GLOBAL EXISTENCE FOR A HIGHLY NONLINEAR TEMPERATURE-DEPENDENT SYSTEM MODELING NONLOCAL ADHESIVE CONTACT

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Abstract. In this paper we analyze a new temperature-dependent model for adhesive contact that encompasses nonlocal adhesive forces and damage effects, as well as nonlocal heat flux contributions on the contact surface. The related PDE system combines heat equations, in the bulk domain and on the contact surface, with mechanical force balances, including micro-forces, that result in the equation for the displacements and in the flow rule for the damage-type internal variable describing the state of the adhesive bonds. Nonlocal effects are accounted for by terms featuring integral operators on the contact surface.

The analysis of this system poses several difficulties due to its overall highly nonlinear character, and in particular to the presence of quadratic terms, in the rates of the strain tensor and of the internal variable, that feature in the bulk and surface heat equations. Another major challenge is related to proving *strict* positivity for the bulk and surface temperatures.

We tackle these issues by very careful estimates that enable us to prove the existence of global-in-time solutions and could be useful in other contexts. All calculations are rigorously rendered on an accurately devised time discretization scheme in which the limit passage is carried out via variational techniques.

Key words: Contact; adhesion; nonlocal effects; temperature; weak solvability; existence results; time discretization.

AMS (MOS) Subject Classification: 35K55, 35Q72, 74A15, 74M15.

1. Introduction

In this paper we investigate a PDE system describing adhesive contact between a thermoviscoelastic body and a rigid support, in the presence of nonlocal thermo-mechanical effects. Its overall highly nonlinear character is in particular manifest in the heat equations in the bulk domain and on the contact surface. To prove the existence of global-in-time solutions we develop some techniques that could be of interest for the analysis of other thermodynamically consistent systems in solid mechanics.

1.1. **The model and the PDE system.** The study of adhesive contact and delamination phenomena is of applicative interest due to the extensive presence of layered structures in several industrial contexts, cf. e.g. [34, 36] and the references therein. In this paper we address a temperature-dependent model for adhesive contact that originates from the theory by M. Frémond [15], in the broader framework of the theory of generalized standard materials [18] (see [9] for a (partial) survey of adhesive contact and delamination models pertaining to this cadre). Our model includes *nonlocal adhesive and damage forces* as originally proposed, in the *isothermal* case, in [16], as well as *nonlocal heat sources* on the contact surface. Its derivation, based on the principle of virtual power also encompassing 'microscopic movements' as in the approach from [15], was carried out in [9].

The motivation for including nonlocal effects in adhesive contact modeling stems from experiments showing that elongation, i.e. a variation of the distance of two distinct points on the contact surface, may have damaging effects on the substance gluing the body to the support along such surface. This is thoroughly illustrated in [16] also via numerical experiments. The analysis of the isothermal model from [16] was first carried out in [5]. In the model derived in [9] we have additionally encompassed a nonlocal interaction between the body and the adhesive substance as far as it concerns heat exchange on the contact surface.

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More precisely, during a time interval $(0,T)$, $T>0$, we consider a thermoviscoelastic body located in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ and lying on a rigid support on a part of its boundary, on which some adhesive substance is present. Hence, $\partial\Omega = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}} \cup \overline{\Gamma}_{\mathrm{C}}$ with (1) Γ_{C} the contact surface, hereafter assumed *flat* and identified with a subset of $\mathbb{R}^2$, (2) Γ_{D} the Dirichlet part of the boundary, with positive measure, on which homogeneous boundary conditions are prescribed, and (3) Γ_{N} the Neumann part, on which a traction is applied. The state variables in the bulk domain Ω are the absolute temperature θ of the body and its displacement $\mathbf{u}$ (at small strains); the state variables defined on the contact surface Γ_{C} are the absolute temperature θ_{s} of the adhesive substance, and a surface damage-type parameter χ representing the fraction of fully effective links in the bonding. As such, χ takes values in $[0,1]$, with $\chi=0$ for completely damaged bonds, $\chi=1$ for fully intact bonds, and $\chi\in(0,1)$ for the intermediate states. The distinction between the temperature θ of the bulk domain and the temperature θ_{s} of the adhesive substance is typical of FRÉMOND's approach to the modeling of thermal effects in rate-dependent adhesive contact, cf. e.g. [6, 8]. Nonetheless, it has also been discussed in the context of the rate-independent modeling of delamination, see [24, Sec. 5.3.3.3].

The evolution of the variables $(\theta,\mathbf{u},\theta_{\mathrm{s}},\chi)$ during the time interval $(0,T)$ is governed by a system of PDEs in the bulk domain and on the contact surface derived from the general laws of Thermomechanics and from suitable choices for the free energy and pseudo-potential of dissipation that also account for nonlocal interactions between the body and its support. Indeed, the principle of virtual power leads to the quasistatic momentum balance for the macroscopic movements

$$-\mathrm{div}(\mathbb{E}\varepsilon(\mathbf{u}) + \mathbb{V}\varepsilon(\mathbf{u}_t) + \theta\mathbb{I}) = \mathbf{f} \qquad \text{in } \Omega\times(0,T), \tag{1.1a}$$

supplemented by the following boundary conditions

$$\mathbf{u} = \mathbf{0} \qquad \text{in } \Gamma_{\mathrm{D}}\times(0,T), \tag{1.1b}$$

$$(\mathbb{E}\varepsilon(\mathbf{u}) + \mathbb{V}\varepsilon(\mathbf{u}_t) + \theta\mathbb{I})\mathbf{n} = \mathbf{g} \qquad \text{in } \Gamma_{\mathrm{N}}\times(0,T), \tag{1.1c}$$

$$(\mathbb{E}\varepsilon(\mathbf{u}) + \mathbb{V}\varepsilon(\mathbf{u}_t) + \theta\mathbb{I})\mathbf{n} + \chi\mathbf{u} + \partial I_{(-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} + \int_{\Gamma_{\mathrm{C}}} \mu(|x-y|)\mathbf{u}(x)\chi(x)\chi(y)\,\mathrm{d}y \ni \mathbf{0} \quad \text{in } \Gamma_{\mathrm{C}}\times(0,T). \tag{1.1d}$$

Here, $\mathbb{E}$ and $\mathbb{V}$ denote the elasticity and the viscosity tensors, $\mathbb{I}$ the identity matrix, $\mathbf{n}$ the outward unit normal vector to the boundary $\partial\Omega$. Moreover, $\mathbf{f}$ is a volume force while $\mathbf{g}$ is a traction applied on Γ_{N}. Condition (1.1d), coupling the evolution of $\mathbf{u}$ (hereafter, we shall denote its trace on Γ_{C} by the same symbol) to that of χ, can be understood as a generalization of the classical *Signorini contact* conditions. Indeed, it features a selection $\boldsymbol{\xi} \in \partial I_{(-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$, with $\partial I_{(-\infty,0]}$ the subdifferential of the indicator function of the interval $(-\infty,0]$, that represents a reaction force activated when the non-interpenetration constraint $\mathbf{u}\cdot\mathbf{n}\leq 0$ on Γ_{C} holds as an equality, i.e. when $\mathbf{u}\cdot\mathbf{n}=0$. The normal reaction on the boundary condition described by (1.1d) also includes the nonlocal contribution

$$\int_{\Gamma_{\mathrm{C}}} \mu(|x-y|)\mathbf{u}(x)\chi(x)\chi(y)\,\mathrm{d}y\,,$$

where the positive function μ accounts for the attenuation of nonlocal interactions as the distance $|x-y|$ between two points x and y on the contact surface increases. Observe that in (1.1d) (and in the forthcoming (1.1e), (1.1i) and (1.1j)), we have written explicitly the dependence of the unknows $(\theta,\mathbf{u},\theta_{\mathrm{s}},\chi)$ on the variable $x\in\Gamma_{\mathrm{C}}$ only in the nonlocal terms involving integrals (with respect to the spatial variable $y\in\Gamma_{\mathrm{C}}$).

As customary in FRÉMOND's approach, the principle of virtual power leads to a micro-force balance on the contact surface that results in the following flow rule for the damage-like parameter χ

$$\begin{aligned} &\chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) + \gamma'(\chi) + \lambda'(\chi)(\theta_{\mathrm{s}} - \theta_*) \\ &\qquad \ni -\frac{1}{2}|\mathbf{u}|^2 - \frac{1}{2}\int_{\Gamma_{\mathrm{C}}} \mu(|x-y|)(|\mathbf{u}(x)|^2 + |\mathbf{u}(y)|^2)\chi(y)\,\mathrm{d}y \qquad \text{in } \Gamma_{\mathrm{C}}\times(0,T), \end{aligned} \tag{1.1e}$$

supplemented by the no-flux boundary condition

$$\partial_{\mathbf{n}_{\mathrm{s}}}\chi = 0 \qquad \text{in } \partial\Gamma_{\mathrm{C}}\times(0,T). \tag{1.1f}$$

In (1.1e)–(1.1f), $\mathbf{n}_\mathrm{s}$ denotes the outward unit normal vector to $\partial\Gamma_\mathrm{C}$, $\theta_* > 0$ is a phase transition temperature, the function λ is related to the latent heat while γ describes possible non-monotone dynamics for χ (it may model some cohesion in the material). Moreover, the subdifferential term $\partial I_{[0,1]}(\chi)$ ($I_{[0,1]}(\cdot)$ denoting the indicator function of the interval $[0,1]$) enforces the physical constraint that χ takes values in $[0,1]$. The source of damage on the right-hand side of (1.1e) features local and nonlocal terms and, in particular, it may differ from zero even in the case in which $\mathbf{u} = \mathbf{0}$, due to the nonlocal, integral contribution that renders the damaging effects of elongation.

The equations for the bulk and surface temperature variables θ and θ_s are recovered from the first principle of Thermodynamics. The internal energy balance equation written in the bulk domain reads

$$\theta_t - \theta\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\alpha(\theta)\nabla\theta) = h + \varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t), \qquad \text{in } \Omega\times(0,T), \tag{1.1g}$$

with prescribed boundary conditions

$$\alpha(\theta)\nabla\theta\cdot\mathbf{n} = 0 \qquad \text{in } (\Gamma_\mathrm{D}\cup\Gamma_\mathrm{N})\times(0,T), \tag{1.1h}$$

$$\alpha(\theta)\nabla\theta\cdot\mathbf{n} = -\theta\left(k(\chi)(\theta-\theta_\mathrm{s}) + \int_{\Gamma_\mathrm{C}} \mu(|x-y|)(\theta(x)-\theta_\mathrm{s}(y))\chi(x)\chi(y)\,\mathrm{d}y\right) \quad \text{in } \Gamma_\mathrm{C}\times(0,T). \tag{1.1i}$$

It is coupled to the internal energy balance equation on the contact surface

$$\begin{aligned}&\partial_t\theta_\mathrm{s} - \theta_\mathrm{s}\lambda'(\chi)\chi_t - \mathrm{div}(\alpha(\theta_\mathrm{s})\nabla\theta_\mathrm{s})\\ &= \ell + |\chi_t|^2 + \theta_\mathrm{s}\left(k(\chi)(\theta-\theta_\mathrm{s}) + \int_{\Gamma_\mathrm{C}} \mu(|x-y|)(\theta(y)-\theta_\mathrm{s}(x))\chi(x)\chi(y)\,\mathrm{d}y\right) \qquad \text{in } \Gamma_\mathrm{C}\times(0,T),\end{aligned} \tag{1.1j}$$

$$\alpha(\theta_\mathrm{s})\nabla\theta_\mathrm{s}\cdot\mathbf{n}_\mathrm{s} = 0 \qquad \text{in } \partial\Gamma_\mathrm{C}\times(0,T). \tag{1.1k}$$

Here, the positive function α represents the heat conductivity coefficient both in the bulk domain and on the contact surface, k is a surface thermal diffusion coefficient, while h and l are volume and surface heat sources. The evolutions of the bulk and surface temperatures θ and θ_s are coupled by the boundary condition (1.1i), featuring two distinct contributions. The first one has a 'local' character, as it depends on the quantity $(\theta-\theta_\mathrm{s})$ evaluated at the *same point* $x \in \Gamma_\mathrm{C}$ (again, we keep on denoting by θ the trace of the absolute temperature on Γ_C). Instead, the second term on the right-hand side of (1.1i) has a nonlocal character as it involves the quantity

$$\int_{\Gamma_\mathrm{C}} \mu(|x-y|)(\theta(x)-\theta_\mathrm{s}(y))\chi(x)\chi(y)\,\mathrm{d}y$$

featuring the thermal gap between *different* points x and y on the contact surface.

Actually, in what follows we will tackle the analysis of a generalized version of system (1.1), in which the nonlocal integral terms are replaced by more general nonlocal operators and where the various, concrete, subdifferential operators are replaced by general maximal monotone nonlinearities. Nonetheless, throughout this Introduction we shall confine the discussion to system (1.1).

1.2. **Analytical difficulties.** The major difficulties related to the analysis of system (1.1) are that

(1) it encompasses both bulk and surface equations. In particular, the evolutions of the displacement variable $\mathbf{u}$ and of the adhesion parameter χ are coupled through the Robin-type boundary condition (1.1d). This prevents us from applying regularity results for elliptic systems that would lead to enhanced spatial (e.g., H^2-) regularity for $\mathbf{u}$ and $\mathbf{u}_t$. In turn, such regularity would be handy, for instance, to better control the right-hand side of the heat equation (1.1g), since, indeed,

(2) the bulk temperature and displacement equation (the surface temperature equation and the flow rule, respectively) are coupled by the quadratic term $\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t)$ (by the term $|\chi_t|^2$, resp.) that is just in $L^1(\Omega\times(0,T))$ (in $L^1(\Gamma_\mathrm{C}\times(0,T))$, resp.) once the basic energy estimates on system (1.1) are performed. Other nonlinear coupling terms between bulk and surface equations occur in (1.1d), (1.1e), (1.1i), and (1.1j), but the L^1-character of the right-hand side of the heat equations poses the most prominent challenge, together with

(3) proving that the temperature variables θ and θ_s are *strictly* positive. As a matter of fact, because of the nonlocal terms in (1.1i) and (1.1j), well-established techniques for proving strict positivity of θ and θ_s, based on comparison arguments, fail.

The presence of quadratic terms in the rate of the internal variable (and in the rate of the strain tensor, when the momentum balance is also included in the system) is typical of *thermodynamically consistent* models; *strict* positivity of the temperature is also a key ingredient for their compliance with the laws of Thermodynamics. In fact, the challenges in items (2) & (3) of the above list transcend the specific problem examined here, and have stimulated the development of a variety of techniques over the last two decades.

The first existence result for the 'full' model by FRÉMOND for solid-liquid phase transitions (here 'full' refers to the fact the quadratic term, in the rate of the phase-field parameter, on the right-hand side of the heat equation is not neglected), dates back to [21,22], for (spatially) *one-dimensional* systems. To our knowledge, the analysis of a 'full' model in the three-dimensional case was first addressed in [3], tackling a thermodynamically consistent PDE system for damage in thermo-visco-elastic materials. Therein, the heat equation featured the quadratic terms $|\chi_t|^2$ and $|\nabla\chi_t|^2$ (with χ the damage parameter), as well as $\varepsilon(\mathbf{u}_t){:}\varepsilon(\mathbf{u}_t)$, on its right-hand side, while the heat conduction coefficient $\alpha = \alpha(\theta)$ was assumed to be constant. In that framework, only a local-in-time existence result was obtained. Ever since, in most of the papers addressing the analysis of thermo-(visco-)elastic models with L^1 right-hand sides in the heat equation, *global-in-time* existence results have been obtained under suitable growth conditions, either on the non-constant heat conductivity, or on a non-constant heat capacity coefficient.

The latter course has been pursued in a series of papers by T. ROUBÍČEK, starting from [30] that address the analysis of a broad class of thermomechanical, thermodynamically consistent, *rate-independent* processes. In [30] and in several subsequent papers covering a wide range of applications (cf., e.g., [25,31–33]; see also [8,28] for applications to adhesive contact and delamination) ROUBÍČEK switches to an alternative thermal variable, the 'enthalpy', defined in terms of a primitive of the heat-capacity. In this way, the nonlinear character of the heat equation is partially 'tamed'; its L^1 right-hand side (featuring quadratic terms in the rates of the strain tensor and of the internal variables of the model) is dealt with by means of BOCCARDO-GALLOUËT type estimates [2], as adapted in [11]. Such estimates yield a limited spatial regularity for the enthalpy/temperature variable, which is estimated in the space $W^{1,r}(\Omega)$ for some specific $r \in (1,2)$. Hence, in the aforementioned papers (global-in-time) existence results are typically obtained for a formulation of the heat/enthalpy equation with spatially smooth test functions.

In turn, [12] pioneered an alternative approach to the analysis of the heat equation with a L^1-right-hand side in the 'full' model for solid-liquid phase transitions by FRÉMOND. The core assumption there is some suitable growth condition on the heat conductivity α. This leads to a H^1-spatial regularity for the temperature variable, albeit in the context of a quite weak formulation for thermal evolution. Specifically, in [12,13] the heat equation is formulated, consistently with the laws of Thermodynamics, in terms of an entropy inequality, involving smooth test functions, and of a total energy balance. The 'entropic' solution concept advanced in [12] has proved to be remarkably flexible. It has been extended to various contexts, from the evolution of non-isothermal nematic liquid crystals [10,14], to models for damage and phase separation in thermo-visco-elastic solids in $\mathbb{R}^d$, $d \in \{2,3\}$, cf. [19,26]. In the latter papers the existence of 'entropic' solutions was proved under the condition that

$$\exists\, c_0,\, c_1 > 0 \ \ \exists\, \mu > 1 \ \ \forall\, \theta \in \mathbb{R}^+ : \quad c_0(1+\theta^\mu) \leq \alpha(\theta) \leq c_1(1+\theta^\mu) \tag{1.2}$$

(cf., e.g., [38] for examples of nonlinear heat conduction). Under the more restrictive condition that

$$\text{the exponent } \mu \text{ in (1.2) satisfies } \mu \begin{cases} \in (1,2), \\ \in \left(1, \frac{5}{3}\right) \end{cases} \quad \text{if the space dimension } d = \begin{cases} 2, \\ 3 \end{cases} \tag{1.3}$$

[26] showed the existence of 'conventional' weak solutions to the PDE system coupling the momentum balance, the flow rule for the damage parameter, and the heat equation, which was formulated in a variational way, with suitable test functions.

Finally, we would like to mention the analysis of a (still thermodynamically consistent) PDE system for thermo-visco-plasticity at small strains from [20]. Via maximal parabolic regularity arguments, the authors succeeded in proving the existence of global-in-time solutions to a suitable weak formulation of the system without resorting to growth conditions on the heat conductity $\alpha(\theta) \equiv 1$.

1.3. **Our results.** With **our main result, Theorem 1** ahead, we are going to prove the existence of global-in-time, weak solutions to system (1.1), under the *more general condition* (1.2): in particular, we are not going to restrict the range of the exponent μ. We highlight, in the similar contexts of [19,26], (1.2) previously granted the existence of 'entropic' solutions, only, with the heat equation formulated via an entropy inequality and an overall energy balance. Therefore here, under the same conditions as in [19,26], we succeed in bypassing entropic solutions and directly conclude the existence of 'conventionally' weak solutions, that we will term *weak energy solutions*. We are also going to obtain the *strict* positivity properties

$$\exists\,\overline{\theta},\,\overline{\theta}_{\rm s}>0: \qquad \theta\geq\overline{\theta} \text{ a.e. in } \Omega\times(0,T), \quad \theta_{\rm s}\geq\overline{\theta}_{\rm s} \text{ a.e. in } \Gamma_{\rm C}\times(0,T), \tag{1.4}$$

for which the comparison arguments often used in the literature are not applicable due to the nonlocal terms in the heat equations and related boundary conditions. The cornerstone of our existence proof for weak solutions, under the *sole* (1.2), is a suitable estimate for the temperature variables, akin to the estimate that lies at the core of the proof of (1.4).

Indeed, for proving (1.4) we will revisit a powerful technique, advanced in [37], that consists in testing the heat equations (1.1g) and (1.1j) by the *negative* powers $-\theta^{-p}$ and $-\theta_{\rm s}^{-p}$, respectively, with an *arbitrary* $p>2$. As it will be shown in Section 3.2, this leads to an estimate for $\frac{1}{\theta}$ and $\frac{1}{\theta_{\rm s}}$ in $L^\infty(0,T;L^{p-1}(\Omega))$ and $L^\infty(0,T;L^{p-1}(\Gamma_{\rm C}))$, respectively. Letting $p\to\infty$ leads to an estimate for the reciprocal temperatures in $L^\infty(\Omega\times(0,T))$ and $L^\infty(\Gamma_{\rm C}\times(0,T))$, which gives (1.4).

It turns out that a closely related idea will also allow us to 'tame' the L^1 right-hand sides of the heat equations. For that, the key issue is estimating the spatial gradient of the temperatures θ and $\theta_{\rm s}$. This will result from testing (1.1g) and (1.1j) by $\theta^{\nu-1}$ and $\theta_{\rm s}^{\nu-1}$, respectively, for an arbitrary $\nu\in(0,1)$ (cf. Section 3.3.3 ahead). This will lead to the bounds

$$\|\theta^{(\mu+\nu)/2}\|_{L^2(0,T;H^1(\Omega))}+\|\theta_{\rm s}^{(\mu+\nu)/2}\|_{L^2(0,T;H^1(\Gamma_{\rm C}))}\leq C \tag{1.5}$$

(recall that the exponent μ featured in the growth condition (1.2)). In turn, via interpolation arguments, (1.5) shall bring to higher integrability estimates for the temperature variables, which will allow us to estimate their derivatives θ_t and $\partial_t\theta_{\rm s}$ in $L^1(0,T;W^{1,3+\epsilon}(\Omega)^*)$ and $L^1(0,T;W^{1,2+\epsilon}(\Gamma_{\rm C})^*)$ for all $\epsilon>0$, respectively. Clearly, from the estimates of the gradients and the time derivatives of θ and $\theta_{\rm s}$ we will extract all the compactness information necessary for dealing with the heat equations. The analysis of the momentum balance and of the flow rule will follow more standard paths.

The estimates described above will be formally developed in Sections 3.2 and 3.3. Making them rigorous in the frame of a time discretization scheme for system (1.1), which might be conducive to its numerical analysis, has been a challenging issue by itself. First of all, in devising the approximation scheme for (1.1) we have had to carefully balance the terms to be kept implicit with those to be kept explicit. In this way, we have ensured the validity of a discrete form of the total energy balance associated with (1.1), whence all the basic energy estimates stem. Secondly, we have had to combine time discretization with an additional regularization obtained by (1) adding the higher order terms $-\rho\mathrm{div}(|\varepsilon(\mathbf{u}_t)|^{\omega-2}\varepsilon(\mathbf{u}_t))$ and $\rho|\chi_t|^{\omega-2}\chi_t\,,\omega>4$, to the momentum balance and to the flow rule for χ; (2) replacing the maximal monotone operators in the flow rule for χ and in the boundary condition for $\mathbf{u}$ on $\Gamma_{\rm C}$ by their Yosida regularizations. The reason for this *threefold* approximation procedure essentially resides in the fact that, on the time-discrete level, we shall not be able to fully carry out the arguments from [37], leading to a uniform, in space and time, estimate for the reciprocal temperatures. Namely, for the discrete bulk and surface temperatures, we shall only prove a strict positivity property, but not a lower bound by a positive constant as in (1.4). Therefore, in order to rigorously perform the test of the temperature equations by negative powers of θ and $\theta_{\rm s}$ that leads to (1.5), we will need to work on the regularized version of system (1.1) described in the above lines, cf. system (3.69) ahead.

We believe that the formal estimates from Sections 3.2 and 3.3, as well as the technical machinery rigorously supporting them developed in Sections 4 and 5, are robust enough to be applied to other thermodynamically consistent models in solid mechanics. In particular, the analysis of a PDE system for damage in thermo-visco-elastic materials will be carried out in future work, in which the issues related to the unidirectionality of the evolution of the damage parameter will also be addressed.

Plan of the paper. In Section 2, after settling some preliminary results and all our conditions on the constitutive functions of the model, and on the problem data, we will consider a generalized version of system (1.1) and introduce our notion of 'weak energy solution' to the associated Cauchy problem. We will then state our main existence result, Theorem 1. Throughout Section 3 we will carry out in a formal way all the calculations that provide the strict positivity properties (1.4), and all the a priori estimates at the core of the proof of Theorem 1. In Section 3.4 we will then introduce the regularized system (3.69) on which all estimates will be rigorously performed. The existence of solutions to the Cauchy problem for system (3.69) will be proved, via a careful time discretization procedure, throughout Sections 4–5. Finally, in Section 6 we will take the limit of the regularized system in two steps, and thus conclude the proof of Theorem 1.

2. The main result

Let us fix some general notation that will be used throughout the paper.

Notation 2.1. For a given a Banach space X, we will denote by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X' and X; to avoid overburdening notation, we shall write $\|\cdot\|_X$ both the norm in X and in any power of it.

We will work with the space

$$H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3) := \{\mathbf{v}\in H^1(\Omega;\mathbb{R}^3)\ :\ \mathbf{v}=\mathbf{0}\ \text{a.e. on}\ \Gamma_{\mathrm{D}}\},$$

endowed with the natural norm induced by $H^1(\Omega;\mathbb{R}^3)$, and denote the Laplace operator with homogeneous boundary conditions by

$$A : H^1(\Gamma_{\mathrm{C}})\to H^1(\Gamma_{\mathrm{C}})^* \qquad \langle A\chi, w\rangle_{H^1(\Gamma_{\mathrm{C}})} := \int_{\Gamma_{\mathrm{C}}} \nabla\chi\nabla w\,\mathrm{d}x \quad \text{for all } \chi,\, w\in H^1(\Gamma_{\mathrm{C}}).$$

Moreover, we shall use special notation for the following function space

$$\mathbf{Y} := H^{1/2}_{00,\Gamma_{\mathrm{D}}}(\Gamma_{\mathrm{C}};\mathbb{R}^3) = \Big\{\mathbf{w}\in H^{1/2}(\Gamma_{\mathrm{C}};\mathbb{R}^3)\ :\ \exists\,\widetilde{\mathbf{w}}\in H^{1/2}(\partial\Omega;\mathbb{R}^3)\ \text{with}\ \widetilde{\mathbf{w}}=\mathbf{w}\ \text{in}\ \Gamma_{\mathrm{C}},\ \widetilde{\mathbf{w}}=\mathbf{0}\ \text{in}\ \Gamma_{\mathrm{D}}\Big\}.$$

Preliminary results. Throughout the paper, we will also use that

$$H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)\subset L^4(\Gamma_{\mathrm{C}})\ \text{continuously, and}\ H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)\Subset L^{4-s}(\Gamma_{\mathrm{C}})\ \text{compactly for all}\ s\in(0,3], \tag{2.1}$$

where the above embeddings have to be understood in the sense of traces.

Finally, we shall resort to the following *nonlinear* Poincaré-type inequality (cf. e.g. [17, Lemma 2.2])

$$\forall\,q>0 \quad \exists\,C_q>0 \quad \forall\,w\in H^1(\Omega)\ : \qquad \||w|^q w\|_{H^1(\Omega)} \leq C_q(\|\nabla(|w|^q w)\|_{L^2(\Omega)} + |m(w)|^{q+1}), \tag{2.2}$$

(with $m(w)$ the mean value of w), and to the well-known interpolation formula for Lebesgue spaces, holding for every measurable $O\subset\mathbb{R}^d$, $d\geq 1$:

$$L^r(0,T;L^s(O))\cap L^p(0,T;L^q(O))\subset L^a(0,T;L^b(O)) \quad \text{with} \quad \begin{cases} \frac{1}{a} = \frac{\vartheta}{r}+\frac{1-\vartheta}{p}, \\ \frac{1}{b} = \frac{\vartheta}{s}+\frac{1-\vartheta}{q}, \end{cases} \quad \text{for some } \vartheta\in(0,1) \tag{2.3}$$

with a continuous embedding.

2.1. Setup and assumptions.

We start by detailing our conditions on the reference configuration:

$$\begin{gathered} \Omega \text{ is a bounded Lipschitz domain in } \mathbb{R}^3, \text{ with} \\ \partial\Omega = \overline{\Gamma}_{\mathrm{D}}\cup\overline{\Gamma}_{\mathrm{N}}\cup\overline{\Gamma}_{\mathrm{C}},\ \ \Gamma_{\mathrm{D}},\,\Gamma_{\mathrm{N}},\,\Gamma_{\mathrm{C}},\ \text{open disjoint subsets in the relative topology of}\ \partial\Omega,\ \text{such that} \\ \mathcal{H}^2(\Gamma_{\mathrm{D}}),\,\mathcal{H}^2(\Gamma_{\mathrm{C}})>0,\ \text{and}\ \Gamma_{\mathrm{C}}\subset\mathbb{R}^2\ \text{a \textit{flat} surface,} \end{gathered} \tag{2.4}$$

which means that Γ_{C} is a subset of a hyperplane of $\mathbb{R}^3$ and on Γ_{C} the Lebesgue and Hausdorff measures $\mathcal{L}^2$ and $\mathcal{H}^2$ coincide.

Let us now fix

(1) the properties of the elasticity and viscosity tensors: we assume that the fourth-order tensors $\mathbb{E}=(e_{ijkh})$ and $\mathbb{V}=(v_{ijkh})$, satisfy the classical symmetry and ellipticity conditions

$$\begin{aligned} & e_{ijkh}=e_{jikh}=e_{khij}\,, \qquad v_{ijkh}=v_{jikh}=v_{khij}\,, \quad i,j,k,h=1,2,3, \\ & \exists\,\varepsilon_0>0 \quad \forall\,\xi_{ij}: \xi_{ij}=\xi_{ji}\,, \quad i,j=1,2,3\ : \qquad e_{ijkh}\xi_{ij}\xi_{kh}\geq\varepsilon_0\xi_{ij}\xi_{ij}\,, \\ & \exists\,\nu_0>0 \quad \forall\,\xi_{ij}: \xi_{ij}=\xi_{ji}\,, \quad i,j=1,2,3\ : \qquad v_{ijkh}\xi_{ij}\xi_{kh}\geq\nu_0\xi_{ij}\xi_{ij}\,, \end{aligned} \tag{2.5a}$$

where the usual summation convention is used. Moreover, we require that

$$e_{ijkh}, v_{ijkh}\in L^\infty(\Omega)\,, \quad i,j,k,h=1,2,3. \tag{2.5b}$$

Observe that conditions (2.5) are compatible with the properties of an anisotropic and inhomogeneous material. They ensure that the associated bilinear forms $\mathsf{e}, \mathsf{v} : H^1(\Omega;\mathbb{R}^3) \times H^1(\Omega;\mathbb{R}^3) \to \mathbb{R}$, defined by

$$\mathsf{e}(\mathbf{u},\mathbf{v}) := \int_\Omega e_{ijkh}\varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}x \quad \text{for all } \mathbf{u},\mathbf{v} \in H^1(\Omega;\mathbb{R}^3),$$
$$\mathsf{v}(\mathbf{u},\mathbf{v}) := \int_\Omega v_{ijkh}\varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}x \quad \text{for all } \mathbf{u},\mathbf{v} \in H^1(\Omega;\mathbb{R}^3)$$

are continuous and symmetric, i.e.

$$\exists\, M > 0 : \ |\mathsf{e}(\mathbf{u},\mathbf{v})| + |\mathsf{v}(\mathbf{u},\mathbf{v})| \le M\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)} \quad \text{for all } \mathbf{u},\mathbf{v} \in H^1(\Omega;\mathbb{R}^3). \tag{2.6}$$

Furthermore, since Γ_{D} has positive measure, by Korn's inequality we deduce that the forms $\mathsf{e}(\cdot,\cdot)$ and $\mathsf{v}(\cdot,\cdot)$ are $H^1(\Omega;\mathbb{R}^3)$-elliptic on $H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3) \times H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$, i.e. there exist $C_{\mathrm{e}}, C_{\mathrm{v}} > 0$ such that

$$\mathsf{e}(\mathbf{u},\mathbf{u}) \ge C_{\mathrm{e}}\|\mathbf{u}\|^2_{H^1(\Omega)}, \qquad \mathsf{v}(\mathbf{u},\mathbf{u}) \ge C_{\mathrm{v}}\|\mathbf{u}\|^2_{H^1(\Omega)} \qquad \text{for all } \mathbf{u} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3). \tag{2.7}$$

We will in fact deal with a extended version of system (1.1), where the subdifferentials $\partial I_{(-\infty,0]}$ and $\partial I_{[0,1]}$ will be replaced by general maximal monotone operators. Namely,

(2) we consider a function

$$\widehat{\eta} : \mathbb{R} \to [0,+\infty] \quad \text{proper, convex, and lower semicontinuous, with } \widehat{\eta}(0) = 0 \tag{2.8}$$

(note that, if $0 \in \mathrm{dom}(\widehat{\eta})$, we can always reduce to the case $\widehat{\eta}(0) = 0$ by a translation). Then, we introduce the proper, convex and lower semicontinuous functional

$$\widehat{\boldsymbol{\eta}} : \mathbf{Y} \to [0,+\infty] \quad \text{defined by} \quad \widehat{\boldsymbol{\eta}}(\mathbf{u}) := \begin{cases} \int_{\Gamma_{\mathrm{C}}} \widehat{\eta}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}x & \text{if } \widehat{\eta}(\mathbf{u}\cdot\mathbf{n}) \in L^1(\Gamma_{\mathrm{C}}), \\ +\infty & \text{otherwise.} \end{cases}$$

We set $\boldsymbol{\eta} := \partial\widehat{\boldsymbol{\eta}} : \mathbf{Y} \rightrightarrows \mathbf{Y}^*$. It follows from (2.8) that $\mathbf{0} \in \boldsymbol{\eta}(\mathbf{0})$. The subdifferential $\boldsymbol{\eta}(\mathbf{u})$ shall replace the term $\partial I_{(-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$ in the boundary condition (1.1d). Observe that the impenetrability condition $\mathbf{u}\cdot\mathbf{n} \le 0$ a.e. on Γ_{C} is rendered as soon as $\mathrm{dom}(\widehat{\eta}) \subset (-\infty,0]$.

(3) We also generalize the subdifferential $\partial I_{[0,1]}$ to the subdiffential of a function

$$\widehat{\beta} : \mathbb{R} \to [0,+\infty] \text{ proper, convex, lower semicontinuous, with } \mathrm{dom}(\widehat{\beta}) \subset [0,1] \text{ and } \widehat{\beta}(0) = 0, \tag{2.9}$$

and set $\beta := \partial\widehat{\beta} : \mathbb{R} \rightrightarrows \mathbb{R}$.

Observe that the integral terms encompassing nonlocal effects in (1.1i), (1.1d), (1.1j), and (1.1e) can be rewritten as

$$\begin{cases} -\theta^2(x)\chi(x)\int_{\Gamma_{\mathrm{C}}} j(x,y)\chi(y)\,\mathrm{d}y + \theta(x)\chi(x)\int_{\Gamma_{\mathrm{C}}} j(x,y)\theta_{\mathrm{s}}(y)\chi(y)\,\mathrm{d}y, \\ \mathbf{u}(x)\chi(x)\int_{\Gamma_{\mathrm{C}}} j(x,y)\chi(y)\,\mathrm{d}y, \\ \theta_{\mathrm{s}}(x)\chi(x)\int_{\Gamma_{\mathrm{C}}} j(x,y)\theta(y)\chi(y)\,\mathrm{d}y - \chi(x)\theta_{\mathrm{s}}^2(x)\int_{\Gamma_{\mathrm{C}}} j(x,y)\chi(y)\,\mathrm{d}y, \\ -\frac{1}{2}|\mathbf{u}(x)|^2\int_{\Gamma_{\mathrm{C}}} j(x,y)\chi(y)\,\mathrm{d}y - \frac{1}{2}\int_{\Gamma_{\mathrm{C}}} j(x,y)|\mathbf{u}(y)|^2\chi(y)\,\mathrm{d}y \end{cases}$$
$$\text{with } j(x,y) := \mu(|x-y|)\,.$$

It is thus natural to generalize these terms by considering

(4) a kernel

$$j : \Gamma_{\mathrm{C}} \times \Gamma_{\mathrm{C}} \to [0,+\infty) \text{ symmetric, positive, with } j \in L^\infty((\Gamma_{\mathrm{C}}\times\Gamma_{\mathrm{C}});\mathbb{R}^+) \tag{2.10}$$

and introducing the associated nonlocal operator

$$\mathcal{J} : L^1(\Gamma_{\mathrm{C}}) \to L^\infty(\Gamma_{\mathrm{C}}) \qquad \mathcal{J}[w](x) := \int_{\Gamma_{\mathrm{C}}} j(x,y)w(y)\,\mathrm{d}y \quad \text{for all } w \in L^1(\Gamma_{\mathrm{C}})\,. \tag{2.11}$$

Lemma 3.1 ahead will provide some key properties of the operator $\mathcal{J}$.

With the above outlined generalizations, system (1.1) turns into the PDE system

$$\theta_t - \theta\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\alpha(\theta)\nabla\theta) = h + \varepsilon(\mathbf{u}_t)\,\mathbb{V}\,\varepsilon(\mathbf{u}_t) \quad \text{in } \Omega\times(0,T), \tag{2.12a}$$

$$\alpha(\theta)\nabla\theta\cdot\mathbf{n} = 0 \quad \text{in } \Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}\times(0,T), \tag{2.12b}$$

$$\alpha(\theta)\nabla\theta\cdot\mathbf{n} = -k(\chi)\theta(\theta-\theta_{\mathrm{s}}) - \mathcal{J}[\chi]\chi\theta^2 + \mathcal{J}[\chi\theta_{\mathrm{s}}]\chi\theta \quad \text{in } \Gamma_{\mathrm{C}}\times(0,T), \tag{2.12c}$$

$$-\,\mathrm{div}(\mathbb{E}\varepsilon(\mathbf{u}) + \mathbb{V}\varepsilon(\mathbf{u}_t) + \theta\mathbb{I}) = \mathbf{f} \quad \text{in } \Omega\times(0,T), \tag{2.12d}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_{\mathrm{D}}\times(0,T), \tag{2.12e}$$

$$(\mathbb{E}\varepsilon(\mathbf{u}) + \mathbb{V}\varepsilon(\mathbf{u}_t) + \theta\mathbb{I})\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_{\mathrm{N}}\times(0,T), \tag{2.12f}$$

$$(\mathbb{E}\varepsilon(\mathbf{u}) + \mathbb{V}\varepsilon(\mathbf{u}_t) + \theta\mathbb{I})\mathbf{n} + \chi\mathbf{u} + \boldsymbol{\eta}(\mathbf{u}) + \mathcal{J}[\chi]\chi\mathbf{u} \ni \mathbf{0} \quad \text{in } \Gamma_{\mathrm{C}}\times(0,T), \tag{2.12g}$$

$$\begin{aligned}&\partial_t\theta_{\mathrm{s}} - \theta_{\mathrm{s}}\lambda'(\chi)\chi_t - \mathrm{div}(\alpha(\theta_{\mathrm{s}})\nabla\theta_{\mathrm{s}})\\ &\quad = \ell + |\chi_t|^2 + k(\chi)(\theta-\theta_{\mathrm{s}})\theta_{\mathrm{s}} + \mathcal{J}[\chi\theta]\chi\theta_{\mathrm{s}} - \mathcal{J}[\chi]\chi\theta_{\mathrm{s}}^2 \quad \text{in } \Gamma_{\mathrm{C}}\times(0,T),\end{aligned} \tag{2.12h}$$

$$\alpha(\theta_{\mathrm{s}})\nabla\theta_{\mathrm{s}}\cdot\mathbf{n}_{\mathrm{s}} = 0 \quad \text{in } \partial\Gamma_{\mathrm{C}}\times(0,T), \tag{2.12i}$$

$$\chi_t - \Delta\chi + \beta(\chi) + \gamma'(\chi) + \lambda'(\chi)\theta_{\mathrm{s}} \ni -\frac{1}{2}|\mathbf{u}|^2 - \frac{1}{2}\mathcal{J}[\chi]|\mathbf{u}|^2 - \frac{1}{2}\mathcal{J}[\chi|\mathbf{u}|^2] \quad \text{in } \Gamma_{\mathrm{C}}\times(0,T), \tag{2.12j}$$

$$\partial_{\mathbf{n}_{\mathrm{s}}}\chi = 0 \quad \text{in } \partial\Gamma_{\mathrm{C}}\times(0,T), \tag{2.12k}$$

that will be studied in the sequel (note that, here in (2.12j), we have incorporated the term $-\lambda'(\chi)\theta_*$, featuring on the left-hand side of the former (1.1e) into the function γ'). Let us finally specify

(5) our requirements on the heat conductivity: the function $\alpha : [0,+\infty) \to \mathbb{R}^+$ is continuous and fulfills

$$\exists\, c_0,\, c_1 > 0 \quad \exists\, \mu > 1 \quad \forall\, \theta\in[0,+\infty) : \qquad c_0(1+\theta^\mu) \le \alpha(\theta) \le c_1(1+\theta^\mu)\,. \tag{2.13}$$

We will work with its primitive $\widehat{\alpha} : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\widehat{\alpha}(r) := \int_0^r \alpha(s)\,\mathrm{d}s\,; \tag{2.14}$$

(6) our conditions on the nonlinear functions k, γ, and λ:

$$\begin{aligned}&k:\mathbb{R}\to[0,+\infty) \quad \text{has polynomial growth, i.e.}\\ &\exists\, s>1\ \exists\, C_k>0\ \forall\, x\in\mathbb{R} : \quad k(x) \le C_k(|x|^s+1)\,;\end{aligned} \tag{2.15}$$

$$\lambda:\mathbb{R}\to\mathbb{R} \quad \text{is Lipschitz continuous and } \delta\text{-concave for some } \delta\in\mathbb{R}; \tag{2.16}$$

$$\begin{aligned}&\gamma\in \mathrm{C}^1(\mathbb{R}) \text{ with } \gamma':\mathbb{R}\to\mathbb{R} \text{ Lipschitz continuous, } \nu\text{-convex for some } \nu\in\mathbb{R}, \text{ and such that}\\ &\quad \exists\, C_W>0\ \forall\, x\in\mathbb{R} : W(x) := \widehat{\beta}(x) + \gamma(x) \ge -C_W\,;\end{aligned} \tag{2.17}$$

(7) our conditions on the heat sources h and ℓ and on the forces $\mathbf{f}$ and $\mathbf{g}$:

$$h\in L^1(0,T;L^1(\Omega))\cap L^2(0,T;H^1(\Omega)^*), \qquad h\ge 0 \text{ a.e. in } \Omega\times(0,T), \tag{2.18a}$$

$$\ell\in L^1(0,T;L^1(\Gamma_{\mathrm{C}}))\cap L^2(0,T;H^1(\Gamma_{\mathrm{C}})^*), \qquad \ell\ge 0 \text{ a.e. in } \Gamma_{\mathrm{C}}\times(0,T), \tag{2.18b}$$

$$\mathbf{f}\in H^1(0,T;H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)^*), \tag{2.18c}$$

$$\mathbf{g}\in H^1(0,T;\mathbf{Y}^*). \tag{2.18d}$$

We then introduce the function

$$\mathbf{F}\in H^1(0,T;H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)^*), \qquad \langle\mathbf{F}(t),\mathbf{v}\rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)} := \langle\mathbf{f}(t),\mathbf{v}\rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)} + \langle\mathbf{g}(t),\mathbf{v}\rangle_{\mathbf{Y}} \quad \text{for a.a. } t\in(0,T). \tag{2.19}$$

(8) As for the initial data θ_0, θ_{s}^0, $\mathbf{u}_0$, χ_0 we suppose that

$$\theta_0\in L^1(\Omega) \qquad \text{with} \qquad \inf_{x\in\Omega}\theta_0(x) \ge \theta^* > 0, \tag{2.20a}$$

$$\theta_{\mathrm{s}}^0\in L^1(\Gamma_{\mathrm{C}}) \qquad \text{with} \qquad \inf_{x\in\Gamma_{\mathrm{C}}}\theta_{\mathrm{s}}^0(x) \ge \theta_{\mathrm{s}}^* > 0, \tag{2.20b}$$

$$\mathbf{u}_0\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3), \qquad \mathbf{u}_0\in\mathrm{dom}(\widehat{\boldsymbol{\eta}}), \tag{2.20c}$$

$$\chi_0\in H^1(\Gamma_{\mathrm{C}}), \qquad \widehat{\beta}(\chi_0)\in L^1(\Gamma_{\mathrm{C}}). \tag{2.20d}$$

Remark 2.2. Observe that the δ-concavity and ν-convexity requirements for λ and γ mean that

$$\text{the function} \quad \begin{cases} r \mapsto \lambda(r) - \frac{\delta}{2} r^2 \text{ is concave;} \\ r \mapsto \gamma(r) + \frac{\nu}{2} r^2 \text{ is convex.} \end{cases}$$

These properties will be used for devising a time-discretization scheme of system (2.12) such that the validity of a discrete form of the total energy inequality is ensured, cf. Remark 4.2 ahead.

Also the growth condition for k from (2.15) is functional to our approximation scheme, or rather serves to the purpose of simplifying it, cf. Remark 3.4 ahead.

2.2. **Our existence result.** We will prove the existence of weak solutions in the sense specified by Definition 2.3 below. We mention in advance that our notion of 'weak energy solution' to (the Cauchy problem for) system (2.12) features

- the weak formulation of the heat equations (1.1g) and (1.1j) with test functions $v \in W^{1,3+\epsilon}(\Omega)$ and $w \in W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})$ for any $\epsilon > 0$;
- the standard weak formulation of the displacement equation (2.12d), with test functions in $H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$;
- the pointwise formulation (a.e. in $\Gamma_{\mathrm{C}} \times (0,T)$) of the flow rule for the adhesion parameter;
- the *total energy balance*

$$\begin{aligned} &\mathcal{E}(\theta(t), \theta_{\mathrm{s}}(t), \mathbf{u}(t), \chi(t)) + \int_s^t \int_{\Gamma_{\mathrm{C}}} k(\chi(x,r))(\theta(x,r) - \theta_{\mathrm{s}}(x,r))^2 \, \mathrm{d}x \, \mathrm{d}r \\ &\quad + \int_s^t \iint_{\Gamma_{\mathrm{C}} \times \Gamma_{\mathrm{C}}} j(x,y) \chi(x,r) \chi(y,r) (\theta(x,r) - \theta_{\mathrm{s}}(y,r))^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}r \\ &= \mathcal{E}(\theta(s), \theta_{\mathrm{s}}(s), \mathbf{u}(s), \chi(s)) + \int_s^t \int_\Omega h \, \mathrm{d}x \, \mathrm{d}r + \int_s^t \int_{\Gamma_{\mathrm{C}}} \ell \, \mathrm{d}x \, \mathrm{d}r + \int_s^t \langle \mathbf{F}, \mathbf{u}_t \rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)} \, \mathrm{d}r \end{aligned} \tag{2.21}$$

for every $0 \le s \le t \le T$, featuring the stored energy of the system

$$\begin{aligned} \mathcal{E}(\theta, \theta_{\mathrm{s}}, \mathbf{u}, \chi) := & \int_\Omega \theta \, \mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s}} \, \mathrm{d}x \\ & + \frac{1}{2} \mathsf{e}(\mathbf{u}, \mathbf{u}) + \widehat{\boldsymbol{\eta}}(\mathbf{u}) + \frac{1}{2} \int_{\Gamma_{\mathrm{C}}} \left(\chi |\mathbf{u}|^2 + \chi |\mathbf{u}|^2 \mathcal{J}[\chi] \right) \mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \left(\frac{1}{2} |\nabla \chi|^2 + W(\chi) \right) \mathrm{d}x. \end{aligned} \tag{2.22}$$

Definition 2.3. *Given initial data* $(\theta_0, \theta^0_{\mathrm{s}}, \mathbf{u}_0, \chi_0)$ *fulfilling* (2.20)*, we call a quadruple* $(\theta, \theta_{\mathrm{s}}, \mathbf{u}, \chi)$ *a* weak energy solution *to the Cauchy problem for system* (2.12) *if*

$$\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap W^{1,1}(0,T;W^{1,3+\epsilon}(\Omega)^*), \tag{2.23a}$$

$$\widehat{\alpha}(\theta) \in L^1(0,T;W^{1,(3+\epsilon)/(2+\epsilon)}(\Omega)), \qquad \textit{for all } \epsilon > 0, \tag{2.23b}$$

$$\theta_{\mathrm{s}} \in L^2(0,T;H^1(\Gamma_{\mathrm{C}})) \cap L^\infty(0,T;L^1(\Gamma_{\mathrm{C}})) \cap W^{1,1}(0,T;W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})^*), \tag{2.23c}$$

$$\widehat{\alpha}(\theta_{\mathrm{s}}) \in L^1(0,T;W^{1,(2+\epsilon)/(1+\epsilon)}(\Gamma_{\mathrm{C}})), \qquad \textit{for all } \epsilon > 0, \tag{2.23d}$$

$$\mathbf{u} \in H^1(0,T;H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)), \tag{2.23e}$$

$$\chi \in L^2(0,T;H^2(\Gamma_{\mathrm{C}})) \cap L^\infty(0,T;H^1(\Gamma_{\mathrm{C}})) \cap H^1(0,T;L^2(\Gamma_{\mathrm{C}})), \tag{2.23f}$$

the quadruple $(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi)$ *comply with the initial conditions*

$$\begin{aligned} &\theta(x,0) = \theta_0(x), \quad \mathbf{u}(x,0) = \mathbf{u}_0(x) && \textit{for a.a. } x \in \Omega, \\ &\theta_{\mathrm{s}}(x,0) = \theta^0_{\mathrm{s}}(x), \quad \chi(x,0) = \chi_0(x) && \textit{for a.a. } x \in \Gamma_{\mathrm{C}}, \end{aligned} \tag{2.24}$$

and with the positivity *properties*

$$\begin{aligned} &\theta(x,t) > 0 \quad \textit{for a.a. } (x,t) \in \Omega \times (0,T), \\ &\theta_{\mathrm{s}}(x,t) > 0 \quad \textit{for a.a. } (x,t) \in \Gamma_{\mathrm{C}} \times (0,T), \end{aligned} \tag{2.25}$$

and there exist

$$\boldsymbol{\zeta} \in L^2(0,T;\mathbf{Y}^*), \qquad \xi \in L^2(0,T;L^2(\Gamma_{\mathrm{C}})) \tag{2.26}$$

such that the functions $(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi, \boldsymbol{\zeta}, \xi)$ *fulfill*

- *the weak formulation of the bulk heat equation*

$$
\begin{aligned}
&\langle\theta_t, v\rangle_{W^{1,3+\epsilon}(\Omega)} - \int_\Omega \theta \mathrm{div}(\mathbf{u}_t) v \,\mathrm{d}x + \int_\Omega \nabla(\widehat{\alpha}(\theta)) \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma_C} k(\chi)\theta(\theta-\theta_s) v \,\mathrm{d}x \\
&\qquad + \int_{\Gamma_C} \mathcal{J}[\chi]\chi\theta^2 v \,\mathrm{d}x - \int_{\Gamma_C} \mathcal{J}[\chi\theta_s]\chi\theta v \,\mathrm{d}x \\
&= \int_\Omega \varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) v \,\mathrm{d}x + \int_\Omega h v \,\mathrm{d}x \qquad \text{for all } v \in W^{1,3+\epsilon}(\Omega),\ \epsilon>0, \qquad \text{a.e. in } (0,T);
\end{aligned} \tag{2.27}
$$

- *the weak formulation of the surface heat equation*

$$
\begin{aligned}
&\langle\partial_t\theta_s, w\rangle_{W^{1,2+\epsilon}(\Gamma_C)} - \int_{\Gamma_C} \theta_s \lambda'(\chi)\chi_t w \,\mathrm{d}x + \int_{\Gamma_C} \nabla(\widehat{\alpha}(\theta_s)) \cdot \nabla w \,\mathrm{d}x \\
&= \int_{\Gamma_C} \ell w \,\mathrm{d}x + \int_{\Gamma_C} |\chi_t|^2 w \,\mathrm{d}x + \int_{\Gamma_C} k(\chi)\theta_s(\theta-\theta_s) w \,\mathrm{d}x + \int_{\Gamma_C} \mathcal{J}[\chi\theta]\chi\theta_s w \,\mathrm{d}x - \int_{\Gamma_C} \mathcal{J}[\chi]\chi\theta_s^2 w \,\mathrm{d}x \\
&\qquad \text{for all } w \in W^{1,2+\epsilon}(\Gamma_C),\ \epsilon>0, \qquad \text{a.e. in } (0,T);
\end{aligned} \tag{2.28}
$$

- *the weak formulation of the displacement equation*

$$
\mathsf{v}(\mathbf{u}_t,\mathbf{v}) + \mathsf{e}(\mathbf{u},\mathbf{v}) + \int_\Omega \theta \mathrm{div}(\mathbf{v}) \,\mathrm{d}x + \int_{\Gamma_C} \chi \mathbf{u}\mathbf{v} \,\mathrm{d}x + \langle \boldsymbol{\zeta}, \mathbf{v}\rangle_{\mathbf{Y}} + \int_{\Gamma_C} \chi\mathbf{u}\mathcal{J}[\chi]\mathbf{v} \,\mathrm{d}x = \langle \mathbf{F}, \mathbf{v}\rangle_{H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)} \tag{2.29a}
$$

for all $\mathbf{v} \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)$, *with*

$$
\boldsymbol{\zeta} \in L^2(0,T;\mathbf{Y}^*) \text{ fulfilling } \boldsymbol{\zeta}(t) \in \boldsymbol{\eta}(\mathbf{u}(t)) \text{ in } \mathbf{Y}^* \text{ for a.a. } t \in (0,T); \tag{2.29b}
$$

- *the pointwise formulation of the flow rule for the adhesion parameter*
- *the* total energy balance (2.21)*.*

We are now in a position to state the main result of the paper: for technical reasons related to our approximation scheme, we will prove the existence of a weak energy solution such that the pointwise flow rule for the adhesion parameter holds with an additional measurable coefficient $\sigma = \sigma(x,t) \in [0,1]$ for the terms on its right-hand side. However, we point out that the function σ can take values different from 1 only on the set $\{\chi = 0\}$.

Theorem 1 (Global existence of weak energy solutions)**.** *Assume* (2.4)–(2.10) *and* (2.13)–(2.18)*. Then, for every quadruple of initial data* $(\theta_0, \theta_s^0, \mathbf{u}_0, \chi_0)$ *as in* (2.20) *there exists a* weak energy solution $(\theta, \theta_s, \mathbf{u}, \chi)$ *to the Cauchy problem for system* (2.12)*, with an associated selection* $\boldsymbol{\zeta}$ *fulfilling* (2.29a)–(2.29b)*, and a pair* $(\xi,\sigma) \in L^2(0,T;L^2(\Gamma_C)) \times L^\infty(\Gamma_C \times (0,T))$ *such that the pointwise formulation of the flow rule for* χ *holds in the following form:*

$$
\chi_t + A\chi + \xi + \gamma'(\chi) + \lambda'(\chi)\theta_s = -\frac{1}{2}|\mathbf{u}|^2\sigma - \frac{1}{2}\mathcal{J}[\chi]\,|\mathbf{u}|^2\sigma - \frac{1}{2}\mathcal{J}[\chi|\mathbf{u}|^2]\,\sigma \qquad \text{a.e. in } \Gamma_C \times (0,T), \tag{2.30a}
$$

$$
\text{with } \xi \in \beta(\chi) \qquad \text{a.e. in } \Gamma_C \times (0,T), \tag{2.30b}
$$

$$
\text{and } \sigma \begin{cases} \equiv 1 & \text{on } \{(x,t) \in \Gamma_C \times (0,T) \,:\, \chi(x,t) > 0\}, \\ \in [0,1] & \text{on } \{(x,t) \in \Gamma_C \times (0,T) \,:\, \chi(x,t) = 0\}. \end{cases} \tag{2.30c}
$$

In addition, θ *and* θ_s *comply with the positivity properties*

$$
\theta \ge \overline{\theta} > 0 \quad \text{a.e. in } \Omega \times (0,T), \qquad \theta_s \ge \overline{\theta}_s > 0 \quad \text{a.e. in } \Gamma_C \times (0,T) \tag{2.31}
$$

for some positive constants $\overline{\theta}$ *and* $\overline{\theta}_s$*.*

3. Formal a priori estimates and strategy of the proof of Theorem 1

In this Section we derive the basic a priori estimates on the solutions to system (2.12), that are at the core of our definition of *weak energy solution*, by carrying out a series of formal calculations in Section 3.3 ahead. Prior to that, we will fix some preliminary results in Sec. 3.1 and, again formally, prove the strict positivity of the temperature variables in Sec. 3.2. All the calculations in Sec. 3.2 and 3.3 will be rendered rigorously in the context of (the time discretization scheme for) a suitable approximation of system (2.12), set forth in Sec. 3.4. Therein, we will also outline the scheme of the proof of Theorem 1.

In what follows, we shall work under the assumptions listed in Section 2.1; in particular, we will omit to explicitly invoke them in the statements of Lemmas 3.1 and 3.2.

Finally, let us point out that throughout the paper, we will use the symbols $c, c', C, C', \ldots$, with meaning that possibly varies in the same line, to denote several positive constants only depending on known quantities. Analogously, with the symbols $I_1, I_2, \ldots$ we will denote several integral terms appearing in the estimates.

3.1. **Preliminaries.** We will extensively use the following result (cf., e.g., [5]), collecting key properties of the nonlocal operator $\mathfrak{J}$ from (2.11).

Lemma 3.1. *The operator $\mathfrak{J} : L^1(\Gamma_C) \to L^\infty(\Gamma_C)$ is well defined, linear and bounded, with*

$$\|\mathfrak{J}[w]\|_{L^\infty(\Gamma_C)} \le \|j\|_{L^\infty(\Gamma_C\times\Gamma_C)}\|w\|_{L^1(\Gamma_C)} \quad \text{for all } w \in L^1(\Gamma_C); \tag{3.1}$$

$\mathfrak{J}$ also enjoys the positivity property

$$w \ge 0 \quad \text{a.e. in } \Gamma_C \quad \Rightarrow \quad \mathfrak{J}[w] \ge 0 \qquad \text{a.e. in } \Gamma_C\,. \tag{3.2}$$

Furthermore, for every $1 \le p < \infty$ the operator $\mathfrak{J}$ is continuous from $L^1(\Gamma_C)$, equipped with the weak topology, to $L^p(\Gamma_C)$ with the strong topology, namely if $w_n \rightharpoonup w$ in $L^1(\Gamma_C)$ then $\mathfrak{J}[w_n] \to \mathfrak{J}[w]$ in $L^p(\Gamma_C)$. Finally, there holds

$$\int_{\Gamma_C} \mathfrak{J}[w_1](x)\, w_2(x)\,\mathrm{d}x = \int_{\Gamma_C} \mathfrak{J}[w_2](x)\, w_1(x)\,\mathrm{d}x \qquad \text{for all } w_1,\, w_2 \in L^1(\Gamma_C)\,. \tag{3.3}$$

Variational formulations of the heat equations. In the following calculations, we shall (formally) use the variational formulation of the boundary-value problem (2.12a)–(2.12c) for the bulk heat equation, namely

$$\begin{aligned}
&\int_\Omega \theta_t v\,\mathrm{d}x - \int_\Omega \theta \mathrm{div}(\mathbf{u}_t) v\,\mathrm{d}x + \int_\Omega \alpha(\theta)\nabla\theta\nabla v\,\mathrm{d}x + \int_{\Gamma_C} k(\chi)\theta(\theta-\theta_s)v\,\mathrm{d}x \\
&\qquad + \int_{\Gamma_C} \mathfrak{J}[\chi]\chi\theta^2 v\,\mathrm{d}x - \int_{\Gamma_C} \mathfrak{J}[\chi\theta_s]\chi\theta v\,\mathrm{d}x \\
&= \int_\Omega \varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) v\,\mathrm{d}x + \int_\Omega h v\,\mathrm{d}x \qquad \text{for all suitable test functions } v, \qquad \text{a.e. in } (0,T)
\end{aligned} \tag{3.4}$$

and of the boundary value problem (2.12h)–(2.12i) for the surface heat equation, namely

$$\begin{aligned}
&\int_{\Gamma_C} \partial_t\theta_s w\,\mathrm{d}x - \int_{\Gamma_C} \theta_s\lambda'(\chi)\chi_t w\,\mathrm{d}x + \int_{\Gamma_C} \alpha(\theta_s)\nabla\theta_s\nabla w\,\mathrm{d}x \\
&= \int_{\Gamma_C} \ell w\,\mathrm{d}x + \int_{\Gamma_C} |\chi_t|^2 w\,\mathrm{d}x + \int_{\Gamma_C} k(\chi)\theta_s(\theta-\theta_s)w\,\mathrm{d}x + \int_{\Gamma_C} \mathfrak{J}[\chi\theta]\chi\theta_s w\,\mathrm{d}x - \int_{\Gamma_C} \mathfrak{J}[\chi]\chi\theta_s^2 w\,\mathrm{d}x \\
&\qquad\qquad \text{for all suitable test functions } w, \qquad \text{a.e. in } (0,T).
\end{aligned} \tag{3.5}$$

We have been purposefully imprecise in (3.4) and (3.5) since, in any case, the choices of the test functions that we will make in the calculations carried out in Sections 3.2 and 3.3 will be only formal.

Derivation of the total energy balance (2.21)**.** We test the bulk heat equation (3.4) by 1, the displacement equation (2.12d) by $\mathbf{u}_t$, the surface heat equation (3.5) by 1, and the flow rule (2.12j) for χ by χ_t. Adding up the resulting relations, observing the cancellation of some terms, and integrating on a time interval $(s,t) \subset (0,T)$, we obtain

$$
\begin{aligned}
&\int_s^t \int_\Omega \theta_t \,\mathrm{d}x\,\mathrm{d}r + \underbrace{\int_s^t \int_{\Gamma_{\mathrm{C}}} k(\chi)(\theta-\theta_{\mathrm{s}})\theta \,\mathrm{d}x\,\mathrm{d}r}_{\doteq I_1} + \underbrace{\int_s^t \iint_{\Gamma_{\mathrm{C}}\times\Gamma_{\mathrm{C}}} j(x,y)\chi(x)\theta(x)\chi(y)(\theta(x)-\theta_{\mathrm{s}}(y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r}_{\doteq I_2} \\
&\quad + \int_s^t \mathsf{e}(\mathbf{u},\mathbf{u}_t)\,\mathrm{d}r + \underbrace{\int_s^t \langle \boldsymbol{\eta}(\mathbf{u}),\mathbf{u}_t\rangle_{\mathbf{Y}}\,\mathrm{d}r}_{\doteq I_3} + \underbrace{\int_s^t \int_{\Gamma_{\mathrm{C}}} \chi\mathbf{u}\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}r}_{\doteq I_4} \\
&\quad + \underbrace{\int_s^t \iint_{\Gamma_{\mathrm{C}}\times\Gamma_{\mathrm{C}}} j(x,y)\chi(x)\mathbf{u}(x)\mathbf{u}_t(x)\chi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r}_{\doteq I_5} + \int_s^t \int_{\Gamma_{\mathrm{C}}} \partial_t\theta_{\mathrm{s}}\,\mathrm{d}x\,\mathrm{d}r - \underbrace{\int_s^t \int_{\Gamma_{\mathrm{C}}} k(\chi)(\theta-\theta_{\mathrm{s}})\theta_{\mathrm{s}}\,\mathrm{d}x\,\mathrm{d}r}_{\doteq I_6} \\
&\quad - \underbrace{\int_s^t \iint_{\Gamma_{\mathrm{C}}} j(x,y)\chi(x)\theta_{\mathrm{s}}(x)\chi(y)(\theta(y)-\theta_{\mathrm{s}}(x))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r}_{\doteq I_7} + \int_s^t \int_{\Gamma_{\mathrm{C}}} \nabla\chi\cdot\nabla\chi_t\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \underbrace{\int_s^t \int_{\Gamma_{\mathrm{C}}} \beta(\chi)\chi_t\,\mathrm{d}x\,\mathrm{d}r}_{\doteq I_8} + \underbrace{\int_s^t \int_{\Gamma_{\mathrm{C}}} \gamma'(\chi)\chi_t\,\mathrm{d}x\,\mathrm{d}r}_{\doteq I_9} + \underbrace{\int_s^t \int_{\Gamma_{\mathrm{C}}} \frac{1}{2}|\mathbf{u}|^2\chi_t\,\mathrm{d}x\,\mathrm{d}r}_{\doteq I_{10}} \\
&\quad + \underbrace{\int_s^t \iint_{\Gamma_{\mathrm{C}}\times\Gamma_{\mathrm{C}}} \frac{1}{2} j(x,y)\chi_t(x)|\mathbf{u}|^2(x)\chi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r}_{\doteq I_{11}} + \underbrace{\int_s^t \iint_{\Gamma_{\mathrm{C}}} \frac{1}{2} j(x,y)\chi_t(x)|\mathbf{u}|^2(y)\chi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r}_{\doteq I_{12}} \\
&= \int_s^t \int_\Omega h\,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_{\Gamma_{\mathrm{C}}} \ell\,\mathrm{d}x\,\mathrm{d}r + \int_s^t \langle \mathbf{F},\mathbf{u}_t\rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)}\,\mathrm{d}r,
\end{aligned}
\tag{3.6}
$$

where we have formally written the subdifferentials $\boldsymbol{\eta}(\mathbf{u})$ and $\beta(\chi)$ as if singletons. We then observe that

$$
\begin{aligned}
& I_1 - I_6 = \int_s^t \int_{\Gamma_{\mathrm{C}}} k(\chi)(\theta-\theta_{\mathrm{s}})^2\,\mathrm{d}x\,\mathrm{d}r, \\
& I_2 - I_7 = \int_s^t \iint_{\Gamma_{\mathrm{C}}\times\Gamma_{\mathrm{C}}} j(x,y)\chi(x)\theta(x)\chi(y)(\theta(x)-\theta_{\mathrm{s}}(y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r \\
& \qquad\qquad - \int_s^t \iint_{\Gamma_{\mathrm{C}}\times\Gamma_{\mathrm{C}}} j(x,y)\chi(y)\theta_{\mathrm{s}}(y)\chi(x)(\theta(x)-\theta_{\mathrm{s}}(y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r \\
& \qquad \overset{(1)}{=} \int_s^t \iint_{\Gamma_{\mathrm{C}}} j(x,y)\chi(x)\chi(y)(\theta(x)-\theta_{\mathrm{s}}(y))^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r \\
& I_3 \overset{(2)}{=} \widehat{\boldsymbol{\eta}}(\mathbf{u}(t)) - \widehat{\boldsymbol{\eta}}(\mathbf{u}(s)), \quad I_8 + I_9 \overset{(3)}{=} \int_{\Gamma_{\mathrm{C}}} W(\chi(t))\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} W(\chi(s))\,\mathrm{d}x, \\
& I_4 + I_{10} = \int_{\Gamma_{\mathrm{C}}} \frac{1}{2}\chi(t)|\mathbf{u}(t)|^2\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} \frac{1}{2}\chi(s)|\mathbf{u}(s)|^2\,\mathrm{d}x, \\
& I_5 + I_{11} + I_{12} = \int_s^t \int_{\Gamma_{\mathrm{C}}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\chi|\mathbf{u}|^2\mathcal{J}[\chi]\right)\mathrm{d}x\,\mathrm{d}r = \int_{\Gamma_{\mathrm{C}}} \frac{1}{2}\chi(t)|\mathbf{u}(t)|^2\mathcal{J}[\chi](t)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} \frac{1}{2}\chi(s)|\mathbf{u}(s)|^2\mathcal{J}[\chi](s)\,\mathrm{d}x,
\end{aligned}
\tag{3.7}
$$

where for (1) we have used that j is symmetric, for (2) and (3) we have applied the chain rule for the subdifferential operators $\boldsymbol{\eta}$ and β. All in all, we conclude (2.21).

Coercivity properties of the energy functional $\mathcal{E}$. For our first a priori estimate we will indeed start from the energy balance (2.21) and derive the energy bound $\sup_{t\in(0,T)} |\mathcal{E}(\theta(t),\theta_{\mathrm{s}}(t),\mathbf{u}(t),\chi(t))| \leq C$ that will be combined with Lemma 3.2 below to derive a series of uniform-in-time estimates for the solutions.

Lemma 3.2. *There exist two constants C_1, $C_2 > 0$ such that for all $\theta \in L^1(\Omega;\mathbb{R}^+)$, $\theta_{\mathrm{s}} \in L^1(\Gamma_{\mathrm{C}};\mathbb{R}^+)$, $\mathbf{u} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$, and $\chi \in H^1(\Gamma_{\mathrm{C}})$, there holds*

$$
\mathcal{E}(\theta,\theta_{\mathrm{s}},\mathbf{u},\chi) \geq C_1\Big(\|\theta\|_{L^1(\Omega)} + \|\theta_{\mathrm{s}}\|_{L^1(\Gamma_{\mathrm{C}})} + \|\mathbf{u}\|^2_{H^1(\Omega)} + \|\chi\|^2_{H^1(\Gamma_{\mathrm{C}})\cap L^\infty(\Gamma_{\mathrm{C}})}\Big) - C_2. \tag{3.8}
$$

Proof. First of all, we may suppose that $\mathcal{E}(\theta,\theta_{\mathrm{s}},\mathbf{u},\chi) < +\infty$, otherwise estimate (3.8) is trivial. Hence, from $\int_{\Gamma_{\mathrm{C}}} W(\chi)\,\mathrm{d}x < +\infty$ we infer that $\chi \in [0,1]$ a.e. in Γ_{C}. Recalling the definition of the stored energy $\mathcal{E}$ stated by (2.22) and taking into account that θ and θ_{s} are positive functions, we have

$$\mathcal{E}(\theta,\theta_{\mathrm{s}},\mathbf{u},\chi) = \|\theta\|_{L^1(\Omega)} + \|\theta_{\mathrm{s}}\|_{L^1(\Gamma_{\mathrm{C}})} + \frac{1}{2}\mathsf{e}(\mathbf{u},\mathbf{u}) + \widehat{\boldsymbol{\eta}}(\mathbf{u}) + \frac{1}{2}\int_{\Gamma_{\mathrm{C}}} \left(\chi|\mathbf{u}|^2 + \chi|\mathbf{u}|^2 \mathcal{J}[\chi]\right)\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \left(\frac{1}{2}|\nabla\chi|^2 + W(\chi)\right)\,\mathrm{d}x\,.$$

By Korn's inequality (2.7) we have that

$$\frac{1}{2}\mathsf{e}(\mathbf{u},\mathbf{u}) \geq \frac{1}{2}C_{\mathrm{e}}\|\mathbf{u}\|^2_{H^1(\Omega)}. \tag{3.9}$$

Since $\chi \geq 0$ and $\mathcal{J}[\chi] \geq 0$ a.e. in Γ_{C} by (3.2), we find that

$$\int_{\Gamma_{\mathrm{C}}} \left(\chi|\mathbf{u}|^2 + \chi|\mathbf{u}|^2\mathcal{J}[\chi]\right)\,\mathrm{d}x \geq 0.$$

We also have $\widehat{\boldsymbol{\eta}}(\mathbf{u}) \geq 0$ while, according to (2.17), we have

$$\int_{\Gamma_{\mathrm{C}}} W(\chi(t)) \geq -C_W|\Gamma_{\mathrm{C}}|. \tag{3.10}$$

Finally, since $\mathcal{E}(\theta,\theta_{\mathrm{s}},\mathbf{u},\chi)$ estimates $\int_{\Gamma_{\mathrm{C}}} |\nabla\chi|^2\,\mathrm{d}x$ and $\int_{\Gamma_{\mathrm{C}}} \widehat{\beta}(\chi)\,\mathrm{d}x$, and taking into account that $\mathrm{dom}(\widehat{\beta}) \subset [0,1]$, we readily conclude the bound for $\|\chi\|_{L^\infty(\Gamma_{\mathrm{C}})}$, and for $\|\chi\|_{H^1(\Gamma_{\mathrm{C}})}$ via the Poincaré inequality. Hence, (3.8) follows. □

3.2. **Strict positivity of θ and θ_{s}.** In the following calculations we will resort to monotonicity arguments that will be repeatedly used throughout the paper.

We test (3.4) and (3.5) by $-\theta^{-p}$ and $-\theta_{\mathrm{s}}^{-p}$, respectively, with $p > 2$.

This choice of these test functions is only formal for a two-fold reason: the powers $-\theta^{-p}$ and $-\theta_{\mathrm{s}}^{-p}$, with $p > 2$ an arbitrary *real* exponent, are well defined only if θ and θ_{s} are strictly positive. Furthermore, $-\theta^{-p}$ and $-\theta_{\mathrm{s}}^{-p}$ lack sufficient spatial regularity to be admissible test functions for the heat equations. Anyhow, these issues will be fixed when performing this estimate on a suitably regularized time-discretization scheme for system (2.12).

Adding up, integrating over $(0,t)$, and recalling (2.7), we obtain that

$$\begin{aligned}
&\frac{1}{p-1}\int_\Omega \theta^{1-p}(t)\,\mathrm{d}x + \frac{1}{p-1}\int_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s}}^{1-p}(t)\,\mathrm{d}x + p\int_0^t\int_\Omega \alpha(\theta)\theta^{-(1+p)}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s + C_{\mathrm{v}}\int_0^t\int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta^p}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_{\Gamma_{\mathrm{C}}} \frac{|\chi_t|^2}{\theta_{\mathrm{s}}^p}\,\mathrm{d}x\,\mathrm{d}s + p\int_0^t\int_{\Gamma_{\mathrm{C}}} \alpha(\theta_{\mathrm{s}})\theta_{\mathrm{s}}^{-(1+p)}|\nabla\theta_{\mathrm{s}}|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^t\int_{\Gamma_{\mathrm{C}}} \mathcal{J}[\chi]\chi\theta^{2-p}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Gamma_{\mathrm{C}}} \mathcal{J}[\chi\theta_{\mathrm{s}}]\chi\theta^{1-p}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Gamma_{\mathrm{C}}} \chi\theta_{\mathrm{s}}^{1-p}\mathcal{J}[\chi\theta]\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_{\Gamma_{\mathrm{C}}} \chi\theta_{\mathrm{s}}^{2-p}\mathcal{J}[\chi]\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_{\Gamma_{\mathrm{C}}} k(\chi)(\theta-\theta_{\mathrm{s}})\Big(-\theta^{1-p} - (-\theta_{\mathrm{s}}^{1-p})\Big)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega h\theta^{-p}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Gamma_{\mathrm{C}}} \ell\theta_{\mathrm{s}}^{-p}\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \frac{1}{p-1}\int_\Omega \theta_0^{1-p}\,\mathrm{d}x + \frac{1}{p-1}\int_{\Gamma_{\mathrm{C}}} (\theta_{\mathrm{s}}^0)^{1-p}\,\mathrm{d}x - \int_0^t\int_\Omega \theta^{1-p}\mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s}}^{1-p}\lambda'(\chi)\chi_t\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned} \tag{3.11}$$

Due to (2.20a)–(2.20b), for the first two terms on the right hand side of (3.11) it holds

$$\int_\Omega \theta_0^{1-p} + \int_{\Gamma_{\mathrm{C}}} (\theta_{\mathrm{s}}^0)^{1-p} \leq \left(\frac{1}{\theta^*}\right)^{p-1}|\Omega| + \left(\frac{1}{\theta_{\mathrm{s}}^*}\right)^{p-1}|\Gamma_{\mathrm{C}}|, \tag{3.12}$$

(recall that $0 < \theta^* \leq \inf_\Omega \theta_0$ and $0 < \theta_{\mathrm{s}}^* \leq \inf_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s}}^0$). Moreover, the third and the fourth term on the right-hand side of (3.11) can be estimated as follows: we have

$$
\begin{aligned}
-\int_0^t \int_\Omega \theta^{1-p} \mathrm{div}(\mathbf{u}_t) \,\mathrm{d}x \,\mathrm{d}s & \overset{(1)}{\leq} \frac{C_{\mathrm{v}}}{2} \int_0^t \int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta^p} \,\mathrm{d}x \,\mathrm{d}s + c \int_0^t \int_\Omega \theta^{2-p} \,\mathrm{d}x \,\mathrm{d}s \\
& \overset{(2)}{\leq} \frac{C_{\mathrm{v}}}{2} \int_0^t \int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta^p} \,\mathrm{d}x \,\mathrm{d}s + C \left(\frac{1}{p-1} + \frac{p-2}{p-1} \int_0^t \int_\Omega \theta^{1-p} \,\mathrm{d}x \,\mathrm{d}s \right) \\
& = \frac{C_{\mathrm{v}}}{2} \int_0^t \int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta^p} \,\mathrm{d}x \,\mathrm{d}s + C \left(\frac{1}{p-1} + \frac{p-2}{p-1} \int_0^t \left\| \frac{1}{\theta}(s) \right\|_{L^{p-1}(\Omega)}^{p-1} \mathrm{d}s \right),
\end{aligned}
\tag{3.13}
$$

where for (1) we have used that

$$
|\mathrm{div}(\mathbf{u}_t)| \leq c_d |\varepsilon(\mathbf{u}_t)|, \tag{3.14}
$$

with $c_d > 0$ a constant only depending on the space dimension $d = 3$. Moreover we have resorted to the Young inequality

$$
ab \leq \frac{(\delta a)^q}{q} + \frac{1}{q'} \left(\frac{b}{\delta} \right)^{q'}, \tag{3.15}
$$

for all $a, b \in \mathbb{R}^+$, $\delta > 0$ and $q, q' > 1$, such that $\frac{1}{q} + \frac{1}{q'} = 1$. In particular for (2) we have used (3.15) with the choices $b = \delta = 1$, $q = (p-1)/(p-2)$, $q' = p-1$ and $a = \theta^{2-p}$. Furthermore,

$$
\begin{aligned}
-\int_0^t \int_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s}}^{1-p} \lambda'(\chi) \chi_t & \overset{(1)}{\leq} \frac{1}{2} \int_0^t \int_{\Gamma_{\mathrm{C}}} \frac{|\chi_t|^2}{\theta_{\mathrm{s}}^p} \,\mathrm{d}x \,\mathrm{d}s + c \int_0^t \int_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s}}^{2-p} \,\mathrm{d}x \,\mathrm{d}s \\
& \overset{(2)}{\leq} \frac{1}{2} \int_0^t \int_{\Gamma_{\mathrm{C}}} \frac{|\chi_t|^2}{\theta_{\mathrm{s}}^p} \,\mathrm{d}x \,\mathrm{d}s + C \left(\frac{1}{p-1} + \frac{p-2}{p-1} \int_0^t \int_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s}}^{1-p} \,\mathrm{d}x \,\mathrm{d}s \right) \\
& = \frac{1}{2} \int_0^t \int_{\Gamma_{\mathrm{C}}} \frac{|\chi_t|^2}{\theta_{\mathrm{s}}^p} \,\mathrm{d}x \,\mathrm{d}s + C \left(\frac{1}{p-1} + \frac{p-2}{p-1} \int_0^t \left\| \frac{1}{\theta_{\mathrm{s}}}(s) \right\|_{L^{p-1}(\Gamma_{\mathrm{C}})}^{p-1} \mathrm{d}s \right),
\end{aligned}
\tag{3.16}
$$

where (1) follows from the Lipschitz continuity of λ stated by (2.16) and (2), again, from (3.15).

As for the left-hand side of (3.11), we observe that all terms from the first to the sixth are positive. We rewrite the sum of the seventh, eighth, ninth and tenth terms as

$$
\begin{aligned}
& -\int_0^t \int_{\Gamma_{\mathrm{C}}} \theta^{1-p} \left(\mathcal{J}[\chi]\chi\theta - \mathcal{J}[\chi\theta_{\mathrm{s}}]\chi \right) \mathrm{d}x \,\mathrm{d}s + \int_0^t \int_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s}}^{1-p} \left(\mathcal{J}[\chi\theta]\chi - \mathcal{J}[\chi]\chi\theta_{\mathrm{s}} \right) \mathrm{d}x \,\mathrm{d}s \\
& \overset{(1)}{=} -\int_0^t \iint_{\Gamma_{\mathrm{C}} \times \Gamma_{\mathrm{C}}} j(x,y) \chi(x) \chi(y) \theta^{1-p}(x) \Big(\theta(x) - \theta_{\mathrm{s}}(y) \Big) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s \\
& \quad + \int_0^t \iint_{\Gamma_{\mathrm{C}} \times \Gamma_{\mathrm{C}}} j(x,y) \chi(x) \chi(y) \theta_{\mathrm{s}}^{1-p}(y) \Big(\theta(x) - \theta_{\mathrm{s}}(y) \Big) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s \\
& = -\int_0^t \iint_{\Gamma_{\mathrm{C}} \times \Gamma_{\mathrm{C}}} j(x,y) \chi(x) \chi(y) \Big(\theta^{1-p}(x) - \theta_{\mathrm{s}}^{1-p}(y) \Big) \Big(\theta(x) - \theta_{\mathrm{s}}(y) \Big) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s \overset{(2)}{\geq} 0
\end{aligned}
\tag{3.17}
$$

where for (1) we have exchanged x and y in the second integral and used that the kernel j is symmetric (cf. also (3.7)), and inequality (2) follows from the fact that the function $(0, +\infty) \ni r \mapsto -r^{1-p}$ is strictly increasing (since $p > 2$) and from the positivity of the kernel j and of χ. By the same monotonicity argument we also have that

$$
\int_0^t \int_{\Gamma_{\mathrm{C}}} k(\chi)(\theta - \theta_{\mathrm{s}}) \big(-\theta^{1-p} - (-\theta_{\mathrm{s}}^{1-p}) \big) \,\mathrm{d}x \,\mathrm{d}s \geq 0. \tag{3.18}
$$

Finally, due to (2.18a) and (2.18b), we have that

$$
\int_0^t \int_\Omega h \theta^{-p} \,\mathrm{d}x \,\mathrm{d}s \geq 0, \qquad \int_0^t \int_{\Gamma_{\mathrm{C}}} \ell \theta_{\mathrm{s}}^{-p} \,\mathrm{d}x \,\mathrm{d}s \geq 0. \tag{3.19}
$$

Combining (3.11) with (3.12)–(3.13) and (3.16)–(3.19) we infer that

$$\begin{aligned}&\left\|\frac{1}{\theta}(t)\right\|_{L^{p-1}(\Omega)}^{p-1}+p(p-1)\int_0^t\int_\Omega \alpha(\theta)\theta^{-(1+p)}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s+\frac{C_\mathrm{v}(p-1)}{2}\int_0^t\int_\Omega\frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta^p}\,\mathrm{d}x\,\mathrm{d}s\\&+\left\|\frac{1}{\theta_\mathrm{s}}(t)\right\|_{L^{p-1}(\Gamma_\mathrm{C})}^{p-1}+p(p-1)\int_0^t\int_{\Gamma_\mathrm{C}}\alpha(\theta_\mathrm{s})\theta_\mathrm{s}^{-(1+p)}|\nabla\theta_\mathrm{s}|^2\,\mathrm{d}x\,\mathrm{d}s+\frac{(p-1)}{2}\int_0^t\int_{\Gamma_\mathrm{C}}\frac{|\chi_t|^2}{\theta_\mathrm{s}^p}\,\mathrm{d}x\,\mathrm{d}s\\&\le\left(\frac{1}{\theta^*}\right)^{p-1}|\Omega|+\left(\frac{1}{\theta_\mathrm{s}^*}\right)^{p-1}|\Gamma_\mathrm{C}|+C\left(1+(p-2)\int_0^t\left(\left\|\frac{1}{\theta}(s)\right\|_{L^{p-1}(\Omega)}^{p-1}+\left\|\frac{1}{\theta_\mathrm{s}}(s)\right\|_{L^{p-1}(\Gamma_\mathrm{C})}^{p-1}\right)\mathrm{d}s\right)\end{aligned}\tag{3.20}$$

with the constant C on the right-hand side of (3.20) independent of p. Thus,

$$\begin{aligned}&\left\|\frac{1}{\theta}(t)\right\|_{L^{p-1}(\Omega)}^{p-1}+\left\|\frac{1}{\theta_\mathrm{s}}(t)\right\|_{L^{p-1}(\Gamma_\mathrm{C})}^{p-1}\\&\le|\Omega|\left(\frac{1}{\theta^*}\right)^{p-1}+|\Gamma_\mathrm{C}|\left(\frac{1}{\theta_\mathrm{s}^*}\right)^{p-1}+C\left(1+(p-2)\int_0^t\left(\left\|\frac{1}{\theta}(s)\right\|_{L^{p-1}(\Omega)}^{p-1}+\left\|\frac{1}{\theta_\mathrm{s}}(s)\right\|_{L^{p-1}(\Gamma_\mathrm{C})}^{p-1}\right)\mathrm{d}s\right).\end{aligned}\tag{3.21}$$

Applying the Gronwall Lemma, we therefore obtain that

$$\left\|\frac{1}{\theta}(t)\right\|_{L^{p-1}(\Omega)}^{p-1}+\left\|\frac{1}{\theta_\mathrm{s}}(t)\right\|_{L^{p-1}(\Gamma_\mathrm{C})}^{p-1}\le\left(|\Omega|\left(\frac{1}{\theta^*}\right)^{p-1}+|\Gamma_\mathrm{C}|\left(\frac{1}{\theta_\mathrm{s}^*}\right)^{p-1}+C\right)\exp(CT(p-2))\,.$$

Therefore,

$$\begin{aligned}&\max\left\{\left\|\frac{1}{\theta}(t)\right\|_{L^{p-1}(\Omega)},\left\|\frac{1}{\theta_\mathrm{s}}(t)\right\|_{L^{p-1}(\Gamma_\mathrm{C})}\right\}\le\left(|\Omega|\left(\frac{1}{\theta^*}\right)^{p-1}+|\Gamma_\mathrm{C}|\left(\frac{1}{\theta_\mathrm{s}^*}\right)^{p-1}+C\right)^{1/(p-1)}\exp\left(CT\frac{(p-2)}{(p-1)}\right)\\&\le\left(|\Omega|^{1/(p-1)}\left(\frac{1}{\theta^*}\right)+|\Gamma_\mathrm{C}|^{1/(p-1)}\left(\frac{1}{\theta_\mathrm{s}^*}\right)+C^{1/(p-1)}\right)\exp(CT)\le\overline{C},\end{aligned}\tag{3.22}$$

where for the last estimate we have used that $|\Omega|^{1/(p-1)}\le|\Omega|+1$ and analogously for $|\Gamma_\mathrm{C}|^{1/(p-1)}$ and $C^{1/(p-1)}$. Since the positive constant $\overline{C}$ is independent of p, we are allowed to conclude that the above estimate holds for arbitrary p. All in all, we find

$$\left\|\frac{1}{\theta}\right\|_{L^\infty(\Omega\times(0,T))}+\left\|\frac{1}{\theta_\mathrm{s}}\right\|_{L^\infty(\Gamma_\mathrm{C}\times(0,T))}\le C.\tag{3.23}$$

Consequently, we infer that the positivity properties (2.31) hold.

3.3. **A priori estimates.** We are now in a position to (formally) derive all of our a priori estimates on the solutions to system (2.12).

3.3.1. **First a priori estimate.** We consider the total energy balance (2.21) on a generic interval $(0,t)$, $t\in(0,T)$. Taking into account the positivity of the second and third terms on the left-hand side, we infer

$$\begin{aligned}\mathcal{E}(\theta(t),\theta_\mathrm{s}(t),\mathbf{u}(t),\chi(t))&\le\mathcal{E}(\theta_0,\theta_\mathrm{s}^0,\mathbf{u}_0,\chi_0)+\int_0^t\int_\Omega h\,\mathrm{d}x\,\mathrm{d}r+\int_0^t\int_{\Gamma_\mathrm{C}}\ell\,\mathrm{d}x\,\mathrm{d}r+\int_0^t\langle\mathbf{F},\mathbf{u}_t\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}\,\mathrm{d}r\\&\doteq I_0+I_1+I_2+I_3\,.\end{aligned}\tag{3.24}$$

Now, by (2.20) and (2.18a)–(2.18b) we have $I_0+I_1+I_2\le C$. Integrating by parts in time, we have

$$\begin{aligned}I_3&=\int_0^t\langle\mathbf{F},\mathbf{u}_t\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}\,\mathrm{d}r=\langle\mathbf{F}(t),\mathbf{u}(t)\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}-\langle\mathbf{F}(0),\mathbf{u}_0\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}-\int_0^t\langle\mathbf{F}_t,\mathbf{u}\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}\,\mathrm{d}r\\&\le\frac{C_1}{2}\|\mathbf{u}(t)\|_{H^1(\Omega)}^2+c\Big(\|\mathbf{F}\|^2_{L^\infty(0,T;H^1_{\Gamma_\mathrm{D}}(\Omega)^*)}+\|\mathbf{F}_t\|^2_{L^2(0,T;H^1_{\Gamma_\mathrm{D}}(\Omega)^*)}+\|\mathbf{u}_0\|^2_{H^1(\Omega)}+\int_0^t\|\mathbf{u}(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s\Big),\end{aligned}\tag{3.25}$$

with $C_1>0$ the constant from the coercivity estimate (3.8). We combine (3.24) and (3.25); taking into account (3.8), we may absorb the term $\frac{C_1}{2}\|\mathbf{u}(t)\|^2_{H^1(\Omega)}$, into the left-hand side of (3.24). Applying the Gronwall Lemma we conclude that $\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^3))}\le C$. Then, a fortiori, the term I_3 on the right-hand side of (3.24) is estimated by a constant. All in all, also taking into account that $\mathcal{E}$ is bounded from below, we conclude that

$$\sup_{t\in(0,T)} |\mathcal{E}(\theta(t),\theta_{\mathrm{s}}(t),\mathbf{u}(t),\chi(t))| \leq C$$

and then, by (3.8), we find that

$$\|\theta\|_{L^\infty(0,T;L^1(\Omega))} + \|\theta_{\mathrm{s}}\|_{L^\infty(0,T;L^1(\Gamma_{\mathrm{C}}))} + \|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega))} + \|\chi\|_{L^\infty(0,T;L^\infty(\Gamma_{\mathrm{C}})\cap H^1(\Gamma_{\mathrm{C}}))} \leq C. \tag{3.26}$$

Remark 3.3. In the calculations for the following a priori estimates we shall not use the $L^\infty(0,T;L^\infty(\Gamma_{\mathrm{C}}))$-bound for χ. The reason is that these computations will be rendered rigorously once performed on a suitable approximation of system (2.12), cf. system (3.69) ahead, in which, in particular, the maximal monotone operator β is replaced by its Yosida regularization β_ς, with primitive $\widehat{\beta}_\varsigma$. Therefore, on the approximate level the bound for $\sup_{t\in(0,T)}\int_{\Gamma_{\mathrm{C}}}\widehat{\beta}_\varsigma(\chi)\,\mathrm{d}x$ will no longer yield the information that χ takes values in the interval $[0,1]$ a.e. in Γ_{C}; in particular, some technical adjustments in devising system (3.69) will be necessary to cope with the lack of positivity of χ.

In any case, the following calculations can be carried out without resorting to the $L^\infty(0,T;L^\infty(\Gamma_{\mathrm{C}}))$-bound for χ. In this way, they will be immediately translated in the context of system (3.69).

3.3.2. **Second a priori estimate.** We first carry out the calculations in the case $\mu\in(1,2)$, then address the cases $\mu>2$ and $\mu=2$.

Case $\mu\in(1,2)$**.** We introduce the function

$$F(v) := v^\nu/\nu, \qquad F'(v) := v^{\nu-1}, \quad \text{with } \nu = 2-\mu \in (0,1). \tag{3.27}$$

Then, we test (3.4) by $F'(\theta)=\theta^{\nu-1}$ and (3.5) by $F'(\theta_{\mathrm{s}})=\theta_{\mathrm{s}}^{\nu-1}$, respectively. Integrating over $(0,t)$ and adding the corresponding equations, with easy calculations (and again recalling (2.7)) we obtain that

$$\begin{aligned}
&\int_0^t\int_{\Gamma_{\mathrm{C}}}|\chi_t|^2F'(\theta_{\mathrm{s}})\,\mathrm{d}x\,\mathrm{d}r + C_{\mathrm{v}}\int_0^t\int_\Omega|\varepsilon(\mathbf{u}_t)|^2F'(\theta)\,\mathrm{d}x\,\mathrm{d}r \\
&\quad+ \underbrace{\int_0^t\int_{\Gamma_{\mathrm{C}}}\mathfrak{I}[\chi\theta_{\mathrm{s}}]\chi\theta F'(\theta)\,\mathrm{d}x\,\mathrm{d}r}_{I_1} + \underbrace{\int_0^t\int_{\Gamma_{\mathrm{C}}}\mathfrak{I}[\chi\theta]\chi\theta_{\mathrm{s}}F'(\theta_{\mathrm{s}})\,\mathrm{d}x\,\mathrm{d}r}_{I_2} \\
&\quad- \underbrace{\int_0^t\int_\Omega\alpha(\theta)\nabla\theta\nabla(F'(\theta))\,\mathrm{d}x\,\mathrm{d}r}_{I_3} - \underbrace{\int_0^t\int_{\Gamma_{\mathrm{C}}}\alpha(\theta_{\mathrm{s}})\nabla\theta_{\mathrm{s}}\nabla(F'(\theta_{\mathrm{s}}))\,\mathrm{d}x\,\mathrm{d}r}_{I_4} \\
&\leq \underbrace{\int_0^t\int_{\Gamma_{\mathrm{C}}}k(\chi)(\theta-\theta_{\mathrm{s}})(\theta F'(\theta)-\theta_{\mathrm{s}}F'(\theta_{\mathrm{s}}))\,\mathrm{d}x\,\mathrm{d}r}_{I_5} + \underbrace{\int_0^t\int_{\Gamma_{\mathrm{C}}}\mathfrak{I}[\chi]\chi\theta^2F'(\theta)\,\mathrm{d}x\,\mathrm{d}r}_{I_6} \\
&\quad+ \underbrace{\int_0^t\int_{\Gamma_{\mathrm{C}}}\mathfrak{I}[\chi]\chi\theta_{\mathrm{s}}^2F'(\theta_{\mathrm{s}})\,\mathrm{d}x\,\mathrm{d}r}_{I_7} + \underbrace{\int_0^t\int_\Omega\theta_tF'(\theta)\,\mathrm{d}x\,\mathrm{d}r}_{I_8} \\
&\quad+ \underbrace{\int_0^t\int_{\Gamma_{\mathrm{C}}}\partial_t\theta_{\mathrm{s}}F'(\theta_{\mathrm{s}})\,\mathrm{d}x\,\mathrm{d}r}_{I_9} - \underbrace{\int_0^t\int_\Omega\theta\,\mathrm{div}(\mathbf{u}_t)F'(\theta)\,\mathrm{d}x\,\mathrm{d}r}_{I_{10}} \\
&\quad- \underbrace{\int_0^t\int_{\Gamma_{\mathrm{C}}}\theta_{\mathrm{s}}\lambda'(\chi)\chi_tF'(\theta_{\mathrm{s}})\,\mathrm{d}x\,\mathrm{d}r}_{I_{11}} - \underbrace{\int_0^t\int_\Omega hF'(\theta)\,\mathrm{d}x\,\mathrm{d}r}_{I_{12}} - \underbrace{\int_0^t\int_{\Gamma_{\mathrm{C}}}\ell F'(\theta_{\mathrm{s}})\,\mathrm{d}x\,\mathrm{d}r}_{I_{13}}\,.
\end{aligned} \tag{3.28}$$

Now, by the previously proved positivity of θ and θ_{s}, it is immediate to see that $I_1\geq 0$ and $I_2\geq 0$. Recalling the growth properties of α (cf. (2.13)), we have that

$$-I_3 = -\int_0^t\int_\Omega\alpha(\theta)\nabla\theta\nabla(F'(\theta))\,\mathrm{d}x\,\mathrm{d}r = (1-\nu)\int_0^t\int_\Omega\alpha(\theta)|\nabla\theta|^2\theta^{\nu-2}\,\mathrm{d}x\,\mathrm{d}r \geq c\int_0^t\int_\Omega|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}r \tag{3.29}$$

since $\nu = 2-\mu < 1$. Analogously, we find that

$$-I_4 = -\int_0^t\int_{\Gamma_C}\alpha(\theta_s)\nabla\theta_s\nabla(F'(\theta_s))\,dx\,dr \geq c\int_0^t\int_{\Gamma_C}|\nabla\theta_s|^2\,dx\,dr. \tag{3.30}$$

As for the terms on the right-hand side of (3.28), we have that

$$\begin{aligned} I_5 &\leq \int_0^t\int_{\Gamma_C}|k(\chi)|(\theta_s+\theta)(\theta_s^\nu+\theta^\nu)\,dx\,dr \\ &= C_k\int_0^t\int_{\Gamma_C}(|\chi|^s+1)(\theta^{\nu+1}+\theta\theta_s^\nu+\theta_s\theta^\nu+\theta_s^{\nu+1})\,dx\,dr \leq C\int_0^t\int_{\Gamma_C}(|\chi|^s+1)\Big(\theta^{\nu+1}+\theta_s^{\nu+1}\Big)\,dx\,dr, \end{aligned}$$

where we have used the polynomial growth of k (cf. (2.15)), and and the previously obtained positivity of θ and θ_s. We have that

$$I_6 \leq \int_0^t\int_{\Gamma_C}|\mathcal{J}[\chi]|\chi\theta^2F'(\theta)\,dx\,dr \leq C\int_0^t\int_{\Gamma_C}(|\chi|^s+1)\theta^{\nu+1}\,dx\,dr,$$

where we have again used (2.15) and the fact that $\|\chi\|_{L^\infty(0,T;L^1(\Gamma_C))} \leq C$, so that $\|\mathcal{J}[\chi]\|_{L^\infty(\Gamma_C\times(0,T))} \leq C$ by Lemma 3.1. Analogously, we have

$$I_7 \leq C\int_0^t\int_{\Gamma_C}(|\chi|^s+1)\theta_s^{\nu+1}\,dx\,dr.$$

We clearly have

$$\begin{aligned} I_8 &= \int_0^t\int_\Omega\theta_tF'(\theta)\,dx\,dr = \int_\Omega(F(\theta(t))-F(\theta_0))\,dx = \frac{1}{\nu}\int_\Omega\theta^\nu(t)\,dx - \frac{1}{\nu}\int_\Omega\theta_0^\nu\,dx, \\ I_9 &= \int_0^t\int_{\Gamma_C}\partial_t\theta_sF'(\theta_s)\,dx\,dr = \int_{\Gamma_C}\big(F(\theta_s(t))-F(\theta_s^0)\big)\,dx = \frac{1}{\nu}\int_{\Gamma_C}\theta_s^\nu(t)\,dx - \frac{1}{\nu}\int_{\Gamma_C}(\theta_s^0)^\nu\,dx. \end{aligned}$$

Applying Young's inequality and recalling (3.14), we obtain that

$$I_{10} \leq \int_0^t\int_\Omega|\theta\mathrm{div}(\mathbf{u}_t)F'(\theta)|\,dx\,dr \leq \frac{C_v}{2}\int_0^t\int_\Omega|\varepsilon(\mathbf{u}_t)|^2F'(\theta)\,dx\,dr + C\int_0^t\int_\Omega\theta^{\nu+1}\,dx\,dr$$

while, due to the Lipschitz continuity of λ, we have that

$$I_{11} \leq \int_0^t\int_{\Gamma_C}\theta_s|\lambda'(\chi)||\chi_t|F'(\theta_s)\,dx\,dr \leq \frac{1}{2}\int_0^t\int_{\Gamma_C}|\chi_t|^2F'(\theta_s)\,dx\,dr + C\int_0^t\int_{\Gamma_C}\theta_s^{\nu+1}\,dx\,dr.$$

Finally, by the positivity assumptions in (2.18a) and (2.18b), we have that

$$I_{12} = -\int_0^t\int_\Omega hF'(\theta)\,dx\,dr \leq 0\,, \qquad I_{13} = -\int_0^t\int_{\Gamma_C}\ell F'(\theta_s)\,dx\,dr \leq 0\,.$$

Collecting all of the above estimates, we arrive at

$$\begin{aligned} &\frac{1}{2}\int_0^t\int_{\Gamma_C}|\chi_t|^2F'(\theta_s)\,dx\,dr + \frac{C_v}{2}\int_0^t\int_\Omega|\varepsilon(\mathbf{u}_t)|^2F'(\theta)\,dx\,dr + c\int_0^t\int_\Omega|\nabla\theta|^2\,dx\,dr + c\int_0^t\int_{\Gamma_C}|\nabla\theta_s|^2\,dx\,dr \\ &\quad + \frac{1}{\nu}\Big(\int_\Omega\theta_0^\nu\,dx + \int_{\Gamma_C}(\theta_s^0)^\nu\,dx\Big) \\ &\leq \frac{1}{\nu}\Big(\int_\Omega\theta^\nu(t)\,dx + \int_{\Gamma_C}\theta_s^\nu(t)\,dx\Big) + C\int_0^t\int_{\Gamma_C}(|\chi|^s+1)\theta^{\nu+1}\,dx\,dr + C\int_0^t\int_{\Gamma_C}(|\chi|^s+1)\theta_s^{\nu+1}\,dx\,dr \\ &\doteq I_{14}+I_{15}+I_{16}+I_{17} \end{aligned} \tag{3.31}$$

Now, since $\nu < 1$, we clearly have

$$I_{14} \leq C\Big(\|\theta(t)\|_{L^1(\Omega)}^\nu+1\Big) \leq C, \qquad I_{15} \leq C\Big(\|\theta_s(t)\|_{L^1(\Gamma_C)}^\nu+1\Big) \leq C, \tag{3.32}$$

where the last estimates are due to the previously obtained (3.26). Furthermore, we have

$$
\begin{aligned}
I_{16} &\overset{(1)}{\leq} C\int_0^t (\|\chi^s\|_{L^{\varrho'}(\Gamma_C)}+1)\|\theta^{\nu+1}\|_{L^{\varrho}(\Gamma_C)}\,\mathrm{d}r = C\int_0^t (\|\chi\|^s_{L^{s\varrho'}(\Gamma_C)}+1)\|\theta\|^{\nu+1}_{L^{(\nu+1)\varrho}(\Gamma_C)}\,\mathrm{d}r \\
&\overset{(2)}{\leq} C\int_0^t \|\theta\|^{\nu+1}_{H^1(\Omega)}\,\mathrm{d}r \\
&\overset{(3)}{\leq} C\left(\int_0^t \|\nabla\theta\|^{\nu+1}_{L^2(\Omega)}\,\mathrm{d}r + \|\theta\|^{\nu+1}_{L^\infty(0,T;L^1(\Omega))}\right) \\
&\overset{(4)}{\leq} \frac{c}{2}\int_0^t\int_\Omega |\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}r + C
\end{aligned}
\tag{3.33}
$$

where for (1) we have used Hölder's inequality with some $\varrho > 1$, chosen in such a way that $(\nu+1)\varrho \leq 4$ so that, by Sobolev embeddings and trace theorems, we have that $\|\theta\|_{L^{(\nu+1)\varrho}(\Gamma_C)} \leq C\|\theta\|_{H^1(\Omega)}$. Then, taking into account the previously proved estimate for χ in $L^\infty(0,T;H^1(\Gamma_C))$ and, a fortiori, in $L^\infty(0,T;L^q(\Gamma_C))$ for all $1 \leq q < \infty$, we conclude (2). Estimate (3) follows from the Poincaré inequality, and (4) from Young's inequality (since $\nu+1 < 2$) and, again, (3.26). In this way, the term $\frac{c}{2}\int_0^t\int_\Omega |\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}r$ can be absorbed into the left-hand side of (3.31). The term $\int_0^t\int_{\Gamma_C}(|\chi|^s+1)\theta_s^{\nu+1}\,\mathrm{d}x\,\mathrm{d}r$ can be treated in a completely analogous way. Hence, from (3.31) we conclude

$$
\|\theta\|_{L^2(0,T;H^1(\Omega))} + \|\theta_s\|_{L^2(0,T;H^1(\Gamma_C))} \leq C. \tag{3.34}
$$

Case $\mu > 2$**.** We test (3.4) and (3.5) by $-\theta^{-q}$ and $-\theta_s^{-q}$, respectively, with $q = \mu - 1$. Adding the resulting relations and integrating over $(0,t)$ we obtain the analogue of (3.11), with q in place of p. We observe that the first two terms on the right-hand side of (3.11) can be estimated as in (3.12), while the last seven terms on the left-hand side of (3.11) can be handled by monotonicity arguments as in (3.17)–(3.19). Since $q = \mu - 1$, in view of the growth properties of α, cf. (2.13), we have that

$$
q\int_0^t\int_\Omega \alpha(\theta)\theta^{-(1+q)}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s \geq c_0 q\int_0^t\int_\Omega \theta^{\mu-(1+q)}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s = c_0 q\int_0^t\int_\Omega |\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s, \tag{3.35}
$$

$$
q\int_0^t\int_{\Gamma_C} \alpha(\theta_s)\theta_s^{-(1+q)}|\nabla\theta_s|^2\,\mathrm{d}x\,\mathrm{d}s \geq c_0 q\int_0^t\int_{\Gamma_C} \theta_s^{\mu-(1+q)}|\nabla\theta_s|^2\,\mathrm{d}x\,\mathrm{d}s = c_0 q\int_0^t\int_{\Gamma_C} |\nabla\theta_s|^2\,\mathrm{d}x\,\mathrm{d}s. \tag{3.36}
$$

Besides, using (3.14) and the Young inequality (3.15), the third term on the right-hand side of (3.11) can be estimated as follows:

$$
-\int_0^t\int_\Omega \theta^{1-q}\mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}s \leq \frac{C_v}{2}\int_0^t\int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta^q}\,\mathrm{d}x\,\mathrm{d}s + c\left(\int_0^t\int_\Omega \theta^{2-q}\,\mathrm{d}x\,\mathrm{d}s\right). \tag{3.37}
$$

Since $\theta \geq \theta^* > 0$ a.e. in $\Omega \times (0,T)$, we have that

$$
\int_0^t\int_\Omega \theta^{2-q}\,\mathrm{d}x\,\mathrm{d}s \leq \int_0^t\int_\Omega (\theta^*)^{2-q}\,\mathrm{d}x\,\mathrm{d}s \leq c \quad \text{whenever } q \geq 2, \tag{3.38}
$$

while

$$
\int_0^t\int_\Omega \theta^{2-q}\,\mathrm{d}x\,\mathrm{d}s \leq \int_0^t\int_\Omega \theta\,\mathrm{d}x\,\mathrm{d}s + |\Omega|T \quad \text{whenever } 1 < q < 2. \tag{3.39}
$$

Combining (3.37) with (3.38)–(3.39) and recalling that $\|\theta\|_{L^\infty(0,T;L^1(\Omega))} \leq c$, by (3.26), we conclude that

$$
-\int_0^t\int_\Omega \theta^{1-q}\mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}s \leq \frac{C_v}{2}\int_0^t\int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta^q}\,\mathrm{d}x\,\mathrm{d}s + c\,. \tag{3.40}
$$

Arguing in a similar way and recalling the bound $\|\theta_s\|_{L^\infty(0,T;L^1(\Gamma_C))} \leq c$, we infer that

$$
\begin{aligned}
-\int_0^t\int_{\Gamma_C} \theta_s^{1-q}\lambda'(\chi)\chi_t\,\mathrm{d}x\,\mathrm{d}s &\leq \frac{1}{2}\int_0^t\int_{\Gamma_C} \frac{|\chi_t|^2}{\theta_s^q}\,\mathrm{d}x\,\mathrm{d}s + c\int_0^t\int_{\Gamma_C} \theta_s^{2-q}\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \frac{1}{2}\int_0^t\int_{\Gamma_C} \frac{|\chi_t|^2}{\theta_s^q}\,\mathrm{d}x\,\mathrm{d}s + c,
\end{aligned}
\tag{3.41}
$$

where we have taken into account the Lipschitz continuity of λ stated by (2.16). Combining the analogue of (3.11) with (3.35)–(3.36), (3.40), (3.41), we obtain that

$$\left\|\frac{1}{\theta}(t)\right\|^{q-1}_{L^{q-1}(\Omega)} + \left\|\frac{1}{\theta_{\mathrm{s}}}(t)\right\|^{q-1}_{L^{q-1}(\Gamma_{\mathrm{C}})} + \int_0^t\int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta^q}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Gamma_{\mathrm{C}}} \frac{|\chi_t|^2}{\theta_{\mathrm{s}}^q}\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t\int_\Omega |\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Gamma_{\mathrm{C}}} |\nabla\theta_{\mathrm{s}}|^2\,\mathrm{d}x\,\mathrm{d}s \le c,$$

whence estimate (3.34) follows.

Case $\mu = 2$. We test (3.4) and (3.5) by $-\theta^{-1}$ and $-\theta_{\mathrm{s}}^{-1}$, respectively. Adding the resulting relations, integrating over $(0,t)$ using (2.13), (3.26), recalling that the kernel j is symmetric and that $h \ge 0$ a.e. in $\Omega\times(0,T)$ and $\ell \ge 0$ a.e. in $\Gamma_{\mathrm{C}}\times(0,T)$, exploiting cancellations we obtain

$$\begin{aligned} &c_0\int_0^t\int_\Omega |\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s + c_0\int_0^t\int_{\Gamma_{\mathrm{C}}} |\nabla\theta_{\mathrm{s}}|^2\,\mathrm{d}x\,\mathrm{d}s + C_{\mathrm{v}}\int_0^t\int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Gamma_{\mathrm{C}}} \frac{|\chi_t|^2}{\theta_{\mathrm{s}}}\,\mathrm{d}x\,\mathrm{d}s \\ &\le -\int_\Omega \ln(\theta^0)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}}\ln(\theta_{\mathrm{s}}^0)\,\mathrm{d}x + \int_\Omega \ln(\theta(t))\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}}\ln(\theta_{\mathrm{s}}(t))\,\mathrm{d}x - \int_0^t\int_\Omega \mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_0^t\int_{\Gamma_{\mathrm{C}}} \lambda'(\chi)\chi_t\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.42}$$

The first two terms on the right-hand side of (3.42) are bounded, due to (2.20a)–(2.20b). Since $\ln(r) \le 0$ whenever $0 < r \le 1$ and $\ln(r) \le r$ for every $r > 1$, estimates (3.26) ensure that the second and the third term on the right hand side of (3.42) can be estimated as follows:

$$\int_\Omega \ln(\theta(t))\,\mathrm{d}x \le \int_{\Omega\cap\{\theta>1\}} \ln(\theta(t))\,\mathrm{d}x \le \int_\Omega \theta(t)\,\mathrm{d}x \le c, \tag{3.43}$$

$$\int_{\Gamma_{\mathrm{C}}} \ln(\theta_{\mathrm{s}}(t))\,\mathrm{d}x \le \int_{\Gamma_{\mathrm{C}}\cap\{\theta_{\mathrm{s}}>1\}} \ln(\theta_{\mathrm{s}}(t))\,\mathrm{d}x \le \int_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s}}(t)\,\mathrm{d}x \le c. \tag{3.44}$$

Finally, the last two terms on the right-hand side can be estimated using (3.14), (3.26), and the Young inequality:

$$\begin{aligned} -\int_0^t\int_\Omega \mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}s \le c_d\int_0^t\int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|}{\theta^{1/2}}\theta^{1/2}\,\mathrm{d}x\,\mathrm{d}s &\le \frac{C_{\mathrm{v}}}{2}\int_0^t\int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}s + c\int_0^t\int_\Omega \theta\,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{C_{\mathrm{v}}}{2}\int_0^t\int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}s + c, \end{aligned} \tag{3.45}$$

$$\begin{aligned} -\int_0^t\int_{\Gamma_{\mathrm{C}}} \lambda'(\chi)\chi_t\,\mathrm{d}x\,\mathrm{d}s \le c\int_0^t\int_{\Gamma_{\mathrm{C}}} \frac{|\chi_t|}{\theta_{\mathrm{s}}^{1/2}}\theta_{\mathrm{s}}^{1/2}\,\mathrm{d}x\,\mathrm{d}s &\le \frac{1}{2}\int_0^t\int_{\Gamma_{\mathrm{C}}} \frac{|\chi_t|^2}{\theta_{\mathrm{s}}}\,\mathrm{d}x\,\mathrm{d}s + c\int_0^t\int_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s}}\,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{1}{2}\int_0^t\int_{\Gamma_{\mathrm{C}}} \frac{|\chi_t|^2}{\theta_{\mathrm{s}}}\,\mathrm{d}x\,\mathrm{d}s + c, \end{aligned} \tag{3.46}$$

again using the Lipschitz continuity of λ ensured by (2.16). Combing (3.42) with (3.43)–(3.46), we infer that

$$c_0\int_0^t\int_\Omega |\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s + c_0\int_0^t\int_{\Gamma_{\mathrm{C}}} |\nabla\theta_{\mathrm{s}}|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C_{\mathrm{v}}}{2}\int_0^t\int_\Omega \frac{|\varepsilon(\mathbf{u}_t)|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_0^t\int_{\Gamma_{\mathrm{C}}} \frac{|\chi_t|^2}{\theta_{\mathrm{s}}}\,\mathrm{d}x\,\mathrm{d}s \le c,$$

whence, recalling (3.26), we have (3.34).

3.3.3. **Third a priori estimate.** We enhance estimate (3.34) by testing (3.4) by $F'(\theta) = \theta^{\nu-1}$ and (3.5) by $F'(\theta_{\mathrm{s}}) = \theta_{\mathrm{s}}^{\nu-1}$, where $\nu \in (0,1)$ is now arbitrary. Hence, for the terms I_3 and I_4 from (3.29) & (3.30) contributing to (3.28) we now find

$$-I_3 \ge c\int_0^t\int_\Omega \theta^{\mu+\nu-2}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}r = c\int_0^t\int_\Omega |\nabla(\theta^{(\mu+\nu)/2})|^2\,\mathrm{d}x\,\mathrm{d}r$$

and, in the same way,

$$-I_4 \ge c\int_0^t\int_{\Gamma_{\mathrm{C}}} |\nabla(\theta_{\mathrm{s}}^{(\mu+\nu)/2})|^2\,\mathrm{d}x\,\mathrm{d}r.$$

With the very same calculations that lead to (3.31), and recalling (3.32),

$$\begin{aligned}&\int_0^t\int_\Omega|\nabla(\theta^{(\mu+\nu)/2})|^2\,\mathrm{d}x\,\mathrm{d}r+\int_0^t\int_\Omega|\nabla(\theta^{(\mu+\nu)/2})|^2\,\mathrm{d}x\,\mathrm{d}r\\&\leq C+C\int_0^t\int_{\Gamma_\mathrm{C}}(|\chi|^s+1)\theta^{\nu+1}\,\mathrm{d}x\,\mathrm{d}r+C\int_0^t\int_{\Gamma_\mathrm{C}}(|\chi|^s+1)\theta_\mathrm{s}^{\nu+1}\,\mathrm{d}x\,\mathrm{d}r\,.\end{aligned}\tag{3.47}$$

Now, as in (3.33) we control $\int_0^t\int_{\Gamma_\mathrm{C}}(|\chi|^s+1)\theta^{\nu+1}\,\mathrm{d}x\,\mathrm{d}r$ by means of $C\|\theta\|^2_{L^2(0,T;H^1(\Omega))}+C$. We proceed analogously for the last term on the right-hand side of (3.47). In view of the previously proved estimate (3.34), we thus conclude that for all $\mu>1$ and $\nu\in(0,1)$ there exists a positive constant C such that

$$\|\theta^{(\mu+\nu)/2}\|_{L^2(0,T;H^1(\Omega))}+\|\theta_\mathrm{s}^{(\mu+\nu)/2}\|_{L^2(0,T;H^1(\Gamma_\mathrm{C}))}\leq C.\tag{3.48}$$

Notice that the estimate for the full H^1-norm of $\theta^{(\mu+\nu)/2}$ (of $\theta_\mathrm{s}^{(\mu+\nu)/2}$, respectively) follows by the fact that $\|\theta\|_{L^\infty(0,T;L^1(\Omega))}\leq C$ and $\|\theta_\mathrm{s}\|_{L^\infty(0,T;L^1(\Gamma_\mathrm{C}))}\leq C$ (cf. (3.26)) via, e.g., the Poincaré-type inequality from (2.2).

3.3.4. **Fourth a priori estimate.** We test the weak formulation (2.29) of the displacement equation by $\mathbf{u}_t$, the weak formulation of the flow rule for the adhesion parameter

$$\chi_t+A\chi+\xi+\gamma'(\chi)+\lambda'(\chi)\theta_\mathrm{s}=-\frac12|\mathbf{u}|^2-\frac12\mathcal{J}[\chi]\,|\mathbf{u}|^2-\frac12\mathcal{J}[\chi|\mathbf{u}|^2]\qquad\text{a.e. in }\Gamma_\mathrm{C}\times(0,T)\tag{3.49}$$

(with $\xi\in\beta(\chi)$ a.e. in $\Gamma_\mathrm{C}\times(0,T)$), by χ_t, add the resulting equations and integrate in time. We thus arrive at

$$\begin{aligned}&\int_0^t\mathsf{v}(\mathbf{u}_t,\mathbf{u}_t)\,\mathrm{d}r+\frac12\mathsf{e}(\mathbf{u}(t),\mathbf{u}(t))+\widehat{\boldsymbol{\eta}}(\mathbf{u}(t))+\frac12\int_{\Gamma_\mathrm{C}}\chi(t)|\mathbf{u}(t)|^2\,\mathrm{d}x+\frac12\int_{\Gamma_\mathrm{C}}\chi(t)|\mathbf{u}(t)|^2\mathcal{J}[\chi(t)]\,\mathrm{d}x\\&+\int_0^t\int_{\Gamma_\mathrm{C}}|\chi_t|^2\,\mathrm{d}x\,\mathrm{d}r+\int_{\Gamma_\mathrm{C}}\left(\frac12|\nabla\chi(t)|^2+W(\chi(t))\right)\mathrm{d}x\\&=\frac12\mathsf{e}(\mathbf{u}_0,\mathbf{u}_0)+\widehat{\boldsymbol{\eta}}(\mathbf{u}_0)+\frac12\int_{\Gamma_\mathrm{C}}\chi_0|\mathbf{u}_0|^2\,\mathrm{d}x+\frac12\int_{\Gamma_\mathrm{C}}\chi_0|\mathbf{u}_0|^2\mathcal{J}[\chi_0]\,\mathrm{d}x+\int_{\Gamma_\mathrm{C}}\left(\frac12|\nabla\chi_0|^2+W(\chi_0)\right)\mathrm{d}x\\&+\int_0^t\langle\mathbf{F},\mathbf{u}_t\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}\,\mathrm{d}r-\int_0^t\int_\Omega\theta\mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}r-\int_0^t\int_{\Gamma_\mathrm{C}}\lambda'(\chi)\theta_\mathrm{s}\chi_t\,\mathrm{d}x\,\mathrm{d}r\,.\end{aligned}\tag{3.50}$$

Now, the first five terms on the right-hand side of (3.50) are estimated by a constant in view of conditions (2.20), also taking into account Lemma 3.1. Furthermore, we find

$$\begin{aligned}&\left|\int_0^t\langle\mathbf{F},\mathbf{u}_t\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}\,\mathrm{d}r\right|\stackrel{(1)}{\leq}C\,\|\mathbf{F}\|^2_{L^2(0,T;H^1_{\Gamma_\mathrm{D}}(\Omega)^*)}+\frac14\int_0^t\mathsf{v}(\mathbf{u}_t,\mathbf{u}_t)\,\mathrm{d}r\\&\left|\int_0^t\int_\Omega\theta\mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}r\right|\stackrel{(2)}{\leq}C\|\theta\|^2_{L^2(0,T;L^2(\Omega))}+\frac14\int_0^t\mathsf{v}(\mathbf{u}_t,\mathbf{u}_t)\,\mathrm{d}r\\&\left|\int_0^t\int_{\Gamma_\mathrm{C}}\lambda'(\chi)\theta_\mathrm{s}\chi_t\,\mathrm{d}x\,\mathrm{d}r\right|\stackrel{(3)}{\leq}C\|\theta_\mathrm{s}\|^2_{L^2(0,T;L^2(\Gamma_\mathrm{C}))}+\frac14\int_0^t\int_{\Gamma_\mathrm{C}}|\chi_t|^2\,\mathrm{d}x\,\mathrm{d}r\end{aligned}$$

where (1) & (2) follow from Korn's inequality (cf. (2.7)), while (3) is due to (2.16). Combining the above estimates with (3.50) we deduce

$$\|\mathbf{u}\|_{H^1(0,T;H^1_{\Gamma_\mathrm{D}}(\Omega))}+\|\chi\|_{H^1(0,T;L^2(\Gamma_\mathrm{C}))}\leq C.\tag{3.51}$$

3.3.5. **Fifth a priori estimate.** Taking into account the previously obtained (3.26), (3.34), and (3.51), it is immediate to see, arguing by comparison in the flow rule for the adhesion parameter, that

$$\|A\chi+\xi\|_{L^2(0,T;L^2(\Gamma_\mathrm{C}))}\leq C.$$

Hence, well-known arguments from theory of maximal monotone operators yield a separate estimate for $A\chi$ and ξ, namely

$$\|A\chi\|_{L^2(0,T;L^2(\Gamma_\mathrm{C}))}+\|\xi\|_{L^2(0,T;L^2(\Gamma_\mathrm{C}))}\leq C\tag{3.52}$$

so that, by elliptic regularity, we infer that

$$\|\chi\|_{L^2(0,T;H^2(\Gamma_\mathrm{C}))}\leq C\,.\tag{3.53}$$

3.3.6. **Sixth a priori estimate.** It follows from (3.48) that $\theta^{(\mu+\nu)/2}$ is estimated in $L^2(0,T;L^6(\Omega))$, namely that

$$\forall\,\mu>1\ \forall\,\nu\in(0,1)\quad \exists C>0:\qquad \|\theta\|_{L^{\mu+\nu}(0,T;L^{3(\mu+\nu)}(\Omega))}\le C\,. \tag{3.54}$$

We combine this with the previously found estimate for $\|\theta\|_{L^\infty(0,T;L^1(\Omega))}$: by (2.3) we have the continuous embedding

$$L^{\mu+\nu}(0,T;L^{3(\mu+\nu)}(\Omega))\cap L^\infty(0,T;L^1(\Omega))\subset L^a(0,T;L^b(\Omega))$$
$$\text{with}\quad \begin{cases}\frac{1}{a}=\frac{\vartheta}{\mu+\nu},\\ \frac{1}{b}=\frac{1}{3a}+1-\vartheta\end{cases}\qquad \text{and } \vartheta=\frac{\mu+\nu}{\mu-\nu+2}$$

(observe that, with such a choice one has $\vartheta\in(0,1)$ since $\nu\in(0,1)$). Therefore, we obtain $a=\mu-\nu+2$ and $b=\frac{3(\mu-\nu+2)}{7-6\nu}$, so that we conclude, from (3.26) and (3.54), the bound

$$\forall\,\mu>1\ \forall\,\nu\in(0,1)\quad \exists C>0:\qquad \|\theta\|_{L^{\mu-\nu+2}(0,T;L^{3(\mu-\nu+2)/(7-6\nu)}(\Omega))}\le C\,. \tag{3.55}$$

Analogously, due to (3.48) and the continuous embedding $H^1(\Gamma_C)\subset L^q(\Gamma_C)$ for all $q<\infty$, we have that $\theta_s^{(\mu+\nu)/2}$ is estimated in $L^2(0,T;L^q(\Gamma_C))$ for every $q\in[1,\infty)$. Thus

$$\forall\,\mu>1\ \forall\,\nu\in(0,1)\ \forall\,q\in[1,\infty)\quad \exists C>0:\qquad \|\theta_s\|_{L^{\mu+\nu}(0,T;L^q(\Gamma_C))}\le C\,. \tag{3.56}$$

We combine this with the previously found estimate for $\|\theta_s\|_{L^\infty(0,T;L^1(\Gamma_C))}$. Indeed, again resorting to by (2.3) we observe that the continuous embedding

$$L^{\mu+\nu}(0,T;L^q(\Gamma_C))\cap L^\infty(0,T;L^1(\Gamma_C))\subset L^a(0,T;L^b(\Gamma_C))$$
$$\text{with}\quad \begin{cases}\frac{1}{a}=\frac{\vartheta}{\mu+\nu},\\ \frac{1}{b}=\frac{\vartheta}{q}+1-\vartheta\end{cases}\qquad \text{and } \vartheta=\frac{\mu+\nu}{\mu-\nu+2}$$

holds. Now, since $q\in[1,\infty)$ is arbitrary, the exponent b can be chosen arbitrarily close to $\frac{\mu-\nu+2}{2-2\nu}$, while $a=\mu-\nu+2$. Therefore, from (3.26) and (3.56) we conclude that

$$\forall\,\mu>1\ \forall\,\nu\in(0,1)\ \forall\,l\in\left(1,\frac{\mu-\nu+2}{2-2\nu}\right)\quad \exists C>0:\qquad \|\theta_s\|_{L^{\mu-\nu+2}(0,T;L^l(\Gamma_C))}\le C\,. \tag{3.57}$$

3.3.7. **Seventh estimate on the bulk heat equation.** In the weak formulation (3.4) of the bulk heat equation we (formally) choose a test function $v\in W^{1,3+\epsilon}(\Omega)\subset C^0(\overline{\Omega})$, with $\epsilon>0$. By comparison, we have that

$$\left|\int_\Omega\theta_t v\,\mathrm{d}x\right|\le\left|\int_\Omega\mathcal{L}_1 v\,\mathrm{d}x\right|+\left|\int_{\Gamma_C}\mathcal{L}_2 v\,\mathrm{d}x\right|+\left|\int_\Omega\alpha(\theta)\nabla\theta\nabla v\,\mathrm{d}x\right|\doteq I_1+I_2+I_3, \tag{3.58}$$

where

$$\mathcal{L}_1:=\theta\mathrm{div}(\mathbf{u}_t)+\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t)+h,\qquad \mathcal{L}_2:=-k(\chi)\theta(\theta-\theta_s)-\mathcal{J}[\chi]\chi\theta^2+\mathcal{J}[\chi\theta_s]\chi\theta.$$

It follows from (2.18a), (3.34) and (3.51) that $\|\mathcal{L}_1\|_{L^1(0,T;L^1(\Omega))}\le C$. In order to estimate $\mathcal{L}_2$, recalling (2.15) we observe that

$$\begin{aligned}\|k(\chi)\theta(\theta-\theta_s)\|_{L^1(\Gamma_C)}&\le C\int_{\Gamma_C}(|\chi|^s+1)\theta(\theta-\theta_s)\,\mathrm{d}x\\&\le C(\|\chi\|^s_{L^{s\varrho'}(\Gamma_C)}+1)(\|\theta\|^2_{L^{2\varrho}(\Gamma_C)}+\|\theta\|_{L^{2\varrho}(\Gamma_C)}\|\theta_s\|_{L^{2\varrho}(\Gamma_C)})\\&\le C(\|\theta\|^2_{H^1(\Omega)}+\|\theta\|^2_{H^1(\Gamma_C)})\,,\end{aligned} \tag{3.59}$$

where the exponent ϱ is chosen such that $\varrho\le2$, so that $\|\theta\|_{L^{2\varrho}(\Gamma_C)}\le C\|\theta\|_{H^1(\Omega)}$. In order to control the terms $\mathcal{J}[\chi]\chi\theta^2$ and $\mathcal{J}[\chi\theta_s]\chi\theta$ we resort to an analogous Hölder estimate, also taking into account that $\|\mathcal{J}[\chi]\|_{L^\infty(0,T;L^\infty(\Gamma_C))}\le C$ and $\|\mathcal{J}[\chi\theta_s]\|_{L^\infty(0,T;L^\infty(\Gamma_C))}\le C$ thanks to (3.26) and Lemma 3.1. All in all, thanks again to (3.34), we conclude that $\|\mathcal{L}_2\|_{L^1(0,T;L^1(\Gamma_C))}\le C$. Therefore, denoting by $L_1:=\|\mathcal{L}_1(t)\|_{L^1(\Omega)}$ and $L_2:=\|\mathcal{L}_2(t)\|_{L^1(\Gamma_C)}$, we obtain

$$I_1\le L_1(t)\|v\|_{L^\infty(\Omega)}\qquad \text{with } L_1\in L^1(0,T), \tag{3.60}$$
$$I_2\le L_2(t)\|v\|_{L^\infty(\Gamma_C)}\qquad \text{with } L_2\in L^1(0,T). \tag{3.61}$$

Now, by the growth condition on α we have that

$$I_3 \leq C \int_\Omega (1+\theta^\mu)|\nabla\theta||\nabla v| \,\mathrm{d}x \doteq I_{3,1} + I_{3,2} \,.$$

Clearly,

$$I_{3,1} \leq C\|\nabla\theta\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \,.$$

In order to estimate the integral term $I_{3,2}$ we resort to estimate (3.55), which yields the bound

$$\|\theta^{(\mu-\nu+2)/2}\|_{L^2(0,T;L^{6/(7-6\nu)}(\Omega))} \leq C \,. \tag{3.62}$$

Therefore,

$$\begin{aligned} I_{3,2} &\leq C\|\theta^{(\mu-\nu+2)/2}\|_{L^{6/(7-6\nu)}(\Omega)}\|\theta^{(\mu+\nu-2)/2}\nabla\theta\|_{L^2(\Omega)}\|\nabla v\|_{L^{3+\epsilon}(\Omega)} \\ &= C\|\theta^{(\mu-\nu+2)/2}\|_{L^{6/(7-6\nu)}(\Omega)}\|\nabla(\theta^{(\mu+\nu)/2})\|_{L^2(\Omega)}\|\nabla v\|_{L^{3+\epsilon}(\Omega)} \end{aligned} \tag{3.63}$$

where we have applied Hölder's inequality, choosing $\nu \in (0,1)$ such that

$$\frac{7-6\nu}{6} + \frac{1}{2} + \frac{1}{3+\epsilon} = 1.$$

Therefore, taking into account the previously obtained (3.48) and (3.62), we conclude that

$$I_3 \leq L_3(t)\|\nabla v\|_{L^{3+\epsilon}(\Omega)} \qquad \text{with } L_3 \in L^1(0,T).$$

All in all, we conclude that

$$\forall\,\epsilon>0 \quad \exists\, C>0: \quad \|\theta_t\|_{L^1(0,T;W^{1,3+\epsilon}(\Omega)^*)} \leq C \,. \tag{3.64}$$

3.3.8. **Seventh estimate on the surface heat equation.** We (formally) test the surface heat equation (3.5) by a function $w \in W^{1,2+\epsilon}(\Gamma_{\mathrm{C}}) \subset \mathrm{C}^0(\overline{\Gamma}_{\mathrm{C}})$, with $\epsilon > 0$. By comparison, we have that

$$\left|\int_{\Gamma_{\mathrm{C}}} \partial_t\theta_{\mathrm{s}} w \,\mathrm{d}x\right| \leq \left|\int_{\Gamma_{\mathrm{C}}} \mathcal{F} w \,\mathrm{d}x\right| + \left|\int_{\Gamma_{\mathrm{C}}} \alpha(\theta_{\mathrm{s}})\nabla\theta_{\mathrm{s}}\nabla w \,\mathrm{d}x\right| \doteq I_1 + I_2, \tag{3.65}$$

where

$$\mathcal{F} := \theta_{\mathrm{s}}\lambda'(\chi)\chi_t + \ell + |\chi_t|^2 + k(\chi)\theta_{\mathrm{s}}(\theta-\theta_{\mathrm{s}}) + \mathcal{J}[\chi\theta]\chi\theta_{\mathrm{s}} - \mathcal{J}[\chi]\chi\theta_{\mathrm{s}}^2.$$

Thanks to (3.34), (3.26), (3.48), (3.51), and arguing as for (3.59), we obtain that $\|\mathcal{F}\|_{L^1(0,T;L^1(\Gamma_{\mathrm{C}}))} \leq C$. Therefore,

$$I_1 \leq F(t)\|w\|_{L^\infty(\Gamma_{\mathrm{C}})}, \qquad \text{where } F(t) := \|\mathcal{F}(t)\|_{L^1(\Gamma_{\mathrm{C}})} \text{ and } F \in L^1(0,T). \tag{3.66}$$

In analogy with the calculations in the previous paragraph, we estimate

$$I_2 \leq C \int_{\Gamma_{\mathrm{C}}} (1+\theta_{\mathrm{s}}^\mu)|\nabla\theta_{\mathrm{s}}||\nabla w| \,\mathrm{d}x \doteq I_{2,1} + I_{2,2} \,,$$

where, again, we trivially estimate

$$I_{2,1} \leq C\|\nabla\theta_{\mathrm{s}}\|_{L^2(\Gamma_{\mathrm{C}})}\|\nabla w\|_{L^2(\Gamma_{\mathrm{C}})} \,.$$

In turn, as in (3.63) we have

$$I_{2,2} \leq C\|\theta_{\mathrm{s}}^{(\mu-\nu+2)/2}\|_{L^l(\Gamma_{\mathrm{C}})}\|\nabla(\theta_{\mathrm{s}}^{(\mu+\nu)/2})\|_{L^2(\Gamma_{\mathrm{C}})}\|\nabla w\|_{L^{2+\epsilon}(\Gamma_{\mathrm{C}})} \tag{3.67}$$

where we have applied Hölder's inequality and chosen $\nu \in (0,1)$ such that the exponent l from (3.57) fulfills

$$\frac{1}{l} + \frac{1}{2} + \frac{1}{2+\epsilon} = 1$$

Hence, we have that

$$I_2 \leq F_2(t)\|\nabla w\|_{L^{2+\epsilon}(\Gamma_{\mathrm{C}})} \qquad \text{with } F_2 \in L^1(0,T).$$

All in all, we conclude that

$$\forall\,\epsilon>0 \quad \exists\, C>0: \quad \|\partial_t\theta_{\mathrm{s}}\|_{L^1(0,T;W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})^*)} \leq C \,. \tag{3.68}$$

3.4. **Outline of the proof of Theorem 1.** As already mentioned, we will rigorously render the calculations in Sec. 3.3, and thus the resulting a priori estimates, by working on a carefully devised time discretization scheme featuring

(1) additional regularizing terms for system (2.12), modulated by a paramer $\rho > 0$,
(2) the Yosida regularizations β_ς and η_ς, $\varsigma > 0$, of the maximal monotone operators in the flow rule for χ, and in the boundary condition for $\mathbf{u}$ on Γ_{C},

and where

(3) several occurrences of the term χ have been replaced by its positive part $(\chi)^+$ and the right-hand side of the flow rule for χ has been modulated by a selection in the subdifferential of the positive part of χ (see (3.69j) and (3.70) below).

On the one hand, the latter changes are motivated by the fact that, since the subdifferential operator $\beta : \mathbb{R} \rightrightarrows \mathbb{R}$, with domain in $[0,1]$, has been replaced by its Yosida regularization, we can no longer exploit the information that $\chi \geq 0$ a.e. $\Gamma_{\mathrm{C}} \times (0,T)$ which, in turn, would be crucial to estimate from below several integral terms in the subsequent estimates. Clearly, upon passing to the limit as $\varsigma \downarrow 0$ we shall recover positivity of χ. Correspondingly, the choice to replace χ by its positive part has led to the presence of the subdifferential of the positive part of χ in the flow rule (for a modelling justification, see, e.g., [7]).

On the other hand, the reason for this threefold approximation procedure (the time discretization of system (3.69) combined with the double-parameter approximation), and hence for a threefold passage to the limit, resides in the fact that we shall not be able to obtain the positivity estimates (2.31) for the temperatures on the time-discrete level. Namely, for the discrete bulk and surface temperatures, we shall only prove a strict positivity property (cf. (4.8) ahead), but not a lower bound by a positive constant as in (2.31); we postpone to Remark 4.4 later on a thorough explanation for this. In turn, recall that the Second a priori estimate involves testing the temperature equations by negative powers of θ and θ_{s}: in order to carry it out rigorously, such powers should be in $H^1(\Omega)/H^1(\Gamma_{\mathrm{C}})$, respectively. By the lack of the *uniform positivity* estimates (2.31), we will not have such information at disposal for the discrete bulk and surface temperatures. Hence, we will not be able to replicate the Second estimate (and, a fortiori, the Third, Fifth, and Sixth estimates) on the time discrete scheme.

We shall be able to rigorously perform the Second estimate only on the time-continuous level, by working with (a weak formulation of) the following regularized system

$$\theta_t - \theta \mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\alpha(\theta)\nabla\theta) = \varepsilon(\mathbf{u}_t)\,\mathbb{V}\,\varepsilon(\mathbf{u}_t) + h \quad \text{in } \Omega \times (0,T), \tag{3.69a}$$

$$\alpha(\theta)\nabla\theta \cdot \mathbf{n} = 0 \quad \text{in } \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} \times (0,T), \tag{3.69b}$$

$$\alpha(\theta)\nabla\theta \cdot \mathbf{n} = -k(\chi)\theta(\theta - \theta_{\mathrm{s}}) - \mathcal{J}[(\chi)^+]\,(\chi)^+\theta^2 + \mathcal{J}[(\chi)^+\theta_{\mathrm{s}}]\,(\chi)^+\theta \quad \text{in } \Gamma_{\mathrm{C}} \times (0,T), \tag{3.69c}$$

$$-\,\mathrm{div}(\mathbb{E}\varepsilon(\mathbf{u}) + \mathbb{V}\varepsilon(\mathbf{u}_t) + \theta\mathbb{I}) - \rho \mathrm{div}(|\varepsilon(\mathbf{u}_t)|^{\omega-2}\varepsilon(\mathbf{u}_t)) = \mathbf{f} \quad \text{in } \Omega \times (0,T), \qquad \omega > 4, \tag{3.69d}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_{\mathrm{D}} \times (0,T), \tag{3.69e}$$

$$(\mathbb{E}\varepsilon(\mathbf{u}) + \mathbb{V}\varepsilon(\mathbf{u}_t) + \theta\mathbb{I})\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_{\mathrm{N}} \times (0,T), \tag{3.69f}$$

$$(\mathbb{E}\varepsilon(\mathbf{u}) + \mathbb{V}\varepsilon(\mathbf{u}_t) + \theta\mathbb{I})\mathbf{n} + (\chi)^+\mathbf{u} + \boldsymbol{\zeta}_\varsigma + \mathcal{J}[(\chi)^+]\,(\chi)^+\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_{\mathrm{C}} \times (0,T), \tag{3.69g}$$

$$\begin{aligned} &\partial_t\theta_{\mathrm{s}} - \theta_{\mathrm{s}}\lambda'(\chi)\chi_t - \mathrm{div}(\alpha(\theta_{\mathrm{s}})\nabla\theta_{\mathrm{s}}) \\ &\quad = \ell + |\chi_t|^2 + k(\chi)(\theta - \theta_{\mathrm{s}})\theta_{\mathrm{s}} + \mathcal{J}[(\chi)^+\theta]\,(\chi)^+\theta_{\mathrm{s}} - \mathcal{J}[(\chi)^+]\,(\chi)^+\theta_{\mathrm{s}}^2 \quad \text{in } \Gamma_{\mathrm{C}} \times (0,T), \end{aligned} \tag{3.69h}$$

$$\alpha(\theta_{\mathrm{s}})\nabla\theta_{\mathrm{s}} \cdot \mathbf{n}_{\mathrm{s}} = 0 \quad \text{in } \partial\Gamma_{\mathrm{C}} \times (0,T), \tag{3.69i}$$

$$\begin{aligned} &\chi_t + \rho|\chi_t|^{\omega-2}\chi_t - \Delta\chi + \beta_\varsigma(\chi) + \gamma'(\chi) + \lambda'(\chi)\theta_{\mathrm{s}} \\ &\quad = -\frac{1}{2}|\mathbf{u}|^2\sigma - \frac{1}{2}\mathcal{J}[(\chi)^+]\,|\mathbf{u}|^2\sigma - \frac{1}{2}\mathcal{J}[(\chi)^+|\mathbf{u}|^2]\sigma \quad \text{in } \Gamma_{\mathrm{C}} \times (0,T), \qquad \omega > 4, \end{aligned} \tag{3.69j}$$

$$\text{with } \sigma \in \partial\varphi(\chi) \quad \text{in } \Gamma_{\mathrm{C}} \times (0,T), \tag{3.69k}$$

$$\partial_{\mathbf{n}_{\mathrm{s}}}\chi = 0 \quad \text{in } \partial\Gamma_{\mathrm{C}} \times (0,T), \tag{3.69l}$$

where $\varphi : \mathbb{R} \to [0, +\infty)$ is defined by $\varphi(x) := (x)^+$ for all $x \in \mathbb{R}$ and hence

$$\partial\varphi(x) = \begin{cases} \{0\} & \text{if } x < 0, \\ [0,1] & \text{if } x = 0, \\ \{1\} & \text{if } x > 0. \end{cases} \tag{3.70}$$

System (3.69) features two parameters ρ, $\varsigma > 0$, where:

(1) the higher order terms $-\rho\mathrm{div}(|\varepsilon(\mathbf{u}_t)|^{\omega-2}\varepsilon(\mathbf{u}_t))$ and $\rho|\chi_t|^{\omega-2}\chi_t$, have been added to the left-hand sides of the momentum balance and of the flow rule for the adhesion parameter in order to compensate the quadratic terms on the right-hand sides of the bulk and surface heat equations. This will pave the way to further estimates, and enhanced regularity, for the temperature variables which, in turn, will enables us to rigorously perform the estimates from Sec. 3.3 on system (3.69);
(2) in place of a selection $\boldsymbol{\zeta} \in \boldsymbol{\eta}(\mathbf{u})$, the boundary condition (3.69g) features the term
$$\boldsymbol{\zeta}_\varsigma = \eta_\varsigma(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \tag{3.71}$$
where η_ς is the Yosida regularization of the subdifferential $\eta = \partial\widehat{\eta} : \mathbb{R} \rightrightarrows \mathbb{R}$ of $\widehat{\eta}$ from (2.8);
(3) in the flow rule (3.69j) we have considered the Yosida regularization β_ς of the subdifferential operator $\beta : \mathbb{R} \rightrightarrows \mathbb{R}$.

The Yosida regularizations in the momentum balance equation and in the flow rule for χ are motivated by the presence of the nonlinear terms in $\varepsilon(\mathbf{u}_t)$ and χ_t in the approximate momentum balance and flow rules. The parameter ρ is kept distinct from the parameter of the Yosida regularization because, for technical reasons it will be necessary to perform two different limit passages, in system (3.69). First, we shall let $\rho \downarrow 0$ with fixed $\varsigma > 0$, while the identification of the maximal monotone operators $\boldsymbol{\eta}$ and β in the momentum balance and in the flow rule will be performed in the limit passage as $\varsigma \downarrow 0$.

Remark 3.4. As we have pointed out in Remark 3.3, by replacing the operator β, with domain in $[0,1]$, by its Yosida regularization in the flow rule (3.69j) we can no longer deduce a uniform-in-time bound for $\|\chi\|_{L^\infty(\Gamma_{\mathrm{C}})}$ from the First a priori estimate. This is the reason why we need to impose the growth condition (2.15) for the function k, that has been indeed used in Sections 3.3.2, 3.3.3, 3.3.7 and 3.3.8.

A close perusal at those calculations reveal that condition (2.15) could be dispensed with at the price of adding an additional approximation to system (3.69). Namely, it should be necessary to truncate the term $k(\chi)$, and remove the truncation in the limit as $\varsigma \downarrow 0$. However, to avoid overburdening the analysis we have chosen not to do so.

We will supplement system (3.69) with initial data

$$\begin{aligned} &(\theta_\rho^0)_\rho \subset L^{\mu+2}(\Omega),\ (\theta_{\mathrm{s},\rho}^0)_\rho \subset L^{\mu+2}(\Gamma_{\mathrm{C}}) \text{ fulfilling (2.20a)–(2.20b), and such that} \\ &\begin{cases} \theta_\rho^0 \to \theta_0 \text{ in } L^1(\Omega), \\ \theta_{\mathrm{s},\rho}^0 \to \theta_{\mathrm{s}}^0 \text{ in } L^1(\Gamma_{\mathrm{C}}) \end{cases} \qquad \text{as } \rho \downarrow 0, \end{aligned} \tag{3.72a}$$

with μ from (2.13),

$$(\mathbf{u}_\rho^0)_\rho \subset W_{\mathrm{D}}^{1,\omega}(\Omega;\mathbb{R}^3) \text{ and such that } \mathbf{u}_\rho^0 \to \mathbf{u}_0 \text{ in } H_{\Gamma_{\mathrm{D}}}^1(\Omega;\mathbb{R}^3) \quad \text{as } \rho \downarrow 0, \tag{3.72b}$$

where we have used the notation $W_{\mathrm{D}}^{1,\omega}(\Omega;\mathbb{R}^3) := W^{1,\omega}(\Omega;\mathbb{R}^3) \cap H_{\Gamma_{\mathrm{D}}}^1(\Omega;\mathbb{R}^3)$.

Our strategy for proving Theorem 1 is the following:

(1) in Section 4 we will devise a careful time discretization scheme for system (3.69), and show that it admits a solution (cf. Proposition 4.3);
(2) in Section 5 we will derive a series of a priori estimates on the discrete solutions, and prove that, as the time step vanishes, they converge to a (weak) solution of system (3.69), cf. Theorem 5.2 ahead, such that the temperature variables θ and θ_{s} enjoy the positivity properties (2.31). This information will enable us to perform the a priori estimates, formally carried out in Section 3.3, in a rigorous way on the solutions to system (3.69);
(3) in Section 6 we will then address the limit passage in system (3.69), first as $\rho \downarrow 0$ and then as $\varsigma \downarrow 0$. In this way, we shall obtain the existence of *weak energy solutions* to system (2.12) and conclude the proof of Theorem 1.

4. Time discretization

Given a time step $\tau > 0$ and an equidistant partition of $[0,T]$ with nodes $t_\tau^k := k\tau$, $k = 0, \dots, K_\tau$, we approximate the data $\mathbf{f}$, $\mathbf{g}$, h, and ℓ by local means, namely we set for $k = 1, \dots, K_\tau$

$$\begin{aligned} \mathbf{f}_\tau^k &:= \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \mathbf{f}(s)\,\mathrm{d}s, \qquad & \mathbf{g}_\tau^k &:= \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \mathbf{g}(s)\,\mathrm{d}s, \\ h_\tau^k &:= \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} h(s)\,\mathrm{d}s, \qquad & \ell_\tau^k &:= \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \ell(s)\,\mathrm{d}s. \end{aligned} \tag{4.1}$$

Accordingly, we will also consider the local means $(\mathbf{F}_\tau^k)_{k=1}^{K_\tau}$ of the function $\mathbf{F}$ from (2.19).

We shall construct discrete solutions to system (3.69) by recursively solving an elliptic system, (4.4) below. In particular, for the discrete version of the flow rule for the adhesion parameter we shall use that, thanks to (2.16) and (2.17), the functions λ and γ decompose as

$$\begin{aligned} \lambda(r) &= \lambda(r) - \frac{\delta}{2}r^2 + \frac{\delta}{2}r^2 \doteq \lambda_\delta(r) + \frac{\delta}{2}r^2 \quad \text{with } \lambda_\delta \quad \text{concave}, \\ \gamma(r) &= \gamma(r) + \frac{\nu}{2}r^2 - \frac{\nu}{2}r^2 \doteq \gamma_\nu(r) - \frac{\nu}{2}r^2 \quad \text{with } \gamma_\nu \quad \text{convex}. \end{aligned} \tag{4.2}$$

We will look for the temperature components θ_τ^k and $\theta_{\mathrm{s},\tau}^k$ of the solutions to the discrete system (4.4) below in the spaces (recall that $\widehat{\alpha}$ is the primitive of α null in 0),

$$\begin{aligned} X &= \{\theta \in H^1(\Omega) : \quad \widehat{\alpha}(\theta) \in H^1(\Omega)\}, \\ X_\mathrm{s} &= \{\theta_\mathrm{s} \in H^1(\Gamma_\mathrm{C}) : \quad \widehat{\alpha}(\theta_\mathrm{s}) \in H^1(\Gamma_\mathrm{C})\}, \end{aligned} \tag{4.3}$$

We are now in a position to introduce our time discretization scheme for system (3.69), postponing to Remark 4.2 below further comments on our choices.

Problem 4.1. *Let $\omega > 4$. Starting from the initial data $(\theta_\tau^0, \mathbf{u}_\tau^0, \theta_{\mathrm{s},\tau}^0, \chi_\tau^0)$ with $\theta_\tau^0 = \theta_\rho^0$, $\mathbf{u}_\tau^0 = \mathbf{u}_\rho^0$, $\theta_{\mathrm{s},\tau}^0 = \theta_{\mathrm{s},\rho}^0$ (cf. (3.72)), and $\chi_\tau^0 = \chi_0$ (with χ_0 from (2.20d)), find*

$$\{(\theta_\tau^k, \mathbf{u}_\tau^k, \theta_{\mathrm{s},\tau}^k, \chi_\tau^k)\}_{k=1}^{K_\tau} \subset X \times W_\mathrm{D}^{1,\omega}(\Omega;\mathbb{R}^3) \times X_\mathrm{s} \times H^2(\Gamma_\mathrm{C}),$$

fulfilling

- *the discrete bulk temperature equation*

$$\begin{aligned} &\int_\Omega \frac{\theta_\tau^k - \theta_\tau^{k-1}}{\tau} v\,\mathrm{d}x - \int_\Omega \theta_\tau^k \mathrm{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) v\,\mathrm{d}x + \int_\Omega \alpha(\theta_\tau^k)\nabla\theta_\tau^k \nabla v\,\mathrm{d}x \\ &+ \int_{\Gamma_\mathrm{C}} k(\chi_\tau^{k-1})\theta_\tau^k(\theta_\tau^k - \theta_{\mathrm{s},\tau}^k) v\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} \mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+ \left(\theta_\tau^k\right)^2 v\,\mathrm{d}x - \int_{\Gamma_\mathrm{C}} \mathcal{J}[(\chi_\tau^{k-1})^+ \theta_{s,\tau}^k]\,(\chi_\tau^{k-1})^+ \theta_\tau^k v\,\mathrm{d}x \\ &= \int_\Omega \varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \mathbb{V}\,\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) v\,\mathrm{d}x + \langle h_\tau^k, v\rangle_{H^1(\Omega)} \end{aligned} \tag{4.4a}$$

for all test functions $v \in H^1(\Omega)$;

- *the discrete momentum balance equation*

$$\begin{aligned} &\mathsf{v}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}, \mathbf{v}\right) + \rho\int_\Omega \left|\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right|^{\omega-2} \varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \varepsilon(\mathbf{v})\,\mathrm{d}x + \mathsf{e}\left(\mathbf{u}_\tau^k, \mathbf{v}\right) \\ &+ \int_\Omega \theta_\tau^k \mathrm{div}(\mathbf{v})\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} (\chi_\tau^k)^+ \mathbf{u}_\tau^k \mathbf{v}\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} \boldsymbol{\zeta}_\tau^k \cdot \mathbf{v}\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} \mathcal{J}[(\chi_\tau^k)^+]\,(\chi_\tau^k)^+ \mathbf{u}_\tau^k \mathbf{v}\,\mathrm{d}x = \langle \mathbf{F}_\tau^k, \mathbf{v}\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}, \end{aligned}$$

with $\boldsymbol{\zeta}_\tau^k = \eta_\varsigma(\mathbf{u}_\tau^k \cdot \mathbf{n})\mathbf{n}$ (4.4b)

for all test functions $\mathbf{v} \in W_\mathrm{D}^{1,\omega}(\Omega;\mathbb{R}^3)$;

- *the discrete surface temperature equation*

$$
\begin{aligned}
&\int_{\Gamma_C} \frac{\theta_{s,\tau}^k - \theta_{s,\tau}^{k-1}}{\tau} v \, dx - \int_{\Gamma_C} \theta_{s,\tau}^k \frac{\lambda(\chi_\tau^k) - \lambda(\chi_\tau^{k-1})}{\tau} v \, dx + \int_{\Gamma_C} \alpha(\theta_{s,\tau}^k) \nabla \theta_{s,\tau}^k \nabla v \, dx \\
&= \int_{\Gamma_C} \left| \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \right|^2 v \, dx + \int_{\Gamma_C} k(\chi_\tau^{k-1})(\theta_\tau^k - \theta_{s,\tau}^k)\theta_{s,\tau}^k v \, dx + \int_{\Gamma_C} \mathfrak{J}[(\chi_\tau^{k-1})^+ \theta_\tau^k] \, (\chi_\tau^{k-1})^+ \theta_{s,\tau}^k v \, dx \qquad (4.4c) \\
&\quad - \int_{\Gamma_C} \mathfrak{J}[(\chi_\tau^{k-1})^+](\chi_\tau^{k-1})^+ \left(\theta_{s,\tau}^k\right)^2 v \, dx + \langle \ell_\tau^k, v \rangle_{H^1(\Gamma_C)}
\end{aligned}
$$

for all test functions $v \in H^1(\Gamma_C)$;

- *the discrete flow rule for the adhesion parameter*

$$
\begin{aligned}
&\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \rho \left| \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \right|^{\omega-2} \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + A\chi_\tau^k + \beta_\varsigma(\chi_\tau^k) + \gamma_\nu'(\chi_\tau^k) - \nu \chi_\tau^{k-1} \\
&= -\lambda_\delta'(\chi_\tau^{k-1})\theta_{s,\tau}^k - \delta \chi_\tau^k \theta_{s,\tau}^k - \frac{1}{2}|\mathbf{u}_\tau^{k-1}|^2 \sigma_\tau^k - \frac{1}{2}\mathfrak{J}[(\chi_\tau^k)^+] \, |\mathbf{u}_\tau^{k-1}|^2 \sigma_\tau^k - \frac{1}{2}\mathfrak{J}[(\chi_\tau^{k-1})^+ |\mathbf{u}_\tau^{k-1}|^2] \sigma_\tau^k \qquad \text{a.e. in } \Gamma_C, \\
&\text{with } \sigma_\tau^k \in \partial\varphi(\chi_\tau^k) \qquad \text{a.e. in } \Gamma_C.
\end{aligned} \tag{4.4d}
$$

Observe that, thanks to the request that $\widehat{\alpha}(\theta_\tau^k) \in H^1(\Omega)$ and $\widehat{\alpha}(\theta_{s,\tau}^k) \in H^1(\Gamma_C)$, the weak formulations for the bulk and surface equation with test functions in $H^1(\Omega)$ and $H^1(\Gamma_C)$, respectively, are appropriately posed. Furthermore, taking into account the growth properties of α (cf. also (4.40) ahead), from $\widehat{\alpha}(\theta_\tau^k) \in H^1(\Omega)$ we conclude that $\theta_\tau^k \in L^{6\mu+6}(\Omega)$. An even higher integrability property holds for $\theta_{s,\tau}^k$, as a consequence of the fact that $\widehat{\alpha}(\theta_{s,\tau}^k) \in H^1(\Gamma_C)$, taking into account that $H^1(\Gamma_C) \subset L^q(\Gamma_C)$ for all $1 \le q < \infty$. Therefore, starting from initial data $(\theta_\tau^0, \mathbf{u}_\tau^0, \theta_{s,\tau}^0, \chi_\tau^0)$ with $\theta_\tau^0 \in L^{\mu+2}(\Omega)$ and $\theta_{s,\tau}^0 \in L^{\mu+2}(\Gamma_C)$, we will gain the same integrability property (an even higher one) also for the discrete solutions. This information shall be used for the rigorous a priori estimates performed on the time-discrete scheme in Section 5.

Remark 4.2. The time-discretization scheme (4.4) has been carefully devised in such a way as to ensure the validity of a form of the total energy balance (cf. (4.22) and (4.9) ahead) for the discrete solutions. This has motivated

- the choice of the terms to be kept implicit instead of explicit;
- the usage of the convex and concave decompositions of the functions γ and λ in the discrete flow rule for χ, which will allow us to exploit suitable convexity/concavity inequalities (cf. (4.24) ahead) that are instrumental to the discrete total energy balance and, likewise,
- the presence of the selections $\sigma_\tau^k \in \partial\varphi(\chi_\tau^k)$ in the terms of the discrete flow rule that are coupled with the terms of the discrete momentum balance featuring the positive parts $(\chi_\tau^k)^+$.

As it turns out, scheme (4.4) is fully implicit, with all equations tightly coupled one with another. Because of this, it will not be possible to prove the existence of solutions to (4.4) by separately solving the discrete bulk temperature equation, the momentum balance equation, the surface temperature equation, and the flow rule for the adhesion parameter. Instead, to prove existence for (a suitably truncated version of) system (4.4) we will resort to a fixed-point type existence result for equations featuring pseudo-monotone operators.

The main result of this section ensures the existence of solutions to scheme (4.4), as well as the strict positivity of the discrete temperatures (cf. (4.8)).

We will also show that the solutions to system (4.4) comply with the total energy inequality (4.9) below, featuring the energy functional $\mathcal{E}_\varsigma : L^1(\Omega) \times L^1(\Gamma_C) \times H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^3) \times H^1(\Gamma_C) \to \mathbb{R}$ defined by

$$
\begin{aligned}
&\mathcal{E}_\varsigma(\theta, \theta_s, \mathbf{u}, \chi) := \int_\Omega \theta \, dx + \int_{\Gamma_C} \theta_s \, dx + \frac{1}{2}\mathsf{e}(\mathbf{u}, \mathbf{u}) \\
&+ \int_{\Gamma_C} \widehat{\eta}_\varsigma(\mathbf{u} \cdot \mathbf{n}) \, dx + \frac{1}{2} \int_{\Gamma_C} \left((\chi)^+ |\mathbf{u}|^2 + (\chi)^+ |\mathbf{u}|^2 \mathfrak{J}[(\chi)^+] \right) dx + \int_{\Gamma_C} \left(\frac{1}{2}|\nabla \chi|^2 + \widehat{\beta}_\varsigma(\chi) + \gamma(\chi) \right) dx,
\end{aligned} \tag{4.5}
$$

with $\widehat{\eta}_\varsigma$ and $\widehat{\beta}_\varsigma$ the Yosida approximations of the functions $\widehat{\eta}$ and $\widehat{\beta}$. In fact, inequality (4.9) will be the starting point for the derivation of the estimates, uniform w.r.t. $\tau > 0$, in Sec. 5.

In the statements of all the following results, we will omit to explicitly invoke the assumptions of Theorem 1.

Proposition 4.3. *Let $\tau > 0$, sufficiently small, be fixed. Start from initial data*

$$(\theta_\tau^0, \mathbf{u}_\tau^0, \theta_{\mathrm{s},\tau}^0, \chi_\tau^0) = (\theta_\rho^0, \mathbf{u}_\rho^0, \theta_{\mathrm{s},\rho}^0, \chi_0) \in L^{\mu+2}(\Omega) \times W_{\mathrm{D}}^{1,\omega}(\Omega;\mathbb{R}^3) \times L^{\mu+2}(\Gamma_{\mathrm{C}}) \times H^2(\Gamma_{\mathrm{C}}) \quad \text{fulfilling (2.20a) and (2.20b).} \tag{4.6}$$

Then, for every $k \in \{1,\ldots,K_\tau\}$ and there exists a quadruple $(\theta_\tau^k, \mathbf{u}_\tau^k, \theta_{\mathrm{s},\tau}^k, \chi_\tau^k) \in X \times W_{\mathrm{D}}^{1,\omega}(\Omega;\mathbb{R}^3) \times X_{\mathrm{s}} \times H^2(\Gamma_{\mathrm{C}})$, with an associated $\sigma_\tau^k \in L^\infty(\Gamma_{\mathrm{C}})$ such that $\sigma_\tau^k \in \partial\varphi(\chi_\tau^k)$ a.e. in Γ_{C}, solving (4.4).

Furthermore, the discrete solutions $(\theta_\tau^k)_{k=1}^{K_\tau}$ and $(\theta_{\mathrm{s},\tau}^k)_{k=1}^{K_\tau}$ enjoy the following estimate

$$\exists\, S_0>0 \quad \forall\, p\in[1,\infty) \quad \exists\, \bar\tau_p>0 \quad \forall\, \tau\in(0,\bar\tau_p) \quad \forall\, k\in\{1,\ldots,K_\tau\}\,:\quad \left\|\frac{1}{\theta_\tau^k}\right\|_{L^p(\Omega)} + \left\|\frac{1}{\theta_{\mathrm{s},\tau}^k}\right\|_{L^p(\Gamma_{\mathrm{C}})} \le S_0. \tag{4.7}$$

In particular,

$$\theta_\tau^k > 0 \quad \text{a.e. in } \Omega, \qquad \theta_{\mathrm{s},\tau}^k > 0 \quad \text{a.e. in } \Gamma_{\mathrm{C}} \qquad \text{for all } k\in\{1,\ldots,K_\tau\}. \tag{4.8}$$

Finally, there holds

$$\begin{aligned} &\mathcal{E}_\varsigma(\theta_\tau^k, \theta_{\mathrm{s},\tau}^k, \mathbf{u}_\tau^k, \chi_\tau^k) + \rho\tau \int_\Omega \left|\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^k}{\tau}\right)\right|^\omega \mathrm{d}x + \rho\tau\int_{\Gamma_{\mathrm{C}}} \left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^\omega \mathrm{d}x \\ &\quad + \tau\int_{\Gamma_{\mathrm{C}}} k(\chi_\tau^{k-1})(\theta_\tau^k - \theta_{\mathrm{s},\tau}^k)^2\,\mathrm{d}x + \tau\int_{\Gamma_{\mathrm{C}}\times\Gamma_{\mathrm{C}}} j(x,y)(\chi_\tau^{k-1}(x))^+(\chi_\tau^{k-1}(y))^+(\theta_\tau^k(x) - \theta_{\mathrm{s},\tau}^k(y))^2\,\mathrm{d}x\,\mathrm{d}y \\ &\le \mathcal{E}_\varsigma(\theta_\tau^{k-1}, \theta_{\mathrm{s},\tau}^{k-1}, \mathbf{u}_\tau^{k-1}, \chi_\tau^{k-1}) + \tau\int_\Omega h_\tau^k\,\mathrm{d}x + \tau\int_{\Gamma_{\mathrm{C}}} \ell_\tau^k\,\mathrm{d}x + \tau\left\langle \mathbf{F}_\tau^k, \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right\rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)}. \end{aligned} \tag{4.9}$$

Remark 4.4. Although the constant S_0 in (4.7) is independent of the exponent p, for any fixed p estimate (4.7) in fact holds for only $\tau < \tau_p$, for a certain threshold τ_p that tends to 0 as $p \to \infty$ (cf. (4.21) below). This is the reason why, unlike in the time-continuous case (cf. the arguments in Section 3.2), from the arbitrariness of p in (4.7) we cannot deduce a uniform $L^\infty(\Omega)$-bound for the the quantities $\frac{1}{\theta_\tau^k}$, which would provide a lower bound for the discrete bulk temperatures $(\theta_\tau^k)_{k=1}^{K_\tau}$ by a strictly positive constant. The same considerations apply to the discrete surface temperatures $(\theta_{\mathrm{s},\tau}^k)_{k=1}^{K_\tau}$.

In any case, the weaker positivity information (4.8) will be sufficient to replicate on the time-discrete level all the estimates needed to prove the existence of solutions to system (3.69).

We will prove Proposition 4.3 by approximating system (4.4) via suitable truncations depending on two parameters $0 < \epsilon \ll 1$ and $M \gg 1$; we postpone to Remark 4.7 the motivation for such truncations, resulting in system (4.13) below. Next, we will pass to the limit in (4.13) as $\epsilon \downarrow 0$, first, and then as $M \to +\infty$. Namely,

(1) in the upcoming Section 4.1 we will address the existence of solutions to the (ϵ, M)-truncated system (4.13);
(2) we shall perform the limit passage as $\epsilon \downarrow 0$ in Section 4.2;
(3) and the limit passage as $M \to +\infty$ in Section 4.3, thus concluding the proof of Prop. 4.3.

In what follows, we will resort to the following *discrete* Gronwall Lemma, whose proof can be found, e.g., in [27, Lemma 4.5].

Lemma 4.5. *Let $K_\tau \in \mathbb{N}$ and b, λ, $\Lambda \in (0,+\infty)$ fulfill $1 - b \ge \frac{1}{\lambda} > 0$; let $(a_k)_{k=1}^{K_\tau} \subset [0,+\infty)$ satisfy*

$$a_k \le \Lambda + b\sum_{j=1}^k a_j \qquad \text{for all } k\in\{1,\ldots,K_\tau\}.$$

Then, there holds

$$a_k \le \lambda\Lambda\exp(\lambda b k) \quad \text{for all } k\in\{1,\ldots,K_\tau\}. \tag{4.10}$$

4.1. **Existence of solutions to the (ϵ, M)-truncated discrete system.** Recall the notation $(r)^+ := \max\{r,0\}$ and $(r)^- := \max\{-r,0\}$ for the positive and negative parts of a real number $r \in \mathbb{R}$. Furthermore, for $\epsilon \in (0,1)$ and $M \geq 1$ we introduce the truncation operators

$$
\begin{aligned}
&\mathcal{T}_\epsilon : \mathbb{R} \to \mathbb{R}, \quad \mathcal{T}_\epsilon(r) := \max\{r, \epsilon\}\,, \\
&\mathcal{T}_M : \mathbb{R} \to \mathbb{R}, \quad \mathcal{T}_M(r) := \min\{\max\{r,0\}, M\} = \begin{cases} 0 & \text{if } r < 0, \\ r & \text{if } 0 \leq r \leq M, \\ M & \text{if } r > M. \end{cases}
\end{aligned}
\tag{4.11}
$$

Accordingly, we define

$$
\alpha_M : \mathbb{R} \to (0, +\infty), \qquad \alpha_M(r) := \alpha\left(\mathcal{T}_M(r)\right)\,.
$$

It follows from (2.13) that

$$
\alpha_M(r) \geq c_0 \qquad \text{for all } r \in \mathbb{R}. \tag{4.12}
$$

We consider the (ϵ, M)-truncated system, consisting of

- the discrete bulk temperature equation

$$
\begin{aligned}
&\int_\Omega \frac{\theta_\tau^k - \mathcal{T}_\epsilon(\theta_\tau^{k-1})}{\tau} v \,\mathrm{d}x - \int_\Omega (\theta_\tau^k)^+ \mathrm{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) v \,\mathrm{d}x + \int_\Omega \alpha_M(\theta_\tau^k) \nabla \theta_\tau^k \nabla v \,\mathrm{d}x \\
&\quad + \int_{\Gamma_{\mathrm{C}}} k(\chi_\tau^{k-1}) \mathcal{T}_\epsilon(\theta_\tau^k)(\theta_\tau^k - \theta_{\mathrm{s},\tau}^k) v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+ \theta_\tau^k \mathcal{T}_\epsilon(\theta_\tau^k) v \,\mathrm{d}x \\
&\quad - \int_{\Gamma_{\mathrm{C}}} \mathcal{J}[(\chi_\tau^{k-1})^+ \theta_{s,\tau}^k]\,(\chi_\tau^{k-1})^+ \mathcal{T}_\epsilon(\theta_\tau^k) v \,\mathrm{d}x \\
&= \int_\Omega \varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \mathbb{V}\, \varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) v \,\mathrm{d}x + \langle h_\tau^k, v \rangle_{H^1(\Omega)} \qquad \text{for all } v \in H^1(\Omega);
\end{aligned}
\tag{4.13a}
$$

- the discrete momentum balance equation

$$
\begin{aligned}
&\mathsf{v}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}, \mathbf{v}\right) + \mathsf{e}\left(\mathbf{u}_\tau^k, \mathbf{v}\right) + \rho \int_\Omega \left|\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right|^{\omega-2} \varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \varepsilon(\mathbf{v}) \,\mathrm{d}x \\
&\quad + \int_\Omega (\theta_\tau^k)^+ \mathrm{div}(\mathbf{v}) \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} (\chi_\tau^k)^+ \mathbf{u}_\tau^k \mathbf{v} \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \boldsymbol{\zeta}_\tau^k \cdot \mathbf{v} \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \mathcal{J}[(\chi_\tau^k)^+]\,(\chi_\tau^k)^+ \mathbf{u}_\tau^k \mathbf{v} \,\mathrm{d}x \\
&= \langle \mathbf{F}_\tau^k \mathbf{v} \rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)}, \\
&\text{with} \quad \boldsymbol{\zeta}_\tau^k = \eta_\varsigma(\mathbf{u}_\tau^k \cdot \mathbf{n})\mathbf{n}, \qquad \text{for all } \mathbf{v} \in W^{1,\omega}_{\mathrm{D}}(\Omega;\mathbb{R}^3);
\end{aligned}
\tag{4.13b}
$$

- the discrete surface temperature equation

$$
\begin{aligned}
&\int_{\Gamma_{\mathrm{C}}} \frac{\theta_{s,\tau}^k - \mathcal{T}_\epsilon(\theta_{s,\tau}^{k-1})}{\tau} v \,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} (\theta_{s,\tau}^k)^+ \frac{\lambda(\chi_\tau^k) - \lambda(\chi_\tau^{k-1})}{\tau} v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \alpha_M(\theta_{s,\tau}^k) \nabla \theta_{s,\tau}^k \nabla v \,\mathrm{d}x \\
&= \int_{\Gamma_{\mathrm{C}}} \left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} k(\chi_\tau^{k-1})(\theta_\tau^k - \theta_{s,\tau}^k) \mathcal{T}_\epsilon(\theta_{s,\tau}^k) v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \mathcal{J}[(\chi_\tau^{k-1})^+ \theta_\tau^k]\,(\chi_\tau^{k-1})^+ \mathcal{T}_\epsilon(\theta_{s,\tau}^k) v \,\mathrm{d}x \\
&\quad - \int_{\Gamma_{\mathrm{C}}} \mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+ \theta_{s,\tau}^k \mathcal{T}_\epsilon(\theta_{s,\tau}^k) v \,\mathrm{d}x + \langle \ell_\tau^k, v \rangle_{H^1(\Gamma_{\mathrm{C}})} \qquad \text{for all } v \in H^1(\Gamma_{\mathrm{C}});
\end{aligned}
\tag{4.13c}
$$

- the discrete flow rule for the adhesion parameter

$$
\begin{aligned}
&\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \rho \left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^{\omega-2} \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + A\chi_\tau^k + \beta_\varsigma(\chi_\tau^k) + \gamma_\nu'(\chi_\tau^k) - \nu \chi_\tau^{k-1} \\
&= -\lambda_\delta'(\chi_\tau^{k-1})(\theta_{s,\tau}^k)^+ - \delta \chi_\tau^k (\theta_{s,\tau}^k)^+ - \frac{1}{2}|\mathbf{u}_\tau^{k-1}|^2 \sigma_\tau^k - \frac{1}{2}\mathcal{J}[(\chi_\tau^k)^+]\,|\mathbf{u}_\tau^{k-1}|^2 \sigma_\tau^k - \frac{1}{2}\mathcal{J}[(\chi_\tau^{k-1})^+ |\mathbf{u}_\tau^{k-1}|^2]\,\sigma_\tau^k \qquad \text{a.e. in } \Gamma_{\mathrm{C}}\,, \\
&\text{with } \sigma_\tau^k \in \partial\varphi(\chi_\tau^k) \qquad \text{a.e. in } \Gamma_{\mathrm{C}}\,.
\end{aligned}
\tag{4.13d}
$$

For notational simplicity, we have not highlighted the dependence of a solution $(\theta_\tau^k, \mathbf{u}_\tau^k, \theta_{\mathrm{s},\tau}^k, \chi_\tau^k)$ to system (4.13) on the parameters ϵ and M, and we shall not do so, with the exception of the statements of Proposition 4.6 and Lemma 4.8 below.

Proposition 4.6. *For any fixed $\tau > 0$, sufficiently small, for every $k \in \{1, \ldots, K_\tau\}$ and $(\theta_\tau^{k-1}, \mathbf{u}_\tau^{k-1}, \theta_{\mathrm{s},\tau}^{k-1}, \chi_\tau^{k-1})$ as in (4.6), there exists a quadruple*

$$(\theta_{\tau,\epsilon,M}^k, \mathbf{u}_{\tau,\epsilon,M}^k, \theta_{\mathrm{s},\tau,\epsilon,M}^k, \chi_{\tau,\epsilon,M}^k) \in H^1(\Omega) \times W_{\mathrm{D}}^{1,\omega}(\Omega;\mathbb{R}^3) \times H^1(\Gamma_{\mathrm{C}}) \times H^2(\Gamma_{\mathrm{C}})$$

with an associated $\sigma_{\tau,\epsilon,M}^k \in \partial\varphi(\chi_{\tau,\epsilon,M}^k)$ a.e. in Γ_{C}, solving (4.13).

Furthermore, for every $k \in \{1, \ldots, K_\tau\}$ we have that

$$\theta_{\tau,\epsilon,M}^k \geq 0 \quad \text{a.e. in } \Omega, \qquad \theta_{\mathrm{s},\tau,\epsilon,M}^k \geq 0 \quad \text{a.e. in } \Gamma_{\mathrm{C}}. \tag{4.14}$$

Finally,

$$\begin{aligned} &\exists\, S_0 > 0 \ \forall\, p \in [1,\infty) \ \exists\, \bar\tau_p > 0 \ \forall\, \epsilon, M > 0 \ \forall\, \tau \in (0,\bar\tau_p) \ \ \forall\, k \in \{1,\ldots,K_\tau\} : \\ &\left\| \frac{1}{\mathcal{T}_\epsilon(\theta_{\tau,\epsilon,M}^k)} \right\|_{L^p(\Omega)} + \left\| \frac{1}{\mathcal{T}_\epsilon(\theta_{\mathrm{s},\tau,\epsilon,M}^k)} \right\|_{L^p(\Gamma_{\mathrm{C}})} \leq S_0. \end{aligned} \tag{4.15}$$

Proof. *Step 1: existence for system (4.13):* We observe that a quadruple $(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi)$ solves the elliptic system (4.13) if and only if it solves

$$\partial\Psi(\theta_\tau^k, \mathbf{u}_\tau^k, \theta_{\mathrm{s},\tau}^k, \chi_\tau^k) + \mathscr{A}(\theta_\tau^k, \mathbf{u}_\tau^k, \theta_{\mathrm{s},\tau}^k, \chi_\tau^k) \ni 0 \qquad \text{in } \mathbf{X}^*, \tag{4.16}$$

where $\mathbf{X}$ is a suitable ambient space $\Psi : \mathbf{X} \to [0,+\infty]$ is a (proper) convex and l.s.c. potential, with subdifferential $\partial\Psi : \mathbf{X} \rightrightarrows \mathbf{X}^*$, and $\mathscr{A} : \mathbf{X} \to \mathbf{X}^*$ an appropriate pseudomonotone operator. As we will see, both Ψ and $\mathscr{A}$ depend on the discrete solutions $(\theta_\tau^{k-1}, \mathbf{u}_\tau^{k-1}, \theta_{\mathrm{s},\tau}^{k-1}, \chi_\tau^{k-1})$ at the previous step, as well as on the parameters ϵ and M. However, we choose not to highlight this in their notation.

Indeed, let us set $\mathbf{X} := H^1(\Omega) \times W_{\mathrm{D}}^{1,\omega}(\Omega;\mathbb{R}^3) \times H^1(\Gamma_{\mathrm{C}}) \times H^1(\Gamma_{\mathrm{C}})$ and define

$$\mathscr{A} : \mathbf{X} \to \mathbf{X}^* \quad \text{by} \quad \mathscr{A}(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) := \begin{pmatrix} \mathscr{A}_1(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) \\ \mathscr{A}_2(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) \\ \mathscr{A}_3(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) \\ \mathscr{A}_4(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) \end{pmatrix}$$

where

(1) $\mathscr{A}_1 : \mathbf{X} \to H^1(\Omega)^*$ is defined via

$$\mathscr{A}_1(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) := \theta - \theta^+ \mathrm{div}(\mathbf{u} - \mathbf{u}_\tau^{k-1}) + \tau\mathcal{A}^M(\theta) + \tau B_1(\theta, \theta_{\mathrm{s}}) - \frac{1}{\tau}\varepsilon(\mathbf{u})\mathbb{V}\varepsilon(\mathbf{u} - 2\mathbf{u}_\tau^{k-1}) - \tau F_1 \tag{4.17a}$$

with

$$\mathcal{A}_M : H^1(\Omega) \to H^1(\Omega)^*, \qquad \langle \mathcal{A}_M(\theta), v \rangle_{H^1(\Omega)} := \int_\Omega \alpha_M(\theta)\nabla\theta \cdot \nabla v \,\mathrm{d}x; \tag{4.17b}$$

$$\begin{aligned} &B_1 : H^1(\Omega) \times H^1(\Gamma_{\mathrm{C}}) \to H^1(\Omega)^*, \\ &\langle B_1(\theta, \theta_{\mathrm{s}}), v \rangle_{H^1(\Omega)} := \\ &\quad \int_{\Gamma_{\mathrm{C}}} \left(k(\chi_\tau^{k-1})\mathcal{T}_\epsilon(\theta)(\theta - \theta_{\mathrm{s}}) + \mathcal{J}[(\chi_\tau^{k-1})^+]\, (\chi_\tau^{k-1})^+ \theta \mathcal{T}_\epsilon(\theta) - \mathcal{J}[(\chi_\tau^{k-1})^+ \theta_{\mathrm{s}}]\, (\chi_\tau^{k-1})^+ \mathcal{T}_\epsilon(\theta) \right) v \,\mathrm{d}x, \end{aligned} \tag{4.17c}$$

$$F_1 := \mathcal{T}_\epsilon(\theta_\tau^{k-1}) + \frac{1}{\tau}\varepsilon(\mathbf{u}_\tau^{k-1})\mathbb{V}\varepsilon(\mathbf{u}_\tau^{k-1}) + h_\tau^k; \tag{4.17d}$$

(2) $\mathscr{A}_2 : \mathbf{X} \to W^{1,\omega}(\Omega;\mathbb{R}^3)^*$ is defined via

$$\mathscr{A}_2(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) := -\mathrm{div}(\mathbb{V}\mathbf{u} + \tau\mathbb{E}\mathbf{u} + \tau^{2-\omega}\rho|\varepsilon(\mathbf{u} - \mathbf{u}_\tau^{k-1})|^{\omega-2}\varepsilon(\mathbf{u} - \mathbf{u}_\tau^{k-1}) + \tau\theta^+\mathbb{I}) + \tau B_2(\mathbf{u}, \chi) - \tau F_2 \tag{4.17e}$$

with

$$\begin{aligned} &B_2 : W^{1,\omega}(\Omega;\mathbb{R}^3) \times H^1(\Gamma_{\mathrm{C}}) \to W^{1,\omega}(\Omega;\mathbb{R}^3)^*, \\ &\langle B_2(\mathbf{u}, \chi), \mathbf{v} \rangle_{W^{1,\omega}(\Omega;\mathbb{R}^3)} := \int_{\Gamma_{\mathrm{C}}} (\chi)^+ \mathbf{u}\mathbf{v} \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \mathcal{J}[(\chi)^+]\, (\chi)^+ \mathbf{u}\mathbf{v} \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \eta_\varsigma(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{v} \,\mathrm{d}x, \end{aligned} \tag{4.17f}$$

$$F_2 := \frac{1}{\tau}\mathrm{div}(\mathbb{V}\mathbf{u}_\tau^{k-1}) + \mathbf{F}_\tau^k; \tag{4.17g}$$

(3) $\mathscr{A}_3 : \mathbf{X} \to H^1(\Gamma_{\mathrm{C}})^*$ is defined via

$$\mathscr{A}_3(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) := \theta_{\mathrm{s}} - (\theta_{\mathrm{s}})^+(\lambda(\chi) - \lambda(\chi_\tau^{k-1})) + \tau \mathcal{A}_{\mathrm{s}}^M(\theta_{\mathrm{s}}) - \tau B_3(\theta, \theta_{\mathrm{s}}) - \frac{1}{\tau}\chi(\chi - 2\chi_\tau^{k-1}) - \tau F_3 \tag{4.17h}$$

with

$$\mathcal{A}_{M,\mathrm{s}} : H^1(\Gamma_{\mathrm{C}}) \to H^1(\Gamma_{\mathrm{C}})^*, \qquad \langle \mathcal{A}_{M,\mathrm{s}}(\theta_{\mathrm{s}}), v \rangle_{H^1(\Gamma_{\mathrm{C}})} := \int_\Omega \alpha_M(\theta_{\mathrm{s}}) \nabla \theta_{\mathrm{s}} \cdot \nabla v \,\mathrm{d}x; \tag{4.17i}$$

$$\begin{aligned} & B_3 : H^1(\Omega) \times H^1(\Gamma_{\mathrm{C}}) \to H^1(\Gamma_{\mathrm{C}})^*, \\ & \langle B_3(\theta, \theta_{\mathrm{s}}), v \rangle_{H^1(\Omega)} := \\ & \quad \int_{\Gamma_{\mathrm{C}}} \left(k(\chi_\tau^{k-1}) \mathcal{T}_\epsilon(\theta_{\mathrm{s}})(\theta - \theta_{\mathrm{s}}) + \mathcal{J}[(\chi_\tau^{k-1})^+ \theta]\,(\chi_\tau^{k-1})^+ \mathcal{T}_\epsilon(\theta_{\mathrm{s}}) - \mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+ \theta_{\mathrm{s}} \mathcal{T}_\epsilon(\theta_{\mathrm{s}}) \right) v \,\mathrm{d}x, \end{aligned} \tag{4.17j}$$

$$F_3 := \mathcal{T}_\epsilon(\theta_{\mathrm{s},\tau}^{k-1}) + \frac{1}{\tau}|\chi_\tau^{k-1}|^2 + \ell_\tau^k; \tag{4.17k}$$

(4) $\mathscr{A}_4 : \mathbf{X} \to H^1(\Gamma_{\mathrm{C}})^*$ is defined via

$$\begin{aligned} \mathscr{A}_4(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) := & \chi + \tau^{2-\omega}\rho|\chi - \chi_\tau^{k-1}|^{\omega-2}(\chi - \chi_\tau^{k-1}) + \tau A\chi + \tau\beta_\varsigma(\chi) \\ & + \tau\gamma_\nu'(\chi) + \tau\lambda_\delta'(\chi_\tau^{k-1})(\theta_{\mathrm{s}})^+ + \tau\delta\chi(\theta_{\mathrm{s}})^+ - \tau F_4 \end{aligned} \tag{4.17l}$$

with

$$F_4 := \frac{1}{\tau}\chi_\tau^{k-1} + \nu\chi_\tau^{k-1}\,. \tag{4.17m}$$

The potential $\Psi : \mathbf{X} \to [0, +\infty)$ featuring in (4.16) is defined by

$$\Psi(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) = \Psi(\chi) := \frac{\tau}{2} \int_{\Gamma_{\mathrm{C}}} \left(|\mathbf{u}_\tau^{k-1}|^2 (\chi)^+ + \mathcal{J}[\chi_\tau^{k-1}|\mathbf{u}_\tau^{k-1}|^2]\,(\chi)^+ + \mathcal{J}[(\chi)^+]\,(\chi)^+ |\mathbf{u}_\tau^{k-1}|^2 \right) \mathrm{d}x\,. \tag{4.18}$$

It can be readily checked that with $\mathscr{A}$ defined by (4.17) and Ψ by (4.18), system (4.16) yields solutions to system (4.13) in which the discrete flow rule for the adhesion parameter holds as a subdifferential inclusion in $H^1(\Gamma_{\mathrm{C}})^*$. However, a comparison argument in (4.13) yields a fortiori that $A\chi_\tau^k \in L^2(\Gamma_{\mathrm{C}})$, and thus $\chi \in H^2(\Gamma_{\mathrm{C}})$ and (4.13) holds a.e. in Γ_{C}. Let us then show that (4.16) does admit solutions.

Standard arguments in the theory of quasilinear elliptic equations (cf. [29, Chap. 2.4]) yield that the operator $\mathscr{A}$ is pseudomonotone. The next step is to verify that $\mathscr{A}$ is coercive, namely that (using the variable $\mathbf{x}$ as a place-holder for $(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi)$)

$$\lim_{\|\mathbf{x}\|_{\mathbf{X}} \to +\infty} \frac{\langle \mathscr{A}(\mathbf{x}), \mathbf{x} \rangle_{\mathbf{X}}}{\|\mathbf{x}\|_{\mathbf{X}}} = +\infty\,.$$

This can be done by the very same calculations that we will carry out in the proof of Lemma 4.9 ahead.

Hence, the existence theorem in [29, Cor. 5.17], yields that (4.16) admits a solution $(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi) =: (\theta_{\tau,\epsilon,M}^k, \mathbf{u}_{\tau,\epsilon,M}^k, \theta_{\mathrm{s},\tau,\epsilon,M}^k, \chi_{\tau,\epsilon,M}^k)$.

Step 2: proof of the non-negativity properties (4.14) We test the discrete bulk heat equation (4.13a) by $-(\theta_\tau^k)^-$ and the discrete surface heat equation (4.13c) by $-(\theta_{\mathrm{s},\tau}^k)^-$ and sum the resulting relations. Thus, we get

$$\begin{aligned}
0 = & \int_\Omega \frac{\theta_\tau^k - \mathcal{T}_\epsilon(\theta_\tau^{k-1})}{\tau}(-(\theta_\tau^k)^-)\,\mathrm{d}x + \int_{\Gamma_C} \frac{\theta_{s,\tau}^k - \mathcal{T}_\epsilon(\theta_{s,\tau}^{k-1})}{\tau}(-(\theta_{s,\tau}^k)^-)\,\mathrm{d}x \\
& + \int_\Omega (\theta_\tau^k)^+ \mathrm{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)(\theta_\tau^k)^-\,\mathrm{d}x + \int_{\Gamma_C} (\theta_{s,\tau}^k)^+ \frac{\lambda(\chi_\tau^k) - \lambda(\chi_\tau^{k-1})}{\tau}(\theta_{s,\tau}^k)^-\,\mathrm{d}x \\
& + \int_\Omega \alpha_M(\theta_\tau^k)|\nabla(\theta_\tau^k)^-|^2\,\mathrm{d}x + \int_{\Gamma_C} \alpha_M(\theta_{s,\tau}^k)|\nabla(\theta_{s,\tau}^k)^-|^2\,\mathrm{d}x \\
& + \int_{\Gamma_C} k(\chi_\tau^{k-1})(\theta_\tau^k - \theta_{s,\tau}^k)\left(\mathcal{T}_\epsilon(\theta_{s,\tau}^k)(\theta_{s,\tau}^k)^- - \mathcal{T}_\epsilon(\theta_\tau^k)(\theta_\tau^k)^-\right)\mathrm{d}x \\
& - \int_{\Gamma_C} (\theta_\tau^k)^- \left(\mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+\theta_\tau^k\mathcal{T}_\epsilon(\theta_\tau^k) - \mathcal{J}[(\chi_\tau^{k-1})^+\theta_{s,\tau}^k]\,(\chi_\tau^{k-1})^+\mathcal{T}_\epsilon(\theta_\tau^k)\right)\mathrm{d}x \\
& + \int_{\Gamma_C} (\theta_{s,\tau}^k)^- \left(\mathcal{J}[(\chi_\tau^{k-1})^+\theta_\tau^k]\,(\chi_\tau^{k-1})^+\mathcal{T}_\epsilon(\theta_{s,\tau}^k) - \mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+\theta_{s,\tau}^k\mathcal{T}_\epsilon(\theta_{s,\tau}^k)\right)\mathrm{d}x \\
& + \int_\Omega \varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\,\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)(\theta_\tau^k)^-\,\mathrm{d}x + \int_\Omega h_\tau^k(\theta_\tau^k)^-\,\mathrm{d}x + \int_{\Gamma_C} \ell_\tau^k(\theta_{s,\tau}^k)^-\,\mathrm{d}x \\
& + \int_{\Gamma_C} \left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 (\theta_{s,\tau}^k)^-\,\mathrm{d}x \\
\doteq\; & I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{4.19}$$

First, we have that $I_1 \geq \frac{1}{\tau}\|(\theta_\tau^k)^-\|_{L^2(\Omega)}^2$ and $I_2 \geq \frac{1}{\tau}\|(\theta_{s,\tau}^k)^-\|_{L^2(\Gamma_C)}^2$. Moreover, since $r^+r^- = 0$ for all $r \in \mathbb{R}$, $I_3 = I_4 = 0$, whereas by (4.12) we find that $I_5 \geq c_0\|\nabla(\theta_\tau^k)^-\|_{L^2(\Omega)}^2$ and $I_6 \geq c_0\|\nabla(\theta_{s,\tau}^k)^-\|_{L^2(\Gamma_C)}^2$. Observing that $\mathcal{T}_\epsilon(r)r^- = \epsilon r^-$ for all $r \in \mathbb{R}$, the function $r \mapsto \mathcal{T}_\epsilon(r)r^-$ is non-increasing and hence $I_7 \geq 0$, while the very same arguments as in (3.17) show that $I_8 + I_9 \geq 0$. Clearly, $I_{10} \geq 0$ and $I_{13} \geq 0$. The positivity of I_{11} is due to the fact that $h_\tau^k \geq 0$ a.e. in Ω by (2.18a); analogously we have that $I_{12} \geq 0$. All in all, from (4.19) we gather that

$$\|(\theta_\tau^k)^-\|_{L^2(\Omega)}^2 + \|(\theta_{s,\tau}^k)^-\|_{L^2(\Gamma_C)}^2 \leq 0, \quad \text{whence} \quad (\theta_\tau^k)^- = 0 \text{ a.e. in}\,\Omega, \quad (\theta_{s,\tau}^k)^- = 0 \text{ a.e. in}\,\Gamma_C,$$

whence the non-negativity properties (4.14).

Step 3: proof of estimate (4.15): Mimicking the calculations from Section 3.2, for $p > 2$ we test the discrete bulk heat equation (4.13a) by $-(\mathcal{T}_\epsilon(\theta_\tau^k))^{-p}$ and the discrete surface heat equation (4.13c) by $-(\mathcal{T}_\epsilon(\theta_{s,\tau}^k))^{-p}$. Summing the resulting relations, we get

$$\begin{aligned}
0 = & \int_\Omega \frac{\theta_\tau^k - \mathcal{T}_\epsilon(\theta_\tau^{k-1})}{\tau}(-(\mathcal{T}_\epsilon(\theta_\tau^k))^{-p})\,\mathrm{d}x + \int_{\Gamma_C} \frac{\theta_{s,\tau}^k - \mathcal{T}_\epsilon(\theta_{s,\tau}^{k-1})}{\tau}(-(\mathcal{T}_\epsilon(\theta_{s,\tau}^k))^{-p})\,\mathrm{d}x \\
& + \int_\Omega \theta_\tau^k \mathrm{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)(\mathcal{T}_\epsilon(\theta_\tau^k))^{-p}\,\mathrm{d}x + \int_{\Gamma_C} \theta_{s,\tau}^k \frac{\lambda(\chi_\tau^k) - \lambda(\chi_\tau^{k-1})}{\tau}(\mathcal{T}_\epsilon(\theta_{s,\tau}^k))^{-p}\,\mathrm{d}x \\
& + p\int_\Omega \alpha_M(\theta_\tau^k)(\mathcal{T}_\epsilon(\theta_\tau^k))^{-(1+p)}|\nabla\theta_\tau^k|^2\,\mathrm{d}x + p\int_{\Gamma_C} \alpha_M(\theta_{s,\tau}^k)(\mathcal{T}_\epsilon(\theta_{s,\tau}^k))^{-(1+p)}|\nabla\theta_{s,\tau}^k|^2\,\mathrm{d}x \\
& + \int_{\Gamma_C} k(\chi_\tau^{k-1})(\theta_\tau^k - \theta_{s,\tau}^k)((\mathcal{T}_\epsilon(\theta_{s,\tau}^k))^{1-p} - (\mathcal{T}_\epsilon(\theta_\tau^k))^{1-p})\,\mathrm{d}x \\
& - \int_{\Gamma_C} \mathcal{T}_\epsilon(\theta_\tau^k)^{1-p}\left(\mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+\theta_\tau^k - \mathcal{J}[(\chi_\tau^{k-1})^+\theta_{s,\tau}^k]\,(\chi_\tau^{k-1})^+\right)\mathrm{d}x \\
& + \int_{\Gamma_C} \mathcal{T}_\epsilon(\theta_{s,\tau}^k)^{1-p}\left(\mathcal{J}[(\chi_\tau^{k-1})^+\theta_\tau^k]\,(\chi_\tau^{k-1})^+ - \mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+\theta_{s,\tau}^k\right)\mathrm{d}x \\
& + \int_\Omega \varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\,\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\frac{1}{(\mathcal{T}_\epsilon(\theta_\tau^k))^p}\,\mathrm{d}x + \int_\Omega h_\tau^k \frac{1}{(\mathcal{T}_\epsilon(\theta_\tau^k))^p}\,\mathrm{d}x + \int_{\Gamma_C} \ell_\tau^k \frac{1}{(\mathcal{T}_\epsilon(\theta_\tau^k))^p}\,\mathrm{d}x \\
& + \int_{\Gamma_C} \left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 \frac{1}{(\mathcal{T}_\epsilon(\theta_\tau^k))^p}\,\mathrm{d}x \\
\doteq\; & I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12} + I_{13}\,,
\end{aligned} \tag{4.20}$$

where we have used that $(\theta_\tau^k)^+ = \theta_\tau^k$ and $(\theta_{s,\tau}^k)^+ = \theta_{s,\tau}^k$ by the previously obtained (4.14). Then, we observe that, as $\theta_\tau^k \le \mathcal{T}_\epsilon(\theta_\tau^k)$ a.e. in Ω, we have

$$I_1 \ge \int_\Omega \frac{\mathcal{T}_\epsilon(\theta_\tau^k) - \mathcal{T}_\epsilon(\theta_\tau^{k-1})}{\tau}(-(\mathcal{T}_\epsilon(\theta_\tau^k))^{-p})\,\mathrm{d}x \ge \frac{1}{(p-1)\tau}\int_\Omega (\mathcal{T}_\epsilon(\theta_\tau^k)^{1-p} - \mathcal{T}_\epsilon(\theta_\tau^{k-1})^{1-p})\,\mathrm{d}x,$$

where the last estimate follows from the convexity inequality $-r^{-p}(r-s) \ge \frac{1}{p-1}(r^{1-p} - s^{1-p})$ for every $r, s \in (0, +\infty)$. Analogously,

$$I_2 \ge \frac{1}{(p-1)\tau}\int_{\Gamma_C} (\mathcal{T}_\epsilon(\theta_{s,\tau}^k)^{1-p} - \mathcal{T}_\epsilon(\theta_{s,\tau}^{k-1})^{1-p})\,\mathrm{d}x.$$

Clearly, $I_5 \ge 0$ and $I_6 \ge 0$. Since the function $r \mapsto \mathcal{T}_\epsilon(r)^{1-p}$ is non-increasing, we have that $I_7 \ge 0$. The very same arguments as for (3.17) show that $I_8 + I_9 \ge 0$, and, again by (2.18a) and (2.18b), we have that $I_{11} \ge 0$ and $I_{12} \ge 0$. Finally, we observe that

$$\begin{aligned}
I_3 + I_{10} &\overset{(1)}{\ge} -c_d \int_\Omega \left|\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right| (\mathcal{T}_\epsilon(\theta_\tau^k))^{1-p}\,\mathrm{d}x + C_{\mathrm{v}} \int_\Omega \left|\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right|^2 (\mathcal{T}_\epsilon(\theta_\tau^k))^{-p}\,\mathrm{d}x \\
&\overset{(2)}{\ge} -C\int_\Omega (\mathcal{T}_\epsilon(\theta_\tau^k))^{2-p}\,\mathrm{d}x + \frac{C_{\mathrm{v}}}{2}\int_\Omega \left|\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right|^2 (\mathcal{T}_\epsilon(\theta_\tau^k))^{-p}\,\mathrm{d}x \\
&\overset{(3)}{\ge} -\frac{C}{p-1} - \frac{C(p-2)}{p-1}\int_\Omega (\mathcal{T}_\epsilon(\theta_\tau^k))^{1-p}\,\mathrm{d}x + \frac{C_{\mathrm{v}}}{2}\int_\Omega \left|\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right|^2 (\mathcal{T}_\epsilon(\theta_\tau^k))^{-p}\,\mathrm{d}x
\end{aligned}$$

where (1) is due to (3.14) and to (2.7), while (2) and (3) are due to Young's inequality, arguing as in (3.13). Analogously, we find that

$$I_4 + I_{12} \ge -\frac{C}{p-1} - \frac{C(p-2)}{p-1}\int_{\Gamma_C} (\mathcal{T}_\epsilon(\theta_{s,\tau}^k))^{1-p}\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_C} \left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 (\mathcal{T}_\epsilon(\theta_{s,\tau}^k))^{-p}\,\mathrm{d}x\,.$$

Combining all of the above calculations we arrive at

$$\begin{aligned}
&\frac{1}{(p-1)\tau}\int_\Omega \mathcal{T}_\epsilon(\theta_\tau^k)^{1-p}\,\mathrm{d}x + \frac{1}{(p-1)\tau}\int_{\Gamma_C} \mathcal{T}_\epsilon(\theta_{s,\tau}^k)^{1-p}\,\mathrm{d}x \\
&\le \frac{1}{(p-1)\tau}\int_\Omega \mathcal{T}_\epsilon(\theta_\tau^{k-1})^{1-p}\,\mathrm{d}x + \frac{1}{(p-1)\tau}\int_{\Gamma_C} \mathcal{T}_\epsilon(\theta_{s,\tau}^{k-1})^{1-p}\,\mathrm{d}x \\
&\quad + \frac{C}{p-1} + \frac{C(p-2)}{p-1}\left(\int_\Omega (\mathcal{T}_\epsilon(\theta_\tau^k))^{1-p}\,\mathrm{d}x + \int_{\Gamma_C} (\mathcal{T}_\epsilon(\theta_{s,\tau}^k))^{1-p}\,\mathrm{d}x\right),
\end{aligned}$$

whence, multiplying the inequality by $(p-1)\tau$ and summing over the index j, for $j \in \{1, \ldots, k\}$ with an arbitrary $k \in \{1, \ldots, K_\tau\}$, we arrive at the relation

$$\mathcal{T}_k \le \mathcal{T}_0 + CT + \sum_{j=1}^k C(p-2)\tau\mathcal{T}_j$$

with the place-holder $\mathcal{T}_k := \int_\Omega \mathcal{T}_\epsilon(\theta_\tau^k)^{1-p}\,\mathrm{d}x + \int_{\Gamma_C} \mathcal{T}_\epsilon(\theta_{s,\tau}^k)^{1-p}\,\mathrm{d}x$. The discrete Gronwall Lemma 4.5 yields,

$$\mathcal{T}_k \le \frac{\mathcal{T}_0 + CT}{1 - C(p-2)\tau}\exp\left(\frac{C(p-2)\tau k}{1 - C(p-2)\tau}\right)$$

for $\tau \in (0, \bar\tau_p)$ with

$$\bar\tau_p = \frac{1}{2C(p-2)}\,. \tag{4.21}$$

Now, since $\tau < \bar\tau_p$, we have that $1 - C(p-2)\tau > \frac{1}{2}$ and thus we get the analogue of estimate (3.22), i.e.

$$\max\left\{\left\|\frac{1}{\mathcal{T}_\epsilon(\theta_\tau^k)}\right\|_{L^{p-1}(\Omega)}, \left\|\frac{1}{\mathcal{T}_\epsilon(\theta_{s,\tau}^k)}\right\|_{L^{p-1}(\Gamma_C)}\right\} \le (2C')^{1/(p-1)}\exp\left(\frac{2CT(p-2)}{(p-1)}\right) \le S_0$$

with $C' = \mathcal{T}_0 + CT$ and $S_0 = 2C'\exp(2CT)$. Clearly, by the arbitrariness of $p > 2$ we conclude (4.15). □

Remark 4.7. A careful perusal of the calculations in Step 3 shows the role of the positive parts $(\theta_\tau^k)^+$ and $(\chi_\tau^{k-1})^+$, as well as of the truncation operator $\mathcal{T}_\epsilon$, in ensuring the positivity of various integral terms that appear in the proof of estimate (4.15).

We conclude this section by showing that the discrete solutions $(\theta^k_{\tau,\epsilon,M}, \mathbf{u}^k_{\tau,\epsilon,M}, \theta^k_{\mathrm{s},\tau,\epsilon,M}, \chi^k_{\tau,\epsilon,M})$ fulfill an energy inequality, involving the stored energy functional $\mathcal{E}_\varsigma$ from (4.5), that will play a crucial role for the limit passage as $\epsilon \downarrow 0$. In the proof of (4.22) below we will use in a key way the convex and concave decompositions from (4.2).

Lemma 4.8. *The functions* $(\theta^k_{\tau,\epsilon,M}, \mathbf{u}^k_{\tau,\epsilon,M}, \theta^k_{\mathrm{s},\tau,\epsilon,M}, \chi^k_{\tau,\epsilon,M})$ *fulfill*

$$
\begin{aligned}
&\mathcal{E}_\varsigma(\theta^k_{\tau,\epsilon,M}, \theta^k_{\mathrm{s},\tau,\epsilon,M}, \mathbf{u}^k_{\tau,\epsilon,M}, \chi^k_{\tau,\epsilon,M}) + \rho\tau \int_\Omega \left| \varepsilon\left(\frac{\mathbf{u}^k_{\tau,\epsilon,M} - \mathbf{u}^{k-1}_\tau}{\tau}\right)\right|^\omega \mathrm{d}x + \rho\tau \int_{\Gamma_{\mathrm{C}}} \left| \frac{\chi^k_{\tau,\epsilon,M} - \chi^{k-1}_\tau}{\tau}\right|^\omega \mathrm{d}x \\
&\quad + \tau \int_{\Gamma_{\mathrm{C}}} k(\chi^{k-1}_\tau)(\theta^k_{\tau,\epsilon,M} - \theta^k_{\mathrm{s},\tau,\epsilon,M})(\mathcal{T}_\epsilon(\theta^k_{\tau,\epsilon,M}) - \mathcal{T}_\epsilon(\theta^k_{\mathrm{s},\tau,\epsilon,M}))\,\mathrm{d}x \\
&\quad + \tau \iint_{\Gamma_{\mathrm{C}}\times\Gamma_{\mathrm{C}}} j(x,y)(\chi^{k-1}_\tau(x))^+(\chi^{k-1}_\tau(y))^+(\mathcal{T}_\epsilon(\theta^k_{\tau,\epsilon,M}(x)) - \mathcal{T}_\epsilon(\theta^k_{\mathrm{s},\tau,\epsilon,M})(y))(\theta^k_{\tau,\epsilon,M}(x) - \theta^k_{\mathrm{s},\tau,\epsilon,M}(y))\,\mathrm{d}x\,\mathrm{d}y \\
&\le \mathcal{E}_\varsigma(\mathcal{T}_\epsilon(\theta^{k-1}_\tau), \mathcal{T}_\epsilon(\theta^{k-1}_{\mathrm{s},\tau}), \mathbf{u}^{k-1}_\tau, \chi^{k-1}_\tau) + \tau \int_\Omega h^k_\tau\,\mathrm{d}x + \tau \int_{\Gamma_{\mathrm{C}}} \ell^k_\tau\,\mathrm{d}x + \tau \left\langle \mathbf{F}^k_\tau, \frac{\mathbf{u}^k_{\tau,\epsilon,M} - \mathbf{u}^{k-1}_\tau}{\tau}\right\rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)}.
\end{aligned}
\tag{4.22}
$$

Proof. We test (4.13a) by τ, (4.13b) by $(\mathbf{u}^k_\tau - \mathbf{u}^{k-1}_\tau)$, (4.13c) by τ, and (4.13d) by $(\chi^k_\tau - \chi^{k-1}_\tau)$, add the resulting relations, and observe the cancellation of the terms

$$
-\tau \int_\Omega (\theta^k_\tau)^+ \mathrm{div}\left(\tfrac{\mathbf{u}^k_\tau - \mathbf{u}^{k-1}_\tau}{\tau}\right) \mathrm{d}x, \qquad \tau \int_\Omega \varepsilon\left(\frac{\mathbf{u}^k_\tau - \mathbf{u}^{k-1}_\tau}{\tau}\right) \mathbb{V}\,\varepsilon\left(\frac{\mathbf{u}^k_\tau - \mathbf{u}^{k-1}_\tau}{\tau}\right) \mathrm{d}x, \qquad \tau \int_{\Gamma_{\mathrm{C}}} \left|\tfrac{\chi^k_\tau - \chi^{k-1}_\tau}{\tau}\right|^2 \mathrm{d}x.
$$

We now manipulate the remaining terms. Elementary convexity inequalities give that

$$
\begin{aligned}
&\mathsf{e}\left(\mathbf{u}^k_\tau, \mathbf{u}^k_\tau - \mathbf{u}^{k-1}_\tau\right) \ge \frac12 \mathsf{e}(\mathbf{u}^k_\tau) - \frac12 \mathsf{e}(\mathbf{u}^{k-1}_\tau), \\
&\int_{\Gamma_{\mathrm{C}}} (\chi^k_\tau)^+ \mathbf{u}^k_\tau (\mathbf{u}^k_\tau - \mathbf{u}^{k-1}_\tau)\,\mathrm{d}x + \frac12 \int_{\Gamma_{\mathrm{C}}} |\mathbf{u}^{k-1}_\tau|^2 \sigma^k_\tau (\chi^k_\tau - \chi^{k-1}_\tau)\,\mathrm{d}x \\
&\ge \frac12 \int_{\Gamma_{\mathrm{C}}} (\chi^k_\tau)^+ |\mathbf{u}^k_\tau|^2\,\mathrm{d}x - \frac12 \int_{\Gamma_{\mathrm{C}}} (\chi^k_\tau)^+ |\mathbf{u}^{k-1}_\tau|^2\,\mathrm{d}x + \frac12 \int_{\Gamma_{\mathrm{C}}} (\chi^k_\tau)^+ |\mathbf{u}^{k-1}_\tau|^2\,\mathrm{d}x - \frac12 \int_{\Gamma_{\mathrm{C}}} (\chi^{k-1}_\tau)^+ |\mathbf{u}^{k-1}_\tau|^2\,\mathrm{d}x \\
&= \frac12 \int_{\Gamma_{\mathrm{C}}} (\chi^k_\tau)^+ |\mathbf{u}^k_\tau|^2\,\mathrm{d}x - \frac12 \int_{\Gamma_{\mathrm{C}}} (\chi^{k-1}_\tau)^+ |\mathbf{u}^{k-1}_\tau|^2\,\mathrm{d}x, \\
&\int_{\Gamma_{\mathrm{C}}} \boldsymbol{\zeta}^k_\tau \cdot (\mathbf{u}^k_\tau - \mathbf{u}^{k-1}_\tau)\,\mathrm{d}x \ge \int_{\Gamma_{\mathrm{C}}} \widehat{\eta}_\varsigma(\mathbf{u}^k_\tau \cdot \mathbf{n})\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} \widehat{\eta}_\varsigma(\mathbf{u}^{k-1}_\tau \cdot \mathbf{n})\,\mathrm{d}x, \\
&\int_{\Gamma_{\mathrm{C}}} \nabla\chi^k_\tau \cdot \nabla(\chi^k_\tau - \chi^{k-1}_\tau)\,\mathrm{d}x \ge \frac12 \int_{\Gamma_{\mathrm{C}}} |\nabla\chi^k_\tau|^2\,\mathrm{d}x - \frac12 \int_{\Gamma_{\mathrm{C}}} |\nabla\chi^{k-1}_\tau|^2\,\mathrm{d}x, \\
&\int_{\Gamma_{\mathrm{C}}} \beta_\varsigma(\chi^k_\tau)(\chi^k_\tau - \chi^{k-1}_\tau)\,\mathrm{d}x \ge \int_{\Gamma_{\mathrm{C}}} \widehat{\beta}_\varsigma(\chi^k_\tau)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} \widehat{\beta}_\varsigma(\chi^{k-1}_\tau)\,\mathrm{d}x.
\end{aligned}
\tag{4.23}
$$

Using that $\lambda_\delta(r) = \lambda(r) - \frac{\delta}{2} r^2$ is concave we infer that

$$
\begin{aligned}
&- \int_{\Gamma_{\mathrm{C}}} (\theta^k_{s,\tau})^+ \left(\lambda(\chi^k_\tau) - \lambda(\chi^{k-1}_\tau) - \lambda'_\delta(\chi^{k-1}_\tau)(\chi^k_\tau - \chi^{k-1}_\tau) - \delta\chi^k_\tau(\chi^k_\tau - \chi^{k-1}_\tau)\right) \mathrm{d}x \\
&\ge - \int_{\Gamma_{\mathrm{C}}} (\theta^k_{s,\tau})^+ \left(\lambda(\chi^k_\tau) - \lambda(\chi^{k-1}_\tau) + \lambda_\delta(\chi^{k-1}_\tau) - \lambda_\delta(\chi^k_\tau) + \tfrac{\delta}{2}|\chi^{k-1}_\tau|^2 - \tfrac{\delta}{2}|\chi^k_\tau|^2\right) \mathrm{d}x = 0.
\end{aligned}
\tag{4.24a}
$$

Analogously, exploiting that $\gamma_\nu(r) = \gamma(r) + \frac{\nu}{2} r^2$ is convex we find that

$$
\begin{aligned}
\int_{\Gamma_{\mathrm{C}}} \gamma'_\nu(\chi^k_\tau)(\chi^k_\tau - \chi^{k-1}_\tau) - \nu \int_{\Gamma_{\mathrm{C}}} \chi^{k-1}_\tau(\chi^k_\tau - \chi^{k-1}_\tau)\,\mathrm{d}x &\ge \int_{\Gamma_{\mathrm{C}}} \left(\gamma_\nu(\chi^k_\tau) - \gamma_\nu(\chi^{k-1}_\tau) + \tfrac{\nu}{2}|\chi^{k-1}_\tau|^2 - \tfrac{\nu}{2}|\chi^k_\tau|^2\right) \mathrm{d}x \\
&\ge \int_{\Gamma_{\mathrm{C}}} \left(\gamma(\chi^k_\tau) - \gamma(\chi^{k-1}_\tau)\right) \mathrm{d}x.
\end{aligned}
\tag{4.24b}
$$

As for the nonlocal terms in the discrete displacement equation, we have

$$\int_{\Gamma_C} \mathcal{J}[(\chi_\tau^k)^+]\,(\chi_\tau^k)^+\mathbf{u}_\tau^k\cdot(\mathbf{u}_\tau^k-\mathbf{u}_\tau^{k-1})\,\mathrm{d}x \geq \frac12\int_{\Gamma_C}\mathcal{J}[(\chi_\tau^k)^+]\,(\chi_\tau^k)^+|\mathbf{u}_\tau^k|^2\,\mathrm{d}x - \frac12\int_{\Gamma_C}\mathcal{J}[(\chi_\tau^k)^+]\,(\chi_\tau^k)^+|\mathbf{u}_\tau^{k-1}|^2\,\mathrm{d}x, \tag{4.25a}$$

whereas the nonlocal terms in the discrete flow rule for the adhesion parameter yield

$$\begin{aligned}&\int_{\Gamma_C}\frac12\mathcal{J}[(\chi_\tau^k)^+]\,\sigma_\tau^k(\chi_\tau^k-\chi_\tau^{k-1})|\mathbf{u}_\tau^{k-1}|^2\,\mathrm{d}x\\ &\geq \frac12\int_{\Gamma_C}\mathcal{J}[(\chi_\tau^k)^+]\,(\chi_\tau^k)^+|\mathbf{u}_\tau^{k-1}|^2\,\mathrm{d}x-\frac12\int_{\Gamma_C}\mathcal{J}[(\chi_\tau^k)^+]\,(\chi_\tau^{k-1})^+|\mathbf{u}_\tau^{k-1}|^2\,\mathrm{d}x,\end{aligned} \tag{4.25b}$$

$$\begin{aligned}&\frac12\int_{\Gamma_C}\mathcal{J}[(\chi_\tau^{k-1})^+|\mathbf{u}_\tau^{k-1}|^2]\,\sigma_\tau^k(\chi_\tau^k-\chi_\tau^{k-1})\,\mathrm{d}x\\ &\geq \frac12\int_{\Gamma_C}\mathcal{J}[(\chi_\tau^{k-1})^+|\mathbf{u}_\tau^{k-1}|^2]\,((\chi_\tau^k)^+-(\chi_\tau^{k-1})^+)\,\mathrm{d}x\\ &=\frac12\int_{\Gamma_C}\mathcal{J}[(\chi_\tau^k)^+]\,(\chi_\tau^{k-1})^+|\mathbf{u}_\tau^{k-1}|^2\,\mathrm{d}x-\frac12\int_{\Gamma_C}\mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+|\mathbf{u}_\tau^{k-1}|^2\,\mathrm{d}x\,,\end{aligned} \tag{4.25c}$$

where we have used the symmetry properties of $\mathcal{J}$. Adding (4.25a), (4.25b), and (4.25c), we obtain

$$\frac12\int_{\Gamma_C}\mathcal{J}[(\chi_\tau^k)^+]\,(\chi_\tau^k)^+|\mathbf{u}_\tau^k|^2\,\mathrm{d}x-\frac12\int_{\Gamma_C}\mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+|\mathbf{u}_\tau^{k-1}|^2\,\mathrm{d}x\,.$$

All in all, summing the terms on the right-hand sides of the inequalities in (4.23)–(4.25) with the temperature terms we obtain $\mathcal{E}_\varsigma(\theta_\tau^k,\theta_{\mathrm{s},\tau}^k,\mathbf{u}_\tau^k,\chi_\tau^k)-\mathcal{E}_\varsigma(\mathcal{T}_\epsilon(\theta_\tau^{k-1}),\mathcal{T}_\epsilon(\theta_{\mathrm{s},\tau}^{k-1}),\mathbf{u}_\tau^{k-1},\chi_\tau^{k-1})$.

Furthermore, the integrals $\tau\int_{\Gamma_C}k(\chi_\tau^{k-1})\mathcal{T}_\epsilon(\theta_\tau^k)(\theta_\tau^k-\theta_{\mathrm{s},\tau}^k)\,\mathrm{d}x$ and $-\tau\int_{\Gamma_C}k(\chi_\tau^{k-1})(\theta_\tau^k-\theta_{s,\tau}^k)\mathcal{T}_\epsilon(\theta_{s,\tau}^k)\,\mathrm{d}x$ combine to give the fourth term on the left-hand side of (4.22). Repeating the same calculations as in (3.7), we see that

$$\begin{aligned}&\tau\int_{\Gamma_C}\mathcal{T}_\epsilon(\theta_\tau^k)\left(\mathcal{J}[\chi_\tau^{k-1}]\chi_\tau^{k-1}\theta_\tau^k-\mathcal{J}[\chi_\tau^{k-1}\theta_{s,\tau}^k]\chi_\tau^{k-1}\right)\,\mathrm{d}x-\tau\int_{\Gamma_C}\mathcal{T}_\epsilon(\theta_{s,\tau}^k)\left(\mathcal{J}[\chi_\tau^{k-1}\theta_\tau^k]\chi_\tau^{k-1}-\mathcal{J}[\chi_\tau^{k-1}]\chi_\tau^{k-1}\theta_{s,\tau}^k\right)\,\mathrm{d}x\\ &=\tau\iint_{\Gamma_C\times\Gamma_C}j(x,y)\chi_\tau^{k-1}(x)\chi_\tau^{k-1}(y)(\mathcal{T}_\epsilon(\theta_{\tau,\epsilon,M}^k(x))-\mathcal{T}_\epsilon(\theta_{\mathrm{s},\tau,\epsilon,M}^k)(y))(\theta_{\tau,\epsilon,M}^k(x)-\theta_{\mathrm{s},\tau,\epsilon,M}^k(y))\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Finally, we find

$$\rho\int_\Omega\left|\varepsilon\left(\frac{\mathbf{u}_\tau^k-\mathbf{u}_\tau^{k-1}}{\tau}\right)\right|^{\omega-2}\varepsilon\left(\frac{\mathbf{u}_\tau^k-\mathbf{u}_\tau^{k-1}}{\tau}\right)\varepsilon(\mathbf{u}_\tau^k-\mathbf{u}_\tau^{k-1})\,\mathrm{d}x=\rho\tau\int_\Omega\left|\varepsilon\left(\frac{\mathbf{u}_\tau^k-\mathbf{u}_\tau^{k-1}}{\tau}\right)\right|^{\omega}\,\mathrm{d}x,$$

$$\rho\int_{\Gamma_C}\left|\frac{\chi_\tau^k-\chi_\tau^{k-1}}{\tau}\right|^{\omega-2}\frac{\chi_\tau^k-\chi_\tau^{k-1}}{\tau}(\chi_\tau^k-\chi_\tau^{k-1})\,\mathrm{d}x=\rho\tau\int_{\Gamma_C}\left|\frac{\chi_\tau^k-\chi_\tau^{k-1}}{\tau}\right|^{\omega}\,\mathrm{d}x\,.$$

Taking into account the above calculations, we conclude (4.22). □

4.2. **Existence of solutions to the M-truncated discrete system.** Now, we perform, for $\tau>0$, $M>0$, and $k\in\{1,\ldots,K_\tau\}$ fixed, the limit passage in system (4.13) as $\epsilon\downarrow 0$. In this way, we will obtain discrete solutions to a discrete system featuring only the M-truncation in α. To shorten notation, throughout this section we will abbreviate the solution quadruple $(\theta_{\tau,\epsilon,M}^k,\mathbf{u}_{\tau,\epsilon,M}^k,\theta_{\mathrm{s},\tau,\epsilon,M}^k,\chi_{\tau,\epsilon,M}^k)$ with $(\theta_\epsilon^k,\mathbf{u}_\epsilon^k,\theta_{\mathrm{s},\epsilon}^k,\chi_\epsilon^k)$. Likewise, we will simply denote by σ_ϵ^k the selections in $\partial\varphi(\chi_\epsilon^k)$, and use the notation $\boldsymbol{\zeta}_\epsilon^k:=\eta_\varsigma(\mathbf{u}_\epsilon^k\cdot\mathbf{n})\mathbf{n}$.

Our first result collects a series of a priori estimate on the sequence $(\theta_\epsilon^k,\mathbf{u}_\epsilon^k,\theta_{\mathrm{s},\epsilon}^k,\chi_\epsilon^k,\sigma_\epsilon^k)_\epsilon$. They hold uniformly w.r.t. τ in $(0,\bar\tau)$ for some $\bar\tau>0$ that shall be specified in the proof), uniformly w.r.t. ϵ and, in fact, w.r.t. $M>0$.

Lemma 4.9. *Let $\tau\in(0,\bar\tau)$, for some $\bar\tau>0$, and $k\in\{1,\ldots,K_\tau\}$ be fixed. There exists a constant $S_1>0$, also* independent *of $M>0$, such that the following estimate holds*

$$\sup_{\epsilon>0}\left(\|\theta_\epsilon^k\|_{H^1(\Omega)}+\|\theta_{\mathrm{s},\epsilon}^k\|_{H^1(\Gamma_C)}+\|\mathbf{u}_\epsilon^k\|_{W^{1,\omega}(\Omega;\mathbb{R}^3)}+\|\chi_\epsilon^k\|_{H^2(\Gamma_C)}+\|\sigma_\epsilon^k\|_{L^\infty(\Gamma_C)}\right)\leq S_1\,, \tag{4.26}$$

in addition to estimate (4.15).

Proof. Observe that the fourth and the fifth terms on the left-hand side of the energy inequality (4.22) are positive. Taking into account that the functions $(\theta_\tau^{k-1}, \mathbf{u}_\tau^{k-1}, \theta_{s,\tau}^{k-1}, \chi_\tau^{k-1})$ are given, and recalling assumptions (2.18) on the problem data, we thus infer from (4.22) that

$$\mathcal{E}_\varsigma(\theta_\epsilon^k, \theta_{s,\epsilon}^k, \mathbf{u}_\epsilon^k, \chi_\epsilon^k) + \rho\tau \int_\Omega \left| \varepsilon\left(\tfrac{\mathbf{u}_\epsilon^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right|^\omega \mathrm{d}x + \rho\tau \int_{\Gamma_\mathrm{C}} \left|\tfrac{\chi_\epsilon^k - \chi_\tau^{k-1}}{\tau}\right|^\omega \mathrm{d}x \le C + C\|\mathbf{u}_\epsilon^k\|_{H^1(\Omega)} .$$

Now, the functional $\mathcal{E}_\varsigma$ enjoys the coercivity properties (3.8) (with the exception of the control of the $\|\cdot\|_{L^\infty(\Gamma_\mathrm{C})}$-norm, and of the enforcement of the positivity, of χ). Taking them into account, we absorb the second term on the right-hand side of the above estimate into the energy term $\mathcal{E}_\varsigma(\theta_\epsilon^k, \theta_{s,\epsilon}^k, \mathbf{u}_\epsilon^k, \chi_\epsilon^k)$, and thus conclude a bound for the whole left-hand side. Again in view of (3.8), we thus conclude that

$$\sup_{\epsilon>0} \left(\|\theta_\epsilon^k\|_{L^1(\Omega)} + \|\theta_{s,\epsilon}^k\|_{L^1(\Gamma_\mathrm{C})} + \|\mathbf{u}_\epsilon^k\|_{W^{1,\omega}(\Omega;\mathbb{R}^3)} + \|\chi_\epsilon^k\|_{H^1(\Gamma_\mathrm{C})} \right) \le C. \tag{4.27}$$

Furthermore, by the definition of $\partial\varphi$ (cf. (3.70)) we have that $\|\sigma_\epsilon^k\|_{L^\infty(\Gamma_\mathrm{C})} \le C$.

Next, we test (4.13a) by θ_ϵ^k and (4.13c) by $\theta_{s,\epsilon}^k$ and add the resulting relations; observe that all the positive parts $(\cdot)^+$ can be removed, in view of (4.14). We thus obtain

$$\begin{aligned} &\|\theta_\epsilon^k\|_{L^2(\Omega)}^2 + \|\theta_{s,\epsilon}^k\|_{L^2(\Gamma_\mathrm{C})}^2 + \tau \int_\Omega \alpha_M(\theta_\epsilon^k)|\nabla\theta_\epsilon^k|^2 \,\mathrm{d}x + \tau \int_{\Gamma_\mathrm{C}} \alpha_M(\theta_{s,\epsilon}^k)|\nabla\theta_{s,\epsilon}^k|^2 \,\mathrm{d}x + I_1 + I_2 \\ &= I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10} \end{aligned} \tag{4.28}$$

with

$$\begin{aligned} I_1 &= \tau \int_{\Gamma_\mathrm{C}} k(\chi_\tau^{k-1})(\mathcal{T}_\epsilon(\theta_\epsilon^k)\theta_\epsilon^k - \mathcal{T}_\epsilon(\theta_{s,\epsilon}^k)\theta_{s,\epsilon}^k)(\theta_\epsilon^k - \theta_{s,\epsilon}^k)\,\mathrm{d}x \ge 0, \\ I_2 &= \tau \int_{\Gamma_\mathrm{C}} \mathcal{T}_\epsilon(\theta_\epsilon^k)\theta_\epsilon^k \left(\mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+\theta_\epsilon^k - \mathcal{J}[(\chi_\tau^{k-1})^+\theta_{s,\epsilon}^k]\,(\chi_\tau^{k-1})^+ \right) \mathrm{d}x \\ &\quad - \tau \int_{\Gamma_\mathrm{C}} \mathcal{T}_\epsilon(\theta_{s,\epsilon}^k)\theta_{s,\epsilon}^k \left(\mathcal{J}[(\chi_\tau^{k-1})^+\theta_\epsilon^k]\,(\chi_\tau^{k-1})^+ - \mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+\theta_{s,\epsilon}^k \right) \mathrm{d}x \\ &\overset{(1)}{=} \tau \iint_{\Gamma_\mathrm{C}\times\Gamma_\mathrm{C}} j(x,y)(\chi_\tau^{k-1}(x))^+(\chi_\tau^{k-1}(y))^+(\theta_\epsilon^k(x) - \theta_{s,\epsilon}^k(y))(\mathcal{T}_\epsilon(\theta_\epsilon^k(x))\theta_\epsilon^k(x) - \mathcal{T}_\epsilon(\theta_{s,\epsilon}^k(y))\theta_{s,\epsilon}^k(y))\,\mathrm{d}x\,\mathrm{d}y \ge 0, \end{aligned}$$

where (1) follows by the very same arguments used for (3.7); both I_1 and I_2 are positive since the function $r \mapsto \mathcal{T}_\epsilon(r)r$ is increasing. As for the terms on the right-hand side, we have

$$\begin{aligned} I_3 &= \int_\Omega \mathcal{T}_\epsilon(\theta_\tau^{k-1})\theta_\epsilon^k \,\mathrm{d}x \le C\|\theta_\epsilon^k\|_{L^2(\Omega)} \le \frac{1}{8}\|\theta_\epsilon^k\|_{L^2(\Omega)}^2 + C, \\ I_4 &= \int_{\Gamma_\mathrm{C}} \mathcal{T}_\epsilon(\theta_{s,\tau}^{k-1})\theta_{s,\epsilon}^k \,\mathrm{d}x \le C\|\theta_{s,\epsilon}^k\|_{L^2(\Gamma_\mathrm{C})} \le \frac{1}{8}\|\theta_{s,\epsilon}^k\|_{L^2(\Gamma_\mathrm{C})}^2 + C, \\ I_5 &= \tau \int_\Omega \mathrm{div}\left(\tfrac{\mathbf{u}_\epsilon^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)|\theta_\epsilon^k|^2 \,\mathrm{d}x \le C\tau \left\|\tfrac{\mathbf{u}_\epsilon^k - \mathbf{u}_\tau^{k-1}}{\tau}\right\|_{W^{1,4}(\Omega)} \|\theta_\epsilon^k\|_{L^4(\Omega)}\|\theta_\epsilon^k\|_{L^2(\Omega)} \\ &\overset{(2)}{\le} C\tau\|\theta_\epsilon^k\|_{L^4(\Omega)}\|\theta_\epsilon^k\|_{L^2(\Omega)} \overset{(3)}{\le} C\tau^2\|\theta_\epsilon^k\|_{H^1(\Omega)}^2 + \frac{1}{8}\|\theta_\epsilon^k\|_{L^2(\Omega)}^2, \\ I_6 &= \tau \int_\Omega \varepsilon\left(\tfrac{\mathbf{u}_\epsilon^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \mathbb{V}\,\varepsilon\left(\tfrac{\mathbf{u}_\epsilon^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\theta_\epsilon^k \,\mathrm{d}x \le C\tau \left\|\varepsilon\left(\tfrac{\mathbf{u}_\epsilon^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right\|_{L^4(\Omega)}^2 \|\theta_\epsilon^k\|_{L^2(\Omega)} \overset{(4)}{\le} \frac{1}{8}\|\theta_\epsilon^k\|_{L^2(\Omega)}^2 + C, \\ I_7 &= \tau\,\langle h_\tau^k, \theta_\epsilon^k\rangle_{H^1(\Omega)} \overset{(5)}{\le} C\tau\|\theta_\epsilon^k\|_{H^1(\Omega)} \le \frac{\tau^2}{2}\|\theta_\epsilon^k\|_{H^1(\Omega)}^2 + C, \\ I_8 &= \tau\,\langle \ell_\tau^k, \theta_{s,\epsilon}^k\rangle_{H^1(\Gamma_\mathrm{C})} \overset{(6)}{\le} C\tau\|\theta_{s,\epsilon}^k\|_{H^1(\Gamma_\mathrm{C})} \le \frac{\tau^2}{2}\|\theta_{s,\epsilon}^k\|_{H^1(\Gamma_\mathrm{C})}^2 + C, \\ I_9 &= \tau \int_{\Gamma_\mathrm{C}} \tfrac{\lambda(\chi_\epsilon^k) - \lambda(\chi_\tau^{k-1})}{\tau}|\theta_{s,\epsilon}^k|^2 \,\mathrm{d}x \overset{(7)}{\le} C\tau \left\|\tfrac{\chi_\epsilon^k - \chi_\tau^{k-1}}{\tau}\right\|_{L^4(\Gamma_\mathrm{C})} \|\theta_{s,\epsilon}^k\|_{L^4(\Gamma_\mathrm{C})}\|\theta_{s,\epsilon}^k\|_{L^2(\Gamma_\mathrm{C})} \\ &\overset{(8)}{\le} \frac{C\tau^2}{2}\|\theta_{s,\epsilon}^k\|_{H^1(\Gamma_\mathrm{C})}^2 + \frac{1}{8}\|\theta_{s,\epsilon}^k\|_{L^2(\Gamma_\mathrm{C})}^2, \\ I_{10} &= \tau \int_{\Gamma_\mathrm{C}} \left|\tfrac{\chi_\epsilon^k - \chi_\tau^{k-1}}{\tau}\right|^2 \theta_{s,\epsilon}^k \,\mathrm{d}x \le C\tau \left\|\tfrac{\chi_\epsilon^k - \chi_\tau^{k-1}}{\tau}\right\|_{L^4(\Gamma_\mathrm{C})}^2 \|\theta_{s,\epsilon}^k\|_{L^2(\Gamma_\mathrm{C})} \overset{(9)}{\le} \frac{1}{8}\|\theta_{s,\epsilon}^k\|_{L^2(\Gamma_\mathrm{C})}^2 + C, \end{aligned}$$

where (2) follows from (4.27), and in (3) we have to choose $\tau > 0$ small enough so that the term $C\tau^2\|\theta_\epsilon^k\|_{H^1(\Omega)}^2$ can be absorbed by the left-hand side of (4.28), taking into account (4.12); (4) also follows from (4.27); (5) and (6) from (2.18a) and (2.18b), respectively; (7) from the Lipschitz continuity of λ and (8) from (4.27), just like (9). Again, in (8) we choose $\tau > 0$ small enough in such a way as to absorb $C\tau^2\|\theta_{\mathrm{s},\epsilon}^k\|_{H^1(\Gamma_{\mathrm{C}})}^2$ into the left-hand side of (4.28). All in all, combining the above calculations with (4.28) we easily conclude that

$$\|\theta_\epsilon^k\|_{L^2(\Omega)}^2 + \|\theta_{\mathrm{s},\epsilon}^k\|_{L^2(\Gamma_{\mathrm{C}})}^2 + \tau\int_\Omega \alpha_M(\theta_\epsilon^k)|\nabla\theta_\epsilon^k|^2\,\mathrm{d}x + \tau\int_{\Gamma_{\mathrm{C}}} \alpha_M(\theta_{\mathrm{s},\epsilon}^k)|\nabla\theta_{\mathrm{s},\epsilon}^k|^2\,\mathrm{d}x \le C,$$

whence the H^1-bounds for θ_ϵ^k and $\theta_{\mathrm{s},\epsilon}^k$.

Since $H^1(\Gamma_{\mathrm{C}})$ embeds continuously in $L^q(\Gamma_{\mathrm{C}})$ for every $q \in [1,\infty)$, the term $\left|\frac{\chi_\epsilon^k-\chi_\tau^{k-1}}{\tau}\right|^{\omega-2}\frac{\chi_\epsilon^k-\chi_\tau^{k-1}}{\tau}$ is estimated in any $L^q(\Gamma_{\mathrm{C}})$; with arguments analogous to those in the previous lines, it is not difficult to check that the right-hand side of the discrete flow rule (4.13d) is estimated in $L^2(\Gamma_{\mathrm{C}})$. Hence, by comparison, $(A\chi_\epsilon^k)_\epsilon$ is also estimated in $L^2(\Gamma_{\mathrm{C}})$, whence the estimate for $(\chi_\epsilon^k)_\epsilon$ in $H^2(\Gamma_{\mathrm{C}})$. □

We are now in a position to pass to the limit as $\epsilon \downarrow 0$, for fixed $M > 0$, $\tau > 0$, and $k \in \{1,\ldots,K_\tau\}$, in system (4.13). In what follows, for convenience we shall suppose that $\frac{1}{\epsilon} \in \mathbb{N}\setminus\{0\}$, so that, up to labelling $(\theta_\epsilon^k, \mathbf{u}_\epsilon^k, \theta_{\mathrm{s},\epsilon}^k, \chi_\epsilon^k, \sigma_\epsilon^k)$ by means of the natural number $m = \frac{1}{\epsilon}$, the functions $(\theta_\epsilon^k, \mathbf{u}_\epsilon^k, \theta_{\mathrm{s},\epsilon}^k, \chi_\epsilon^k, \sigma_\epsilon^k)_\epsilon$ form a sequence.

Lemma 4.10. *There exist a (not relabeled) subsequence of* $(\theta_\epsilon^k, \mathbf{u}_\epsilon^k, \theta_{\mathrm{s},\epsilon}^k, \chi_\epsilon^k, \sigma_\epsilon^k)_\epsilon$ *and a quintuple*

$$(\theta_{\tau,M}^k, \mathbf{u}_{\tau,M}^k, \theta_{\mathrm{s},\tau,M}^k, \chi_{\tau,M}^k, \sigma_{\tau,M}^k) \in H^1(\Omega) \times W_{\mathrm{D}}^{1,\omega}(\Omega;\mathbb{R}^3) \times H^1(\Gamma_{\mathrm{C}}) \times H^2(\Gamma_{\mathrm{C}}) \times L^\infty(\Gamma_{\mathrm{C}})$$

such that the following convergences hold as $\epsilon \downarrow 0$*:*

$$\begin{aligned} &\theta_\epsilon^k \rightharpoonup \theta_{\tau,M}^k \quad \text{in } H^1(\Omega), && \theta_{\mathrm{s},\epsilon}^k \rightharpoonup \theta_{\mathrm{s},\tau,M}^k \quad \text{in } H^1(\Gamma_{\mathrm{C}}), \\ &\mathbf{u}_\epsilon^k \rightharpoonup \mathbf{u}_{\tau,M}^k \quad \text{in } W^{1,\omega}(\Omega;\mathbb{R}^3), && \chi_\epsilon^k \rightharpoonup \chi_{\tau,M}^k \quad \text{in } H^2(\Gamma_{\mathrm{C}}), \quad \sigma_\epsilon^k \overset{*}{\rightharpoonup} \sigma_{\tau,M}^k \quad \text{in } L^\infty(\Gamma_{\mathrm{C}}), \end{aligned} \tag{4.29}$$

and the functions $(\theta_{\tau,M}^k, \mathbf{u}_{\tau,M}^k, \theta_{\mathrm{s},\tau,M}^k, \chi_{\tau,M}^k, \sigma_{\tau,M}^k)$ *fulfill*

$$\theta_{\tau,M}^k > 0 \quad \text{a.e. in } \Omega, \qquad \theta_{\mathrm{s},\tau,M}^k > 0 \quad \text{a.e. in } \Gamma_{\mathrm{C}}, \tag{4.30}$$

as well as

- *the discrete bulk temperature equation*

$$\begin{aligned} &\int_\Omega \frac{\theta_{\tau,M}^k - \theta_\tau^{k-1}}{\tau} v\,\mathrm{d}x - \int_\Omega \theta_{\tau,M}^k \mathrm{div}\left(\frac{\mathbf{u}_{\tau,M}^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) v\,\mathrm{d}x + \int_\Omega \alpha_M(\theta_{\tau,M}^k)\nabla\theta_{\tau,M}^k \nabla v\,\mathrm{d}x \\ &\quad + \int_{\Gamma_{\mathrm{C}}} k(\chi_\tau^{k-1})\theta_{\tau,M}^k(\theta_{\tau,M}^k - \theta_{\mathrm{s},\tau,M}^k) v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \mathfrak{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+(\theta_{\tau,M}^k)^2 v\,\mathrm{d}x \\ &\quad - \int_{\Gamma_{\mathrm{C}}} \mathfrak{J}[(\chi_\tau^{k-1})^+\theta_{\mathrm{s},\tau,M}^k]\,(\chi_\tau^{k-1})^+\theta_{\tau,M}^k v\,\mathrm{d}x \\ &= \int_\Omega \varepsilon\left(\frac{\mathbf{u}_{\tau,M}^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\,\varepsilon\left(\frac{\mathbf{u}_{\tau,M}^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) v\,\mathrm{d}x + \langle h_\tau^k, v\rangle_{H^1(\Omega)} \qquad \text{for all } v \in H^1(\Omega); \end{aligned} \tag{4.31a}$$

- *the discrete momentum balance equation* (4.4b)*, with* $\boldsymbol{\zeta}_{\tau,M}^k = \eta_\varsigma(\mathbf{u}_{\tau,M}^k \cdot \mathbf{n})\mathbf{n}$*;*
- *the discrete surface temperature equation*

$$\begin{aligned} &\int_{\Gamma_{\mathrm{C}}} \frac{\theta_{\mathrm{s},\tau,M}^k - \theta_{s,\tau}^{k-1}}{\tau} v\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} \theta_{\mathrm{s},\tau,M}^k \frac{\lambda(\chi_{\tau,M}^k) - \lambda(\chi_\tau^{k-1})}{\tau} v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \alpha_M(\theta_{\mathrm{s},\tau,M}^k)\nabla\theta_{\mathrm{s},\tau,M}^k \nabla v\,\mathrm{d}x \\ &= \int_{\Gamma_{\mathrm{C}}} \left|\frac{\chi_{\tau,M}^k - \chi_\tau^{k-1}}{\tau}\right|^2 v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} k(\chi_\tau^{k-1})(\theta_{\tau,M}^k - \theta_{\mathrm{s},\tau,M}^k)\theta_{\mathrm{s},\tau,M}^k v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \mathfrak{J}[(\chi_\tau^{k-1})^+\theta_{\tau,M}^k]\,(\chi_\tau^{k-1})^+\theta_{\mathrm{s},\tau,M}^k v\,\mathrm{d}x \\ &\quad - \int_{\Gamma_{\mathrm{C}}} \mathfrak{J}[(\chi_\tau^{k-1})^+](\chi_\tau^{k-1})^+(\theta_{\mathrm{s},\tau,M}^k)^2 v\,\mathrm{d}x + \langle \ell_\tau^k, v\rangle_{H^1(\Gamma_{\mathrm{C}})} \qquad \text{for all } v \in H^1(\Gamma_{\mathrm{C}}); \end{aligned} \tag{4.31b}$$

- *the discrete flow rule for the adhesion parameter* (4.4d) *a.e. in* Γ_{C}*.*

Finally,

$$\exists\, S_0>0\ \forall\, p\in[1,\infty)\ \exists\, \bar\tau_p>0\ \forall\, M>0\ \forall\, \tau\in(0,\bar\tau_p)\ \ \forall\, k\in\{1,\ldots,K_\tau\}\ :$$
$$\left\|\frac{1}{\theta^k_{\tau,M}}\right\|_{L^p(\Omega)}+\left\|\frac{1}{\theta^k_{\mathrm{s},\tau,M}}\right\|_{L^p(\Gamma_{\mathrm{C}})}\le S_0, \tag{4.31c}$$

Proof. Convergences (4.29) are an immediate consequence of estimates (4.26). Besides, from (4.14) it immediately follows that

$$\theta^k_{\tau,M}\ge 0\quad \text{a.e. in } \Omega,\qquad \theta^k_{\mathrm{s},\tau,M}\ge 0\quad \text{a.e. in } \Gamma_{\mathrm{C}}.$$

Furthermore, there exists $\mathsf{E}\in L^{\omega/(\omega-1)}(\Omega;\mathbb{R}^{3\times 3})$ such that

$$\left|\varepsilon\left(\tfrac{\mathbf{u}^k_\epsilon-\mathbf{u}^{k-1}_\tau}{\tau}\right)\right|^{\omega-2}\varepsilon\left(\tfrac{\mathbf{u}^k_\epsilon-\mathbf{u}^{k-1}_\tau}{\tau}\right)\rightharpoonup \mathsf{E}\qquad \text{in } L^{\omega/(\omega-1)}(\Omega;\mathbb{R}^{3\times 3})\,. \tag{4.32}$$

Taking into account that $W^{1,\omega}(\Omega;\mathbb{R}^3)$ continuously embeds (in the sense of traces) in $L^\infty(\Gamma_{\mathrm{C}})$, we also observe that $(\chi^k_\epsilon)^+\mathbf{u}^k_\epsilon\to(\chi^k_{\tau,M})^+\mathbf{u}^k_{\tau,M}$ and that, by Lemma 3.1, $\mathcal{J}[(\chi^k_\epsilon))^+]\,(\chi^k_\epsilon)^+\mathbf{u}^k_\epsilon\to\mathcal{J}[(\chi^k_{\tau,M})^+]\,\chi^k_{\tau,M}\mathbf{u}^k_{\tau,M}$ in $L^\infty(\Gamma_{\mathrm{C}})$. Since η_ς is Lipschitz continuous, we ultimately infer that

$$\boldsymbol{\zeta}^k_\epsilon=\eta_\varsigma(\mathbf{u}^k_\epsilon\cdot\mathbf{n})\mathbf{n}\to\eta_\varsigma(\mathbf{u}^k_{\tau,M}\cdot\mathbf{n})\mathbf{n}\doteq\boldsymbol{\zeta}^k_{\tau,M}\qquad \text{in } L^\infty(\Gamma_{\mathrm{C}}).$$

Hence, we readily pass to the limit in the discrete momentum balance (4.13b) and conclude that the quintuple $(\theta^k_{\tau,M},\mathbf{u}^k_{\tau,M},\theta^k_{\mathrm{s},\tau,M},\boldsymbol{\zeta}^k_{\tau,M},\mathsf{E})$ fulfills

$$\begin{aligned}&\mathsf{v}\left(\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau},\mathbf{v}\right)+\rho\int_\Omega \mathsf{E}:\varepsilon(\mathbf{v})\,\mathrm{d}x+\mathsf{e}\left(\mathbf{u}^k_{\tau,M},\mathbf{v}\right)+\int_\Omega\theta^k_{\tau,M}\mathrm{div}(\mathbf{v})\,\mathrm{d}x+\int_{\Gamma_{\mathrm{C}}}(\chi^k_{\tau,M})^+\mathbf{u}^k_{\tau,M}\mathbf{v}\,\mathrm{d}x\\&\quad+\int_{\Gamma_{\mathrm{C}}}\boldsymbol{\zeta}^k_{\tau,M}\mathbf{v}\,\mathrm{d}x+\int_{\Gamma_{\mathrm{C}}}\mathcal{J}[(\chi^k_{\tau,M})^+]\,(\chi^k_{\tau,M})^+\mathbf{u}^k_{\tau,M}\mathbf{v}\,\mathrm{d}x=\langle\mathbf{F}^k,\mathbf{v}\rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)}\qquad\text{for all } \mathbf{v}\in W^{1,\omega}_{\mathrm{D}}(\Omega;\mathbb{R}^3)\,.\end{aligned}$$

Furthermore, testing (4.13b) by $\frac{\mathbf{u}^k_\epsilon-\mathbf{u}^{k-1}_\tau}{\tau}$ and passing to the limit in the equation we readily show that

$$\begin{aligned}&\limsup_{\epsilon\downarrow 0}\left(\rho\int_\Omega\left|\varepsilon\left(\tfrac{\mathbf{u}^k_\epsilon-\mathbf{u}^{k-1}_\tau}{\tau}\right)\right|^\omega\right)\mathrm{d}x\\&\le-\mathsf{v}\left(\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau},\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau}\right)-\mathsf{e}\left(\mathbf{u}^k_{\tau,M},\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau},\right)-\int_\Omega\theta^k_{\tau,M}\mathrm{div}(\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau})\,\mathrm{d}x\\&\quad-\int_{\Gamma_{\mathrm{C}}}(\chi^k_{\tau,M})^+\mathbf{u}^k_{\tau,M}\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau}\,\mathrm{d}x-\int_{\Gamma_{\mathrm{C}}}\mathcal{J}[(\chi^k_{\tau,M})^+]\,(\chi^k_{\tau,M})^+\mathbf{u}^k_{\tau,M}\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau}\,\mathrm{d}x-\langle\mathbf{F}^k,\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau}\rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)}\\&\quad-\int_{\Gamma_{\mathrm{C}}}\boldsymbol{\zeta}^k_{\tau,M}\cdot\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau}\,\mathrm{d}x\\&=\rho\int_\Omega\mathsf{E}\cdot\varepsilon(\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau})\,\mathrm{d}x\end{aligned}$$

Thus, by standard results on the theory of maximal monotone operators (cf., e.g., [1, Lemma 1.3, p. 42]), we infer that

$$\mathsf{E}=\left|\varepsilon\left(\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau}\right)\right|^{\omega-2}\varepsilon\left(\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau}\right).$$

A fortiori, we conclude that

$$\varepsilon\left(\tfrac{\mathbf{u}^k_\epsilon-\mathbf{u}^{k-1}_\tau}{\tau}\right)\to\varepsilon\left(\tfrac{\mathbf{u}^k_{\tau,M}-\mathbf{u}^{k-1}_\tau}{\tau}\right)\qquad\text{in } L^\omega(\Omega;\mathbb{R}^{3\times 3})\,. \tag{4.33}$$

This completes the limit passage in (4.13b), leading to the discrete momentum balance (4.4b).

We now address the limit passage in the discrete truncated bulk heat equation (4.13a). First of all, we observe that, as $\epsilon\downarrow 0$,

$$\mathcal{T}_\epsilon(\theta^k_\epsilon)\to(\theta^k_{\tau,M})^+=\theta^k_{\tau,M}\qquad \text{in } L^q(\Omega) \text{ for every } q\in[1,6), \tag{4.34}$$

and that the traces of $\mathcal{T}_\epsilon(\theta^k_\epsilon)$ strongly converge to the trace of $\theta^k_{\tau,M}$ in $L^q(\Gamma_{\mathrm{C}})$ for every $q\in[1,4)$. Furthermore, taking into account that $\|(\chi^{k-1}_\tau)^+\|_{L^\infty(\Gamma_{\mathrm{C}})}\le C\|\chi^{k-1}_\tau\|_{H^2(\Gamma_{\mathrm{C}})}\le C$, $\|k(\chi^{k-1}_\tau)\|_{L^\infty(\Gamma_{\mathrm{C}})}\le C$, that $\|\mathcal{J}[(\chi_\tau^{k-1})^+\theta_{\mathrm{s},\epsilon}^k]\|_{L^\infty(\Gamma_{\mathrm{C}})} \leq C\|(\chi_\tau^{k-1})^+\theta_{\mathrm{s},\epsilon}^k\|_{L^1(\Gamma_{\mathrm{C}})} \leq C$ by Lemma 3.1, and recalling (4.33), by a comparison in (4.13a) we see that

$$\exists\, C>0\ \forall\, \epsilon>0 \quad \sup_{v\in H^1(\Omega)} \left| \int_\Omega \alpha_M(\theta_\epsilon^k)\nabla\theta_\epsilon^k\cdot\nabla v\,\mathrm{d}x \right| \leq C. \tag{4.35}$$

The above estimate can be rephrased, in terms of the operator $\mathcal{A}_M$ from (4.17b), as $\sup_{\epsilon>0}\|\mathcal{A}_M(\theta_\epsilon^k)\|_{H^1(\Omega)^*} \leq C$. We combine this with the facts that $\nabla\theta_\epsilon^k \rightharpoonup \nabla\theta_{\tau,M}^k$ in $L^2(\Omega;\mathbb{R}^3)$ and that $\alpha_M(\theta_\epsilon^k) \to \alpha_M(\theta_{\tau,M}^k)$ in $L^q(\Omega)$ for every $1\leq q<\infty$ (since $\theta_\epsilon^k\to\theta_{\tau,M}^k$ a.e. in Ω and the function $\alpha_M:\mathbb{R}\to(0,+\infty)$ is bounded). Therefore, $\mathcal{A}_M(\theta_\epsilon^k)\rightharpoonup\mathcal{A}_M(\theta_{\tau,M}^k)$ in $W^{1,s}(\Omega)^*$ for every $s>2$. Thanks to (4.35), we conclude that

$$\mathcal{A}_M(\theta_\epsilon^k)\rightharpoonup\mathcal{A}_M(\theta_{\tau,M}^k) \qquad \text{in } H^1(\Omega)^*. \tag{4.36}$$

We also use the strong convergences

$$\begin{aligned}
&\theta_\epsilon^k \mathrm{div}\left(\tfrac{\mathbf{u}_\epsilon^k-\mathbf{u}_\tau^{k-1}}{\tau}\right) \to \theta_{\tau,M}^k\mathrm{div}\left(\tfrac{\mathbf{u}_{\tau,M}^k-\mathbf{u}_\tau^{k-1}}{\tau}\right) && \text{in } L^q(\Omega) \text{ for all } q\in[1,\tfrac32),\\
&k(\chi_\tau^{k-1})\mathcal{T}_\epsilon(\theta_\epsilon^k)(\theta_\epsilon^k-\theta_{\mathrm{s},\epsilon}^k) \to k(\chi_\tau^{k-1})\theta_{\tau,M}^k(\theta_{\tau,M}^k-\theta_{\mathrm{s},\tau,M}^k) && \text{in } L^q(\Gamma_{\mathrm{C}}) \text{ for all } q\in[1,2),\\
&\mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+\theta_\epsilon^k\mathcal{T}_\epsilon(\theta_\epsilon^k) \to \mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+(\theta_{\tau,M}^k)^2 && \text{in } L^q(\Gamma_{\mathrm{C}}) \text{ for all } q\in[1,2),\\
&\mathcal{J}[(\chi_\tau^{k-1})^+\theta_{\mathrm{s},\epsilon}^k]\,(\chi_\tau^{k-1})^+\mathcal{T}_\epsilon(\theta_\epsilon^k) \to \mathcal{J}[(\chi_\tau^{k-1})^+\theta_{\mathrm{s},\tau,M}^k]\,(\chi_\tau^{k-1})^+\theta_{\tau,M}^k && \text{in } L^q(\Gamma_{\mathrm{C}}) \text{ for all } q\in[1,4),\\
&\varepsilon\left(\tfrac{\mathbf{u}_\epsilon^k-\mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\tfrac{\mathbf{u}_\epsilon^k-\mathbf{u}_\tau^{k-1}}{\tau}\right) \to \varepsilon\left(\tfrac{\mathbf{u}_{\tau,M}^k-\mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\tfrac{\mathbf{u}_{\tau,M}^k-\mathbf{u}_\tau^{k-1}}{\tau}\right) && \text{in } L^{\omega/2}(\Omega),
\end{aligned} \tag{4.37}$$

due to convergences (4.29), to the properties of $\mathcal{J}$ (cf. Lemma 3.1), and to the previously proved (4.33). Combining (4.36) and (4.37) we pass to the limit in (4.13a), thus obtaining the discrete bulk heat equation (4.31a).

The limit passage in the discrete flow rule (4.13d) as $\epsilon\downarrow0$ is an easy consequence of convergences (4.29), which in particular imply that $\chi_\varepsilon^k\to\chi_{\tau,M}^k$ strongly in $L^\infty(\Gamma_{\mathrm{C}})$. Hence, by the Lipschitz continuity of β_ς and γ_ν', we conclude that $\beta_\varsigma(\chi_\varepsilon^k)\to\beta_\varsigma(\chi_\tau^k)$ and $\gamma_\nu'(\chi_\varepsilon^k)\to\gamma_\nu'(\chi_\tau^k)$ in $L^\infty(\Gamma_{\mathrm{C}})$. Furthermore, $\sigma_\varepsilon^k\rightharpoonup^*\sigma_{\tau,M}^k$ in $L^\infty(\Gamma_{\mathrm{C}})$ and, by the strong weak closedness (in the sense of graphs) of the maximal monotone operator (induced by) $\partial\varphi$, we readily conclude that

$$\sigma_{\tau,M}^k\in\partial\varphi(\chi_{\tau,M}^k) \qquad \text{a.e. in}\,\Gamma_{\mathrm{C}}\,.$$

Hence, the triple $(\theta_{\mathrm{s},\tau,M}^k,\chi_{\tau,M}^k,\sigma_{\tau,M}^k)$ fulfills the discrete flow rule (4.4d).

Finally, we address the passage to the limit in the discrete truncated surface temperature equation (4.13c): it is based on the fact that

$$\mathcal{T}_\epsilon(\theta_{\mathrm{s},\epsilon}^k)\to(\theta_{\mathrm{s},\tau,M}^k)^+=\theta_{\mathrm{s},\tau,M}^k \qquad \text{in } L^q(\Gamma_{\mathrm{C}}) \text{ for every } q\in[1,\infty), \tag{4.38}$$

on the convergence

$$\mathcal{A}_M(\theta_{\mathrm{s},\epsilon}^k)\rightharpoonup\mathcal{A}_M(\theta_{\mathrm{s},\tau,M}^k) \qquad \text{in } H^1(\Gamma_{\mathrm{C}})^*, \tag{4.39}$$

(which can be inferred by the same arguments as (4.36)), on the analogues of convergences (4.37) for the terms on the right-hand side of (4.13c), and on the fact that

$$\left|\frac{\chi_\epsilon^k-\chi_\tau^{k-1}}{\tau}\right|^{\omega-2}\frac{\chi_\epsilon^k-\chi_\tau^{k-1}}{\tau} \to \left|\frac{\chi_{\tau,M}^k-\chi_\tau^{k-1}}{\tau}\right|^{\omega-2}\frac{\chi_{\tau,M}^k-\chi_\tau^{k-1}}{\tau} \qquad \text{in } L^\infty(\Gamma_{\mathrm{C}})$$

as $\chi_\varepsilon^k\to\chi_{\tau,M}^k$ strongly in $L^\infty(\Gamma_{\mathrm{C}})$. We thus obtain the discrete surface heat equation (4.4c).

Finally, to prove the strict positivity (4.30) and estimate (4.31c), we combine estimate (4.15) with convergences (4.34) & (4.38). Let us just detail the argument for $\theta_{\tau,M}^k$ (the positivity property for $\theta_{\mathrm{s},\tau,M}^k$ follows analogously). On the one hand, from (4.15) we infer that

$$\int_\Omega \liminf_{\epsilon\downarrow0}\frac{1}{\mathcal{T}_\epsilon(\theta_\epsilon^k)}\,\mathrm{d}x \leq \sup_{\epsilon>0}\int_\Omega\frac{1}{\mathcal{T}_\epsilon(\theta_\epsilon^k)}\,\mathrm{d}x\leq S_0,$$

so that $\liminf_{\epsilon\downarrow0}\frac{1}{\mathcal{T}_\epsilon(\theta_\epsilon^k(x))}\,\mathrm{d}x<+\infty$ a.e. in Ω. On the other hand, from (4.34) we have that

$$\frac{1}{\mathcal{T}_\epsilon(\theta_\epsilon^k(x))}\longrightarrow\begin{cases}\frac{1}{\theta_{\tau,M}^k(x)} & \text{if } \theta_{\tau,M}^k(x)>0,\\ +\infty & \text{otherwise}\end{cases} \qquad \text{for a.a. } x\in\Omega.$$

Therefore, we conclude that $\theta_{\tau,M}^k > 0$ a.e. in Ω, and estimate (4.31c) follows by lower semicontinuity. This finishes the proof. $\square$

4.3. **Proof of Proposition 4.3.** We shall carry out the proof of Proposition 4.3 by passing to the limit as $M \to +\infty$, for fixed $\tau > 0$, and $k \in \{1, \ldots, K_\tau\}$, in system (4.31). In what follows, we shall suppose that $M \in \mathbb{N}\setminus\{0\}$. For simplicity, we shall drop the parameter τ and just denote by $(\theta_M^k, \mathbf{u}_M^k, \theta_{\mathrm{s},M}^k, \chi_M^k)_M$, with associated selections $\sigma_M^k \in \partial\varphi(\chi_M^k)$ a.e. in Γ_{C}, the sequence of solutions to system (4.31). We split the proof of the limit passage in some steps.

Preliminarily, we will need the following result (whose proof is left to the reader) collecting properties of the primitives of α; observe that (4.40) is a consequence of (2.13).

Lemma 4.11. *Define*

$$\widehat{\widehat{\alpha}} : \mathbb{R}^+ \to \mathbb{R}^+ \qquad \widehat{\widehat{\alpha}}(r) := \int_0^r \widehat{\alpha}(s)\,\mathrm{d}s,$$

(with $\widehat{\alpha}$ from (2.14)*). The function $\widehat{\alpha}$ is (strictly) increasing and thus $\widehat{\widehat{\alpha}}$ is (strictly) convex. Furthermore,*

$$\begin{aligned} & c_0\left(r + \frac{r^{\mu+1}}{\mu+1}\right) \leq \widehat{\alpha}(r) \leq c_1\left(r + \frac{r^{\mu+1}}{\mu+1}\right), \\ & c_0\left(\frac{r^2}{2} + \frac{r^{\mu+2}}{(\mu+2)(\mu+1)}\right) \leq \widehat{\widehat{\alpha}}(r) \leq c_1\left(\frac{r^2}{2} + \frac{r^{\mu+2}}{(\mu+2)(\mu+1)}\right). \end{aligned} \tag{4.40}$$

The functions

$$\mathbb{R}^+ \ni r \mapsto \widehat{\alpha}(\mathcal{T}_M(r)) \qquad \text{and} \qquad \mathbb{R}^+ \ni r \mapsto r\widehat{\alpha}(\mathcal{T}_M(r)) \ \text{are non-decreasing.} \tag{4.41}$$

Finally, the function

$$\widehat{\widehat{\alpha}}_M : \mathbb{R}^+ \to \mathbb{R}^+ \qquad \widehat{\widehat{\alpha}}_M(r) := \int_0^r \widehat{\alpha}(\mathcal{T}_M(s))\,\mathrm{d}s \tag{4.42}$$

is convex, and

$$\exists\, C_1',\, C_2' > 0 \ \ \forall\, M > 0 \ \ \forall\, r \in \mathbb{R}^+ : \quad \widehat{\alpha}(\mathcal{T}_M(r)) \leq C_1'\widehat{\widehat{\alpha}}_M(r) + C_2'. \tag{4.43}$$

An analogous estimate holds with $\widehat{\alpha}$ in place of $\widehat{\alpha} \circ \mathcal{T}_M$ and $\widehat{\widehat{\alpha}}$ in place of $\widehat{\widehat{\alpha}}_M$.

Proof of Proposition 4.3. *Step 1: a priori estimates.* Since the constant in estimate (4.26) was independent of the parameter M, by virtue of convergences (4.29) and lower semicontinuity arguments we immediately conclude that

$$\sup_{M>0}\left(\|\theta_M^k\|_{H^1(\Omega)} + \|\theta_{\mathrm{s},M}^k\|_{H^1(\Gamma_{\mathrm{C}})} + \|\mathbf{u}_M^k\|_{W^{1,\omega}(\Omega;\mathbb{R}^3)} + \|\chi_M^k\|_{H^2(\Gamma_{\mathrm{C}})} + \|\sigma_M^k\|_{L^\infty(\Gamma_{\mathrm{C}})}\right) \leq S_1. \tag{4.44}$$

Next, we test (4.31a) by $\tau\widehat{\alpha}(\mathcal{T}_M(\theta_M^k))$, (4.31b) by $\tau\widehat{\alpha}(\mathcal{T}_M(\theta_{\mathrm{s},M}^k))$ and add the resulting equations (it is a standard matter to check that $\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)) \in H^1(\Omega)$ and $\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)) \in H^1(\Gamma_{\mathrm{C}})$). By convexity of $\widehat{\widehat{\alpha}}_M$, we have that

$$\begin{aligned} & \int_\Omega \widehat{\widehat{\alpha}}_M(\theta_M^k)\,\mathrm{d}x - \int_\Omega \widehat{\widehat{\alpha}}_M(\theta_\tau^{k-1})\,\mathrm{d}x \leq \int_\Omega (\theta_M^k - \theta_\tau^{k-1})\widehat{\alpha}(\mathcal{T}_M(\theta_M^k))\,\mathrm{d}x, \\ & \int_{\Gamma_{\mathrm{C}}} \widehat{\widehat{\alpha}}_M(\theta_{\mathrm{s},M}^k)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} \widehat{\widehat{\alpha}}_M(\theta_{s,\tau}^{k-1})\,\mathrm{d}x \leq \int_{\Gamma_{\mathrm{C}}} (\theta_{\mathrm{s},M}^k - \theta_{s,\tau}^{k-1})\widehat{\alpha}(\mathcal{T}_M(\theta_{\mathrm{s},M}^k))\,\mathrm{d}x. \end{aligned}$$

Using that

$$\nabla(\widehat{\alpha}(\mathcal{T}_M(\theta_M^k))) = \alpha_M(\theta_M^k)\nabla(\mathcal{T}_M(\theta_M^k)) \tag{4.45}$$

(cf., e.g., [23]), it is immediate to check that

$$\tau\int_\Omega \alpha_M(\theta_M^k)\nabla\theta_M^k\nabla(\hat{\alpha}(\mathcal{T}_M(\theta_M^k)))\,\mathrm{d}x = \tau\int_\Omega \left|\nabla(\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)))\right|^2\,\mathrm{d}x$$

and we deal with the term $\tau\int_{\Gamma_{\mathrm{C}}} \alpha_M(\theta_{\mathrm{s},M}^k)\nabla\theta_{\mathrm{s},M}^k\nabla(\widehat{\alpha}(\mathcal{T}_M(\theta_{\mathrm{s},M}^k)))\,\mathrm{d}x$ analogously. By the second monotonicity property in (4.41), we have that

$$\tau\int_{\Gamma_{\mathrm{C}}} k(\chi_\tau^{k-1})(\theta_M^k - \theta_{\mathrm{s},M}^k)(\theta_M^k\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)) - \theta_{\mathrm{s},M}^k\widehat{\alpha}(\mathcal{T}_M(\theta_{\mathrm{s},M}^k)))\,\mathrm{d}x \geq 0,$$

while with calculations completely analogous to those for (3.17) we can check that

$$
\begin{aligned}
&-\tau\int_{\Gamma_C}\widehat{\alpha}(\mathcal{T}_M(\theta_M^k))\left(\mathcal{J}[(\chi_\tau^{k-1})^+]\,(\chi_\tau^{k-1})^+(\theta_M^k)^2-\mathcal{J}[(\chi_\tau^{k-1})^+\theta_{s,M}^k]\,(\chi_\tau^{k-1})^+\theta_M^k\right)\,\mathrm{d}x\\
&\quad+\tau\int_{\Gamma_C}\widehat{\alpha}(\mathcal{T}_M(\theta_{s,M}^k))\left(\mathcal{J}[\chi_\tau^{k-1}\theta_M^k]\chi_\tau^{k-1}\theta_{s,M}^k-\mathcal{J}[(\chi_\tau^{k-1})^+](\chi_\tau^{k-1})^+(\theta_{s,M}^k)^2\right)\,\mathrm{d}x\geq 0\,.
\end{aligned}
$$

Taking into account the above calculations, we end up with the following estimate

$$
\begin{aligned}
&\int_\Omega\widehat{\widehat{\alpha}}_M(\theta_M^k)\,\mathrm{d}x+\int_{\Gamma_C}\widehat{\widehat{\alpha}}_M(\theta_{s,M}^k)\,\mathrm{d}x+\tau\|\nabla(\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)))\|_{L^2(\Omega)}^2+\tau\|\nabla(\widehat{\alpha}(\mathcal{T}_M(\theta_{s,M}^k)))\|_{L^2(\Gamma_C)}^2\\
&\leq\int_\Omega\widehat{\widehat{\alpha}}_M(\theta_\tau^{k-1})\,\mathrm{d}x+\int_{\Gamma_C}\widehat{\widehat{\alpha}}_M(\theta_{s,\tau}^{k-1})\,\mathrm{d}x+\tau\int_\Omega\theta_M^k\widehat{\alpha}(\mathcal{T}_M(\theta_M^k))\mathrm{div}\left(\tfrac{\mathbf{u}_M^k-\mathbf{u}_\tau^{k-1}}{\tau}\right)\,\mathrm{d}x\\
&\quad+\tau\int_\Omega\varepsilon\left(\tfrac{\mathbf{u}_M^k-\mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\,\varepsilon\left(\tfrac{\mathbf{u}_M^k-\mathbf{u}_\tau^{k-1}}{\tau}\right)\widehat{\alpha}(\mathcal{T}_M(\theta_M^k))\,\mathrm{d}x\\
&\quad+\tau\,\langle h_\tau^k,\widehat{\alpha}(\mathcal{T}_M(\theta_M^k))\rangle_{H^1(\Omega)}+\tau\,\langle\ell_\tau^k,\widehat{\alpha}(\mathcal{T}_M(\theta_{s,M}^k))\rangle_{H^1(\Gamma_C)}\\
&\quad+\tau\int_{\Gamma_C}\theta_{s,M}^k\widehat{\alpha}(\mathcal{T}_M(\theta_{s,M}^k))\frac{\lambda(\chi_M^k)-\lambda(\chi_\tau^{k-1})}{\tau}v\,\mathrm{d}x+\tau\int_{\Gamma_C}\left|\frac{\chi_{\tau,M}^k-\chi_\tau^{k-1}}{\tau}\right|^2\widehat{\alpha}(\mathcal{T}_M(\theta_{s,M}^k))\,\mathrm{d}x\\
&\doteq I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8\,.
\end{aligned}
\tag{4.46}
$$

Now, taking into account (4.40), it is not difficult to check that

$$
I_1+I_2\leq C\left(1+\|\theta_\tau^{k-1}\|_{L^{\mu+2}(\Omega)}^{\mu+2}+\|\theta_\tau^{k-1}\|_{L^{\mu+2}(\Gamma_C)}^{\mu+2}\right)\leq C,
$$

since, by assumption (cf. (4.6)), $\theta_\tau^{k-1}\in L^{\mu+2}(\Omega)$ and $\theta_{s,\tau}^{k-1}\in L^{\mu+2}(\Gamma_C)$. In order to estimate the ensuing terms, we will use that

$$
\begin{aligned}
\|\widehat{\alpha}(\mathcal{T}_M(\theta_M^k))\|_{H^1(\Omega)}&\overset{(1)}{\leq}C\left(\|\nabla(\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)))\|_{L^2(\Omega)}+\|\widehat{\alpha}(\mathcal{T}_M(\theta_M^k))\|_{L^1(\Omega)}\right)\\
&\overset{(2)}{\leq}C'\left(\|\nabla(\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)))\|_{L^2(\Omega)}+\|\widehat{\widehat{\alpha}}_M(\theta_M^k)\|_{L^1(\Omega)}+1\right),
\end{aligned}
\tag{4.47}
$$

where (1) follows from the Poincaré inequality, and (2) from (4.43). Clearly, an analogous estimate holds for $\|\widehat{\alpha}(\mathcal{T}_M(\theta_{s,M}^k))\|_{H^1(\Gamma_C)}$. Hence

$$
\begin{aligned}
|I_3|&\leq C\tau\|\theta_M^k\|_{L^4(\Omega)}\|\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)))\|_{L^2(\Omega)}\left\|\mathrm{div}\left(\tfrac{\mathbf{u}_M^k-\mathbf{u}_\tau^{k-1}}{\tau}\right)\right\|_{L^4(\Omega)}\\
&\overset{(3)}{\leq}C'\tau\|\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)))\|_{L^2(\Omega)}\\
&\overset{(4)}{\leq}\frac{\tau}{4}\|\nabla(\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)))\|_{L^2(\Omega)}^2+\overline{C}_1\tau\|\widehat{\widehat{\alpha}}_M(\theta_M^k)\|_{L^1(\Omega)}+C,
\end{aligned}
\tag{4.48}
$$

where for (3) we have exploited the previously observed estimate (4.44), while (4) follows from (4.47) and Young's inequality. With analogous calculations, again taking into account (4.44) and combining it with (2.16) and (2.18a)–(2.18b), we easily conclude that

$$
\begin{aligned}
I_4+I_5+I_6+I_7+I_8\leq&\frac{\tau}{4}\|\nabla(\widehat{\alpha}(\mathcal{T}_M(\theta_M^k)))\|_{L^2(\Omega)}^2\frac{\tau}{4}\|\nabla(\widehat{\alpha}(\mathcal{T}_M(\theta_{s,M}^k)))\|_{L^2(\Gamma_C)}^2\\
&+\overline{C}_2\tau\|\widehat{\widehat{\alpha}}_M(\theta_M^k)\|_{L^1(\Omega)}+\overline{C}_2\tau\|\widehat{\widehat{\alpha}}_M(\theta_{s,M}^k)\|_{L^1(\Gamma_C)}+C\,.
\end{aligned}
\tag{4.49}
$$

Hence, we choose $\tau>0$ sufficiently small such that $\tau(\overline{C}_1+\overline{C}_2)<\frac{1}{2}$, so that the terms with $\|\widehat{\widehat{\alpha}}_M(\theta_M^k)\|_{L^1(\Omega)}$ and $\|\widehat{\widehat{\alpha}}_M(\theta_{s,M}^k)\|_{L^1(\Gamma_C)}$ can be absorbed into the right-hand side of (4.46). Them, combining (4.46) with (4.48) and (4.49), and again taking into account (4.47), we infer that

$$
\sup_{M>0}\left(\|\widehat{\widehat{\alpha}}_M(\theta_M^k)\|_{L^1(\Omega)}+\|\widehat{\widehat{\alpha}}_M(\theta_{s,M}^k)\|_{L^1(\Gamma_C)}+\|\widehat{\alpha}(\mathcal{T}_M(\theta_M^k))\|_{H^1(\Omega)}+\|\widehat{\alpha}(\mathcal{T}_M(\theta_{s,M}^k))\|_{H^1(\Gamma_C)}\right)\leq S_2\,.
\tag{4.50}
$$

In particular, in view of (4.40) we conclude that

$$
\exists\,C>0\ \ \forall\,M>0:\quad\|\theta_M^k\|_{L^{\mu+2}(\Omega)}+\|\theta_{s,M}^k\|_{L^{\mu+2}(\Gamma_C)}\leq C\,.
$$

Step 2: limit passage as $M \uparrow \infty$. In view of estimates (4.44), there exist a (not relabeled) subsequence and functions $(\theta_\tau^k, \mathbf{u}_\tau^k, \theta_{\mathrm{s},\tau}^k, \chi_\tau^k, \sigma_\tau^k)$ that

$$\begin{aligned} &\theta_M^k \rightharpoonup \theta_\tau^k \quad \text{in } H^1(\Omega) \cap L^{\mu+2}(\Omega), \quad \theta_{\mathrm{s},M}^k \rightharpoonup \theta_{\mathrm{s},\tau}^k \quad \text{in } H^1(\Gamma_{\mathrm{C}}), \\ &\mathbf{u}_M^k \rightharpoonup \mathbf{u}_\tau^k \quad \text{in } W^{1,\omega}(\Omega;\mathbb{R}^3), \qquad \chi_M^k \rightharpoonup \chi_\tau^k \quad \text{in } H^2(\Gamma_{\mathrm{C}}), \quad \sigma_M^k \rightharpoonup^* \sigma_\tau^k \quad \text{in } L^\infty(\Gamma_{\mathrm{C}}). \end{aligned} \tag{4.51}$$

With the very same arguments as in the proof of Lemma 4.10, from estimate (4.31c) we conclude that $\theta_\tau^k > 0$ a.e. in Ω and $\theta_{\mathrm{s},\tau}^k > 0$ a.e. in Γ_{C} and the validity of estimate (4.7).

The limit passage in the momentum balance and in the flow rule for the adhesion parameter in system (4.31) follows by the very same arguments as in the proof of Lemma 4.10. In this way, we conclude that the quintuple $(\theta_\tau^k, \mathbf{u}_\tau^k, \theta_{\mathrm{s},\tau}^k, \chi_\tau^k, \sigma_\tau^k)$ solve (4.4b) and (4.4d).

Therefore, repeating the arguments in the proof of Lemma 4.10, we pass to the limit in the discrete bulk heat equation (4.31a) and in the surface heat equations (4.31b). We only detail the passage to the limit in the term featuring the operator $\mathcal{A}(\theta_M^k) : H^1(\Omega) \to H^1(\Omega)^*$ defined by $\langle \mathcal{A}(\theta_M^k), v \rangle_{H^1(\Omega)} := \int_\Omega \alpha_M(\theta_M^k) \nabla \theta_M^k \cdot \nabla v \,\mathrm{d}x$. On the one hand, the sequence of operators $(\mathcal{A}(\theta_M^k))_M \subset H^1(\Omega)^*$ is bounded, by comparison in (4.31a). On the other hand, we observe that $\mathcal{T}_M(\theta_M^k) \to \theta_\tau^k$ a.e. in Ω and hence $\alpha(\mathcal{T}_M(\theta_M^k)) \to \alpha(\theta_\tau^k)$ a.e. in Ω, whereas estimates (4.50), combined with the growth properties of $\widehat{\alpha}$, guarantee that $(\mathcal{T}_M(\theta_M^k))_M$ is bounded in $L^{6(\mu+1)}(\Omega)$, so that $\alpha_M(\theta_M^k) = \alpha(\mathcal{T}_M(\theta_M^k)) \to \alpha(\theta_\tau^k)$ in $L^{6s}(\Omega)$ for every $1 \le s < \frac{\mu+1}{\mu}$. Therefore, the sequence $(\alpha_M(\theta_M^k)\nabla\theta_M^k)_M$ weakly converges to $\alpha(\theta_\tau^k)\nabla\theta_\tau^k$ in $L^{3/2}(\Omega;\mathbb{R}^3)$. This is sufficient to conclude that

$$\mathcal{A}(\theta_M^k) \rightharpoonup \mathcal{A}_M(\theta_\tau^k) \qquad \text{in } H^1(\Omega)^*. \tag{4.52}$$

With the same argument we perform the passagge to the limit in the analogous term for $\theta_{\mathrm{s},\tau}^k$. All in all, we deduce that $(\theta_\tau^k, \mathbf{u}_\tau^k, \theta_{\mathrm{s},\tau}^k, \chi_\tau^k)$ fulfill the discrete bulk and surface heat equations (4.4a) and (4.4c).

Step 3: proof of (4.9). The total energy inequality (4.9) follows by repeating the very same calculations as for (4.22).

This finishes the proof of Proposition 4.3. □

5. Existence for the regularized system

In this section we address the limit passage in the discrete system (4.4) (formulated in terms of suitable interpolants of the discrete solutions, cf. (5.4) below) as the time step $\tau \downarrow 0$, and in this way we shall conclude the existence of (weak) solutions to the Cauchy problem for the regularized system (3.69). Prior to that, let us set up some notation.

Notation and preliminaries. For a given K_τ-uple of discrete elements $(\mathfrak{h}_\tau^k)_{k=0}^{K_\tau} \subset \mathbf{B}$, with $\mathbf{B}$ a given Banach space, we introduce the (left-continuous and right-continuous) piecewise constant, and the piecewise linear interpolants $\overline{\mathfrak{h}}_\tau$, $\underline{\mathfrak{h}}_\tau$, $\widehat{\mathfrak{h}}_\tau : [0,T] \to X$ defined by $\overline{\mathfrak{h}}_\tau(0) = \underline{\mathfrak{h}}_\tau(0) = \widehat{\mathfrak{h}}_\tau(0) := \mathfrak{h}_\tau^0$ and by

$$\begin{aligned} &\overline{\mathfrak{h}}_\tau(t) := \mathfrak{h}_\tau^k \text{ for } t \in (t_\tau^{k-1}, t_\tau^k], \qquad \underline{\mathfrak{h}}_\tau(t) := \mathfrak{h}_\tau^{k-1} \text{ for } t \in [t_\tau^{k-1}, t_\tau^k), \\ &\widehat{\mathfrak{h}}_\tau(t) := \frac{t - t_\tau^{k-1}}{\tau} \mathfrak{h}_\tau^k + \frac{t_\tau^k - t}{\tau} \mathfrak{h}_\tau^{k-1} \text{ for } t \in (t_\tau^{k-1}, t_\tau^k]\,. \end{aligned} \tag{5.1}$$

For later use, we also recall that

$$\|\overline{\mathfrak{h}}_\tau - \widehat{\mathfrak{h}}_\tau\|_{L^2(0,T;\mathbf{B})} \le \|\overline{\mathfrak{h}}_\tau - \underline{\mathfrak{h}}_\tau\|_{L^2(0,T;\mathbf{B})} \quad \le \quad \tau \|\partial_t \widehat{\mathfrak{h}}_\tau\|_{L^2(0,T;\mathbf{B})}, \tag{5.2a}$$

$$\|\overline{\mathfrak{h}}_\tau - \widehat{\mathfrak{h}}_\tau\|_{L^\infty(0,T;\mathbf{B})} \le \|\overline{\mathfrak{h}}_\tau - \underline{\mathfrak{h}}_\tau\|_{L^\infty(0,T;\mathbf{B})} \quad \le \quad \sqrt{\tau} \|\partial_t \widehat{\mathfrak{h}}_\tau\|_{L^2(0,T;\mathbf{B})}, \tag{5.2b}$$

as well as the well-known discrete by-part integration formula

$$\sum_{i=1}^{j} \tau \, \langle \mathfrak{l}_\tau^i, \tfrac{\mathfrak{h}_\tau^i - \mathfrak{h}_\tau^{i-1}}{\tau} \rangle_{\mathbf{B}} = \langle \mathfrak{l}_\tau^j, \mathfrak{h}_\tau^j \rangle_{\mathbf{B}} - \langle \mathfrak{l}_\tau^0, \mathfrak{h}_\tau^0 \rangle_{\mathbf{B}} - \sum_{i=1}^{j} \tau \, \langle \tfrac{\mathfrak{l}_\tau^k - \mathfrak{l}_\tau^{k-1}}{\tau}, \mathfrak{h}_\tau^i \rangle_{\mathbf{B}} \tag{5.3}$$

for all K_τ-uples $(\mathfrak{h}_\tau^k)_{k=1}^{K_\tau} \subset \mathbf{B}$, $(\mathfrak{l}_\tau^k)_{k=1}^{K_\tau} \subset \mathbf{B}^*$.

Approximate solutions. With $\mathfrak{h} \in \{\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi, \sigma, \mathbf{F}, h, \ell\}$, we thus obtain the piecewise constant and linear interpolants of the discrete solution quadruples $(\theta_\tau^k, \mathbf{u}_\tau^k, \theta_{\mathrm{s},\tau}^k, \chi_\tau^k, \sigma_\tau^k)$ and of the discrete data $(\mathbf{F}_\tau^k, h_\tau^k, \ell_\tau^k)$. Finally, we also introduce the piecewise constant interpolants $\overline{\mathsf{t}}_\tau : [0,T] \to [0,T]$ and $\underline{\mathsf{t}}_\tau : [0,T] \to [0,T]$ associated with the partition $\mathscr{P}_\tau = (t_\tau^k)_{k=1}^{N_\tau}$ and defined by $\overline{\mathsf{t}}_\tau(0) = \underline{\mathsf{t}}_\tau(0) := 0$, $\overline{\mathsf{t}}_\tau(T) = \underline{\mathsf{t}}_\tau(T) := T$, and

$$\overline{\mathsf{t}}_\tau(t) = t_\tau^k \quad \text{for } t \in (t_\tau^{k-1}, t_\tau^k], \qquad \overline{\mathsf{t}}_\tau(t) = t_\tau^{k-1} \quad \text{for } t \in [t_\tau^{k-1}, t_\tau^k).$$

In terms of the above interpolants, the discrete system (4.4) rephrases as

$$\begin{aligned}
&\int_\Omega \widehat{\theta}_\tau'(t) v \,\mathrm{d}x - \int_\Omega \overline{\theta}_\tau(t) \mathrm{div}(\widehat{\mathbf{u}}_\tau'(t)) v \,\mathrm{d}x + \int_\Omega \alpha(\overline{\theta}_\tau(t)) \nabla \overline{\theta}_\tau(t) \nabla v \,\mathrm{d}x \\
&\quad + \int_{\Gamma_{\mathrm{C}}} k(\underline{\chi}_\tau(t)) \overline{\theta}_\tau(t) (\overline{\theta}_\tau(t) - \overline{\theta}_{\mathrm{s},\tau}(t)) v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \mathfrak{J}[(\underline{\chi}_\tau(t))^+] \, (\underline{\chi}_\tau(t))^+ \left(\overline{\theta}_\tau(t)\right)^2 v \,\mathrm{d}x \\
&\quad - \int_{\Gamma_{\mathrm{C}}} \mathfrak{J}[(\underline{\chi}_\tau(t))^+ \overline{\theta}_{\mathrm{s},\tau}(t)] \, (\underline{\chi}_\tau(t))^+ \overline{\theta}_\tau(t) v \,\mathrm{d}x \\
&= \int_\Omega \varepsilon\left(\widehat{\mathbf{u}}_\tau'(t)\right) \mathbb{V} \varepsilon\left(\widehat{\mathbf{u}}_\tau'(t)\right) v \,\mathrm{d}x + \langle \overline{h}_\tau(t), v \rangle_{H^1(\Omega)} \qquad \text{for all } v \in H^1(\Omega),
\end{aligned} \tag{5.4a}$$

$$\begin{aligned}
&\mathsf{v}\left(\widehat{\mathbf{u}}_\tau'(t), \mathbf{v}\right) + \rho \int_\Omega \left|\varepsilon\left(\widehat{\mathbf{u}}_\tau'(t)\right)\right|^{\omega-2} \varepsilon\left(\widehat{\mathbf{u}}_\tau'(t)\right) \varepsilon(\mathbf{v}) \,\mathrm{d}x + \mathsf{e}\left(\overline{\mathbf{u}}_\tau(t), \mathbf{v}\right) \\
&\quad + \int_\Omega \overline{\theta}_\tau(t) \mathrm{div}(\mathbf{v}) \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} (\overline{\chi}_\tau(t))^+ \overline{\mathbf{u}}_\tau(t) \mathbf{v} \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \overline{\boldsymbol{\zeta}}_\tau(t) \cdot \mathbf{v} \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \mathfrak{J}[(\overline{\chi}_\tau(t))^+] \, (\overline{\chi}_\tau(t))^+ \overline{\mathbf{u}}_\tau(t) \mathbf{v} \,\mathrm{d}x \\
&= \langle \overline{\mathbf{F}}_\tau(t), \mathbf{v} \rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)} \qquad \text{for all } \mathbf{v} \in W^{1,\omega}_{\mathrm{D}}(\Omega;\mathbb{R}^3), \\
&\text{with } \overline{\boldsymbol{\zeta}}_\tau(t) := \eta_\varsigma(\overline{\mathbf{u}}_\tau(t) \cdot \mathbf{n}) \mathbf{n},
\end{aligned} \tag{5.4b}$$

$$\begin{aligned}
&\int_{\Gamma_{\mathrm{C}}} \widehat{\theta}_{\mathrm{s},\tau}'(t) v \,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} \overline{\theta}_{\mathrm{s},\tau}(t) \frac{\lambda(\overline{\chi}_\tau(t)) - \lambda(\underline{\chi}_\tau(t))}{\tau} v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \alpha(\overline{\theta}_{\mathrm{s},\tau}(t)) \nabla \overline{\theta}_{\mathrm{s},\tau}(t) \nabla v \,\mathrm{d}x \\
&= \int_{\Gamma_{\mathrm{C}}} \left|\widehat{\chi}_\tau'(t)\right|^2 v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} k(\underline{\chi}_\tau(t)) (\overline{\theta}_\tau(t) - \overline{\theta}_{\mathrm{s},\tau}(t)) \overline{\theta}_{\mathrm{s},\tau}(t) v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \mathfrak{J}[(\underline{\chi}_\tau(t))^+ \overline{\theta}_\tau(t)] \, (\underline{\chi}_\tau(t))^+ \overline{\theta}_{\mathrm{s},\tau}(t) v \,\mathrm{d}x \\
&\quad - \int_{\Gamma_{\mathrm{C}}} \mathfrak{J}[(\underline{\chi}_\tau(t))^+] \, (\underline{\chi}_\tau(t))^+ \left(\overline{\theta}_{\mathrm{s},\tau}(t)\right)^2 v \,\mathrm{d}x + \langle \overline{\ell}_\tau, v \rangle_{H^1(\Gamma_{\mathrm{C}})} \qquad \text{for all } v \in H^1(\Gamma_{\mathrm{C}}).
\end{aligned} \tag{5.4c}$$

$$\begin{aligned}
&\widehat{\chi}_\tau'(t) + \rho \left|\widehat{\chi}_\tau'(t)\right|^{\omega-2} \widehat{\chi}_\tau'(t) + A\overline{\chi}_\tau(t) + \beta_\varsigma(\overline{\chi}_\tau(t)) + \gamma_\nu'(\overline{\chi}_\tau(t)) - \nu \underline{\chi}_\tau(t) \\
&= -\lambda_\delta'(\underline{\chi}_\tau(t)) \overline{\theta}_{\mathrm{s},\tau}(t) - \delta \overline{\chi}_\tau(t) \overline{\theta}_{\mathrm{s},\tau}(t) \\
&\quad - \frac{1}{2} |\underline{\mathbf{u}}_\tau(t)|^2 \overline{\sigma}_\tau(t) - \frac{1}{2} \mathfrak{J}[(\overline{\chi}_\tau(t))^+] \, |\underline{\mathbf{u}}_\tau(t)|^2 \overline{\sigma}_\tau(t) - \frac{1}{2} \mathfrak{J}[(\underline{\chi}_\tau(t))^+ |\underline{\mathbf{u}}_\tau(t)|^2] \, \overline{\sigma}_\tau(t) \qquad \text{a.e. in } \Gamma_{\mathrm{C}} \\
&\text{with } \overline{\sigma}_\tau(t) \in \partial\varphi(\overline{\chi}_\tau(t)) \qquad \text{a.e. in } \Gamma_{\mathrm{C}}
\end{aligned} \tag{5.4d}$$

for almost all $t \in (0,T)$, supplemented with the Cauchy data $(\theta_\rho^0, \mathbf{u}_\rho^0, \theta_{\mathrm{s},\rho}^0, \chi_0)$ as in (3.72) and (2.20d). It is in system (5.4) that we shall pass to the limit as $\tau \downarrow 0$, thus proving the existence of solutions to (the weak formulation of) the Cauchy problem for system (3.69). In the following results we shall omit to specify the standing assumptions on the problem and on the Cauchy data.

5.1. **A priori estimates.** The following result collects a series of a priori estimates on the approximate solutions that will be at the basis of the limit passage procedure as $\tau \downarrow 0$ performed in Sec. 5.2 and leading to the proof of the existence of solutions to system (3.69). In view of the further limit passages as $\rho \downarrow 0$ and $\varsigma \downarrow 0$ carried out in Sec. 6, we shall distinguish the estimates that hold *uniformly* w.r.t. τ *and* ρ, ς, from those that are not uniform w.r.t. ρ, ς. In the statement of Lemma 5.1 below we will use the notation

$$\mathrm{Var}_{\mathbf{B}}(\mathfrak{h}; [0,T]) := \sup \left\{ \sum_{i=1}^m \|\mathfrak{h}(\sigma_i) - \mathfrak{h}(\sigma_{i-1})\|_{\mathbf{B}} \; : \; (\sigma_i)_{i=1}^m \text{ partition of } [0,T] \right\}$$

for the total variation of a function $\mathfrak{h} : [0,T] \to \mathbf{B}$.

Lemma 5.1 (A priori estimates)**.** *There exists a constant $S > 0$ such that the following estimates hold for every $\tau > 0$ and $\rho, \varsigma > 0$:*

$$\|\overline{\theta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + \|\widehat{\theta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} \leq S, \tag{5.5a}$$

$$\|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^3))} + \|\widehat{\mathbf{u}}_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^3))} \leq S, \tag{5.5b}$$

$$\|\overline{\theta}_{\mathrm{s},\tau}\|_{L^\infty(0,T;L^1(\Gamma_\mathrm{C}))} + \|\widehat{\theta}_{\mathrm{s},\tau}\|_{L^\infty(0,T;L^1(\Gamma_\mathrm{C}))} \leq S, \tag{5.5c}$$

$$\|\overline{\chi}_\tau\|_{L^\infty(0,T;H^1(\Gamma_\mathrm{C}))} + \|\widehat{\chi}_\tau\|_{L^\infty(0,T;H^1(\Gamma_\mathrm{C}))} \leq S, \tag{5.5d}$$

$$\|\overline{\sigma}_\tau\|_{L^\infty(0,T;L^\infty(\Gamma_\mathrm{C}))} \leq S\,. \tag{5.5e}$$

Furthermore, for every $\rho > 0$ there exists a constant $S_\rho > 0$ such that for every $\tau > 0$ and $\varsigma > 0$:

$$\|\overline{\theta}_\tau\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))} + \|\widehat{\alpha}(\overline{\theta}_\tau)\|_{L^2(0,T;H^1(\Omega))} + \|\widehat{\theta}_\tau\|_{L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^1(\Omega)^*)} \leq S_\rho, \tag{5.6a}$$

$$\mathrm{Var}_{H^1(\Omega)^*}(\overline{\theta}_\tau;[0,T]) \leq S_\rho, \tag{5.6b}$$

$$\|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;W^{1,\omega}(\Omega;\mathbb{R}^3))} + \|\widehat{\mathbf{u}}_\tau\|_{W^{1,\omega}(0,T;W^{1,\omega}(\Omega;\mathbb{R}^3))} \leq S_\rho, \tag{5.6c}$$

$$\begin{aligned}&\|\overline{\theta}_{\mathrm{s},\tau}\|_{L^2(0,T;H^1(\Gamma_\mathrm{C}))\cap L^\infty(0,T;L^2(\Gamma_\mathrm{C}))} + \|\widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau})\|_{L^2(0,T;H^1(\Gamma_\mathrm{C}))}\\ &\quad + \|\widehat{\theta}_{\mathrm{s},\tau}\|_{L^2(0,T;H^1(\Gamma_\mathrm{C}))\cap H^1(0,T;H^1(\Gamma_\mathrm{C})^*)} \leq S_\rho,\end{aligned} \tag{5.6d}$$

$$\mathrm{Var}_{H^1(\Gamma_\mathrm{C})^*}(\overline{\theta}_{\mathrm{s},\tau};[0,T]) \leq S_\rho, \tag{5.6e}$$

$$\|\widehat{\chi}_\tau\|_{W^{1,\omega}(0,T;L^\omega(\Gamma_\mathrm{C}))}S_\rho. \tag{5.6f}$$

Moreover, for every $\rho, \varsigma > 0$ there exists a constant $S_{\rho,\varsigma} > 0$ such that for every $\tau > 0$:

$$\|A\overline{\chi}_\tau\|_{L^{\omega/(\omega-1)}(\Gamma_\mathrm{C}\times(0,T))} \leq S_{\rho,\varsigma}. \tag{5.7}$$

Finally,

$$\exists S_0 > 0\; \forall p \in [1,\infty)\;\; \exists\, \bar\tau_p > 0\;\; \forall \tau \in (0,\bar\tau_p)\;\;\; \forall \rho,\varsigma > 0: \quad \left\|\frac{1}{\overline{\theta}_\tau}\right\|_{L^\infty(0,T;L^p(\Omega))} + \left\|\frac{1}{\overline{\theta}_{\mathrm{s},\tau}}\right\|_{L^\infty(0,T;L^p(\Gamma_\mathrm{C}))} \leq S_0. \tag{5.8}$$

Proof. Step 1: energy estimates. We sum the discrete total energy inequality (4.9) over the index $k \in \{1,\ldots,J\}$, for every $J \in \{1,\ldots,K_\tau\}$. Applying the discrete by-part integration formula (5.3) to the term $\sum_{k=1}^J \tau\, \langle \mathbf{F}_\tau^k, \dfrac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}$ we obtain for all $0 \leq s < t \leq T$

$$\begin{aligned}
&\mathcal{E}_\varsigma(\overline{\theta}_\tau(t),\overline{\theta}_{\mathrm{s},\tau}(t),\overline{\mathbf{u}}_\tau(t),\overline{\chi}_\tau(t)) + \rho\int_{\underline{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega |\varepsilon(\widehat{\mathbf{u}}_\tau')|^\omega \,\mathrm{d}x\,\mathrm{d}r + \rho\int_{\underline{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_{\Gamma_\mathrm{C}} \left|\widehat{\chi}_\tau'\right|^\omega \mathrm{d}x\,\mathrm{d}r\\
&\quad + \int_{\underline{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_{\Gamma_\mathrm{C}} k(\underline{\chi}_\tau)(\overline{\theta}_\tau - \overline{\theta}_{\mathrm{s},\tau})^2 \,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}x + \int_{\underline{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\iint_{\Gamma_\mathrm{C}\times\Gamma_\mathrm{C}} j(x,y)(\underline{\chi}_\tau(x))^+(\underline{\chi}_\tau(y))^+(\overline{\theta}_\tau(x) - \overline{\theta}_{\mathrm{s},\tau}(y))^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\\
&\leq \mathcal{E}_\varsigma(\underline{\theta}_\tau(s),\underline{\theta}_{\mathrm{s},\tau}(s),\underline{\mathbf{u}}_\tau(s),\underline{\chi}_\tau(s)) + \int_{\underline{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \overline{h}_\tau \,\mathrm{d}x\,\mathrm{d}r + \int_{\underline{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_{\Gamma_\mathrm{C}} \overline{\ell}_\tau \,\mathrm{d}x\,\mathrm{d}r\\
&\quad + \langle \overline{\mathbf{F}}_\tau(\bar{\mathsf{t}}_\tau(t)), \overline{\mathbf{u}}_\tau(\bar{\mathsf{t}}_\tau(t))\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)} - \langle \overline{\mathbf{F}}_\tau(\underline{\mathsf{t}}_\tau(s)), \underline{\mathbf{u}}_\tau(\bar{\mathsf{t}}_\tau(s))\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)} - \int_{\underline{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \langle \widehat{\mathbf{F}}_\tau', \overline{\mathbf{u}}_\tau\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)} \,\mathrm{d}r\,.
\end{aligned} \tag{5.9}$$

We now take $s = 0$ in (5.9), and observe that $\mathcal{E}_\varsigma(\underline{\theta}_\tau(0),\underline{\theta}_{\mathrm{s},\tau}(0),\underline{\mathbf{u}}_\tau(0),\underline{\chi}_\tau(0)) = \mathcal{E}_\varsigma(\theta_\rho^0,\theta_{\mathrm{s},\rho}^0,\mathbf{u}_\rho^0,\chi_0) \leq C$ thanks to (2.20d) and (3.72). For the second and third integrals on the right-hand side, we use that $\|\overline{h}_\tau\|_{L^1(0,T;L^1(\Omega))} \leq C$ by (2.18a) and that $\|\overline{\ell}_\tau\|_{L^1(0,T;L^1(\Gamma_\mathrm{C}))} \leq C$ by (2.18b), respectively, while we deal with the last three terms by mimicking the calculations from (3.25). Namely, they can be controlled by the left-hand side of (5.9) thanks to the coercivity estimate (3.8). All in all, as in Section 3.3.1 we conclude that

$$\sup_{t\in(0,T)} \left|\mathcal{E}_\varsigma(\overline{\theta}_\tau(t),\overline{\theta}_{\mathrm{s},\tau}(t),\overline{\mathbf{u}}_\tau(t),\overline{\chi}_\tau(t))\right| \leq C. \tag{5.10}$$

An analogue of the coercivity estimate (3.8) holds for the functional $\mathcal{E}_\varsigma$: the only difference is that, since $\mathcal{E}_\varsigma$ features $\widehat{\beta}_\varsigma$ in place of $\widehat{\beta}$, it does no longer control the $L^\infty(\Gamma_\mathrm{C})$-norm of χ. However, it is not difficult to see that $\int_{\Gamma_C} \widehat{\beta}_\varsigma(\chi)\,\mathrm{d}x \geq c\|\chi\|^2_{L^2(\Gamma_C)} - C$. Therefore, from (5.10) we deduce estimates (5.5a), (5.5b), (5.5c), and (5.5d).

Furthermore, also taking into account the positivity of the fourth and fifth terms on the left-hand side of (5.9), we deduce a bound for the second and third summands, which gives (5.6c) and (5.6f).

Finally, estimate (5.5e) follows from the fact that $\overline{\sigma}_\tau \in \partial\varphi(\overline{\chi}_\tau)$ a.e. in $\Gamma_C \times (0,T)$, cf. (3.70).

Step 2: Estimates for the temperature variables. We test (4.4a) by $\tau\theta^k_\tau$, (4.4c) by $\tau\theta^k_{\mathrm{s},\tau}$, add the resulting relations and sum over the index $k \in \{1,\ldots,J\}$ for an arbitrary $J \in \{1,\ldots,K_\tau\}$. Let $t \in (0,T]$ satisfy $(J-1)\tau \leq t \leq J\tau$: we obtain

$$\sum_{k=1}^J \tau \int_\Omega \left(\frac{\theta^k_\tau - \theta^{k-1}_\tau}{\tau}\right)\theta^k_\tau \,\mathrm{d}x \geq \frac{1}{2}\|\overline{\theta}_\tau(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\theta^0_\rho\|^2_{L^2(\Omega)},$$
$$\sum_{k=1}^J \tau \int_{\Gamma_C} \left(\frac{\theta^k_{\mathrm{s},\tau} - \theta^{k-1}_{\mathrm{s},\tau}}{\tau}\right)\theta^k_{\mathrm{s},\tau} \,\mathrm{d}x \geq \frac{1}{2}\|\overline{\theta}_{\mathrm{s},\tau}(t)\|^2_{L^2(\Gamma_C)} - \frac{1}{2}\|\theta^0_{\mathrm{s},\rho}\|^2_{L^2(\Gamma_C)}.$$

Mimicking the calculations in Lemma 4.9 (cf. (4.28)) we obtain

$$\begin{aligned}&\frac{1}{2}\|\overline{\theta}_\tau(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\overline{\theta}_{\mathrm{s},\tau}(t)\|^2_{L^2(\Gamma_C)} + c_0\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega |\nabla\overline{\theta}_\tau|^2\,\mathrm{d}x\,\mathrm{d}r + c_0\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_{\Gamma_C}|\nabla\overline{\theta}_{\mathrm{s},\tau}|^2\,\mathrm{d}x\,\mathrm{d}r + I_1 + I_2\\ &\leq I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10}\end{aligned} \tag{5.11}$$

with

$$\begin{aligned}I_1 &= \int_0^{\bar{\mathsf{t}}_\tau(t)}\int_{\Gamma_C} k(\underline{\chi}_\tau)((\overline{\theta}_\tau)^2 - (\overline{\theta}_{\mathrm{s},\tau})^2)(\overline{\theta}_\tau - \overline{\theta}_{\mathrm{s},\tau})\,\mathrm{d}x\,\mathrm{d}r \overset{(1)}{\geq} 0,\\ I_2 &= \int_0^{\bar{\mathsf{t}}_\tau(t)}\int_{\Gamma_C} (\overline{\theta}_\tau)^2\left(\mathcal{J}[(\underline{\chi}_\tau)^+]\,(\underline{\chi}_\tau)^+\overline{\theta}_\tau - \mathcal{J}[(\underline{\chi}_\tau)^+\overline{\theta}_{\mathrm{s},\tau}]\,(\underline{\chi}_\tau)^+\right)\mathrm{d}x\,\mathrm{d}r\\ &\quad - \int_0^{\bar{\mathsf{t}}_\tau(t)}\int_{\Gamma_C} (\overline{\theta}_{\mathrm{s},\tau})^2\left(\mathcal{J}[(\underline{\chi}_\tau)^+\overline{\theta}_\tau]\,(\underline{\chi}_\tau)^+ - \mathcal{J}[(\underline{\chi}_\tau)^+]\,(\underline{\chi}_\tau)^+\overline{\theta}_{\mathrm{s},\tau}\right)\mathrm{d}x\,\mathrm{d}r \overset{(2)}{\geq} 0\end{aligned}$$

where for (1) we have used the estimate $(a^2-b^2)(a-b) = (a+b)(a-b)^2 \geq 0$ for all $a,\, b \in [0,\infty)$ and for (2) the very same monotonicity arguments used for (3.7), based on the fact that $[0,+\infty) \ni r \mapsto r^2$ is non-decreasing. As for the terms on the right-hand side of (5.11), we have

$$\begin{aligned}I_3 &= \frac{1}{2}\|\theta^0_\rho\|^2_{L^2(\Omega)} \leq C, \qquad I_4 = \frac{1}{2}\|\theta^0_{\mathrm{s},\rho}\|^2_{L^2(\Gamma_C)} \leq C,\\ I_5 &= \int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \mathrm{div}\,(\widehat{\mathbf{u}}'_\tau)\,|\overline{\theta}_\tau|^2\,\mathrm{d}x\,\mathrm{d}r\\ &\leq C\int_0^{\bar{\mathsf{t}}_\tau(t)} \|\widehat{\mathbf{u}}'_\tau\|_{W^{1,4}(\Omega)}\,\|\overline{\theta}_\tau\|_{L^4(\Omega)}\|\overline{\theta}_\tau\|_{L^2(\Omega)}\,\mathrm{d}r\\ &\overset{(3)}{\leq} C\int_0^{\bar{\mathsf{t}}_\tau(t)} \|\widehat{\mathbf{u}}'_\tau\|_{W^{1,4}(\Omega)}\,\|\nabla\overline{\theta}_\tau\|_{L^2(\Omega;\mathbb{R}^3)}\|\overline{\theta}_\tau\|_{L^2(\Omega)}\,\mathrm{d}r + \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\widehat{\mathbf{u}}'_\tau\|_{W^{1,4}(\Omega)}\,\|\overline{\theta}_\tau\|_{L^1(\Omega;\mathbb{R}^3)}\|\overline{\theta}_\tau\|_{L^2(\Omega)}\,\mathrm{d}r\\ &\overset{(4)}{\leq} \frac{c_0}{4}\int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla\overline{\theta}_\tau\|^2_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}r + C\int_0^{\bar{\mathsf{t}}_\tau(t)} \|\widehat{\mathbf{u}}'_\tau\|^2_{W^{1,4}(\Omega)}\,\|\overline{\theta}_\tau\|^2_{L^2(\Omega)}\,\mathrm{d}r + C\end{aligned}$$

where (3), with c_0 the constant from (2.13), follows from the continuous embedding $H^1(\Omega) \subset L^4(\Omega)$ and from the Poincaré inequality, and (4) from the previously proved (5.5a),

$$
\begin{aligned}
I_6 &= \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \varepsilon(\widehat{\mathbf{u}}'_\tau)\, \mathbb{V}\varepsilon(\widehat{\mathbf{u}}'_\tau)\, \overline{\theta}_\tau \,\mathrm{d}x\,\mathrm{d}r \le C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\varepsilon(\widehat{\mathbf{u}}'_\tau)\|_{L^4(\Omega)}^2 \|\overline{\theta}_\tau\|_{L^2(\Omega)} \,\mathrm{d}r \\
&\le \frac{1}{2} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\theta}_\tau\|_{L^2(\Omega)}^2 \,\mathrm{d}r + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\widehat{\mathbf{u}}'_\tau\|_{W^{1,4}(\Omega;\mathbb{R}^3)}^4 \,\mathrm{d}r \\
I_7 &= \int_0^{\bar{\mathsf{t}}_\tau(t)} \langle \overline{h}_\tau, \overline{\theta}_\tau \rangle_{H^1(\Omega)} \,\mathrm{d}r \\
&\overset{(5)}{\le} \frac{c_0}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla\overline{\theta}_\tau\|_{L^2(\Omega)}^2 \,\mathrm{d}r + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{h}_\tau\|_{H^1(\Omega)^*}^2 \,\mathrm{d}r + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{h}_\tau\|_{H^1(\Omega)^*} \|\overline{\theta}_\tau\|_{L^1(\Omega)} \,\mathrm{d}r \\
&\le \frac{c_0}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla\overline{\theta}_\tau\|_{L^2(\Omega)}^2 \,\mathrm{d}r + C
\end{aligned}
$$

where (5) again follows from Poincaré inequality, and analogously

$$
\begin{aligned}
I_8 &= \int_0^{\bar{\mathsf{t}}_\tau(t)} \langle \overline{\ell}_\tau, \overline{\theta}_{\mathrm{s},\tau} \rangle_{H^1(\Gamma_{\mathrm{C}})} \,\mathrm{d}r \\
&\le \frac{c_0}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla\overline{\theta}_{\mathrm{s},\tau}\|_{L^2(\Gamma_{\mathrm{C}})}^2 \,\mathrm{d}r + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\ell}_\tau\|_{H^1(\Gamma_{\mathrm{C}})^*}^2 \,\mathrm{d}r + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\ell}_\tau\|_{H^1(\Gamma_{\mathrm{C}})^*} \|\overline{\theta}_{\mathrm{s},\tau}\|_{L^1(\Gamma_{\mathrm{C}})} \,\mathrm{d}r \\
&\le \frac{c_0}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla\overline{\theta}_{\mathrm{s},\tau}\|_{L^2(\Gamma_{\mathrm{C}})}^2 \,\mathrm{d}r + C \\
I_9 &= \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_{\Gamma_{\mathrm{C}}} \tfrac{\lambda(\overline{\chi}_\tau) - \lambda(\underline{\chi}_\tau)}{\tau} (\overline{\theta}_{\mathrm{s},\tau})^2 \,\mathrm{d}x\,\mathrm{d}r \\
&\overset{(6)}{\le} C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\widehat{\chi}'_\tau\|_{L^4(\Gamma_{\mathrm{C}})} \|\overline{\theta}_{\mathrm{s},\tau}\|_{L^4(\Gamma_{\mathrm{C}})} \|\overline{\theta}_{\mathrm{s},\tau}\|_{L^2(\Gamma_{\mathrm{C}})} \,\mathrm{d}r \\
&\overset{(7)}{\le} \frac{c_0}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla\overline{\theta}_{\mathrm{s},\tau}\|_{L^2(\Gamma_{\mathrm{C}};\mathbb{R}^2)}^2 \,\mathrm{d}r + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \left\|\widehat{\chi}'_\tau\right\|_{L^4(\Gamma_{\mathrm{C}})}^2 \|\overline{\theta}_{\mathrm{s},\tau}\|_{L^2(\Gamma_{\mathrm{C}})}^2 \,\mathrm{d}r + C
\end{aligned}
$$

where (6) follows from the Lipschitz continuity of λ, and (7) from the same arguments as for inequality (3), and, finally,

$$
I_{10} = \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_{\Gamma_{\mathrm{C}}} (\widehat{\chi}'_\tau)^2 \overline{\theta}_{\mathrm{s},\tau} \,\mathrm{d}x\,\mathrm{d}r \le \frac{1}{2} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\theta}_{\mathrm{s},\tau}\|_{L^2(\Gamma_{\mathrm{C}})}^2 \,\mathrm{d}r + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\widehat{\chi}'_\tau\|_{L^4(\Gamma_{\mathrm{C}})}^4 \,\mathrm{d}r\,.
$$

All in all, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\|\overline{\theta}_\tau(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\overline{\theta}_{\mathrm{s},\tau}(t)\|_{L^2(\Gamma_{\mathrm{C}})}^2 + \frac{c_0}{2} \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\nabla\overline{\theta}_\tau|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{c_0}{2} \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_{\Gamma_{\mathrm{C}}} |\nabla\overline{\theta}_{\mathrm{s},\tau}|^2 \,\mathrm{d}x\,\mathrm{d}r \\
&\le C + C \int_0^{\bar{\mathsf{t}}_\tau(t)} (\|\widehat{\mathbf{u}}'_\tau\|_{W^{1,4}(\Omega;\mathbb{R}^3)}^2 + 1) \|\overline{\theta}_\tau\|_{L^2(\Omega)}^2 \,\mathrm{d}r + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \left(\left\|\widehat{\chi}'_\tau\right\|_{L^4(\Gamma_{\mathrm{C}})}^2 + 1 \right) \|\overline{\theta}_{\mathrm{s},\tau}\|_{L^2(\Gamma_{\mathrm{C}})}^2 \,\mathrm{d}r\,.
\end{aligned}
$$

Applying the discrete Gronwall Lemma 4.5 and taking into account the previously proved (5.6c) and (5.6f), we conclude that

$$
\|\overline{\theta}_\tau\|_{L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))} + \|\overline{\theta}_{\mathrm{s},\tau}\|_{L^2(0,T;H^1(\Gamma_{\mathrm{C}})) \cap L^\infty(0,T;L^2(\Gamma_{\mathrm{C}}))} \le C_\rho, \tag{5.12}
$$

with the constant C_ρ depending on the parameter ρ.

Step 3: Further estimates for the temperature variables. We test (4.4a) by $\tau\widehat{\alpha}(\theta_\tau^k)$, (4.4c) by $\tau\widehat{\alpha}(\theta_{\mathrm{s},\tau}^k)$, add the resulting relations and sum over the index $k \in \{1, \ldots, J\}$ for an arbitrary $J \in \{1, \ldots, K_\tau\}$. Let $t \in (0, T]$ satisfy $(J-1)\tau \le t \le J\tau$. By the convexity and positivity of $\widehat{\widehat{\alpha}}$ (cf. (4.40)), we have

$$
\begin{aligned}
\sum_{k=1}^J \int_\Omega (\theta_\tau^k - \theta_\tau^{k-1}) \widehat{\alpha}(\theta_\tau^k) \,\mathrm{d}x \quad &\ge \quad \sum_{k=1}^J \left(\|\widehat{\widehat{\alpha}}(\theta_\tau^k)\|_{L^1(\Omega)} - \|\widehat{\widehat{\alpha}}(\theta_\tau^{k-1})\|_{L^1(\Omega)} \right) \\
&= \quad \|\widehat{\widehat{\alpha}}(\overline{\theta}_\tau(t))\|_{L^1(\Omega)} - \|\widehat{\widehat{\alpha}}(\theta_\rho^0)\|_{L^1(\Omega)},
\end{aligned} \tag{5.13}
$$

and, analogously,

$$\sum_{k=1}^{J}\int_{\Gamma_C}(\theta_{s,\tau}^k-\theta_{s,\tau}^{k-1})\widehat{\alpha}(\theta_{s,\tau}^k)\,\mathrm{d}x \quad\geq\quad \|\widehat{\widehat{\alpha}}(\overline{\theta}_{s,\tau}(t))\|_{L^1(\Gamma_C)}-\|\widehat{\widehat{\alpha}}(\theta_{s,\rho}^0)\|_{L^1(\Gamma_C)}.$$

Since the function $\mathbb{R}^+\ni r\mapsto r\widehat{\alpha}(r)$ is increasing, we have that

$$\int_0^{\overline{\mathsf{t}}_\tau(t)}\int_{\Gamma_C}k(\underline{\chi}_\tau)(\overline{\theta}_\tau-\overline{\theta}_{s,\tau})(\overline{\theta}_\tau\widehat{\alpha}(\overline{\theta}_\tau)-\overline{\theta}_{s,\tau}\widehat{\alpha}(\overline{\theta}_{s,\tau}))\,\mathrm{d}x\,\mathrm{d}r\geq 0, \tag{5.14}$$

while, using that $\widehat{\alpha}$ is (strictly) increasing and mimicking the calculations in (3.17), we observe that

$$\begin{aligned}&\int_0^{\overline{\mathsf{t}}_\tau(t)}\int_{\Gamma_C}\widehat{\alpha}(\overline{\theta}_\tau)(\underline{\chi}_\tau)^+\overline{\theta}_\tau\Big(\mathfrak{J}[(\underline{\chi}_\tau)^+]\,\overline{\theta}_\tau-\mathfrak{J}[(\underline{\chi}_\tau)^+\overline{\theta}_{s,\tau}]\Big)\,\mathrm{d}x\,\mathrm{d}r\\ &\quad-\int_0^{\overline{\mathsf{t}}_\tau(t)}\int_{\Gamma_C}\widehat{\alpha}(\overline{\theta}_{s,\tau})(\underline{\chi}_\tau)^+\overline{\theta}_{s,\tau}\Big(\mathfrak{J}[(\underline{\chi}_\tau)^+\overline{\theta}_\tau]-\mathfrak{J}[(\underline{\chi}_\tau)^+]\,\overline{\theta}_{s,\tau}\Big)\,\mathrm{d}x\,\mathrm{d}r\geq 0.\end{aligned} \tag{5.15}$$

Combining (5.13)–(5.15) and observing that $\nabla\widehat{\alpha}(\theta_\tau^k)=\alpha(\theta_\tau^k)\nabla\theta_\tau^k$ and $\nabla\widehat{\alpha}(\theta_{s,\tau}^k)=\alpha(\theta_{s,\tau}^k)\nabla\theta_{s,\tau}^k$, we arrive at

$$\begin{aligned}&\|\widehat{\widehat{\alpha}}(\overline{\theta}_\tau(t))\|_{L^1(\Omega)}+\|\widehat{\widehat{\alpha}}(\overline{\theta}_{s,\tau}(t))\|_{L^1(\Gamma_C)}+\int_0^{\overline{\mathsf{t}}_\tau(t)}\|\nabla\widehat{\alpha}(\overline{\theta}_\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}r+\int_0^{\overline{\mathsf{t}}_\tau(t)}\|\nabla\widehat{\alpha}(\overline{\theta}_{s,\tau})\|_{L^2(\Gamma_C)}^2\,\mathrm{d}r\\ &\leq C\Big(1+\|\widehat{\widehat{\alpha}}(\theta_\rho^0)\|_{L^1(\Omega)}+\|\widehat{\widehat{\alpha}}(\theta_{s,\rho}^0)\|_{L^1(\Gamma_C)}\Big)+\int_0^{\overline{\mathsf{t}}_\tau(t)}\int_\Omega\overline{\theta}_\tau\mathrm{div}\,(\widehat{\mathbf{u}}_\tau')\,\widehat{\alpha}(\overline{\theta}_\tau)\,\mathrm{d}x\,\mathrm{d}r\\ &\quad+\int_0^{\overline{\mathsf{t}}_\tau(t)}\langle\overline{h}_\tau,\widehat{\alpha}(\overline{\theta}_\tau)\rangle_{H^1(\Omega)}\,\mathrm{d}r+\int_0^{\overline{\mathsf{t}}_\tau(t)}\langle\overline{\ell}_\tau,\widehat{\alpha}(\overline{\theta}_{s,\tau})\rangle_{H^1(\Gamma_C)}\,\mathrm{d}r\\ &\quad+\int_0^{\overline{\mathsf{t}}_\tau(t)}\int_\Omega\varepsilon(\widehat{\mathbf{u}}_\tau')\,\mathbb{V}\varepsilon(\widehat{\mathbf{u}}_\tau')\,\widehat{\alpha}(\overline{\theta}_\tau)\,\mathrm{d}x\,\mathrm{d}r+\int_0^{\overline{\mathsf{t}}_\tau(t)}\int_{\Gamma_C}(\widehat{\chi}_\tau')^2\widehat{\alpha}(\overline{\theta}_{s,\tau})\,\mathrm{d}x\,\mathrm{d}r\\ &\quad+\int_0^{\overline{\mathsf{t}}_\tau(t)}\int_{\Gamma_C}\tfrac{\lambda(\overline{\chi}_\tau)-\lambda(\underline{\chi}_\tau)}{\tau}\overline{\theta}_{s,\tau}\widehat{\alpha}(\overline{\theta}_{s,\tau})\,\mathrm{d}x\,\mathrm{d}r\doteq I_1+I_2+I_3+I_4+I_5+I_6+I_7.\end{aligned} \tag{5.16}$$

Thanks to (3.72) and (4.40), we have

$$I_1\leq C''\Big(1+\|\theta_\rho^0\|_{L^{\mu+2}(\Omega)}^{\mu+2}+\|\theta_{s,\rho}^0\|_{L^{\mu+2}(\Gamma_C)}^{\mu+2}\Big)\leq C. \tag{5.17}$$

With calculations analogous to those of (4.47) we infer that

$$\begin{aligned}\|\widehat{\alpha}(\overline{\theta}_\tau)\|_{H^1(\Omega)}&\leq C\left(\|\nabla(\widehat{\alpha}(\overline{\theta}_\tau))\|_{L^2(\Omega)}+\|\widehat{\alpha}(\overline{\theta}_\tau)\|_{L^1(\Omega)}\right)\\ &\leq C'\left(\|\nabla(\widehat{\alpha}(\overline{\theta}_\tau))\|_{L^2(\Omega)}+\|\widehat{\widehat{\alpha}}(\overline{\theta}_\tau)\|_{L^1(\Omega)}+1\right).\end{aligned} \tag{5.18}$$

By virtue of estimate (5.12), we have that $(\overline{\theta}_\tau)_\tau$ is bounded in $L^\infty(0,T;L^1(\Omega))\cap L^2(0,T;L^6(\Omega))$ and, a fortiori, in $L^4(0,T;L^{12/7}(\Omega))$ by interpolation. Therefore,

$$\begin{aligned}I_2&\leq\int_0^{\overline{\mathsf{t}}_\tau(t)}\int_\Omega\overline{\theta}_\tau|\mathrm{div}\,(\widehat{\mathbf{u}}_\tau')\,|\widehat{\alpha}(\overline{\theta}_\tau)\,\mathrm{d}x\,\mathrm{d}r\\ &\overset{(1)}{\leq}\int_0^{\overline{\mathsf{t}}_\tau(t)}\|\overline{\theta}_\tau\|_{L^{12/7}(\Omega)}\|\mathrm{div}\,(\widehat{\mathbf{u}}_\tau')\,\|_{L^4(\Omega)}\|\widehat{\alpha}(\overline{\theta}_\tau)\|_{L^6(\Omega)}\,\mathrm{d}r\\ &\overset{(2)}{\leq}C\int_0^{\overline{\mathsf{t}}_\tau(t)}\|\overline{\theta}_\tau\|_{L^{12/7}(\Omega)}\|\mathrm{div}\,(\widehat{\mathbf{u}}_\tau')\,\|_{L^4(\Omega)}\left(\|\widehat{\widehat{\alpha}}(\overline{\theta}_\tau)\|_{L^1(\Omega)}+1\right)\,\mathrm{d}r\\ &\qquad+C\int_0^{\overline{\mathsf{t}}_\tau(t)}\|\overline{\theta}_\tau\|_{L^{12/7}(\Omega)}^2\|\mathrm{div}\,(\widehat{\mathbf{u}}_\tau')\,\|_{L^4(\Omega)}^2\,\mathrm{d}r+\frac14\int_0^{\overline{\mathsf{t}}_\tau(t)}\|\nabla\widehat{\alpha}(\overline{\theta}_\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}r\end{aligned} \tag{5.19}$$

with (1) due to the Hölder inequality and (2) to (5.18) and the Young inequality. Analogously, we have

$$
\begin{aligned}
I_5 &\leq C \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\varepsilon(\widehat{\mathbf{u}}_\tau')|^2 \widehat{\alpha}(\overline{\theta}_\tau) \,\mathrm{d}x \,\mathrm{d}r \\
&\leq C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\varepsilon(\widehat{\mathbf{u}}_\tau')\|_{L^4(\Omega)}^2 \left(\|\nabla(\widehat{\alpha}(\overline{\theta}_\tau))\|_{L^2(\Omega)} + \|\widehat{\widehat{\alpha}}(\overline{\theta}_\tau)\|_{L^1(\Omega)} + 1 \right) \mathrm{d}r \\
&\leq \frac{1}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla \widehat{\alpha}(\overline{\theta}_\tau)\|_{L^2(\Omega)}^2 \,\mathrm{d}r + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\varepsilon(\widehat{\mathbf{u}}_\tau')\|_{L^4(\Omega)}^2 \left(\|\widehat{\widehat{\alpha}}(\overline{\theta}_\tau)\|_{L^1(\Omega)} + 1 \right) \mathrm{d}r \\
&\quad + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\varepsilon(\widehat{\mathbf{u}}_\tau')\|_{L^4(\Omega)}^4 \,\mathrm{d}r \,,
\end{aligned}
\tag{5.20}
$$

$$
\begin{aligned}
I_6 &= \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_{\Gamma_{\mathrm{C}}} (\widehat{\chi}_\tau')^2 \widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau}) \,\mathrm{d}x \,\mathrm{d}r \\
&\leq \frac{1}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla \widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau})\|_{L^2(\Gamma_{\mathrm{C}})}^2 \,\mathrm{d}r + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\widehat{\chi}_\tau'\|_{L^4(\Gamma_{\mathrm{C}})}^2 \left(\|\widehat{\widehat{\alpha}}(\overline{\theta}_{\mathrm{s},\tau})\|_{L^1(\Gamma_{\mathrm{C}})} + 1 \right) \mathrm{d}r \\
&\quad + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\widehat{\chi}_\tau'\|_{L^4(\Gamma_{\mathrm{C}})}^4 \,\mathrm{d}r.
\end{aligned}
\tag{5.21}
$$

Furthermore, again observing that $(\overline{\theta}_{\mathrm{s},\tau})_\tau$ is bounded in $L^4(0,T;L^{12/7}(\Gamma_{\mathrm{C}}))$ (a higher integrability estimate actually holds) by interpolation and arguing as for (5.19), we find that

$$
\begin{aligned}
I_7 &\leq C \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_{\Gamma_{\mathrm{C}}} \frac{|\overline{\chi}_\tau - \underline{\chi}_\tau|}{\tau} \overline{\theta}_{\mathrm{s},\tau} \widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau}) \,\mathrm{d}x \,\mathrm{d}r \\
&\leq C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\widehat{\chi}_\tau'\|_{L^4(\Gamma_{\mathrm{C}})} \|\overline{\theta}_{\mathrm{s},\tau}\|_{L^{12/7}(\Gamma_{\mathrm{C}})} \|\widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau})\|_{H^1(\Gamma_{\mathrm{C}})} \,\mathrm{d}r \\
&\leq C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\theta}_{\mathrm{s},\tau}\|_{L^{12/7}(\Gamma_{\mathrm{C}})} \|\widehat{\chi}_\tau'\|_{L^4(\Gamma_{\mathrm{C}})} \left(\|\widehat{\widehat{\alpha}}(\overline{\theta}_{\mathrm{s},\tau})\|_{L^1(\Gamma_{\mathrm{C}})} + 1 \right) \mathrm{d}r \\
&\quad + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\theta}_{\mathrm{s},\tau}\|_{L^{12/7}(\Gamma_{\mathrm{C}})}^2 \|\widehat{\chi}_\tau'\|_{L^4(\Gamma_{\mathrm{C}})}^2 \,\mathrm{d}r + \frac{1}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla \widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau})\|_{L^2(\Gamma_{\mathrm{C}})}^2 \,\mathrm{d}r \,.
\end{aligned}
\tag{5.22}
$$

Finally, we observe that

$$
I_3 \leq \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{h}_\tau\|_{H^1(\Omega)^*} \left(\|\widehat{\widehat{\alpha}}(\overline{\theta}_\tau)\|_{L^1(\Gamma_{\mathrm{C}})} + 1 \right) \mathrm{d}r + \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{h}_\tau\|_{H^1(\Omega)^*}^2 \,\mathrm{d}r + \frac{1}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla \widehat{\alpha}(\overline{\theta}_\tau)\|_{L^2(\Omega)}^2 \,\mathrm{d}r \,, \tag{5.23}
$$

and we estimate in the very same way the term I_4. All in all, from (5.17) and (5.19)–(5.23), also taking into account the previously obtained estimates, we conclude

$$
\begin{aligned}
&\|\widehat{\alpha}(\overline{\theta}_\tau(t))\|_{L^1(\Omega)} + \|\widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau}(t))\|_{L^1(\Gamma_{\mathrm{C}})} + \frac{1}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla \widehat{\alpha}(\overline{\theta}_\tau)\|_{L^2(\Omega)}^2 \,\mathrm{d}r + \frac{1}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\nabla \widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau})\|_{L^2(\Gamma_{\mathrm{C}})}^2 \,\mathrm{d}r \\
&\leq C + C \int_0^{\bar{\mathsf{t}}_\tau(t)} m_\tau \left(\|\widehat{\widehat{\alpha}}(\overline{\theta}_\tau)\|_{L^1(\Omega)} + \|\widehat{\widehat{\alpha}}(\overline{\theta}_{\mathrm{s},\tau})\|_{L^1(\Gamma_{\mathrm{C}})} \right) \mathrm{d}r \,,
\end{aligned}
\tag{5.24}
$$

with $(m_\tau)_\tau$ a sequence bounded in $L^1(0,T)$. Therefore, applying the discrete Gronwall Lemma 4.5, we conclude that

$$
\|\widehat{\alpha}(\overline{\theta}_\tau)\|_{L^2(0,T;H^1(\Omega))} + \|\widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau})\|_{L^2(0,T;H^1(\Gamma_{\mathrm{C}}))} + \|\widehat{\widehat{\alpha}}(\overline{\theta}_\tau)\|_{L^\infty(0,T;L^1(\Omega))} + \|\widehat{\widehat{\alpha}}(\overline{\theta}_{\mathrm{s},\tau})\|_{L^\infty(0,T;L^1(\Gamma_{\mathrm{C}}))} \leq C_\rho. \tag{5.25}
$$

Step 4: Comparison estimates. Taking into account estimates (5.25) and the previously found bounds, by comparison in the heat equations (5.4a) and (5.4c), respectively, we infer that

$$
\|\widehat{\theta}_\tau\|_{H^1(0,T;H^1(\Omega)^*)} + \|\widehat{\theta}_{\mathrm{s},\tau}\|_{H^1(0,T;H^1(\Gamma_{\mathrm{C}})^*)} \leq C_\rho \,, \tag{5.26}
$$

so that (5.6a) and (5.6d) immediately follow.

The total variation estimate (5.6b) then immediately ensues, taking into account that

$$
\mathrm{Var}_{H^1(\Omega)^*}(\overline{\theta}_\tau; [0,T]) \leq \|\widehat{\theta}_\tau'\|_{L^1(0,T;H^1(\Omega)^*)};
$$

the very same arguments also yield (5.6e).

Moreover, taking into account the previous estimates, the Lipschitz continuity of β_ς, γ' and λ, by comparison in the flow rule (5.4d) we find estimate (5.7), with a constant also depending on ς. Finally, estimate (5.8) follows from (4.7). □

5.2. **Limit passage as $\tau \downarrow 0$.** Lemma 5.2 ahead fixes the compactness properties of the sequences of approximate solutions for which the estimates of Lemma 5.1 hold.

The most delicate point is the proof of the relative compactness, a.e. in $\Omega\times(0,T)$ and a.e. in $\Gamma_{\mathrm{C}}\times(0,T)$, of the families of functions $(\overline{\theta}_\tau)_\tau$ and $(\overline{\theta}_{\mathrm{s},\tau})_\tau$; from this information we can indeed infer the compactness, a.e. $\Omega\times(0,T)$ and a.e. in $\Gamma_{\mathrm{C}}\times(0,T)$, of the sequences $\left(\frac{1}{\overline{\theta}_\tau}\right)_\tau$ and $\left(\frac{1}{\overline{\theta}_{\mathrm{s},\tau}}\right)_\tau$, which, combined with estimates (5.8), ultimately yields (5.32) below. In fact, for proving the pointwise (in space and time) convergence of $(\overline{\theta}_\tau)_\tau$ and $(\overline{\theta}_{\mathrm{s},\tau})_\tau$ we shall resort to the following Helly-type compactness result, which we quote from [26] in a slightly simplified version. In the statement we will use the following space

$$\mathrm{B}([0,T];\mathbf{Y}^*) := \{\mathbf{h} : [0,T] \to \mathbf{Y}^* : \text{ measurable and such that } \mathbf{h}(t) \text{ is defined at every } t \in [0,T]\}.$$

Theorem 5.1 (Theorem A.5, [26])**.** *Let $\mathbf{V}$ and $\mathbf{Y}$ be two (separable) reflexive Banach spaces such that $\mathbf{V} \subset \mathbf{Y}^*$ continuously. Let $(\mathfrak{h}_n)_n \subset L^p(0,T;\mathbf{V}) \cap \mathrm{B}([0,T];\mathbf{Y}^*)$ be bounded in $L^p(0,T;\mathbf{V})$ and suppose in addition that*

$$(\mathfrak{h}_n(0))_n \subset \mathbf{Y}^* \text{ is bounded,} \tag{5.27}$$

$$\exists\, C>0 \quad \forall\, n \in \mathbb{N}: \quad \mathrm{Var}_{\mathbf{Y}^*}(\mathfrak{h}_n;[0,T]) \leq C. \tag{5.28}$$

Then, there exists a subsequence $(\mathfrak{h}_{n_k})_k$ of $(\mathfrak{h}_n)_n$ and a function $\mathfrak{h} \in L^p(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{Y}^*)$ such that as $k \to \infty$

$$\mathfrak{h}_{n_k} \rightharpoonup^* \mathfrak{h} \quad \text{in } L^p(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{Y}^*), \tag{5.29}$$

$$\mathfrak{h}_{n_k}(t) \rightharpoonup \mathfrak{h}(t) \quad \text{in } \mathbf{V} \quad \text{for a.a. } t \in (0,T). \tag{5.30}$$

Lemma 5.2 (Compactness results)**.** *Let $\rho, \varsigma > 0$ be fixed. For any sequence $(\tau_k)_k \subset (0,+\infty)$ with $\tau_k \downarrow 0$ as $k \to +\infty$ there exist a (not relabeled) subsequence and a quintuple $(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi, \sigma)$ with*

$$\begin{cases} \theta \in L^2(0,T;H^1(\Omega)) \cap \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)^*), \\ \mathbf{u} \in W^{1,\omega}(0,T;W^{1,\omega}(\Omega;\mathbb{R}^3)), \\ \theta_{\mathrm{s}} \in L^2(0,T;H^1(\Gamma_{\mathrm{C}})) \cap \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Gamma_{\mathrm{C}})) \cap H^1(0,T;H^1(\Gamma_{\mathrm{C}})^*), \\ \chi \in \mathrm{C}^0_{\mathrm{weak}}([0,T];H^1(\Gamma_{\mathrm{C}})) \cap W^{1,\omega}(0,T;L^\omega(\Gamma_{\mathrm{C}})), \quad A\chi \in L^{\omega/(\omega-1)}(\Gamma_{\mathrm{C}}\times(0,T)), \\ \sigma \in L^\infty(\Gamma_{\mathrm{C}}\times(0,T)), \end{cases}$$

such that the following weak and strong converges hold as $k \to \infty$

$$\widehat{\theta}_{\tau_k} \rightharpoonup^* \theta \quad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)^*), \tag{5.31a}$$

$$\overline{\theta}_{\tau_k} \rightharpoonup^* \theta \quad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \tag{5.31b}$$

$$\widehat{\theta}_{\tau_k} \to \theta \quad \text{in } L^2(0,T;L^{6-\epsilon}(\Omega)) \quad \text{for all } \epsilon \in (0,5] \tag{5.31c}$$

$$\overline{\theta}_{\tau_k}(t) \rightharpoonup \theta(t) \quad \text{in } H^1(\Omega) \text{ for a.a. } t \in (0,T), \tag{5.31d}$$

$$\overline{\theta}_{\tau_k} \to \theta \quad \text{a.e. in } \Omega \times (0,T), \tag{5.31e}$$

$$\widehat{\theta}_{\mathrm{s},\tau_k} \rightharpoonup^* \theta_{\mathrm{s}} \quad \text{in } L^2(0,T;H^1(\Gamma_{\mathrm{C}})) \cap L^\infty(0,T;L^2(\Gamma_{\mathrm{C}})) \cap H^1(0,T;H^1(\Gamma_{\mathrm{C}})^*), \tag{5.31f}$$

$$\overline{\theta}_{\mathrm{s},\tau_k} \rightharpoonup^* \theta_{\mathrm{s}} \quad \text{in } L^2(0,T;H^1(\Gamma_{\mathrm{C}})) \cap L^\infty(0,T;L^2(\Gamma_{\mathrm{C}})), \tag{5.31g}$$

$$\widehat{\theta}_{\mathrm{s},\tau_k} \to \theta_{\mathrm{s}} \quad \text{in } L^2(0,T;L^q(\Gamma_{\mathrm{C}})) \text{ for all } 1 \le q < \infty, \tag{5.31h}$$

$$\overline{\theta}_{\mathrm{s},\tau_k}(t) \rightharpoonup \theta_{\mathrm{s}}(t) \quad \text{in } H^1(\Gamma_{\mathrm{C}}) \text{ for a.a. } t \in (0,T), \tag{5.31i}$$

$$\overline{\theta}_{\mathrm{s},\tau_k} \to \theta_{\mathrm{s}} \quad \text{a.e. in } \Gamma_{\mathrm{C}} \times (0,T), \tag{5.31j}$$

$$\widehat{\mathbf{u}}_{\tau_k} \rightharpoonup \mathbf{u} \quad \text{in } W^{1,\omega}(0,T;W^{1,\omega}(\Omega;\mathbb{R}^3)), \tag{5.31k}$$

$$\widehat{\mathbf{u}}_{\tau_k} \to \mathbf{u} \quad \text{in } \mathrm{C}^0([0,T];\mathrm{C}^0(\overline{\Omega};\mathbb{R}^3)), \tag{5.31l}$$

$$\overline{\mathbf{u}}_{\tau_k},\, \underline{\mathbf{u}}_{\tau_k} \rightharpoonup^* \mathbf{u} \quad \text{in } L^\infty(0,T;W^{1,\omega}(\Omega;\mathbb{R}^3)), \tag{5.31m}$$

$$\overline{\mathbf{u}}_{\tau_k},\, \underline{\mathbf{u}}_{\tau_k} \to \mathbf{u} \quad \text{in } L^\infty(0,T;\mathrm{C}^0(\overline{\Omega};\mathbb{R}^3)), \tag{5.31n}$$

$$\widehat{\chi}_{\tau_k} \rightharpoonup^* \chi \quad \text{in } L^\infty(0,T;H^1(\Gamma_{\mathrm{C}})) \cap W^{1,\omega}(0,T;L^\omega(\Gamma_{\mathrm{C}})), \tag{5.31o}$$

$$\widehat{\chi}_{\tau_k} \to \chi \quad \text{in } \mathrm{C}^0([0,T];L^q(\Gamma_{\mathrm{C}})) \text{ for all } 1 \le q < \infty. \tag{5.31p}$$

$$\overline{\chi}_{\tau_k} \rightharpoonup^* \chi \quad \text{in } L^\infty(0,T;H^1(\Gamma_{\mathrm{C}})), \tag{5.31q}$$

$$\overline{\chi}_{\tau_k},\, \underline{\chi}_{\tau_k} \to \chi \quad \text{in } L^\infty(0,T;L^q(\Gamma_{\mathrm{C}})) \text{ for all } 1 \le q < \infty, \tag{5.31r}$$

$$\overline{\sigma}_{\tau_k} \rightharpoonup^* \sigma \quad \text{in } L^\infty(\Gamma_{\mathrm{C}} \times (0,T)). \tag{5.31s}$$

Furthermore,

$$\left\| \frac{1}{\theta} \right\|_{L^\infty(\Omega\times(0,T))} + \left\| \frac{1}{\theta_{\mathrm{s}}} \right\|_{L^\infty(\Gamma_{\mathrm{C}}\times(0,T))} \le S_0, \tag{5.32}$$

with the constant S_0 from (4.7). Therefore, the functions θ and θ_{s} enjoy the positivity properties (2.31) with constants $\bar{\theta}$ and $\bar{\theta}_{\mathrm{s}}$ independent of ρ and ς.

Proof. Convergences (5.31a), (5.31f), (5.31k), (5.31o), and (5.31s) follow from estimates (5.5) and (5.6) by weak and weak* compactness arguments. In view of (5.7), we also have

$$A\overline{\chi}_{\tau_k} \rightharpoonup A\chi \qquad \text{in } L^{\omega/(\omega-1)}(\Gamma_{\mathrm{C}}\times(0,T)). \tag{5.33}$$

As for (5.31b), combining estimates (5.2) and the fact that the sequence $(\widehat{\theta}'_{\tau_k})_k$ is bounded in $L^2(0,T;H^1(\Omega)^*)$, we conclude that $\|\widehat{\theta}_{\tau_k} - \overline{\theta}_{\tau_k}\|_{L^\infty(0,T;H^1(\Omega)^*)} \to 0$ as $\tau_k \downarrow 0$. This identifies θ as the weak* limit of $(\overline{\theta}_{\tau_k})_k$ in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. With the same argument we also infer convergence (5.31g). Clearly, we have that $\theta \ge 0$ a.e. in $\Omega \times (0,T)$ and $\theta_{\mathrm{s}} \ge 0$ a.e. in $\Gamma_{\mathrm{C}} \times (0,T)$. Analogously, (5.31m) and (5.31n) ((5.31q) & (5.31r), respectively) shall follow from (5.31l) ((5.31p), resp.).

For (5.31c) we apply, e.g., the compactness result [35, Cor. 4], which ensures that $(\widehat{\theta}_{\tau_k})_k$ is relatively compact in $L^2(0,T;X) \cap \mathrm{C}^0([0,T];Y)$ for any Banach spaces X and Y such that $H^1(\Omega) \Subset X \subset L^2(\Omega)$ and $L^2(\Omega) \Subset Y \subset H^1(\Omega)^*$. In the same way, (5.31h) follows, recalling that $H^1(\Gamma_{\mathrm{C}}) \Subset L^q(\Gamma_{\mathrm{C}})$ for every $1 \le q < \infty$. The strong convergence (5.31l) can be deduced by the same result, taking into account that $W^{1,\omega}(\Omega;\mathbb{R}^3) \Subset \mathrm{C}^0(\overline{\Omega})$ since $\omega > 4$, by the Rellich-Kondrachov Theorem. Analogously, we have (5.31p).

Finally, applying to the sequences $(\overline{\theta}_{\tau_k})_k$ and $(\widehat{\theta}_{\tau_k})_k$ the compactness Theorem 5.1 (also recalling estimate (5.6b)), we infer the *pointwise-in-time* convergences whence, in particular, (5.31e) (since $H^1(\Omega) \Subset L^p(\Omega)$ for all $1 \le p < 6$). We recover (5.31j) in the very same way.

Therefore,

$$\frac{1}{\overline{\theta}_{\tau_k}(x,t)} \to L(x,t) := \begin{cases} \frac{1}{\theta(x,t)} & \text{if } \theta(x,t)>0, \\ +\infty & \text{otherwise} \end{cases} \qquad \text{for a.a. } (x,t) \in \Omega\times(0,T)$$

In turn, combining the Fatou Lemma with estimate (5.8) for, e.g., $p=2$ we infer that

$$\int_{\Omega\times(0,T)} L^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leq \liminf_{k\to\infty} \int_0^T \int_\Omega \frac{1}{\overline{\theta}_{\tau_k}^2(x,t)}\,\mathrm{d}x\,\mathrm{d}t \leq S_0 T,$$

so that $L(x,t)<\infty$ for a.a. $(x,t)\in\Omega\times(0,T)$. Hence, $\theta>0$ a.e. in $\Omega\times(0,T)$ and $L=\frac{1}{\theta}$. A fortiori, again in view of (5.8) and the Fatou Lemma we conclude that

$$\int_{t_0-r}^{t_0+r} \left\| \frac{1}{\theta(s)} \right\|_{L^p(\Omega)} \mathrm{d}s \leq \liminf_{k\to\infty} \int_{t_0-r}^{t_0+r} \left\| \frac{1}{\overline{\theta}_{\tau_k}(s)} \right\|_{L^p(\Omega)} \mathrm{d}s \leq 2S_0 r$$

for every $t_0\in(0,T)$ and $r\in(t_0,T-t_0)$. In particular, picking a Lebesgue point for $\|\frac{1}{\theta(\cdot)}\|_{L^p(\Omega)}$ we gather that

$$\left\| \frac{1}{\theta(t_0)} \right\|_{L^p(\Omega)} \leq S_0 \qquad \text{for a.a. } t_0\in(0,T) \text{ and for all } p\in[1,\infty),$$

whence estimate (5.32) for $\frac{1}{\theta}$. We conclude the estimate for $\left\|\frac{1}{\theta_{\mathrm{s}}}\right\|_{L^\infty(\Gamma_{\mathrm{C}}\times(0,T))}$ in the very same way. This finishes the proof. □

We are now in a position to prove our existence result for the Cauchy problem for (a weak formulation of) the regularized system (3.69).

Theorem 5.2. *Assume* (2.4)–(2.10) *and* (2.13)–(2.18). *Let* $\rho,\varsigma>0$ *be fixed. Then, for any quadruple* $(\theta_\rho^0,\mathbf{u}_\rho^0,\theta_{\mathrm{s},\rho}^0,\chi_0)$ *as in* (3.72) *and* (2.20d), *there exists a quintuple* $(\theta,\mathbf{u},\theta_{\mathrm{s}},\chi,\sigma)$, *with*

$$\begin{aligned}
&\theta\in L^2(0,T;H^1(\Omega))\cap \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega))\cap H^1(0,T;H^1(\Omega))^* \quad \textit{and} \quad \widehat{\alpha}(\theta)\in L^2(0,T;H^1(\Omega)),\\
&\mathbf{u}\in W^{1,\omega}(0,T;W^{1,\omega}(\Omega;\mathbb{R}^3)),\\
&\theta_{\mathrm{s}}\in L^2(0,T;H^1(\Gamma_{\mathrm{C}}))\cap \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Gamma_{\mathrm{C}}))\cap H^1(0,T;H^1(\Gamma_{\mathrm{C}}))^* \quad \textit{and} \quad \widehat{\alpha}(\theta_{\mathrm{s}})\in L^2(0,T;H^1(\Gamma_{\mathrm{C}})),\\
&\chi\in L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H^1(\Gamma_{\mathrm{C}}))\cap W^{1,\omega}(0,T;L^\omega(\Omega)),\\
&\sigma\in L^\infty(\Gamma_{\mathrm{C}}\times(0,T)),
\end{aligned}$$

fulfilling the initial conditions

$$\theta(0)=\theta_\rho^0 \text{ a.e. in } \Omega,\ \theta_{\mathrm{s}}(0)=\theta_{\mathrm{s},\rho}^0 \text{ a.e. in } \Gamma_{\mathrm{C}},\ \mathbf{u}(0)=\mathbf{u}_\rho^0 \text{ a.e. in } \Omega,\ \chi(0)=\chi_0 \text{ a.e. in } \Gamma_{\mathrm{C}}, \tag{5.34}$$

and the weak formulation of system (3.69), *consisting of*

(1) the weak formulation of the bulk heat equation for almost all $t\in(0,T)$

$$\begin{aligned}
&\langle\theta_t,v\rangle_{H^1(\Omega)} - \int_\Omega \theta\mathrm{div}(\mathbf{u}_t)v\,\mathrm{d}x + \int_\Omega \alpha(\theta)\nabla\theta\nabla v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} k(\chi)\theta(\theta-\theta_{\mathrm{s}})v\,\mathrm{d}x\\
&\quad + \int_{\Gamma_{\mathrm{C}}} \mathcal{J}[(\chi)^+]\,(\chi)^+\theta^2 v\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} \mathcal{J}[(\chi)^+\theta_{\mathrm{s}}]\,(\chi)^+\theta v\,\mathrm{d}x = \int_\Omega \varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t)v\,\mathrm{d}x + \int_\Omega hv\,\mathrm{d}x
\end{aligned} \tag{5.35a}$$

for all $v\in H^1(\Omega)$;

(2) the weak formulation of the displacement equation for almost all $t\in(0,T)$

$$\begin{aligned}
&\mathsf{v}(\mathbf{u}_t,\mathbf{v}) + \rho\int_\Omega |\varepsilon(\mathbf{u}_t)|^{\omega-2}\varepsilon(\mathbf{u}_t)\varepsilon(\mathbf{v})\,\mathrm{d}x + \mathsf{e}(\mathbf{u},\mathbf{v}) + \int_\Omega \theta\mathrm{div}(\mathbf{v})\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} (\chi)^+\mathbf{u}\mathbf{v}\,\mathrm{d}x\\
&\quad + \int_{\Gamma_{\mathrm{C}}} \boldsymbol{\zeta}\cdot\mathbf{v}\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} (\chi)^+\mathcal{J}[(\chi)^+]\,\mathbf{u}\mathbf{v}\,\mathrm{d}x = \langle\mathbf{F},\mathbf{v}\rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)}
\end{aligned} \tag{5.35b}$$

for all $\mathbf{v}\in W^{1,\omega}_{\mathrm{D}}(\Omega;\mathbb{R}^3)$, *with* $\boldsymbol{\zeta}(t)=\eta_\varsigma(\mathbf{u}(t)\cdot\mathbf{n})\mathbf{n}$ *for a.a.* $t\in(0,T)$;

(3) the weak formulation of the surface heat equation for almost all $t \in (0,T)$

$$
\begin{aligned}
&\langle \partial_t \theta_s, w \rangle_{H^1(\Gamma_C)} - \int_{\Gamma_C} \theta_s \lambda'(\chi)\chi_t w \,\mathrm{d}x + \int_{\Gamma_C} \alpha(\theta_s)\nabla\theta_s \nabla w \,\mathrm{d}x \\
&= \int_{\Gamma_C} \ell w \,\mathrm{d}x + \int_{\Gamma_C} |\chi_t|^2 w \,\mathrm{d}x + \int_{\Gamma_C} k(\chi)\theta_s(\theta-\theta_s) w \,\mathrm{d}x + \int_{\Gamma_C} \mathcal{J}[(\chi)^+\theta]\,(\chi)^+\theta_s w \,\mathrm{d}x - \int_{\Gamma_C} \mathcal{J}[(\chi)^+]\,(\chi)^+\theta_s^2 w \,\mathrm{d}x
\end{aligned} \tag{5.35c}
$$

with test functions $w \in H^1(\Gamma_C)$, a.e. in $(0,T)$;

(4) the flow rule for the adhesion parameter

$$
\begin{aligned}
&\chi_t + \rho|\chi_t|^{\omega-2}\chi_t + A\chi + \beta_\varsigma(\chi) + \gamma'(\chi) + \lambda'(\chi)\theta_s \\
&\quad = -\frac{1}{2}|\mathbf{u}|^2\sigma - \frac{1}{2}\mathcal{J}[(\chi)^+]\,|\mathbf{u}|^2\sigma - \frac{1}{2}\mathcal{J}[(\chi)^+|\mathbf{u}|^2]\,\sigma \quad \text{a.e. in } \Gamma_C \times (0,T)
\end{aligned} \tag{5.35d}
$$

with $\sigma \in \partial\varphi(\chi)$ a.e. in Γ_C.

Furthermore, estimate (5.32) *holds and the quadruple $(\theta, \mathbf{u}, \theta_s, \chi)$ satisfies the total energy balance (with the stored energy $\mathcal{E}_\varsigma$ from* (4.5)*)*

$$
\begin{aligned}
&\mathcal{E}_\varsigma(\theta(t),\theta_s(t),\mathbf{u}(t),\chi(t)) + \rho\int_s^t\int_\Omega |\varepsilon(\mathbf{u}_t)|^\omega \,\mathrm{d}x\,\mathrm{d}r + \rho\int_s^t\int_{\Gamma_C} |\chi_t|^\omega \,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \int_s^t\int_{\Gamma_C} k(\chi)(\theta-\theta_s)^2 \,\mathrm{d}x\,\mathrm{d}r \int_s^t\int_{\Gamma_C} j(x,y)(\chi(x))^+(\chi(y))^+(\theta(x)-\theta_s(y))^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r \\
&= \mathcal{E}_\varsigma(\theta(s),\theta_s(s),\mathbf{u}(s),\chi(s)) + \int_s^t\int_\Omega h \,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_{\Gamma_C} \ell \,\mathrm{d}x\,\mathrm{d}r + \int_s^t \langle \mathbf{F}, \mathbf{u}_t\rangle_{H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)} \,\mathrm{d}r
\end{aligned} \tag{5.35e}
$$

for all $0 \le s \le t \le T$.

Proof. We shall pass to the limit in system (5.4) relying on convergences (5.31) for the approximate solutions.

Step 1: limit passage in the momentum balance: First of all, we focus on the limit passage in equation (5.4b), which we integrate in time. Thanks to convergences (5.31b), (5.31k), and (5.31m) we pass to the limit in the first, third and fourth integral terms on the left-hand side of (5.4b). As for the second term, we observe that there exists $\mathbf{E} \in L^{\omega/(\omega-1)}(\Omega\times(0,T);\mathbb{R}^{3\times 3})$ such that

$$
\left|\varepsilon\left(\widehat{\mathbf{u}}'_{\tau_k}\right)\right|^{\omega-2}\varepsilon\left(\widehat{\mathbf{u}}'_{\tau_k}\right) \rightharpoonup \mathbf{E} \quad \text{in } L^{\omega/(\omega-1)}(\Omega\times(0,T);\mathbb{R}^{3\times 3}) \tag{5.36}
$$

as $\tau_k \downarrow 0$. We also use that

$$
\begin{cases}
(\overline{\chi}_{\tau_k})^+\overline{\mathbf{u}}_{\tau_k} \to (\chi)^+\mathbf{u}, \\
\mathcal{J}[(\overline{\chi}_{\tau_k})^+]\,(\overline{\chi}_{\tau_k})^+\overline{\mathbf{u}}_{\tau_k} \to \mathcal{J}[(\chi)^+](\chi)^+\mathbf{u}
\end{cases} \quad \text{in } L^\infty(0,T;L^q(\Gamma_C)) \quad \text{for all } 1 \le q < \infty \tag{5.37a}
$$

as $\tau_k \downarrow 0$ thanks to convergences (5.31n), (5.31r), and Lemma 3.1. By the Lipschitz continuity of η_ς we readily have that

$$
\overline{\boldsymbol{\zeta}}_{\tau_k} \to \boldsymbol{\zeta} := \eta_\varsigma(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \qquad \text{in } \mathrm{C}^0(0,T;\mathrm{C}^0(\overline{\Gamma}_C)). \tag{5.37b}
$$

Finally, we use that

$$
\overline{\mathbf{F}}_{\tau_k} \to \mathbf{F} \qquad \text{in } L^2(0,T;H^1(\Omega,\mathbb{R}^3)^*). \tag{5.38}
$$

All in all, we conclude that

$$
\begin{aligned}
&\int_s^t \left(\mathsf{v}(\mathbf{u}_t,\mathbf{v}) + \rho\,\mathbf{E} : \varepsilon(\mathbf{v}) \,\mathrm{d}x + \mathsf{e}(\mathbf{u},\mathbf{v}) + \int_\Omega \theta \mathrm{div}(\mathbf{v}) \,\mathrm{d}x + \int_{\Gamma_C} (\chi)^+\mathbf{u}\mathbf{v} \,\mathrm{d}x \right) \mathrm{d}r \\
&\quad + \int_s^t \left(\int_{\Gamma_C} \boldsymbol{\zeta}\cdot\mathbf{v} \,\mathrm{d}x + \int_{\Gamma_C} \mathcal{J}[(\chi)^+]\,(\chi)^+\mathbf{u}\mathbf{v} \,\mathrm{d}x \right) \mathrm{d}r = \int_s^t \langle \mathbf{F}, \mathbf{v}\rangle_{H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)} \,\mathrm{d}r
\end{aligned} \tag{5.39}
$$

for every test function $\mathbf{v} \in W_{\mathrm{D}}^{1,\omega}(\Omega;\mathbb{R}^3)$ and every $0 \le s \le t \le T$. Then, testing (5.4b) by $\widehat{\mathbf{u}}_{\tau_k}'$ we infer that for every $(s,t) \subset (0,T)$ there holds

$$
\begin{aligned}
&\limsup_{\tau_k \searrow 0} \rho \int_s^t \int_\Omega \left|\varepsilon\left(\widehat{\mathbf{u}}_{\tau_k}'\right)\right|^\omega \,\mathrm{d}x\,\mathrm{d}r \\
&\le \limsup_{k\to\infty} \Big(- \int_s^t \left(\mathsf{v}(\widehat{\mathbf{u}}_{\tau_k}', \widehat{\mathbf{u}}_{\tau_k}') + e(\overline{\mathbf{u}}_{\tau_k}, \widehat{\mathbf{u}}_{\tau_k}') + \int_\Omega \overline{\theta}_{\tau_k} \mathrm{div}(\widehat{\mathbf{u}}_{\tau_k}')\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} (\overline{\chi}_{\tau_k})^+ \overline{\mathbf{u}}_{\tau_k} \widehat{\mathbf{u}}_{\tau_k}'\,\mathrm{d}x \right) \mathrm{d}r \\
&\qquad - \int_s^t \left(\int_{\Gamma_{\mathrm{C}}} \langle \overline{\boldsymbol{\zeta}}_{\tau_k} \cdot \widehat{\mathbf{u}}_{\tau_k}'\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \partial[(\overline{\chi}_{\tau_k})^+]\,(\overline{\chi}_{\tau_k})^+ \overline{\mathbf{u}}_{\tau_k} \widehat{\mathbf{u}}_{\tau_k}'\,\mathrm{d}x - \langle \overline{\mathbf{F}}_{\tau_k}(t), \widehat{\mathbf{u}}_{\tau_k}' \rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)} \right) \mathrm{d}r \Big) \\
&\overset{(1)}{\le} - \int_s^t \left(\mathsf{v}(\mathbf{u}_t, \mathbf{u}_t) + \mathsf{e}(\mathbf{u}, \mathbf{u}_t) + \int_\Omega \theta \mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} (\chi)^+ \mathbf{u}\mathbf{u}_t\,\mathrm{d}x \right) \mathrm{d}r \\
&\qquad - \int_s^t \left(\int_{\Gamma_{\mathrm{C}}} \boldsymbol{\zeta} \cdot \mathbf{u}_t\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} \partial[(\chi)^+]\,(\chi)^+ \mathbf{u}\mathbf{u}_t\,\mathrm{d}x - \langle \mathbf{F}, \mathbf{u}_t \rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)} \right) \mathrm{d}r \\
&\overset{(2)}{=} \rho \int_s^t \mathbf{E} : \varepsilon(\mathbf{u}_t)\,\mathrm{d}r\,,
\end{aligned}
\tag{5.40}
$$

where (1) ensues from convergences (5.31) (which, in particular, yield that $\overline{\theta}_{\tau_k} \to \theta$ in $L^2(0,T;L^2(\Omega))$, for instance) and (5.37), while (2) follows from the previously obtained (5.39). Hence, [1, Lemma 1.3, p. 42] yields that

$$
\mathbf{E} = |\varepsilon(\mathbf{u}')|^{\omega-2}\varepsilon(\mathbf{u}') \text{ a.e. in } \Omega\times(0,T) \quad \text{ and } \quad \varepsilon(\widehat{\mathbf{u}}_{\tau_k}') \to \varepsilon(\mathbf{u}_t) \text{ strongly in } L^\omega(\Omega\times(0,T);\mathbb{R}^{3\times3})\,, \tag{5.41}
$$

and, since the interval (s,t) in (5.39) is chosen arbitrarily, we thus conclude the momentum balance equation (5.35b). We remark for later use that (5.41) yields that

$$
\varepsilon(\widehat{\mathbf{u}}_{\tau_k}'(t))\mathbb{V}\varepsilon(\widehat{\mathbf{u}}_{\tau_k}'(t)) \to \varepsilon(\mathbf{u}_t(t))\mathbb{V}\varepsilon(\mathbf{u}_t(t)) \quad \text{strongly in } L^2(\Omega) \text{ for a.e. } t\in(0,T). \tag{5.42}
$$

Step 2: limit passage in the flow rule (5.4d)**:** We now address the limit passage in the approximate flow rule (5.4d), integrated on a generic interval $(s,t) \subset (0,T)$. We use that there exists $\Lambda \in L^{\omega/(\omega-1)}(\Gamma_{\mathrm{C}}\times(0,T))$ such that

$$
|\widehat{\chi}_{\tau_k}'|^{\omega-2}\widehat{\chi}_{\tau_k'}' \rightharpoonup \Lambda \quad \text{ in } L^{\omega/(\omega-1)}(\Gamma_{\mathrm{C}}\times(0,T)),
$$

and that $\beta_\varsigma(\overline{\chi}_{\tau_k}) \to \beta_\varsigma(\chi)$ and $\gamma_\nu'(\overline{\chi}_{\tau_k}) \to \gamma_\nu'(\chi)$ in $L^\infty(0,T;L^q(\Gamma_{\mathrm{C}}))$ for every $1 \le q < \infty$ by the Lipschitz continuity of β_ς and γ_ν'. Also taking into account convergence (5.31r) for the right-continuous piecewise constant interpolants $(\overline{\chi}_{\tau_k})_k$, we carry out the limit passage for the terms on the left-hand side of (5.4d). As for the right-hand side, we use that λ_δ' is Lipschitz continuous and that, for instance, $\overline{\theta}_{\mathrm{s},\tau_k} \to \theta_{\mathrm{s}}$ in $L^2(0,T;L^2(\Gamma_{\mathrm{C}}))$, so that

$$
-\lambda_\delta'(\underline{\chi}_{\tau_k})\overline{\theta}_{\mathrm{s},\tau_k} - \delta\overline{\chi}_{\tau_k}\overline{\theta}_{\mathrm{s},\tau_k} \to -\lambda'(\chi)\theta_{\mathrm{s}} \qquad \text{in } L^2(0,T;L^{\omega/(\omega-1)}(\Gamma_{\mathrm{C}}\times(0,T)).
$$

We combine (5.31q), (5.31r), and (5.31s) yielding that

$$
\begin{cases}
-\frac12|\underline{\mathbf{u}}_{\tau_k}|^2\overline{\sigma}_{\tau_k} \rightharpoonup^* -\frac12|\mathbf{u}|^2\sigma, \\
-\frac12\partial[(\overline{\chi}_{\tau_k})^+]\,|\underline{\mathbf{u}}_{\tau_k}|^2\overline{\sigma}_{\tau_k} \rightharpoonup^* -\frac12\partial[(\chi)^+]|\mathbf{u}|^2\sigma, \\
-\frac12\partial[(\underline{\chi}_{\tau_k})^+|\underline{\mathbf{u}}_{\tau_k}|^2]\,\overline{\sigma}_{\tau_k} \rightharpoonup^* -\frac12\partial[(\chi)^+|\mathbf{u}|^2]\,\sigma
\end{cases}
\qquad \text{in } L^\infty(\Gamma_{\mathrm{C}}\times(0,T))
$$

also in view of Lemma 3.1. All in all, also recalling (5.33) we take the limit of (5.4d) and obtain that

$$
\begin{aligned}
&\int_s^t \left(\int_{\Gamma_{\mathrm{C}}} \chi_t v\,\mathrm{d}x + \rho\int_{\Gamma_{\mathrm{C}}} \Lambda v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} A\chi v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} (\beta_\varsigma(\chi) + \gamma'(\chi) + \lambda'(\chi)\theta_{\mathrm{s}})v\,\mathrm{d}x \right) \mathrm{d}r \\
&= -\frac12 \int_s^t \left(\int_{\Gamma_{\mathrm{C}}} \left(|\mathbf{u}|^2\sigma + \partial[(\chi)^+]|\mathbf{u}|^2\sigma + \partial[(\chi)^+|\mathbf{u}|^2]\sigma\right) v\,\mathrm{d}x \right) \mathrm{d}r
\end{aligned}
$$

for every $v \in L^\omega(\Gamma_{\mathrm{C}})$ and every sub-interval $[s,t] \subset (0,T)$. By the strong-weak closedness in the sense of graphs of (the maximal monotone operator induced by) $\partial\varphi$, we have that $\sigma \in \partial\varphi(\chi)$ a.e. in $\Gamma_{\mathrm{C}} \times (0,T)$. In order to identify the weak limit Λ, we proceed as for the weak limit $\mathbf{E}$ from (5.36) and conclude that that

$$
\Lambda = |\chi_t|^{\omega-2}\chi_t \text{ and } \widehat{\chi}_{\tau_k}' \to \chi_t \text{ strongly in } L^\omega(\Gamma_{\mathrm{C}}\times(0,T))\,. \tag{5.43}
$$

In particular,

$$|\widehat{\chi}'_{\tau_k}(t)|^2 \to |\chi_t(t)|^2 \quad \text{strongly in } L^2(\Gamma_{\mathrm{C}}) \text{ for a.a. } t \in (0,T)\,. \tag{5.44}$$

From (5.43) we deduce the validity of the flow rule for the adhesion parameter integrated along an arbitrary time interval, and with arbitrary test functions $v \in L^\omega(\Gamma_{\mathrm{C}})$. Then, (5.35d) ensues.

Step 3: limit passage in the bulk heat equation: We are now in a position to perform the limit passage in (5.4a), integrated in time. For this, we need to refine the convergences available for the sequences $(\overline{\theta}_{\tau_k})$ and $(\overline{\theta}_{\mathrm{s},\tau_k})$.

(1) In order to pass to the limit in the elliptic operator, we will use that $\nabla\widehat{\alpha}(\overline{\theta}_{\tau_k}) = \alpha(\overline{\theta}_{\tau_k})\nabla\overline{\theta}_{\tau_k}$ a.e. in $\Omega\times(0,T)$. First of all, we notice that, by (5.6a) there exists $\phi \in L^2(0,T;H^1(\Omega))$ such that, along a not relabeled subsequence,

$$\widehat{\alpha}(\overline{\theta}_{\tau_k}) \rightharpoonup \phi \quad \text{in } L^2(0,T;H^1(\Omega)).$$

On the other hand, $\widehat{\alpha}(\overline{\theta}_{\tau_k}) \to \widehat{\alpha}(\theta)$ a.e. in $\Omega\times(0,T)$ thanks to (5.31e). We combine this with the fact that $(\overline{\theta}_{\tau_k})_k$ is bounded in $L^\infty(0,T;L^{\mu+2}(\Omega))$, thanks to (5.25) and the growth properties of $\widehat{\widehat{\alpha}}$, to deduce that $\widehat{\alpha}(\overline{\theta}_{\tau_k}) \rightharpoonup^* \widehat{\alpha}(\theta)$ in $L^\infty(0,T;L^{(\mu+2)/(\mu+1)}(\Omega))$ (by the growth properties of $\widehat{\alpha}$). Therefore, we ultimately conclude that $\phi = \widehat{\alpha}(\theta)$, so that

$$\widehat{\alpha}(\overline{\theta}_{\tau_k}) \rightharpoonup \widehat{\alpha}(\theta) \quad \text{in } L^2(0,T;H^1(\Omega)). \tag{5.45}$$

(2) In order to identify the elliptic operator featuring in the bulk heat equation, we argue in a similar way as we did in the proof of Proposition 4.3. Indeed, from the fact that $(\widehat{\alpha}(\overline{\theta}_{\tau_k}))_k$ is bounded in $L^2(0,T;H^1(\Omega))$ and from the growth properties of $\widehat{\alpha}$ we deduce that $(\overline{\theta}_{\tau_k})_k$ is bounded in $L^{2(\mu+1)}(0,T;L^{6(\mu+1)}(\Omega))$, with $\mu > 1$. Taking into account (5.31e) we deduce, a fortiori, that

$$\overline{\theta}_{\tau_k} \to \theta \quad \text{in } L^4(\Omega\times(0,T)), \tag{5.46}$$

as well as $\alpha(\overline{\theta}_{\tau_k}) \to \alpha(\theta)$ in $L^{2s}(0,T;L^{6s}(\Omega))$ for every $1 \le s < 1+\frac{1}{\mu}$. This is enough to pass to the limit in the relation $\int_\Omega \nabla\widehat{\alpha}(\overline{\theta}_{\tau_k})\cdot\nabla v\,\mathrm{d}x = \int_\Omega \alpha(\overline{\theta}_{\tau_k})\nabla\overline{\theta}_{\tau_k}\cdot\nabla v\,\mathrm{d}x$ for every $v \in H^1(\Omega)$ by suitably adapting the arguments developed at the end of the proof of Prop. 4.3.

(3) It follows from (5.31d), combined with the fact that $H^1(\Omega) \Subset L^p(\Gamma_{\mathrm{C}})$ (in the sense of traces) for every $1 \le p < 4$, that

$$\overline{\theta}_{\tau_k} \to \theta \qquad \text{a.e. in } \Gamma_{\mathrm{C}}\times(0,T).$$

In turn, from the fact that $(\widehat{\alpha}(\overline{\theta}_{\tau_k}))_k$ is bounded in $L^2(0,T;L^4(\Gamma_{\mathrm{C}}))$ we gather that $(\overline{\theta}_{\tau_k})_k$ is bounded in $L^{2(\mu+1)}(0,T;L^{4(\mu+1)}(\Gamma_{\mathrm{C}}))$. Combining this with the above pointwise convergence we immediately infer that

$$\overline{\theta}_{\tau_k} \to \theta \qquad \text{in } L^{4+\epsilon}(\Gamma_{\mathrm{C}}\times(0,T)) \qquad \text{for all } \epsilon \in (0,2\mu-2) \tag{5.47}$$

(so that $4+\epsilon < 2\mu+2$).

(4) Analogously, combining (5.31i) with the estimate for $(\widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau_k}))_k$ is bounded in $L^2(0,T;H^1(\Gamma_{\mathrm{C}}))$, which continuously embeds into $L^2(0,T;L^q(\Gamma_{\mathrm{C}}))$ for all $1 \le q < \infty$, we deduce, for instance, that

$$\overline{\theta}_{\mathrm{s},\tau_k} \to \theta_{\mathrm{s}} \qquad \text{in } L^{4+\epsilon}(\Gamma_{\mathrm{C}}\times(0,T)) \qquad \text{for all } \epsilon \in (0,2\mu-2). \tag{5.48}$$

In view of the enhanced convergences (5.46)–(5.48), we infer that

$$\begin{aligned}
&\overline{\theta}_{\tau_k}\mathrm{div}(\widehat{\mathbf{u}}'_{\tau_k}) \rightharpoonup \theta\mathrm{div}(\mathbf{u}') && \text{in } L^2(\Omega\times(0,T)),\\
&k(\overline{\chi}_{\tau_k})\overline{\theta}_{\tau_k}(\overline{\theta}_{\tau_k}-\overline{\theta}_{\mathrm{s},\tau_k}) \to k(\chi)\theta(\theta-\theta_{\mathrm{s}}) && \text{in } L^2(\Gamma_{\mathrm{C}}\times(0,T)),\\
&\mathcal{J}[(\underline{\chi}_{\tau_k})^+]\,(\underline{\chi}_{\tau_k})^+\overline{\theta}_{\tau_k}^2 \to \mathcal{J}[(\chi)^+]\,(\chi)^+\theta^2 && \text{in } L^2(\Gamma_{\mathrm{C}}\times(0,T)),\\
&\mathcal{J}[(\underline{\chi}_{\tau_k})^+\overline{\theta}_{\mathrm{s},\tau_k}]\,(\underline{\chi}_{\tau_k})^+\overline{\theta}_{\tau_k} \to \mathcal{J}[(\chi)^+\theta_{\mathrm{s}}]\,(\chi)^+\theta && \text{in } L^2(\Gamma_{\mathrm{C}}\times(0,T)),
\end{aligned} \tag{5.49}$$

where we have also used that $k(\underline{\chi}_{\tau_k}) \to k(\chi)$ in $L^q(\Gamma_{\mathrm{C}}\times(0,T))$ for all $1 \le q < \infty$ thanks to (5.31r) and the polynomial growth of k, that $\mathcal{J}[\underline{\chi}_{\tau_k}] \to \mathcal{J}[\chi]$ in $L^\infty(\Gamma_{\mathrm{C}}\times(0,T))$ by Lemma 3.1, as well as that $\mathcal{J}[\underline{\chi}_{\tau_k}\overline{\theta}_{\mathrm{s},\tau_k}] \to \mathcal{J}[\chi\theta]$ in $L^4(\Gamma_{\mathrm{C}}\times(0,T))$.

Also recalling (5.42) and the fact that

$$\overline{h}_{\tau_k} \to h \qquad \text{in } L^2(0,T;H^1(\Omega)^*) \tag{5.50}$$

we conclude the limit passage in (5.4a). This yields the weak formulation (5.35a) of the bulk heat equation, with test functions $v \in H^1(\Omega)$, a.e. in $(0,T)$.

Step 4: limit passage in the surface heat equation: For the limit passage in (5.4c) we use that

$$\widehat{\alpha}(\overline{\theta}_{\mathrm{s},\tau_k}) \rightharpoonup \widehat{\alpha}(\theta_{\mathrm{s}}) \quad \text{in } L^2(0,T;H^1(\Gamma_{\mathrm{C}}))$$

(which can be shown by the very same arguments as for (5.45)). Arguing as we did for the bulk heat equation, we identify the elliptic operator featuring in the limiting surface heat equation. Furthermore, we observe that

$$\begin{aligned} \frac{\lambda(\overline{\chi}_{\tau_k}) - \lambda(\underline{\chi}_{\tau_k})}{\tau_k} & = \frac{1}{\tau_k}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} f(r)\,\mathrm{d}r \qquad && \text{with } f(r) = \lambda(\underline{\chi}_{\tau_k} + r(\overline{\chi}_{\tau_k} - \underline{\chi}_{\tau_k})) \\ & = \int_0^1 \lambda'(\underline{\chi}_{\tau_k} + r(\overline{\chi}_{\tau_k} - \underline{\chi}_{\tau_k}))\frac{1}{\tau_k}(\overline{\chi}_{\tau_k} - \underline{\chi}_{\tau_k})\,\mathrm{d}r \\ & \longrightarrow \int_0^1 \lambda'(\chi)\chi_t\,\mathrm{d}r = \lambda'(\chi)\chi_t && \text{strongly in } L^2(\Gamma_{\mathrm{C}}\times(0,T)), \end{aligned} \tag{5.51}$$

thanks to the Lipschitz continuity of λ' combined with convergences (5.31r) and (5.44). The latter convergence also allows us to pass to the limit in the first term on the right-hand side of (5.4c); the limit passage in the second, third, and fourth terms follows by the same arguments leading to (5.49). Finally, we observe that

$$\overline{\ell}_{\tau_k} \to \ell \quad \text{in } L^2(0,T;H^1(\Gamma_{\mathrm{C}})^*).$$

All in all, we deduce that the triple $(\theta, \theta_{\mathrm{s}}, \chi)$ fulfills the weak formulation (5.35c) of the surface heat equation, with test functions $w \in H^1(\Gamma_{\mathrm{C}})$, a.e. in $(0,T)$.

Finally, (5.35e) follows from testing the weak formulation (5.35a) of the bulk heat equation by 1, the weak momentum balance (5.35b) by $\mathbf{u}_t$, the weak surface heat equation (5.35c) by 1, the flow rule (5.35d) by χ_t, adding the resulting relations, and integrating them over the generic interval $[s,t] \subset [0,T]$. This concludes the proof. □

6. Proof of Theorem 1

In order to prove Theorem 1, we will perform a double limit passage in system (3.69) (more precisely, in its weak formulation that was specified in Theorem 5.2). We shall first pass to the limit as $\rho \downarrow 0$, with the parameter $\varsigma > 0$ fixed, and then as $\varsigma \downarrow 0$. Let us thus consider a family $(\theta_{\rho,\varsigma}, \mathbf{u}_{\rho,\varsigma}, \boldsymbol{\zeta}_{\rho,\varsigma}, \theta_{\mathrm{s},\rho,\varsigma}, \chi_{\rho,\varsigma}, \sigma_{\rho,\varsigma}, \xi_{\rho,\varsigma})_{\rho,\varsigma}$, with

$$\boldsymbol{\zeta}_{\rho,\varsigma} := \eta_\varsigma(\mathbf{u}_{\rho,\varsigma}\cdot\mathbf{n})\mathbf{n}, \qquad \xi_{\rho,\varsigma} = \beta_\varsigma(\chi_{\rho,\varsigma}),$$

of weak solutions to the Cauchy problem for the approximate system (3.69); the first result of this Section collects all the a priori estimates, uniform w.r.t. ρ *and* ς, on which our compactness arguments shall rely. As we will see, these estimates can be obtained by replicating the formal estimates carried out in Section 3.3 on the level of system (3.69).

Proposition 6.1. *There holds for every ρ, $\varsigma > 0$*

$$\theta_{\rho,\varsigma} \geq \frac{1}{S_0} > 0 \quad \textit{a.e. in}\, \Omega\times(0,T), \qquad \theta_{\mathrm{s},\rho,\varsigma} \geq \frac{1}{S_0} > 0 \quad \textit{a.e. in}\, \Gamma_{\mathrm{C}}\times(0,T), \tag{6.1}$$

with S_0 from (5.32). *Furthermore, exists a constant $\overline{S} > 0$ such that the following estimates hold for all ρ, $\varsigma > 0$:*

$$\|\theta_{\rho,\varsigma}\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^1(\Omega))\cap W^{1,1}(0,T;W^{1,3+\epsilon}(\Omega)^*)} \leq \overline{S}, \tag{6.2a}$$

$$\|(\theta_{\rho,\varsigma})^{(\mu+\nu)/2}\|_{L^2(0,T;H^1(\Omega))} \leq \overline{S}, \tag{6.2b}$$

$$\|\theta_{\mathrm{s},\rho,\varsigma}\|_{L^2(0,T;H^1(\Gamma_{\mathrm{C}}))\cap L^\infty(0,T;L^1(\Gamma_{\mathrm{C}}))\cap W^{1,1}(0,T;W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})^*)} \leq \overline{S}, \tag{6.2c}$$

$$\|(\theta_{\mathrm{s},\rho,\varsigma})^{(\mu+\nu)/2}\|_{L^2(0,T;H^1(\Gamma_{\mathrm{C}}))} \leq \overline{S}, \tag{6.2d}$$

$$\|\mathbf{u}_{\rho,\varsigma}\|_{H^1(0,T;H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3))} + \rho^{1/\omega}\|\varepsilon(\partial_t\mathbf{u}_{\rho,\varsigma})\|_{L^\omega(\Omega\times(0,T);\mathbb{R}^{3\times3})} \leq \overline{S}, \tag{6.2e}$$

$$\|\boldsymbol{\zeta}_{\rho,\varsigma}\|_{L^{\omega/(\omega-1)}(0,T;W^{1,\omega}(\Omega;\mathbb{R}^3)^*)} \leq \overline{S}, \tag{6.2f}$$

$$\|\chi_{\rho,\varsigma}\|_{L^\infty(0,T;H^1(\Gamma_{\mathrm{C}}))\cap H^1(0,T;L^2(\Gamma_{\mathrm{C}}))} + \rho^{1/\omega}\|\partial_t\chi_{\rho,\varsigma}\|_{L^\omega(\Omega\times(0,T))} \leq \overline{S}, \tag{6.2g}$$

$$\|A\chi_{\rho,\varsigma} + \xi_{\rho,\varsigma}\|_{L^{\omega/(\omega-1)}(\Gamma_{\mathrm{C}}\times(0,T))} \leq \overline{S}, \tag{6.2h}$$

$$\|\sigma_{\rho,\varsigma}\|_{L^\infty(\Gamma_{\mathrm{C}}\times(0,T))} \leq \overline{S}. \tag{6.2i}$$

Proof. The positivity property (6.1) clearly follows from estimate (5.32). The bounds for $(\theta_{\rho,\varsigma})_{\rho,\varsigma}$ in $L^\infty(0,T;L^1(\Omega))$, for $(\theta_{\mathrm{s},\rho,\varsigma})_{\rho,\varsigma}$ in $L^\infty(0,T;L^1(\Gamma_\mathrm{C})$, for $(\rho^{1/\omega}\varepsilon(\partial_t\mathbf{u}_{\rho,\varsigma}))_{\rho,\varsigma}$ in $L^\omega(\Omega\times(0,T);\mathbb{R}^{3\times3})$, for $(\chi_{\rho,\varsigma})_{\rho,\varsigma}$ in $L^\infty(0,T;H^1(\Gamma_\mathrm{C}))$, and for $(\rho^{1/\omega}\partial_t\chi_{\rho,\varsigma})_{\rho,\varsigma}$ in $L^\omega(\Gamma_\mathrm{C}\times(0,T))$ follow from the total energy balance (5.35e), arguing in the very same way as in Section 3.3.1. Estimate (6.2i) simply follows from the fact that $\sigma_{\rho,\varsigma}\in\partial\varphi(\chi_{\rho,\varsigma})$ a.e. in $\Gamma_\mathrm{C}\times(0,T)$.

We then proceed to the Second a priori estimate (cf. Sec. 3.3.2) and test the weak formulations (3.4) and (3.5) of the heat equations

1. by $\theta_{\rho,\varsigma}^{\nu-1}$ and $\theta_{\mathrm{s},\rho,\varsigma}^{\nu-1}$, with $\nu=2-\mu$, in the case $\mu\in(1,2)$;
2. by $-\theta_{\rho,\varsigma}^{-1}$ and $-\theta_{\mathrm{s},\rho,\varsigma}^{-1}$ in the case $\mu=2$;
3. by $-\theta_{\rho,\varsigma}^{-q}$ and $-\theta_{\mathrm{s},\rho,\varsigma}^{-q}$, with $q=\mu-1$, in the case $\mu>2$.

Observe that in all of the above cases the test functions are admissible (namely, they belong to $H^1(\Omega)$ and $H^1(\Gamma_\mathrm{C})$, respectively), thanks to (6.1), combined with the fact that $\theta_{\rho,\varsigma}\in L^2(0,T;H^1(\Omega))$ and $\theta_{\mathrm{s},\rho,\varsigma}\in L^2(0,T;H^1(\Gamma_\mathrm{C}))$, respectively. We then add the resulting relations, integrate in time, and perform, in the three cases $\mu\in(1,2)$, $\mu=2$, and $\mu>2$, the very same calculations as in Section 3.3.2. In this way, we conclude that $\|\theta_{\rho,\varsigma}\|_{L^2(0,T;H^1(\Omega))}\le C$ and $\|\theta_{\mathrm{s},\rho,\varsigma}\|_{L^2(0,T;H^1(\Gamma_\mathrm{C}))}\le C$. These estimates are enhanced to (6.2b) and (6.2d) by repeating the calculations from Section 3.3.3.

In order to replicate the Fourth a priori estimate from Section 3.3.4, we subtract from the total energy balance (5.35e) the bulk and surface heat equations tested by 1 and integrated in time. This leads to the analogue of the *mechanical energy inequality* (3.50), additionally featuring the integrals $\rho\int_0^t\int_\Omega|\varepsilon(\partial_t\mathbf{u}_{\rho,\varsigma})|^\omega\,\mathrm{d}x\,\mathrm{d}r$ and $\rho\int_0^t\int_{\Gamma_\mathrm{C}}|\partial_t\chi_{\rho,\varsigma}|^\omega\,\mathrm{d}x\,\mathrm{d}r$ on the left-hand side. Repeating the very same calculations as in Sec. 3.3.4, we conclude the estimates for $\|\mathbf{u}_{\rho,\varsigma}\|_{H^1(0,T;H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3))}$ and $\|\chi_{\rho,\varsigma}\|_{H^1(0,T;L^2(\Gamma_\mathrm{C}))}$.

Relying on the Sixth a priori estimate (cf. Sec. 3.3.6), which yields the bounds (3.55) and (3.57) for the sequences $(\theta_{\rho,\varsigma})_{\rho,\varsigma}$ and $(\theta_{\mathrm{s},\rho,\varsigma})_{\rho,\varsigma}$, we are in a position to rigorously render the calculations for Seventh a priori estimate, cf. Sec. 3.3.7. Thus, we deduce the bounds for $(\partial_t\theta_{\rho,\varsigma})_{\rho,\varsigma}\subset L^1(0,T;W^{1,3+\epsilon}(\Omega)^*)$ and $(\partial_t\theta_{\mathrm{s},\rho,\varsigma})_{\rho,\varsigma}\subset L^1(0,T;W^{1,2+\epsilon}(\Gamma_\mathrm{C})^*)$ for every $\epsilon>0$.

Finally, estimates (6.2f) and (6.2h) follow from a comparison in the momentum balance equation and in the flow rule for the adhesion parameter. □

We shall now prove **Theorem 1** in two main steps, carried out in the ensuing Sections 6.1 and 6.2. More precisely,

1. in Sec. 6.1 we will pass to the limit in system (3.69) as $\rho\downarrow0$ and $\varsigma>0$ is kept fixed; in this way, we shall prove the existence of *weak energy* solutions (in the sense of Definition 2.3) of system (3.69), in which ρ is set equal to 0;
2. in Sec. 6.2 we will finally perform the limit passage as $\varsigma\downarrow0$, thus concluding the proof of Thm. 1.

6.1. **Limit passage as $\rho\downarrow0$, for fixed $\varsigma>0$.** Since the parameter $\varsigma>0$ is kept fixed, we shall not highlight the dependence on ς of the solutions to system (3.69) and just denote them by $(\theta_\rho,\mathbf{u}_\rho,\theta_{\mathrm{s},\rho},\chi_\rho,\sigma_\rho)$.

Let $(\rho_j)_j\subset(0,+\infty)$ be a null sequence and, correspondingly, let $(\theta_{\rho_j},\mathbf{u}_{\rho_j},\theta_{\mathrm{s},\rho_j},\chi_{\rho_j},\sigma_{\rho_j})_j$ be a sequence of solutions to system (3.69), formulated as in the statement of Thm. 5.2 and supplemented by the initial conditions (5.34), with sequences $(\theta^0_{\rho_j})_j$, $(\theta^0_{\mathrm{s},\rho_j})_j$ and $(\mathbf{u}^0_{\rho_j})_j$ of initial data fulfilling (3.72); set $\boldsymbol{\zeta}_{\rho_j}:=\eta_\varsigma(\mathbf{u}_{\rho_j}\cdot\mathbf{n})\mathbf{n}$, $\xi_{\rho_j}:=\beta_\varsigma(\chi_{\rho_j})$, $\sigma_{\rho_j}:=\sigma_{\rho_j,\varsigma}$. In what follows we will show that, up to a subsequence, the quintuples $(\theta_{\rho_j},\mathbf{u}_{\rho_j},\theta_{\mathrm{s},\rho_j},\chi_{\rho_j},\sigma_{\rho_j})_j$ converge to a 'weak energy solution' $(\theta,\mathbf{u},\theta_\mathrm{s},\chi,\sigma)$ to the Cauchy problem for system (3.69), in which $\rho=0$. Namely, we will prove that $(\theta,\mathbf{u},\theta_\mathrm{s},\chi)$

- enjoy the integrability and regularity properties (2.23), and the positivity property (2.31);
- fulfill the weak formulations (5.35a) and (5.35c) of the bulk and surface heat equations, with test functions $v\in W^{1,3+\epsilon}(\Omega)$ and $w\in W^{1,2+\epsilon}(\Gamma_\mathrm{C})$, respectively, for every $\epsilon>0$;
- fulfill the weak formulation of the displacement equation (2.29a), with $\boldsymbol{\zeta}\in\mathrm{C}^0([0,T];L^4(\Gamma_\mathrm{C};\mathbb{R}^3))$ given by $\boldsymbol{\zeta}=\eta_\varsigma(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$;
- fulfill the pointwise formulation (5.35d) of the flow rule in which ρ is set equal to 0.

We shall split the argument into some steps.

Step 1.0**: compactness.** There exist a (not relabeled) subsequence and a quintuple $(\theta, \mathbf{u}, \theta_s, \chi, \sigma)$, with

$$\begin{aligned} &\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)), && \mathbf{u} \in H^1(0,T;H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)), \\ &\theta_s \in L^2(0,T;H^1(\Gamma_C)) \cap L^\infty(0,T;L^1(\Gamma_C)), && \chi \in L^\infty(0,T;H^1(\Gamma_C)) \cap H^1(0,T;L^2(\Gamma_C)), \\ &\sigma \in L^\infty(\Gamma_C \times (0,T)), \end{aligned} \tag{6.3}$$

such that the following weak and strong convergences hold as $j \to \infty$:

$$\theta_{\rho_j} \overset{*}{\rightharpoonup} \theta \quad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,3+\epsilon}(\Omega)^*) \quad \text{for all } \epsilon > 0, \tag{6.4a}$$

$$\theta_{\rho_j}(t) \rightharpoonup \theta(t) \quad \text{in } H^1(\Omega) \quad \text{for a.a. } t \in (0,T), \tag{6.4b}$$

$$\theta_{\rho_j} \to \theta \quad \text{in } L^2(0,T;L^p(\Omega)) \cap L^q(0,T;L^1(\Omega)) \quad \text{for all } p \in [1,6) \text{ and } q \in [1,\infty), \tag{6.4c}$$

$$\theta_{s,\rho_j} \overset{*}{\rightharpoonup} \theta_s \quad \text{in } L^2(0,T;H^1(\Gamma_C)) \cap L^\infty(0,T;W^{1,2+\epsilon}(\Gamma_C)^*) \quad \text{for all } \epsilon > 0, \tag{6.4d}$$

$$\theta_{s,\rho_j}(t) \rightharpoonup \theta_s(t) \quad \text{in } H^1(\Gamma_C) \quad \text{for a.a. } t \in (0,T), \tag{6.4e}$$

$$\theta_{s,\rho_j} \to \theta_s \quad \text{in } L^2(0,T;L^q(\Gamma_C)) \cap L^q(0,T;L^1(\Omega)) \quad \text{for all } q \in [1,\infty), \tag{6.4f}$$

$$\mathbf{u}_{\rho_j} \rightharpoonup \mathbf{u} \quad \text{in } H^1(0,T;H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)) \tag{6.4g}$$

$$\mathbf{u}_{\rho_j} \to \mathbf{u} \quad \text{in } \mathrm{C}^0([0,T];H^{1-\epsilon}(\Omega;\mathbb{R}^3)) \quad \text{for all } \epsilon \in (0,1) \tag{6.4h}$$

$$\rho_j^\upsilon \mathbf{u}_{\rho_j} \to 0 \quad \text{in } W^{1,\omega}(0,T;W^{1,\omega}(\Omega;\mathbb{R}^3)) \quad \text{for all } \upsilon > \frac{1}{\omega}, \tag{6.4i}$$

$$\chi_{\rho_j} \overset{*}{\rightharpoonup} \chi \quad \text{in } L^\infty(0,T;H^1(\Gamma_C)) \cap H^1(0,T;L^2(\Gamma_C)), \tag{6.4j}$$

$$\chi_{\rho_j} \to \chi \quad \text{in } \mathrm{C}^0(0,T;L^q(\Gamma_C)) \quad \text{for all } q \in [1,\infty), \tag{6.4k}$$

$$\rho_j^\upsilon \chi_{\rho_j} \to 0 \quad \text{in } W^{1,\omega}(0,T;L^\omega(\Gamma_C)) \quad \text{for all } \upsilon > \frac{1}{\omega}, \tag{6.4l}$$

$$\sigma_{\rho_j} \overset{*}{\rightharpoonup} \sigma \quad \text{in } L^\infty(\Gamma_C \times (0,T)). \tag{6.4m}$$

Indeed, convergences (6.4a), (6.4d), (6.4g), (6.4j), and (6.4m) immediately follow from estimates (6.2) via weak compactness arguments. Convergence (6.4i) is a straightforward consequence of the second of (6.2e) also in view of Korn's inequality. Analogously, (6.4l) follows from estimate (6.2g). Arguing as in the proof of Lemma 5.2 and resorting to the aforementioned results from [35] we deduce the strong convergences (6.4c), (6.4f), (6.4h), and (6.4k). Likewise, the pointwise convergences (6.4b) and (6.4e) ensue from combining estimates (6.2) with Theorem 5.1.

Combining the estimates for $(\theta_{\rho_j})_j$ and $(\theta_{s,\rho_j})_j$ in $L^\infty(0,T;L^1(\Omega))$ and $L^\infty(0,T;L^1(\Gamma_C))$ with the pointwise convergences (6.4b) and (6.4e) we immediately deduce that $\theta \in L^\infty(0,T;L^1(\Omega))$ and $\theta_s \in L^\infty(0,T;L^1(\Gamma_C))$.

Clearly, from the strong convergences (6.4c) and (6.4f) and the strict positivity properties (6.1) we conclude that the limiting temperatures θ and θ_s also fulfill

$$\theta \geq \frac{1}{S_0} > 0 \quad \text{a.e. in } \Omega \times (0,T), \qquad \theta_s \geq \frac{1}{S_0} > 0 \quad \text{a.e. in } \Gamma_C \times (0,T). \tag{6.5}$$

Furthermore, from convergences (6.4h) (which in particular yields, for the traces $(\mathbf{u}_{\rho_j})_j$, the convergence $\mathbf{u}_j \to \mathbf{u}$ in $\mathrm{C}^0([0,T];L^q(\Gamma_C;\mathbb{R}^3))$ for every $1 \leq q < 4$ since $H^1_{\Gamma_D}(\Omega;\mathbb{R}^3) \Subset L^q(\Gamma_C;\mathbb{R}^3)$ in the sense of traces), and (6.4k) we derive, taking into account that the functions $\eta_\varsigma : \mathbb{R} \to \mathbb{R}$ and $\beta_\varsigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous, that

$$\boldsymbol{\zeta}_{\rho_j} \to \boldsymbol{\zeta} := \eta_\varsigma(\mathbf{u} \cdot \mathbf{n})\mathbf{n} \quad \text{in } \mathrm{C}^0([0,T];L^q(\Gamma_C;\mathbb{R}^3)), \quad \text{for all } q \in [1,4), \tag{6.6a}$$

$$\xi_{\rho_j} \to \xi := \beta_\varsigma(\chi) \quad \text{in } \mathrm{C}^0([0,T];L^q(\Gamma_C)) \quad \text{for all } q \in [1,\infty). \tag{6.6b}$$

Step 1.1**: limit passage in the momentum balance.** We integrate the weak formulation (5.35b) of the momentum balance over an arbitrary time interval (s,t) and pass to the limit as $j \to \infty$ in (5.35b). We handle the first, second, third, fourth and sixth integrals on the left-hand side by resorting to convergences (6.4a), (6.4g), (6.4i), and (6.6a). For the remaining terms, we use that

$$\begin{cases} (\chi_{\rho_j})^+ \mathbf{u}_{\rho_j} \to (\chi)^+ \mathbf{u}, \\ (\chi_{\rho_j})^+ \mathbf{u}_{\rho_j} \daleth[(\chi_{\rho_j})^+] \to (\chi)^+ \mathbf{u} \daleth[(\chi)^+] \end{cases} \quad \text{in } \mathrm{C}^0([0,T];L^q(\Gamma_C)) \quad \text{for all } 1 \leq q < 4 \tag{6.7}$$

which follow from the strong convergences (6.4h) and (6.4k), also by Lemma 3.1. All in all, we conclude that the quadruple $(\theta, \mathbf{u}, \chi, \boldsymbol{\zeta})$, with $\boldsymbol{\zeta} = \eta_\varsigma(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$ fulfill

$$\begin{aligned} &\int_s^t \left(\mathsf{v}(\mathbf{u}_t, \mathbf{v}) + \mathsf{e}(\mathbf{u}, \mathbf{v}) + \int_\Omega \theta \mathrm{div}(\mathbf{v})\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} (\chi)^+ \mathbf{u}\mathbf{v}\,\mathrm{d}x \right) \mathrm{d}r \\ &\quad + \int_s^t \left(\int_{\Gamma_\mathrm{C}} \boldsymbol{\zeta}\cdot \mathbf{v}\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} (\chi)^+ \mathbf{u} \mathcal{J}[(\chi)^+]\mathbf{v}\,\mathrm{d}x \right) \mathrm{d}r = \int_s^t \langle \mathbf{F}, \mathbf{v}\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}\,\mathrm{d}r \end{aligned} \tag{6.8}$$

for all $\mathbf{v} \in W^{1,\omega}(\Omega;\mathbb{R}^3) \cap \in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)$, which translates into a relation holding at almost all $t \in (0,T)$ by the arbitrariness of the interval (s,t). Furthermore, taking into account the integrability properties of $\mathbf{u}$, θ and χ, it is immediate to see that (6.8) extends to all test functions $\mathbf{v} \in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)$. Therefore, we have proved (2.29a) (where χ is replaced by $(\chi)^+$).

Lastly, in view of the limit passage in the bulk heat equation, let us improve the weak convergence $\varepsilon(\partial_t \mathbf{u}_{\rho_j}) \rightharpoonup \varepsilon(\mathbf{u}_t)$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times3}))$ to a strong one. To this end, we revert to (5.35b), test it by $\partial_t \mathbf{u}_{\rho_j}$ and integrate it in time. Passing to the limit as $j\to\infty$ we find that

$$\begin{aligned} &\limsup_{j\to\infty} \left(\int_s^t \mathsf{v}(\partial_t \mathbf{u}_{\rho_j}, \partial_t \mathbf{u}_{\rho_j})\,\mathrm{d}r + \rho_j \int_s^t \int_\Omega |\varepsilon(\partial_t \mathbf{u}_{\rho_j})|^\omega \,\mathrm{d}x\,\mathrm{d}r \right) \\ &\le -\liminf_{j\to\infty} \int_s^t \mathsf{e}(\mathbf{u}_{\rho_j}, \partial_t \mathbf{u}_{\rho_j})\,\mathrm{d}r - \liminf_{j\to\infty} \int_s^t \int_\Omega \theta_{\rho_j} \mathrm{div}(\partial_t \mathbf{u}_{\rho_j})\,\mathrm{d}x\,\mathrm{d}r \\ &\quad - \liminf_{j\to\infty} \int_s^t \int_{\Gamma_\mathrm{C}} (\chi_{\rho_j})^+ \mathbf{u}_{\rho_j}\cdot \partial_t \mathbf{u}_{\rho_j}\,\mathrm{d}x\,\mathrm{d}r - \liminf_{j\to\infty} \int_s^t \int_{\Gamma_\mathrm{C}} \boldsymbol{\zeta}_{\rho_j}\cdot \partial_t \mathbf{u}_{\rho_j}\,\mathrm{d}x\,\mathrm{d}r \\ &\quad - \liminf_{j\to\infty} \int_s^t \int_{\Gamma_\mathrm{C}} (\chi_{\rho_j})^+ \mathcal{J}[(\chi_{\rho_j})^+]\,\mathbf{u}_{\rho_j}\cdot \partial_t \mathbf{u}_{\rho_j}\,\mathrm{d}x\,\mathrm{d}r + \lim_{j\to\infty} \int_s^t \langle \mathbf{F}, \partial_t \mathbf{u}_{\rho_j}\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)}\,\mathrm{d}r \\ &\overset{(1)}{\le} -\int_s^t \left(\mathsf{e}(\mathbf{u}, \mathbf{u}_t) + \int_\Omega \theta \mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} (\chi)^+ \mathbf{u}\mathbf{u}_t\,\mathrm{d}x \right) \mathrm{d}r \\ &\quad - \int_s^t \left(\int_{\Gamma_\mathrm{C}} \boldsymbol{\zeta}\cdot \mathbf{u}_t\,\mathrm{d}x \int_{\Gamma_\mathrm{C}} (\chi)^+ \mathbf{u}\mathcal{J}[(\chi)^+]\,\mathbf{u}_t\,\mathrm{d}x + \langle \mathbf{F}, \mathbf{u}_t\rangle_{H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)} \right) \mathrm{d}r \overset{(6.8)}{=} \int_s^t \mathsf{v}(\mathbf{u}_t, \mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}r\,. \end{aligned}$$

For (1), we have used that

$$\begin{aligned} \liminf_{j\to\infty} \int_s^t \mathsf{e}(\mathbf{u}_{\rho_j}, \partial_t \mathbf{u}_{\rho_j})\,\mathrm{d}r &= \liminf_{j\to\infty} \left(\frac{1}{2}\mathsf{e}(\mathbf{u}_{\rho_j}(t), \mathbf{u}_{\rho_j}(t)) - \frac{1}{2}\mathsf{e}(\mathbf{u}_{\rho_j}(s), \mathbf{u}_{\rho_j}(s)) \right) \\ &\ge \frac{1}{2}\mathsf{e}(\mathbf{u}(t), \mathbf{u}(t)) - \frac{1}{2}\mathsf{e}(\mathbf{u}(s), \mathbf{u}(s)) = \int_s^t \mathsf{e}(\mathbf{u}, \mathbf{u}_t)\,\mathrm{d}r \end{aligned}$$

thanks to (6.4h), as well as the strong convergences (6.4c) and (6.6a). All in all, we conclude that

$$\int_s^t \mathsf{v}(\partial_t \mathbf{u}_{\rho_j}, \partial_t \mathbf{u}_{\rho_j})\,\mathrm{d}r \to \int_s^t \mathsf{v}(\mathbf{u}_t, \mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}r,$$

which, combined with (6.4h), immediately gives the desired strong convergence

$$\mathbf{u}_{\rho_j} \to \mathbf{u} \qquad \text{in } H^1(0,T;H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)) \qquad \text{as } j\to\infty. \tag{6.9}$$

Step 1.2: limit passage in the flow rule. We take the limit as $j\to\infty$ of (5.35d) integrated on an arbitrary time interval $(s,t)\subset(0,T)$. For the left-hand side we use convergences (6.4j), (6.4l), (6.4k) (which also yields strong convergences for the terms $\gamma'(\chi_{\rho_j})$ and $\lambda'(\chi_{\rho_j})$ by the Lipschitz continuity of γ' and λ), (6.4f), and (6.6b). We also use that, in view of estimate (6.2h) and the previously observed (6.6b), there holds

$$A\chi_{\rho_j} \rightharpoonup A\chi \text{ in } L^{\omega/(\omega-1)}(\Gamma_\mathrm{C}\times(0,T)) \qquad \text{as } j\to\infty. \tag{6.10}$$

As for the right-hand side, we use that

$$\begin{cases} -\frac{1}{2}|\mathbf{u}_{\rho_j}|^2\sigma_{\rho_j} \rightharpoonup^* -\frac{1}{2}|\mathbf{u}|^2\sigma & \text{in } L^\infty(0,T;L^q(\Gamma_\mathrm{C})) \text{ for all } 1\le q<2, \\ -\frac{1}{2}\mathcal{J}[(\chi_{\rho_j})^+]\,|\mathbf{u}_{\rho_j}|^2\sigma_{\rho_j} \rightharpoonup^* -\frac{1}{2}\mathcal{J}[(\chi)^+]\,|\mathbf{u}|^2\sigma & \text{in } L^\infty(0,T;L^q(\Gamma_\mathrm{C})) \text{ for all } 1\le q<2, \\ -\frac{1}{2}\mathcal{J}[(\chi_{\rho_j})^+|\mathbf{u}_{\rho_j}|^2]\sigma_{\rho_j} \rightharpoonup^* -\frac{1}{2}\mathcal{J}[(\chi)^+|\mathbf{u}|^2]\,\sigma & \text{in } L^\infty(\Gamma_\mathrm{C}\times(0,T)) \end{cases}$$

also in view of Lemma 3.1. All in all, we conclude the validity of (5.35d) with $\rho=0$. Again by the strong weak closedness of the graph of the (operator induced by) $\partial\varphi$, we have $\sigma\in\partial\varphi(\chi)$ a.e. in $\Gamma_\mathrm{C}\times(0,T)$.

A comparison argument in (5.35d) immediately yields that $A\chi \in L^2(\Gamma_{\mathrm{C}}\times(0,T))$, so that we ultimately infer that $\chi \in L^2(0,T;H^2(\Gamma_{\mathrm{C}}))$.

Lastly, in view of the limit passage in the surface heat equation, let us enhance the weak convergence of $\partial_t\chi_{\rho_j}$ to a strong one. With this aim, we test (5.35b) by $\partial_t\mathbf{u}_{\rho_j}$, (5.35d) by $\partial_t\chi_{\rho_j}$, add the resulting relations, and integrate in time (cf. (3.50)). Taking the limit as $j\to\infty$ we have

$$
\begin{aligned}
&\limsup_{j\to\infty}\left(\int_0^t\int_{\Gamma_{\mathrm{C}}}|\partial_t\chi_{\rho_j}|^2\,\mathrm{d}x\,\mathrm{d}r+\rho_j\int_0^t\int_{\Gamma_{\mathrm{C}}}|\partial_t\chi_{\rho_j}|^{\omega}\,\mathrm{d}x\,\mathrm{d}r\right)\\
&\leq -\lim_{j\to\infty}\Big(\int_0^t\left(\mathsf{v}(\partial_t\mathbf{u}_{\rho_j},\partial_t\mathbf{u}_{\rho_j})+\int_\Omega\theta_{\rho_j}\mathrm{div}(\partial_t\mathbf{u}_{\rho_j})\,\mathrm{d}x\right)\mathrm{d}r+\frac12\mathsf{e}(\mathbf{u}_{\rho_j}(t),\mathbf{u}_{\rho_j}(t))+\int_{\Gamma_{\mathrm{C}}}\widehat{\eta}_{\varsigma}(\mathbf{u}_{\rho_j}(t)\cdot\mathbf{n})\,\mathrm{d}x\\
&\qquad\qquad+\frac12\int_{\Gamma_{\mathrm{C}}}(\chi_{\rho_j}(t))^+|\mathbf{u}_{\rho_j}(t)|^2\,\mathrm{d}x+\frac12\int_{\Gamma_{\mathrm{C}}}(\chi_{\rho_j}(t))^+|\mathbf{u}_{\rho_j}(t)|^2\mathcal{J}[(\chi_{\rho_j}(t))^+]\,\mathrm{d}x\Big)\\
&\quad+\lim_{j\to\infty}\int_0^t\langle\mathbf{F},\partial_t\mathbf{u}_{\rho_j}\rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)}\,\mathrm{d}r-\liminf_{j\to\infty}\left(\int_{\Gamma_{\mathrm{C}}}\left(\frac12|\nabla\chi_{\rho_j}(t)|^2+\widehat{\beta}_{\varsigma}(\chi_{\rho_j}(t))+\gamma(\chi_{\rho_j}(t))\right)\mathrm{d}x\right)\\
&\quad+\lim_{j\to\infty}\Big(\frac12\mathsf{e}(\mathbf{u}_0^{\rho_j},\mathbf{u}_0^{\rho_j})+\int_{\Gamma_{\mathrm{C}}}\widehat{\eta}_{\varsigma}(\mathbf{u}_0^{\rho_j}\cdot\mathbf{n})\,\mathrm{d}x+\frac12\int_{\Gamma_{\mathrm{C}}}(\chi_0)^+|\mathbf{u}_0^{\rho_j}|^2\,\mathrm{d}x+\frac12\int_{\Gamma_{\mathrm{C}}}(\chi_0)^+|\mathbf{u}_0^{\rho_j}|^2\mathcal{J}[(\chi_0)^+]\,\mathrm{d}x\\
&\qquad\qquad+\int_{\Gamma_{\mathrm{C}}}\big(\frac12|\nabla\chi_0|^2+\widehat{\beta}_{\varsigma}(\chi_0)+\gamma(\chi_0)\big)\,\mathrm{d}x\Big)-\liminf_{j\to\infty}\int_0^t\int_{\Gamma_{\mathrm{C}}}\lambda'(\chi_{\rho_j})\theta_{\mathrm{s},\rho_j}\partial_t\chi_{\rho_j}\,\mathrm{d}x\,\mathrm{d}r\\
&\overset{(1)}{\leq}-\int_0^t\left(\mathsf{v}(\mathbf{u}_t,\mathbf{u}_t)+\int_\Omega\theta\mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x\right)\mathrm{d}r+\frac12\mathsf{e}(\mathbf{u}(t),\mathbf{u}(t))+\int_{\Gamma_{\mathrm{C}}}\widehat{\eta}_{\varsigma}(\mathbf{u}(t)\cdot\mathbf{n})\,\mathrm{d}x\\
&\quad+\frac12\int_{\Gamma_{\mathrm{C}}}(\chi(t))^+|\mathbf{u}(t)|^2\,\mathrm{d}x+\frac12\int_{\Gamma_{\mathrm{C}}}(\chi(t))^+|\mathbf{u}(t)|^2\mathcal{J}[(\chi(t))^+]\,\mathrm{d}x+\int_0^t\langle\mathbf{F},\mathbf{u}_t\rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)}\,\mathrm{d}r\\
&\quad-\int_{\Gamma_{\mathrm{C}}}\left(\frac12|\nabla\chi(t)|^2+\widehat{\beta}_{\varsigma}(\chi(t))+\gamma(\chi(t))\right)\mathrm{d}x+\frac12\mathsf{e}(\mathbf{u}_0,\mathbf{u}_0)+\int_{\Gamma_{\mathrm{C}}}\widehat{\eta}_{\varsigma}(\mathbf{u}_0\cdot\mathbf{n})\,\mathrm{d}x\\
&\quad+\frac12\int_{\Gamma_{\mathrm{C}}}(\chi_0)^+|\mathbf{u}_0|^2\,\mathrm{d}x+\frac12\int_{\Gamma_{\mathrm{C}}}(\chi_0)^+|\mathbf{u}_0|^2\mathcal{J}[(\chi_0)^+]\,\mathrm{d}x+\int_{\Gamma_{\mathrm{C}}}\big(\frac12|\nabla\chi_0|^2+\widehat{\beta}_{\varsigma}(\chi_0)+\gamma(\chi_0)\big)\,\mathrm{d}x\Big)\\
&\quad-\int_0^t\int_{\Gamma_{\mathrm{C}}}\lambda'(\chi)\theta_{\mathrm{s}}\partial_t\chi\,\mathrm{d}x\,\mathrm{d}r\overset{(2)}{=}\int_0^t\int_{\Gamma_{\mathrm{C}}}|\chi_t|^2\,\mathrm{d}x\,\mathrm{d}r
\end{aligned}
\tag{6.11}
$$

where for (1) we have used the previously found convergences properties, while (2) follows from testing the weak momentum balance (6.8) by $\mathbf{u}_t$, the flow rule (5.35d) by χ_t, adding the resulting relations and integrating in time. All in all, from the above chain of inequalities we infer

$$
\chi_{\rho_j}\to\chi\qquad\text{in } H^1(0,T;L^2(\Gamma_{\mathrm{C}}))\qquad\text{as } j\to\infty. \tag{6.12}
$$

Step 1.3**: limit passage in the bulk heat equation.** We shall pass to the limit in (5.35a) with test functions $v\in W^{1,3+\epsilon}(\Omega)$, for an arbitrary $\epsilon>0$. In analogy with (3.58), we rewrite the bulk heat equation by grouping its terms in the following way:

$$
\partial_t\theta_{\rho_j}(t)=\mathcal{L}_{1,j}(t)-\mathcal{A}(\theta_{\rho_j}(t))+\mathcal{L}_{2,j}(t)\text{ in }W^{1,3+\epsilon}(\Omega)^*\qquad\text{for a.a. } t\in(0,T), \tag{6.13}
$$

with (omitting the t-dependence of the operators below to simplify notation)

$$
\begin{cases}
\mathcal{L}_{1,j}:=\theta_{\rho_j}\mathrm{div}(\partial_t\mathbf{u}_{\rho_j})+\varepsilon(\partial_t\mathbf{u}_{\rho_j})\mathbb{V}\varepsilon(\partial_t\mathbf{u}_{\rho_j})+h\in L^1(\Omega)\\
\mathcal{L}_{2,j}\in W^{1,3+\epsilon}(\Omega)^*\text{ defined by}\\
\qquad\langle\mathcal{L}_{2,j},v\rangle_{W^{1,3+\epsilon}(\Omega)}:=\int_{\Gamma_{\mathrm{C}}}\Big(\mathcal{J}[(\chi_{\rho_j})^+\theta_{\mathrm{s},\rho_j}]\,(\chi_{\rho_j})^+\theta_{\rho_j}-k(\chi_{\rho_j})\theta_{\rho_j}(\theta_{\rho_j}-\theta_{\mathrm{s},\rho_j})-\mathcal{J}[(\chi_{\rho_j})^+]\,(\chi_{\rho_j})^+\theta_{\rho_j}^2\Big)v\,\mathrm{d}x,\\
\mathcal{A}(\theta_{\rho_j})\in W^{1,3+\epsilon}(\Omega)^*\text{ defined by}\\
\qquad\langle\mathcal{A}(\theta_{\rho_j}),v\rangle_{W^{1,3+\epsilon}(\Omega)}:=\int_\Omega\alpha(\theta_{\rho_j})\nabla\theta_{\rho_j}\cdot\nabla v\,\mathrm{d}x=\int_\Omega\nabla(\widehat{\alpha}(\theta_{\rho_j}))\cdot\nabla v\,\mathrm{d}x
\end{cases}
$$

with $\widehat{\alpha}$ from (2.14).

Now, it follows from (6.9) and (6.4c) that

$$
\mathcal{L}_{1,j}\to\mathcal{L}_1\text{ in }L^1(0,T;L^1(\Omega))\quad\text{with }\mathcal{L}_1(t):=\theta(t)\mathrm{div}(\mathbf{u}_t(t))+\varepsilon(\mathbf{u}_t(t))\mathbb{V}\varepsilon(\mathbf{u}_t(t))+h(t) \tag{6.14}
$$

for a.a. $t\in(0,T)$.

Let us now address the limit of the operators $\mathcal{A}(\theta_{\rho_j})$: for this, we rely on the interpolation estimate (3.55) implying, for all $\mu > 1$ and $0 < \nu < 1$, that the sequence

$$(\theta_{\rho_j})_j \text{ is bounded in } L^{\mu-\nu+2}(0,T;L^{3(\mu-\nu+2)/(7-6\nu)}(\Omega)). \tag{6.15}$$

Choosing $\frac{2}{3} < \nu < 1$, we have that $\mu - \nu + 2 > \mu + 1$ and $\frac{3(\mu-\nu+2)}{7-6\nu} > \mu + 1$ and hence, from (6.15), we infer the estimate

$$\sup_j \|\theta_{\rho_j}^{\mu+1}\|_{L^{1+\delta}(\Omega\times(0,T))} \leq C \qquad \text{for some } \delta > 0. \tag{6.16}$$

Now, we use (6.16) to settle the compactness properties of the sequence $(\widehat{\alpha}(\theta_{\rho_j}))_j$. First of all, it follows from (6.4b) that $\theta_{\rho_j} \to \theta$ a.e. in $\Omega \times (0,T)$ and hence $\widehat{\alpha}(\theta_{\rho_j}) \to \widehat{\alpha}(\theta)$ a.e. in $\Omega \times (0,T)$. Combining this information with the fact that $|\widehat{\alpha}(\theta_{\rho_j})| \leq C(\theta_{\rho_j}^{\mu+1} + 1)$ (cf. (4.40)) and with estimate (6.16), we ultimately infer that

$$\widehat{\alpha}(\theta_{\rho_j}) \to \widehat{\alpha}(\theta) \qquad \text{in } L^1(\Omega\times(0,T)).$$

Therefore, $\nabla(\widehat{\alpha}(\theta_{\rho_j})) \to \nabla(\widehat{\alpha}(\theta))$ in the sense of distributions on $\Omega \times (0,T)$.

Now, we need to improve the convergence properties of the sequence $(\nabla(\widehat{\alpha}(\theta_{\rho_j})))_j$. We again interpolate estimates (6.2a) and (6.2b) and (cf. (3.55)) deduce that

$$\sup_j \|\theta_{\rho_j}\|_{L^r(0,T;L^s(\Omega))} \leq C \quad \text{for some } r > \mu - \nu + 2 \text{ and } s < \frac{3(\mu-\nu+2)}{7-6\nu}. \tag{6.17}$$

Therefore, the sequence

$$(\theta_{\rho_j}^{(\mu-\nu+2)/2})_j \text{ is bounded in } L^{2r/(\mu-\nu+2)}(0,T;L^{2s/(\mu-\nu+2)}(\Omega)).$$

In turn, mimicking the calculations from Sec. 3.3.7 we find that

$$\begin{aligned} &\left| \langle \mathcal{A}(\theta_{\rho_j})(t), v\rangle_{W^{1,3+\epsilon}(\Omega)} \right| \\ &\leq C\|\nabla\theta_{\rho_j}(t)\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + C\|\theta_{\rho_j}^{(\mu-\nu+2)/2}(t)\|_{L^{2s/(\mu-\nu+2)}(\Omega)}\|\nabla(\theta_{\rho_j}^{(\mu+\nu)/2}(t))\|_{L^2(\Omega)}\|\nabla v\|_{L^{3+\epsilon}(\Omega)} \end{aligned} \tag{6.18}$$

where we have applied Hölder's inequality, choosing $\nu \in (\frac{2}{3}, 1)$ such that

$$\frac{\mu-\nu+2}{2s} + \frac{1}{2} + \frac{1}{3+\epsilon} = 1.$$

Hence, from (6.17) and (6.18) we deduce that the sequence

$$(\mathcal{A}(\theta_{\rho_j}))_j \text{ is bounded in } L^{1+\delta}(0,T;W^{1,3+\epsilon}(\Omega)^*) \text{ for some } \delta > 0. \tag{6.19}$$

Therefore,

$$\begin{aligned} &\mathcal{A}(\theta_{\rho_j}) \rightharpoonup \mathcal{A}(\theta) \quad \text{in } L^1(0,T;W^{1,3+\epsilon}(\Omega)^*), \qquad \text{with} \\ &\langle \mathcal{A}(\theta(t)), v\rangle_{W^{1,3+\epsilon}(\Omega)} := \int_\Omega \nabla(\widehat{\alpha}(\theta(t))) \cdot \nabla v \,\mathrm{d}x \quad \text{for a.a. } t \in (0,T). \end{aligned} \tag{6.20}$$

Finally, in order to take the limit of the operators $(\mathcal{L}_{2,j})_j$ we need to refine the convergences available for the traces of $(\theta_{\rho_j})_j$. Indeed, taking into account that the sequence $(\theta_{\rho_j}^{(\mu+\nu)/2})_j$ is bounded in $L^2(0,T;H^1(\Omega))$ for every $\nu \in (0,1)$ and that, a fortiori, its traces are bounded in $L^2(0,T;L^4(\Gamma_{\mathrm{C}}))$, we infer that $(\theta_{\rho_j})_j$ is bounded in $L^{\mu+\nu}(0,T;L^{2(\mu+\nu)}(\Gamma_{\mathrm{C}}))$ for every $\nu \in (0,1)$. Since $\mu > 1$, we may choose $\nu \in (\frac{2}{3}, 1)$ such that $\mu + \nu > 2$. Thus, from this estimate we improve the weak convergence of $(\theta_{\rho_j})_j$ in $L^2(0,T;L^4(\Gamma_{\mathrm{C}}))$ to a strong convergence, i.e.

$$\theta_{\rho_j} \to \theta \quad \text{in } L^2(0,T;L^4(\Gamma_{\mathrm{C}})). \tag{6.21}$$

Therefore, in view of (6.4k) we find that $(\chi_{\rho_j})^+\theta_{\rho_j} \to (\chi)^+\theta$ in $L^2(0,T;L^q(\Gamma_{\mathrm{C}}))$ for every $q \in [1,4)$. We now use that $(\chi_{\rho_j})^+\theta_{\mathrm{s},\rho_j} \to (\chi)^+\theta_{\mathrm{s}}$ in $L^2(0,T;L^s(\Gamma_{\mathrm{C}}))$ for every $s \in [1,\infty)$ thanks to (6.4f) so that, by Lemma (3.1), $\mathcal{J}[\chi_{\rho_j}\theta_{\mathrm{s},\rho_j}] \to \mathcal{J}[\chi\theta_{\mathrm{s}}]$ in $L^2(0,T;L^\infty(\Gamma_{\mathrm{C}}))$. Hence we have that, as $j \to \infty$,

$$\mathcal{J}[(\chi_{\rho_j})^+\theta_{\mathrm{s},\rho_j}]\,(\chi_{\rho_j})^+\theta_{\rho_j} \to \mathcal{J}[(\chi)^+\theta_{\mathrm{s}}]\,(\chi)^+\theta \qquad \text{in } L^1(0,T;L^q(\Gamma_{\mathrm{C}})) \quad \text{for all } q \in [1,4).$$

In order to pass to the limit in the second contribution to $\mathcal{L}_{2,j}$, we recall that $k(\chi_{\rho_j}) \to k(\chi)$ in $L^\infty(0,T;L^q(\Gamma_{\mathrm{C}}))$ for every $q \in [1,\infty)$ by (6.4k) and the polynomial growth of k. Hence, in view of (6.21) and (6.4f) we have that

$$k(\chi_{\rho_j})\theta_{\rho_j}(\theta_{\rho_j} - \theta_{\mathrm{s},\rho_j}) \to k(\chi)\theta(\theta - \theta_{\mathrm{s}}) \qquad \text{in } L^1(0,T;L^1(\Gamma_{\mathrm{C}})).$$

Analogously, we find that

$$\mathfrak{J}[(\chi_{\rho_j})^+](\chi_{\rho_j})^+\theta_{\rho_j}^2 \to \mathfrak{J}[(\chi)^+]\,(\chi)^+\theta^2 \qquad \text{in } L^1(0,T;L^1(\Gamma_{\mathrm{C}})).$$

All in all, we have that

$$\begin{aligned} &\mathcal{L}_{2,j} \rightharpoonup \mathcal{L}_2 \qquad \text{in } L^1(0,T;W^{1,3+\epsilon}(\Omega)^*) \qquad \text{with} \\ &\langle \mathcal{L}_2(t), v\rangle_{W^{1,3+\epsilon}(\Omega)} := \int_{\Gamma_{\mathrm{C}}} \big(\mathfrak{J}[(\chi(t))^+\theta_{\mathrm{s}}(t)](\chi(t))^+\theta(t) - k(\chi(t))\theta(t)(\theta(t)-\theta_{\mathrm{s}}(t)) - \mathfrak{J}[(\chi(t))^+](\chi(t))^+\theta(t)^2\big)\, v \,\mathrm{d}x \end{aligned} \tag{6.22}$$

for a.a. $t \in (0,T)$.

Combining (6.13) with (6.14), (6.20), and (6.22) we ultimately conclude, by comparison in the bulk heat equation, that, a fortiori, $\theta \in W^{1,1}(0,T;W^{1,3+\epsilon}(\Omega)^*)$ for every $\epsilon > 0$ and

$$\partial_t \theta_{\rho_j} \rightharpoonup \theta_t \quad \text{in } L^1(0,T;W^{1,3+\epsilon}(\Omega)^*)\,. \tag{6.23}$$

This concludes the limit passage in the bulk heat equation (5.35a).

Remark 6.2. We have not succeeded in showing that the elliptic operator $\mathcal{A}(\theta)$ from (6.20) satisfies $\langle \mathcal{A}(\theta), v\rangle_{W^{1,3+\epsilon}(\Omega)} = \int_\Omega \alpha(\theta)\nabla\theta\cdot\nabla v\,\mathrm{d}x$ for every $v \in W^{1,3+\epsilon}(\Omega)$. Indeed, from (6.16) we are just in a position to infer that $\alpha(\theta_{\rho_j}) \to \alpha(\theta)$ in $L^{1+\zeta}(\Omega\times(0,T))$ for some $\zeta > 0$, which is not sufficient to identify the weak limit of the sequence $(\alpha(\theta_{\rho_j})\nabla\theta_{\rho_j})_j$ in any L^p space. Thus, we are not in a position to pass to the limit in the relation $\langle \mathcal{A}(\theta_{\rho_j}), v\rangle_{W^{1,3+\epsilon}(\Omega)} := \int_\Omega \alpha(\theta_{\rho_j})\nabla\theta_{\rho_j}\cdot\nabla v\,\mathrm{d}x$.

Step 1.4**: limit passage in the surface heat equation.** We pass to the limit in (5.35c), written for test functions $w \in W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})$ for all $\epsilon > 0$ as

$$\partial_t \theta_{\mathrm{s},\rho_j}(t) = \mathcal{F}_j(t) - \mathcal{A}_{\mathrm{s}}(\theta_{\mathrm{s},\rho_j}(t)) \qquad \text{in } W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})^* \text{for a.a. } t \in (0,T), \tag{6.24}$$

with

$$\begin{aligned} \mathcal{F}_j &:= \theta_{\mathrm{s},\rho_j}\lambda'(\chi_{\rho_j})\partial_t\chi_{\rho_j} + \ell + |\partial_t\chi_{\rho_j}|^2 + k(\chi_{\rho_j})\theta_{\mathrm{s},\rho_j}(\theta_{\rho_j} - \theta_{\mathrm{s},\rho_j}) + \mathfrak{J}[(\chi_{\rho_j})^+\theta_{\rho_j}]\,(\chi_{\rho_j})^+\theta_{\mathrm{s},\rho_j} \\ &\qquad - \mathfrak{J}[(\chi_{\rho_j})^+]\,(\chi_{\rho_j})^+\theta_{\mathrm{s},\rho_j}^2, \\ \mathcal{A}_{\mathrm{s}}&(\theta_{\mathrm{s},\rho_j}) \in W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})^* \text{ defined by} \\ &\langle \mathcal{A}_{\mathrm{s}}(\theta_{\mathrm{s},\rho_j}), w\rangle_{W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})} := \int_{\Gamma_{\mathrm{C}}} \alpha(\theta_{\mathrm{s},\rho_j})\nabla\theta_{\mathrm{s},\rho_j}\cdot\nabla w\,\mathrm{d}x = \int_{\Gamma_{\mathrm{C}}} \nabla(\widehat{\alpha}(\theta_{\mathrm{s},\rho_j}))\cdot\nabla w\,\mathrm{d}x\,. \end{aligned} \tag{6.25}$$

Taking into account convergences (6.4), (6.12), and (6.21), the Lipschitz continuity of λ and the polynomial growth of k, it is easy to show that

$$\begin{aligned} &\mathcal{F}_j \to \mathcal{F} \quad \text{in } L^1(0,T;L^1(\Gamma_{\mathrm{C}})) \\ &\text{with } \mathcal{F} := \theta_{\mathrm{s}}\lambda'(\chi)\chi_t + \ell + |\chi_t|^2 + k(\chi)\theta_{\mathrm{s}}(\theta-\theta_{\mathrm{s}}) + \mathfrak{J}[(\chi)^+\theta]\,(\chi)^+\theta_{\mathrm{s}} - \mathfrak{J}[(\chi)^+]\,(\chi)^+\theta_{\mathrm{s}}^2\,. \end{aligned} \tag{6.26}$$

In order to pass to the limit in the elliptic operators $(\mathcal{A}_{\mathrm{s}}(\theta_{\mathrm{s},\rho_j}))_j$ we adapt the very same arguments for the operators $(\mathcal{A}(\theta_{\rho_j}))_j$, cf. (6.15)–(6.16). Namely, on the one hand, arguing by interpolation we deduce from the bound for $(\theta_{\mathrm{s},\rho_j})_j \subset L^{\mu+\nu}(0,T;L^q(\Gamma_{\mathrm{C}}))\cap L^\infty(0,T;L^1(\Gamma_{\mathrm{C}}))$ that $\widehat{\alpha}(\theta_{\mathrm{s},\rho_j}) \to \widehat{\alpha}(\theta)$ in $L^1(0,T;L^1(\Gamma_{\mathrm{C}}))$, and thus $\nabla\widehat{\alpha}(\theta_{\mathrm{s},\rho_j}) \to \nabla\widehat{\alpha}(\theta_{\mathrm{s}})$ in the sense of distributions on $(0,T)\times\Gamma_{\mathrm{C}}$. On the other hand, relying on the estimates in Sec. 3.3.8 in the same way as we have done in Step 1.3, we show that the sequence $(\mathcal{A}_{\mathrm{s}}(\theta_{\mathrm{s},\rho_j}))_j$ is bounded in $L^{1+\delta}(0,T;W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})^*)$ for some $\delta > 0$, so that

$$\begin{aligned} &\mathcal{A}_{\mathrm{s}}(\theta_{\mathrm{s},\rho_j}) \rightharpoonup \mathcal{A}_{\mathrm{s}}(\theta_{\mathrm{s}}) \quad \text{in } L^1(0,T;W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})^*), \qquad \text{with} \\ &\langle \mathcal{A}_{\mathrm{s}}(\theta_{\mathrm{s}}(t)), w\rangle_{W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})} := \int_{\Gamma_{\mathrm{C}}} \nabla(\widehat{\alpha}(\theta_{\mathrm{s}}(t)))\cdot\nabla w\,\mathrm{d}x \quad \text{for a.a. } t \in (0,T). \end{aligned} \tag{6.27}$$

By comparison in (6.24) and convergences (6.26), (6.27) we deduce that, a fortiori, $\theta_{\mathrm{s}} \in W^{1,1}(0,T;W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})^*)$ for every $\epsilon > 0$, and

$$\partial_t\theta_{\mathrm{s},\rho_j} \rightharpoonup \partial_t\theta_{\mathrm{s}} \quad \text{in } L^1(0,T;W^{1,2+\epsilon}(\Gamma_{\mathrm{C}})^*).$$

Hence, we pass to the limit in the surface heat equation (5.35c).

We have thus shown that the quintuple $(\theta,\mathbf{u},\theta_{\mathrm{s}},\chi,\sigma)$ is a 'weak energy' solution to the Cauchy problem for system (3.69) with $\rho = 0$.

6.2. **Limit passage as $\varsigma \downarrow 0$ and conclusion of the proof of Theorem 1.** We shall only sketch the argument for the limit passage, as it is completely analogous to that carried out in Section 6.1 up to the identification of this maximal monotone operators in the momentum balance and in the flow rule for the adhesion parameter.

Let $(\theta_{\varsigma_n}, \mathbf{u}_{\varsigma_n}, \theta_{\mathrm{s},\varsigma_n}, \chi_{\varsigma_n}, \boldsymbol{\zeta}_{\varsigma_n}, \xi_{\varsigma_n}, \sigma_{\varsigma_n})_n$ be a sequence of weak energy solutions to the Cauchy problem for system (3.69), in which $\rho = 0$ and $\varsigma = \varsigma_n$ with $\varsigma_n \downarrow 0$ as $n \to \infty$; we have set $\boldsymbol{\zeta}_{\varsigma_n} := \eta_{\varsigma_n}(\mathbf{u}_{\varsigma_n} \cdot \mathbf{n})\mathbf{n}$ and $\xi_{\varsigma_n} := \beta_{\varsigma_n}(\chi_{\varsigma_n})$. We suppose that for every $n \in \mathbb{N}$ the seventuple $(\theta_{\varsigma_n}, \mathbf{u}_{\varsigma_n}, \theta_{\mathrm{s},\varsigma_n}, \chi_{\varsigma_n}, \boldsymbol{\zeta}_{\varsigma_n}, \xi_{\varsigma_n}, \sigma_{\varsigma_n})$ has been obtained by the limiting procedure described in Sec. 6.1, so that, by lower semicontinuity arguments, estimates (6.2) hold for the sequence $(\theta_{\varsigma_n}, \mathbf{u}_{\varsigma_n}, \theta_{\mathrm{s},\varsigma_n}, \chi_{\varsigma_n}, \boldsymbol{\zeta}_{\varsigma_n}, \xi_{\varsigma_n}, \sigma_{\varsigma_n})_n$, uniformly w.r.t. n. Therefore, there exists a quintuple $(\theta, \mathbf{u}, \theta_{\mathrm{s}}, \chi, \sigma)$ as in (6.3) such that convergences (6.4) hold, as $n \to \infty$, along a not relabeled subsequence. Then, the limiting temperatures θ and θ_{s} enjoy the positivity properties (6.5).

In turn, we are in a position to improve estimates (6.2f) and (6.2h) for the sequences $(\boldsymbol{\zeta}_{\varsigma_n})_n$ and $(\xi_{\varsigma_n})_n$. Indeed, a comparison argument in the momentum balance (2.29a) shows that the sequence $(\boldsymbol{\zeta}_{\varsigma_n})_n$ is bounded in $L^2(0,T;\mathbf{Y}^*)$. Analogously, by comparison in the pointwise flow rule (5.35d) (cf. Sec. 3.3.5) we infer that the sequence $(A\chi_{\varsigma_n}+\xi_{\varsigma_n})_n$ is bounded in $L^2(0,T;L^2(\Gamma_{\mathrm{C}}))$ and then, a fortiori, we easily deduce that the sequence $(\xi_{\varsigma_n})_n$ is bounded in $L^2(0,T;L^2(\Gamma_{\mathrm{C}}))$. Hence, there exist $\boldsymbol{\zeta} \in L^2(0,T;\mathbf{Y}^*)$ and $\xi \in L^2(0,T;L^2(\Gamma_{\mathrm{C}}))$ such that, up to a subsequence, there holds

$$
\begin{aligned}
&\boldsymbol{\zeta}_{\varsigma_n} \rightharpoonup \boldsymbol{\zeta} \quad \text{in } L^2(0,T;\mathbf{Y}^*),\\
&\xi_{\varsigma_n} \rightharpoonup \xi \quad \text{in } L^2(0,T;L^2(\Gamma_{\mathrm{C}})).
\end{aligned}
\tag{6.28}
$$

Finally, since $(\chi_{\varsigma_n})_n$ is bounded in $L^2(0,T;H^2(\Gamma_{\mathrm{C}}))$, we ultimately have that

$$
\chi_{\varsigma_n} \rightharpoonup \chi \quad \text{in } L^2(0,T;H^2(\Gamma_{\mathrm{C}})). \tag{6.29}
$$

Let us now outline the argument for the limit passage in the weak formulation of system (3.69).

Step 2.1: limit passage in the momentum balance. Thanks to convergences (6.4a) and (6.28), with the very same arguments as in Sec. 6.1 we conclude that the quadruple $(\theta, \mathbf{u}, \chi, \boldsymbol{\zeta})$ fulfills (6.8) for every $\mathbf{v} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$, namely the weak formulation (2.29a) of the momentum balance. It remains to show that $\boldsymbol{\zeta}(t) \in \boldsymbol{\eta}(\mathbf{u}(t))$ in $\mathbf{Y}^*$ for almost all $t \in (0,T)$. With this aim, we test (2.29a) by $\mathbf{u}_{\varsigma_n}$ and integrate it in time. Passing to the limit as $n \to \infty$ we find that

$$
\begin{aligned}
&\limsup_{n\to\infty} \int_s^t \int_{\Gamma_{\mathrm{C}}} \boldsymbol{\zeta}_{\varsigma_n} \cdot \mathbf{u}_{\varsigma_n} \,\mathrm{d}x\,\mathrm{d}r\\
&\leq -\liminf_{n\to\infty} \int_s^t \mathsf{v}(\partial_t \mathbf{u}_{\varsigma_n}, \mathbf{u}_{\varsigma_n})\,\mathrm{d}r - \liminf_{n\to\infty} \int_s^t \mathsf{e}(\mathbf{u}_{\varsigma_n}, \mathbf{u}_{\varsigma_n})\,\mathrm{d}r - \liminf_{n\to\infty} \int_s^t \int_\Omega \theta_{\varsigma_n} \mathrm{div}(\mathbf{u}_{\varsigma_n})\,\mathrm{d}x\,\mathrm{d}r\\
&\quad -\liminf_{n\to\infty} \int_s^t \int_{\Gamma_{\mathrm{C}}} (\chi_{\varsigma_n})^+ |\mathbf{u}_{\varsigma_n}|^2 \,\mathrm{d}x\,\mathrm{d}r - \liminf_{n\to\infty} \int_s^t \int_{\Gamma_{\mathrm{C}}} (\chi_{\varsigma_n})^+ \mathcal{J}[(\chi_{\varsigma_n})^+]\, |\mathbf{u}_{\varsigma_n}|^2 \,\mathrm{d}x\,\mathrm{d}r\\
&+ \lim_{n\to\infty} \int_s^t \langle \mathbf{F}, \mathbf{u}_{\varsigma_n} \rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)} \,\mathrm{d}r\\
&\leq -\int_s^t \left(\mathsf{v}(\mathbf{u}_t, \mathbf{u}) + \mathsf{e}(\mathbf{u}, \mathbf{u}) + \int_\Omega \theta \mathrm{div}(\mathbf{u})\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} (\chi)^+ \mathbf{u}\mathbf{u}\,\mathrm{d}x \right) \mathrm{d}r\\
&\qquad -\int_s^t \left(\int_{\Gamma_{\mathrm{C}}} (\chi)^+ \mathbf{u} \mathcal{J}[(\chi)^+]\, \mathbf{u}\,\mathrm{d}x + \langle \mathbf{F}, \mathbf{u} \rangle_{H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)} \right) \mathrm{d}r \overset{(2.29a)}{=} \int_s^t \langle \boldsymbol{\zeta}, \mathbf{u} \rangle_{\mathbf{Y}} \,\mathrm{d}r
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
\liminf_{n\to\infty} \int_s^t \mathsf{v}(\partial_t \mathbf{u}_{\varsigma_n}, \mathbf{u}_{\varsigma_n})\,\mathrm{d}r &= \liminf_{n\to\infty} \left(\frac{1}{2}\mathsf{v}(\mathbf{u}_{\varsigma_n}(t), \mathbf{u}_{\varsigma_n}(t)) - \frac{1}{2}\mathsf{v}(\mathbf{u}_{\varsigma_n}(s), \mathbf{u}_{\varsigma_n}(s)) \right)\\
&\geq \frac{1}{2}\mathsf{v}(\mathbf{u}(t), \mathbf{u}(t)) - \frac{1}{2}\mathsf{v}(\mathbf{u}(s), \mathbf{u}(s)) = \int_s^t \mathsf{v}(\partial_t \mathbf{u}, \mathbf{u})\,\mathrm{d}r
\end{aligned}
$$

by the chain rule and convergence (6.4h), and that

$$\lim_{n\to\infty}\int_s^t\int_{\Gamma_C}(\chi_{\varsigma_n})^+|\mathbf{u}_{\varsigma_n}|^2\,\mathrm{d}x\,\mathrm{d}r=\int_s^t\int_{\Gamma_C}(\chi)^+|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}r,$$
$$\lim_{n\to\infty}\int_s^t\int_{\Gamma_C}(\chi_{\varsigma_n})^+\mathcal{J}[(\chi_{\varsigma_n})^+]|\mathbf{u}_{\varsigma_n}|^2\,\mathrm{d}x\,\mathrm{d}r=\int_s^t\int_{\Gamma_C}(\chi)^+\mathcal{J}[(\chi)^+]|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}r$$

by well-known lower semicontinuity results. Therefore, we conclude that for every $\mathbf{v}\in H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)$ such that $\widehat{\eta}(\mathbf{v}\cdot\mathbf{n})\in L^1(\Gamma_C)$ there holds

$$\begin{aligned}\int_s^t(\widehat{\boldsymbol{\eta}}(\mathbf{v})-\widehat{\boldsymbol{\eta}}(\mathbf{u}))\,\mathrm{d}r&=\int_s^t\int_{\Gamma_C}(\widehat{\eta}(\mathbf{v}\cdot\mathbf{n})-\widehat{\eta}(\mathbf{u}\cdot\mathbf{n}))\,\mathrm{d}x\,\mathrm{d}r\\&\overset{(1)}{\geq}\limsup_{n\to\infty}\int_s^t\int_{\Gamma_C}(\widehat{\eta}_{\varsigma_n}(\mathbf{v}\cdot\mathbf{n})-\widehat{\eta}_{\varsigma_n}(\mathbf{u}_{\varsigma_n}\cdot\mathbf{n}))\,\mathrm{d}x\,\mathrm{d}r\\&\overset{(2)}{\geq}\limsup_{n\to\infty}\int_s^t\int_{\Gamma_C}\eta_{\varsigma_n}(\mathbf{u}_{\varsigma_n}\cdot\mathbf{n})\mathbf{n}\cdot(\mathbf{v}-\mathbf{u}_{\varsigma_n})\,\mathrm{d}x\,\mathrm{d}r\\&\overset{(3)}{\geq}\int_s^t\langle\boldsymbol{\zeta},\mathbf{v}-\mathbf{u}\rangle_{\mathbf{Y}}\,\mathrm{d}r,\end{aligned}$$

which yields the desired (2.29b). We have thus shown that the quadruple $(\theta,\mathbf{u},\chi,\boldsymbol{\zeta})$ fulfills (2.29) (where χ is, momentarily, replaced by $(\chi)^+$).

Step 2.2**: limit passage in the flow rule.** With convergences (6.4a) and (6.28) and the arguments developed for the limit passage in system (3.69) as $\rho_j\downarrow 0$ we show that the quintuple $(\mathbf{u},\theta_s,\chi,\xi,\sigma)$ fulfills (2.30). Combining the weak convergence and the strong convergence of $(\xi_{\varsigma_n})_n$ and $(\chi_{\varsigma_n})_n$ in $L^2(0,T;L^2(\Gamma_C))$ we infer that

$$\lim_{n\to\infty}\int_s^t\int_{\Gamma_C}\xi_{\varsigma_n}\chi_{\varsigma_n}\,\mathrm{d}x\,\mathrm{d}r=\int_s^t\int_{\Gamma_C}\xi\chi\,\mathrm{d}x\,\mathrm{d}r\qquad\text{for all }(s,t)\subset(0,T),$$

whence we deduce that $\xi\in\beta(\chi)$ a.e. in $\Omega\times(0,T)$ so that, in particular,

$$\chi\geq 0\qquad\text{a.e. in}\,\Gamma_C\times(0,T)\,.$$

All in all, the quintuple $(\mathbf{u},\theta_s,\chi,\xi,\sigma)$ fulfills (2.30).

Steps 2.3 & 2.4**: limit passage in the bulk and surface equations.** These limit procedures can be performed by the very same arguments as in Steps 1.3 and 1.4. We thus obtain the weak formulations of the bulk and surface equations (2.27) and (2.28).

Conclusion of the proof. We have shown that the seventuple $(\theta,\mathbf{u},\theta_s,\chi,\boldsymbol{\zeta},\xi,\sigma)$

(1) enjoy the regularity, integrability, and positivity properties (2.23), (2.25), and (2.26);
(2) fulfill the Cauchy conditions (2.24) as a trivial consequences of convergences (6.4);
(3) fulfill the weak formulation of system (2.12) consisting of (2.27)–(2.29a) and (2.30).

The total energy balance (2.21) follows by testing (2.27) by 1, (2.28) by 1, (2.29a) by $\mathbf{u}_t$, (2.30) by χ_t, and carrying out the same calculations as in Sec. 3.1.

This finishes the proof of Theorem 1. ∎

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