

THE ARONSSON EQUATION FOR ABSOLUTE MINIMIZERS OF SUPREMAL FUNCTIONALS IN CARNOT-CARATHÉODORY SPACES

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ABSTRACT. Given a C^2 family of vector fields X_1, \dots, X_m which induces a continuous Carnot-Carathéodory distance, we show that any absolute minimizer of a supremal functional defined by a C^2 quasiconvex Hamiltonian $f(x, s, p)$, allowing s -variable dependence, is a viscosity solution to the Aronsson equation

$$-\sum_{i=1}^m X_i(f(x, u(x), Xu(x))) \frac{\partial f}{\partial p_i}(x, u(x), Xu(x)) = 0,$$

1. INTRODUCTION

The study of variational problems in L^∞ is very often a good starting point to set up problems coming both from theoretical issues and from real applications. The earliest works in this direction are due to Aronsson ([A1, A2]). In these seminal papers, the author studied the connection between Lipschitz extension problems and PDEs, introducing the notion of *absolute minimizing Lipschitz extension (AMLE)* and showing that a C^2 function is an AMLE if and only if it satisfies the *infinity Laplace equation*

$$-\sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0. \tag{1.1}$$

Aronsson observed ([A3]) that there are examples of AMLE which are not C^2 , and thus solving equation (1.1) only in a formal sense. The problem was solved by Jensen in [J], where he exploited the machinery of viscosity solutions introduced by Crandall and Lions in [CL] (cf. also [CIL] for an exhaustive account on the topic), and showed that being an AMLE is equivalent to being a viscosity solution to (1.1). Moreover, he showed that viscosity solutions to problem (1.1) are unique, provided a Dirichlet boundary datum is assigned.

One step further was made by Barron, Jensen and Wang ([BJW]), who started the study of L^∞ variational functionals F which are usually known as *supremal functionals*, that is

$$F(u, V) := \|f(x, u(x), Du(x))\|_{L^\infty(V)} \quad u \in W^{1,\infty}(U), V \in \mathcal{A}.$$

where throughout the paper U is an open and connected subset of \mathbb{R}^n , \mathcal{A} is the class of all open subsets of U and f is a suitable continuous non-negative function. In particular, they generalized the notion of AMLE to the one of *absolute minimizer* of the functional F , that is a function $u \in W^{1,\infty}(U)$ such that

$$F(u, V) \leq F(v, V)$$

for any $V \Subset U$ and for any $v \in W^{1,\infty}(V)$ with $v|_{\partial V} = u|_{\partial V}$. The authors of [BJW] showed that any absolute minimizer of F is a solution, in the viscosity sense, of the so-called *Aronsson equation*

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i} (f(x, u(x), Du(x))) \frac{\partial f}{\partial p_i}(x, u(x), Du(x)) = 0,$$

provided that, among the other things, f is C^2 and $p \mapsto f(x, s, p)$ is *strictly quasiconvex*, where we call a function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ (strictly) quasiconvex whenever

$$g(tp_1 + (1-t)p_2) \leq (<) \max\{g(p_1), g(p_2)\}$$

for any $p_1, p_2 \in \mathbb{R}^m$ with $p_1 \neq p_2$ and $t \in (0, 1)$. This result generalizes the previous ones, in the sense that, in the particular case in which $f(p) = |p|^2$, the notion of absolute minimizer reduces to the one of AMLE and the Aronsson equation becomes the infinity Laplace equation. Many improvements of the results in [BJW] have been achieved by Crandall ([Cr]), both weakening some assumptions and exploiting a concise and elegant proof, and by Crandall, Wang and Yu ([CYW]), dealing with the more natural assumption of C^1 Hamiltonians.

More recently, Bieske and Capogna ([B, BC]) studied the derivation of the Aronsson equation, and the question of uniqueness of absolute minimizers, in the setting of Carnot groups and for the case $f(p) = |p|^2$. Later, Wang ([W]) moved the focus on the possibility to extend the previous results to more general frameworks, and started the study of supremal functionals defined in the setting of *Carnot-Carathéodory spaces*. We stress that this point of view is pretty general and encompasses, among other things, the Euclidean setting and many interesting sub-Riemannian manifolds. On the other hand its rich analytical structure allows to study many interesting problems in great generality (see for example [EPV, MSC, MPSC, MPSC2] and references therein).

In order to better introduce this issue we recall some terminology and some well known facts. Given a family $X = (X_1, \dots, X_m)$ of locally Lipschitz vector fields defined on U , we say that an absolutely continuous curve $\gamma : [0, \delta] \rightarrow U$ is *horizontal* when there are measurable functions $a_1(t), \dots, a_m(t)$ with

$$\dot{\gamma}(t) = \sum_{j=1}^m a_j(t) X_j(\gamma(t)) \quad \text{for a.e. } t \in [0, \delta], \quad (1.2)$$

and we say that it is *subunit* whenever it is horizontal with $\sum_{j=1}^m a_j^2(t) \leq 1$ for a.e. $t \in [0, \delta]$. Moreover, we define the *Carnot-Carathéodory distance* on U as

$$d_X(x, y) := \inf\{T : \gamma : [0, T] \rightarrow U \text{ is subunit, } \gamma(0) = x \text{ and } \gamma(T) = y\}$$

If d_X is a (finite) distance on U , we say that (U, d_X) is a *Carnot-Carathéodory space*. Moreover, we denote by $C(x)$ the $m \times n$ matrix defined by

$$C(x) := [c_{j,i}(x)]_{\substack{i=1, \dots, n \\ j=1, \dots, m}},$$

where for each $j = 1, \dots, m$ we have $X_j := \sum_{i=1}^n c_{j,i} \frac{\partial}{\partial x_i}$. If $u \in L^1_{loc}(U)$, we define the distributional X -gradient (or *horizontal gradient*) of u by

$$\langle Xu, \varphi \rangle := - \int_U u \operatorname{div}(\varphi \cdot C(x)) dx \quad \text{for any } \varphi \in C_c^\infty(U, \mathbb{R}^m).$$

Finally, if $p \in [1, +\infty]$, we define the *horizontal Sobolev spaces* as

$$W_X^{1,p}(U) := \{u \in L^p(U) : Xu \in L^p(U, \mathbb{R}^m)\}$$

and

$$W_{X,loc}^{1,p}(U) := \{u \in L^p_{loc}(U) : u|_V \in W_X^{1,p}(V), \quad \forall V \Subset U\}.$$

In [W] the author adapted in the obvious way the notion of absolute minimizer to this framework. He showed, under mild assumptions on the vector fields, that any absolute minimizer of the supremal functional defined by

$$F(u, V) := \|f(x, Xu(x))\|_{L^\infty(V)}$$

is a viscosity solution to

$$-\sum_{i=1}^m X_i(f(x, Xu(x))) \frac{\partial f}{\partial p_i}(x, Xu(x)) = 0,$$

provided that $p \mapsto f(x, p)$ is quasiconvex, f is homogeneous of degree $\alpha \geq 1$ and $D_p f(0, 0) = 0$. Finally, Wang and Yu ([WY]) improved the previous result by requiring only C^1 regularity for f and dropping the assumption that $D_p f(0, 0) = 0$ (see also [DMV] for some more specific results for the case $f(p) = |p|^2$). However, neither [W] nor [WY] studied the problem for Hamiltonian functions f that allow s -variable dependence.

In the present paper we generalize the results in [Cr] and [W], showing that any absolute minimizer of the functional

$$F(u, V) := \|f(x, u(x), Xu(x))\|_{L^\infty(V)}$$

is a viscosity solution to the Aronsson equation

$$-\sum_{i=1}^m X_i(f(x, u(x), Xu(x))) \frac{\partial f}{\partial p_i}(x, u(x), Xu(x)) = 0,$$

provided that the following conditions hold.

- (X1) d_X is a distance on U , and it is continuous with respect to the Euclidean topology.
- (X2) X_i is a C^2 vector field defined on U , for any $i = 1, \dots, m$.
- (f1) $f \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^m, [0, \infty))$.
- (f2) $p \mapsto f(x, s, p)$ is quasiconvex for any $x \in \Omega$ and for any $s \in \mathbb{R}$.

The strategy of our proof, strongly inspired by [Cr], is divided into five steps.

Step 1. Arguing by contradiction, we assume that there is an absolute minimizer which fails to be a viscosity subsolution to the Aronsson equation. Therefore, without loss of generality, we assume that there exists a function $\phi \in C^2(U)$, which touches u from above in 0, such that

$$-\sum_{i=1}^m X_i(f(0, \phi(0), X\phi(0))) \frac{\partial f}{\partial p_i}(0, \phi(0), X\phi(0)) > 0.$$

Step 2. Exploiting ideas from [Cr, W], we build a family $(\Psi_\varepsilon)_\varepsilon$ of classical solutions to the Hamilton-Jacobi equation

$$f(x, \Psi_\varepsilon(x), X\Psi_\varepsilon(x)) = f(0, \phi(0) - \varepsilon, X\phi(0)),$$

in order to approximate in a suitable way the behavior of ϕ in 0. We stress that, since this passage strongly relies on the arguments in [Cr, pages 275-276], the C^2 regularity of f is crucial to guarantee that the Ψ_ε 's are classical C^2 solutions.

Step 3. We find an open set \mathcal{N}_ε which allows to consider Ψ_ε as a competitor in the definition of absolute minimizer.

Step 4. By an appropriate change of variables we reduce to the case in which $s \mapsto f(x, s, p)$ is non-decreasing in a neighborhood of $(0, \phi(0), X\phi(0))$.

Step 5. We show the solvability of a suitable system of ODEs to get a family of C^1 curves $(\gamma_\varepsilon)_\varepsilon$, and we show that there is a choice among such curves which allows to reach a contradiction.

The previous scheme is formally analogous to the one employed in [Cr]. Nevertheless, our non-Euclidean framework presents some technical difficulties that required the introduction of some new tools. In particular, the last step requires some preliminary results about differentiability in Carnot-Carathéodory spaces which as far as we know are new, and which we tackled, inspired again by [Cr], by suitably adapting the notion of *subdifferential* introduced in [Cl]. Moreover,

differently from [Cr], the aforementioned system of ODEs cannot be solved by means of the classical Cauchy-Lipschitz existence theorem.

From one hand, our result generalizes [Cr] to the more general setting of Carnot-Carathéodory spaces. Moreover, differently from [W], we allow also the function dependence of the Hamiltonian and we drop the requirement $D_p f(0,0) = 0$. Finally, the results in [WY], apart from not allowing the function dependence of the Hamiltonian, are achieved under the Hörmander condition, which is known to be stronger than (X1). On the other hand the construction of $(\Psi_\varepsilon)_\varepsilon$, according to [Cr, W], strongly relies on the C^2 regularity of the Hamiltonian, which, on the contrary, is weakened in [WY]. We point out that our assumptions are too general to ensure uniqueness for the associated Dirichlet problem, as shown in [JWY] in the Euclidean setting. Nevertheless, many uniqueness results are available in particular settings and under suitable hypotheses on the Hamiltonian (cf. for instance [J, W, JWY]). The paper is organized as follows. In Section 2, we recall some preliminaries about Carnot Carathéodory spaces, viscosity solutions, absolute minimizers and quasiconvex functions, we introduce the aforementioned notion of subdifferential and we show some useful properties of differentiability along horizontal curves. In Section 3, we state and prove the main result of this paper.

2. PRELIMINARIES

2.1. Notation. Unless otherwise specified, we let $m, n \in \mathbb{N} \setminus \{0\}$ with $m \leq n$, we denote by U an open and connected subset of \mathbb{R}^n and by \mathcal{A} the class of all open subsets of U . Given two open sets A and B , we write $A \Subset B$ whenever $\bar{A} \subseteq B$. If $E \subseteq \mathbb{R}^n$, we set $\bar{co}E$ to be the closure of

$$coE := \bigcap \{C : C \text{ is convex and } E \subseteq C\}.$$

It is easy to see that coE is convex and that $\bar{co}E$ is closed and convex. Moreover we set

$$\Lambda_n := \{(\lambda_1, \dots, \lambda_n) : 0 \leq \lambda_j \leq 1, \sum_{j=1}^n \lambda_j = 1\}. \quad (2.1)$$

For any $v \in \mathbb{R}^n$, we denote by $|v|$ the Euclidean norm of v . We let S^m be the class of all $m \times m$ symmetric matrices with real coefficients. Moreover, if A is a $p \times q$ matrix and B is a $q \times r$ matrix, we let $A \cdot B$ be the usual matrix product. If $A, B \in S^m$, we write $A \leq B$ whenever $p \cdot A \cdot p^T \leq p \cdot B \cdot p^T$ for any $p \in \mathbb{R}^m$. We denote by \mathcal{L}^n the restriction to U of the n -th dimensional Lebesgue measure, and for any set $E \subseteq U$ we write $|E| := \mathcal{L}^n(E)$. If $|E| = 0$, we say that E is null or Lebesgue-null. Given $x \in \mathbb{R}^n$ and $R > 0$ we let $B_R(x) := \{y \in \mathbb{R}^n : |x - y| < R\}$. If we have a function $g \in L^1_{loc}(U)$ and $x \in U$ is a Lebesgue point of g , when we write $g(x)$ we always mean that

$$g(x) = \lim_{r \rightarrow 0^+} \int_{B_r(x)} g(y) dy.$$

If $f(x, s, p)$ is a regular function defined on $U \times \mathbb{R} \times \mathbb{R}^m$, we denote by $D_x f = (D_{x_1} f, \dots, D_{x_n} f)$, $D_s f$ and $D_p f = (D_{p_1} f, \dots, D_{p_m} f)$ the partial gradients of f with respect to the variables x, s and p respectively. In general we mean gradients as row vectors.

2.2. Carnot-Carathéodory spaces. Assume that we have a family X_1, \dots, X_m of locally Lipschitz vector fields defined on U . Given $k \geq 1$, we define $C_X^k(U)$ by

$$C_X^k(U) := \{u \in C(U) : \exists X_{i_1} \cdots X_{i_s} u \in C(U) \text{ for any } (i_1, \dots, i_s) \in \{1, \dots, m\}^s \text{ and } 1 \leq s \leq k\}.$$

Therefore, whenever we have a function $u \in C_X^2(U)$, we can define its *horizontal Hessian* $X^2 u \in C(U, S^m)$ as

$$X^2 u(x)_{ij} := \frac{X_i X_j u(x) + X_j X_i u(x)}{2}$$

for any $x \in U$ and $i, j = 1, \dots, m$. When in addition (U, d_X) is a Carnot-Carathéodory space, we can define the *Horizontal Lipschitz space* as

$$\text{Lip}_X(U) := \left\{ u : U \longrightarrow \overline{\mathbb{R}} : \sup_{x \neq y} \frac{u(x) - u(y)}{d_X(x, y)} < +\infty \right\}.$$

In this paper, unless otherwise specified, we assume that

(X1) d_X is a distance on U , and it is continuous with respect to the Euclidean topology.

When (X1) holds, we know from [GN] that

$$W_{X,loc}^{1,\infty}(U) = \text{Lip}_{X,loc}(U). \quad (2.2)$$

In particular, each function $u \in W_{X,loc}^{1,\infty}(U)$ admits a continuous representative, that is

$$W_{X,loc}^{1,\infty}(U) \subseteq C(U). \quad (2.3)$$

Indeed, if $u \in W_{X,loc}^{1,\infty}(U)$ and $x \neq y \in K \Subset U$, then thanks to (2.2) it holds that

$$|u(x) - u(y)| = d_X(x, y) \frac{|u(x) - u(y)|}{d_X(x, y)} \leq d_X(x, y) \sup_{z \neq w \in K} \frac{|u(z) - u(w)|}{d_\Omega(z, w)},$$

and the right side goes to zero as $x \rightarrow y$ in virtue of (X1). Therefore, in the following we identify $u \in W_{X,loc}^{1,\infty}(U)$ with its continuous representative. As it is well known, assumption (X1) is quite mild in this framework, since it includes many relevant situations. Just to mention the most famous instance, we recall that a family X_1, \dots, X_m satisfies the *Hörmander condition* whenever each X_j is a smooth vector field and it holds that

$$\text{span}\{\text{Lie}(X_1(x), \dots, X_m(x))\} = \mathbb{R}^n \quad \text{for any } x \in U,$$

where $\text{Lie}(X_1(x), \dots, X_m(x))$ denotes the Lie algebra generated by $X_1(x), \dots, X_m(x)$. From [Ch, NSW, G] we know the following result.

Proposition 2.1. *Assume that X satisfies the Hörmander condition. Then the following properties hold.*

- (i) (U, d_X) is a Carnot-Carathéodory space.
- (i) For any compact set $K \subseteq U$ there exists a positive constant C_K such that

$$C_K^{-1}|x - y| \leq d_X(x, y) \leq C_K|x - y|^{\frac{1}{r}} \quad \text{for any } x, y \in K,$$

r being the nilpotency step of $\text{Lie}(X_1, \dots, X_m)$.

Hence Hörmander vector fields are examples of vector fields satisfying (X1). Nevertheless, there are relevant classes of vector fields which satisfy (X1) but not the Hörmander condition. We refer to [CGPV, FL] for some examples.

We stress that, according to (1.2), an absolutely continuous curve $\gamma : [0, \delta] \longrightarrow \mathbb{R}$ is horizontal if and only if there exists a measurable function $A : [0, \delta] \longrightarrow \mathbb{R}^m$ with

$$\dot{\gamma}(t) = C(\gamma(t))^T \cdot A(t) \quad (2.4)$$

for a.e. $t \in [0, \delta]$.

2.3. Subgradient in Carnot-Carathéodory spaces. When $u \in W_{X,loc}^{1,\infty}(U)$ and $N \subseteq U$ is any Lebesgue-null set which contains all the non-Lebesgue points of Xu , we define the (X, N) -subgradient of u as

$$\partial_{X,N}u(x) := \overline{\text{co}}\left\{ \lim_{n \rightarrow \infty} Xu(y_n) : y_n \rightarrow x, y_n \notin N \text{ and } \exists \lim_{n \rightarrow \infty} Xu(y_n) \right\}$$

for any $x \in U$. This notion generalizes the classical subdifferential introduced in [Cl]. Indeed, let us fix $u \in W_{loc}^{1,\infty}(U)$. Then Morrey's inequality implies that any Lebesgue point of ∇u is a point of differentiability of u (cf. [L, Corollary 11.36]). Therefore, when $X = (\partial_1, \dots, \partial_n)$,

$\partial_{X,N}u$ coincides with Clarke's subdifferential. We begin by proving some properties of the (X, N) -subgradient with the help of the two following lemmas, whose proof can be found at the end of this paper.

Lemma 2.2. *Let*

$$S := \left\{ \lim_{n \rightarrow \infty} Xu(y_n) : y_n \rightarrow x, y_n \notin N \text{ and } \exists \lim_{n \rightarrow \infty} Xu(y_n) \right\}$$

and, for any $k \geq 1$, let

$$A_k = \{Xu(y) : y \in B_{1/k}(x) \setminus N\}.$$

Then it follows that

$$\bigcap_{k=1}^{\infty} \overline{A}_k \subseteq S.$$

Lemma 2.3. *Let $(A_k)_k$ be a decreasing sequence of non-empty bounded subsets of \mathbb{R}^m , and let S be a non-empty, bounded subset of \mathbb{R}^m . Assume that*

$$\bigcap_{k=1}^{\infty} \overline{A}_k \subseteq S.$$

Then it follows that

$$\bigcap_{k=1}^{\infty} \overline{\text{co}}(\overline{A}_k) \subseteq \overline{\text{co}}(S).$$

Proposition 2.4. *Let u and N be as above. Then the following facts hold.*

- (i) $\partial_{X,N}u(x)$ is a non-empty, convex, closed and bounded subset of \mathbb{R}^m for any $x \in U$;
- (ii) for any $x \in U$

$$\partial_{X,N}u(x) = \bigcap_{k=1}^{\infty} \overline{\text{co}}\{Xu(y) : y \in B_{1/k}(x) \setminus N\};$$

- (iii) if $u \in C_X^1(U)$, then

$$\partial_{X,N}u(x) = \{Xu(x)\}$$

for any $x \in U$.

Proof. We start by proving (i). We fix $x \in U$ and show that $\partial_{X,N}u(x) \neq \emptyset$. Let $r > 0$ be small enough to have $B_r(x) \Subset U$. Then $u \in W_X^{1,\infty}(B_r(x))$. So we set $L := \|Xu\|_{L^\infty(B_r(x))}$. Let $(r_n)_n \subseteq (0, r)$ with $r_n \searrow 0$. Then, for any $n \in \mathbb{N}$, take $y_n \in B_{r_n}(x) \setminus N$. Then clearly y_n tends to x . Moreover, being y_n a Lebesgue point of Xu , it follows that

$$|Xu(y_n)| = \left| \lim_{s \rightarrow 0^+} \int_{B_s(y_n)} Xu(z) dz \right| \leq \lim_{s \rightarrow 0^+} \int_{B_s(y_n)} |Xu(z)| dz \leq L,$$

and so $(Xu(y_n))_n$ is bounded in \mathbb{R}^m . Therefore, up to a subsequence, we can assume that its limit exists, that is $\partial_{X,N}u(x)$ is non-empty. From the above proof it is easy to see that $\partial_{X,N}u(x)$ is bounded, while convexity and closure follows directly from its definition. Let us prove (ii). We fix $x \in U$ and start by proving the left-to-right inclusion. As the right set is convex and closed, it is sufficient to show that any z of the form

$$z = \lim_{n \rightarrow \infty} Xu(y_n),$$

with $y_n \rightarrow x$ and $y_n \notin N$, belongs to

$$\overline{\text{co}}\{Xu(y) : y \in B_{1/k}(x) \setminus N\}$$

for any $k \in \mathbb{N} \setminus \{0\}$. As y_n tends to x we get that $y_n \in B_{1/k}(x) \setminus N$ for n sufficiently large. Therefore, as the conclusion follows for each $X(y_n)$ and the right set is closed, we have proved the desired inclusion. The proof of the converse inclusion follows from Lemma 2.2 and Lemma

2.3. Now we prove (iii). Let $x \in U$ and let $(y_n)_n \subseteq U \setminus N$ converges to x . Then from the continuity of Xu it follows that $\lim_{n \rightarrow \infty} Xu(y_n) = Xu(x)$. Since $\{Xu(x)\}$ is convex and closed, this implies that $\partial_{X,N}u(x) \subseteq \{Xu(x)\}$. Conversely, being N null, there exists a sequence $(y_n)_n \subseteq U \setminus N$ which converges to x . Again thanks to the continuity of Xu , the converse inclusion follows. \square

With the following proposition we see that the notion of (X, N) -subgradient, in analogy with the Euclidean setting, is the right tool to deal with differentiability of X -Lipschitz functions along horizontal curves.

Proposition 2.5. *Assume that X satisfies (X1). Let $1 \leq p \leq +\infty$, let $u \in W_{X,loc}^{1,\infty}(U)$ and let $\gamma \in \text{AC}([-\beta, \beta], U)$ be a horizontal curve with*

$$\dot{\gamma}(t) = C(\gamma(t))^T \cdot A(t)$$

and $A \in L^p((-\beta, \beta), \mathbb{R}^m)$. Then the curve $t \mapsto u(\gamma(t))$ belongs to $W^{1,p}(-\beta, \beta)$, and there exists a function $g \in L^\infty((-\beta, \beta), \mathbb{R}^m)$ such that

$$\frac{du(\gamma(t))}{dt} = g(t) \cdot A(t)$$

for a.e. $t \in (-\beta, \beta)$. Moreover

$$g(t) \in \partial_{X,N}u(\gamma(t))$$

for a.e. $t \in (-\beta, \beta)$.

Proof. Let $(\varrho_\delta)_\delta$ be a sequence of spherically symmetric mollifiers, and let N be any null set which contains all the non-Lebesgue points of Xu . If δ is sufficiently small and we define u_δ and $(Xu)_\delta$ to be the standard convolutions, we have that these functions are smooth on a bounded open set, say V , such that $V \Subset U$ and V contains the support of γ . Moreover, as X satisfies (X1), from [W] we know that there exists a non-negative and non-decreasing function $w(\delta)$ (depending on the chosen function u) defined in a right neighborhood of 0, such that

$$\lim_{\delta \rightarrow 0^+} w(\delta) = 0$$

and moreover

$$|X(u_\delta)(x) - (Xu)_\delta(x)| \leq w(\delta) \tag{2.5}$$

for any $x \in V$. As u_δ is C^1 and γ is absolutely continuous, from standard calculus we have that

$$\begin{aligned} u_\delta(\gamma(t)) - u_\delta(\gamma(0)) &= \int_0^t D(u_\delta)(\gamma(s)) \cdot \dot{\gamma}(s) ds \\ &= \int_0^t D(u_\delta)(\gamma(s)) \cdot C(\gamma(s))^T \cdot A(s) ds \\ &= \int_0^t X(u_\delta)(\gamma(s)) \cdot A(s) ds. \end{aligned} \tag{2.6}$$

Let us consider now the sequence of functions $X(u_{1/n})(\gamma(\cdot))$. It is easy to see that it is bounded in $L^\infty((-\beta, \beta), \mathbb{R}^m)$. Therefore (up to a subsequence) there exists a function $g \in L^\infty((-\beta, \beta), \mathbb{R}^m)$ such that

$$X(u_{1/n})(\gamma(\cdot)) \rightharpoonup^* g(\cdot) \quad \text{in } L^\infty((-\beta, \beta), \mathbb{R}^m) \tag{2.7}$$

as n goes to infinity, and so in particular

$$X(u_{1/n})(\gamma(\cdot)) \rightharpoonup g(\cdot) \quad \text{in } L^2((-\beta, \beta), \mathbb{R}^m) \tag{2.8}$$

as n goes to infinity. Since u is continuous, then by well known results we have that u_δ converges uniformly to u on V . Therefore, passing to the limit in (2.6), noticing in particular that $A \in L^1((-\beta, \beta), \mathbb{R}^m)$ and exploiting (2.7), we obtain that

$$u(\gamma(t)) - u(\gamma(0)) = \int_0^t g(s) \cdot A(s) ds.$$

We are left to show that $g(t) \in \partial_{X, Nu}(\gamma(t))$ for a.e. $t \in (-\beta, \beta)$. Let us notice that, since for any $x \in V$ we have that

$$(Xu)_\delta(x) = \int_{B_\delta(x) \setminus N} \rho_\delta(y-x) Xu(y) dy,$$

it follows that

$$(Xu)_\delta(x) \in \overline{\text{co}}\{Xu(y) : y \in B_\delta(x) \setminus N\} \quad (2.9)$$

for any $x \in V$. Indeed, recalling that $Xu \in L^\infty(B_\delta(x) \setminus N)$ for δ small enough, setting $m := \inf_{B_\delta(x) \setminus N} Xu$ and $M := \sup_{B_\delta(x) \setminus N} Xu$, it holds that

$$m = m \int_{B_\delta(x) \setminus N} \rho_\delta(x-y) dy \leq (Xu)_\delta(x) \leq M \int_{B_\delta(x) \setminus N} \rho_\delta(x-y) dy = M,$$

and so $(Xu)_\delta(x) \in [m, M]$. Therefore, noticing that

$$[m, M] = \overline{\text{co}}\{m, M\} \subseteq \overline{\text{co}}\{Xu(y) : y \in B_\delta(x) \setminus N\} \subseteq [m, M],$$

then (2.9) follows. Thanks to (2.8) and Mazur's Lemma (cf. e.g. [Br, Corollary 3.9]), for each $m \in \mathbb{N}$ there are convex combinations of $X(u_{1/n})(\gamma(\cdot))$ converging strongly to g in $L^2((-\beta, \beta), \mathbb{R}^m)$, that is

$$v_m(\cdot) := \sum_{n=M_m}^{N_m} a_{m,n} X(u_{1/n})(\gamma(\cdot)) \longrightarrow g(\cdot) \quad \text{in } L^2((-\beta, \beta), \mathbb{R}^m),$$

with $M_m < N_m$ and $\lim_{m \rightarrow \infty} M_m = +\infty$. Moreover (again up to a subsequence) we can assume that the above convergence holds pointwise for a.e. $t \in (-\beta, \beta)$. Let us define now

$$z_m(\cdot) := \sum_{n=M_m}^{N_m} a_{m,n} (Xu)_{1/n}(\gamma(\cdot)).$$

Then, (2.5) implies that

$$\begin{aligned} |z_m(t) - g(t)| &\leq \sum_{n=M_m}^{N_m} a_{m,n} |X(u_{1/n})(\gamma(t)) - (Xu)_{1/n}(\gamma(t))| + |v_m(t) - g(t)| \\ &\leq \sum_{n=M_m}^{N_m} a_{m,n} w(1/n) + |v_m(t) - g(t)| \\ &\leq \sum_{n=M_m}^{N_m} a_{m,n} w(1/M_m) + |v_m(t) - g(t)| \\ &= w(1/M_m) + |v_m(t) - g(t)|, \end{aligned}$$

which implies that z_m converges to g pointwise for a.e. $t \in (-\beta, \beta)$ as $m \rightarrow \infty$. Moreover, from (2.9) and the definition of z_m it follows easily that

$$z_m(t) \in \overline{\text{co}}\{Xu(y) : y \in B_{1/M_m}(\gamma(t)) \setminus N\} \subseteq \overline{\text{co}}\{Xu(y) : y \in B_{1/k}(\gamma(t)) \setminus N\}$$

for any $t \in (-\beta, \beta)$ and for any $k \leq M_m$. Therefore, thanks to the pointwise convergence as $m \rightarrow \infty$, we get that

$$g(t) \in \bigcap_{k=1}^{\infty} \overline{\text{co}}\{Xu(y) : y \in B_{1/k}(\gamma(t)) \setminus N\}.$$

for a.e. $t \in (-\beta, \beta)$. Finally, from Proposition 2.4, the thesis follows. \square

As a corollary of the previous proposition we have the following result.

Proposition 2.6. *Assume that X satisfies (X1). Let $u \in C_X^1(U)$ and let $\gamma \in C^1([-\beta, \beta], U)$ be a horizontal curve with*

$$\dot{\gamma}(t) = C(\gamma(t))^T \cdot A(t)$$

and $A \in C([-\beta, \beta], \mathbb{R}^m)$. Then the curve $t \mapsto u(\gamma(t))$ belongs to $C^1(-\beta, \beta)$ and

$$\frac{du(\gamma(t))}{dt} = Xu(\gamma(t)) \cdot A(t)$$

for any $t \in (-\beta, \beta)$.

We conclude this section with a useful property which links subgradients and quasiconvex functions.

Lemma 2.7. *Let $f \in C(U \times \mathbb{R} \times \mathbb{R}^m)$ be a non-negative function which satisfies (f2). Let $u \in W_{X, \text{loc}}^{1, \infty}(U)$, $V \in \mathcal{A}$ and $K \geq 0$ such that*

$$f(x, u(x), Xu(x)) \leq K \tag{2.10}$$

for a.e. $x \in V$. Let N be a Lebesgue-null subset of V containing all the points where (2.10) fails and all the non-Lebesgue points of Xu . Then it follows that

$$f(x, u(x), w) \leq K$$

for any $x \in V$ and for any $w \in \partial_{X, N}u(x)$.

Proof. Let $x \in V$ be fixed and let $w \in \partial_{X, N}u(x)$. Then there exists a sequence

$$(w_h)_h \subseteq \text{co} \left\{ \lim_{n \rightarrow \infty} Xu(y_n) : y_n \rightarrow x, y_n \notin N \text{ and } \exists \lim_{n \rightarrow \infty} Xu(y_n) \right\}$$

converging to w in \mathbb{R}^m . If we are able to prove the claim for each w_h , the thesis follows from the continuity of f in the third argument. Fix then h . Thanks to Carathéodory Theorem (cf. [D, Theorem 1.2]) there are $(\lambda_1^h, \dots, \lambda_{n+1}^h) \in \Lambda_{n+1}$ and w_1^h, \dots, w_{n+1}^h such that

$$w_j^h \subseteq \left\{ \lim_{n \rightarrow \infty} Xu(y_n) : y_n \rightarrow x, y_n \notin N \text{ and } \exists \lim_{n \rightarrow \infty} Xu(y_n) \right\}$$

for any $j = 1, \dots, n+1$ and

$$w^h = \sum_{j=1}^{n+1} \lambda_j^h w_j^h.$$

Again, if we are able to show the claim for each w_j^h , we are done because of the convexity of sublevel sets of f . Let us fix j and take a sequence $(y_s)_s \subseteq V \setminus N$ converging to x and such that $w_j^h = \lim_{s \rightarrow \infty} Xu(y_s)$. As the map $(x, \eta) \mapsto f(x, u(x), \eta)$ is continuous, and thanks again to the global continuity of f , we conclude that

$$f(x, u(x), w_j^h) = \lim_{s \rightarrow \infty} f(x, u(x), Xu(y_s)) = \lim_{s \rightarrow \infty} f(y_s, u(y_s), Xu(y_s)) \leq K.$$

□

2.4. Supremal functionals, absolute minimizers and Aronsson equation. For sake of completeness we make explicit the definition of supremal functional and of absolute minimizer in the framework of Carnot-Carathéodory spaces. Indeed, given a non-negative function $f \in C(U \times \mathbb{R} \times \mathbb{R}^m)$, we define its associated *supremal functional* $F : W_X^{1, \infty}(U) \times \mathcal{A} \rightarrow [0, +\infty]$ as

$$F(u, V) := \|f(x, u, Xu)\|_{L^\infty(V)}$$

for any $V \in \mathcal{A}$, $u \in W_X^{1, \infty}(V)$, and we say that $u \in W_X^{1, \infty}(U)$ is an *absolute minimizer* of F if

$$F(u, V) \leq F(v, V)$$

for any $V \Subset U$ and for any $v \in W_X^{1,\infty}(V)$ with $v|_{\partial V} = u|_{\partial V}$. Moreover, according to [W], we say that a function $A \in C(U \times \mathbb{R} \times \mathbb{R}^m \times S^m)$ is *horizontally elliptic* if

$$A(x, s, p, Z) \leq A(x, s, p, Y)$$

whenever $x \in U$, $s \in \mathbb{R}$, $p \in \mathbb{R}^m$ and $Z, Y \in S^m$ with $Y \leq Z$. If f as above belongs to $C^1(U \times \mathbb{R} \times \mathbb{R}^m)$, we can define $A_f : U \times \mathbb{R} \times \mathbb{R}^m \times S^m \rightarrow \mathbb{R}$ as

$$A_f(x, s, p, Z) := -(Xf(x, s, p) + D_s f(x, s, p)p + D_p f(x, s, p) \cdot Z) \cdot D_p f(x, s, p)^T,$$

and we say that

$$A_f[\phi](x) := A_f(x, \phi(x), X\phi(x), X^2\phi(x)) = 0 \tag{2.11}$$

is the *Aronsson equation* associated to F . It is easy to check that A_f is continuous and horizontally elliptic. Moreover, for any $\phi \in C^2(U)$ and $x \in U$ it holds that

$$A_f[\phi](x) = -X(f(x, \phi, X\phi)) \cdot D_p f(x, \phi, X\phi)^T.$$

According to [CIL, W] we can now recall the notion of *viscosity solution* to the Aronsson equation. Therefore, we say that a function $u \in C(U)$ is a *viscosity subsolution* to the Aronsson equation if

$$A_f[\phi](x_0) \leq 0$$

for any $x_0 \in U$ and for any $\phi \in C^2(U)$ such that

$$0 = \phi(x_0) - u(x_0) \leq \phi(x) - u(x) \tag{2.12}$$

for any x in a neighbourhood of x_0 . Moreover we say that u is a *viscosity supersolution* if $-u$ is a viscosity subsolution, and finally we say that u is a *viscosity solution* if it is both a subsolution and a supersolution.

We end this section with a straightforward property satisfied by quasiconvex function.

Proposition 2.8. *Let $g \in C^1(\mathbb{R}^m)$ be a quasiconvex function. Then it holds that*

$$g(p) \geq g(q) \implies D_p g(p) \cdot (q - p) \leq 0$$

for any $p, q \in \mathbb{R}^m$.

3. THE MAIN THEOREM

We are ready to state and prove the main theorem of this paper.

Theorem 3.1. *Assume that (X1), (X2), (f1), (f2) hold. Then any absolute minimizer of F is a viscosity solution to the Aronsson equation.*

Proof. We divide the proof into several steps:

Step 1. Let u be an absolute minimizer for F . It suffices to show that u is a viscosity subsolution to (2.11), the other half of the proof being completely analogous. Arguing by contradiction, we assume that u fails to be a subsolution, that is there exists $x_0 \in U$, $R_1 > 0$ and $\phi \in C^2(U)$ such that (2.12) holds for any $x \in \overline{B_{R_1}(x_0)}$ and

$$A_f[\phi](x_0) > 0. \tag{3.1}$$

Without loss of generality we assume that $x_0 = 0 \in U$.

Step 2. We combine ideas from [Cr] and [W] to achieve the following

Lemma 3.2. *There exist $0 < R_2 < R_1$, $\epsilon_1 > 0$, $\mu > 0$ and a continuous function $\Psi : [0, \epsilon_1] \times B_{R_2}(0) \rightarrow \mathbb{R}$ such that, if we denote $\Psi(\epsilon, x)$ by $\Psi_\epsilon(x)$, it holds that $x \rightarrow \Psi_\epsilon(x) \in C^2(B_{R_2}(0))$ for any $\epsilon \in [0, \epsilon_1]$ and*

$$D\Psi_\epsilon \text{ is continuous in } (x, \epsilon) = (0, 0). \quad (3.2)$$

Moreover, it holds that

$$\begin{aligned} \Psi_\epsilon(0) &= \phi(0) - \epsilon, & D\Psi_\epsilon(0) &= D\phi(0), & D^2\Psi_\epsilon(0) - D^2\phi(0) &> 2\mu I_n, \\ f(x, \Psi_\epsilon(x), X\Psi_\epsilon(x)) &= f(0, \phi(0) - \epsilon, X\phi(0)), \end{aligned} \quad (3.3)$$

for any $x \in B_{R_2}(0)$.

Proof of Lemma 3.2. Let us define a new function \bar{f} on $U \times \mathbb{R} \times \mathbb{R}^n$ by

$$\bar{f}(x, s, \xi) := f(x, s, C(x) \cdot \xi) \quad (3.4)$$

for any $x \in U$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Then, since f and X are C^2 , it follows that $\bar{f} \in C^2(U \times \mathbb{R} \times \mathbb{R}^n)$. Moreover, trivial computations show that

$$D_\xi \bar{f}(x, u, \xi) = D_p f(x, u, C(x) \cdot \xi) \cdot C(x), \quad (3.5)$$

and that

$$f(x, \varphi(x), X\varphi(x)) = \bar{f}(x, \varphi(x), D\varphi(x)) \quad (3.6)$$

for any $x \in U$ and any $\varphi \in C^2(U)$. Finally, if we let $\bar{A}_{\bar{f}} \in C(U \times \mathbb{R} \times \mathbb{R}^n \times S^n)$ be the Euclidean Aronsson operator associated to \bar{f} , i.e.

$$\bar{A}_{\bar{f}}(x, s, \xi, Z) := -(D_x \bar{f}(x, s, \xi) + D_s \bar{f}(x, s, \xi)\xi + D_\xi \bar{f}(x, s, \xi) \cdot Z) \cdot D_\xi \bar{f}(x, s, \xi)^T,$$

it follows from (3.5) and (3.6) that

$$\begin{aligned} \bar{A}_{\bar{f}}[\varphi](x) &= D_x(\bar{f}(x, \varphi(x), D\varphi(x))) \cdot D_\xi \bar{f}(x, s, D\varphi)^T \\ &= D_x(f(x, \varphi(x), X\varphi(x))) \cdot (D_p f(x, \varphi(x), X\varphi(x)) \cdot C(x))^T \\ &= D_x(f(x, \varphi(x), X\varphi(x))) \cdot C(x)^T \cdot D_p f(x, \varphi(x), X\varphi(x))^T \\ &= X(f(x, \varphi(x), X\varphi(x))) \cdot D_p f(x, \varphi(x), X\varphi(x))^T = A_f[\varphi](x), \end{aligned}$$

whence $\bar{A}_{\bar{f}}[\varphi](0) > 0$. The claim then follows as in [Cr, Theorem 1] and thanks to (3.6). \square

Step 3. Now we want to exploit Ψ_ϵ as a test function in the definition of absolute minimizer on a suitable neighbourhood of 0. For doing this let us notice that (3.3) implies that

$$\begin{aligned} \Psi_\epsilon(x) &= \Psi_\epsilon(0) + D\Psi_\epsilon(0) \cdot x + x^T \cdot D^2\Psi_\epsilon(0) \cdot x + o(|x|^2) \\ &= \phi(0) - \epsilon + D\phi(0) \cdot x + x^T \cdot D^2\Psi_\epsilon(0) \cdot x + o(|x|^2) \\ &> \phi(0) - \epsilon + D\phi(0) \cdot x + x^T \cdot D^2\phi(0) \cdot x + 2\mu|x|^2 + o(|x|^2) \\ &= \phi(x) - \epsilon + 2\mu|x|^2 + o(|x|^2) \end{aligned}$$

as x goes to zero. Therefore we have that

$$\Psi_\epsilon(x) > \phi(x) - \epsilon + \mu|x|^2 \quad (3.7)$$

for any $x \in \overline{B_{R_3}(0)} \setminus \{0\}$, for any $\epsilon \in [0, \epsilon_1]$ and for some $R_3 < R_2$ sufficiently small. Let now $0 < \epsilon_2 < \epsilon_1$ small enough such that $\sqrt{\frac{\epsilon}{\mu}} < R_3$ for any $\epsilon \in [0, \epsilon_2]$ and define \mathcal{N}_ϵ as the connected component of

$$\{x \in B_{R_3}(0) : \Psi_\epsilon(x) < u(x)\}$$

containing zero (note that $\Psi_\epsilon(0) = u(0) - \epsilon < u(0)$ if $\epsilon > 0$). Therefore \mathcal{N}_ϵ is an open and connected neighborhood of 0 for any $\epsilon \in (0, \epsilon_2]$. Moreover, since (3.7) implies that

$$\Psi_\epsilon(x) > \phi(x) \geq u(x) \quad \text{on } \partial B_{\sqrt{\frac{\epsilon}{\mu}}}(0),$$

it follows that

$$\mathcal{N}_\epsilon \subseteq B_{\sqrt{\frac{\epsilon}{\mu}}}(0) \subsetneq B_{R_3}(0), \quad (3.8)$$

which implies that

$$u|_{\partial\mathcal{N}_\epsilon} = \Psi_\epsilon|_{\partial\mathcal{N}_\epsilon}.$$

Being u an absolute minimizer, and recalling (3.3), we conclude that

$$f(x, u(x), Xu(x)) \leq F(u, \mathcal{N}_\epsilon) \leq F(\Psi_\epsilon, \mathcal{N}_\epsilon) = f(0, \phi(0) - \epsilon, X\phi(0)) = f(x, \Psi_\epsilon(x), X\Psi_\epsilon(x)) \quad (3.9)$$

for a.e. $x \in \mathcal{N}_\epsilon$ and for any $\epsilon \in [0, \epsilon_2]$.

Step 4. At this point we wish to achieve the situation in which $s \mapsto f(x, s, p)$ is non-decreasing locally in a neighborhood of $(0, \phi(0), X\phi(0))$. Therefore we follow the strategy of [Cr] and we show that, via a suitable change of variables, this assumption is possible. Let us define then a new function g by

$$g(x, s, p) := f(x, u(0) + q \cdot x + G(s), q \cdot C(0)^T + G'(s)p)$$

for any (x, s, p) in a suitable neighborhood of $(0, \phi(0), X\phi(0))$, where $q \in \mathbb{R}^n$ has to be determined and $G \in C^\infty(-\delta, \delta)$ is a local increasing diffeomorphism such that $G(0) = 0$ and $G'(0) > 0$. Let us notice that g is C^2 and quasiconvex in the third argument. Moreover, if we define \bar{u} and $\bar{\phi}$ in a neighborhood of 0 by requiring that

$$\begin{aligned} u(x) &= u(0) + q \cdot x + G(\bar{u}(x)), \\ \phi(x) &= \phi(0) + q \cdot x + G(\bar{\phi}(x)), \end{aligned} \quad (3.10)$$

it is easy to see that (2.12) holds for \bar{u} and $\bar{\phi}$ and that $\bar{\phi}(0) = \bar{u}(0) = 0$. If H is the supremal functional associated to g it is easy to see that \bar{u} is an absolute minimizer for H (we stress that we are working in a suitable neighborhood of 0). Easy computations show that

$$D_x g = D_x f + D_s f q, \quad D_s g = G'(s)D_s f + G''(s)D_p f \cdot p^T, \quad D_p g = G'(s)D_p f.$$

Therefore, noticing that

$$g(x, \bar{\phi}(x), X\bar{\phi}(x)) = f(x, \phi(x), X\phi(x))$$

for any x in the usual neighborhood of 0, we have that

$$\begin{aligned} A_g[\bar{\phi}](x) &= -X(g(x, \bar{\phi}(x), X\bar{\phi}(x))) \cdot D_p g(x, \bar{\phi}(x), X\bar{\phi}(x))^T \\ &= -X(f(x, \phi(x), X\phi(x))) \cdot D_p g(x, \bar{\phi}(x), X\bar{\phi}(x))^T \\ &= -X(f(x, \phi(x), X\phi(x))) \cdot (G'(\bar{\phi}(x))D_p f(x, \phi(x), X\phi(x))^T) = G'(\bar{\phi}(x))A_f[\phi](x), \end{aligned}$$

and so $A_g[\bar{\phi}](0) = G'(0)A_f[\phi](0) > 0$. Moreover, (3.10) implies that

$$X\bar{\phi}(0) = \frac{X\phi(0) - q \cdot C(0)^T}{G'(0)}.$$

Therefore we have that

$$D_s g(0, \bar{\phi}(0), X\bar{\phi}(0)) = G'(0)D_s f(0, \phi(0), X\phi(0)) + \frac{G''(0)}{G'(0)}(X\phi(0) - q \cdot C(0)^T) \cdot D_p f(0, \phi(0), X\phi(0))^T.$$

Hence, if we choose G as $G(s) = s + \frac{\beta}{2}s^2$, where $\beta > 0$, and we choose q as

$$q := D\phi(0) + D_x f(0, \phi(0), X\phi(0)) + D_s f(0, \phi(0), X\phi(0))D\phi(0) + D_p f(0, \phi(0), X\phi(0)) \cdot B,$$

where B is the $m \times n$ matrix defined by

$$B_{ij} := \left. \frac{\partial}{\partial x_j} X_i \phi(x) \right|_{x=0}$$

for any $i = 1, \dots, m$ and $j = 1, \dots, n$, and noticing that

$$p \cdot B \cdot C(0)^T \cdot p^T = p \cdot X^2 \phi(0) \cdot p^T$$

for any $p \in \mathbb{R}^m$, thanks to (3.1) we conclude that

$$D_s g(0, \bar{\phi}(0), X \bar{\phi}(0)) = D_s f(0, \phi(0), X \phi(0)) + \beta A_f[\phi](0) > 0,$$

provided we choose β sufficiently big. Therefore in this new setting we can assume that $s \mapsto f(x, s, p)$ is increasing in a neighborhood of $(0, \phi(0), X \phi(0))$. This fact and (3.9) allow to find $0 < \epsilon_3 < \epsilon_2$ such that

$$f(x, u(x), Xu(x)) \leq f(x, u(x), X\Psi_\epsilon(x)) \quad (3.11)$$

for any $\epsilon \in (0, \epsilon_3]$ and for a.e. $x \in \mathcal{N}_\epsilon$.

Step 5. We are going to exploit (3.11), together with Proposition 2.8, in a suitable way. For doing this let us consider the first-order system of ODEs

$$\begin{cases} \dot{\gamma}(t) = -C(\gamma(t))^T \cdot D_p f(\gamma(t), u(\gamma(t), X\Psi_\epsilon(\gamma(t))))^T \\ \gamma(0) = 0 \end{cases} \quad (3.12)$$

and, for any $\epsilon \in [0, \epsilon_3]$ and a suitable $R_4 < R_3$, we define $g_\epsilon : B_{R_4}(0) \rightarrow \mathbb{R}^n$ as

$$g_\epsilon(x) := -C(x)^T \cdot D_p f(x, u(x), X\Psi_\epsilon(x))^T.$$

It is easy to see (recall (2.3)) that $g_\epsilon \in C(B_{R_4}(0), \mathbb{R}^n)$. If we define

$$\mathcal{C} := \max_{i,j} \left\{ \sup_{B_{R_4}(0)} |c_{ij}| \right\},$$

it follows from our assumptions that $0 < \mathcal{C} < +\infty$. Moreover, thanks to (2.3) and (3.2), there exist $0 < \epsilon_4 < \epsilon_3$ and $0 < R_5 < R_4$ such that

$$\begin{aligned} |D\Psi_\epsilon(x) - D\phi(0)| &\leq 1 \\ |u(x) - u(0)| &\leq 1 \end{aligned}$$

for any $x \in \overline{B_{R_5}(0)}$ and $\epsilon \in [0, \epsilon_4]$. Therefore, if we let $M_\epsilon := \max\{g_\epsilon(x) : x \in \overline{B_{R_5}(0)}\}$, it follows that

$$\begin{aligned} \|g_\epsilon(x)\|_{L^\infty(B_{R_5}(0))} &\leq \mathcal{C} \|D_p f(x, u(x), X\Psi_\epsilon(x))\|_{L^\infty(B_{R_5}(0))} \\ &\leq \mathcal{C} \|D_p f(x, s, p)\|_{L^\infty(B_{R_5}(0) \times B_1(u(0)) \times B_{\mathcal{C}}(D\phi(0)))} := M \end{aligned}$$

for any $\epsilon \in [0, \epsilon_4]$. Since (3.1) implies that $M_\epsilon > 0$, we conclude that $0 < M_\epsilon < M$ for any $\epsilon \in [0, \epsilon_4]$. Therefore, if we let

$$\epsilon_5 := \min \left\{ \epsilon_4, \frac{R_5}{M} \right\},$$

Peano's Theorem (cf. e.g. [T, Theorem 2.19]) guarantees the existence, for any $\epsilon \in [0, \epsilon_5]$, of a curve $\gamma_\epsilon \in C^1((-\epsilon_5, \epsilon_5), \mathbb{R}^n)$ which solves (3.12). Moreover, from (2.4) and the first line of (3.12) it follows that γ_ϵ is a horizontal curve. Then, Propositions 2.8 and 2.5, together with Lemma 2.7 and (3.11), imply that

$$\left. \frac{d}{dt} (\Psi_\epsilon(\gamma_\epsilon(t)) - u(\gamma_\epsilon(t))) \right|_{t=t_0} = D_p f(\gamma_\epsilon(t_0), u(\gamma_\epsilon(t_0)), X\Psi_\epsilon(\gamma_\epsilon(t_0))) \cdot (g(t_0) - X\Psi_\epsilon(\gamma_\epsilon(t_0))) \leq 0$$

for a.e. $t_0 \in (-\epsilon_5, \epsilon_5)$ and for any $\epsilon \in [0, \epsilon_5)$, and where $g(t_0)$ is as in Proposition 2.5. Therefore, if we fix $t_0 \in (0, \epsilon_5)$, the previous inequality implies that

$$\begin{aligned}\Psi_\epsilon(\gamma_\epsilon(t_0)) &= \Psi_\epsilon(0) + \int_0^{t_0} \frac{d\Psi_\epsilon(\gamma_\epsilon(t))}{dt} dt \\ &\leq u(0) - \epsilon + \int_0^{t_0} \frac{du(\gamma_\epsilon(t))}{dt} dt \\ &= u(\gamma_\epsilon(t_0)) - \epsilon < u(\gamma_\epsilon(t_0)),\end{aligned}$$

hence we conclude that $\gamma_\epsilon(t_0) \in \mathcal{N}_\epsilon$, which implies, together with (3.8), that

$$\gamma_\epsilon(t_0) \in B_{\sqrt{\frac{\epsilon}{\mu}}}(0) \quad (3.13)$$

for any $t_0 \in [0, \epsilon_5)$ and any $\epsilon \in (0, \epsilon_5)$. On the other hand, the classical Taylor's formula applied to γ_ϵ implies that

$$\gamma_\epsilon(t) = -C(0)^T \cdot (D_p f(0, \phi(0), X\phi(0)))^T t + o(t) \quad (3.14)$$

as t tends to zero and for any $\epsilon \in (0, \epsilon_5)$. If we let $2K := |C(0)^T \cdot (D_p f(0, \phi(0), X\phi(0)))^T|$, (3.1) says that $2K > 0$. Therefore, thanks to (3.14), we know that there exists $0 < \epsilon_6 < \epsilon_5$ such that

$$|\gamma_\epsilon(t)| \geq Kt \quad (3.15)$$

for any for any $t, \epsilon \in (0, \epsilon_6)$. Let us choose $\bar{\epsilon} \in (0, \epsilon_6)$ such that

$$t_0 := \frac{2}{K} \sqrt{\frac{\bar{\epsilon}}{\mu}} < \epsilon_6.$$

Then (3.15) yields that $|\gamma_{\bar{\epsilon}}(t_0)| \geq 2\sqrt{\frac{\bar{\epsilon}}{\mu}}$, which is a clear contradiction with (3.13). \square

4. APPENDIX

Proof of Lemma 2.2. Let $z \in \bar{A}_k$ for any $k \geq 1$. Then for any $k \geq 1$ there exists a sequence $(z_h^k)_h \subseteq A_k$ converging to z as h goes to infinity. Therefore we can select a subsequence $(z^k)_k \subseteq (z_h^k)_h$ which converges to z as k goes to infinity and such that $z^k \in A_k$ for any $k \geq 1$. Since $z^k \in A_k$, then there exists $y^k \in B_{1/k}(x) \setminus N$ such that $Xu(y^k) = z^k$. It follows that y^k converges to x as k goes to infinity, $y^k \notin N$ and

$$z = \lim_{k \rightarrow \infty} z^k = \lim_{k \rightarrow \infty} Xu(y^k).$$

We conclude that $z \in S$. \square

Proof of Lemma 2.3. Let $z \in \overline{co}(\bar{A}_k)$ for any $k \geq 1$. Then for any $k \geq 1$ there exists a sequence $(z_h^k)_h \subseteq co(\bar{A}_k)$ converging to z as h goes to infinity. As in the previous proof, let $(z^k)_k \subseteq (z_h^k)_h$ be a sequence which converges to z as k goes to infinity and such that $z^k \in co(\bar{A}_k)$ for any $k \geq 1$. Therefore, for each $k \geq 1$, there exist $(\lambda_1^k, \dots, \lambda_{m+1}^k) \in \Lambda_{m+1}$, where Λ_{m+1} is as in (2.1), and y_1^k, \dots, y_{m+1}^k belonging to \bar{A}_k such that

$$z^k = \sum_{j=1}^{m+1} \lambda_j^k y_j^k.$$

Up to subsequences, we assume that

$$\lambda_j^k \rightarrow \lambda_j \quad \text{as } k \rightarrow \infty$$

and

$$y_j^k \rightarrow y_j \quad \text{as } k \rightarrow \infty$$

for any $j = 1, \dots, m+1$. It is easy to see that $(\lambda_1, \dots, \lambda_{m+1}) \in \Lambda_{m+1}$ and that y_j belongs to \overline{A}_k for any $k \geq 1$. Therefore, thanks to our hypotheses, we have that $y_j^h \in S$. If we set

$$x := \sum_{j=1}^{m+1} \lambda_j y_j,$$

then $x \in \text{co}(S)$. Moreover, it holds that

$$x = \sum_{j=1}^{m+1} \lambda_j y_j = \sum_{j=1}^{m+1} \lim_{k \rightarrow \infty} \lambda_j^k y_j^k = \lim_{k \rightarrow \infty} \sum_{j=1}^{m+1} \lambda_j^k y_j^k = \lim_{k \rightarrow \infty} z^k = z,$$

which implies that $z \in \text{co}(S)$. □

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