (1997).

[40] I. Fonseca, J. Malý, G. Mingione, Scalar minimizers with fractal singular sets. *Arch. Rat. Mech. Anal.* 172, 295-307, (2004).

[41] M. Giaquinta, G. Modica, Remarks on the regularity of the minimizers of certain degenerate functionals. *manuscripta math.* 57, no. 1, 55-99, (1986).

[42] E. Giusti, Direct Methods in the calculus of variations. *World Scientific Publishing Co., Inc., River Edge.* (2003).

[43] F. Gmeineder, Partial regularity for symmetric quasiconvex functionals on BD. *J. Math. Pures Appl.* 145, 83-129, (2021).

[44] F. Gmeineder, The Regularity of Minima for the Dirichlet Problem on BD. *Arch. Rational Mech. Anal.* 1099–1171, 237(3), (2020).

[45] F. Gmeineder, J. Kristensen, Partial Regularity for BV Minimizers. *Arch. Rational Mech. Anal.* 232, 1429-1473, (2019).

[46] F. Gmeineder, J. Kristensen, Quasiconvex functionals of (p,q)-growth and the partial regularity of relaxed minimizers. *Preprint* (2022). arXiv:2209.01613

[47] V. P. Havin, V. G. Maz'ya, A nonlinear potential theory. *Uspehi Mat. Nauk* 27, 6, 67-138, (1972).

[48] J. Hirsch, M. Schäffner, Growth conditions and regularity, an optimal local boundedness result. *Comm. Cont. Math* 23, nr. 03, 2050029, (2021).

[49] C. Irving, BMO ε-regularity results for solutions to Legendre-Hadamard elliptic systems. *Preprint* (2021). arXiv:2109.07265v1

[50] C. Irving, Partial regularity for minima of higher-order quasiconvex integrands with natural Orlicz growth. *Preprint* (2021). arXiv:2111.14740

[51] J. Kristensen, Lower semicontinuity of quasi-convex integrals in BV. *Calc. Var. & PDE* 7, 3, 249-261, (1998).

[52] J. Kristensen, On the nonlocality of quasiconvexity. *Ann. I. H. Poincaré - AN* 16, 1, 1-13, (1999).

[53] J. Kristensen, G. Mingione: The singular set of Lipschitzian minima of multiple integrals. *Arch. Ration. Mech. Anal.* 184, 341-369, (2007).

[54] J. Kristensen, A. Taheri, Partial regularity of strong local minimizers in the multi-dimensional calculus of variations, *Arch. Ration. Mech. Anal.* 170, 63-89, (2003).

[55] L. Koch, Global higher integrability for minimisers of convex functionals with (p,q)-growth. *Calc. Var. & PDE* 60, 2, 63, (2021).

[56] L. Koch, Global higher integrability for minimisers of convex obstacle problems with (p,q)-growth. *Calc. Var. & PDE* 61(3), 88, (2022).

[57] T. Kuusi, G. Mingione, A nonlinear Stein Theorem. *Calc. Var. & PDE* 51, 1-2, 45-86, (2014).

[58] T. Kuusi, G. Mingione, Guide to nonlinear potential estimates. *Bull. Math. Sci.* 4, 1-82, (2014).
[59] T. Kuusi, G. Mingione, Partial regularity and potentials. *J. École polytechnique math.* 3, 309-363, (2016).
[60] T. Kuusi, G. Mingione, Vectorial nonlinear potential theory. *J. Eur. Math. Soc.* 20, 929-1004, (2018).
[61] Z. Li, Partial regularity for BV ω-minimizers of quasiconvex functionals. *Calc. Var. & PDE* 61:178, (2022).
[62] P. Marcellini, Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals. *manuscripta math.* 51, 1-3, (1985).
[63] P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic PDEs. *J. Math. Anal. Appl.* 501, 1, 1, 124408, (2021).
[64] P. Marcellini, On the definition and the lower semicontinuity of certain quasiconvex integrals. *Annales de l'I.H.P. Analyse non linéaire* 3, nr. 5, 391-409, (1986).
[65] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Diff. Equ.* 90, 1-30, (1991).
[66] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. *Arch. Rat. Mech. Anal.* 105, 267-284, (1989).
[67] P. Marcellini, The stored-energy for some discontinuous deformations in nonlinear elasticity. *Partial Differential Equations and the Calculus of Variations* vol. II, Birkhäuser Boston Inc., (1989).
[68] C. Mooney, O. Savin, Some singular minimizers in low dimensions in the calculus of variations. *Arch. Ration. Mech. Anal.* 221:1–22, (2016).
[69] C. B. Morrey, Quasiconvexity and the lower semicontinuity of multiple integrals. *Pac. J. Math.* 2, 25-53, (1952).
[70] S. Müller, V. Šverák, Convex integration for Lipschitz mappings and counterexamples to regularity. *Ann. of Math. (2)* 157, 715-742, (2003).
[71] Q.-H. Nguyen, N. C. Phuc, Pointwise gradient estimates for a class of singular quasilinear equation with measure data. *J. Funct. Anal.* 278, 5, 108391, (2020).
[72] Q.-H. Nguyen, N. C. Phuc, A comparison estimate for singular p-Laplace equations and its consequences. *Preprint* (2022). arXiv:2202.11318
[73] J. Ok, Partial Hölder regularity for elliptic systems with non-standard growth. *J. Funct. Anal.* 274(3), 723-768, (2018).
[74] M. Schäffner, Higher integrability for variational integrals with non-standard growth. *Calc. Var. & PDE* 60, 77, (2021).
[75] T. Schmidt, Regularity of minimizers of $W^{1,p}$-quasiconvex variational integrals with (p, q)-growth. *Calc. Var. & PDE* 32, 1-24, (2008).
[76] T. Schmidt, Regularity of relaxed minimizers of quasiconvex variational integrals with (p, q)-growth. *Arch. Rational Mech. Anal.* 193, 311-337, (2009).
[77] T. Schmidt, Regularity theorems for degenerate quasiconvex energies with (p, q)-growth. *Adv. Calc. Var.* 1, 241-270, (2008).
[78] G. Scilla, B. Stroffolini, Partial regularity for steady double phase fluids. *Preprint* (2022). arXiv:2211.05438
[79] S. Sil, Nonlinear Stein theorem for differential forms. *Calc. Var. & PDE* 58(4), 154, (2019).
[80] E. M. Stein, Editor's note: the differentiability of functions in $\mathbb{R}^n$. *Ann. of Math. (2)* 113, 383-385, (1981).
[81] V. Šverák, Quasiconvex functions with subquadratic growth. *Proc. Royal Soc. London Ser. A* 433, 723-725, (1991).
[82] V. Šverák, X. Yan, Non-Lipschitz minimizers of smooth uniformly convex variational integrals. *Proc. Natl. Acad. Sci. USA* 99, 15269-15276, 2002.
[83] K. Uhlenbeck, Regularity for a class of non-linear elliptic systems. *Acta Math.* 138, 3-4, 219-240, (1977).
[84] N. N. Ural'tseva, Degenerate quasilinear elliptic systems. *Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI)* 7, 184-222, (1968).

Cristiana De Filippis, Dipartimento SMFI, Universitá di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy
Email address: cristiana.defilippis@unipr.it

Bianca Stroffolini, Dipartimento di Ingegneria Elettrica e delle Tecnologie dell'Informazione, University of Napoli "Federico II", Via Claudio, 80125 Napoli, Italy
Email address: bstroffo@unina.it