UNIFORM STABILITY IN THE EUCLIDEAN ISOPERIMETRIC PROBLEM FOR THE ALLEN–CAHN ENERGY

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ABSTRACT. We consider the isoperimetric problem defined on the whole $\mathbb{R}^n$ by the Allen–Cahn energy functional. For non-degenerate double well potentials, we prove sharp quantitative stability inequalities of quadratic type which are uniform in the length scale of the phase transitions. We also derive a rigidity theorem for critical points analogous to the classical Alexandrov's theorem for constant mean curvature boundaries.

CONTENTS

1. INTRODUCTION

1.1. **Overview.** In this paper we study the following family of "Euclidean isoperimetric problems" on $\mathbb{R}^n$, $n \geq 2$,

$$\Psi(\sigma, m) = \inf \Big\{ \mathcal{AC}_\sigma(u) : \int_{\mathbb{R}^n} V(u) = m \,, u \in H^1(\mathbb{R}^n; [0,1]) \Big\} , \qquad \sigma\,, m > 0 \,, \tag{1.1}$$

associated to the Allen-Cahn energy functionals of a non-degenerate double-well potential W (see (1.11) and (1.12) below)

$$\mathcal{AC}_\sigma(u) = \sigma \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{\sigma} \int_{\mathbb{R}^n} W(u) \,, \qquad \sigma > 0 \,. \tag{1.2}$$

We analyze in particular the relation of these problems to the classical Euclidean isoperimetric problem

$$\Psi_{\rm iso}(m) = \inf \Big\{ P(E) : E \subset \mathbb{R}^n \,, |E| = m \Big\} = n\, \omega_n^{1/n}\, m^{(n-1)/n} \,, \qquad m > 0 \,, \tag{1.3}$$

in the natural regime where the phase transition length scale σ and the volume constraint m satisfy

$$0 < \sigma < \varepsilon_0\, m^{1/n} \,, \tag{1.4}$$

for some sufficiently small (dimensionless) constant $\varepsilon_0 = \varepsilon_0(n, W)$. The volume constraint in $\Psi(\sigma, m)$ is prescribed by means of the potential $V(t) = (\int_0^t \sqrt{W})^{n/(n-1)}$. This specific choice is natural in light of the classical estimate obtained by combining Young's inequality with the BV-Sobolev inequality/Euclidean isoperimetry, and showing that, if $u \in H^1(\mathbb{R}^n;[0,1])$, then, for $\Phi(t) = \int_0^t \sqrt{W}$,

$$\mathcal{AC}_\sigma(u) \ge 2\int_{\mathbb{R}^n} |\nabla u|\sqrt{W(u)} = 2\int_{\mathbb{R}^n} |\nabla \Phi(u)| > 2\,n\,\omega_n^{1/n}\Big(\int_{\mathbb{R}^n} V(u)\Big)^{(n-1)/n}. \tag{1.5}$$

In particular, by our choice of V, $\Psi(\sigma,m)$ is always non-trivial[1], with

$$\Psi(\sigma,m) > 2\,\Psi_{\rm iso}(m)\,, \qquad \forall \sigma,\, m > 0\,. \tag{1.6}$$

(The strict sign does not follow from (1.5) alone, but also requires the existence of minimizers in (1.5).) By combining (1.6) with a standard construction of competitors for $\Psi(\sigma,m)$, one sees immediately that

$$\lim_{\sigma\to 0^+} \Psi(\sigma,m) = 2\,\Psi_{\rm iso}(m)\,, \qquad \forall m > 0\,. \tag{1.7}$$

The relation between the Allen–Cahn energy and the perimeter functional is of course a widely explored subject (without trying to be exhaustive, see, for example, [MM77, Mod87a, Ste88, LM89a, HT00, RT08, Le11, TW12, DMFL15, Le15, Gas20]), and so is the the relation between the "volume constrained" minimization of $\mathcal{AC}_\sigma$ and relative isoperimetry/capillarity theory in bounded or periodic domains (e.g. [Mod87b, SZ98, SZ99, PR03, CCE+06, BGLN06, LM16b]). The goal of this paper is exploring in detail the proximity of $\Psi(\sigma,m)$ to the classical Euclidean isoperimetric problem $\Psi_{\rm iso}(m)$ in connection with two fundamental properties of the latter:

(i) the validity of the sharp quantitative Euclidean isoperimetric inequality [FMP08]: if $E \subset \mathbb{R}^n$ has finite perimeter $P(E)$ and positive and finite volume (Lebesgue measure) $\mathcal{L}^n(E)$, then

$$C(n)\sqrt{\frac{P(E)}{n\,\omega_n^{1/n}\,\mathcal{L}^n(E)^{(n-1)/n}} - 1} \ge \inf_{x_0\in\mathbb{R}^n} \frac{\mathcal{L}^n(E\Delta B_r(x_0))}{\mathcal{L}^n(E)}\,, \qquad r = \Big(\frac{\mathcal{L}^n(E)}{\omega_n}\Big)^{1/n}, \tag{1.8}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$;

(ii) Alexandrov's theorem [Ale62] (see [DM19] for a distributional version): a bounded open set whose boundary is smooth and has constant mean curvature is a ball; in other words, among bounded sets, the only volume-constrained critical points of the perimeter functional are its (global) volume-constrained minimizers.

Concerning property (i), the natural question in relation to $\Psi(\sigma,m)$ is if a sharp stability estimate similar to (1.8) holds *uniformly* with respect to the ratio $\sigma/m^{1/n} \in (0,\varepsilon_0)$ for $\Psi(\sigma,m)$. Uniformity in $\sigma/m^{1/n}$ seems indeed a necessary feature for a stability estimate of this kind to be physically meaningful and interesting.

Concerning property (ii), we notice that the notion of smooth, volume-constrained critical point of $\Psi(\sigma,m)$ is that of a non-zero function $u \in C^2(\mathbb{R}^n;[0,1])$ such that the semilinear PDE

$$-2\,\sigma^2\,\Delta u = \sigma\,\lambda\,V'(u) - W'(u) \qquad \text{on } \mathbb{R}^n\,, \tag{1.9}$$

holds for a Lagrange multiplier $\lambda \in \mathbb{R}$. The boundedness assumption in Alexandrov's theorem is crucial to avoid examples of non-spherical constant mean curvature boundaries, like cylinders and unduloids. This is directly translated, for solutions of (1.9), into the requirement that $u(x) \to 0$ as $|x| \to \infty$, without which semilinear PDEs like (1.9) are known

[1]Obviously, this is not always true with others choices of V. For example, setting $V(t) = t$ in (1.1), which is the most common choice in addressing diffuse interface capillarity problems in bounded containers, one has $\Psi(\sigma,m) = 0$ by a simple scaling argument. Among the possible choices that make $\Psi(\sigma,m)$ non-trivial, our has of course the advantage of appearing naturally in the lower bound (1.5). For this reason, and in the interest of definiteness and simplicity, we have not considered more general options here.

to possess non-radial solutions modeled on the aforementioned examples of unbounded constant mean curvature boundaries, see e.g. [PR03].

Under the decay assumption $u(x) \to 0$ as $|x| \to \infty$, and without further constraints on σ and λ, every solution of (1.9) will be radial symmetric thanks to the moving planes method [GNN81]. However, even in presence of symmetry, possible solutions to (1.9) will have a geometric meaning (and thus a chance of being exhausted by the family of global minimizers of $\Psi(\sigma, m)$) only if the parameters σ and λ are taken in the "geometric regime" where $\sigma\,\lambda$ is small. To explain why we consider such regime geometrically significant, we notice that the Lagrange multiplier λ in (1.9) has the dimension of an inverse length, which, geometrically, is the dimensionality of curvature. For σ to be the length of a phase transition around an interface of curvature λ, it must be that

$$0 < \sigma\,\lambda < \nu_0\,, \tag{1.10}$$

for some sufficiently small (dimensionless) constant $\nu_0 = \nu_0(n, W)$, Notice that since inverse length is volume$^{-1/n} = m^{-1/n}$, (1.10) is compatible with (1.4). We conclude that a natural generalization of Alexandrov's theorem to the Allen–Cahn setting is showing the existence of constants ε_0 and ν_0, depending on n and W only, such that, if $u \in C^2(\mathbb{R}^n; [0,1])$ vanishes at infinity and solves (1.9) for σ and λ as in (1.10), then u is a minimizer of $\Psi(\sigma, m)$ for some value m such that (1.4) holds.

1.2. **Statement of the main theorem.** We start by setting the following notation and conventions:

Assumptions on W: The double-well potential $W \in C^{2,1}[0,1]$ satisfies the standard set of non-degeneracy assumptions

$$W(0) = W(1) = 0\,, \quad W > 0 \text{ on } (0,1)\,, \quad W''(0), W''(1) > 0\,, \tag{1.11}$$

as well as the normalization

$$\int_0^1 \sqrt{W} = 1\,. \tag{1.12}$$

Correspondingly to W, we introduce the potential V used in imposing the volume constraint in $\Psi(\sigma, m)$, by setting

$$V(t) = \Phi(t)^{n/(n-1)}\,, \qquad \Phi(t) = \int_0^t \sqrt{W}\,, \qquad t \in [0,1]\,. \tag{1.13}$$

Notice that both V and Φ are strictly increasing on $[0,1]$, with $V(1) = \Phi(1) = 1$ and $\Phi(t) \approx t^2$ and $V(t) \approx t^{2n/(n-1)}$ as $t \to 0^+$. All the relevant properties of W, Φ and V are collected in section A.3.

Classes of radial decreasing functions: We say that $u : \mathbb{R}^n \to \mathbb{R}$ is radial if $u(x) = \zeta(|x|)$ for some $\zeta : [0,\infty) \to \mathbb{R}$, and that u is radial decreasing if, in addition, ζ is decreasing. We denote by

$$\mathcal{R}_0\,, \qquad \mathcal{R}_0^*\,,$$

the family of radial decreasing and radial strictly decreasing functions. For the sake of simplicity, when u is radial we shall simply write u in place of ζ, that is, we shall use indifferently $u(x)$ and $u(r)$ to denote the value of u at x with $|x| = r$. Similarly, we shall write u', u'', etc. for the radial derivatives of u.

Universal constants and rates: We say that a real number is a **universal constant** it is positive and can be defined in terms of the dimension n and of the double-well potential W only. Following a widely used convention, we will use the latter C for a generically "large" universal constant, and $1/C$ for a generically "small" one. We will use ε_0, δ_0, ν_0, ℓ_0, etc. for small universal constants whose value will be typically "chosen" at the end of an argument to make products like $C\,\varepsilon_0$ "sufficiently small". Finally, given $k \in \mathbb{N}$, we will write "$f(\varepsilon)=\mathrm{O}(\varepsilon^k)$ as $\varepsilon\to 0^+$" if there exists a universal constant C such that $|f(\varepsilon)|\le C\,\varepsilon^k$ for every $\varepsilon\in(0,1/C)$; similar definitions are given for "$\mathrm{O}(t)$ as $t\to\infty$", etc.

Theorem 1.1 (Main theorem)**.** *If $n\ge 2$ and $W\in C^{2,1}[0,1]$ satisfies (1.11) and (1.12), then there exists a universal constant ε_0 such that, setting,*

$$\mathcal{X}(\varepsilon_0)=\Big\{(\sigma,m):0<\sigma<\varepsilon_0\,m^{1/n}\Big\}\,,$$

the following holds:

(i): *for every $(\sigma,m)\in\mathcal{X}(\varepsilon_0)$ there exists a minimizer $u_{\sigma,m}$ of $\Psi(\sigma,m)$ such that $u_{\sigma,m}\in\mathcal{R}_0^*\cap C^2(\mathbb{R}^n;(0,1))$, every other minimizer of $\Psi(\sigma,m)$ is obtained from $u_{\sigma,m}$ by translation, and the Euler–Lagrange equation*

$$-2\,\sigma^2\,\Delta u_{\sigma,m}=\sigma\,\Lambda(\sigma,m)\,V'(u_{\sigma,m})-W'(u_{\sigma,m}) \tag{1.14}$$

holds on $\mathbb{R}^n$ for some $\Lambda(\sigma,m)>0$.

(ii): *Ψ is continuous on $\mathcal{X}(\varepsilon_0)$ and*

$$\begin{aligned}
&\Psi(\sigma,\cdot)\text{ is strictly concave, strictly increasing,} &(1.15)\\
&\qquad\text{and continuously differentiable on }((\sigma/\varepsilon_0)^n,\infty)\,,\\
&\Lambda(\sigma,\cdot)=\tfrac{\partial\Psi}{\partial m}(\sigma,\cdot)\text{ is strictly decreasing and continuous on }((\sigma/\varepsilon_0)^n,\infty)\,, &(1.16)\\
&\Psi(\cdot,m)\text{ is strictly increasing on }(0,\varepsilon_0\,m^{1/n})\,. &(1.17)
\end{aligned}$$

Moreover, setting $\varepsilon=\sigma/m^{1/n}$, we have

$$\begin{aligned}
\frac{\Psi(\sigma,m)}{m^{(n-1)/n}} &= 2\,n\,\omega_n^{1/n}+2\,n\,(n-1)\,\omega_n^{2/n}\,\kappa_0\,\varepsilon+\mathrm{O}(\varepsilon^2)\,, &(1.18)\\
m^{1/n}\,\Lambda(\sigma,m) &= 2\,(n-1)\,\omega_n^{1/n}+\mathrm{O}(\varepsilon)\,, &(1.19)
\end{aligned}$$

as $\varepsilon\to 0^+$ with $(\sigma,m)\in\mathcal{X}(\varepsilon_0)$. Here κ_0 is the universal constant defined by

$$\kappa_0=\int_{\mathbb{R}}\big(V'(\eta)\,\eta'+W(\eta)\big)\,s\,ds\,, \tag{1.20}$$

and η is the unique solution to $\eta'=-\sqrt{W(\eta)}$ on $\mathbb{R}$ with $\eta(0)=1/2$.

(iii)-uniform stability: *for every $(\sigma,m)\in\mathcal{X}(\varepsilon_0)$ and $u\in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n}V(u)=m$ we have, for a universal constant C,*

$$C\,\sqrt{\frac{\mathcal{AC}_\sigma(u)}{\Psi(\sigma,m)}-1}\ge\inf_{x_0\in\mathbb{R}^n}\frac{1}{m}\int_{\mathbb{R}^n}\big|\Phi(u)-\Phi(T_{x_0}u_{\sigma,m})\big|^{n/(n-1)} \tag{1.21}$$

where $T_{x_0}u_{\sigma,m}(x)=u_{\sigma,m}(x-x_0)$, $x\in\mathbb{R}^n$;

(iv)-rigidity of critical points: *there exists a universal constant ν_0 such that, if $\sigma>0$, $u\in C^2(\mathbb{R}^n;[0,1])$, $u(x)\to 0^+$ as $|x|\to\infty$, and u is a solution of*

$$-2\,\sigma^2\,\Delta u=\sigma\,\lambda\,V'(u)-W'(u)\qquad\text{on }\mathbb{R}^n\,, \tag{1.22}$$

for a parameter λ such that

$$0<\sigma\,\lambda<\nu_0\,, \tag{1.23}$$

then there exist $x_0\in\mathbb{R}^n$ and $m>0$ such that

$$\sigma<\varepsilon_0\,m^{1/n}\,,\qquad\lambda=\Lambda(\sigma,m)\,,\qquad u=T_{x_0}u_{\sigma,m}\,.$$

In particular, u is a minimizer of $\Psi(\sigma,m)$.

1.3. **Relation of Theorem 1.1–(iii) to Euclidean isoperimetric stability.** We start with some remarks connecting the (σ,m)-uniform stability estimate (1.21) to the sharp quantitative Euclidean isoperimetric inequality (1.8). To this end, it will be convenient to introduce the unit volume problem

$$\psi(\varepsilon)=\Psi(\varepsilon,1)=\inf\Big\{\mathcal{AC}_\varepsilon(u):\int_{\mathbb{R}^n}V(u)=1\,,u\in H^1(\mathbb{R}^n;[0,1])\Big\}\,,\qquad \varepsilon>0\,,$$

and correspondingly set

$$\lambda(\varepsilon)=\Lambda(\varepsilon,1)=\frac{\partial\Psi}{\partial m}(\varepsilon,1)\,,\qquad u_\varepsilon=u_{\varepsilon,1}\,,\qquad \varepsilon>0\,.$$

Notice that all the information about $\Psi(\sigma,m)$, $u_{\sigma,m}$, and $\Lambda(\sigma,m)$, is contained in $\psi(\varepsilon)$, u_ε and $\lambda(\varepsilon)$, thanks to the identities

$$\frac{\Psi(\sigma,m)}{m^{(n-1)/n}}=\psi\Big(\frac{\sigma}{m^{1/n}}\Big)\,,\qquad m^{1/n}\,\Lambda(\sigma,m)=\lambda\Big(\frac{\sigma}{m^{1/n}}\Big)\,,\qquad u_{\sigma,m}(x)=u_{\sigma/m^{1/n}}\Big(\frac{x}{m^{1/n}}\Big)\,,$$

which are easily proved by a scaling argument (see (A.1) and (A.2)).

With this terminology at hand, we start by noticing that the right-hand side of (1.21) is bounded from above by $C(n)$ thanks to the volume constraint $\int_{\mathbb{R}^n}V(u)=m$. Therefore, in proving (1.21) with, say, $(\sigma,m)=(\varepsilon,1)$, one can directly assume that u is a "low energy competitor for $\psi(\varepsilon)$", in the sense that, for a suitably small universal constant ℓ_0,

$$\mathcal{AC}_\varepsilon(u)\le\psi(\varepsilon)+\ell_0\,. \tag{1.24}$$

Now, if u is such a low energy competitor u, then $f=\Phi(u)$ is $(\ell_0+C\,\varepsilon)$-close to be an equality case for BV-Sobolev inequality

$$|Df|(\mathbb{R}^n)\ge n\,\omega_n^{1/n}\,,\qquad \text{if } \textstyle\int_{\mathbb{R}^n}|f|^{n/(n-1)}=1\,, \tag{1.25}$$

(where $|Df|$ denotes the total variation measure of $f\in BV(\mathbb{R}^n)$, and $|Df|=|\nabla f|\,dx$ if $f\in W^{1,1}(\mathbb{R}^n)$; see [AFP00]). Indeed, by an elementary comparison argument, we have

$$\psi(\varepsilon)\le 2\,n\,\omega_n^{1/n}+C\,\varepsilon\,,\qquad \forall\varepsilon<\varepsilon_0\,, \tag{1.26}$$

while (1.5) gives

$$\mathcal{AC}_\varepsilon(u)-2\,n\,\omega_n^{1/n}=\int_{\mathbb{R}^n}\Big(\sqrt{\varepsilon}\,|\nabla u|-\sqrt{\frac{W(u)}{\varepsilon}}\Big)^2+2\Big\{\int_{\mathbb{R}^n}|\nabla[\Phi(u)]|-n\,\omega_n^{1/n}\Big\}\,, \tag{1.27}$$

so that the combination of (1.24), (1.26) and (1.27) gives

$$\int_{\mathbb{R}^n}|\nabla[\Phi(u)]|-n\,\omega_n^{1/n}\le C\,\big(\ell_0+\varepsilon\big)\,,$$

while, clearly, $\int_{\mathbb{R}^n}f^{n/(n-1)}=\int_{\mathbb{R}^n}V(u)=1$.

It is well-known that (1.25) boils down to the Euclidean isoperimetric inequality if $f=1_E$ is the characteristic function of $E\subset\mathbb{R}^n$, and that equality holds in (1.25) if and only if $f=a\,1_{B_r(x_0)}$ for some $r,a\ge0$. A sharp quantitative version of (1.25) was proved in [FMP08] on sets, and then in [FMP07, Theorem 1.1] on functions, and takes the following form: if $n\ge2$, $f\in BV(\mathbb{R}^n)$, $f\ge0$, and $\int_{\mathbb{R}^n}f^{n/(n-1)}=1$, then there exists $x_0\in\mathbb{R}^n$ and $r>0$ such that

$$C(n)\,\sqrt{|Df|(\mathbb{R}^n)-n\,\omega_n^{1/n}}\ge\inf_{x_0\in\mathbb{R}^n,r>0}\int_{\mathbb{R}^n}\big|f-a(r)\,1_{B_r}(x_0)\big|^{n/(n-1)}\,, \tag{1.28}$$

where $a(r)$ is defined by $\omega_n\,r^n\,a(r)^{n/(n-1)}=1$. The uniform stability estimate (1.21) is thus modeled after (1.28), where of course one is working with a different "deficit", namely, $\mathcal{AC}_\varepsilon(u)-\psi(\varepsilon)$ rather than $|Df|(\mathbb{R}^n)-n\,\omega_n^{1/n}$ for $f=\Phi(u)$, and with a different "asymmetry", namely, the $n/(n-1)$-power of the distance of $\Phi(u)$ from Φ composed with u_ε rather than with the multiple of the characteristic function of a ball.

The key result behind (1.21) is the following *Fuglede-type estimate* for $\psi(\varepsilon)$ (Theorem 4.1): there exist universal constants δ_0 and ε_0 such that if $\varepsilon < \varepsilon_0$, $u \in H^1(\mathbb{R}^n;[0,1])$ is a radial (but not necessarily radial decreasing) function, $\int_{\mathbb{R}^n} V(u) = 1$ and

$$\int_{\mathbb{R}^n} |u-u_\varepsilon|^2 \le C\,\varepsilon\,, \qquad \|u-u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \le \delta_0\,, \tag{1.29}$$

then

$$C\left(\mathcal{AC}_\varepsilon(u)-\psi(\varepsilon)\right) \ge \int_{\mathbb{R}^n} \varepsilon\,|\nabla(u-u_\varepsilon)|^2 + \frac{(u-u_\varepsilon)^2}{\varepsilon}\,. \tag{1.30}$$

Note carefully the restriction here to *radial* functions. The right-hand side of (1.30) is the natural ε-dependent Hilbert norm associated to $\mathcal{AC}_\varepsilon$. By the usual trick based on Young's inequality, (1.30) implies

$$C\left(\mathcal{AC}_\varepsilon(u)-\psi(\varepsilon)\right) \ge \int_{\mathbb{R}^n} |\nabla[(u-u_\varepsilon)^2]|\,, \qquad \forall\, u \text{ radial, } \textstyle\int_{\mathbb{R}^n} V(u)=1\,, \tag{1.31}$$

and, then, thanks to the H^1-Sobolev inequality,

$$C\left(\mathcal{AC}_\varepsilon(u)-\psi(\varepsilon)\right) \ge \Big(\int_{\mathbb{R}^n} |u-u_\varepsilon|^{2n/(n-1)}\Big)^{(n-1)/n}\,, \qquad \forall\, u \text{ radial, } \textstyle\int_{\mathbb{R}^n} V(u)=1\,. \tag{1.32}$$

The ε-independent stability estimate (1.32) (and, *a fortiori*, the stronger estimate (1.31)) cannot hold on general $u \in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n} V(u) = 1$: indeed, if this were the case, one could take in (1.32) $u = v_\varepsilon$ to be a family of smoothings of 1_E for any set $E \subset \mathbb{R}^n$, and then let $\varepsilon \to 0^+$, to find a version of (1.8) with linear rather than quadratic rate. However, such linear estimate is well-known to be false, since the rate in (1.8) is saturated, for example, by a family of ellipsoids converging to a ball.

We conclude that, on radial functions, one can get estimates, like (1.30), (1.31) and (1.32), that are stronger than what is available for generic functions. We notice in this regard that the validity of stronger stability estimates in presence of symmetries is well-known. For example, in the case of the BV-Sobolev inequality, it was proved in [FMP07, Theorem 3.1] that if $f \in BV(\mathbb{R}^n)$ is radial decreasing, $f \ge 0$, and $\int_{\mathbb{R}^n} f^{n/(n-1)} = 1$, then (1.28) can be improved to

$$C(n)\left(|Df|(\mathbb{R}^n) - n\,\omega_n^{1/n}\right) \ge \int_{\mathbb{R}^n} \big|f-a(r)\,1_{B_r}\big|^{n/(n-1)}\,, \tag{1.33}$$

i.e., the quadratic rate in (1.28) is refined into a linear rate.

We finally notice that (1.21) implies the sharp quantitative form of the Euclidean isoperimetric inequality (1.8) by a standard approximation argument. However, since our proof of (1.21) exploits (1.8), we are not really providing a new proof of (1.8). We approach the proof of (1.21) as follows. Adopting the general selection principle strategy of Cicalese and Leonardi [CL12] we start by deducing (1.21) on radial functions from the Fuglede-type inequality (1.30). Then we adapt to our setting the quantitative symmetrization method from the proof of (1.8) originally devised in [FMP08], and thus reduce the proof of (1.21) from the general case to the radial decreasing case. (It is in this reduction step, see in particular Theorem 5.4, that we exploit (1.8).) In principle, one could have tried to approach (1.21) by working on general functions in both the selection principle and in the Fuglede-type estimate steps. This approach does not seem convenient, however, since it would not save the work needed to implement the selection principle and the Fuglede-type estimates on radial functions, while, at the same time, it would still require the repetition of all the work done in [CL12] to prove (1.8). In other words, an advantage of the approach followed here is that it separates neatly the two stability mechanisms at work in (1.21), the one related to the relation with the Euclidean isoperimetric problem, and the one specific to optimal transition profile problem (which is entirely captured by working with radial functions).

1.4. **Remarks on the Alexandrov-type result.** We now make some comments on the proof of Theorem 1.1-(iv), and explain why this result is closely related to the stability problem addressed in Theorem 1.1-(iii).

We start noticing that any $u \in C^2(\mathbb{R}^n;[0,1])$, with $u(x) \to 0$ as $|x| \to \infty$, and solving (1.22) for some $\sigma > 0$ and $\lambda \in \mathbb{R}$, will necessarily be a radial function by the moving planes method of [GNN81]; see Theorem 6.2-(i) below.

However, as explained in the overview, there is no clear reason to expect these solutions to have a geometric meaning unless σ and λ are in a meaningful geometric relation, which, interpreting λ as a curvature and σ as a phase transition length, must take the form of $0 < \sigma\,\lambda < \nu_0$ for some sufficiently small ν_0, see (1.10). In Theorem 6.2-(ii) we apply to (1.22) a classical result of Peletier and Serrin [PS83] about the uniqueness of radial solutions of semilinear PDEs on $\mathbb{R}^n$. Interestingly, the condition $0 < \sigma\,\lambda < \nu_0$, which was introduced because its natural geometric interpretation, plays a crucial role in checking the validity of one of the assumptions of the Peletier–Serrin's uniqueness theorem[2].

Once symmetry and uniqueness have been addressed by means of classical results like [GNN81] and [PS83], proving Theorem 1.1-(iv) essentially amounts to answering the following question: what is the range of values of λ in (1.22) corresponding to the minimizers $u_{\sigma,m}$ of $\Psi(\sigma,m)$ (with $0 < \sigma < \varepsilon_0\, m^{1/n}$)? Can we show that every λ satisfying $0 < \sigma\,\lambda < \nu_0$ for a sufficiently small universal ν_0 falls in that range?

Looking back at (1.14) we are thus trying to identify the range of $m \mapsto \Lambda(\sigma,m) = (\partial\Psi/\partial m)(\sigma,m)$ for $m > (\sigma/\varepsilon_0)^n$, and to show that it contains an interval of the form $(0,\nu_0/\sigma)$. Such range is indeed proved to be an interval in Theorem 1.1-(ii), where we show that $\Lambda(\sigma,\cdot)$ is decreasing and continuous. The fact that this interval contains a sub-interval of the form $(0,\nu_0/\sigma)$ is also something that is established in Theorem 1.1-(ii), specifically when we analyze the asymptotic behavior of $\Lambda(\sigma,m)$ as $\sigma/m^{1/n} \to 0$, see (1.19). Here we want to stress, however, the role of the *continuity* of $\Lambda(\sigma,\cdot)$, which is of course crucial in showing that $\{\Lambda(\sigma,m)\}_{m>(\sigma/\varepsilon_0)^n}$ covers the *interval* of values between the end-points $\Lambda(\sigma,+\infty) = 0$ and $\Lambda(\sigma,(\sigma/\varepsilon_0)^n)$. In turn, the Fuglede-type stability estimate (1.30) plays a crucial role in our proof of this continuity property: see step three in the proof of Corollary 4.2.

The importance of the Fuglede-type estimate (1.30) in answering both questions of uniform stability and of Alexandrov-type rigidity is the main reason why both problems have been addressed in a same paper.

1.5. **Organization of the paper and proof of Theorem 1.1.** The existence of minimizers of $\psi(\varepsilon)$ (for $\varepsilon < \varepsilon_0$) and the fact that such minimizers must be radial decreasing (although not necessarily unique up to translations) is established in section 2, see Theorem 2.1, through a careful concentration-compactness argument, which exploits both the quantitative stability for the BV-Sobolev inequality (in ruling out vanishing) and the specific properties of the Allen-Cahn energy (in ruling out dichotomy). After deducing the validity of the Euler–Lagrange equation (which, because of the range constraint $0 \le u \le 1$, holds initially only as a system of variational inequalities), the radial decreasing rearrangement of a minimizer is proved to be *strictly* decreasing, so that the Brothers–Ziemer theorem [BZ88] can be used to infer that generic minimizers belong to $\mathcal{R}_0^*$. This existence argument

[2]In particular, it is not obvious to us if, outside of the "geometrically natural" regime defined by (1.10), we should expect uniqueness of radial solutions of (1.22) with decay at infinity.

is then adapted to a more general family of perturbations of $\psi(\varepsilon)$, which later plays a crucial role in obtaining the main stability estimates (1.21) on radial decreasing functions, see Theorem 2.2. Here the notion of "critical sequence" for $\psi(\varepsilon_j)$, $\varepsilon_j \in (0,\varepsilon_0)$, which mixes the notion of "low-energy sequence" to that of "Palais–Smale sequence", is introduced.

In section 3 we prove a resolution result for minimizers of $\psi(\varepsilon)$ (and, more generally, for the above mentioned notion of critical sequence). In particular, in Theorem 3.1, we show, quantitatively in ε, that minimizers u_ε of $\psi(\varepsilon)$ in $\mathcal{R}_0$ are close to an *Ansatz* which is well-known in the literature (see, e.g., [Nie95, LM16b]), and is given by

$$u_\varepsilon(x) \approx \eta\Big(\frac{|x|-R_0}{\varepsilon}-\tau_0\Big)\,,\qquad R_0=\frac{1}{\omega_n^{1/n}}\,,\qquad \tau_0=\int_{\mathbb{R}}\eta'\,V'(\eta)\,s\,ds,$$

where η is the unique solution of $\eta'=-\sqrt{W(\eta)}$ on $\mathbb{R}$ with $\eta(0)=1/2$. Exponential decay rates against this *Ansatz* are then obtained in that same theorem. Our analysis is comparably simpler to that of [LM16b], because our solutions are monotonic decreasing, and, in particular, cannot exhibit the oscillatory behavior at infinity also described, for positive solution of general semilinear PDEs like (1.22), in [Ni83].

Section 4 is devoted to the proof of the Fuglede-type estimate (1.30). This is crucially based on the resolution theorem and on a careful contradiction argument based on the concentration-compactness principle. The Fuglede-type estimate is then shown to imply the uniqueness of radial minimizers (in particular, there is a unique minimizer u_ε of $\psi(\varepsilon)$ in $\mathcal{R}_0$, and every other minimizer of $\psi(\varepsilon)$ is obtained from u_ε by translation), the continuity of $\lambda(\varepsilon)$ on $\varepsilon<\varepsilon_0$, and the expansions as $\varepsilon\to 0^+$ for $\psi(\varepsilon)$ and $\lambda(\varepsilon)$ (which, by scaling, imply (1.18) and (1.19)).

In section 5 we prove the uniform stability inequality (1.21). As explained in the remarks above, we first prove (1.21) on radial decreasing functions by means of the selection principle method of Cicalese and Leonardi [CL12] (this is where Theorem 2.2 and the above mentioned notion of critical sequence are used), and then reduce the proof of (1.21) from the general case to the radial decreasing case by adapting to our setting the quantitative symmetrization method introduced in [FMP08] for proving (1.8).

In section 6 we finally prove the Alexandrov-type result along the lines already illustrated in section 1.4.

Finally, in appendix A we collect, for ease of reference, some basic facts and results which are frequently used throughout the paper. Readers are recommended to quickly familiarize themselves with the basic estimates for the potentials W, Φ and V contained therein before entering into the technical aspects of our proofs.

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2. Existence and radial decreasing symmetry of minimizers

We begin by proving the following existence and symmetry result for minimizers of $\psi(\varepsilon)$.

Theorem 2.1. *If $n\ge 2$ and $W\in C^{2,1}[0,1]$ satisfies* (1.11) *and* (1.12)*, then there exists a universal constant ε_0 such that ψ is continuous on $(0,\varepsilon_0)$ and, for every $\varepsilon<\varepsilon_0$, there exist minimizers of $\psi(\varepsilon)$. Moreover, if u_ε is a minimizer of $\psi(\varepsilon)$ with $\varepsilon<\varepsilon_0$, then, up to a translation, $u_\varepsilon\in\mathcal{R}_0^*\cap C^{2,\alpha}_{\rm loc}(\mathbb{R}^n)$ for every $\alpha\in(0,1)$, $0<u_\varepsilon<1$ on $\mathbb{R}^n$, and for some $\lambda\in\mathbb{R}$, u_ε solves*

$$-2\,\varepsilon^2\,\Delta u_\varepsilon=\varepsilon\,\lambda\,V'(u_\varepsilon)-W'(u_\varepsilon)\qquad \text{on } \mathbb{R}^n\,, \tag{2.1}$$

where λ satisfies

$$\lambda = \frac{(n-1)}{n}\,\psi(\varepsilon) + \frac{1}{n}\left\{\frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u_\varepsilon) - \varepsilon\int_{\mathbb{R}^n}|\nabla u_\varepsilon|^2\right\}. \tag{2.2}$$

Finally, λ obeys the bound

$$\left|\lambda - 2\,(n-1)\,\omega_n^{1/n}\right| \le C\,\sqrt{\varepsilon}\,, \qquad \forall \varepsilon < \varepsilon_0\,, \tag{2.3}$$

so that, in particular, $0 < 1/C \le \lambda \le C$ for a universal constant C.

Proof. Step one: We show the existence of universal constants ℓ_0, M_0, and C such that if $\varepsilon < \varepsilon_0$ and $u \in H^1(\mathbb{R}^n;[0,1])$ satisfies

$$\mathcal{AC}_\varepsilon(u) \le 2\,n\,\omega_n^{1/n} + \ell\,, \qquad \int_{\mathbb{R}^n} V(u) = 1\,, \tag{2.4}$$

for some $\ell < \ell_0$, then, up to a translation,

$$\int_{B_{M_0}} V(u) \ge 1 - C\,\sqrt{\ell}\,. \tag{2.5}$$

Moreover, in the special case when $u \in \mathcal{R}_0$, the factor $\sqrt{\ell}$ in (2.5) can replaced by ℓ.

Indeed, by applying (1.28), to $f = \Phi(u)$, we deduce that, up to a translation of u, we have

$$\int_{\mathbb{R}^n} |\Phi(u) - (\omega_n^{1/n}\,r)^{1-n}\,1_{B_r}|^{n/(n-1)} \le C(n)\left(\frac{\mathcal{AC}_\varepsilon(u)}{2} - n\,\omega_n^{1/n}\right)^{1/2} \le C\,\sqrt{\ell}\,, \tag{2.6}$$

for suitable $r > 0$ (and with ℓ in place of $\sqrt{\ell}$ if $u \in \mathcal{R}_0$). Clearly, (2.6) implies

$$\int_{B_r^c} V(u) \le C\,\sqrt{\ell}\,. \tag{2.7}$$

Let us now define M_0 by setting

$$\Phi(1/4)\,[\omega_n^{1/n}\,M_0]^{n-1} = 1\,.$$

Clearly, if $r \le M_0$, then (2.7) gives

$$\int_{B_{M_0}^c} V(u) \le C\,\sqrt{\ell}\,,$$

and (2.5) follows. Assuming by contradiction that $r > M_0$, by definition of M_0 we find

$$[\omega_n^{1/n}\,r]^{1-n} < [\omega_n^{1/n}\,M_0]^{1-n} = \Phi(1/4) < \Phi(1/2)\,,$$

so that

$$\int_{\{u\ge 1/2\}\cap B_r} \left|\Phi(1/2) - [\omega_n^{1/n}\,r]^{1-n}\right|^{n/(n-1)} \le \int_{\{u\ge 1/2\}} \left|\Phi(u) - [\omega_n^{1/n}\,r]^{1-n}\,1_{B_r}\right|^{n/(n-1)}.$$

In particular, (2.6) and the fact that $\Phi(1/2) - \Phi(1/4)$ is a universal constant imply

$$\big|\{u \ge 1/2\} \cap B_r\big| \le C\,\sqrt{\ell_0}\,. \tag{2.8}$$

At the same time (A.13) gives

$$\int_{\{u<1/2\}} V(u) \le C\int_{\{u<1/2\}} W(u) \le C\,\varepsilon\,\mathcal{AC}_\varepsilon(u) \le C\,\varepsilon\,. \tag{2.9}$$

By using, in the order, (2.9), the fact that V is increasing with $V(1) = 1$, (2.8) and (2.7), we conclude

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} V(u) \le \int_{\{u\ge 1/2\}} V(u) + C\,\varepsilon \le |\{u \ge 1/2\} \cap B_r| + \int_{B_r^c} V(u) + C\,\varepsilon \\ &\le C\left(\sqrt{\ell_0} + \varepsilon_0\right) \end{aligned}$$

which is a contradiction provided we take ℓ_0 and ε_0 small enough.

Step two: We show the existence of a universal constant ℓ_0 such that, if $\varepsilon < \varepsilon_0$ and $\{u_j\}_j$ is a sequence in $H^1(\mathbb{R}^n;[0,1])$ with

$$\mathcal{AC}_\varepsilon(u_j) \le \psi(\varepsilon) + \ell_0\,, \qquad \int_{\mathbb{R}^n} V(u_j) = 1\,, \qquad \forall j\,, \tag{2.10}$$

then there exists $u \in H^1(\mathbb{R}^n;[0,1])$ such that, up to extracting subsequences and up to translations, $\Phi(u_j) \to \Phi(u)$ in $L^{n/(n-1)}(\mathbb{R}^n)$ and, in particular, $\int_{\mathbb{R}^n} V(u) = 1$.

We first notice that, by the elementary upper bound (1.26) and by (2.10), we have $\mathcal{AC}_\varepsilon(u_j) \le C$ for every j. Next, we apply the concentration-compactness principle (see appendix A.2) to $\{V(u_j)\,dx\}_j$. By (2.5) in step one, we find that

$$\int_{B_{M_0}} V(u_j) \ge 1 - C\sqrt{\ell_0}\,, \qquad \forall j\,. \tag{2.11}$$

This rules out the vanishing case. We consider the case that the dichotomy case occurs. To that end, it will be convenient to notice the validity of the Lipschitz estimate

$$|\mathcal{AC}_\varepsilon(u) - \mathcal{AC}_{\nu\,\varepsilon}(u)| \le C\,|1-\nu|\,\mathcal{AC}_\varepsilon(u)\,, \qquad \forall \nu \ge \frac{1}{C}\,,\ u \in H^1(\mathbb{R}^n;[0,1])\,, \tag{2.12}$$

which is deduced immediately from

$$\mathcal{AC}_{\nu\,\varepsilon}(u) - \mathcal{AC}_\varepsilon(u) = (\nu-1)\,\varepsilon\int_{\mathbb{R}^n}|\nabla u|^2 + \Big(\frac{1}{\nu}-1\Big)\frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u)\,.$$

By (2.11), if we are in the dichotomy case, then there exists

$$\alpha \in \Big(1 - C\sqrt{\ell_0}, 1\Big)\,, \tag{2.13}$$

such that for every $\tau \in (0,\alpha/2)$ we can find $S(\tau) > 0$ and $S_j(\tau) \to \infty$ as $j \to \infty$ such that

$$\Big|\alpha - \int_{B_{S(\tau)}} V(u_j)\Big| < \tau\,, \qquad \Big|(1-\alpha) - \int_{B^c_{S_j(\tau)}} V(u_j)\Big| < \tau \qquad \forall j\,. \tag{2.14}$$

We now pick a cut-off function[3] φ between $B_{S(\tau)}$ and $B_{S_j(\tau)}$, so that $\varphi \in C^\infty_c(B_{S_j(\tau)})$ with $0 \le \varphi \le 1$ and $|\nabla\varphi| \le (S_j(\tau) - S(\tau))^{-1} \le 2\,S_j(\tau)^{-1}$ on $\mathbb{R}^n$, and with $\varphi = 1$ on $B_{S(\tau)}$. We notice that (2.14) and the monotonicity of V entail

$$\Big|\alpha - \int_{\mathbb{R}^n} V(\varphi\,u_j)\Big| < 2\,\tau\,, \qquad \Big|(1-\alpha) - \int_{\mathbb{R}^n} V\big((1-\varphi)\,u_j\big)\Big| < 2\,\tau \qquad \forall j\,. \tag{2.15}$$

We compute that

$$\begin{aligned}
\mathcal{AC}_\varepsilon(u_j) &= \mathcal{AC}_\varepsilon(\varphi\,u_j) + \mathcal{AC}_\varepsilon((1-\varphi)\,u_j) + a_j + b_j\,,\\
a_j &= 2\,\varepsilon\int_{B_{S_j(\tau)}\setminus B_{S(\tau)}} \varphi\,(1-\varphi)\,|\nabla u_j|^2 - u_j^2\,|\nabla\varphi|^2 - (1-2\,\varphi)\,u_j\,\nabla u_j\cdot\nabla\varphi\,,\\
b_j &= \frac{1}{\varepsilon}\int_{B_{S_j(\tau)}\setminus B_{S(\tau)}} W(u_j) - W(\varphi\,u_j) - W\big((1-\varphi)\,u_j\big)\,,
\end{aligned}$$

where we have taken into account that $\varphi\,(1-\varphi)$ and $\nabla\varphi$ are supported in $B_{S_j(\tau)} \setminus B_{S(\tau)}$, as well as that $W(0) = 0$. Let us now set, for $\sigma \in (0,1)$,

$$\Gamma_j^+(\tau,\sigma) = \big(B_{S_j(\tau)} \setminus B_{S(\tau)}\big) \cap \{u_j > \sigma\}\,, \qquad \Gamma_j^-(\tau,\sigma) = \big(B_{S_j(\tau)} \setminus B_{S(\tau)}\big) \cap \{u_j < \sigma\}\,.$$

[3]Notice that φ depends on both j and τ. We will not stress this dependency in the notation.

By (2.14), we have

$$V(\sigma)\,\mathcal{L}^n\big(\Gamma_j^+(\tau,\sigma)\big)\le\int_{B_{S_j(\tau)}\setminus B_{S(\tau)}}V(u_j)\le C\,\tau\,,\qquad\forall j\,.$$

Taking into account (A.11), if $\sigma<\delta_0$, then we have

$$\mathcal{L}^n\big(\Gamma_j^+(\tau,\sigma)\big)\le C\,\frac{\tau}{V(\sigma)}\le C\,\frac{\tau}{\sigma^{2\,n/(n-1)}}\,,\qquad\forall j\,.$$

Provided $\tau\le\tau_*$ for a suitable small universal constant τ_* we can thus guarantee that

$$\sigma(\tau):=\tau^{1/[1+(2n/(n-1))]}=\tau^{(n-1)/(3\,n-1)}<\delta_0\,,\tag{2.16}$$

and, therefore, that, setting for brevity $\sigma=\sigma(\tau)$ as in (2.16),

$$\mathcal{L}^n\big(\Gamma_j^+(\tau,\sigma)\big)\le C\,\tau^{(n-1)/(3\,n-1)}=C\,\sigma\,,\qquad\forall j\,.$$

At the same time, by applying (A.5) with $b=u_j$ and $a=0$ to get

$$\Big|W(u_j)-W''(0)\frac{u_j^2}{2}\Big|\le C\,u_j^3\le C\,\sigma\,u_j^2\,,\qquad\text{on }\Gamma_j^-(\tau,\sigma)\,,\tag{2.17}$$

and identical inequalities with $\varphi\,u_j$ and $(1-\varphi)\,u_j$ in place of u_j, thus finding

$$\begin{aligned}
b_j\ &\ge\ \frac{W''(0)}{2\,\varepsilon}\int_{\Gamma_j^-(\tau,\sigma)}u_j^2-(\varphi\,u_j)^2-\big((1-\varphi)\,u_j\big)^2-\frac{C\,\sigma}{\varepsilon}\int_{\Gamma_j^-(\tau,\sigma)}u_j^2-\frac{C}{\varepsilon}\,\mathcal{L}^n\big(\Gamma_j^+(\tau,\sigma)\big)\\
&\ge\ \frac{W''(0)}{\varepsilon}\int_{\Gamma_j^-(\tau,\sigma)}\varphi\,(1-\varphi)\,u_j^2-\frac{C\,\sigma}{\varepsilon}\int_{\Gamma_j^-(\tau,\sigma)}u_j^2-C\,\frac{\sigma}{\varepsilon}\\
&\ge\ -\frac{C\,\sigma}{\varepsilon}\int_{\mathbb{R}^n}W(u_j)-C\,\frac{\sigma}{\varepsilon}\ge-C\,\frac{\sigma}{\varepsilon}\,.
\end{aligned}$$

where, in the last line, we have used $W''(0)\ge 0$, $\varepsilon^{-1}\int_{\mathbb{R}^n}W(u_j)\le\mathcal{AC}_\varepsilon(u_j)\le C$, and the fact that (A.6) and $u_j\le\sigma\le\delta_0$ on $\Gamma_j^-(\tau,\sigma)$ give us

$$u_j^2\le C\,W(u_j)\qquad\text{on }\Gamma_j^-(\tau,\sigma)\,.\tag{2.18}$$

Similarly, if we discard the first term in the expression for a_j (which is, indeed, non-negative), we find

$$\begin{aligned}
a_j\ &\ge\ -2\,\varepsilon\int_{B_{S_j(\tau)}\setminus B_{S(\tau)}}u_j^2\,|\nabla\varphi|^2+u_j\,|\nabla u_j|\,|\nabla\varphi|\\
&\ge\ -C\,\varepsilon\,\|\nabla\varphi\|_{C^0(\mathbb{R}^n)}\int_{B_{S_j(\tau)}\setminus B_{S(\tau)}}\varepsilon\,|\nabla u_j|^2+\frac{u_j^2}{\varepsilon}\ge-\frac{C}{S_j(\tau)}\,,
\end{aligned}$$

where we have used $\|\nabla\varphi\|_{C^0(\mathbb{R}^n)}\le 2\,S_j(\tau)^{-1}$, as well as have noticed that

$$\begin{aligned}
\varepsilon\int_{\mathbb{R}^n}|\nabla u_j|^2\ &\le\ C\,\mathcal{AC}_\varepsilon(u_j)\le C\,,\\
\int_{\mathbb{R}^n}u_j^2\ &\le\ \mathcal{L}^n(\{u_j\ge\delta_0\})+C\int_{\{u_j\le\delta_0\}}W(u_j)\\
&\le\ C\int_{\{u_j\ge\delta_0\}}V(u_j)+C\,\varepsilon\,\mathcal{AC}_\varepsilon(u_j)\le C\,,
\end{aligned}$$

thanks to $V(t)\ge 1/C$ for $t\in(\delta_0,1)$ and to $W(t)\ge t^2/C$ on for $t\in(0,\delta_0)$, see (A.6) and (A.14). Combining the lower bounds for a_j and b_j, we have thus proved

$$\mathcal{AC}_\varepsilon(u_j)\ge\mathcal{AC}_\varepsilon(\varphi\,u_j)+\mathcal{AC}_\varepsilon((1-\varphi)\,u_j)-C\,\Big(\frac{\sigma}{\varepsilon}+\frac{1}{S_j(\tau)}\Big)\,.\tag{2.19}$$

If we set

$$m_j=\int_{\mathbb{R}^n}V(\varphi\,u_j)\,,\qquad n_j=\int_{\mathbb{R}^n}V\big((1-\varphi)\,u_j\big)\,,$$

and define

$$v_j(x)=(\varphi\,u_j)(m_j^{1/n}\,x)\,,\qquad w_j(x)=\big((1-\varphi)\,u_j\big)(n_j^{1/n}\,x)\,,\qquad x\in\mathbb{R}^n\,,\tag{2.20}$$

then by (A.1) and (A.2) we find

$$\int_{\mathbb{R}^n}V(v_j)=1\,,\qquad \mathcal{AC}_{\varepsilon/m_j^{1/n}}(v_j)=m_j^{(1-n)/n}\,\mathcal{AC}_\varepsilon(\varphi\,u_j)\,,\tag{2.21}$$

with analogous identities for w_j. By (2.15) and (2.12), and keeping in mind (2.13), we find

$$\begin{aligned}\mathcal{AC}_\varepsilon(\varphi\,u_j)&=m_j^{(n-1)/n}\,\mathcal{AC}_{\varepsilon/m_j^{1/n}}(v_j)\\&\ge(\alpha-C\,\tau)^{(n-1)/n}\Big(1-C\,\big|m_j^{-1/n}-1\big|\Big)\,\mathcal{AC}_\varepsilon(v_j)\\&\ge(\alpha-C\,\tau)^{(n-1)/n}\big(1-C\,|\alpha-1|-C\,\tau\big)\,\psi(\varepsilon)\,.\end{aligned}\tag{2.22}$$

Similarly, taking τ small enough with respect to $1-\alpha$, since $\int_{\mathbb{R}^n}V(w_j)=1$ we have that

$$\mathcal{AC}_\varepsilon((1-\varphi)\,u_j)=n_j^{(n-1)/n}\,\mathcal{AC}_{\varepsilon/n_j^{1/n}}(w_j)\ge((1-\alpha)-C\,\tau)^{(n-1)/n}\,2\,n\,\omega_n^{1/n}\,.\tag{2.23}$$

By combining (2.22) and (2.23) with (2.19) we get

$$\begin{aligned}\frac{\mathcal{AC}_\varepsilon(u_j)}{\psi(\varepsilon)}&\ge(\alpha-C\,\tau)^{(n-1)/n}\big(1-C\,|\alpha-1|-C\,\tau\big)\\&\quad+\frac{c(n)}{\psi(\varepsilon)}\,((1-\alpha)-C\,\tau)^{(n-1)/n}-\frac{C}{\psi(\varepsilon)}\Big(\frac{\sigma}{\varepsilon}+\frac{1}{S_j(\tau)}\Big)\,.\end{aligned}$$

Considering that $\psi(\varepsilon)\le C$ for $\varepsilon<\varepsilon_0$, we let first $j\to\infty$ and then $\tau\to0^+$ (recall that $\sigma\to0^+$ as $\tau\to0^+$) to find

$$\begin{aligned}1&\ge\big(1-C\,|\alpha-1|\big)\,\alpha^{(n-1)/n}+c(n)\,(1-\alpha)^{(n-1)/n}\\&\ge 1-C\,|\alpha-1|+c(n)\,(1-\alpha)^{(n-1)/n}\,.\end{aligned}\tag{2.24}$$

Since $1>\alpha>1-C\,\sqrt{\ell_0}$, by taking ℓ_0 small enough we can make α arbitrarily close to 1 in terms of n and W, thus obtaining a contradiction with (2.24). This proves that $\{V(u_j)\,dx\}_j$ is in the compactness case of the concentration–compactness principle. Since (2.10) implies that $\{\Phi(u_j)\}_j$ has bounded total variation on $\mathbb{R}^n$ and since $V(u_j)=\Phi(u_j)^{n/(n-1)}$ does not concentrate mass at infinity, the compactness statement now follows by standard considerations.

Step three: Let $\{u_j\}_j$ be a minimizing sequence of $\psi(\varepsilon)$, for some $\varepsilon<\varepsilon_0$. By (1.26) we can assume that for every j

$$\mathcal{AC}_\varepsilon(u_j)\le\psi(\varepsilon)+C\,\varepsilon\le 2\,n\,\omega_n^{1/n}+C\,\varepsilon\,.$$

We can then apply the compactness statement of step two to deduce the existence of minimizers of $\psi(\varepsilon)$. To prove the continuity of ψ on $(0,\varepsilon_0)$, let $\varepsilon_j\to\varepsilon_*\in(0,\varepsilon_0)$ as $j\to\infty$, and, for each ε_j, let u_j be a minimizer of $\psi(\varepsilon_j)$. By (1.26) we can apply step two to $\{u_j\}_j$ and deduce the existence, up to translations and up to extracting subsequences, of $u_*\in H^1(\mathbb{R}^n;[0,1])$ such that $\Phi(u_j)\to\Phi(u_*)$ in $L^{n/(n-1)}(\mathbb{R}^n)$ as $j\to\infty$. If $v\in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n}V(v)=1$, then

$$\mathcal{AC}_{\varepsilon_j}(u_j)\le\mathcal{AC}_{\varepsilon_j}(v)$$

so that, letting $j\to\infty$ and using lower semicontinuity,

$$\mathcal{AC}_{\varepsilon_*}(u_*)\le\liminf_{j\to\infty}\mathcal{AC}_{\varepsilon_j}(u_j)\le\lim_{j\to\infty}\mathcal{AC}_{\varepsilon_j}(v)=\mathcal{AC}_{\varepsilon_*}(v)\,.$$

Since $\int_{\mathbb{R}^n} V(u_*) = 1$, we conclude that u_* is a minimizer of $\psi(\varepsilon_*)$; and by plugging $v = u_*$ in the previous chain of inequalities, we find that $\psi(\varepsilon_j) \to \psi(\varepsilon_*)$ as $j \to \infty$.

Step four: We now notice that, by the Pólya–Szegö inequality [BZ88], once there is a minimizer of $\psi(\varepsilon)$, there is also a minimizer of $\psi(\varepsilon)$ which belongs to $\mathcal{R}_0$, or, in brief, a *radial decreasing minimizer* (more precisely: a radial decreasing minimizer with maximum at 0). In this step we prove that every radial decreasing minimizer u_ε of $\psi(\varepsilon)$ satisfies $0 < u_\varepsilon < 1$ on $\mathbb{R}^n$ and $u_\varepsilon \in C^{2,\alpha}_{\rm loc}(\mathbb{R}^n)$, and that in correspondence of u_ε one can find $\lambda \in \mathbb{R}$ such that

$$-2\,\varepsilon^2\,\Delta u_\varepsilon = \varepsilon\,\lambda V'(u_\varepsilon) - W'(u_\varepsilon) \qquad \text{on } \mathbb{R}^n\,. \tag{2.25}$$

To begin with, since u_ε is radial decreasing and has finite Dirichlet energy, u_ε is continuous on $\mathbb{R}^n$. In particular, there exist $0 \le a < b \le +\infty$ such that

$$\{u_\varepsilon > 0\} = B_b\,, \qquad \{u_\varepsilon < 1\} = \mathbb{R}^n \setminus \overline{B_a} = \Big\{x : |x| > a\Big\}\,.$$

A standard first variation argument shows the existence of $\lambda \in \mathbb{R}$ such that

$$-2\,\varepsilon^2\,\Delta u_\varepsilon = \varepsilon\,\lambda\,V'(u_\varepsilon) - W'(u_\varepsilon) \qquad \text{in } \mathcal{D}'(\Omega),\ \Omega = B_b \setminus \overline{B_a}\,. \tag{2.26}$$

Since (2.26) implies that Δu_ε is bounded in Ω, by the Calderon–Zygmund theorem we find that $u_\varepsilon \in {\rm Lip}_{\rm loc}(\Omega)$. As a consequence, (2.26) gives that $-2\,\varepsilon^2\,\Delta u_\varepsilon = f(u_\varepsilon)$ for some $f \in C^1(0,1)$, and thus, by Schauder's theory, $u_\varepsilon \in C^{2,\alpha}_{\rm loc}(\Omega)$ for every $\alpha \in (0,1)$. We complete this step by showing that $\Omega = \mathbb{R}^n$.

Proof that $\Omega = \mathbb{R}^n$: Considering functions of the form $u + t\,\varphi$ with $t \ge 0$ and either $\varphi \in C^\infty_c(\mathbb{R}^n \setminus \overline{B_a})$, $\varphi \ge 0$, or $\varphi \in C^\infty_c(B_b)$, $\varphi \le 0$, and then adjusting the volume constraint by a suitable variation localized in $B_b \setminus \overline{B_a}$, we also obtain the validity, in distributional sense, of the inequalities

$$-2\,\varepsilon^2\,\Delta u_\varepsilon \ge \varepsilon\,\lambda\,V'(u_\varepsilon) - W'(u_\varepsilon) \qquad \text{in } \mathcal{D}'(\mathbb{R}^n \setminus \overline{B_a})\,. \tag{2.27}$$

$$-2\,\varepsilon^2\,\Delta u_\varepsilon \le \varepsilon\,\lambda\,V'(u_\varepsilon) - W'(u_\varepsilon) \qquad \text{in } \mathcal{D}'(B_b)\,. \tag{2.28}$$

We now stress that, in the rest of the argument, the only property of

$$f(t) = \varepsilon\,\lambda\,V'(t) - W'(t)\,, \qquad t \in [0,1]\,,$$

that will be used is the validity of the bound

$$|f(t)| \le C\,(1 + |\lambda|)\,t\,(1-t)\,, \qquad \forall t \in [0,1]\,. \tag{2.29}$$

This remark will be useful to avoid repetitions when we come to step two of the proof of Theorem 2.2. Notice that (2.29) indeed holds true thanks to (A.6) and (A.11), and that in (2.29) we cannot absorb $|\lambda|$ into C since we do not know yet that $|\lambda|$ admits a universal bound (this will actually be proved in step five below).

By (2.29), (2.27) implies

$$-2\,\varepsilon^2\,\Big\{u_\varepsilon'' + (n-1)\,\frac{u_\varepsilon'}{r}\Big\} \ge -C\,(1 + |\lambda|)\,u_\varepsilon \qquad \text{in } \mathcal{D}'(a,\infty)\,. \tag{2.30}$$

Assuming by contradiction that $b < \infty$, let $r \in (a,b)$, s be such that $(r-s, r+s) \subset (a,b)$, and ζ_s be the Lipschitz function with $\zeta_s = 0$ on $(0, r-s)$, $\zeta_s = 1$ on $(r+s,\infty)$, and $\zeta_s' = 1/2s$ on $(r-s, r+s)$. Testing (2.30) with $-u_\varepsilon'\,\zeta_s \ge 0$ (which is compactly supported in (a,∞)) we find that

$$\varepsilon^2\,\int_a^\infty [(u_\varepsilon')^2]'\,\zeta_s + 2\,(n-1)\frac{(u_\varepsilon')^2}{t}\,\zeta_s \ge C\,(1+|\lambda|)\,\int_a^\infty u_\varepsilon\,u_\varepsilon'\,\zeta_s$$

so that, after integration by parts, we obtain

$$2\,(n-1)\,\varepsilon^2\,\int_a^\infty \frac{(u_\varepsilon')^2}{t}\,\zeta_s + \frac{C\,(1+|\lambda|)}{2\,s}\int_{r-s}^{r+s} \frac{u_\varepsilon^2}{2} \ge \frac{\varepsilon^2}{2\,s}\int_{r-s}^{r+s} (u_\varepsilon')^2\,.$$

Letting $s \to 0^+$ we obtain

$$2\,(n-1)\,\varepsilon^2 \int_r^b \frac{(u_\varepsilon')^2}{t} + C\,(1+|\lambda|)\frac{u_\varepsilon(r)^2}{2} \ge \varepsilon^2\,u_\varepsilon'(r)^2\,.$$

Finally letting $r \to b^-$ we conclude that $u_\varepsilon'(b^-)=0$. This fact, combined with $u_\varepsilon(b)=0$ and the uniqueness theorem for the second order ODE (2.26), implies that $u_\varepsilon = 0$ on (a,b), which is in contradiction with the continuity of u_ε if $a>0$, and with $\int_{\mathbb{R}^n} V(u_\varepsilon)=1$ if $a=0$. This proves that $b=+\infty$ (and thus that $u_\varepsilon>0$ on $\mathbb{R}^n$).

The proof of $a=0$ (that is, of $u_\varepsilon<1$ on $\mathbb{R}^n$) is analogous. After the change of variables $v=1-u_\varepsilon$, we have $v\ge 0$, $v'\ge 0$, $v=0$ on $(0,a)$, and, thanks to (2.28),

$$-2\,\varepsilon^2\left\{v''+(n-1)\,\frac{v'}{r}\right\} \ge -C\,(1+|\lambda|)\,v \qquad \text{in } \mathcal{D}'(0,\infty)\,. \tag{2.31}$$

Notice that (2.31) is identical to (2.30), and that an even reflection by $r=a$ maps the boundary conditions of v into those of u_ε: the same argument used for proving $u_\varepsilon'(b^-)=0$ will thus show that $v'(a^+)=0$. For the sake of clarity we give some details. We pick $r>a$, introduce a Lipschitz function $\bar\zeta_s$ with $\bar\zeta_s=1$ on $(0,r-s)$, $\bar\zeta_s=0$ on $(r+s,\infty)$, and $\bar\zeta_s'=-1/2s$ on $(r-s,r+s)$, and test (2.31) with $v'\,\bar\zeta_s\ge 0$, to get

$$-\varepsilon^2\int_0^\infty [(v')^2]'\,\bar\zeta_s + 2\,(n-1)\frac{(v')^2}{t}\,\bar\zeta_s \ge -C\,(1+|\lambda|)\int_0^\infty v\,v'\,\bar\zeta_s\,.$$

Integration by parts now gives

$$-\frac{\varepsilon^2}{2s}\int_{r-s}^{r+s}(v')^2 - 2\,(n-1)\,\varepsilon^2\int_a^{r+s}\frac{(v')^2}{t}\,\bar\zeta_s \ge -\frac{C\,(1+|\lambda|)}{2s}\int_{r-s}^{r+s}\frac{v^2}{2}\,,$$

so that in the limit $s\to 0^+$, and then $r\to a^+$, we find $v'(a^+)=0$, that is to say, $u_\varepsilon'(a^+)=0$. If $a>0$ and thus $u_\varepsilon(a)=1$, this, combined with (2.26), implies $u_\varepsilon=1$ on $\mathbb{R}^n$, a contradiction.

Step five: Given a radial decreasing minimizer u_ε of $\psi(\varepsilon)$, we prove that the corresponding $\lambda\in\mathbb{R}$ such that (2.25) holds satisfies

$$n\,\lambda = (n-1)\,\mathcal{AC}_\varepsilon(u_\varepsilon) + \frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u_\varepsilon) - \varepsilon\int_{\mathbb{R}^n}|\nabla u_\varepsilon|^2\,, \tag{2.32}$$

as well as

$$\left|\lambda - 2\,(n-1)\,\omega_n^{1/n}\right| \le C\,\sqrt{\varepsilon}\,. \tag{2.33}$$

In particular, up to decrease the value of ε_0, we always have $1/C\le\lambda\le C$ for a universal constant C. To prove (2.32), following [LM89b], we test the distributional form of (2.25) with $\varphi = X\cdot\nabla u_\varepsilon$, for some $X\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, and get

$$\begin{aligned} 2\,\varepsilon\int_{\mathbb{R}^n}\nabla u_\varepsilon\cdot\nabla X[\nabla u_\varepsilon] &= -\int_{\mathbb{R}^n}\left\{2\,\varepsilon\,\nabla^2 u_\varepsilon[\nabla u_\varepsilon] + \Big(\frac{W'(u_\varepsilon)}{\varepsilon} - \lambda\,V'(u_\varepsilon)\Big)\nabla u_\varepsilon\right\}\cdot X \\ &= \int_{\mathbb{R}^n}\left\{\varepsilon\,|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} - \lambda\,V(u_\varepsilon)\right\}\mathrm{div}\,X\,. \end{aligned} \tag{2.34}$$

We now pick $\eta\in C_c^\infty(B_2)$ with $0\le\eta\le 1$ on B_2 and $\eta=1$ in B_1. We set $\eta_R(x)=\eta(x/R)$ and test (2.34) with $X(x)=\eta_R(x)\,x$. We notice that $\mathrm{div}\,X = n\,\eta_R + (x/R)\cdot(\nabla\eta)_R$, and that, by dominated convergence,

$$\lim_{R\to\infty}\int_{\mathbb{R}^n}\Big\{\varepsilon\,|\nabla u_\varepsilon|^2+\frac{W(u_\varepsilon)}{\varepsilon}-\lambda\,V(u_\varepsilon)\Big\}\,n\,\eta_R=n\,\big(\mathcal{AC}_\varepsilon(u_\varepsilon)-\lambda\big)\,,$$
$$\lim_{R\to\infty}\int_{\mathbb{R}^n}\Big\{\varepsilon\,|\nabla u_\varepsilon|^2+\frac{W(u_\varepsilon)}{\varepsilon}-\lambda\,V(u_\varepsilon)\Big\}\,\frac{x}{R}\cdot(\nabla\eta)_R=0\,,$$
$$\lim_{R\to\infty}\int_{\mathbb{R}^n}\nabla u_\varepsilon\cdot\Big(\eta_R\,\mathrm{Id}+\frac{x}{R}\otimes(\nabla\eta)_R\Big)[\nabla u_\varepsilon]=\int_{\mathbb{R}^n}|\nabla u_\varepsilon|^2\,.$$

In particular, (2.34) implies

$$n\,\lambda=n\,\mathcal{AC}_\varepsilon(u_\varepsilon)-2\,\varepsilon\int_{\mathbb{R}^n}|\nabla u_\varepsilon|^2\,,$$

which can be easily rearranged into (2.32). At the same time, by (1.26) we find

$$\begin{aligned}
&\int_{\mathbb{R}^n}\Big|\varepsilon\,|\nabla u_\varepsilon|^2-\frac{W(u_\varepsilon)}{\varepsilon}\Big|\\
\le\;&\Big(\int_{\mathbb{R}^n}\Big|\sqrt{\varepsilon}\,|\nabla u_\varepsilon|-\sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\Big|^2\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\Big|\sqrt{\varepsilon}\,|\nabla u_\varepsilon|+\sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\Big|^2\Big)^{1/2}\\
=\;&\Big(\mathcal{AC}_\varepsilon(u_\varepsilon)-\int_{\mathbb{R}^n}|\nabla\Phi(u_\varepsilon)|\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\Big|\sqrt{\varepsilon}\,|\nabla u_\varepsilon|+\sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\Big|^2\Big)^{1/2}\\
\le\;&C\,\sqrt{\varepsilon}\,\sqrt{\mathcal{AC}_\varepsilon(u_\varepsilon)}\le C\,\sqrt{\varepsilon}\,,
\end{aligned}$$

which can be combined with (2.32) and with (1.26) to deduce (2.33).

Step six: We are left to prove that every minimizer of $\psi(\varepsilon)$ is radial decreasing. Indeed, let u be a generic, possibly non-radial, minimizer of $\psi(\varepsilon)$, and let $v\in\mathcal{R}_0$ denote its radial decreasing rearrangement. By standard properties of rearrangements, $\int_{\mathbb{R}^n}V(u)=\int_{\mathbb{R}^n}V(v)=1$, while by the Pólya-Szegö inequality $\mathcal{AC}_\varepsilon(u)\ge\mathcal{AC}_\varepsilon(v)$, so that v is a minimizer of $\psi(\varepsilon)$ and equality holds in the Pólya-Szegö inequality for u, that is

$$\int_{\mathbb{R}^n}|\nabla u|^2=\int_{\mathbb{R}^n}|\nabla v|^2\,. \tag{2.35}$$

By step four and five, v solves the ODE

$$2\,\varepsilon^2\Big\{v''+(n-1)\,\frac{v'}{r}\Big\}=W'(v)-\lambda\,\varepsilon\,V'(v)\,,\qquad\text{on }(0,\infty)\,, \tag{2.36}$$

with $0<1/C\le\lambda\le C$. Multiplying in (2.36) by v' and integrating over $(0,r)$ for some $r>0$ we obtain

$$\varepsilon^2\,v'(r)^2+2\,(n-1)\int_0^r\frac{(v')^2}{t}=W(v(r))-\lambda\,\varepsilon\,V(v(r))+\lambda\,\varepsilon\,V(v(0))\,,\qquad\forall r>0\,, \tag{2.37}$$

where we have used $v'(0)=0$, $v(1)=1$, and $W(1)=0$. If r is such that $v(r)\le\delta_0$, then by (A.6), (A.11) and (2.37) we find

$$\varepsilon^2\,v'(r)^2\ge W(v)-C\,\varepsilon V(v)\ge\frac{v(r)^2}{C}-C\,\varepsilon\,\frac{v(r)^{2\,n/(n-1)}}{C}\ge\frac{v(r)^2}{C}\,,$$

which gives, in particular, $v'(r)<0$; if r is such that $v(r)\in(\delta_0,1-\delta_0)$, then, by the same method and thanks to $\inf_{(\delta_0,1-\delta_0)}W\ge 1/C$, we find that

$$\varepsilon^2\,v'(r)^2\ge W(v)-C\,\varepsilon V(v)\ge\frac{1}{C}-C\,\varepsilon\ge\frac{1}{C}\,,$$

so that, once again, $v'(r) < 0$; finally, if the interval $\{v \geq 1 - \delta_0\}$ is non-empty, then it has the form $(0, a]$ for some $a > 0$; multiplying (2.36) by r^{n-1}, integrating over $(0, r)$, and taking into account that $W' < 0$ on $(1 - \delta_0, 1)$, $V' > 0$ on $(0, 1)$ and $\lambda > 0$, we find

$$2\,\varepsilon^2\, r^{n-1}\, v'(r) = \int_0^r [W'(v) - \lambda\, \varepsilon\, V'(v)]\, r^{n-1}\, dr < 0\,,$$

that is, once again $v'(r) < 0$. We have thus proved that $v' < 0$ on $(0, \infty)$. This information, combined with (2.35), allows us to exploit the Brothers–Ziemer theorem [BZ88] to conclude that u is a translation of v. This shows that every minimizer of $\psi(\varepsilon)$ is in $\mathcal{R}_0^*$, and concludes the proof of the theorem. □

The compactness argument used in the proof of Theorem 2.1 is relevant also in the implementation of the selection principle used in the proof of the stability estimate (1.21) in the radial decreasing case. Specifically, an adaptation of that argument is needed in showing the existence of minimizers in the variational problems used in the selection principle strategy. In the interest of clarity, it thus seems convenient to discuss this adaptation in this same section. We thus turn to the proof of Theorem 2.2 below. In the statement of this theorem we use for the first time the quantity

$$d_\Phi(u, v) = \int_{\mathbb{R}^n} |\Phi(u) - \Phi(v)|^{n/(n-1)}\,, \tag{2.38}$$

which is finite whenever $u, v \in H^1(\mathbb{R}^n; [0, 1])$ (indeed, $u \in H^1(\mathbb{R}^n; [0, 1])$ and $W(t) \leq C\, t^2$ for $t \in [0, 1]$ imply $\mathcal{AC}_\varepsilon(u) < \infty$, thus $|D(\Phi(u))|(\mathbb{R}^n) < \infty$, and hence $\Phi(u) \in L^{n/(n-1)}(\mathbb{R}^n)$ by the BV-Sobolev inequality).

Theorem 2.2. *If $n \geq 2$ and $W \in C^{2,1}[0, 1]$ satisfies* (1.11) *and* (1.12)*, then there exist universal constant ε_0, a_0, ℓ_0 and C with the following properties.*

(i): *If $a \in (0, a_0)$, $\varepsilon < \varepsilon_0$, u_ε is a minimizer of $\psi(\varepsilon)$, and $v_\varepsilon \in H^1(\mathbb{R}^n; [0, 1])$ is such that*

$$\int_{\mathbb{R}^n} V(v_\varepsilon) = 1\,, \qquad \mathcal{AC}_\varepsilon(v_\varepsilon) \leq \psi(\varepsilon) + a\, \ell_0\,, \qquad d_\Phi(v_\varepsilon, u_\varepsilon) \leq \ell_0\,, \tag{2.39}$$

then the variational problem

$$\gamma(\varepsilon, a, v_\varepsilon) = \inf\Big\{\mathcal{AC}_\varepsilon(w) + a\, d_\Phi(w, v_\varepsilon) : w \in H^1(\mathbb{R}^n; [0, 1])\,, \int_{\mathbb{R}^n} V(w) = 1\Big\}\,,$$

admits minimizers.

(ii): *If, in addition, $v_\varepsilon \in \mathcal{R}_0$, then $\gamma(\varepsilon, a, v_\varepsilon)$ admits a minimizer $w_\varepsilon \in \mathcal{R}_0$. Every such minimizer satisfies $w_\varepsilon \in \mathcal{R}_0^* \cap C^{2,1/(n-1)}_{\rm loc}(\mathbb{R}^n)$, $0 < w_\varepsilon < 1$ on $\mathbb{R}^n$, and solves*

$$-2\,\varepsilon^2\, \Delta w_\varepsilon = \varepsilon\, w_\varepsilon\, (1 - w_\varepsilon)\, \mathrm{E}_\varepsilon - W'(w_\varepsilon) \qquad \text{on } \mathbb{R}^n\,, \tag{2.40}$$

where E_ε is a continuous radial function on $\mathbb{R}^n$ with

$$\sup_{\mathbb{R}^n} |\mathrm{E}_\varepsilon| \leq C\,. \tag{2.41}$$

Proof. Step one: Set $\gamma = \gamma(\varepsilon, a, v_\varepsilon)$ for the sake of brevity, and let $\{u_j\}_j$ be a minimizing sequence for γ. Since $a > 0$, we can assume that

$$\mathcal{AC}_\varepsilon(u_j) + a\, d_\Phi(u_j, v_\varepsilon) \leq \gamma + a\, \ell_0\,, \qquad \forall j\,. \tag{2.42}$$

In particular, comparing u_j by means of (2.42) with v_ε and u_ε respectively, we obtain the two basic bounds

$$\mathcal{AC}_\varepsilon(u_j) + a\, d_\Phi(u_j, v_\varepsilon) \leq \mathcal{AC}_\varepsilon(v_\varepsilon) + a\, \ell_0 \leq \psi(\varepsilon) + 2\, \ell_0\,, \tag{2.43}$$

$$\mathcal{AC}_\varepsilon(u_j) + a\, d_\Phi(u_j, v_\varepsilon) \leq \psi(\varepsilon) + a\, d_\Phi(u_\varepsilon, v_\varepsilon) + a\, \ell_0\,. \tag{2.44}$$

Subtracting $\psi(\varepsilon)$ from (2.44), noticing that $\mathcal{AC}_\varepsilon(u_j) \ge \psi(\varepsilon)$, and using (2.39), we also find

$$d_\Phi(u_j, v_\varepsilon) \le d_\Phi(u_\varepsilon, v_\varepsilon) + \ell_0 \le 2\,\ell_0\,, \tag{2.45}$$

and hence, using again (2.39),

$$d_\Phi(u_j, u_\varepsilon) \le C\,\ell_0\,. \tag{2.46}$$

Finally, by (2.39), (2.43), and $\psi(\varepsilon) \le 2\,n\,\omega_n^{1/n} + C\,\varepsilon$, we can apply step one of the proof of Theorem 2.1 to u_j, u_ε and v_ε, to find

$$\min\Big\{\int_{B_{M_0}} V(u_j)\,, \int_{B_{M_0}} V(u_\varepsilon)\,, \int_{B_{M_0}} V(v_\varepsilon)\Big\} \ge 1 - C\,\sqrt{\ell_0 + \varepsilon_0}\,, \qquad \forall j\,, \tag{2.47}$$

where M_0 is a universal constant. Since (2.47) rules out the possibility of the vanishing case for $\{V(u_j)\,dx\}_j$, we can directly assume that the dichotomy case occurs, and in particular that there exists

$$\alpha \in \Big(1 - C\,\sqrt{\ell_0 + \varepsilon_0}, 1\Big)\,, \tag{2.48}$$

such that for every $\tau \in (0, \min\{\alpha/2, \tau_*\})$ (here τ_* is as in (2.16)) we can find $S(\tau) > 0$, $S_j(\tau) \to \infty$ and a cut-off function φ between $B_{S(\tau)}$ and $B_{S_j(\tau)}$ such that $|\nabla\varphi| \le 2\,S_j(\tau)^{-1}$ on $\mathbb{R}^n$, and

$$\begin{aligned}
\alpha - C\,\tau &\le \int_{B_{S(\tau)}} V(u_j), \int_{\mathbb{R}^n} V(\varphi\,u_j) \le \alpha + C\,\tau\,,\\
(1-\alpha) - C\,\tau &\le \int_{B^c_{S_j(\tau)}} V(u_j), \int_{\mathbb{R}^n} V((1-\varphi)\,u_j) \le (1-\alpha) + C\,\tau\,.
\end{aligned} \tag{2.49}$$

We can now *verbatim* repeat the argument used in step two of the proof of Theorem 2.1 to deduce (2.19), and find that, if $\sigma = \tau^{(n-1)/(3\,n-1)}$ as in (2.16), then

$$\mathcal{AC}_\varepsilon(u_j) \ge \mathcal{AC}_\varepsilon(\varphi\,u_j) + \mathcal{AC}_\varepsilon((1-\varphi)\,u_j) - C\,\Big(\frac{\sigma}{\varepsilon} + \frac{1}{S_j(\tau)}\Big)\,; \tag{2.50}$$

in the same vein, by exactly the same argument used to deduce (2.23), we also have

$$\mathcal{AC}_\varepsilon((1-\varphi)\,u_j) \ge c(n)\,((1-\alpha) - C\,\tau)^{(n-1)/n}\,. \tag{2.51}$$

We now need to show that the $\mathcal{AC}_\varepsilon(\varphi\,u_j)$ term is larger than γ up to $\mathrm{O}(1-\alpha)$ and $\mathrm{O}(\tau)$ errors, but, for reasons that will become clearer in a moment, we cannot do this by just taking a rescaling of $\varphi\,u_j$ as done in Theorem 2.1. We will rather need to introduce the "localized" family of rescalings which we now pass to describe.

We let $\zeta \in C^\infty_c(B_{2\,M_0}; [0,1]) \cap \mathcal{R}_0$ with $\zeta = 1$ on B_{M_0} and $|\zeta'| \le 2/M_0$. In particular,

$$|x|\,|\zeta'| \le 2 \qquad \text{on } \mathbb{R}^n\,. \tag{2.52}$$

Next, we set $f_t(x) = x + t\,\zeta(|x|)\,x$ and $\hat{x} = x/|x|$ for $x \in \mathbb{R}^n$ and $t > 0$. By (2.52), if $|t| \le t_0 = t_0(n) < 1$, then $f_t : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism with

$$\begin{aligned}
f_t(x) &= x\,, && \text{on } B^c_{2\,M_0}\,,\\
f_t(x) &= (1+t)\,x && \text{on } B_{M_0}\,,\\
\nabla f_t(x) &= (1 + t\,\zeta)\,\mathrm{Id} + t\,|x|\,\zeta'\,\hat{x}\otimes\hat{x}\,,\\
Jf_t(x) &= (1+t\,\zeta)^{n-1}\big(1 + t\,(\zeta + |x|\,\zeta')\big) = 1 + \big(n\,\zeta + |x|\,\zeta'\big)\,t + \mathrm{O}(t^2)\,.
\end{aligned}$$

We set $v_j(t) = (\varphi\,u_j)\circ f_t$, so that $v_j(0) = \varphi\,u_j$, and consider the functions

$$b_j(t) = \int_{\mathbb{R}^n} V\big(v_j(t)\big) = \int_{\mathbb{R}^n} V(\varphi\,u_j)\,Jf_t\,, \qquad |t| \le t_0\,.$$

Clearly we have

$$b_j(0) = \int_{\mathbb{R}^n} V(\varphi\, u_j) \in [\alpha - C\,\tau, \alpha + C\,\tau]\,, \tag{2.53}$$

$$|b_j''(t)| = \int_{\mathbb{R}^n} V(\varphi\, u_j)\Big|\frac{d^2(Jf_t)}{dt^2}\Big| \le C\,, \qquad \forall |t| \le t_0\,; \tag{2.54}$$

more crucially, if we choose ε_0 and ℓ_0 small enough, then by (2.47) and (2.52) we find

$$b_j'(0) = \int_{\mathbb{R}^n} V(\varphi\, u_j)\,\big(n\,\zeta + |x|\,\zeta'\big) \ge n\int_{B_{M_0}} V(u_j) - (n+2)\int_{B_{2\,M_0}\setminus B_{M_0}} V(u_j) \ge \frac{n}{2}\,.$$

As a consequence, by (2.54), we can find a universal constant t_1 such that

$$b_j'(t) \ge \frac{n}{3} \qquad \forall |t| \le t_1\,. \tag{2.55}$$

In particular, b_j is strictly increasing on $[-t_1, t_1]$, with

$$b_j(t_1) \ge b_j(0) + \frac{n}{3}\,t_1 \ge \alpha - C\,\tau + \frac{n}{3}\,t_1 > 1 - C\,(\ell_0 + \varepsilon_0 + \tau) + \frac{n}{3}\,t_1 > 1\,,$$
$$b_j(-t_1) \le b_j(0) - \frac{n}{3}\,t_1 \le \alpha + C\,\tau - \frac{n}{3}\,t_1 \le 1 + C\,(\ell_0 + \varepsilon_0 + \tau) - \frac{n}{3}\,t_1 < 1 - \frac{n}{4}\,t_1\,,$$

so that, for every j, there exists $t_j \in (-t_1, t_1)$ such that $b_j(t_j) = 1$: in other words,

$$\int_{\mathbb{R}^n} V(v_j(t_j)) = 1\,. \tag{2.56}$$

We now compare the energy of $v_j(t_j) = (\varphi\, u_j)\circ f_{t_j}$ to that of $\varphi\, u_j$. To this end, we first notice that, by comparing $b_j(0) = \int_{\mathbb{R}^n} V(\varphi\, u_j) = \alpha + \mathrm{O}(\tau)$ to $b_j(1) = 1$, thanks to (2.55) we conclude that

$$|t_j| \le C\,\big((1-\alpha) + \tau\big)\,, \qquad \forall j\,. \tag{2.57}$$

Denoting by $\|A\|$ the operator norm of a linear map A, we have

$$\|\nabla f_t(x) - \mathrm{Id}\,\| \le C\,|t|\,, \qquad |Jf_t(x) - 1| \le C\,|t|\,, \qquad \forall x \in \mathbb{R}^n\,,$$

so that

$$\begin{aligned}\mathcal{AC}_\varepsilon(v_j(t)) &= \int_{\mathbb{R}^n}\Big\{\varepsilon\,\Big|\Big(\nabla f_t \circ f_t^{-1}\Big)[\nabla(\varphi\, u_j)]\Big|^2 + \frac{W(\varphi\, u_j)}{\varepsilon}\Big\}\,Jf_t \\ &\le \int_{\mathbb{R}^n}\Big\{\varepsilon\,(1 + C\,|t|)^2|\nabla(\varphi\, u_j)|^2 + \frac{W(\varphi\, u_j)}{\varepsilon}\Big\}\,(1 + C\,|t|) \\ &\le (1 + C\,|t|)\,\mathcal{AC}_\varepsilon(\varphi\, u_j)\,.\end{aligned}$$

Therefore if we combine (2.50), (2.51), and (2.57) with this last estimate, and take into account that $\mathcal{AC}_\varepsilon(u_j), \mathcal{AC}_\varepsilon(\varphi\, u_j) \le C$, then we obtain

$$\begin{aligned}\mathcal{AC}_\varepsilon(u_j) + a\,d_\Phi(u_j, v_\varepsilon) \ \ge\ & \mathcal{AC}_\varepsilon(v_j(t_j)) + a\,d_\Phi(v_j(t_j), v_\varepsilon) \\ & + a\,\Big(d_\Phi(u_j, v_\varepsilon) - d_\Phi(v_j(t_j), v_\varepsilon)\Big) \\ & + c(n)\,((1-\alpha) - C\,\tau)^{(n-1)/n} - C\,\Big((1-\alpha) + \tau + \frac{1}{S_j(\tau)} + \frac{\sigma}{\varepsilon}\Big)\,.\end{aligned} \tag{2.58}$$

We notice that for every $u, v \in H^1(\mathbb{R}^n; [0,1])$, thanks to the triangular inequality in $L^{n/(n-1)}$ and to $|b^{1/n'} - a^{1/n'}| \ge c(n)\,b^{-1/n}(b-a)$ for $0 < a < b$, we have

$$c(n)\,\frac{|d_\Phi(u, v_\varepsilon) - d_\Phi(v, v_\varepsilon)|}{\max\{d_\Phi(u, v_\varepsilon), d_\Phi(v, v_\varepsilon)\}^{1/n}} \le d_\Phi(u, v)^{(n-1)/n}\,. \tag{2.59}$$

We apply (2.59) with $u = u_j$ and $v = u_j\,\varphi$ to find

$$\begin{aligned}\big|d_\Phi(u_j,v_\varepsilon)-d_\Phi(\varphi\,u_j,v_\varepsilon)\big| &\le C\int_{\mathbb{R}^n}|\Phi(u_j)-\Phi(\varphi\,u_j)|^{n/(n-1)}\\ &\le \int_{\mathbb{R}^n\setminus B_{S(\tau)}} V(u_j)\le C\big((1-\alpha)+\tau\big)\end{aligned}$$

where we have used (2.49). Similarly, noticing that

$$\begin{aligned}\frac{d}{ds}\,\Phi(v_j(s)) &= \sqrt{W(v_j(s))}\,\big[\nabla(\varphi\,u_j)\circ f_s\big]\cdot\frac{d}{ds}\,f_s\\ &= \sqrt{W(v_j(s))}\,\big[\nabla(\varphi\,u_j)\circ f_s\big]\cdot\big(\zeta(|x|)\,x\big)\,,\end{aligned}$$

with $\zeta(x)\,|x|\le 2\,M_0$ for every $x\in\mathbb{R}^n$ by (2.52), we find[4]

$$\begin{aligned}&\big|d_\Phi(v_j(t_j),v_\varepsilon)-d_\Phi(\varphi\,u_j,v_\varepsilon)\big|\\ \le\;& C\int_{\mathbb{R}^n}|\Phi(v_j(t_j))-\Phi(\varphi\,u_j)|^{n/(n-1)}\le C\int_{\mathbb{R}^n}|\Phi(v_j(t_j))-\Phi(\varphi\,u_j)|\\ \le\;& C\Big|\int_0^{t_j}ds\int_{\mathbb{R}^n}\sqrt{W(v_j(s))}\,\big[\nabla(\varphi\,u_j)\circ f_s\big]\cdot\big(\zeta(|x|)\,x\big)\Big|\\ \le\;& C\Big|\int_0^{t_j}ds\int_{\mathbb{R}^n}\sqrt{W(\varphi\,u_j)}\,\nabla(\varphi\,u_j)\cdot\big(\zeta(|f_s^{-1}|)\,f_s^{-1}\big)\,Jf_s\Big|\\ \le\;& C\,M_0\,|t_j|\int_{\mathbb{R}^n}\sqrt{W(\varphi\,u_j)}\,|\nabla(\varphi\,u_j)|\le C\,|t_j|\,\mathcal{AC}_\varepsilon(\varphi\,u_j)\,.\end{aligned}\tag{2.60}$$

We finally combine (2.57), (2.58), (2.60),and the fact that $v_j(t_j)$ is a competitor for γ to conclude that

$$\mathcal{AC}_\varepsilon(u_j)+a\,d_\Phi(u_j,v_\varepsilon)\ge\gamma+c(n)\,((1-\alpha)-C\,\tau)^{(n-1)/n}-C\Big((1-\alpha)+\tau+\frac{\sigma}{\varepsilon}+\frac{1}{S_j(\tau)}\Big)\,.$$

Letting $j\to\infty$ and then $\tau\to 0^+$ (so that $\sigma\to 0^+$ thanks to (2.16)), we finally conclude

$$0\ge c(n)\,(1-\alpha)^{(n-1)/n}-C\,(1-\alpha)\,,$$

which gives a contradiction with (2.48) if ε_0 and ℓ_0 are small enough. Having excluded vanishing and dichotomy, by a standard argument we deduce the existence of a minimizer of γ.

Step two: We now assume that $v_\varepsilon\in\mathcal{R}_0$. Since Φ is an increasing function on $[0,1]$, if u^* denotes the radial decreasing rearrangement of $u:\mathbb{R}^n\to[0,\infty)$, then $\Phi(u^*)=\Phi(u)^*$. In particular, by a standard property of rearrangements,

$$d_\Phi(u,v)=\int_{\mathbb{R}^n}|\Phi(u)-\Phi(v)|^{n/(n-1)}\ge\int_{\mathbb{R}^n}|\Phi(u)^*-\Phi(v)^*|^{n/(n-1)}=d_\Phi(u^*,v^*)\,.$$

This fact, combined with the Pólya–Szegö inequality and the fact that $v_\varepsilon^*=v_\varepsilon$, implies that the radial decreasing rearrangement of a minimizer of γ is also a minimizer of γ (in brief, a radial decreasing minimizer).

We now show that every radial decreasing minimizer w_ε of γ satisfies $0<w_\varepsilon<1$ on $\mathbb{R}^n$, that $w_\varepsilon\in C^{2,1/(n-1)}_{\rm loc}(\mathbb{R}^n)$, and that (2.40) holds for a radial continuous function E_ε

[4] This is the key step where using $f_t(x)$ rather than $(1+t)\,x$ (as done when proving Theorem 2.1) makes a substantial difference. Indeed, by using a global rescaling to fix the volume constraint of $\varphi\,u_j$, we end up having to control, in the analogous estimate to (2.60), the first moment of the energy density of $\varphi\,u_j$, i.e.

$$\int_{\mathbb{R}^n}|x|\,\Big(\varepsilon\,|\nabla(\varphi\,u_j)|^2+\frac{W(\varphi\,u_j)}{\varepsilon}\Big)\,,$$

rather than the trivially bounded quantity $M_0\,\mathcal{AC}_\varepsilon(u_j)$.

bounded by a universal constant. Arguing as in step four of the proof of Theorem 2.1, with $0 \le a < b \le +\infty$ and $\Omega = B_b \setminus \overline{B}_a = \{0 < w_\varepsilon < 1\}$, we see that w_ε solves

$$-2\,\varepsilon^2\,\Delta w_\varepsilon = \varepsilon\,\lambda V'(w_\varepsilon) - W'(w_\varepsilon) - a\,\varepsilon\,Z_\varepsilon(x,w_\varepsilon) \qquad \text{in } \mathcal{D}'(\Omega)\,, \tag{2.61}$$

where, for $x \in \mathbb{R}^n$ and $t \in [0,1]$, we have set

$$Z_\varepsilon(x,t) = \frac{n}{n-1}\,\big|\Phi(t)-\Phi(v_\varepsilon)\big|^{(n/(n-1))-2}\,\big(\Phi(t)-\Phi(v_\varepsilon)\big)\,\sqrt{W(t)}\,.$$

By (2.61), Δw_ε is bounded in Ω, and thus, by the Calderon–Zygmund theorem, $w_\varepsilon \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. This implies that $Z_\varepsilon(x,t) \in C^{0,1/(n-1)}_{\mathrm{loc}}(\Omega)$, and thus, by Schauder's theory, that $w_\varepsilon \in C^{2,1/(n-1)}_{\mathrm{loc}}(\Omega)$. We now want to prove that $\Omega = \mathbb{R}^n$. By the same variational arguments used in deriving (2.27) and (2.28), we have that

$$-2\,\varepsilon^2\,\Delta w_\varepsilon \ge f(x,t) \qquad \text{in } \mathcal{D}'(\mathbb{R}^n \setminus \overline{B_a})\,, \tag{2.62}$$

$$-2\,\varepsilon^2\,\Delta w_\varepsilon \le f(x,t) \qquad \text{in } \mathcal{D}'(B_b)\,, \tag{2.63}$$

where $f(x,t)$ satisfies

$$|f(x,t)| \le C\;t\,(1-t)\,, \qquad \forall (x,t) \in \mathbb{R}^n \times [0,1]\,, \tag{2.64}$$

thanks to (A.6) and (A.11) (which, in particular, give $|Z_\varepsilon(x,t)| \le C\,t\,(1-t)$ for every $(x,t) \in \mathbb{R}^n \times [0,1]$). By repeating the same argument used in step four of the proof of Theorem 2.1, we thus see that $\Omega = \mathbb{R}^n$. Finally, it is easily seen that (2.61) with $\Omega = \mathbb{R}^n$ and $w_\varepsilon \in C^2(\mathbb{R}^n)$, takes the form

$$-2\,\varepsilon^2\,\Delta w_\varepsilon = \varepsilon\,w_\varepsilon\,(1-w_\varepsilon)\,\mathrm{E}_\varepsilon - W'(w_\varepsilon) \qquad \text{on } \mathbb{R}^n\,, \tag{2.65}$$

for a radial function E_ε bounded by a universal constant on $\mathbb{R}^n$, as claimed. □

3. Resolution of almost-minimizing sequences

In the main result of this section, Theorem 3.1 below, we provide a sharp description, up to first order as $\varepsilon \to 0^+$, of the minimizers of $\psi(\varepsilon)$. This resolution result is proved not only for minimizers of $\psi(\varepsilon)$, but also for a general notion of "critical sequence for $\psi(\varepsilon_j)$ as $\varepsilon_j \to 0^+$" modeled after the selection principle minimizers of Theorem 2.2.

In the following statement, η is the solution of $\eta' = -\sqrt{W(\eta)}$ on $\mathbb{R}$ with $\eta(0) = 1/2$,

$$\tau_0 = \int_{\mathbb{R}} \eta'\,V'(\eta)\,s\,ds, \qquad \tau_1 = \int_{\mathbb{R}} W(\eta)\,s\,ds\,,$$

and $R_0 = \omega_n^{-1/n}$. Relevant properties of η are collected in section A.4.

Theorem 3.1. *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1.11) *and* (1.12)*, then there exist universal constants ε_0, δ_0, and ℓ_0 with the following properties:*

Ansatz: *for every $\varepsilon < \varepsilon_0$ there exists a unique $\tau_\varepsilon \in \mathbb{R}$ such that if we set*

$$z_\varepsilon(x) = \eta\Big(\frac{|x|-R_0}{\varepsilon} - \tau_\varepsilon\Big)\,, \tag{3.1}$$

then

$$\int_{\mathbb{R}^n} V(z_\varepsilon) = 1\,. \tag{3.2}$$

Moreover, we have $|\tau_\varepsilon - \tau_0| \le C\,\varepsilon$ and, in the limit as $\varepsilon \to 0^+$,

$$\mathcal{AC}_\varepsilon(z_\varepsilon) = 2\,n\,\omega_n^{1/n} + 2\,n\,(n-1)\,\omega_n^{2/n}\,\big(\tau_0+\tau_1\,\big)\,\varepsilon + \mathrm{O}(\varepsilon^2)\,. \tag{3.3}$$

Resolution of critical sequences: *if $\varepsilon_j \to 0^+$ as $j \to \infty$, $\{v_j\}_j$ is a sequence in $C^2(\mathbb{R}^n;[0,1]) \cap \mathcal{R}_0$ such that*

$$\int_{\mathbb{R}^n} V(v_j) = 1\,, \tag{3.4}$$

$$\mathcal{AC}_{\varepsilon_j}(v_j) \le 2\,n\,\omega_n^{1/n} + \ell_0\,, \tag{3.5}$$

and $\{\mathrm{E}_j\}_j$ is a sequence of radial continuous functions on $\mathbb{R}^n$ with

$$-2\,\varepsilon_j^2\,\Delta v_j = \varepsilon_j\,v_j\,(1-v_j)\,\mathrm{E}_j - W'(v_j) \qquad \text{on } \mathbb{R}^n\,, \tag{3.6}$$

$$\sup_j \|\mathrm{E}_j\|_{C^0(\mathbb{R}^n)} \le C\,, \tag{3.7}$$

then, for j large enough, we have

$$v_j(x) = z_{\varepsilon_j}(x) + f_j\Big(\frac{|x|-R_0}{\varepsilon_j}\Big)\,, \qquad x \in \mathbb{R}^n\,, \tag{3.8}$$

where $f_j \in C^2(-R_0/\varepsilon_j, \infty)$, and

$$|f_j(s)| \le C\,\varepsilon_j\,e^{-|s|/C}\,, \qquad \forall s \ge -R_0/\varepsilon_j\,. \tag{3.9}$$

Moreover, for j large enough, there exist positive constants b_j and c_j such that

$$v_j(R_0+c_j) = \delta_0\,, \qquad v_j(R_0-b_j) = 1-\delta_0\,, \tag{3.10}$$

and b_j and c_j satisfy

$$\frac{\varepsilon_j}{C} \le b_j\,, c_j \le C\,\varepsilon_j\,. \tag{3.11}$$

Finally, one has

$$\frac{C}{\varepsilon_j} \ge -v_j'(r) \ge \frac{1}{C\,\varepsilon_j}\,, \qquad \forall r \in [R_0-b_j, R_0+c_j]\,, \tag{3.12}$$

$$\begin{cases} v_j(r) \le C\,e^{-(r-R_0)/C\,\varepsilon_j}\,, \\ |v_j^{(k)}(r)| \le \dfrac{C}{\varepsilon_j^k}\,e^{-(r-R_0)/C\,\varepsilon_j}\,, \end{cases} \qquad \forall r \in [R_0+c_j,\infty)\,,\ k=1,2\,, \tag{3.13}$$

$$\begin{cases} 1-v_j(r) \le C\,e^{-(R_0-r)/C\,\varepsilon_j}\,, \\ |v_j'(r)| \le C\,\min\Big\{\dfrac{r}{\varepsilon_j^2}, \dfrac{1}{\varepsilon_j}\Big\}\,e^{-(R_0-r)/C\,\varepsilon_j}\,, \\ |v_j''(r)| \le \dfrac{C}{\varepsilon_j^2}\,e^{-(R_0-r)/C\,\varepsilon_j}\,, \end{cases} \qquad \forall r \in (0, R_0-b_j)\,. \tag{3.14}$$

Proof. The first two steps of the proof take care of the *Ansatz*-part of the statement, while starting from step three we address the resolution result. We premise the remark that, if we set $z_\tau(x) = \eta([(|x|-R_0)/\varepsilon] - \tau)$, then $f(\tau) = \int_{\mathbb{R}^n} V(z_\tau)$ is strictly increasing in τ with $f(-\infty) = 0$ and $f(+\infty) = +\infty$. For this reason, τ_ε is indeed uniquely defined by (3.2).

Step one: We prove that if $\{w_\varepsilon\}_{\varepsilon>0}$ is defined by

$$w_\varepsilon(x) = \eta\Big(\frac{|x|-R_0}{\varepsilon} - t_\varepsilon\Big) + f_\varepsilon\Big(\frac{|x|-R_0}{\varepsilon}\Big)\,, \qquad x \in \mathbb{R}^n\,, \varepsilon > 0\,,$$

for some $t_\varepsilon \in \mathbb{R}$ and some functions $f_\varepsilon \in C^2(-R_0/\varepsilon, \infty)$ such that

$$\int_{\mathbb{R}^n} V(w_\varepsilon) = 1\,, \tag{3.15}$$

$$|f_\varepsilon(s)| \le C\,\varepsilon\,e^{-|s|/C}\,, \qquad \forall s \ge -R_0/\varepsilon\,, \tag{3.16}$$

then

$$|t_\varepsilon - \tau_0| \le C\,\varepsilon\,, \qquad \forall \varepsilon < \varepsilon_0\,, \tag{3.17}$$

Of course, in the particular case when $f_\varepsilon \equiv 0$, we have $w_\varepsilon = z_\varepsilon$ and $t_\varepsilon = \tau_\varepsilon$ thanks to (3.1) and (3.2).

Indeed, setting $z_0(x) = \eta([(|x| - R_0)/\varepsilon] - \tau_0)$ for $x \in \mathbb{R}^n$, and recalling (3.2) and (3.15), we consider the quantity

$$\kappa_\varepsilon = \int_{\mathbb{R}^n} V(1_{B_{R_0}}) - V(z_0) = \int_{\mathbb{R}^n} V(w_\varepsilon) - V(z_0)\,. \tag{3.18}$$

We look at the first expression for κ_ε, passing first to the radial coordinate $r = |x|$ and then changing variables into $s = (r - R_0)/\varepsilon$. By taking into account the fact that τ_0 satisfies

$$\int_{\mathbb{R}} \Big(1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))\Big)\,ds = 0\,,$$

see (A.19), we find

$$\begin{aligned}
\frac{\kappa_\varepsilon}{n\,\omega_n} &= \varepsilon \int_{-R_0/\varepsilon}^{\infty} \Big(1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))\Big)\,(R_0+\varepsilon\,s)^{n-1}\,ds \\
&= \varepsilon\,R_0^{n-1} \int_{-R_0/\varepsilon}^{\infty} \Big(1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))\Big)\,ds \\
&\quad + \varepsilon \int_{-R_0/\varepsilon}^{\infty} \Big(1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))\Big)\,[(R_0+\varepsilon\,s)^{n-1} - R_0^{n-1}]\,ds \\
&= -\varepsilon\,R_0^{n-1} \int_{-\infty}^{-R_0/\varepsilon} \Big(1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))\Big)\,ds \\
&\quad + \varepsilon \sum_{k=0}^{n-2} a_k \int_{-R_0/\varepsilon}^{\infty} \Big(1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))\Big)\,R_0^k\,(s\,\varepsilon)^{n-1-k}\,ds,
\end{aligned}$$

with $a_k = \binom{n-1}{k}$. Since $\tau_0 = \tau_0(W)$, by the decay properties (A.16) of η, we have

$$|\,1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))| \le C\,e^{-|s|/C} \qquad \forall s \in \mathbb{R}\,, \tag{3.19}$$

so that

$$\Big|\int_{-\infty}^{-R_0/\varepsilon} \Big(1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))\Big)\,ds\Big| \le C \int_{-\infty}^{-R_0/\varepsilon} e^{-|s|/C}\,ds \le C\,e^{-R_0/C\,\varepsilon}\,,$$

and, recalling that $\omega_n\,R_0^n = 1$,

$$|\kappa_\varepsilon| \le C\,\varepsilon\,e^{-R_0/C\,\varepsilon} + C\,\varepsilon^2 \sum_{j=1}^{n-1} \int_{-R_0/\varepsilon}^{\infty} \Big|1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))\Big|\,|s|^j\,ds \le C\,\varepsilon^2\,,$$

where in the last inequality we have used (3.19) again. Taking into account the second formula for κ_ε in (3.18), we have thus proved

$$C\,\varepsilon^2 \ge \Big|\int_{\mathbb{R}^n} V(w_\varepsilon) - V(z_0)\Big|\,. \tag{3.20}$$

With the same change of variables used before we have

$$C\,\varepsilon \ge \Big|\int_{-R_0/\varepsilon}^{\infty} \Big\{V\big(\eta(s-t_\varepsilon) + f_\varepsilon(s)\big) - V(\eta(s-\tau_0))\Big\}\,(R_0+\varepsilon\,s)^{n-1}\,ds\Big|\,,$$

while the decay properties of f_ε assumed in (3.16) give

$$\begin{aligned}
&\Big|\int_{-R_0/\varepsilon}^{\infty} \Big\{V\big(\eta(s-t_\varepsilon) + f_\varepsilon(s)\big) - V(\eta(s-t_\varepsilon))\Big\}\,(R_0+\varepsilon\,s)^{n-1}\,ds\Big| \\
&\le \int_{-R_0/\varepsilon}^{\infty} f_\varepsilon(s)\,(R_0+\varepsilon\,s)^{n-1}\,ds \int_0^1 V'\big(\eta(s-t_\varepsilon) + r\,f_\varepsilon(s)\big)\,dr \,\le C\,\varepsilon\,;
\end{aligned}$$

by combining the last two inequalities we thus find

$$\begin{aligned} C\,\varepsilon &\ge \Big|\int_{-R_0/\varepsilon}^{\infty}\Big\{V(\eta(s-t_\varepsilon))-V(\eta(s-\tau_0))\Big\}\,(R_0+\varepsilon\,s)^{n-1}\,ds\Big| \\ &= \int_{-R_0/\varepsilon}^{\infty}|V(\eta(s-t_\varepsilon))-V(\eta(s-\tau_0))|\,(R_0+\varepsilon\,s)^{n-1}\,ds\,, \end{aligned} \tag{3.21}$$

where in the last step we have used that $\tau\to V(\eta(.-\tau))$ is strictly increasing in τ. Since (3.21) implies $t_\varepsilon\to\tau_0$ as $\varepsilon\to 0^+$, we can choose $\varepsilon_0=\varepsilon_0(n,W)$ so that $|t_\varepsilon-\tau_0|\le 1$ and $R_0+\varepsilon\,(\tau_0-1)\ge R_0/2$. Since $V\circ\eta$ is strictly decreasing on $\mathbb{R}$, we have $|(V\circ\eta)'|\ge 1/C$ on $[-2,2]$, and noticing that if $|s-\tau_0|\le 1$, then $|s-t_\varepsilon|<2$, we finally conclude

$$C\,\varepsilon\ge\int_{\tau_0-1}^{\tau_0+1}\frac{|(s-t_\varepsilon)-(s-\tau_0)|}{C}\,(R_0+\varepsilon\,s)^{n-1}\,ds\ge\frac{|\tau_0-t_\varepsilon|}{C}\,,$$

thus proving (3.17).

Step two: We compute $\mathcal{AC}_\varepsilon(z_\varepsilon)$. Passing to the radial coordinate $r=|x|$, setting first $r=R_0+\varepsilon\,s$ and then $t=s-\tau_\varepsilon$, recalling that $\eta'=-\sqrt{W(\eta)}$, and exploiting the decay property (A.16) of η at $-\infty$, we find that, as $\varepsilon\to 0^+$,

$$\begin{aligned} \mathcal{AC}_\varepsilon(z_\varepsilon) &= n\,\omega_n\int_{-R_0/\varepsilon}^{\infty}\Big(\eta'(s-\tau_\varepsilon)^2+W(\eta(s-\tau_\varepsilon))\Big)\,(R_0+\varepsilon\,s)^{n-1}\,ds \\ &= 2\,n\,\omega_n\int_{-\tau_\varepsilon-R_0/\varepsilon}^{\infty}W(\eta(t))\,\,(R_0+\varepsilon(t+\tau_\varepsilon))^{n-1}\,dt \\ &= 2\,n\,\omega_n\int_{-\infty}^{\infty}W(\eta(t))\,\,(R_0+\varepsilon(t+\tau_\varepsilon))^{n-1}\,dt+\mathrm{O}(e^{-C/\varepsilon}) \\ &= 2\,n\,\omega_n\int_{-\infty}^{\infty}W(\eta(t))\,\,(R_0+\varepsilon(t+\tau_0))^{n-1}\,dt+\mathrm{O}(\varepsilon^2)\,, \end{aligned} \tag{3.22}$$

where in the last step we have used $\tau_\varepsilon=\tau_0+\mathrm{O}(\varepsilon)$. Recalling that, by (1.12),

$$\int_{\mathbb{R}}W(\eta)=-\int_{\mathbb{R}}\sqrt{W(\eta)}\,\eta'=-\int_{\mathbb{R}}\Phi'(\eta)\,\eta'=\Phi(\eta(-\infty))-\Phi(\eta(+\infty))=\Phi(1)=1\,,$$

as well as that $\omega_n\,R_0^n=1$, we find

$$\mathcal{AC}_\varepsilon(z_\varepsilon)=2\,n\,\omega_n^{1/n}+2\,n\,(n-1)\,\omega_n^{2/n}\,(\tau_0+\tau_1)\,\varepsilon+\mathrm{O}(\varepsilon^2)\,,$$

as $\varepsilon\to 0^+$, that is (3.3). This proves the first part of the statement of the theorem.

Step three: In preparation to the proof of the second part of the statement, we show that if $\varepsilon<\varepsilon_0$ and $u\in H^1(\mathbb{R}^n;[0,1])$ satisfies

$$\mathcal{AC}_\varepsilon(u)\le 2\,n\,\omega_n^{1/n}+\ell_0\,,\qquad\int_{\mathbb{R}^n}V(u)=1\,, \tag{3.23}$$

then

$$\int_{\mathbb{R}^n}|\Phi(u)-1_{B_{R_0}}|^{n/(n-1)}\le C\,\big((\sqrt{\ell_0})^{(n-1)/n}+\varepsilon\big)\,. \tag{3.24}$$

Moreover, if $u\in\mathcal{R}_0$, then $\sqrt{\ell_0}$ can be replaced by ℓ_0 in (3.24).

Indeed, by (3.23), as seen in step one of the proof of Theorem 2.1, we have

$$\int_{\mathbb{R}^n}|\Phi(u)-(\omega_n^{1/n}\,r(u))^{1-n}\,1_{B_{r(u)}}|^{n/(n-1)}\le C\,\sqrt{\ell_0}\,, \tag{3.25}$$

for some $r(u)\in(0,M_0]$, where M_0 is a universal constant. Setting $f(r)=(\omega_n^{1/n}\,r)^{1-n}$, and noticing that $f(R_0)=1$, it is enough to prove that

$$|r(u)-R_0|\le C\,\big((\sqrt{\ell_0})^{(n-1)/n}+\varepsilon\big)\,,\qquad\big|f(r(u))-1\big|\le C\,\big((\sqrt{\ell_0})^{(n-1)/n}+\varepsilon\big)\,. \tag{3.26}$$

Since $\mathrm{Lip}(f,[R_0/2,2R_0])\le C$ and $f(R_0)=1$, it is enough to prove the first estimate in (3.26). To this end, we start noticing that if $r(u)<R_0$, then $f(r(u))>f(R_0)=1\ge\Phi(u)$, and (3.25) gives

$$\begin{aligned}C\sqrt{\ell_0}&\ge\int_{B_{r(u)}}|\Phi(u)-f(r(u))|^{n/(n-1)}\ge\omega_n\,r(u)^n\,\big(f(r(u))-1\big)^{n/(n-1)}\\&=\omega_n\,r(u)^n\,\big(f(r(u))-f(R_0)\big)^{n/(n-1)}=c(n)\Big(1-\big(r(u)/R_0\big)^{n-1}\Big)^{n/(n-1)}\\&\ge c(n)\,(R_0-r(u))^{n/(n-1)}\,,\end{aligned}$$

as desired. If, instead $r(u)>R_0$, then by $\int_{\mathbb{R}^n}W(u)\le\varepsilon\,\mathcal{AC}_\varepsilon(u)\le C$, $f(r(u))\in(0,1)$ and (A.8) (that is, $\Phi(b)-\Phi(a)\ge(b-a)^2/C$ if $0\le a\le b\le1$), we deduce that

$$\begin{aligned}C\,\varepsilon&\ge\int_{B_{R_0}}W(u)\ge\int_{B_{R_0}}W\Big(\Phi^{-1}\big(f(r(u))\big)\Big)-C\int_{B_{R_0}}\big|u-\Phi^{-1}(f(r(u)))\big|\\&\ge\int_{B_{R_0}}W\Big(\Phi^{-1}\big(f(r(u))\big)\Big)-C\int_{B_{R_0}}\big|\Phi(u)-f(r(u))\big|^2\\&\ge\int_{B_{R_0}}W\Big(\Phi^{-1}\big(f(r(u))\big)\Big)-C\int_{B_{R_0}}\big|\Phi(u)-f(r(u))\big|^{n/(n-1)}\,,\end{aligned}$$

where in the last inequality we have just used $2\ge n/(n-1)$ and the fact that $\Phi(u)$ and $f(r(u))$ lie in $[0,1]$. Hence, by $B_{R_0}\subset B_{r(u)}$, (3.25) and $\omega_n\,R_0^n=1$,

$$W\big(\Phi^{-1}(f(r(u)))\big)\le C\,\big((\sqrt{\ell_0})^{(n-1)/n}+\varepsilon\big)\,.$$

Now, $R_0<r(u)\le M_0$ implies $1>f(r(u))\ge f(M_0)\ge\delta_0$ (provided we further decrease the value of δ_0). In particular, by $W(t)\ge(1-t)^2/C$ on $(\delta_0,1)$ (which can be assumed as done with (A.13)), we have

$$C\,\big((\sqrt{\ell_0})^{(n-1)/n}+\varepsilon\big)\ge\Big(1-\Phi^{-1}\big(f(r(u))\big)\Big)^2\,.$$

By (A.7), we have

$$1-\Phi^{-1}(s)\ge\frac{\sqrt{1-s}}{C}\,,\qquad\forall s\in(0,1)\,,$$

thus concluding

$$\begin{aligned}C\,\big((\sqrt{\ell_0})^{(n-1)/n}+\varepsilon\big)&\ge1-f(r(u))=c(n)\,\big(R_0^{1-n}-r(u)^{1-n}\big)\\&\ge\frac{c(n)}{r(u)^{n-1}}\Big(\Big(\frac{r(u)}{R_0}\Big)^{n-1}-1\Big)\ge\frac{c(n)}{M_0^{n-1}}\,(r(u)-R_0)\,.\end{aligned}$$

This completes the proof of (3.26), and thus of (3.24).

Step four: We now consider $\{\varepsilon_j,v_j,\mathrm{E}_j\}_j$ as in the statement, and begin the proof of the resolution result. We introduce the radius $R_j(t)$ by setting $v_j(R_j(t))=t$ for every t in the range of v_j. In this step we prove that both δ_0 (defined in section A.3) and $1-\delta_0$ belong to the range of each v_j, that

$$3\,R_0\ge R_j(\delta_0)\ge R_j(1-\delta_0)\ge\frac{R_0}{3}\,,\tag{3.27}$$

$$\frac{\varepsilon_j}{C}\le R_j(\delta_0)-R_j(1-\delta_0)\le C\,\varepsilon_j\,,\tag{3.28}$$

and that

$$-\frac{C}{\varepsilon_j}\le v_j'\le-\frac{1}{C\varepsilon_j}\qquad\text{on }(R_j(1-\delta_0),R_j(\delta_0))\,.\tag{3.29}$$

In particular, the constants b_j and c_j introduced in (3.10) are well-defined, they satisfy

$$c_j=R_j(\delta_0)-R_0\,,\qquad b_j=R_0-R_j(1-\delta_0)\,,\tag{3.30}$$

and property (3.12) in the statement boils down to (3.29).

By step three, for j large enough and considering that $v_j \in \mathcal{R}_0$, we have

$$\int_{\mathbb{R}^n} |1_{B_{R_0}} - \Phi(v_j)|^{n/(n-1)} \le C\left(\ell_0^{(n-1)/n} + \varepsilon_0\right). \tag{3.31}$$

By (3.31), if ℓ_0 and ε_0 are small enough, then both δ_0 and $1-\delta_0$ must belong to the range of each v_j. Now, if $R_j(\delta_0) \le R_0$, then

$$\int_{B_{R_0}\setminus B_{R_j(\delta_0)}} |1_{B_{R_0}} - \Phi(v_j)|^{n/(n-1)} \ge \omega_n\,(R_0^n - R_j(\delta_0)^n)\,(1-\Phi(\delta_0))^{n/(n-1)} \ge \frac{R_0^n - R_j(\delta_0)^n}{C}\,,$$

and $R_j(\delta_0) \ge R_0/2$ follows by (3.31) for ℓ_0 and ε_0 small enough; if, instead, $R_j(\delta_0) \ge R_0$, then

$$\int_{B_{R_j(\delta_0)}\setminus B_{R_0}} |1_{B_{R_0}} - \Phi(v_j)|^{n/(n-1)} \ge \omega_n\,(R_j(\delta_0)^n - R_0^n)\,\Phi(\delta_0)^{n/(n-1)} \ge \frac{R_j(\delta_0)^n - R_0^n}{C}\,,$$

and $R_j(\delta_0) \le 2\,R_0$ follows, again, for ℓ_0 and ε_0 small enough; we have thus proved $R_0/2 \le R_j(\delta_0) \le 2\,R_0$. Since (3.5) implies $\mathcal{AC}_{\varepsilon_j}(v_j) \le C$ we also have

$$C\,\varepsilon_j \ge \int_{\mathbb{R}^n} W(v_j) \ge \frac{R_j(\delta_0)^n - R_j(1-\delta_0)^n}{C} \ge \frac{R_j(\delta_0) - R_j(1-\delta_0)}{C}$$

where in the last inequality we have used $R_j(\delta_0) \ge R/2$. Thus, we have so far proved (3.27) and the upper bound in (3.28). Before proving the lower bound in (3.28), we prove (3.29). To this end, we multiply (3.6) by v_j', and then integrate over an arbitrary interval $(0,r)$, to get

$$\varepsilon_j^2\Big((v_j')^2 + 2\,(n-1)\int_0^r \frac{v_j'(t)^2}{t}\,dt\Big) = W(v_j) - W(v_j(0)) - \varepsilon_j\int_0^r v_j\,(1-v_j)\,\mathrm{E}_j\,v_j'\,. \tag{3.32}$$

By (3.7), the right-hand side of (3.32) is bounded in terms of n and W, so that (3.32) implies $\varepsilon_j^2\,(v_j')^2 \le C$ on $(0,\infty)$; the lower bound in (3.29) then follows by $v_j' \le 0$. To obtain the upper bound in (3.29), we multiply again (3.6) by v_j', but this time we integrate over (r,∞) for $r \in (R_j(1-\delta_0), R_j(\delta_0))$, thus obtaining

$$\varepsilon_j^2\Big(-v_j'(r)^2 + 2\,(n-1)\int_r^\infty \frac{v_j'(t)^2}{t}\,dt\Big) = -W(v_j(r)) - \varepsilon_j\int_r^\infty v_j\,(1-v_j)\,\mathrm{E}_j\,v_j'\,. \tag{3.33}$$

By $W(v_j(r)) \ge \inf_{[\delta_0,1-\delta_0]} W \ge 1/C$, (3.7), and the non-negativity of the integral on the left-hand side of (3.33), we deduce that

$$2\,\varepsilon_j^2\,v_j'(r)^2 \ge W(v_j(r)) - C\,\varepsilon_j \ge \frac{1}{C}\,, \qquad \forall r \in (R_j(1-\delta_0), R_j(\delta_0))\,,$$

which, again by $v_j' \le 0$, implies the upper bound in (3.29). To finally prove the lower bound in (3.28), we notice that thanks to the lower bound in (3.29) we have

$$\frac{C}{\varepsilon_j}\,(R_j(\delta_0) - R_j(1-\delta_0)) \ge \int_{R_j(1-\delta_0)}^{R_j(\delta_0)} (-v_j') = 1 - 2\,\delta_0\,.$$

We have completed the proofs of (3.27), (3.28) and (3.29).

Step five: We obtain sharp estimates for v_j as $r\to\infty$: precisely, we prove that for every $r \ge R_j(\delta_0)$ one has

$$v_j(r) \le C\,e^{-(r-R_j(\delta_0))/C\,\varepsilon_j}\,, \tag{3.34}$$

$$|v_j^{(k)}(r)| \le \frac{C}{\varepsilon_j^k}\,e^{-(r-R_j(\delta_0))/C\,\varepsilon_j}\,, \qquad k = 1,2\,. \tag{3.35}$$

By (A.6), (3.6) and (3.7) we have that

$$2\,\varepsilon_j^2\left\{v_j''+(n-1)\,\frac{v_j'}{r}\right\}\ge\frac{v_j}{C}-C\,\varepsilon_j\,v_j\ge\frac{v_j}{C}\,,\qquad\text{on }(R_j(\delta_0),\infty)\,.\tag{3.36}$$

Multiplying by v_j' and integrating over $(r,\infty)\subset(R_j(\delta_0),\infty)$ we obtain

$$-\varepsilon_j^2\,v_j'(r)^2\ge-\frac{v_j(r)^2}{C}\,,\qquad\forall r\ge R_j(\delta_0)\,.\tag{3.37}$$

Plugging (3.37) into (3.36) we reabsorb the v_j'/r term and obtain

$$\varepsilon_j^2\,v_j''\ge\frac{v_j}{C_*}\,,\qquad\text{on }(R_j(\delta_0),\infty)\,,\tag{3.38}$$

for a universal constant C_*. We now notice that

$$v_*(r)=\delta_0\,e^{-(r-R_j(\delta_0))/\sqrt{C_*}\,\varepsilon_j}\,,$$

satisfies $\varepsilon_j^2\,v_*''=v_*/C_*$ and

$$v_*(R_j(\delta_0))=\delta_0=v_j(R_j(\delta_0))\,.$$

Therefore $v_j\le v_*$ on $(R_j(\delta_0),\infty)$, and (3.34) follows. The case $k=1$ of (3.35) follows from (3.34) and (3.37). The case $k=2$ of (3.35) follows by noticing that (3.6), (3.7), (3.37) and (A.6) imply

$$\varepsilon_j^2\,|v_j''|\le C\left(\varepsilon_j^2\,|v_j'|+|W'(v_j)|+\varepsilon_j\,v_j\right)\le C\left(\varepsilon_j^2\,|v_j'|+v_j\right)\le C\,v_j\,,$$

and then by using (3.34) again.

Step six: We obtain sharp estimates for $v_j(r)$ when $r\to0^+$: precisely, we prove that for every $r\le R_j(1-\delta_0)$ one has

$$\begin{aligned}1-v_j(r)&\le C\,e^{-(R_j(1-\delta_0)-r)/C\,\varepsilon_j}\,,&(3.39)\\|v_j'(r)|&\le C\,\min\left\{\frac{r}{\varepsilon_j^2},\frac{1}{\varepsilon_j}\right\}\,e^{-(R_j(1-\delta_0)-r)/C\,\varepsilon_j}\,,&(3.40)\\|v_j''(r)|&\le\frac{C}{\varepsilon_j^2}\,e^{-(R_j(1-\delta_0)-r)/C\,\varepsilon_j}\,.&(3.41)\end{aligned}$$

To this end, it is convenient to recast (3.6) in terms of $w_j=1-v_j$, so that

$$2\,\varepsilon_j^2\left\{w_j''+(n-1)\,\frac{w_j'}{r}\right\}=-W'(1-w_j)+\varepsilon_j\,w_j\,(1-w_j)\,\mathrm{E}_j\,.\tag{3.42}$$

By (A.6) and (3.7), if $r\le R_j(1-\delta_0)$, then

$$-W'(1-w_j)+\varepsilon_j\,w_j\,(1-w_j)\,\mathrm{E}_j\le C\,(1-w_j)\tag{3.43}$$

so that (3.42) implies in particular

$$2\,\varepsilon_j^2\left\{w_j''+(n-1)\,\frac{w_j'}{r}\right\}\le C\,w_j\,,\qquad\text{on }(0,R_j(1-\delta_0))\,.\tag{3.44}$$

Multiplying by $w_j'\ge0$ and integrating on $(0,r)\subset(0,R_j(1-\delta_0))$ we deduce

$$\varepsilon_j^2\Big\{w_j'(r)^2+\int_0^r\frac{(w_j')^2}{t}\Big\}\le C\,(w_j(r)^2-w_j(0)^2)\le C\,w_j(r)^2\,,$$

that is,

$$\varepsilon_j\,w_j'\le C\,w_j\,,\qquad\text{on }(0,R_j(1-\delta_0))\,.\tag{3.45}$$

Combining (3.45) with (3.42), (A.6) and (3.7), we find that

$$\begin{aligned} 2\,\varepsilon_j^2\,w_j''+C\,\varepsilon_j\,w_j &\ge 2\,\varepsilon_j^2\Big\{w_j''+\frac{n-1}{r}\,w_j'\Big\}\\ &= -W'(1-w_j)+\varepsilon_j\;w_j\,(1-w_j)\,\mathrm{E}_j\ge\frac{w_j}{C}-C\,\varepsilon_j\,w_j\,, \end{aligned}$$

on $[R_0/4,R_j(1-\delta_0))$, so that, for j large enough and for a constant C_* depending on n and W only, we have

$$\varepsilon_j^2\,w_j''\ge\frac{w_j}{C_*}\,,\qquad\text{on }[R_0/4,R_j(1-\delta_0))\,. \tag{3.46}$$

Correspondingly to C_*, we introduce the barrier

$$w_*(r)=\delta_0\left\{e^{((R_0/4)-r)/\sqrt{C_*\,\varepsilon_j^2}}+e^{(r-R_j(1-\delta_0))/\sqrt{C_*\,\varepsilon_j^2}}\right\},\qquad r>0\,.$$

By the monotonicity of w_j and by $R_j(1-\delta_0)\ge R_0/3$ (recall (3.27)),

$$\begin{aligned} &w_*(R_0/4)\ge\delta_0=w_j(R_j(1-\delta_0))\ge w_j(R_0/4)\,,\\ &w_*(R_j(1-\delta_0))\ge\delta_0=w_j(R_j(1-\delta_0))\,,\\ &\varepsilon_j^2\,w_*''=\frac{w_*}{C_*}\qquad\text{on }[0,\infty)\,. \end{aligned}$$

We thus find $w_j\le w_*$ on $[R_0/4,R_j(1-\delta_0))$, that is, for every $R_0/4\le r\le R_j(1-\delta_0)$,

$$1-v_j(r)\le\delta_0\left\{e^{((R_0/4)-r)/\sqrt{C_*\,\varepsilon_j^2}}+e^{(r-R_j(1-\delta_0))/\sqrt{C_*\,\varepsilon_j^2}}\right\}. \tag{3.47}$$

By testing (3.47) with

$$r_*=\frac{R_0/4+R_0/3}{2}$$

and exploiting the monotonicity of v_j, we find that for $r\in(0,r_*]$

$$1-v_j(r)\le\delta_0\,e^{-1/C\,\varepsilon_j}\qquad\forall r\in(0,r_*]\,, \tag{3.48}$$

(thus obtaining the crucial information that, for j large enough and for every $k\in\mathbb{N}$, $\|1-v_j\|_{C^0[0,r_*]}=\mathrm{o}(\varepsilon_j^k)$ as $j\to\infty$). At the same time, for $r_*\le r\le R_j(1-\delta_0)$, the second exponential in (3.47) is bounded from below in terms of a universal constant, while the first exponential is bounded from above by $e^{-1/C\,\varepsilon_j}$, so that (3.47) and (3.48) can be combined into

$$1-v_j(r)\le C\,e^{-(R_j(1-\delta_0)-r)/C\,\varepsilon_j}\,,\qquad\forall r\in(0,R_j(1-\delta_0)]\,,$$

that is (3.39). By combining (3.39) and (3.45) we also find

$$-v_j'(r)\le\frac{C}{\varepsilon_j}\,e^{-(R_j(1-\delta_0)-r)/C\,\varepsilon_j}\,,\qquad\forall r\in(0,R_j(1-\delta_0)]\,, \tag{3.49}$$

which is half of the estimate for $|v_j'|$ in (3.40). Multiplying (3.44) by r^{n-1} we find

$$2\,\varepsilon_j^2\,\big(r^{n-1}\,w_j'\big)'\le C\,r^{n-1}\,w_j\,,\qquad\forall r\in(0,R_j(1-\delta_0)]\,,$$

which we integrate over $(0,r)\subset(0,R_j(1-\delta_0))$ to conclude that

$$\varepsilon_j^2\,r^{n-1}\,(-v_j'(r))\le C\int_0^r w_j(t)\,t^{n-1}dt\le C\,(1-v_j(r))\,r^n\,,\qquad\forall r\in(0,R_j(1-\delta_0)]\,;$$

in particular, by combining this last inequality with (3.39) we find

$$-v_j'(r)\le C\,\frac{r}{\varepsilon_j^2}\,e^{-(R_j(1-\delta_0)-r)/C\,\varepsilon_j}\,,\qquad\forall r\in(0,R_j(1-\delta_0)]\,,$$

that is the missing half of (3.40). Finally, by (3.42) with (3.43) we find

$$\varepsilon_j^2\,|v_j''|\le C\left\{(1-v_j)+\frac{|v_j'|}{r}\right\}\qquad\text{on }(0,R_j(1-\delta_0))\,,$$

and then (3.41) follows from (3.39) and (3.40).

Step seven: We now improve the first set of inequalities in (3.27), and show that

$$R_0-C\,\varepsilon_j\le R_j(1-\delta_0)<R_j(\delta_0)\le R_0+C\,\varepsilon_j\,. \tag{3.50}$$

Let us set

$$\alpha_j=\int_{B_{R_j(1-\delta_0)}}V(v_j)\,,\qquad\beta_j=\int_{B_{R_j(\delta_0)}\setminus B_{R_j(1-\delta_0)}}V(v_j)\,,\qquad\gamma_j=\int_{B_{R_j(\delta_0)^c}}V(v_j)\,.$$

By (A.11), (3.39) and (3.27) we have

$$\begin{aligned}\big|\alpha_j-\omega_n\,R_j(1-\delta_0)^n\big| &= \int_{B_{R_j(1-\delta_0)}}1-V(v_j)\le C\int_{B_{R_j(1-\delta_0)}}(1-v_j)^2\\ &\le C\int_{B_{R_j(1-\delta_0)}}e^{-(R_j(1-\delta_0)-|x|)/C\,\varepsilon_j}\,dx\\ &= C\int_0^{R_j(1-\delta_0)}e^{-(R_j(1-\delta_0)-r)/C\,\varepsilon_j}\,r^{n-1}\,dr\\ &= C\,\varepsilon_j\int_{-R_j(1-\delta_0)/\varepsilon_j}^0 e^{s/C}\,(R_j(1-\delta_0)+\varepsilon_j\,s)^{n-1}\,ds\le C\,\varepsilon_j\,.\end{aligned}$$

Similarly, by (A.11), (3.27) and (3.34) we find

$$\begin{aligned}|\gamma_j| &= \int_{B^c_{R_j(\delta_0)}}V(v_j)\le C\int_{B_{R_j(\delta_0)^c}}v_j^{2\,n/(n-1)}\le C\int_{R_j(\delta_0)}^\infty e^{-(r-R_j(\delta_0))/C\,\varepsilon_j}\,r^{n-1}\,dr\\ &= C\,\varepsilon_j\int_0^\infty e^{-s/C}\,(R_j(\delta_0)+\varepsilon_j\,s)^{n-1}\,ds\le C\,\varepsilon_j\,.\end{aligned}$$

Finally, thanks to (3.27),

$$|\beta_j| = \int_{B_{R_j(\delta_0)}\setminus B_{R_j(1-\delta_0)}}V(v_j)\le C\,(R_j(\delta_0)-R_j(1-\delta_0))\le C\,\varepsilon_j\,.$$

Combining the estimates for α_j, β_j and γ_j with the fact that

$$\omega_n\,R_0^n=1=\int_{\mathbb{R}^n}V(v_j)=\alpha_j+\beta_j+\gamma_j\,,$$

we conclude that

$$C\,\varepsilon_j\ge\omega_n\,|R_0^n-R_j(1-\delta_0)^n|\le\frac{|R_0-R_j(1-\delta_0)|}{C}\,,$$

so that (3.50) follows by (3.27).

Step eight: We conclude the proof of the theorem: (3.29), (3.30) and (3.50) imply (3.10) and (3.12), as well as

$$|b_j|\,,|c_j|\le C\,\varepsilon_j\,, \tag{3.51}$$

which is a weaker form of (3.11); (3.34) and (3.35) imply (3.13), while (3.39), (3.40), and (3.41) imply (3.14). We are thus left to prove the full form of (3.11) (which includes a positive lower bound in the from ε_j/C for both b_j and c_j), as well as (3.8): that is, we want to show that if v_j satisfies (3.4), (3.5), (3.6) and (3.7), then, for every $x \in \mathbb{R}^n$ and j large enough, we have

$$v_j(x) = z_{\varepsilon_j}(x) + f_j\Big(\frac{|x|-R_0}{\varepsilon_j}\Big) = \eta\Big(\frac{|x|-R_0}{\varepsilon_j} - \tau_j\Big) + f_j\Big(\frac{|x|-R_0}{\varepsilon_j}\Big)\,, \tag{3.52}$$

with functions $f_j \in C^2(I_j)$ such that

$$|f_j(s)| \le C\,\varepsilon_j\,e^{-|s|/C}\,, \qquad \forall s \in I_j = (-R_0/\varepsilon_j, \infty)\,, \tag{3.53}$$

and with $\tau_j = \tau_{\varepsilon_j}$ for τ_ε defined by (3.1) and (3.2). In fact, (3.52) and (3.53) imply the full form of (3.11): for example, combined with (3.12) and (3.17), they give

$$\begin{aligned} C\,\frac{b_j}{\varepsilon_j} &\ge \int_{R_0-b_j}^{R_0} (-v_j') = v_j(R_0-b_j) - v_j(R_0) = (1-\delta_0) - \eta(-\tau_j) - f_j(0) \\ &\ge 1-\delta_0 - \eta(-\tau_0) - C\,\varepsilon_j \end{aligned}$$

where the latter quantity is positive provided j is large enough and we further decrease the value of δ_0 to have $\delta_0 < 1-\eta(-\tau_0)$.

We can thus focus on (3.52) and (3.53), which we recast by looking at the functions

$$\eta_j(s) = v_j(R_0 + \varepsilon_j\,s)\,, \qquad s \in I_j\,,$$

in terms of which $f_j(s) = \eta_j(s) - \eta(s-\tau_j)$. Thus, our goal becomes proving that

$$|\eta_j(s) - \eta(s-\tau_j)| \le C\,\varepsilon_j\,e^{-|s|/C}\,, \qquad \forall s \in I_j\,. \tag{3.54}$$

We start noticing that, by (3.12), (3.13) and (3.14), we have

$$C \ge -\eta_j'(s) \ge \frac{1}{C}\,, \qquad \forall s \in (-b_j/\varepsilon_j, c_j/\varepsilon_j)\,, \tag{3.55}$$

$$\eta_j^{(k)}(s) \le C\,e^{-s/C}\,, \qquad \forall s \in (c_j/\varepsilon_j, \infty)\,, k = 0,1,2, \tag{3.56}$$

$$\begin{cases} (1-\eta_j(s)) + |\eta_j''(s)| \le C\,e^{s/C}\,, \\ |\eta_j'| \le C\,\min\Big\{\dfrac{R_0+\varepsilon_j\,s}{\varepsilon_j}, 1\Big\}\,e^{s/C}\,, \end{cases} \qquad \forall s \in (-R_0/\varepsilon_j, -b_j/\varepsilon_j)\,, \tag{3.57}$$

(while the analogous estimates for η are found in (A.16) and (A.18)). In order to estimate $f_j(s) = \eta_j(s) - \eta(s-\tau_j)$, we introduce

$$g_j(s) = \eta_j(s) - \eta(s-t_j)\,,$$

for t_j defined by the identity

$$\eta\big(-(b_j/\varepsilon_j) - t_j\big) = 1-\delta_0\,. \tag{3.58}$$

(Notice that the definition is well-posed by $\eta' < 0$ and $\eta(\mathbb{R}) = (0,1)$.) We claim that the proof of (3.53) can be reduced to that of

$$|g_j(s)| \le C\,\varepsilon_j\,e^{-|s|/C}\,, \qquad \forall s \in I_j\,. \tag{3.59}$$

Indeed, by (3.4), if (3.59) holds, then we are in the position to apply step one, and deduce from (3.17) that $|t_j - \tau_0| \le C\,\varepsilon_j$. Having also (by the same argument) $|\tau_j - \tau_0| \le C\,\varepsilon_j$, we deduce that

$$|\tau_j - t_j| \le C\,\varepsilon_j\,,$$

that we exploit in combination with (3.56) and (3.57) to deduce

$$\begin{aligned} |f_j(s) - g_j(s)| &= |\eta(s-t_j) - \eta(s-\tau_j)| \le C\int_0^1 \big|\eta'\big(s-\tau_j - t\,(t_j-\tau_j)\big)\big|\,dt \\ &\le C\,\varepsilon_j\,e^{-|s|/C}\,, \qquad \forall s \in I_j\,. \end{aligned}$$

We are thus left to prove (3.59). To this end, we preliminarily notice that, since $\eta_j(-b_j/\varepsilon_j) = v_j(R_0 - b_j) = 1 - \delta_0$, the definition of t_j is such that

$$g_j\big(-b_j/\varepsilon_j\big) = 0\,. \tag{3.60}$$

Moreover, by the decay properties (A.16) of η and by $|b_j| \le C\,\varepsilon_j$, (3.58) implies

$$|t_j| \le C\,. \tag{3.61}$$

We now divide the proof of (3.59) in three separate arguments:

We prove (3.59) *for* $|s| \ge C\,\log(1/\varepsilon_j)$: This is trivial from the decay properties of η and η_j. Indeed, by (A.16), (3.61), (3.56) and (3.57) we find that

$$|g_j(s)| \le K_1\, e^{-|s|/K_1}\,, \qquad \forall s \in I_j\,. \tag{3.62}$$

for a universal constant K_1. In particular, we trivially have

$$|g_j(s)| \le K_1\,\varepsilon_j\, e^{-|s|/2\,K_1}\,, \qquad \forall s \in I_j\,, |s| \ge 2\,K_1 \log\Big(\frac{1}{\varepsilon_j}\Big)\,. \tag{3.63}$$

We will later increase the value of K_1 in (3.62) so that (3.74) below holds too.

We prove (3.59) *on arbitrary compact subsets of* I_j: More precisely, we show that for every $K > 0$ we can find $C_K = C_K(n, W)$ (that is, a constant that depends on n, W and K only) such that

$$|g_j(s)| \le C_K\,\varepsilon_j\,, \qquad \forall s \in I_j\,, |s| \le K\,. \tag{3.64}$$

To this end, setting $\mathrm{E}_j^*(s) = \mathrm{E}_j(R_0 + \varepsilon_j\, s)$, we deduce from (3.6) that η_j satisfies the ODE

$$2\,\eta_j'' + 2\,\varepsilon_j\,\frac{n-1}{R_0 + \varepsilon_j\, s}\,\eta_j' = W'(\eta_j) - \varepsilon_j\,\eta_j\,(1-\eta_j)\,\mathrm{E}_j^* \qquad \text{on } I_j\,. \tag{3.65}$$

Multiplying (3.65) by $-\eta_j'$ and integrating over (s, ∞) we find

$$\eta_j'(s)^2 - 2\,\varepsilon_j\,(n-1)\int_s^\infty \frac{\eta_j'(t)^2}{R_0 + \varepsilon_j\, t}\,dt = W(\eta_j(s)) + \varepsilon_j \int_s^\infty \eta_j\,(1-\eta_j)\,\eta_j'\,\mathrm{E}_j^*\,. \tag{3.66}$$

Since $\eta'(s - t_j)^2 = W(\eta(s - t_j))$ for every $s \in \mathbb{R}$, we find that

$$\eta_j'(s)^2 - \eta'(s - t_j)^2 \;=\; W(\eta_j(s)) - W(\eta(s - t_j)) + \varepsilon_j\, L_j(s)\,, \tag{3.67}$$

$$\text{where} \qquad L_j(s) = \int_s^\infty \Big(2\,(n-1)\,\frac{\eta_j'(t)^2}{R_0 + \varepsilon_j\, t} + \eta_j\,(1-\eta_j)\,\eta_j'\,\mathrm{E}_j^*\Big)\,dt\,.$$

Setting

$$\ell_j(s) = \frac{W(\eta_j(s)) - W(\eta(s - t_j))}{\eta_j(s) - \eta(s - t_j)}\,, \qquad d_j(s) = \eta_j'(s) + \eta'(s - t_j)\,, \qquad \Gamma_j(s) = \frac{\ell_j(s)}{d_j(s)}\,,$$

and noticing that $d_j < 0$ on I_j, (3.67) takes the form

$$g_j'(s) - \Gamma_j(s)\,g_j(s) = \frac{\varepsilon_j\, L_j(s)}{d_j(s)}\,, \qquad \forall s \in I_j\,. \tag{3.68}$$

Multiplying (3.68) by $\exp(-\int_0^s \Gamma_j)$, integrating over an interval $(-b_j/\varepsilon_j, s)$, and taking into account (3.60), we find

$$g_j(s)\, e^{-\int_0^s \Gamma_j} = \varepsilon_j \int_{-b_j/\varepsilon_j}^s \frac{e^{-\int_0^t \Gamma_j}}{d_j(t)}\, L_j(t)\,dt\,, \qquad \forall s \in I_j\,. \tag{3.69}$$

We now notice that by (3.7), (3.56) and (3.57),

$$|L_j(s)| \le C\,\min\{1, e^{-s/C}\}\,, \qquad \forall s \in I_j\,. \tag{3.70}$$

Moreover, by $\operatorname{Lip} W \le C$ we have $|\ell_j| \le C$ on I_j, while $\eta_j' \le 0$ and (3.61) give

$$d_j(s) \le \eta'(s-t_j) \le -\frac{1}{C_K}\,, \qquad \forall |s| \le K\,, \tag{3.71}$$

and, in particular, $|\Gamma_j(s)| \le C_K$ for $|s| \le K$. Now, assuming without loss of generality that K is large enough to entail $K \ge |b_j|/\varepsilon_j$ (as we can do since $|b_j| \le C\,\varepsilon_j$ for a universal constant C), then (3.69), (3.70), (3.71) and $|\Gamma_j| \le C_K$ on $[-K,K]$ combined give (3.64).

Finally, we prove (3.59) *in the remaining case*: Having in mind (3.63) and (3.64) we are left to prove the existence of a sufficiently large universal constant K_2 such that (3.59) holds (provided j is large enough) for every $s \in I_j$ with $K_2 \le |s| \le 2\,K_1 \log(1/\varepsilon_j)$. To this end, we start by subtracting $2\,\eta'' = W(\eta)$ from (3.65), and obtain

$$2g_j'' - m_j\,g_j = \varepsilon_j\,\Big\{\eta_j\,(1-\eta_j)\,\mathrm{E}_j^* - 2\,(n-1)\,\frac{\eta_j'}{R_0+\varepsilon_j\,s}\Big\}\,, \qquad \forall s \in I_j\,, \tag{3.72}$$

where

$$m_j(s) = \frac{W'(\eta_j(s)) - W'(\eta(s-t_j))}{\eta_j(s) - \eta(s-t_j)}\,, \qquad s \in I_j\,.$$

The coefficient m_j is uniformly positive: indeed, the decay properties of η and η_j at infinity, combined with $|t_j| \le C$, imply the existence of a universal constant K_2 such that if $|s| \ge K_2$, then $\eta_j(s)$ and $\eta(s-t_j)$ are both at distance at most δ_0 from $\{0,1\}$: and since $W'' \ge 1/C$ on $(0,\delta_0)\cup(1-\delta_0,1)$ by (A.6), we conclude that, up to further increase the value of K_2,

$$m_j(s) \ge \frac{1}{K_2} \qquad \forall s \in I_j\,, |s| \ge K_2\,. \tag{3.73}$$

At the same time, the right-hand side of (3.72) has exponential decay: indeed, by (3.7), (3.55), (3.56) and (3.57), if $|s| \le \log(1/\varepsilon_j)$, $s \in I_j$, then we get

$$\Big|\eta_j\,(1-\eta_j)\,\mathrm{E}_j^* - 2\,(n-1)\,\frac{\eta_j'}{R_0+\varepsilon_j\,s}\Big| \le K_1\,\varepsilon_j\,e^{-|s|/K_1}\,, \tag{3.74}$$

up to further increase the value of the universal constant K_1 introduced in (3.63). Let us thus consider

$$g_*(s) = C_1\,\varepsilon_j\,e^{-|s|/\sqrt{2\,C_2}}\,, \qquad s \in \mathbb{R}\,,$$

for C_1 and C_2 universal constants to be determined. By combining (3.72) with (3.73) and (3.74) we find that, if $s \in I_j$ with $K_2 \le |s| \le 2\,K_1 \log(1/\varepsilon_j)$, then

$$\begin{aligned} 2\,(g_j - g_*)'' - m_j\,(g_j - g_*) &\ge m_j\,g_* - 2\,g_*'' - K_1\,\varepsilon_j\,e^{-|s|/K_1} \\ &\ge \Big(\frac{1}{K_2} - \frac{1}{C_2}\Big)\,g_* - K_1\,\varepsilon_j\,e^{-|s|/K_1} \\ &= \varepsilon_j\,\Big\{\frac{C_1}{K_1}\,\Big(\frac{1}{K_2} - \frac{1}{C_2}\Big)\,e^{[(1/K_1)-(1/\sqrt{2\,C_2})]\,|s|} - 1\Big\}\,K_1\,e^{-|s|/K_1}\,, \end{aligned}$$

where the latter quantity is non-negative for every $|s| \ge K_2$ provided

$$C_1 \ge 3\,K_1\,K_2\,e^{-K_0/2\,K_1}\,, \qquad C_2 \ge \max\{2\,K_2, 2\,K_1^2\}\,. \tag{3.75}$$

At the same time, by (3.63),

$$\big|g_j\big(\pm\,2\,K_1 \log(1/\varepsilon_j)\big)\big| \le K_1\,\varepsilon_j^2\,,$$

while $C_2 \ge 2\,K_1^2$ gives

$$g_*\big(\pm\,2\,K_1\,\log(1/\varepsilon_j)\big) = C_1\,\varepsilon_j\,e^{-2\,K_1\,\log(1/\varepsilon_j)/\sqrt{2\,C_2}} \ge C_1\,\varepsilon_j^2\,.$$

Upon further requiring $C_1 \ge K_1$ we thus have

$$g_*(s) \ge |g_j(s)| \qquad \text{at } s = \pm\,2\,K_1\,\log(1/\varepsilon_j)\,. \tag{3.76}$$

Similarly, by (3.64),

$$|g_j(\pm K_2)| \le C_{K_2}\,\varepsilon_j\,,$$

while $C_{\ge}2\,K_2$ gives

$$g_*(\pm K_2) = C_1\,\varepsilon_j\,e^{-K_2/\sqrt{2\,C_2}} \ge C_1\,\varepsilon_j\,e^{-\sqrt{K_2}/2}\,.$$

Upon requiring that $C_1 \ge C_{K_2}\,e^{\sqrt{K_2}/2}$, we find that

$$g_*(s) \ge |g_j(s)| \qquad \text{at } s = \pm\,K_2\,. \tag{3.77}$$

In summary, we have proved that if K_1 satisfies (3.62) and (3.74), K_2 satisfies (3.73), and C_1 and C_2 are taken large enough in terms of K_1 and K_2, then (3.76) and (3.77) holds. In particular, $h_j = g_j - g_*$ is non-positive on the boundary of the intervals $[-2\,K_1\,\log(1/\varepsilon_j), -K_2]$ and $[K_2, 2\,K_1\,\log(1/\varepsilon_j)]$, with $h_j'' - m_j\,h \ge 0$, $m_j \ge 0$, on those intervals thanks to (3.75) and (3.73); correspondingly, by the maximum principle, $h_j \le 0$ there, that is,

$$g_j(s) \le C_1\,\varepsilon_j\,e^{-|s|/\sqrt{2\,C_2}}\,, \qquad \forall s \in I_j\,, K_2 \le |s| \le 2\,K_1\,\log(1/\varepsilon_j)\,.$$

To get the matching lower bound we notice that, again by (3.74),

$$(-g_* - g_j)'' - m_j\,(-g_* - g_j) \ge m_j\,g_* - g_*'' - K_1\,\varepsilon_j\,e^{-|s|/K_1}$$

so that, by the same considerations made before, the maximum principle can be applied to $k_j = -g_* - g_j$ on $[-2\,K_1\,\log(1/\varepsilon_j), -K_2] \cup [K_2, 2\,K_1\,\log(1/\varepsilon_j)]$, to deduce $g_j \ge -g_*$. This completes the proof of (3.59). □

4. Strict stability among radial functions

In this section we are going to exploit the resolution result in Theorem 3.1 to deduce a stability estimate for $\psi(\varepsilon)$ on radial (not necessarily decreasing) functions. More precisely, we shall prove the following statement.

Theorem 4.1 (Flugede type estimate)**.** *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies (1.11) and (1.12), then there exist universal constants δ_0 and ε_0 with the following property: if $\varepsilon < \varepsilon_0$, $u_\varepsilon \in \mathcal{R}_0$ is a minimizer of $\psi(\varepsilon)$, and $u \in H^1(\mathbb{R}^n;[0,1])$ is radial and such that*

$$\int_{\mathbb{R}^n} V(u) = 1\,, \tag{4.1}$$

$$\int_{\mathbb{R}^n} (u - u_\varepsilon)^2 \le C\,\varepsilon\,, \tag{4.2}$$

$$\|u - u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \le \delta_0\,, \tag{4.3}$$

then, setting $h = u - u_\varepsilon$,

$$\mathcal{AC}_\varepsilon(u) - \psi(\varepsilon) \ge \frac{1}{C}\int_{\mathbb{R}^n} \varepsilon\,|\nabla h|^2 + \frac{h^2}{\varepsilon}\,. \tag{4.4}$$

Before entering into the proof of Theorem 4.1, we show how it can be used to improve on the conclusions of Theorem 2.1. In particular, it gives the uniqueness of minimizers in $\psi(\varepsilon)$ and, together with the resolution result in Theorem 3.1, allows to compute the precise asymptotic behavior of $\psi(\varepsilon)$ and $\lambda(\varepsilon)$ up to second and first order in $\varepsilon \to 0^+$ respectively. Notice in particular that (4.7) sharply improves (2.3).

Corollary 4.2. *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies (1.11) and (1.12), then there exists a universal constant ε_0 such that, if $\varepsilon < \varepsilon_0$, then $\psi(\varepsilon)$ admits a unique minimizer (modulo translations). In particular, for every $\varepsilon < \varepsilon_0$, $\lambda(\varepsilon)$ is unambiguously defined as the Lagrange multiplier of the unique minimizer $u_\varepsilon \in \mathcal{R}_0$ of $\psi(\varepsilon)$ by the identity (2.2), i.e.*

$$\lambda(\varepsilon) = \Big(1 - \frac{1}{n}\Big)\,\psi(\varepsilon) + \frac{1}{n}\Big\{\frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u_\varepsilon) - \varepsilon\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\Big\}\,. \tag{4.5}$$

Finally, $\varepsilon \in (0,\varepsilon_0) \mapsto \lambda(\varepsilon)$ is continuous and the following expansions hold as $\varepsilon \to 0^+$,

$$\psi(\varepsilon) = 2\,n\,\omega_n^{1/n} + 2\,n\,(n-1)\,\omega_n^{2/n}\,\kappa_0\,\varepsilon + \mathrm{O}(\varepsilon^2) \tag{4.6}$$

$$\lambda(\varepsilon) = 2\,(n-1)\,\omega_n^{1/n} + \mathrm{O}(\varepsilon)\,, \tag{4.7}$$

where $\kappa_0 = \tau_0 + \tau_1 = \int_{\mathbb{R}}[\eta'\,V'(\eta) + W(\eta)]\,s\,ds$.

Proof of Corollary 4.2. Step one: Let $\varepsilon \in (0,\varepsilon_0)$ and let u_ε and v_ε be two minimizers of $\psi(\varepsilon)$, so that, up to translations, $u_\varepsilon, v_\varepsilon \in \mathcal{R}_0^*$ thanks to Theorem 2.1. By Theorem 3.1, if we set $h_\varepsilon = v_\varepsilon - u_\varepsilon$, then

$$h_\varepsilon(x) = f_\varepsilon\Big(\frac{|x| - R_0}{\varepsilon}\Big)\,,$$

where $f_\varepsilon \in C^2(-R_0/\varepsilon, \infty)$, and

$$|f_\varepsilon(s)| \le C\,\varepsilon\,e^{-s/C}\,, \qquad \forall s \ge -R_0/\varepsilon\,. \tag{4.8}$$

We thus see that $u = v_\varepsilon$ satisfies (4.1) and (4.3). Moreover, by (4.8),

$$\int_{\mathbb{R}^n} h_\varepsilon^2 = n\,\omega_n \int_{-R_0/\varepsilon}^{\infty} f_\varepsilon(s)^2\,(R_0 + \varepsilon\,s)^{n-1}\,ds \le C\,\varepsilon^2\,,$$

so that (4.2) holds too. We can thus apply (4.4) with $u = v_\varepsilon$, and exploit the minimality of v_ε to deduce that

$$0 = \mathcal{AC}_\varepsilon(v_\varepsilon) - \psi(\varepsilon) \ge \frac{1}{C}\int_{\mathbb{R}^n} \varepsilon\,|\nabla h_\varepsilon|^2 + \frac{h_\varepsilon^2}{\varepsilon}\,,$$

that is $h_\varepsilon = 0$ on $\mathbb{R}^n$, as claimed.

Step two: We prove (4.6) and (4.7). If u_ε is the minimizer of $\psi(\varepsilon)$ in $\mathcal{R}_0$, then by Theorem 3.1 we have $u_\varepsilon(x) = z_\varepsilon(x) + f_\varepsilon((|x| - R_0)/\varepsilon)$ for every $x \in \mathbb{R}^n$, and with f_ε satisfying (4.8). Moreover, as proved in (3.3), we have

$$\mathcal{AC}_\varepsilon(z_\varepsilon) = 2\,n\,\omega_n^{1/n} + 2\,n\,(n-1)\,\omega_n^{2/n}\,\kappa_0 + \mathrm{O}(\varepsilon^2)\,.$$

Since $\mathcal{AC}_\varepsilon(u_\varepsilon) \le \mathcal{AC}_\varepsilon(z_\varepsilon)$, we are left to prove that $\mathcal{AC}_\varepsilon(u_\varepsilon) \ge \mathcal{AC}_\varepsilon(z_\varepsilon) - C\,\varepsilon^2$. Setting $|x| = R_0 + \varepsilon\,s$ we have

$$u_\varepsilon(x) = \eta(s - \tau_\varepsilon) + f_\varepsilon(s)\,, \qquad \nabla u_\varepsilon(x) = \frac{\eta'(s - \tau_\varepsilon) + f_\varepsilon'(s)}{\varepsilon}\,\frac{x}{|x|}\,,$$

while z_ε satisfies the same identities with $f_\varepsilon = 0$, so that

$$\mathcal{AC}_\varepsilon(u_\varepsilon) - \mathcal{AC}_\varepsilon(z_\varepsilon) = \int_{-R_0/\varepsilon}^{\infty} \Big(2\,\eta'(s - \tau_\varepsilon)\,f_\varepsilon'(s) + f_\varepsilon'(s)^2\Big)\,(R_0 + \varepsilon\,s)^{n-1}\,ds \tag{4.9}$$
$$+ \int_{-R_0/\varepsilon}^{\infty} \Big(W\big(\eta(s - \tau_\varepsilon) + f_\varepsilon(s)\big) - W(\eta(s - \tau_\varepsilon))\Big)\,(R_0 + \varepsilon\,s)^{n-1}\,ds\,.$$

Integration by parts and $2\,\eta'' = W'(\eta)$ give

$$\begin{aligned}
&\int_{-R_0/\varepsilon}^{\infty} 2\,\eta'(s - \tau_\varepsilon)\,f_\varepsilon'(s)\,(R_0 + \varepsilon\,s)^{n-1}\,ds \\
=\;& -\int_{-R_0/\varepsilon}^{\infty} W'\big(\eta(s - \tau_\varepsilon)\big)\,f_\varepsilon(s)\,(R_0 + \varepsilon\,s)^{n-1}\,ds \\
& -2\,(n-1)\,\varepsilon \int_{-R_0/\varepsilon}^{\infty} \eta'(s - \tau_\varepsilon)\,f_\varepsilon(s)\,(R_0 + \varepsilon\,s)^{n-2}\,ds\,.
\end{aligned}$$

Dropping the non-negative term with $f_\varepsilon'(s)^2$ in (4.9), and noticing that, by (A.5) and (4.8), we have

$$\Big|W\big(\eta(s - \tau_\varepsilon) + f_\varepsilon(s)\big) - W(\eta(s - \tau_\varepsilon)) - W'\big(\eta(s - \tau_\varepsilon)\big)\,f_\varepsilon(s)\Big| \le C\,f_\varepsilon(s)^2\,,$$

for every $s > -R_0/\varepsilon$, we thus find

$$\begin{aligned}\mathcal{AC}_\varepsilon(u_\varepsilon) - \mathcal{AC}_\varepsilon(z_\varepsilon) \;&\geq\; -2\,(n-1)\,\varepsilon\int_{-R_0/\varepsilon}^{\infty}\eta'(s-\tau_\varepsilon)\,f_\varepsilon(s)\,(R_0+\varepsilon\,s)^{n-2}\,ds\\ &\quad -C\int_{-R_0/\varepsilon}^{\infty} f_\varepsilon(s)^2\,(R_0+\varepsilon\,s)^{n-1}\,ds \geq -C\,\varepsilon^2\,,\end{aligned}$$

where in the last inequality we have used (4.8), $|\tau_\varepsilon|\leq C$ and the decay estimate for η' in (A.18). Coming to (4.7), rearranging terms in (4.5) we have

$$\lambda(\varepsilon) = \Big(1-\frac{2}{n}\Big)\,\psi(\varepsilon) + \frac{2}{n}\,\frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u_\varepsilon)\,. \tag{4.10}$$

By (4.8)

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u_\varepsilon) = \frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(z_\varepsilon) + \mathrm{O}(\varepsilon) = \frac{\psi(\varepsilon)}{2} + \mathrm{O}(\varepsilon)$$

where in the second identity we have used (3.22). Hence $\lambda(\varepsilon) = (1-(1/n))\,\psi(\varepsilon) + \mathrm{O}(\varepsilon)$ and (4.7) follows from (4.6).

Step three: We prove the continuity of λ on $(0,\varepsilon_0)$. Let $\varepsilon_j \to \varepsilon_* \in (0,\varepsilon_0)$ as $j\to\infty$ and set $h_j = u_{\varepsilon_j} - u_{\varepsilon_*}$. By the resolution formula (3.8) we have

$$\begin{aligned}|u_{\varepsilon_j}(x) - u_{\varepsilon_*}(x)| \;&\leq\; \Big|\eta\Big(\frac{|x|-R_0}{\varepsilon_j} - \tau_{\varepsilon_j}\Big) - \eta\Big(\frac{|x|-R_0}{\varepsilon_*} - \tau_{\varepsilon_*}\Big)\Big|\\ &\quad + \Big|f_{\varepsilon_j}\Big(\frac{|x|-R_0}{\varepsilon_j}\Big) - f_{\varepsilon_*}\Big(\frac{|x|-R_0}{\varepsilon_*}\Big)\Big|\\ &\leq\; C\,\varepsilon_*\,e^{-(|x|-R_0)/C\,\varepsilon_*} + \Big|\eta\Big(\frac{|x|-R_0}{\varepsilon_j} - \tau_0\Big) - \eta\Big(\frac{|x|-R_0}{\varepsilon_*} - \tau_0\Big)\Big|\end{aligned}$$

where we have used (3.17), (3.9) and (A.16). Similarly, since $\varepsilon_j \to \varepsilon_* > 0$, for j large enough we see that

$$\begin{aligned}&\Big|\eta\Big(\frac{|x|-R_0}{\varepsilon_j} - \tau_0\Big) - \eta\Big(\frac{|x|-R_0}{\varepsilon_*} - \tau_0\Big)\Big|\\ \leq\;& \int_0^1 \Big|\eta'\Big(\frac{|x|-R_0}{\varepsilon_* + t(\varepsilon_j-\varepsilon_*)} - \tau_0\Big)\Big|\,\frac{||x|-R_0|}{(\varepsilon_* + t\,(\varepsilon_j-\varepsilon_*))^2}\,|\varepsilon_j-\varepsilon_*|\\ \leq\;& C\,\frac{|\varepsilon_j-\varepsilon_*|}{\varepsilon_*^2}\,e^{-(|x|-R_0)/C\,\varepsilon_*}\,||x|-R_0| \leq C\,\varepsilon_*\,e^{-(|x|-R_0)/C\,\varepsilon_*}\,.\end{aligned}$$

Setting $h_j = u_{\varepsilon_j} - u_{\varepsilon_*}$ we see that (4.1), (4.2) and (4.3) hold with $\varepsilon=\varepsilon_*$ and for j large enough, thus deducing that

$$\begin{aligned}\frac{1}{C}\int_{\mathbb{R}^n} \varepsilon_*\,|\nabla h_j|^2 + \frac{h_j^2}{\varepsilon_*} \;&\leq\; \mathcal{AC}_{\varepsilon_*}(u_{\varepsilon_j}) - \psi(\varepsilon_*)\\ &\leq\; \max\Big\{\frac{\varepsilon_j}{\varepsilon_*},\frac{\varepsilon_*}{\varepsilon_j}\Big\}\,\psi(\varepsilon_j) - \psi(\varepsilon_*)\,.\end{aligned}$$

From the continuity of ψ on $(0,\varepsilon_0)$ (Theorem 2.1) we conclude that

$$\lim_{j\to\infty}\int_{\mathbb{R}^n}|\nabla u_{\varepsilon_j} - \nabla u_{\varepsilon_*}|^2 = 0\,,\qquad \lim_{j\to\infty}\int_{\mathbb{R}^n} W(u_{\varepsilon_j}) = \int_{\mathbb{R}^n} W(u_{\varepsilon_*})\,,$$

and thus λ is continuous on $(0,\varepsilon_0)$ thanks to (4.10). □

We now turn to the proof of Theorem 4.1. This is based on a series of three lemmas, each containing a different stability estimate, coming increasingly closer to (4.4).

Lemma 4.3 (First stability lemma)**.** *Let $n \ge 2$, let $W \in C^{2,1}[0,1]$ satisfy* (1.11) *and* (1.12)*, and let*

$$Q(u) = \int_{\mathbb{R}} 2\,(u')^2 + W''(\eta)\,u^2\,, \qquad u \in H^1(\mathbb{R})\,.$$

Then $Q(u) \ge 0$ on $H^1(\mathbb{R})$, and $Q(u) = 0$ if and only if $u = t\,\eta'$ for some $t \in \mathbb{R}$.

Proof. Let us consider the variational problem

$$\gamma = \inf\Big\{Q(u) : \int_{\mathbb{R}} u^2 = 1\Big\}\,.$$

By (A.18) we have $\eta' \in H^1(\mathbb{R})$. Differentiating $2\,\eta'' = W'(\eta)$ we find $2\,(\eta')'' = W''(\eta)\,\eta'$, and then integration by parts gives $Q(\eta') = 0$. At the same time we clearly have $Q(u) \ge -\|W''\|_{C^0(0,1)}\int_{\mathbb{R}} u^2$ for every $u \in H^1(\mathbb{R})$, so that

$$-\|W''\|_{C^0(0,1)} \le \gamma \le 0\,.$$

We now prove that γ is attained. Let $\{w_j\}_j$ be a minimizing sequence for γ. By the concentration-compactness principle, $\{w_j^2\,dx\}_j$ is in the vanishing case if

$$\lim_{j\to\infty}\int_{I_R} w_j^2 = 0\,, \qquad \forall R > 0\,, \tag{4.11}$$

where we have set $I_R = (-R,R)$. By (A.16) and (A.6) there exists S_0 such that

$$W''(\eta) \ge \frac{1}{C}\,, \qquad \text{on } \mathbb{R}\setminus I_{S_0}\,. \tag{4.12}$$

Therefore by applying (4.11) twice with $R = S_0$ we find

$$\begin{aligned}\limsup_{j\to\infty}\int_{\mathbb{R}} w_j^2 &= \limsup_{j\to\infty}\int_{\mathbb{R}\setminus I_{S_0}} w_j^2 \le C\,\limsup_{j\to\infty}\int_{\mathbb{R}\setminus I_{S_0}} W''(\eta)\,w_j^2\\ &= C\,\limsup_{j\to\infty}\int_{\mathbb{R}} W''(\eta)\,w_j^2 \le \lim_{j\to\infty} Q(w_j) = \gamma \le 0\end{aligned}$$

a contradiction to $\int_{\mathbb{R}} w_j^2 = 1$. If, instead, $\{w_j^2\,dx\}_j$ is in the dichotomy case, then there is $\alpha \in (0,1)$ such that for every $\tau \in (0,\alpha/2)$ there exist $R > 0$ and $R_j \to \infty$ as $j \to \infty$ such that

$$\Big|1 - \alpha - \int_{I_R} w_j^2\Big| < \tau\,, \qquad \Big|\alpha - \int_{\mathbb{R}\setminus I_{R_j}} w_j^2\Big| < \tau\,, \tag{4.13}$$

where, without loss of generality, we can assume $R \ge S_0$ for S_0 as in (4.12). In particular, if φ is a cut-off function between I_R and I_{R_j}, then we have

$$Q(w_j) = Q(\varphi\,w_j) + Q\big((1-\varphi)\,w_j\big) + E_j\,, \tag{4.14}$$

where, taking into account that φ' and $(1-\varphi)\,\varphi$ are supported in $I_{R_j}\setminus I_R$, we have

$$E_j = 2\int_{I_{R_j}\setminus I_R} W''(\eta)\,(1-\varphi)\,\varphi\,w_j^2 + 4\int_{I_{R_j}\setminus I_R} (\varphi w_j)'((1-\varphi)w_j)'\,. \tag{4.15}$$

The first integral in (4.15) is non-negative by (4.12), while the second integral contains a non-negative term of the form $\varphi\,(1-\varphi)\,(w_j')^2$: therefore, by (4.13),

$$\begin{aligned}E_j &\ge 4\int_{I_{R_j}\setminus I_R} w_j\,w_j'\,(1-\varphi)\,\varphi' - w_j\,w_j'\,\varphi\,\varphi' - w_j^2\,(\varphi')^2\\ &\ge -C\int_{I_{R_j}\setminus I_R} w_j^2 - C\Big(\int_{I_{R_j}\setminus I_R} w_j^2\Big)^{1/2}\Big(\int_{\mathbb{R}}(w_j')^2\Big)^{1/2} \ge -C\,\sqrt{\tau}\,,\end{aligned} \tag{4.16}$$

where we have also used $Q(w_j) \to \gamma$ as $j \to \infty$ to infer

$$\int_{\mathbb{R}} (w_j')^2 \le Q(w_j) + \|W\|_{C^0[0,1]} \le C\,.$$

We can take φ supported in I_{R+1}. In this way, up to extracting a subsequence, we have that $\varphi\, w_j$ admits a weak limit w in $H^1(\mathbb{R})$. By lower semicontinuity, homogeneity of Q and (4.13) we have that

$$\liminf_{j\to\infty} Q(\varphi\, w_j) \ge Q(w) \ge \gamma \int_{\mathbb{R}} w^2 \ge\ (1-\alpha)\,\gamma - C\,\tau\,. \tag{4.17}$$

Finally, since $(1-\varphi)$ is supported on $\mathbb{R}\setminus I_{S_0}$, by (4.12) we have

$$\int_{\mathbb{R}} Q\big((1-\varphi)\,v_j\big) \ge \frac{1}{C}\int_{\mathbb{R}} (1-\varphi)^2\, w_j^2 \ge \frac{\alpha}{C} - C\,\tau\,,$$

so that, combining (4.14), (4.16), and (4.17) we find

$$\gamma \ge (1-\alpha)\,\gamma + \frac{\alpha}{C} - C\,\sqrt{\tau}\,.$$

Letting $\tau \to 0^+$ we find a contradiction with $\gamma \le 0$ and $\alpha > 0$. Having excluded vanishing and dichotomy, we have proved the existence of minimizers of γ.

Let now u be a minimizer of γ. Up to replace with u with $|u|$ we can assume $u \ge 0$. By a standard variational argument there exists $\lambda \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} 2u'\,v' + W''(\eta)\,u\,v = \lambda \int_{\mathbb{R}} u\,v\,, \qquad \forall v \in H^1(\mathbb{R})\,. \tag{4.18}$$

Testing with $v = \eta'$ and recalling that $2\,(\eta')'' = W''(\eta)\eta'$, we deduce that

$$\lambda \int_{\mathbb{R}} \eta' u = 0,$$

and, since $u \ge 0$, $\int_{\mathbb{R}} u^2 = 1$, and $\eta' < 0$, we find $\lambda = 0$. Thus u is a C^2-solution of the ODE $2\,u'' = W''(\eta)\,u$ on $\mathbb{R}$. If $u(r_0) = 0$ for some $r_0 \in \mathbb{R}$, then $u'(r_0) \ne 0$ (otherwise we would have $u = 0$ on $\mathbb{R}$, against $\int_{\mathbb{R}} u^2 = 1$), and $u'(r_0) \ne 0$ contradicts $u \ge 0$ on $\mathbb{R}$. Hence, $u > 0$ on $\mathbb{R}$.

Having proved that every minimizer of γ is either positive or negative on the whole $\mathbb{R}$, we deduce that $\gamma = 0$. Indeed, if u_1 and u_2 are minimizers of γ and, say, they are both positive on $\mathbb{R}$, then they solve (4.18) with $\lambda = 0$, and thus

$$2\,\gamma = Q(u_1) + Q(u_2) = Q(u_1 - u_2) \ge \gamma \int_{\mathbb{R}} (u_1 - u_2)^2 = 2\,\gamma\Big(1 - \int_{\mathbb{R}} u_1\,u_2\Big)\,.$$

Since $\int_{\mathbb{R}} u_1\,u_2 \in (0,1)$, $\gamma < 0$ would give a contradiction. Having established that $\gamma = 0$, we now know that η' is a minimizer of γ. If u is also minimizer of γ, then, again by (4.18),

$$Q(u + s\,\eta') = Q(u) + s^2\,Q(\eta') = 0 \qquad \forall s \in \mathbb{R}\,.$$

In particular, if $s \in \mathbb{R}$ is such that $u + s\,\eta'$ is not identically zero on $\mathbb{R}$, then $(u + s\,\eta')/\|u + s\,\eta'\|^2_{L^2(\mathbb{R})}$ is a minimizer of γ, and thus $u + s\,\eta'$ is either positive or negative on the whole $\mathbb{R}$. Let $s_0 = \inf\{s : u + s\,\eta' < 0 \text{ on } \mathbb{R}\}$. If, say, u is a negative minimizer (like η' is), then $s_0 \le 0$; while, clearly, $s_0 > -\infty$, since, for s negative enough, we must have $u + s\,\eta' > 0$ at at least one point, and thus everywhere. Since $u + s_0\,\eta' \le 0$ on $\mathbb{R}$ with $u + s_0\,\eta' = 0$ at at least one point, we deduce that $u + s_0\,\eta' = 0$ on $\mathbb{R}$. $\square$

Lemma 4.4 (Second stability lemma). *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1.11) *and* (1.12)*, then there exists a universal constant ε_0 with the following property. If $u_\varepsilon \in \mathcal{R}_0^*$ is a minimizer of $\psi(\varepsilon)$ for $\varepsilon < \varepsilon_0$ and $h \in H^1(\mathbb{R}^n)$ is a radial function such that*

$$\int_{\mathbb{R}^n} V'(u_\varepsilon)\, h = 0\,, \tag{4.19}$$

then

$$\int_{\mathbb{R}^n} 2\,\varepsilon\,|\nabla h|^2 + \Big(\frac{W''(u_\varepsilon)}{\varepsilon} - \lambda(\varepsilon)V''(u_\varepsilon)\Big)\, h^2 \ge \frac{1}{C}\int_{\mathbb{R}^n} \varepsilon\,|\nabla h|^2 + \frac{h^2}{\varepsilon}\,, \tag{4.20}$$

where $\lambda(\varepsilon)$ is the Lagrange multiplier of u_ε as in (4.5).

Proof. Step one: We show that is enough to prove the lemma with

$$\int_{\mathbb{R}^n} 2\,\varepsilon\,|\nabla h|^2 + \Big(\frac{W''(u_\varepsilon)}{\varepsilon} - \lambda(\varepsilon)V''(u_\varepsilon)\Big)\, h^2 \ge \frac{1}{C}\int_{\mathbb{R}^n} \frac{h^2}{\varepsilon}\,, \tag{4.21}$$

in place of (4.20). Indeed, if ε_0 is small enough, then $|\lambda(\varepsilon)| \le c(n)$ thanks to (2.3), and thus we can find a universal constant C_* such that

$$\int_{\mathbb{R}^n} \Big|\frac{1}{\varepsilon}\, W''(u_\varepsilon) - \lambda(\varepsilon)V''(u_\varepsilon)\Big|\, h^2 \le C_* \int_{\mathbb{R}^n} \frac{h^2}{\varepsilon}\,,$$

whenever u_ε is a minimizer of $\psi(\varepsilon)$, $\varepsilon < \varepsilon_0$, and $h \in H^1(\mathbb{R}^n)$. Let us now fix a radial function $h \in H^1(\mathbb{R}^n)$ satisfying (4.19). If $C_* \int_{\mathbb{R}^n} h^2/\varepsilon \le \int_{\mathbb{R}^n} \varepsilon\,|\nabla h|^2$, then we trivially have

$$\int_{\mathbb{R}^n} 2\,\varepsilon|\nabla h|^2 + \Big(\frac{W''(u_\varepsilon)}{\varepsilon} - \lambda(\varepsilon)\, V''(\zeta_\varepsilon)\Big)\, h^2 \ge \int_{\mathbb{R}^n} 2\,\varepsilon|\nabla h|^2 - C_*\int_{\mathbb{R}^n} \frac{h^2}{\varepsilon} \ge \int_{\mathbb{R}^n} \varepsilon\,|\nabla h|^2\,;$$

if, instead, $C_* \int_{\mathbb{R}^n} h^2/\varepsilon \ge \int_{\mathbb{R}^n} \varepsilon\,|\nabla h|^2$, then we deduce from (4.21)

$$\int_{\mathbb{R}^n} 2\,\varepsilon|\nabla h|^2 + \Big(\frac{W''(u_\varepsilon)}{\varepsilon} - \lambda(\varepsilon)\, V''(u_\varepsilon)\Big)\, h^2 \ge \frac{1}{C}\int_{\mathbb{R}^n} \frac{h^2}{\varepsilon} \ge \frac{1}{C\,C_*}\int_{\mathbb{R}^n} \varepsilon\,|\nabla h|^2\,.$$

In both cases, (4.20) is easily deduced.

Step two: We prove (4.21). We argue by contradiction, and consider $\varepsilon_j \to 0^+$ as $j \to \infty$, $u_j \in \mathcal{R}_0^*$ minimizers of $\psi(\varepsilon_j)$, and radial functions $h_j \in H^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} V'(u_j)\, h_j = 0\,, \tag{4.22}$$

$$\int_{\mathbb{R}^n} 2\,\varepsilon_j\,|\nabla h_j|^2 + \Big(\frac{W''(u_j)}{\varepsilon_j} - \lambda_j\, V''(u_j)\Big)\, h_j^2 < \frac{1}{j}\int_{\mathbb{R}^n} \frac{h_j^2}{\varepsilon_j}\,. \tag{4.23}$$

where λ_j are the Lagrange multipliers corresponding to u_j. By homogeneity of (4.22) and (4.23) we can also assume that

$$\int_{\mathbb{R}^n} \frac{h_j^2}{\varepsilon_j} = 1\,. \tag{4.24}$$

Therefore, setting

$$\eta_j(s) = u_j(R_0 + \varepsilon_j\, s)\,, \qquad \beta_j(s) = h_j(R_0 + \varepsilon_j\, s)\,, \qquad s \ge -\frac{R_0}{\varepsilon_j}\,,$$

we can recast (4.23) and (4.24) as

$$\int_{-R_0/\varepsilon_j}^{\infty} \Big(2\,(\beta_j')^2 + (W''(\eta_j) - \varepsilon_j\,\lambda_j\, V''(\eta_j))\,\beta_j^2\Big)\,(R_0 + \varepsilon_j\, s)^{n-1}\, ds \le \frac{1}{j}\,, \tag{4.25}$$

$$\int_{-R_0/\varepsilon_j}^{\infty} \beta_j(s)^2\,(R_0 + \varepsilon_j\, s)^{n-1}\, ds = 1\,. \tag{4.26}$$

By $\varepsilon_j \to 0^+$ and by (2.3) we know $\lambda_j \to c(n)$ as $j \to \infty$, which combined with $\|V''\|_{C^0[0,1]} \le C$ and $\varepsilon_j \to 0^+$ shows that (4.25) and (4.26) imply

$$\limsup_{j\to\infty} \int_{-R_0/\varepsilon_j}^{\infty} \Big\{2\,(\beta_j')^2 + W''(\eta_j)\,\beta_j^2\Big\}\,(R_0+\varepsilon_j\,s)^{n-1}\,ds \le 0\,. \tag{4.27}$$

Since W'' is bounded on $[0,1]$, by (4.26) and (4.27) we deduce that $\{\beta_j\}_j$ is bounded in $H^1(-s_0,s_0)$ for every $s_0>0$. In particular there exists $\beta \in H^1_{\rm loc}(\mathbb{R})$ such that, up to extracting subsequences, β is the weak limit of $\{\beta_j\}_j$ in $H^1(-s_0,s_0)$ for every $s_0>0$. By $\beta_j' \rightharpoonup \beta'$ in $L^2(-s_0,s_0)$ for every $s_0>0$ we easily find

$$\liminf_{j\to\infty} \int_{-R_0/\varepsilon_j}^{\infty} 2\,\beta_j'(s)^2 (R_0+\varepsilon_j\,s)^{n-1}\,ds \ge R_0^{n-1}\int_{\mathbb{R}} 2\,(\beta')^2\,. \tag{4.28}$$

We now apply the concentration-compactness principle to the sequence of measures

$$\mu_j = 1_{(-R_0/\varepsilon_j,\infty)}(s)\,\beta_j(s)^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds\,,$$

which satisfy $\mu_j(\mathbb{R})=1$ thanks to (4.24). We claim that, if the compactness case hold, and thus

$$\lim_{s_0\to+\infty}\sup_j \mu_j(\mathbb{R}\setminus[-s_0,s_0]) = 0\,, \tag{4.29}$$

then we can reach a contradiction, and complete the proof of the lemma. To prove this claim, let us set

$$\eta_0(s) = \eta(s-\tau_0)\,,$$

for τ_0 as in (A.19), and let us notice that, for every $s_0>0$ we have

$$\begin{aligned}
&\limsup_{j\to\infty}\Big|\int_{-R_0/\varepsilon_j}^{\infty} W''(\eta_j)\beta_j(s)^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds - R_0^{n-1}\int_{\mathbb{R}} W''(\eta_0)\,\beta^2\Big|\\
\le\;& \limsup_{j\to\infty}\int_{-s_0}^{s_0}\Big|W''(\eta_j)\beta_j(s)^2\,(R_0+\varepsilon_j\,s)^{n-1} - R_0^{n-1}\,W''(\eta_0)\,\beta^2\Big| \qquad (4.30)\\
&+\|W''\|_{C^0[0,1]}\sup_{j\in\mathbb{N}}\mu_j(\mathbb{R}\setminus[-s_0,s_0]) + R_0^{n-1}\,\|W''\|_{C^0[0,1]}\int_{\mathbb{R}\setminus[-s_0,s_0]}\beta^2\,.
\end{aligned}$$

Since $\beta_j \to \beta$ in $L^2_{\rm loc}(\mathbb{R})$ and $\eta_j \to \eta_0$ locally uniformly on $\mathbb{R}$ thanks to Theorem 3.1, the first term on the right-hand side of (4.30) is equal to zero. Letting now $s_0\to\infty$, the second term goes to zero thanks to (4.29), while the third terms goes to zero thanks to the fact that (4.29) implies in particular

$$R_0^{n-1}\int_{\mathbb{R}}\beta^2 = 1\,. \tag{4.31}$$

We can combine this information with (4.28) and finally deduce from (4.27) that

$$\int_{\mathbb{R}} 2\,(\beta')^2 + W''(\eta_0)\,\beta^2 \le 0\,. \tag{4.32}$$

By Lemma 4.3 we deduce that, if we set $\beta_0(s)=\beta(s+\tau_0)$, then $\beta_0 = t\,\eta'$ for some $t\ne 0$ ($t=0$ being ruled out by (4.31)). In particular, $\beta = t\,\eta_0'$, and therefore

$$\int_{\mathbb{R}} V'(\eta_0)\,\beta = t\,V(\eta_0)|_{-\infty}^{+\infty} = t\,V(1) = t \ne 0\,.$$

However, by (4.22), we see that

$$0 = \int_{\mathbb{R}^n} V'(u_j)\,h_j = \int_{-R_0/\varepsilon_j}^{\infty} V'(\eta_j)\,\beta_j(s)\,(R_0+s\,\varepsilon_j)^{n-1}\,ds\,, \qquad \forall j\,,$$

and we can thus obtain a contradiction by showing that

$$\lim_{j\to\infty}\int_{-R_0/\varepsilon_j}^{\infty} V'(\eta_j)\,\beta_j(s)\,(R_0+s\,\varepsilon_j)^{n-1}\,ds = R_0^{n-1}\int_{\mathbb{R}} V'(\eta_0)\,\beta\,. \tag{4.33}$$

This is proved by noticing that (A.11), (A.16), (3.56) and (3.57) give

$$0\le \max\{V'(\eta_j),V'(\eta_0)\}\le C\,e^{-|s|/C}\,,$$

for every $s\in\mathbb{R}$ (or for every $s\ge -R_0/\varepsilon_j$, in the case of η_j). In particular,

$$\begin{aligned}
&\lim_{s_0\to\infty}\limsup_{j\to\infty}\Big[\int_{-R_0/\varepsilon_j}^{-s_0}+\int_{s_0}^{\infty}\Big]V'(\eta_j)\,|\beta_j|\,(R_0+s\,\varepsilon_j)^{n-1}\,ds\\
&\le C\lim_{s_0\to\infty}\limsup_{j\to\infty}\Big(\int_{\{|s|>s_0\}} e^{-|s|/C}\,(R_0+s\,\varepsilon_j)^{n-1}\,ds\Big)^{1/2}\mu_j(\mathbb{R}\setminus[-s_0,s_0])^{1/2}=0\,,
\end{aligned}$$

so that a similar argument to the one used in (4.30) can be repeated to prove (4.33).

We are thus left to prove that the sequence of probability measures $\{\mu_j\}_j$ cannot be in the vanishing case nor in the dichotomy case of the concentration-compactness principle.

To exclude that $\{\mu_j\}_j$ is in the vanishing case: Since $\eta_j\to\eta$ locally uniformly on $\mathbb{R}$, up to take j large enough and for S_0 as in (4.12) we have $W''(\eta_j(s))\ge 1/C$ for $|s|\ge S_0$, $s\ge -R_0/\varepsilon_j$. Since we are in the vanishing case, it holds

$$\lim_{j\to\infty}\int_{-S_0}^{S_0}\beta_j(s)^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds=0\,, \tag{4.34}$$

so that, by using first the lower bound on W'', and then (4.34), we get

$$\begin{aligned}
&\frac{1}{C}\limsup_{j\to\infty}\Big[\int_{-R_0/\varepsilon_j}^{-S_0}+\int_{S_0}^{\infty}\Big]\beta_j(s)^2\;(R_0+\varepsilon_j\,s)^{n-1}\,ds\\
\le\;&\limsup_{j\to\infty}\Big[\int_{-R_0/\varepsilon_j}^{-S_0}+\int_{S_0}^{\infty}\Big]W''(\eta_j)\,\beta_j(s)^2\;(R_0+\varepsilon_j\,s)^{n-1}\,ds\\
=\;&\limsup_{j\to\infty}\int_{-R_0/\varepsilon_j}^{\infty}W''(\eta_j)\,\beta_j(s)^2\;(R_0+\varepsilon_j\,s)^{n-1}\,ds\le 0
\end{aligned}$$

where in the last inequality we have used (4.27). Combining this information with (4.34) we obtain a contradiction to (4.26), thus excluding the vanishing case.

To exclude that $\{\mu_j\}_j$ is in the dichotomy case: With S_0 as above, if we are in the dichotomy case, then there exists $\alpha\in(0,1)$ such that for every $\tau\in(0,\alpha/2)$ there exist $R>S_0$ and $R_j\to\infty$ such that

$$|\mu_j(I_R)-(1-\alpha)|<\tau\,,\qquad |\mu_j(\mathbb{R}\setminus I_{R_j})-\alpha|<\tau\,,\qquad \forall j\,. \tag{4.35}$$

Setting $A_j=\varphi\,\beta_j$, $B_j=(1-\varphi)\,\beta_j$, where φ is a cut-off function between B_R and B_{R+1}, and setting for the sake of brevity,

$$Q_j(A,B)=\int_{-R_0/\varepsilon_j}^{\infty}\Big\{2\,A'\,B'+W''(\eta_j)\,A\,B\Big\}\,(R_0+\varepsilon_j\,s)^{n-1}\,ds\,,\qquad Q_j(A)=Q_j(A,A)\,,$$

we can rewrite (4.27) as

$$\limsup_{j\to\infty}Q_j(A_j)+Q_j(B_j)+2\,Q_j(A_j,B_j)\le 0\,, \tag{4.36}$$

Now, since φ' and $(1-\varphi)\,\varphi$ are supported in $I_{R+1}\setminus I_R$, we see that

$$
\begin{aligned}
Q_j(A_j,B_j) \;&\geq\; 2\int_{I_{R+1}\setminus I_R}(1-2\,\varphi)\,\varphi'\,\beta_j\,\beta_j'\,(R_0+\varepsilon_j\,s)^{n-1}\,ds\\
&\quad+\int_{I_{R+1}\setminus I_R}\Big\{W''(\eta_j)-(\varphi')^2\Big\}\,\beta_j^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds
\end{aligned}
$$

where, thanks to (4.27) and Hölder inequality,

$$
\int_{I_{R+1}\setminus I_R}(1-2\,\varphi)\,\varphi'\,\beta_j\,\beta_j'\,(R_0+\varepsilon_j\,s)^{n-1}\,ds\leq C\,\mu_j(I_{R+1}\setminus I_R)^{1/2}\leq C\,\sqrt{\tau}
$$

$$
\int_{I_{R+1}\setminus I_R}\Big\{W''(\eta_j)-(\varphi')^2\Big\}\,\beta_j^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds\leq C\,\mu_j(I_{R+1}\setminus I_R)\leq C\,\tau\,.
$$

We thus conclude that $Q_j(A_j,B_j)\geq -C\,\sqrt{\tau}$ for every j, and thus, by (4.36), that

$$
\limsup_{j\to\infty}Q_j(A_j)+Q_j(B_j)\leq C\,\sqrt{\tau}\,. \tag{4.37}
$$

Now, since the supports of the A_j's are uniformly bounded, we easily see that there exists $A\in H^1(\mathbb{R})$ such that $A_j\rightharpoonup A$ weakly in $H^1(\mathbb{R})$; in particular,

$$
\liminf_{j\to\infty}Q_j(A_j)\geq\int_{\mathbb{R}}2\,(A')^2+W''(\eta_0)\,A^2\geq 0\,,
$$

where in the last inequality we have used Lemma 4.3. By combining this last inequality with (4.37), $W''(\eta_j)\geq 1/C$ on $\mathbb{R}\setminus I_{S_0}$, and $R\geq S_0$, we conclude that

$$
C\,\sqrt{\tau}\geq\limsup_{j\to\infty}Q_j(B_j)\geq\frac{1}{C}\,\limsup_{j\to\infty}\int_{-R_0/\varepsilon_j}^{\infty}(1-\varphi)^2\,\beta_j^2\,(R+s\,\varepsilon_j)^{n-1}\,ds
$$

and thus, by (4.35), that $C\,\sqrt{\tau}\geq(\alpha/C)-C\,\tau$. Letting $\tau\to 0^+$ we obtain a contradiction with $\alpha>0$. □

Lemma 4.5 (Third stability lemma)**.** *If $n\geq 2$ and $W\in C^{2,1}[0,1]$ satisfies* (1.11) *and* (1.12)*, then there exist universal constants δ_0 and ε_0 such that, if $u_\varepsilon\in\mathcal{R}_0^*$ is a minimizer of $\psi(\varepsilon)$ for $\varepsilon<\varepsilon_0$ and $u\in H^1(\mathbb{R}^n;[0,1])$ is a radial function with*

$$
\int_{\mathbb{R}^n}V(u)=1\,, \tag{4.38}
$$

$$
\int_{\mathbb{R}^n}(u-u_\varepsilon)^2\leq C\,\varepsilon\,, \tag{4.39}
$$

$$
\|u-u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\leq\delta_0\,, \tag{4.40}
$$

then, setting $h=u-u_\varepsilon$,

$$
\int_{\mathbb{R}^n}2\,\varepsilon\,|\nabla h|^2+\Big(\frac{W''(u_\varepsilon)}{\varepsilon}-\lambda(\varepsilon)V''(u_\varepsilon)\Big)\,h^2\geq\frac{1}{C}\int_{\mathbb{R}^n}\varepsilon\,|\nabla h|^2+\frac{h^2}{\varepsilon}\,. \tag{4.41}
$$

where $\lambda(\varepsilon)$ is the Lagrange multiplier of u_ε as in (4.5)*.*

Proof. It will be convenient to set

$$
\begin{aligned}
P_\varepsilon(u,v)\;&=\;\int_{\mathbb{R}^n}\varepsilon\,\nabla u\cdot\nabla v+\frac{u\,v}{\varepsilon}\,,\\
Q_\varepsilon(u,v)\;&=\;\int_{\mathbb{R}^n}\varepsilon\,\nabla u\cdot\nabla v+\Big(\frac{W''(u_\varepsilon)}{\varepsilon}-\lambda(\varepsilon)\,V''(u_\varepsilon)\Big)\,u\,v\,,
\end{aligned}
$$

as well as $P_\varepsilon(u)=P_\varepsilon(u,u)$ and $Q_\varepsilon(u)=Q_\varepsilon(u,u)$. Let us start noticing that by Theorem 3.1 we have

$$
\lim_{\sigma\to 0}\sup_{\varepsilon<\sigma}\sup_{v_\varepsilon}\Big|\int_{\mathbb{R}^n}V'(v_\varepsilon)\,v_\varepsilon-R_0^{n-1}\int_{\mathbb{R}}V'(\eta)\,\eta\Big|=0\,,
$$

where v_ε runs over all radial minimizers of $\psi(\varepsilon)$. Since $\int_{\mathbb{R}} V'(\eta)\,\eta$ is a positive constant depending on n and W only this shows in particular that

$$\frac{1}{C} \le \int_{\mathbb{R}^n} V'(u_\varepsilon)\, u_\varepsilon \le C\,, \qquad \forall \varepsilon < \varepsilon_0\,. \tag{4.42}$$

By (4.42), given $h = u - u_\varepsilon$ as in the statement, we can always find $t \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} V'(u_\varepsilon)\,(h + t\,u_\varepsilon) = 0\,, \qquad \text{i.e.} \quad t = -\frac{\int_{\mathbb{R}^n} V'(u_\varepsilon)\,h}{\int_{\mathbb{R}^n} V'(u_\varepsilon)\,u_\varepsilon}\,. \tag{4.43}$$

By (A.12), (4.40), and since $0 \le u_\varepsilon + h \le 1$, we have that, on $\mathbb{R}^n$,

$$\Big|V(u_\varepsilon + h) - V(u_\varepsilon) - V'(u_\varepsilon)\,h - V''(u_\varepsilon)\,\frac{h^2}{2}\Big| \le C\,\delta_0\,h^2\,, \tag{4.44}$$

so that, by (4.38),

$$\Big|\int_{\mathbb{R}^n} V'(u_\varepsilon)\,h + V''(u_\varepsilon)\,\frac{h^2}{2}\Big| \le C\,\delta_0 \int_{\mathbb{R}^n} h^2\,, \tag{4.45}$$

and thus, thanks to $\|V''\|_{C^0[0,1]} \le C$, (4.42), (4.39), and (4.43),

$$|t| \le C \int_{\mathbb{R}^n} h^2 \le C\,\varepsilon\,\min\{P_\varepsilon(h), 1\}\,, \tag{4.46}$$

By (4.43) we can apply Lemma 4.4 to $u_\varepsilon + t\,h$ and find that

$$Q_\varepsilon(h + t\,u_\varepsilon) \ge \frac{P_\varepsilon(h + t\,u_\varepsilon)}{C}\,,$$

which can be more conveniently rewritten as

$$Q_\varepsilon(h) \ge \frac{P_\varepsilon(h)}{C} + 2\,t\left\{\frac{P_\varepsilon(h, u_\varepsilon)}{C} - Q_\varepsilon(h, u_\varepsilon)\right\} + t^2\left\{\frac{P_\varepsilon(u_\varepsilon)}{C} - Q_\varepsilon(u_\varepsilon)\right\}. \tag{4.47}$$

By Theorem 3.1, we see that $P_\varepsilon(u_\varepsilon) + |Q_\varepsilon(u_\varepsilon)| \le C$ (uniformly on $\varepsilon < \varepsilon_0$), so that (4.47) and (4.46) give

$$Q_\varepsilon(h) \ge \frac{P_\varepsilon(h)}{C} + 2\,t\left\{\frac{P_\varepsilon(h, u_\varepsilon)}{C} - Q_\varepsilon(h, u_\varepsilon)\right\}. \tag{4.48}$$

By Hölder inequality, $ab \le (a^2 + b^2)/2$, $P_\varepsilon(u_\varepsilon) \le C$, and (4.46) we see that

$$|t|\,P_\varepsilon(h, u_\varepsilon) \le \frac{|t|}{2}\big(P_\varepsilon(h) + P_\varepsilon(u_\varepsilon)\big) \le C\,\varepsilon\,P_\varepsilon(h)\,, \tag{4.49}$$

while by $|V'| + |W''| \le C$ and $|\lambda(\varepsilon)| \le C$ for $\varepsilon < \varepsilon_0$ we find, arguing as in (4.49),

$$|t|\,Q_\varepsilon(h, u_\varepsilon) \le |t|\left\{\varepsilon \int_{\mathbb{R}^n} |\nabla h|\,|\nabla u_\varepsilon| + \frac{C}{\varepsilon}\int_{\mathbb{R}^n} |h|\,u_\varepsilon\right\} \le C\,\varepsilon\,P_\varepsilon(h)\,. \tag{4.50}$$

By combining (4.48), (4.49), and (4.50) we conclude that $Q_\varepsilon(h) \ge P_\varepsilon(h)/C$, as desired. □

We are finally ready to prove Theorem 4.1.

Proof of Theorem 4.1. We are given u_ε and h as in Lemma 4.5, and now want to prove that

$$\mathcal{AC}_\varepsilon(u_\varepsilon + h) - \psi(\varepsilon) \ge \frac{1}{C}\int_{\mathbb{R}^n} \varepsilon\,|\nabla h|^2 + \frac{h^2}{\varepsilon}\,, \tag{4.51}$$

holds. By (A.5) and (4.40) we have that

$$\Big|W(u_\varepsilon + h) - W(u_\varepsilon) - W'(u_\varepsilon)\,h - W''(u_\varepsilon)\,\frac{h^2}{2}\Big| \le C\,\delta_0\,h^2\,, \qquad \text{on } \mathbb{R}^n\,,$$

therefore

$$\begin{aligned}\mathcal{AC}_\varepsilon(u_\varepsilon+h)-\mathcal{AC}_\varepsilon(u_\varepsilon) &\ge \int_{\mathbb{R}^n} 2\,\varepsilon\,\nabla u_\varepsilon\cdot\nabla h+\frac{W'(u_\varepsilon)}{\varepsilon}\,h\\ &\quad+\int_{\mathbb{R}^n}\varepsilon\,|\nabla h|^2+\frac{W''(u_\varepsilon)}{2\,\varepsilon}\,h^2-C\,\delta_0\int_{\mathbb{R}^n}h^2\,. \end{aligned}\tag{4.52}$$

By the Euler–Lagrange equation for u_ε, see (2.1), we have

$$\int_{\mathbb{R}^n} 2\,\varepsilon\,\nabla u_\varepsilon\cdot\nabla h+\frac{W'(u_\varepsilon)}{\varepsilon}\,h=\lambda(\varepsilon)\int_{\mathbb{R}^n}V'(u_\varepsilon)\,h\,. \tag{4.53}$$

Moreover, by (4.45),

$$\Big|\int_{\mathbb{R}^n}V'(u_\varepsilon)h+\int_{\mathbb{R}^n}V''(u_\varepsilon)\frac{h^2}{2}\Big|\le C\,\delta_0\int_{\mathbb{R}^n}h^2\,. \tag{4.54}$$

On combining (4.52), (4.53), and (4.54) with (4.41) we find that

$$\begin{aligned}\mathcal{AC}_\varepsilon(u_\varepsilon+h)-\psi(\varepsilon) &\ge \frac12\int_{\mathbb{R}^n}2\,\varepsilon\,|\nabla h|^2+\Big\{\frac1\varepsilon\,W''(u_\varepsilon)-\lambda(\varepsilon)\,V''(u_\varepsilon)\Big\}\,h^2-C\,\delta_0\int_{\mathbb{R}^n}h^2\\ &\ge \int_{\mathbb{R}^n}\varepsilon\,|\nabla h|^2+\frac{h^2}{\varepsilon}-C\,\delta_0\int_{\mathbb{R}^n}h^2\,,\end{aligned}$$

so that (4.51) follows by taking δ_0 small enough. □

5. Proof of the uniform stability theorem

In this section we prove Theorem 1.1-(iii), i.e., we prove (1.21). We focus directly on the case $(\sigma,m)=(\varepsilon,1)$, from which the general case follows immediately by scaling.

Theorem 5.1. *If $n\ge 2$ and $W\in C^{2,1}[0,1]$ satisfies (1.11) and (1.12), then there exist universal constants $\varepsilon_0>0$ and C such that, if $\varepsilon<\varepsilon_0$ and $u\in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n}V(u)=1$, then*

$$C\,\sqrt{\mathcal{AC}_\varepsilon(u)-\psi(\varepsilon)}\ge\inf_{x_0\in\mathbb{R}^n}\int_{\mathbb{R}^n}\big|\Phi(u)-\Phi(T_{x_0}u_\varepsilon)\big|^{n/(n-1)} \tag{5.1}$$

where $T_{x_0}u_\varepsilon(x)=u_\varepsilon(x-x_0)$, $x\in\mathbb{R}^n$, and u_ε denotes the unique minimizer of $\psi(\varepsilon)$ in $\mathcal{R}_0$.

In order to streamline the exposition of the proof of Theorem 5.1, we introduce the isoperimetric deficit and asymmetry of $u\in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n}V(u)=1$, by setting

$$\begin{aligned}\delta_\varepsilon(u)&=\mathcal{AC}_\varepsilon(u)-\psi(\varepsilon)\,,\\ \alpha_\varepsilon(u)&=\inf_{x_0\in\mathbb{R}^n}d_\Phi(u,T_{x_0}u_\varepsilon)\,.\end{aligned}$$

Here, as in Theorem 2.2,

$$d_\Phi(u,v)=\int_{\mathbb{R}^n}|\Phi(u)-\Phi(v)|^{n/(n-1)}\,,\qquad\forall u,v\in H^1(\mathbb{R}^n;[0,1])\,.$$

With this notation, Theorem 5.1 states the existence of universal constants C and ε_0 such that, if $\varepsilon<\varepsilon_0$, then

$$C\,\sqrt{\delta_\varepsilon(u)}\ge\alpha_\varepsilon(u)\,,\qquad\forall u\in H^1(\mathbb{R}^n;[0,1])\,,\int_{\mathbb{R}^n}V(u)=1\,. \tag{5.2}$$

In the following subsections we discuss some key steps of the proof of Theorem 5.1, which is then presented at the end of this section.

5.1. **Reduction to the small asymmetry case.** Thanks to the volume constraint $\int_{\mathbb{R}^n} V(u) = 1$ and to the triangular inequality in $L^{n/(n-1)}$, we always have $\alpha_\varepsilon(u) \le 2^{n/(n-1)}$. In particular, in proving (5.2), we can always assume that $\delta_\varepsilon(u) \le \delta_0$ for a universal constant δ_0. This is useful because, by the following lemma, by assuming $\delta_\varepsilon(u) \le \delta_0$ we can take $\alpha_\varepsilon(u)$ as small as needed in dependence of n and W.

Lemma 5.2 (ε-uniform qualitative stability). *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1.11) *and* (1.12)*, then there exists a universal constant ε_0 with the following property: for every $\alpha > 0$ there exists $\delta > 0$ such that*

$$u \in H^1(\mathbb{R}^n;[0,1])\,, \qquad \int_{\mathbb{R}^n} V(u) = 1\,, \qquad \varepsilon < \varepsilon_0\,, \qquad \delta_\varepsilon(u) \le \delta\,,$$

imply

$$\alpha_\varepsilon(u) \le \alpha\,.$$

Proof. We pick ε_0 such that Theorem 2.1 and Corollary 4.2 hold. If the lemma is false for such ε_0, then there exists $\alpha_* > 0$ and a sequence $\{u_j\}_j$ in $H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n} V(u_j) = 1$ such that

$$\delta_{\varepsilon_j}(u_j) \to 0^+ \qquad \text{as } j \to \infty\,, \tag{5.3}$$

for some $\varepsilon_j \to \varepsilon_* \in [0,\varepsilon_0]$ and with $\alpha_{\varepsilon_j}(u_j) \ge \alpha_*$. By (5.3), there is $\ell_j \to 0^+$ as $j \to \infty$ such that

$$\mathcal{AC}_{\varepsilon_j}(u_j) \le \psi(\varepsilon_j) + \ell_j\,, \qquad \forall j\,, \tag{5.4}$$

We now distinguish two cases:

Case one, $\varepsilon_* > 0$: In this case, by continuity of ψ (see Theorem 2.1) and since

$$\mathcal{AC}_{\varepsilon_*}(u_j) - \psi(\varepsilon_*) \le b_j\,\big(\mathcal{AC}_{\varepsilon_j}(u_j) - \psi(\varepsilon_j)\big) + b_j\,\psi(\varepsilon_j) - \psi(\varepsilon_*)\,, \qquad b_j = \max\Big\{\frac{\varepsilon_j}{\varepsilon_*}, \frac{\varepsilon_*}{\varepsilon_j}\Big\}\,,$$

we can assume that $\mathcal{AC}_{\varepsilon_*}(u_j) - \psi(\varepsilon_*) \le \ell_0$ for ℓ_0 as in step two of the proof of Theorem 2.1. We can thus apply that statement and conclude that, up to translations and up to subsequences, there is $u \in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n} V(u) = 1$ such that $d_\Phi(u_j,u) \to 0$ as $j \to \infty$. In particular, u is a minimizer of $\psi(\varepsilon_*)$, and therefore, up to a translation, we can assume that $u = u_{\varepsilon_*} \in \mathcal{R}_0$. Now, by repeating this same argument with the minimizers u_{ε_j} of $\psi(\varepsilon_j)$ in $\mathcal{R}_0$ in place of u_j, we see that

$$d_\Phi(u_{\varepsilon_j}, u_{\varepsilon_*}) \to 0 \qquad \text{as } j \to \infty\,,$$

so that, thanks to (2.59), we find the contradiction

$$\alpha_* \le \alpha_{\varepsilon_j}(u_j) \le d_\Phi(u_j, u_{\varepsilon_j}) \le d_\Phi(u_j, u_{\varepsilon_*}) + C\,d_\Phi(u_{\varepsilon_j}, u_{\varepsilon_*})^{(n-1)/n} \to 0^+\,,$$

as $j \to \infty$.

Case two, $\varepsilon_* = 0$: In this case, thanks to (5.4),

$$2\,|D[\Phi(u_j)]|(\mathbb{R}^n) \le \mathcal{AC}_{\varepsilon_j}(u_j) \le \psi(\varepsilon_j) + \ell_j \le 2\,n\,\omega_n^{1/n} + C\,\varepsilon_j + \ell_j\,,$$

so that $\{\Phi(u_j)\}_j$ is asymptotically optimal for the sharp BV-Sobolev inequality. By the concentration-compactness principle (see, e.g., [FMP07, Theorem A.1]), up to subsequences and up to translations, $\Phi(u_j) \to a\,1_{B_r}$ in $L^{n/(n-1)}(\mathbb{R}^n)$ as $j \to \infty$, for some a and r such that $a^{n/(n-1)}\,\omega_n\,r^n = 1$. The fact that $\mathcal{AC}_{\varepsilon_j}(v_j)$ is bounded implies that $v_j \to \{0,1\}$ a.e. on $\mathbb{R}^n$, therefore, by $\Phi(0) = 0$ and $\Phi(1) = 1$, it must be $a = 1$ and $R = R_0$ for $\omega_n\,R_0^n = 1$. By Theorem 3.1, if u_{ε_j} is a the minimizer of $\psi(\varepsilon_j)$ in $\mathcal{R}_0$, then

$$d_\Phi(u_{\varepsilon_j}, 1_{B_{R_0}}) \to 0 \qquad \text{as } j \to \infty\,,$$

which gives the contradiction

$$\alpha_* \le \alpha_{\varepsilon_j}(u_j) \le d_\Phi(u_j, u_{\varepsilon_j}) \le d_\Phi(u_j, 1_{B_{R_0}}) + C\,d_\Phi(u_{\varepsilon_j}, 1_{B_{R_0}})^{(n-1)/n} \to 0^+\,,$$

as $j \to \infty$. □

5.2. **Proof of Theorem 5.1 in the radial decreasing case.** We start by noticing that, thanks to the results proved in the previous sections, we can quickly prove Theorem 5.1 for functions in $\mathcal{R}_0$.

Theorem 5.3. *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies (1.11) and (1.12), then there exist universal constants C and ε_0 such that, for every $\varepsilon < \varepsilon_0$, denoting by u_ε the unique minimizer of $\psi(\varepsilon)$ in $\mathcal{R}_0$, one has*

$$C\sqrt{\delta_\varepsilon(u)} \ge d_\Phi(u, u_\varepsilon)\,, \tag{5.5}$$

whenever $u \in H^1(\mathbb{R}^n;[0,1]) \cap \mathcal{R}_0$ with $\int_{\mathbb{R}^n} V(u) = 1$.

Proof. Arguing by contradiction, we can find $\varepsilon_j \to 0^+$ and $\{v_j\}_j$ in $H^1(\mathbb{R}^n;[0,1]) \cap \mathcal{R}_0$ with

$$\int_{\mathbb{R}^n} V(v_j) = 1\,, \qquad a_j = \frac{\mathcal{AC}_{\varepsilon_j}(v_j) - \psi(\varepsilon_j)}{d_\Phi(v_j,u_j)^2} \to 0 \qquad \text{as } j \to \infty\,,$$

where $u_j = u_{\varepsilon_j}$ and, thanks to Lemma 5.2 and to $a_j \to 0^+$, we have

$$\lim_{j\to\infty} d_\Phi(v_j,u_j) = 0\,. \tag{5.6}$$

Correspondingly we consider the variational problems

$$\gamma_j = \gamma(\varepsilon_j, a_j, v_j) = \inf\Big\{\mathcal{AC}_{\varepsilon_j}(w) + a_j\, d_\Phi(w,v_j) : w \in H^1(\mathbb{R}^n;[0,1])\,, \int_{\mathbb{R}^n} V(w) = 1\Big\}\,.$$

With a_0, ℓ_0 and ε_0 as in Theorem 2.2, we notice that, for j large enough, we have $a_j \in (0,a_0)$, $\varepsilon_j < \varepsilon_0$, and

$$\mathcal{AC}_{\varepsilon_j}(v_j) \le \psi(\varepsilon_j) + a_j\,\ell_0\,, \qquad d_\Phi(v_j,u_j) \le \ell_0\,. \tag{5.7}$$

In particular we can apply Theorem 2.2, and deduce the existence of minimizers w_j of γ_j. We claim that, as $j \to \infty$,

$$\lim_{j\to\infty} \frac{\mathcal{AC}_{\varepsilon_j}(w_j) - \psi(\varepsilon_j)}{d_\Phi(w_j,u_j)^2} = 0\,. \tag{5.8}$$

To show this, we first notice that, by comparing w_j to u_j we have

$$\mathcal{AC}_{\varepsilon_j}(w_j) + a_j\, d_\Phi(w_j,v_j) \le \psi(\varepsilon_j) + a_j\, d_\Phi(u_j,v_j)\,,$$

so that (5.6) gives $\delta_{\varepsilon_j}(w_j) \to 0$, and then Lemma 5.2 implies

$$\lim_{j\to\infty} d_\Phi(w_j,u_j) = 0\,. \tag{5.9}$$

Next, comparing w_j to v_j we find that

$$\mathcal{AC}_{\varepsilon_j}(w_j) + a_j\, d_\Phi(w_j,v_j) \le \mathcal{AC}_{\varepsilon_j}(v_j)\,,$$

so that $\psi(\varepsilon_j) \le \mathcal{AC}_{\varepsilon_j}(w_j)$ and the definition of a_j give

$$d_\Phi(w_j,v_j) \le \frac{\mathcal{AC}_{\varepsilon_j}(v_j) - \psi(\varepsilon_j)}{a_j} = d_\Phi(v_j,u_j)^2\,. \tag{5.10}$$

By (2.59), (5.6), (5.9), and (5.10) we find

$$\begin{aligned}\big|d_\Phi(w_j,u_j) - d_\Phi(v_j,u_j)\big| &\le C\,\max\big\{d_\Phi(w_j,u_j), d_\Phi(v_j,u_j)\big\}^{1/n}\, d_\Phi(w_j,v_j)^{(n-1)/n}\\ &= \mathrm{o}\big(d_\Phi(v_j,u_j)^{2(n-1)/n}\big)\,,\end{aligned}$$

where $2(n-1)/n \ge 1$ thanks to $n \ge 2$. Thus, $d_\Phi(w_j,u_j) \ge d_\Phi(v_j,u_j)/C$ for j large enough, and $\mathcal{AC}_{\varepsilon_j}(w_j) \le \mathcal{AC}_{\varepsilon_j}(v_j)$ gives

$$\frac{\mathcal{AC}_{\varepsilon_j}(w_j) - \psi(\varepsilon_j)}{d_\Phi(w_j,u_j)^2} \le C\,\frac{\mathcal{AC}_{\varepsilon_j}(v_j) - \psi(\varepsilon_j)}{d_\Phi(v_j,u_j)^2} \to 0^+\,,$$

as claimed in (5.8).

We now derive a contradiction to (5.8). By Theorem 2.2, we know that $w_j \in \mathcal{R}_0^* \cap C^{2,1/(n-1)}_{\rm loc}(\mathbb{R}^n)$, $0 < w_j < 1$ on $\mathbb{R}^n$, and

$$-2\,\varepsilon_j^2\,\Delta w_j = \varepsilon_j\,w_j\,(1-w_j)\,{\rm E}_j - W'(w_j) \qquad \text{on } \mathbb{R}^n\,, \tag{5.11}$$

where ${\rm E}_j$ is a continuous radial function on $\mathbb{R}^n$ with

$$\sup_{\mathbb{R}^n} |{\rm E}_j| \le C\,. \tag{5.12}$$

We can thus apply Theorem 3.1 to w_j. In particular, since both u_j and w_j obey the resolution formula (3.8), we have that $h_j = w_j - u_j$ satisfies

$$|h_j(R_0 + \varepsilon_j\,s)| \le C\,\varepsilon_j\,e^{-|s|/C} \qquad \forall s \ge -\frac{R_0}{\varepsilon_j}\,. \tag{5.13}$$

In particular,

$$\|h_j\|_{L^\infty(\mathbb{R}^n)} \le C\,\varepsilon_j\,, \qquad \int_{\mathbb{R}^n} h_j^2 \le C\,\varepsilon_j\,.$$

and we can thus apply Theorem 4.1 to deduce

$$\begin{aligned}\mathcal{AC}_{\varepsilon_j}(w_j) - \psi(\varepsilon_j) \;&\ge\; \frac{1}{C}\int_{\mathbb{R}^n} \varepsilon_j\,|\nabla h_j|^2 + \frac{h_j^2}{\varepsilon_j}\\ &\ge\; \frac{1}{C}\int_{\mathbb{R}^n} |\nabla(h_j^2)| \ge \frac{1}{C}\Big(\int_{\mathbb{R}^n}|h_j|^{2n/(n-1)}\Big)^{(n-1)/n}\,,\end{aligned} \tag{5.14}$$

where we have also used the BV-Sobolev inequality. By (5.13), and by applying (3.14) to u_j in combination with (A.6), we find that, if $A_j = B_{R_0+c_j} \setminus B_{R_0-b_j}$, then, for every $x \in \mathbb{R}^n \setminus A_j$ we have

$$|\Phi(u_j(x)) - \Phi(w_j(x))| \le |h_j(x)|\int_0^1 \sqrt{W(u_j(x) + t\,h_j(x))}\,dt \le C\,|h_j(x)|\,e^{-||x|-R_0|/C\,\varepsilon_j}\,,$$

and, therefore,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus A_j} |\Phi(u_j) - \Phi(w_j)|^{n/(n-1)} \;&\le\; C\int_{\mathbb{R}^n\setminus A_j} |h_j|^{n/(n-1)}\,e^{-||x|-R_0|/C\,\varepsilon_j}\\ &\le\; C\,\sqrt{\varepsilon_j}\,\Big(\int_{\mathbb{R}^n}|h_j|^{2\,n/(n-1)}\Big)^{1/2}.\end{aligned} \tag{5.15}$$

If, instead, $x \in A_j$, then by $|\Phi(u_j) - \Phi(w_j)| \le C\,|h_j|$ and $\mathcal{L}^n(A_j) \le C\,\varepsilon_j$ we find

$$\int_{A_j} |\Phi(u_j) - \Phi(w_j)|^{n/(n-1)} \le C\,\sqrt{\varepsilon_j}\,\Big(\int_{\mathbb{R}^n}|h_j|^{2\,n/(n-1)}\Big)^{1/2}. \tag{5.16}$$

By combining (5.14), (5.15) and (5.16), and thanks to $\varepsilon_j \le 1$, $n/(n-1) \ge 1$, and $\delta_{\varepsilon_j}(w_j) \le 1$, we conclude that

$$d_\Phi(u_j, w_j) \le C\,\sqrt{\varepsilon_j}\,\delta_{\varepsilon_j}(w_j)^{n/2\,(n-1)} \le C\,\sqrt{\delta_{\varepsilon_j}(w_j)}\,,$$

in contradiction to (5.8). $\square$

Remark 5.1. The argument we have just presented provides further indication that (5.5) should not provide a sharp rate on radial decreasing functions. The sharp stability estimate on small radial perturbations of u_ε is clearly given in Theorem 4.1, but it is not clear what form the sharp stability estimate should take on $\mathcal{R}_0$ (or, more generally, on arbitrary radial functions).

5.3. **Reduction to radial decreasing functions.** We now discuss the reduction of (5.2) to the case of radial decreasing functions. We do this by adapting to our setting the "quantitative symmetrization" strategy developed in [FMP08, FMP07] in the study of Euclidean isoperimetry.

Given $n \geq 2$ and $k \in \{1,...,n\}$ we say that $u : \mathbb{R}^n \to \mathbb{R}$ is **k-symmetric** if there exists k mutually orthogonal hyperplanes such that u is symmetric by reflection through each of these hyperplanes. The class of n-symmetric functions is particularly convenient when it comes to quantify sharp inequalities involving radial decreasing rearrangements. Consider for example the Pólya-Szegö inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 \geq \int_{\mathbb{R}^n} |\nabla u^*|^2 \,, \tag{5.17}$$

where u^* is the radial decreasing rearrangement of u. A classical result of Brothers and Ziemer [BZ88] shows that equality can hold in (5.17) without u being a translation of u^*; in general, the additional condition that $(u^*)' < 0$ a.e. must be assumed to deduce symmetry from equality in (5.17) (compare with step six in the proof of Theorem 2.1). However, if u is n-symmetric, then equality in (5.17) automatically implies that u is radial decreasing. A quantitative version of this statement is proved in [FMP07, Theorem 2.2] in the BV-case of (5.17), and in [CFMP09, Theorem 3] in the Sobolev case. The following theorem is an adaptation of those results to our setting.

Theorem 5.4 (Reduction from n-symmetric to radial decreasing functions)**.** *If $n \geq 2$ and $W \in C^{2,1}[0,1]$ satisfies (1.11) and (1.12), then there exists a universal constant C with the following property. If $u \in H^1(\mathbb{R}^n;[0,1])$ is a n-symmetric function with $\int_{\mathbb{R}^n} V(u) = 1$ and u^* is its radial decreasing rearrangement, then*

$$d_\Phi(u,u^*) \leq C \Big(\int_{\mathbb{R}^n} W(u) \Big)^{1/2} \Big(\int_{\mathbb{R}^n} |\nabla u|^2 - \int_{\mathbb{R}^n} |\nabla u^*|^2 \Big)^{1/2} . \tag{5.18}$$

Moreover, for every $\varepsilon > 0$ we have

$$\alpha_\varepsilon(u) \leq C \Big(\alpha_\varepsilon(u^*) + \big(\mathcal{AC}_\varepsilon(u)\, \delta_\varepsilon(u) \big)^{1/2} \Big) . \tag{5.19}$$

Proof. We first claim that

$$d_\Phi(u,u^*) \;\leq\; \frac{n}{n-1} \int_0^1 \mathcal{L}^n(E_t) \Phi(t)^{1/(n-1)} \sqrt{W(t)}\, dt \,, \tag{5.20}$$

$$\int_{\mathbb{R}^n} |\nabla u|^2 - \int_{\mathbb{R}^n} |\nabla u^*|^2 \;\geq\; \frac{1}{C(n)} \int_0^1 \Big(\frac{\mathcal{L}^n(E_t)}{\mu(t)} \Big)^2 \frac{\mu(t)^{2\,(n-1)/n}}{-\mu'(t)}\, dt \,, \tag{5.21}$$

where $E_t = \{u > t\}\Delta\{u^* > t\}$, $\mu(t) = \mathcal{L}^n(\{u > t\})$, and $\mu'(t)$ denotes the absolutely continuous part of the distributional derivative of the decreasing function μ. To prove (5.20) we recall that, by [CFMP09, Lemma 5], we have

$$d_\Phi(u,u^*) \leq \frac{n}{n-1} \int_0^1 \mathcal{L}^n(F_s)\, s^{1/(n-1)}\, ds \,,$$

provided $F_s = \{\Phi(u) > s\}\Delta\{\Phi(u^*) > s\}$. Since Φ is strictly increasing, we have $F_{\Phi(t)} = E_t$, so that the change of variables $s = \Phi(t)$ gives (5.20). To prove (5.21) we just notice that this is [CFMP09, Equation (3.18)]. Now, by Hölder inequality and (5.20), we find that

$$\begin{aligned} \int_0^1 \mathcal{L}^n(E_s)\, \Phi^{1/(n-1)} \sqrt{W} \;&=\; \int_0^1 \frac{\mathcal{L}^n(E_s)}{\mu} \frac{\mu^{(n-1)/n}}{(-\mu')^{1/2}} \frac{(-\mu')^{1/2}}{\mu^{-1/n}} \Phi^{1/(n-1)} \sqrt{W} \\ &\leq\; \Big(\int_0^1 \Big(\frac{\mathcal{L}^n(E_s)}{\mu} \Big)^2 \frac{\mu^{2\,(n-1)/n}}{-\mu'} \Big)^{1/2} \Big(\int_0^1 \frac{-\mu'}{\mu^{-2/n}} \Phi^{2/(n-1)}\, W \Big)^{1/2} . \end{aligned}$$

By $1 = \int_{\mathbb{R}^n} V(u) \geq V(t)\,\mu(t)$ for every $t \in (0,1)$, we have

$$\int_0^1 \frac{-\mu'}{\mu^{-2/n}}\,\Phi^{2/(n-1)}\,W \leq \int_0^1 -\mu'\,\big(V\,\mu\big)^{2/n}\,W \leq \int_0^1 -\mu'\,W \leq \int_{\mathbb{R}^n} W(u)\,,$$

where in the last inequality we have used $-\mu'\,d\mathcal{L}^1 \leq -D\mu$, integration by parts and Fubini's theorem to deduce

$$-\int_0^1 W\,d[D\mu] = \int_0^1 W'(t)\,\mu(t)\,dt = \int_{\mathbb{R}^n} dx \int_0^{u(x)} W'(t)\,dt = \int_{\mathbb{R}^n} W(u)\,.$$

By combining (5.20), (5.21) and these estimates we find (5.18). To prove (5.19), we notice that, by $\int_{\mathbb{R}^n} W(u) = \int_{\mathbb{R}^n} W(u^*)$ and $\int_{\mathbb{R}^n} V(u^*) = 1$, (5.18) gives

$$d_\Phi(u,u^*) \leq C\,\mathcal{AC}_\varepsilon(u)^{1/2}\Big(\mathcal{AC}_\varepsilon(u) - \mathcal{AC}_\varepsilon(u^*)\Big)^{1/2} \leq C\,\mathcal{AC}_\varepsilon(u)^{1/2}\,\delta_\varepsilon(u)^{1/2} \tag{5.22}$$

and then (5.19) follows by the triangular inequality in $L^{n/(n-1)}(\mathbb{R}^n)$. □

Next we discuss the reduction from generic functions to n-symmetric ones.

Theorem 5.5 (Reduction to n-symmetric functions)**.** *If $n \geq 2$ and $W \in C^{2,1}[0,1]$ satisfies (1.11) and (1.12), then there exist universal constants ε_0 and δ_0 with the following property. If $u \in H^1(\mathbb{R}^n;[0,1])$, $\int_{\mathbb{R}^n} V(u) = 1$ and $\delta_\varepsilon(u) \leq \delta_0$ for some $\varepsilon < \varepsilon_0$, then there exists $v \in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n} V(v) = 1$ such that v is n-symmetric and*

$$\alpha_\varepsilon(u) \leq C\,\alpha_\varepsilon(v)\,, \qquad \delta_\varepsilon(v) \leq C\,\delta_\varepsilon(u)\,. \tag{5.23}$$

Proof. Without loss of generality we can assume that $\delta_\varepsilon(u) \leq \delta_0$ for a universal constant δ_0. By Lemma 5.2 we can choose δ_0 so that $\alpha_\varepsilon(u) \leq \alpha_0$ for α_0 a universal constant of our choice. We divide the proof into a few steps.

Step one: We prove that, if u is k-symmetric, $\{H_i\}_{i=1}^k$ are the mutually orthogonal hyperplanes of symmetry of u, and $J = \bigcap_{i=1}^k H_i$, then

$$\alpha_\varepsilon(u;J) = \inf_{x\in J} d_\Phi(u, T_x u_\varepsilon) \leq C(n)\,\alpha_\varepsilon(u)\,. \tag{5.24}$$

In other words, in computing the asymmetry of u in the proof of an estimate like (5.2), we can compare u with a translation of u_ε with maximum on J.

Indeed, let $x_0 \in \mathbb{R}^n$ be such that $\alpha_\varepsilon(u) = d_\Phi(u, T_{x_0}u_\varepsilon)$. Without loss of generality, we can assume $x_0 \notin J$. In particular, if y_0 denotes the reflection of x_0 with respect to J, then $y_0 \neq x_0$ and

$$d_\Phi(u, T_{y_0}u_\varepsilon) = d_\Phi(u, T_{x_0}u_\varepsilon) = \alpha_\varepsilon(u)\,, \tag{5.25}$$

that is, also y_0 is an optimal center for computing $\alpha_\varepsilon(u)$. Let $z_0 = (x_0+y_0)/2$, so that $z_0 \in J$, let $\nu = (x_0-y_0)/|x_0-y_0|$ (which is well defined by $x_0 \neq y_0$), and let H be the open half-space orthogonal to ν, containing x_0, and such that $z_0 \in \partial H$. By $T_{z_0+t\,\nu}u_\varepsilon(x) = u_\varepsilon(x - z_0 - t\,\nu)$, we have that

$$\frac{d}{dt} T_{z_0+t\,\nu}u_\varepsilon(x) = -\nu\cdot\frac{x-z_0-t\,\nu}{|x-z_0-t\,\nu|}\,u_\varepsilon'(|x-x_0-t\,\nu|) > 0\,, \qquad \forall x \in H\,, t<0\,,$$

since $u_\varepsilon' < 0$, and since the fact that ν points inside H gives

$$\begin{aligned}
&(z-z_0)\cdot\nu > 0\,, \qquad \forall z \in H\,,\\
&z = x - t\,\nu \in H\,, \qquad \forall x \in H\,, t<0\,.
\end{aligned}$$

We thus find that, if $t<0$,

$$\begin{aligned}
&\frac{d}{dt}\int_H |\Phi(T_{x_0}u) - \Phi(T_{z_0+t\,\nu}u_\varepsilon)|^{n/(n-1)}\\
=\;& \frac{n}{n-1}\int_H |\Phi(u) - \Phi(T_{z_0+t\,\nu}u_\varepsilon)|^{1/(n-1)}\sqrt{W(T_{z_0+t\,\nu}u_\varepsilon)}\,\frac{d}{dt}T_{z_0+t\,\nu}u_\varepsilon > 0\,,
\end{aligned}$$

so that

$$
\begin{aligned}
\int_H |\Phi(T_{x_0}u)-\Phi(T_{y_0}u_\varepsilon)|^{n/(n-1)} &= \int_{H^-} |\Phi(T_{x_0}u)-\Phi(T_{z_0+t\,\nu}u_\varepsilon)|^{n/(n-1)}\Big|_{t=-|x_0-y_0|/2}\\
&\le \int_H |\Phi(T_{x_0}u)-\Phi(T_{z_0+t\,\nu}u_\varepsilon)|^{n/(n-1)}\Big|_{t=0}\\
&\le \int_H |\Phi(T_{x_0}u)-\Phi(T_{z_0}u_\varepsilon)|^{n/(n-1)} \qquad (5.26)
\end{aligned}
$$

Now, since both u and $T_{z_0}u_\varepsilon$ are symmetric by reflection with respect to ∂H, we have that

$$
\int_{\mathbb{R}^n} |\Phi(u)-\Phi(T_{z_0}u_\varepsilon)|^{n/(n-1)} = 2\int_H |\Phi(u)-\Phi(T_{z_0}u_\varepsilon)|^{n/(n-1)}\,, \qquad (5.27)
$$

therefore, by (5.25), (5.26) and (5.27) we conclude that

$$
\begin{aligned}
\alpha_\varepsilon(u;J) &\le d_\Phi(u,T_{z_0}u_\varepsilon) = 2\int_H |\Phi(u)-\Phi(T_{z_0}u_\varepsilon)|^{n/(n-1)}\\
&\le C(n)\Big(\int_H |\Phi(u)-\Phi(T_{x_0}u_\varepsilon)|^{n/(n-1)} + \int_H |\Phi(T_{x_0}u_\varepsilon)-\Phi(T_{z_0}u_\varepsilon)|^{n/(n-1)}\Big)\\
&\le C(n)\Big(\alpha_\varepsilon(u) + \int_H |\Phi(T_{y_0}u_\varepsilon)-\Phi(T_{x_0}u_\varepsilon)|^{n/(n-1)}\Big)\\
&\le C(n)\Big(\alpha_\varepsilon(u) + d_\Phi(T_{y_0}u_\varepsilon,T_{x_0}u_\varepsilon)\Big)\\
&\le C(n)\Big(\alpha_\varepsilon(u) + d_\Phi(T_{y_0}u_\varepsilon,u) + d_\Phi(u,T_{x_0}u_\varepsilon)\Big) = C(n)\,\alpha_\varepsilon(u)\,,
\end{aligned}
$$

that is (5.24).

Step two: Let H_1 and H_2 be two orthogonal hyperplanes through the origin, let $H_i^\pm$ be the half-spaces defined by H_i, and let $x_i^\pm \in \partial H_i$. For $i=1,2$, consider the functions

$$
U[u_\varepsilon,H_i,x_i^+,x_i^-] = 1_{H_i^+}\,T_{x_i^+}u_\varepsilon + 1_{H_i^-}\,T_{x_i^-}u_\varepsilon\,,
$$

obtained by "gluing" the restriction of u_ε to H_i^+ translated by x_1^+ to the restriction of u_ε to H_i^- translated by x_1^+ (notice that translating by $x_i^\pm$ brings H_i^+ and H_i^- into themselves). Setting for brevity

$$
U_{\varepsilon,i} = U[u_\varepsilon,H_i,x_i^+,x_i^-]
$$

we claim that, for every $a\in(0,1)$ there is $\kappa=\kappa(a,n,W)>0$ such that if

$$
\max\Big\{|x_1^+-x_1^-|,|x_2^+-x_2^-|,|x_1^+-x_2^+|\Big\} \le \kappa\,, \qquad (5.28)
$$

then, for every $\varepsilon<\varepsilon_0$,

$$
\max\Big\{d_\Phi(T_{x_1^+}u_\varepsilon,T_{x_1^-}u_\varepsilon),d_\Phi(T_{x_2^+}u_\varepsilon,T_{x_2^-}u_\varepsilon)\Big\} \le \frac{8}{1-a}\,d_\Phi(U_{\varepsilon,1},U_{\varepsilon,2})\,. \qquad (5.29)
$$

Indeed, since H_1 and H_2 are hyperplanes through the origin and $u_\varepsilon\in\mathcal{R}_0$, we have

$$
\int_{H_1^\pm} V(T_{x_1^\pm}u_\varepsilon) = \frac12\,, \qquad \int_{H_2^\pm} V(T_{x_2^\pm}u_\varepsilon) = \frac12\,.
$$

It is in general not true that, say, $H_1^+\cap H_2^+$ has measure $1/4$ for either $V(T_{x_1^\pm}u_\varepsilon)\,dx$ or $V(T_{x_1^\pm}u_\varepsilon)\,dx$. However, provided we choose κ sufficiently small, thanks to Theorem 3.1, we can definitely ensure that, for every $\varepsilon<\varepsilon_0$ and $\beta,\gamma\in\{+,-\}$, we have

$$
\int_{H_1^\beta\cap H_2^\gamma} |\Phi(T_{x_1^\beta}u_\varepsilon)-\Phi(T_{x_2^\gamma}u_\varepsilon)|^{n/(n-1)} \ge \frac{1-a}{4}\,d_\Phi(T_{x_1^\beta}u_\varepsilon,T_{x_2^\gamma}u_\varepsilon)\,.
$$

Correspondingly,

$$
\begin{aligned}
d_\Phi(U_{\varepsilon,1},U_{\varepsilon,2}) &\ge \int_{H_1^\beta\cap H_2^\gamma}|\Phi(U_{\varepsilon,1})-\Phi(U_{\varepsilon,2})|^{n/(n-1)}\\
&= \int_{H_1^\beta\cap H_2^\gamma}|\Phi(T_{x_1^\beta}u_\varepsilon)-\Phi(T_{x_2^\gamma}u_\varepsilon)|^{n/(n-1)}\ge\frac{1-a}{4}\,d_\Phi(T_{x_1^\beta}u_\varepsilon,T_{x_2^\gamma}u_\varepsilon)\,,
\end{aligned}
$$

and thus

$$
\begin{aligned}
d_\Phi(T_{x_1^+}u_\varepsilon,T_{x_1^-}u_\varepsilon)^{(n-1)/n} &\le d_\Phi(T_{x_1^+}u_\varepsilon,T_{x_2^+}u_\varepsilon)^{(n-1)/n}+d_\Phi(T_{x_2^+}u_\varepsilon,T_{x_1^-}u_\varepsilon)^{(n-1)/n}\\
&\le \Big(\frac{8}{1-a}\,d_\Phi(U_{\varepsilon,1},U_{\varepsilon,2})\Big)^{(n-1)/n}\,,
\end{aligned}
$$

as claimed.

Step three: Given $u\in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n}V(u)=1$, we now consider an hyperplane H such that, if H^+ and H^- denote the two open half-spaces defined by H, then

$$
\int_{H^+}V(u)=\int_{H^-}V(u)=\frac{1}{2}\,.
$$

Denoting by ρ_H the reflection with respect to H, we let

$$
u^+=1_{H^+}\,u+1_{H^-}\,(u\circ\rho_H)\,,\qquad u^-=1_{H^-}\,u+1_{H^+}\,(u\circ\rho_H)\,,\tag{5.30}
$$

and notice that $u^\pm\in H^1(\mathbb{R}^n;[0,1])$, with

$$
2\,\mathcal{AC}_\varepsilon(u)=\mathcal{AC}_\varepsilon(u^+)+\mathcal{AC}_\varepsilon(u^-)\,,\qquad \int_{\mathbb{R}^n}V(u^+)=\int_{\mathbb{R}^n}V(u^-)=1\,.\tag{5.31}
$$

We claim that

$$
\max\{\delta_\varepsilon(u^+),\delta_\varepsilon(u^-)\}\le 2\,\delta_\varepsilon(u)\,,\qquad \alpha_\varepsilon(u)\le C(n)\Big\{\alpha_\varepsilon(u^+)+\alpha_\varepsilon(u^-)+d_\Phi(T_{x^+}u_\varepsilon,T_{x^-}u_\varepsilon)\Big\}\,,\tag{5.32}
$$

provided $T_{x^+}u_\varepsilon=$ and $T_{x^-}u_\varepsilon=T_{x^-}u_\varepsilon$ are such that $x^+,x^-\in H$ with

$$
\alpha_\varepsilon(u^+;H)=d_\Phi(u^+,T_{x^+}u_\varepsilon)\,,\qquad \alpha_\varepsilon(u^-;H)=d_\Phi(u^-,T_{x^-}u_\varepsilon)\,.
$$

The first inequality in (5.32) is obvious from (5.31). To prove the second one we notice that

$$
\begin{aligned}
\alpha_\varepsilon(u) &\le d_\Phi(u,T_{x^+}u_\varepsilon)=\int_{H^+}|\Phi(u)-\Phi(T_{x^+}u_\varepsilon)|^{n/(n-1)}+\int_{H^-}|\Phi(u)-\Phi(T_{x^+}u_\varepsilon)|^{n/(n-1)}\\
&= \int_{H^+}|\Phi(u^+)-\Phi(T_{x^+}u_\varepsilon)|^{n/(n-1)}+\int_{H^-}|\Phi(u^-)-\Phi(T_{x^+}u_\varepsilon)|^{n/(n-1)}\\
&\le C(n)\Big\{d_\Phi(u^+,T_{x^+}u_\varepsilon)+d_\Phi(u^-,T_{x^-}u_\varepsilon)+d_\Phi(T_{x^-}u_\varepsilon,T_{x^+}u_\varepsilon)\Big\}\,,
\end{aligned}
$$

that is the second inequality in (5.32).

With these preliminary considerations in place, we now prove that if $u\in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n}V(u)=1$, if H_1 and H_2 are orthogonal hyperplanes such that the corresponding half-spaces $H_i^\pm$ satisfy

$$
\int_{H_i^\pm}V(u)=\frac{1}{2}\,,
$$

if $u_i^\pm$ as in (5.30) starting from H_i, then there is at least one $v\in\{u_1^+,u_1^-,u_2^+,u_2^-\}$ such that (5.23) holds. Given that $\delta_\varepsilon(v)\le 2\,\delta_\varepsilon(u)$ for every $v\in\{u_1^+,u_1^-,u_2^+,u_2^-\}$, we need to show that

$$
\exists v\in\{u_1^+,u_1^-,u_2^+,u_2^-\}\text{ such that }\alpha_\varepsilon(u)\le C\,\alpha_\varepsilon(v)\,.\tag{5.33}
$$

Denoting by $x_i^\pm$ the points in H_i such that

$$
\alpha_\varepsilon(u_i^\pm;H_i)=d_\Phi(u_i^\pm,T_{x_i^\pm}u_\varepsilon)\,,
$$

we notice that (5.33) follows if we can show that, provided α_0 is small enough, then

$$\text{either} \qquad d_\Phi(T_{x_1^+}u_\varepsilon, T_{x_1^-}u_\varepsilon) \le M\left\{\alpha_\varepsilon(u_1^+;H_1)+\alpha_\varepsilon(u_1^-;H_1)\right\} \tag{5.34}$$

$$\text{or} \qquad d_\Phi(T_{x_2^+}u_\varepsilon, T_{x_2^-}u_\varepsilon) \le M\left\{\alpha_\varepsilon(u_2^+;H_2)+\alpha_\varepsilon(u_2^-;H_2)\right\}, \tag{5.35}$$

for a constant M (as it turns out, any $M>16$ works). Indeed, if, for example, (5.34) holds, then (5.24) and (5.32) with $H=H_1$ give

$$\alpha_\varepsilon(u) \le C\left\{\alpha_\varepsilon(u_1^+)+\alpha_\varepsilon(u_1^-)+\alpha_\varepsilon(u_1^+;H_1)+\alpha_\varepsilon(u_1^-;H_1)\right\} \le C\left\{\alpha_\varepsilon(u_1^+)+\alpha_\varepsilon(u_1^-)\right\},$$

and then either $C\,\alpha_\varepsilon(u_1^+)\ge\alpha_\varepsilon(u)$ or $C\,\alpha_\varepsilon(u_2^+)\ge\alpha_\varepsilon(u)$; in particular, (5.33) holds. We now want to prove that either (5.34) or (5.35) holds. We argue by contradiction. Recalling that $\alpha_\varepsilon(u_i^\pm;H_i)=d_\Phi(u_i^\pm,T_{x_i^\pm}u_\varepsilon)$, let us thus assume that both

$$d_\Phi(T_{x_1^+}u_\varepsilon, T_{x_1^-}u_\varepsilon) > M\left\{d_\Phi(u_1^+,T_{x_1^+}u_\varepsilon)+d_\Phi(u_1^-,T_{x_1^-}u_\varepsilon)\right\}, \tag{5.36}$$

$$d_\Phi(T_{x_2^+}u_\varepsilon, T_{x_2^-}u_\varepsilon) > M\left\{d_\Phi(u_2^+,T_{x_2^+}u_\varepsilon)+d_\Phi(u_2^-,T_{x_2^-}u_\varepsilon)\right\}, \tag{5.37}$$

hold for M to be determined. In particular, if $U_{\varepsilon,i}$, $i=1,2$, are defined as in step two, and α_0 is small enough that (5.28) holds, then, by (5.29), we have

$$\begin{aligned}
&\max\left\{d_\Phi(T_{x_1^+}u_\varepsilon,T_{x_1^-}u_\varepsilon),d_\Phi(T_{x_2^+}u_\varepsilon,T_{x_2^-}u_\varepsilon)\right\}^{(n-1)/n}\\
\le\ &\Big(\frac{8}{1-a}\,d_\Phi(U_{\varepsilon,1},U_{\varepsilon,2})\Big)^{(n-1)/n} \le \Big(\frac{8}{1-a}\Big)^{(n-1)/n}\sum_{i=1}^2 d_\Phi(U_{\varepsilon,i},u)^{(n-1)/n}\\
=\ &\Big(\frac{8}{1-a}\Big)^{(n-1)/n}\sum_{i=1}^2\Big(\sum_{\beta=+,-}\int_{H_i^\beta}|\Phi(T_{x_i^\beta}u_\varepsilon)-\Phi(u_i^\beta)|^{n/(n-1)}\Big)^{(n-1)/n}\\
\le\ &\Big(\frac{8}{M\,(1-a)}\Big)^{(n-1)/n}\sum_{i=1}^2\Big(d_\Phi(T_{x_i^+}u_\varepsilon,T_{x_i^-}u_\varepsilon)\Big)^{(n-1)/n}\\
\le\ &\Big(\frac{16}{M\,(1-a)}\Big)^{(n-1)/n}\max\left\{d_\Phi(T_{x_1^+}u_\varepsilon,T_{x_1^-}u_\varepsilon),d_\Phi(T_{x_2^+}u_\varepsilon,T_{x_2^-}u_\varepsilon)\right\}^{(n-1)/n}.
\end{aligned}$$

We fix $M>16$ and apply the above with $a\in(0,1)$ such that $M\,(1-a)>16$. We find that either $x_1^+=x_1^-$ (a contradiction to (5.36)), or $x_2^+=x_2^-$ (a contradiction to (5.37)).

Step four: We now pick a family of n mutually orthogonal hyperplanes $\{H_i\}_{i=1}^n$ such that, denoting by $H_i^\pm$ the corresponding half-spaces, we have

$$\int_{H_i^\pm}V(u)=\frac{1}{2}\qquad\forall i=1,...,n.$$

Considering the hyperplanes in pairs and arguing inductively on step three, up to a relabeling we reduce to a situation where there exists a function v, symmetric by reflection with respect to each H_i, $i=1,...,n-1$, and such that

$$\alpha_\varepsilon(u)\le C\,\alpha_\varepsilon(v),\qquad \delta_\varepsilon(v)\le 2^n\,\delta_\varepsilon(v),\qquad \int_{H_n^\pm}V(v)=\frac{1}{2}.$$

We can thus consider the functions $v^\pm$ obtained by reflecting v with respect to H_n as in step three. By (5.32) we have

$$\max\{\delta_\varepsilon(v^+),\delta_\varepsilon(v^-)\}\le 2\,\delta_\varepsilon(v),\qquad \alpha_\varepsilon(u)\le C(n)\left\{\alpha_\varepsilon(v^+)+\alpha_\varepsilon(v^-)+d_\Phi(T_{x^+}u_\varepsilon,T_{x^-}u_\varepsilon)\right\},$$

where x^+ and x^- are optimal centers for $\alpha_\varepsilon(v^+;\bigcap_{i=1}^n H_i)$ and $\alpha_\varepsilon(v^-;\bigcap_{i=1}^n H_i)$. However, $\bigcap_{i=1}^n H_i$ *is a point*, therefore $x^+=x^-$ and we have actually proved

$$\alpha_\varepsilon(u)\le C(n)\Big\{\alpha_\varepsilon(v^+)+\alpha_\varepsilon(v^-)\Big\}.$$

Either v^+ or v^- is an n-symmetric function with the required properties. □

5.4. **Proof of Theorem 5.1.** We finally prove Theorem 5.1. By Theorem 5.5 we can directly assume that u is n-symmetric. Hence, by Theorem 5.4, we can directly assume that $u\in\mathcal{R}_0$. For $u\in\mathcal{R}_0$, the conclusion follows from Theorem 5.3. Theorem 5.1 is proved.

6. Proof of the Alexandrov-type theorem

In this section we complete the proof of Theorem 1.1, including in particular proof of the Alexandrov-type result of part (iv) of the statement. We begin by proving some of the properties of $\Psi(\sigma,m)$ stated in Theorem 1.1-(i) and not yet discussed. We then review, in section 6.2, some classical uniqueness and symmetry results for semilinear PDEs in relation to our setting. Finally, in section 6.3 we review how the various results of the paper combines into Theorem 1.1.

6.1. **Some properties of $\Psi(\sigma,m)$.** We prove here the properties of $\Psi(\sigma,m)$ stated in Theorem 1.1–(ii). As explained in the introduction, these properties will be crucial in proving Theorem 1.1-(iv).

Theorem 6.1. *If $n\ge 2$ and $W\in C^{2,1}[0,1]$ satisfies* (1.11) *and* (1.12)*, then there exists a universal constant ε_0 such that, setting*

$$\mathcal{X}(\varepsilon_0)=\big\{(\sigma,m):0<\sigma<\varepsilon_0\,m^{1/n}\big\},$$

the following holds:

(i): for every $\sigma>0$, $\Psi(\sigma,\cdot)$ is concave on $(0,\infty)$; it is strictly concave on $(0,\infty)$ in $n\ge 3$ and on $((\sigma/\varepsilon_0)^n,\infty)$ if $n=2$;

(ii): $\Lambda(\sigma,m)$ is continuous on $\mathcal{X}(\varepsilon_0)$ and

$$\Big|m^{1/n}\,\Lambda(\sigma,m)-2\,(n-1)\,\omega_n^{1/n}\Big|\le C\,\frac{\sigma}{m^{1/n}}\,,\qquad\forall(\sigma,m)\in\mathcal{X}(\varepsilon_0)\,.\tag{6.1}$$

(iii): $\Psi(\sigma,\cdot)$ is differentiable with

$$\frac{\partial\Psi}{\partial m}(\sigma,m)=\Lambda(\sigma,m)\qquad\forall(\sigma,m)\in\mathcal{X}(\varepsilon_0)\,.\tag{6.2}$$

In particular, for every $\sigma>0$

$$\Psi(\sigma,\cdot)\text{ is strictly increasing on }((\sigma/\varepsilon_0)^n,\infty)\,,$$
$$\Lambda(\sigma,\cdot)\text{ is strictly decreasing }((\sigma/\varepsilon_0)^n,\infty)\,.$$

(iv): for every $m>0$, $\Psi(\cdot,m)$ is increasing on $(0,\varepsilon_0\,m^{1/n})$.

Proof. We recall for convenience the scaling formulas

$$\int_{\mathbb{R}^n}f(\rho_t u)=\frac{1}{t}\int_{\mathbb{R}^n}f(u)\,,\tag{6.3}$$

$$\int_{\mathbb{R}^n}|\nabla(\rho_t u)|^2=t^{(2/n)-1}\int_{\mathbb{R}^n}|\nabla u|^2\,,$$

$$\mathcal{AC}_\varepsilon(\rho_t u)=\varepsilon\,t^{(2/n)-1}\int_{\mathbb{R}^n}|\nabla u|^2+\frac{1}{\varepsilon\,t}\int_{\mathbb{R}^n}W(u)=\frac{\mathcal{AC}_{\varepsilon\,t^{1/n}}(u)}{t^{(n-1)/n}}\,.\tag{6.4}$$

$$\Psi(\sigma,m)=m^{(n-1)/n}\,\psi\Big(\frac{\sigma}{m^{1/n}}\Big)\,,$$

where $\rho_t u(x) = u(t^{1/n}\,x)$ for $x \in \mathbb{R}^n$ and $t > 0$, and the divide the argument in a few steps.

Step one: We prove the concavity of $\Psi(\sigma,\cdot)$. Given $m_2 > m_1 > 0$, $t \in (0,1)$, $\sigma > 0$, and a minimizing sequence $\{w_j\}_j$ for $\Psi(\sigma, t\,m_1 + (1-t)\,m_2)$, we set

$$\alpha_1 = \frac{t\,m_1 + (1-t)\,m_2}{m_1}\,, \qquad \alpha_2 = \frac{t\,m_1 + (1-t)\,m_2}{m_2}\,,$$

so that $t/\alpha_1 + (1-t)/\alpha_2 = 1$. Since $\rho_{\alpha_1} w_j$ and $\rho_{\alpha_2} w_j$ are competitors for $\Psi(\sigma, m_1)$ and $\Psi(\sigma, m_2)$ respectively, by the concavity of $t \mapsto t^{(n-2)/n}$ (strict if $n \geq 3$), we see that

$$\begin{aligned}
&t\,\Psi(\sigma, m_1) + (1-t)\,\Psi(\sigma, m_2) \leq t\,\mathcal{AC}_\sigma(\rho_{\alpha_1} w_j) + (1-t)\,\mathcal{AC}_\sigma(\rho_{\alpha_2} w_j) &(6.5)\\
&= \frac{t}{\alpha_1}\Big(\int_{\mathbb{R}^n} \sigma\,\alpha_1^{2/n}\,|\nabla w_j|^2 + \frac{W(w_j)}{\sigma}\Big) + \frac{1-t}{\alpha_2}\Big(\int_{\mathbb{R}^n} \sigma\,\alpha_2^{2/n}\,|\nabla w_j|^2 + \frac{W(w_j)}{\sigma}\Big)\\
&= \mathcal{AC}_\sigma(w_j) + \Big(t\,\Big(\frac{1}{\alpha_1}\Big)^{(n-2)/n} + (1-t)\,\Big(\frac{1}{\alpha_2}\Big)^{(n-2)/n} - 1\Big)\,\sigma \int_{\mathbb{R}^n} |\nabla w_j|^2 &(6.6)\\
&\leq \mathcal{AC}_\sigma(w_j)\,. &(6.7)
\end{aligned}$$

Letting $j \to \infty$ we deduce the concavity of $\Psi(\sigma,\cdot)$ on $(0,\infty)$ (strict, if $n \geq 3$). If $n = 2$ and $m_1 \geq (\sigma/\varepsilon_0)^n$, then by Theorem 2.1 we can replace the minimizing sequence $\{w_j\}_j$ in the above argument with a minimizer w of $\Psi(\sigma, t\,m_1 + (1-t)\,m_2)$. Since w solves the Euler-Lagrange equation (1.9), there cannot be a $t \neq 1$ such that $\rho_t w$ solves (1.9) with the same σ and some $t \in \mathbb{R}$. Thus, $\rho_{\alpha_i} w$ cannot be a minimizer of $\Psi(\sigma, m_i)$, and therefore we have a strict inequality in (6.5), and no need to take a limit in (6.7) (since $\mathcal{AC}_\sigma(w) = \Psi(\sigma, t\,m_1 + (1-t)\,m_2)$).

Step two: By Theorem 2.1 and Corollary 4.2 for every $m > 0$ and $\sigma < \varepsilon_0\,m^{1/n}$ there exists a unique $u_{\sigma,m} \in \mathcal{R}_0$ such that $u_{\sigma,m}$ is a minimizer of $\Psi(\sigma, m)$ and every other minimizer of $\Psi(\sigma, m)$ is a translation of $u_{\sigma,m}$. Moreover, for some $\Lambda(\sigma, m) > 0$ such that

$$-2\,\sigma^2\,\Delta u_{\sigma,m} = \sigma\,\Lambda(\sigma, m)\,V'(u_{\sigma,m}) - W'(u_{\sigma,m})\,, \qquad \text{on } \mathbb{R}^n\,.$$

If u_ε denotes as usual the unique minimizer of $\psi(\varepsilon)$ in $\mathcal{R}_0$, then by (6.3) and (6.4) we find

$$u_{\sigma,m} = \rho_{1/m}\,u_\varepsilon\,, \qquad \varepsilon = \frac{\sigma}{m^{1/n}}\,,$$

and thus

$$\Lambda(\sigma, m) = \frac{\lambda(\varepsilon)}{m^{1/n}}\,, \qquad \varepsilon = \frac{\sigma}{m^{1/n}}\,. \tag{6.8}$$

By combining (6.8) with Corollary 4.2 and with (4.7) we thus find that Λ is continuous on $\mathcal{X}(\varepsilon_0)$, with

$$\Big|\Lambda(\sigma, m) - \frac{2\,(n-1)\,\omega_n^{1/n}}{m^{1/n}}\Big| \leq C\,\frac{\sigma}{m^{2/n}}\,. \tag{6.9}$$

Step three: We prove statement (iii). For $(\sigma, m) \in \mathcal{X}(\varepsilon_0)$, is we set

$$a(t) = \mathcal{AC}_\sigma((1+t)\,u_{\sigma,m})\,, \qquad m(t) = \int_{\mathbb{R}^n} V((1+t)\,u_{\sigma,m})$$

then

$$m'(0) = \int_{\mathbb{R}^n} \Phi(u_{\sigma,m})^{1/(n-1)}\,\sqrt{W(u_{\sigma,m})}\,u_{\sigma,m} > 0$$

and thus there exist $t_* > 0$ and an open interval I of m such that m is strictly increasing from $(-t_*, t_*)$ to I with $m(0) = m$. From $\Psi(\sigma, m(t)) \leq a(t)$ for every $|t| < t_*$ and from that fact that a is differentiable on $(-t_*, t_*)$ we deduce that, if m is such that $\Psi(\sigma,\cdot)$ is differentiable at m, then

$$\frac{\partial\Psi}{\partial m}(\sigma, m) = \frac{a'(0)}{m'(0)} = \frac{\int_{\mathbb{R}^n} 2\,\nabla u_{\sigma,m}\cdot\nabla u_{\sigma,m} + W'(u_{\sigma,m})\,u_{\sigma,m}}{\int_{\mathbb{R}^n} V'(u_{\sigma,m})\,u_{\sigma,m}} = \Lambda(\sigma, m)\,.$$

Now, by statement (i), $\Psi(\sigma,\cdot)$ is differentiable a.e. on $((\sigma/\varepsilon_0)^n,\infty)$, as well as absolutely continuous, while $\Lambda(\sigma,\cdot)$ is continuous on $((\sigma/\varepsilon_0)^n,\infty)$: by the fundamental theorem of calculus we thus conclude that $(\partial\Psi/\partial m)(\sigma,\cdot)$ exists for every $m>(\sigma/\varepsilon_0)^n$ and agrees with $\Lambda(\sigma,m)$.

Step four: We prove statement (iv). Recalling that

$$\Psi(\sigma,m)=m^{(n-1)/n}\,\psi\Big(\frac{\sigma}{m^{1/n}}\Big)\,,\qquad \forall\sigma,m>0\,,\tag{6.10}$$

we see that, since $\Psi(\sigma,\cdot)$ is differentiable on $((\sigma/\varepsilon_0)^n,\infty)$, then ψ is differentiable on $(0,\varepsilon_0)$. Since ψ is differentiable on $(0,\varepsilon_0$, by (6.10) we see that $\Psi(\cdot,m)$ is differentiable on $(0,\varepsilon_0\,m^{1/n})$ for every $m>0$, with

$$\frac{\partial\Psi}{\partial\sigma}=m^{(n-2)/n}\,\psi'\Big(\frac{\sigma}{m^{1/n}}\Big)\,.$$

Statement (iv) will thus follow by proving that $\psi'>0$ on $(0,\varepsilon_0)$. To derive a useful formula for ψ we differentiate (6.10) in m and use (6.2) and $\lambda(\sigma/m^{1/n})=m^{1/n}\Lambda(\sigma,m)$ to find that

$$\frac{n-1}{n}\,\frac{1}{m^{1/n}}\,\psi\Big(\frac{\sigma}{m^{1/n}}\Big)-\frac{1}{n}\,\frac{\sigma}{m^{2/n}}\,\psi'\Big(\frac{\sigma}{m^{1/n}}\Big)=\frac{\lambda(\sigma/m^{1/n})}{m^{1/n}}\,.$$

In particular, by (4.5),

$$\varepsilon\,\psi'(\varepsilon)=(n-1)\,\psi(\varepsilon)-n\,\lambda(\varepsilon)=\varepsilon\int_{\mathbb{R}^n}|\nabla u_\varepsilon|^2-\frac{1}{\varepsilon}\int_{\mathbb{R}^n}W(u_\varepsilon)\,.$$

By (3.8), if we set $\eta_\varepsilon(s)=\eta(s-\tau_\varepsilon)$ and change variables according to $|x|=R_0+\varepsilon\,s$ we find

$$\varepsilon\,\psi'(\varepsilon)=\int_{-R_0/\varepsilon}^{\infty}\Big\{\Big(\eta_\varepsilon'+f_\varepsilon'\Big)^2-W\big(\eta_\varepsilon+f_\varepsilon\big)\Big\}\,(R_0+\varepsilon\,s)^{n-1}\,ds\,.\tag{6.11}$$

Multiplying by u_ε' and then integrating on (r,∞) the Euler-Lagrange equation

$$-2\,\varepsilon^2\Big\{u_\varepsilon''+(n-1)\,\frac{u_\varepsilon'}{r}\Big\}=\varepsilon\,\lambda(\varepsilon)\,V'(u_\varepsilon)-W'(u_\varepsilon)\,,$$

we obtain as usual

$$\varepsilon^2\,(u_\varepsilon')^2-2\,(n-1)\,\varepsilon^2\int_r^\infty\frac{(u_\varepsilon')^2}{\rho}\,d\rho=W(u_\varepsilon)-\varepsilon\,\lambda(\varepsilon)\,V(u_\varepsilon)\,,$$

for every $r>0$; by the change of variables $r=R_0+\varepsilon\,s$ we thus find

$$(\eta_\varepsilon'+f_\varepsilon')^2-2\,(n-1)\,\varepsilon\int_s^\infty\frac{(\eta_\varepsilon'+f_\varepsilon')^2}{R_0+\varepsilon\,t}\,dt=W(\eta_\varepsilon+f_\varepsilon)-\lambda(\varepsilon)\,\varepsilon\,V(\eta_\varepsilon+f_\varepsilon)$$

for every $s\in(-R_0/\varepsilon,\infty)$. We combine this identity into (6.11) to find

$$\varepsilon\,\psi'(\varepsilon)=\int_{-R_0/\varepsilon}^{\infty}\Big\{2\,(n-1)\,\varepsilon\int_s^\infty\frac{(\eta_\varepsilon'+f_\varepsilon')^2}{R_0+\varepsilon\,t}\,dt-\lambda(\varepsilon)\,\varepsilon\,V(\eta_\varepsilon+f_\varepsilon)\Big\}\,(R_0+\varepsilon\,s)^{n-1}\,ds\tag{6.12}$$

We now notice that, by (A.16), (A.18), and (3.9) (that is, by the exponential decay of η, η', η'' and by $|f_\varepsilon(s)|\le C\,\varepsilon\,e^{-|s|/C\,\varepsilon}$ for $s>-R_0/\varepsilon$), we have

$$\begin{aligned}\int_s^\infty\frac{(\eta_\varepsilon'+f_\varepsilon')^2-(\eta_\varepsilon')^2}{R_0+\varepsilon\,t}\,dt\ &\ge\ 2\int_s^\infty\frac{\eta_\varepsilon'\,f_\varepsilon'}{R_0+\varepsilon\,t}\,dt\\ &=\ -2\,\frac{\eta_\varepsilon'(s)\,f_\varepsilon(s)}{R_0+\varepsilon\,s}-2\int_s^\infty f_\varepsilon(s)\,\Big(\frac{\eta_\varepsilon'}{R_0+\varepsilon\,t}\Big)'\,dt\\ &\ge\ -C\,\varepsilon\,e^{-|s|/C}\end{aligned}$$

so that (6.12) gives

$$\varepsilon\,\psi'(\varepsilon)\ge\int_{-R_0/\varepsilon}^{\infty}\Big\{2\,(n-1)\,\varepsilon\int_s^\infty\frac{(\eta_\varepsilon')^2\,dt}{R_0+\varepsilon\,t}-\lambda(\varepsilon)\,\varepsilon\,V(\eta_\varepsilon+f_\varepsilon)\Big\}\,(R_0+\varepsilon\,s)^{n-1}\,ds-C\,\varepsilon^2\,. \tag{6.13}$$

By (4.7), (3.9), $R_0=\omega_n^{-1/n}$ and (6.13), we have

$$\psi'(\varepsilon)\ge 2\,(n-1)\,\omega_n^{1/n}\int_{-R_0/\varepsilon}^{\infty}\Big\{\int_s^\infty(\eta_\varepsilon')^2\,dt-V(\eta_\varepsilon)\Big\}\,(R_0+\varepsilon\,s)^{n-1}\,ds-C\,\varepsilon\,. \tag{6.14}$$

Since $\int_s^\infty(\eta_\varepsilon')^2=\Phi(\eta_\varepsilon(s))$ thanks to $\eta_\varepsilon'=-\sqrt{W(\eta_\varepsilon)}=-\Phi'(\eta_\varepsilon)$, by (6.14) we have

$$\begin{aligned}\psi'(\varepsilon)&\ge 2\,(n-1)\,\omega_n^{1/n}\int_{\mathbb{R}}\big(\Phi(\eta_\varepsilon)-V(\eta_\varepsilon)\big)\,(R_0+\varepsilon\,s)^{n-1}\,ds-C\,\varepsilon\\&\ge 2\,(n-1)\,\omega_n^{1/n}\,R_0^{n-1}\int_{\mathbb{R}}\big(\Phi(\eta)-V(\eta)\big)\,ds-C\,\varepsilon\,.\end{aligned}$$

Since Φ takes values in $(0,1)$, $V=\Phi^{n/(n-1)}<\Phi$ on $(0,1)$, and

$$\int_{\mathbb{R}}\big(\Phi(\eta)-V(\eta)\big)\,ds$$

is a universal constant. In particular, $\psi'(\varepsilon)\ge 1/C$ for every $\varepsilon<\varepsilon_0$. □

6.2. **General criteria for radial symmetry and uniqueness.** In this brief section we exploit two classical results from [GNN81] and [PS83] to deduce a symmetry and uniqueness result for the kind of semilinear PDE arising as the Euler-Lagrange equation of $\Psi(\sigma,m)$.

Theorem 6.2. *Let $n\ge 2$, let $W\in C^{2,1}[0,1]$ satisfy* (1.11) *and* (1.12)*, and consider $\ell\in\mathbb{R}$ and $\sigma>0$.*

(i): *if $u\in C^2(\mathbb{R}^n;[0,1])$ is a non-zero solution to*

$$-2\,\sigma^2\,\Delta u=\sigma\,\ell\,V'(u)-W'(u)\qquad\text{on }\mathbb{R}^n\,, \tag{6.15}$$

with $u(x)\to 0$ as $|x|\to\infty$, then $0<u<1$ on $\mathbb{R}^n$ and $u\in\mathcal{R}_0^*$.

(ii): *there exists a universal constant ν_0 such that, if $0<\sigma\,\ell<\nu_0$, then, modulo translation,* (6.15) *has a unique solution among functions $u\in\mathcal{R}_0^*$, with $u(x)\to 0$ as $|x|\to\infty$ and $0<u<1$ on $\mathbb{R}^n$.*

Remark 6.1. Notice that the smallness of $\sigma\,\ell$ is required only for proving statement (ii).

Proof. Step one: We prove statement (i). We intend to apply the following particular case of [GNN81, Theorem 2]: *if $n\ge 2$, $u\in C^2(\mathbb{R}^n;[0,1])$, $u>0$ on $\mathbb{R}^n$, $u(x)\to 0$ as $|x|\to\infty$, $-\Delta u+m\,u=g(u)$ on $\mathbb{R}^n$ with $m>0$ and $g\in C^1[0,1]$ with $g(t)=\mathrm{O}(t^{1+\alpha})$ as $t\to 0^+$ for some $\alpha>0$, then, up to translations, $u\in\mathcal{R}_0^*$.*

To this end we reformulate (6.15) as

$$-\Delta u+m\,u=g(u)\qquad\text{on }\mathbb{R}^n\,, \tag{6.16}$$

where $m=W''(0)/2\,\sigma^2>0$ and

$$g(t)=\frac{\ell\,V'(t)}{2\,\sigma}+\frac{W''(0)\,t-W'(t)}{2\,\sigma^2}\,,\qquad t\in[0,1]\,.$$

As noticed in section A.3, $V\in C^{2,\gamma}[0,1]$ for some $\gamma\in(0,1]$, while $W\in C^{2,1}[0,1]$: in particular $g\in C^1[0,1]$. By $W\in C^{2,1}[0,1]$ with $W'(0)=0$ we have $|W'(t)-W''(0)\,t||\le C\,t^2$, while (A.11) states that $|V'(t)|\le C\,t^{1+\alpha}$ for $t\in[0,1]$ for some $\alpha>0$, so that

$$|g(t)|\le C(n,W,\ell,\sigma)\,t^{1+\alpha}\,,\qquad\forall t\in[0,1]\,. \tag{6.17}$$

To check that $u>0$ on $\mathbb{R}^n$, we notice that, by (6.17), for every $m'\in(0,m)$, we can find $t_0>0$ such that (6.16) implies that $-\Delta u+m'\,u\ge 0$ on the open set $\{u<t_0\}$. Since $u\ge 0$ and u is non-zero, we conclude by the strong maximum principle that $u>0$ on $\{u<t_0\}$, and thus, on $\mathbb{R}^n$. We are thus in the position to apply the stated particular case of [GNN81, Theorem 2] and conclude that $u\in\mathcal{R}_0^*$.

We prove that $u<1$ on $\mathbb{R}^n$. Let us set

$$f(t)=\frac{\ell\,V'(t)}{2\,\sigma}-\frac{W'(t)}{2\,\sigma^2}\,,\qquad t\in[0,1]\,,\tag{6.18}$$

and notice that (6.15) is equivalent to $-\Delta u=f(u)$ on $\mathbb{R}^n$. Since f is a Lipschitz function on $[0,1]$ with $f(1)=0$, we can find $c>0$ such that $f(t)+c\,t$ is increasing on $[0,1]$, and rewrite $-\Delta u=f(u)$ as

$$-\Delta\,(1-u)+c\,(1-u)=(f(t)+c\,t)\Big|_{t=u}^{t=1}\ge 0\,.$$

We thus conclude that $v=1-u$ is non-negative on $\mathbb{R}^n$ and such that $-\Delta\,v+c\,v\ge 0$. Since v is non-zero (thanks to $u(x)\to 0$ as $|x|\to\infty$), by the strong maximum principle we conclude that $v>0$ on $\mathbb{R}^n$, i.e. $u<1$ on $\mathbb{R}^n$.

Step two: We prove statement (ii). We intend to use [PS83, Theorem 2]: *if*

(a) *f locally Lipschitz on $(0,\infty)$;*
(b) *$f(t)/t\to -m$ as $t\to 0^+$ where $m>0$;*
(c) *setting $F(t)=\int_0^t f(s)\,ds$, there exists $\delta>0$ such that $F(\delta)>0$;*
(d) *setting $\beta=\inf\{t>0:F(t)>0\}$ (so that by (b) and (c), $\beta\in(0,\delta)$), the function $t\mapsto f(t)/(t-\beta)$ is decreasing on $(\beta,\infty)\cap\{f>0\}$;*

then there is at most one $u\in C^2(\mathbb{R}^n)\cap\mathcal{R}_0$, with $u>0$ on $\mathbb{R}^n$ and $u(x)\to 0$ as $|x|\to\infty$, solving $-\Delta u=f(u)$ on $\mathbb{R}^n$.

Since, by statement (i), solutions to (6.15) satisfy $0<u<1$ on $\mathbb{R}^n$, in checking that f as in (6.18) satisfies the above assumptions it is only the behavior of f on $(0,1)$ (and not on $(0,\infty)$) that matters. Evidently (a) holds, since $f\in C^{1,\alpha}[0,1]$ for some $\alpha\in(0,1)$. Assumption (b) holds with $m=W''(0)/2\,\sigma^2$. Property (c) holds (with $\delta\in(0,1)$) since

$$F(t)=\int_0^t f(s)\,ds=\frac{\ell\,V(t)}{2\,\sigma}-\frac{W(t)}{2\,\sigma^2}\,,\qquad t\in[0,1]\,,$$

and $F(1)=(\ell\,V(1)/2\,\sigma)=\ell/2\,\sigma>0$ by $\ell>0$ and $W(1)=0$. We finally prove (d). Notice that, clearly, $\beta\in(0,1)$ and, by continuity of F, $F(\beta)=0$, so that, taking (A.3) and (A.6) into account, and using $\sigma\,\ell<\nu_0$ and $V(1)=1$,

$$\frac{\min\{\beta^2,(1-\beta)^2\}}{C}\le W(\beta)=\sigma\,\ell\,V(\beta)\le\nu_0\,.\tag{6.19}$$

If $\nu_0<1$, then by (A.6) and (A.11) we find

$$2\,\sigma^2\,F(t)=\sigma\,\ell\,V(t)-W(t)\le V(t)-W(t)\le C\,t^{2n/(n-1)}-\frac{t^2}{C}<0\,,\qquad\forall t\in(0,\delta_0)\,.\tag{6.20}$$

By (6.20) it must be $\beta\ge\delta_0$. Hence, by (6.19), if ν_0 is sufficiently small, then $(1-\beta)^2\le C\,\nu_0$. Up to further decrease the value of ν_0, we can finally entail that $(\beta,1)\subset(1-\delta_0,1)$, with δ_0 as in section A.3.

We are now going to check property (d) by showing that

$$f'(t)\,(t-\beta)\le f(t)\qquad\forall t\in(\beta,1)\,,\tag{6.21}$$

(recall that $0<u<1$ on $\mathbb{R}^n$, so we can use a version of [PS83, Theorem 2] localized to $(0,1)$). Using the explicit formula for f, (6.21) is equivalent to

$$\sigma\,\ell\,V''(t)\,(t-\beta)\le\sigma\,\ell\,V'(t)-W'(t)+W''(t)\,(t-\beta)\,,\qquad\forall t\in(\beta,1)\,.\tag{6.22}$$

By (A.6), we have $W''(t)(t-\beta) > 0$ on $(\beta, 1) \subset (1-\delta_0, 1)$, and since $V' \geq 0$ on $[0,1]$, (6.22) is implied by checking that, for every $t \in (\beta, 1)$,

$$\begin{aligned} -W'(t) &\geq \sigma\,\ell\,V''(t) \\ &= \sigma\,\ell\Big\{\frac{n}{(n-1)^2}\,\frac{W(t)}{\Big(\int_0^t \sqrt{W}\Big)^{(n-2)/(n-1)}} + \frac{n}{n-1}\Big(\int_0^t \sqrt{W}\Big)^{1/(n-1)}\,\frac{W'(t)}{2\sqrt{W(t)}}\Big\}. \end{aligned}$$

In turn, since $W' < 0$ on $(1-\delta_0, 1)$ and $\sigma\,\ell < \nu_0 < 1$, it is actually enough to check that

$$-W'(t) \geq \frac{n}{(n-1)^2}\,\frac{W(t)}{\Big(\int_0^t \sqrt{W}\Big)^{(n-2)/(n-1)}}\,, \qquad \forall t \in (1-\delta_0, 1)\,. \tag{6.23}$$

But, up to further decreasing the value of δ_0, this is obvious: indeed (A.6) gives $-W'(t) \geq (1-t)/C$ and $W(t) \leq C\,(1-t)^2$ for every $t \in (1-\delta_0, 1)$. ☐

6.3. **Proof of Theorem 1.1.** Theorem 2.1, Corollary 4.2, Theorem 6.1 and a scaling argument show the validity of statements (i) and (ii), while statement (iii) follows similarly by scaling and by Theorem 5.1. To prove the Alexandrov-type theorem, that is, statement (iv)[5] we consider $u \in C^2(\mathbb{R}^n; [0,1])$, with $u(x) \to 0$ as $|x| \to \infty$, and solving

$$-2\,\sigma^2\,\Delta u = \sigma\,\ell\,V'(u) - W'(u) \qquad \text{on } \mathbb{R}^n\,, \tag{6.24}$$

for some σ and ℓ with $0 < \sigma\,\ell < \nu_0$. By Theorem 6.2–(i), $u \in \mathcal{R}_0^*$, and by Theorem 6.2, provided ν_0 is small enough, we know that there is at most one radial solution to (6.24). Since we know that $u_{\sigma,m}$ is a radial solution of (6.24) with $\ell = \Lambda(\sigma, m)$, we are left to prove that for every $\ell \in (0, \nu_0/\sigma)$ there exists a unique $m \in ((\sigma/\varepsilon_0)^n, \infty)$ such that $\Lambda(\sigma, m) = \ell$.

To this end, we first notice that, by (4.7) and by scaling, for every $\sigma > 0$ we have

$$\Lambda(\sigma, m) = \frac{1}{m^{1/n}}\,\lambda\big(\sigma/m^{1/n}\big) \to 0^+ \qquad \text{as } m \to +\infty\,.$$

In particular, since, by Theorem 6.1, $\Lambda(\sigma, \cdot)$ is continuous and strictly decreasing on $((\sigma/\varepsilon_0)^n, \infty)$, we have that

$$\Big\{\Lambda(\sigma, m) : m > \Big(\frac{\sigma}{\varepsilon_0}\Big)^n\Big\} = \Big(0, \Lambda\big(\sigma, (\sigma/\varepsilon_0)^n\big)\Big)\,.$$

Now, setting $m = (\sigma/\varepsilon_0)^n$ in (6.1), that is, in

$$\Big|m^{1/n}\,\Lambda(\sigma, m) - 2\,(n-1)\,\omega_n^{1/n}\Big| \leq C\,\frac{\sigma}{m^{1/n}}\,,$$

we find that

$$\Big|\sigma\,\Lambda\big(\sigma, (\sigma/\varepsilon_0)^n\big) - 2\,(n-1)\,\omega_n^{1/n}\,\varepsilon_0\Big| \leq C\,\varepsilon_0^2\,,$$

which implies

$$\Lambda\big(\sigma, (\sigma/\varepsilon_0)^n\big) \geq \frac{(n-1)\,\omega_n^{1/n}\,\varepsilon_0}{\sigma}\,, \qquad \forall \sigma > 0\,,$$

provided ε_0 is small enough. Up to further decrease the value of ν_0 so to have $\nu_0 \leq (n-1)\,\omega_n^{1/n}\,\varepsilon_0$, we have proved that

$$(0, \nu_0/\sigma) \subset \Big\{\Lambda(\sigma, m) : m > \Big(\frac{\sigma}{\varepsilon_0}\Big)^n\Big\}\,,$$

and that for each $\ell \in (0, \nu_0/\sigma)$ there is a unique $m > (\sigma/\varepsilon_0)^n$ such that $\ell = \Lambda(\sigma, m)$, as claimed. This completes the proof of Theorem 1.1.

[5] Notice that we are using ℓ in (6.24) rather than λ (as done in (1.22)) to denote the Lagrange multiplier of u. This is meant to avoid confusion with the function $\lambda(\varepsilon) = (\partial\Psi/\partial m)(\varepsilon, 1)$ appearing in the argument.

Appendix A. Frequently used auxiliary facts

A.1. **Scaling identities.** If $u \in H^1(\mathbb{R}^n;[0,\infty))$, $t>0$, we set

$$\rho_t\, u(x) = u(t^{1/n}\, x)\,, \qquad x \in \mathbb{R}^n\,,$$

and notice that

$$\int_{\mathbb{R}^n} f(\rho_t u) \;=\; \frac{1}{t}\int_{\mathbb{R}^n} f(u)\,, \qquad \int_{\mathbb{R}^n} |\nabla(\rho_t u)|^2 = t^{(2/n)-1}\int_{\mathbb{R}^n}|\nabla u|^2\,, \tag{A.1}$$

$$\mathcal{AC}_\varepsilon(\rho_t u) \;=\; \varepsilon\, t^{(2/n)-1}\int_{\mathbb{R}^n}|\nabla u|^2 + \frac{1}{\varepsilon\, t}\int_{\mathbb{R}^n} W(u) = \frac{\mathcal{AC}_{\varepsilon\, t^{1/n}}(u)}{t^{(n-1)/n}}\,. \tag{A.2}$$

whenever $f:\mathbb{R}\to\mathbb{R}$ is continuous.

A.2. **Concentration-compactness principle.** Denoting by $B_r(x)$ the ball of center x and radius r in $\mathbb{R}^n$, and setting $B_r = B_r(0)$ when $x=0$, we provide a reference statement for Lions' concentration-compactness criterion, which is repeatedly used in our arguments: *if $\{\mu_j\}_j$ is a sequence of probability measures in $\mathbb{R}^n$, then, up to extract subsequences and compose each μ_j with a translation, one the following mutually excluding possibilities holds:*

Compactness case: *for every $\tau>0$ there exists $R>0$ such that*

$$\inf_j\, \mu_j(B_R) \ge 1-\tau\,;$$

Vanishing case: *for every $R>0$,*

$$\lim_{j\to\infty}\, \sup_{x\in\mathbb{R}^n} \mu_j(B_R(x)) = 0\,;$$

Dichotomy case: *there exists $\alpha\in(0,1)$ such that for every $\tau>0$ one can find $S>0$ with $S_j\to\infty$ such that*

$$\sup_j\, |\alpha - \mu_j(B_S)| < \tau\,, \qquad \sup_j\, \big|(1-\alpha) - \mu_j(\mathbb{R}^n\setminus B_{S_j})\big| < \tau\,.$$

Notice that the formulation of the dichotomy case used here is a bit more descriptive than the original one presented in [Lio84, Lemma I]. Its validity is inferred by a quick inspection of the proof presented in the cited reference.

A.3. **Estimates for W, Φ and V.** Throughout the paper we work with a double well potential $W\in C^{2,1}[0,1]$ satisfying (1.11) and (1.12), that is,

$$W(0)=W(1)=0\,, \quad W>0 \text{ on } (0,1)\,, \quad W''(0), W''(1)>0\,, \tag{A.3}$$

$$\int_0^1 \sqrt{W} = 1\,. \tag{A.4}$$

Frequently used properties of W are the validity, for a universal constant C, of the expansion

$$\Big|W(b)-W(a)-W'(a)(b-a)-W''(a)\frac{(b-a)^2}{2}\Big| \le C\,|b-a|^3\,, \qquad \forall a,b\in[0,1]\,, \tag{A.5}$$

and the existence of a universal constant $\delta_0<1/2$ such that

$$\begin{aligned} &\frac{1}{C}\le \frac{W}{t^2}\,,\, \frac{W'}{t}\,, W''\le C \qquad && \text{on } (0,\delta_0]\,,\\ &\frac{1}{C}\le \frac{W}{(1-t)^2}\,,\, \frac{-W'}{1-t}\,, W''\le C && \text{on } [1-\delta_0,1)\,. \end{aligned} \tag{A.6}$$

We can use (A.6) to quantify the behaviors near the wells of Φ and, crucially, of V. We first notice that, by (A.3), $\Phi \in C^3_{\rm loc}(0,1)$, with

$$\Phi' = \sqrt{W}\,, \quad \Phi'' = \frac{W'}{2\sqrt{W}}\,, \quad \Phi''' = \frac{W''}{2\sqrt{W}} - \frac{(W')^2}{4\,W^{3/2}}\,, \qquad \text{on } (0,1)\,.$$

By (A.6) and (A.4) we thus see that Φ satisfies

$$\begin{aligned} &\frac{1}{C} \le \frac{\Phi}{t^2}\,, \frac{\Phi'}{t}\,, \Phi'' \le C\,, \qquad \text{on } (0,\delta_0]\,,\\ &\frac{1}{C} \le \frac{1-\Phi}{(1-t)^2}\,, \frac{\Phi'}{1-t}\,, -\Phi'' \le C\,, \qquad \text{on } [1-\delta_0,1)\,, \end{aligned} \tag{A.7}$$

from which we easily deduce

$$|\Phi(b)-\Phi(a)| \ge \frac{(b-a)^2}{C}\,, \qquad \forall a,b \in [0,1]\,. \tag{A.8}$$

Moreover, by exploiting (A.7) and setting for brevity $a = W''(0)$, we see that as $t \to 0^+$

$$\begin{aligned} \Phi''' &= \frac{2\,W''\,W - (W')^2}{4\,W^{3/2}} = \frac{2\,(a+\mathrm{O}(t))\,(a\,(t^2/2)+\mathrm{O}(t^3)) - (a\,t + \mathrm{O}(t^2))^2}{4\,(a\,(t^2/2)+\mathrm{O}(t^3))^{3/2}}\\ &= \frac{\mathrm{O}(t^3)}{4\,a^{3/2}\,t^3 + \mathrm{o}(t^3)}\,, \end{aligned}$$

and a similar computation holds for $t \to 1^-$, so that

$$|\Phi'''| \le C\,, \qquad \text{on } (0,\delta_0) \cup (1-\delta_0,1)\,. \tag{A.9}$$

By (A.7) and (A.9) we see that $\Phi \in C^{2,1}[0,1]$ with a universal estimate on its $C^{2,1}[0,1]$-norm: in particular,

$$\left|\Phi(b) - \Phi(a) - \Phi'(a)(b-a) - \Phi''(a)\frac{(b-a)^2}{2}\right| \le C\,|b-a|^3\,, \qquad \forall a,b \in (0,1)\,. \tag{A.10}$$

Since $V = \Phi^{1+\alpha}$ for $\alpha = 1/(n-1) \in (0,1]$ (recall that $n \ge 2$) and $\Phi(t) = 0$ if and only if $t = 0$, we easily see that $V \in C^3_{\rm loc}(0,1)$, with

$$\begin{aligned} &V' = (1+\alpha)\,\Phi^\alpha\,\Phi'\,, \qquad V'' = (1+\alpha)\Big\{\alpha\,\frac{(\Phi')^2}{\Phi^{1-\alpha}} + \Phi^\alpha\,\Phi''\Big\}\,,\\ &|V'''| \le C(\alpha)\,\Big\{\frac{(\Phi')^3}{\Phi^{2-\alpha}} + \frac{\Phi'\,|\Phi''|}{\Phi^{1-\alpha}} + \Phi^\alpha\,|\Phi'''|\Big\}\,. \end{aligned}$$

By (A.10), and keeping track of the sign of Φ'' and of the fact that negative powers of $\Phi(t)$ are large only near $t = 0$, but are bounded near $t = 1$, we find that

$$\begin{aligned} &\frac{1}{C} \le \frac{V}{t^{2+2\alpha}}\,, \frac{V'}{t^{1+2\alpha}}\,, \frac{V''}{t^{2\alpha}} \le C\,, \qquad |V'''| \le \frac{C}{t^{1-2\alpha}} \qquad \text{on } (0,\delta_0]\,,\\ &\frac{1}{C} \le \frac{1-V}{(1-t)^2}\,, \frac{V'}{1-t} \le C\,, \quad |V''|\,, |V'''| \le C\,, \qquad \text{on } [1-\delta_0,1)\,. \end{aligned} \tag{A.11}$$

In particular, $V \in C^{2,\gamma(n)}[0,1]$, $\gamma(n) = \min\{1, 2/(n-1)\} \in (0,1]$, with second order Taylor expansions of the form

$$\left|V(b) - V(a) - V'(a)(b-a) - V''(a)\frac{(b-a)^2}{2}\right| \le C\,|b-a|^{2+\gamma(n)}\,, \qquad \forall a,b \in (0,1)\,. \tag{A.12}$$

We finally notice that we can find a universal constant C such that

$$\frac{t^2}{C} \le W(t)\,, \qquad V(t) \le C\,t^2\,, \qquad V(t) \le C\,W(t)\,, \qquad \forall t \in (0,1-\delta_0)\,, \tag{A.13}$$

(as it is easily deduced from the bounds on W and V in (A.6) and (A.11) and from the fact that $W > 0$ on $(0,1)$); and that we can also find C so that

$$V(t) \geq \frac{1}{C}\,, \qquad \forall t \in (\delta_0, 1)\,. \tag{A.14}$$

A.4. **Estimates for the optimal transition profile η.** A crucial object in the analysis of the Allen-Cahn energy is of course the optimal transition profile η, defined by the first order ODE

$$\begin{cases} \eta' = -\sqrt{W(\eta)} & \text{on } \mathbb{R}\,, \\ \eta(0) = \dfrac{1}{2}\,, \end{cases} \tag{A.15}$$

which can be seen to satisfy (see, e.g. [LM16a]) $\eta \in C^{2,1}(\mathbb{R})$, $\eta' < 0$ on $\mathbb{R}$ (and $-C \leq \eta' \leq -1/C$ for $|s| \leq 1$), $\eta(-\infty) = 1$, and $\eta(+\infty) = 0$, with the exponential decay properties

$$1 - \eta(s) \leq C\,e^{s/C} \qquad \forall s < 0\,, \qquad \eta(s) \leq C\,e^{-s/C} \qquad \forall s > 0\,, \tag{A.16}$$

for a universal constant C. Similarly, by combining (A.16) with (A.15), with the second order ODE satisfied by η, namely,

$$2\,\eta'' = W'(\eta) \qquad \text{on } \mathbb{R}\,, \tag{A.17}$$

and with (A.6) we see that also the first and second derivatives of η decay exponentially

$$|\eta'(s)|\,, |\eta''(s)| \leq C\,e^{-|s|/C} \qquad \forall s \in \mathbb{R}\,. \tag{A.18}$$

Combining again (A.16) and (A.6) we also see that

$$s \in \mathbb{R} \mapsto 1_{(-\infty,0)}(s) - V(\eta(s-\tau))$$

belongs to $L^1(\mathbb{R})$ for every $\tau \in \mathbb{R}$, with

$$\tau \in \mathbb{R} \mapsto \int_{-\infty}^{\infty} \Big(1_{(-\infty,0)}(s) - V(\eta(s-\tau))\Big)\,ds$$

increasing in τ and converging to $\mp\infty$ as $\tau \to \pm\infty$. In particular, there is a unique universal constant τ_0 such that

$$\int_{-\infty}^{\infty} \Big(1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))\Big)\,ds = 0\,. \tag{A.19}$$

The constant τ_0 appears in the computation of the first order expansion of $\psi(\varepsilon)$ as $\varepsilon \to 0^+$ and can be characterized, equivalently, to be

$$\tau_0 = \int_{\mathbb{R}} \eta'\,V'(\eta)\,s\,ds\,. \tag{A.20}$$

Indeed, (A.19) gives

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \Big(1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))\Big)\,ds \\ &= \int_{-\infty}^{0} \Big(1 - V(\eta(s-\tau_0))\Big)\,ds - \int_0^{\infty} V(\eta(s-\tau_0))\,ds \\ &= -\int_{-\infty}^{0} ds \int_{-\infty}^{s-\tau_0} \eta'(t)\,V'(\eta(t))\,dt + \int_0^{\infty} ds \int_{s-\tau_0}^{\infty} \eta'(t)\,V'(\eta(t))\,dt\,. \end{aligned}$$

Both integrands are non-negative, therefore by Fubini's theorem

$$\begin{aligned} 0 &= -\int_{-\infty}^{-\tau_0} dt \int_{t+\tau_0}^{0} \eta'(t)\,V'(\eta(t))\,ds - \int_{-\tau_0}^{\infty} dt \int_0^{t+\tau_0} \eta'(t)\,V'(\eta(t))\,ds \\ &= \int_{-\infty}^{-\tau_0} (t+\tau_0)\,\eta'(t)\,V'(\eta(t))\,dt + \int_{-\tau_0}^{\infty} (t+\tau_0)\eta'(t)\,V'(\eta(t))\,dt \end{aligned}$$

that is

$$\int_{\mathbb{R}} \eta'\, V'(\eta)\, t\, dt = -\tau_0 \int_{\mathbb{R}} \eta' V'(\eta) = V(1)\, \tau_0 = \tau_0\,,$$

as claimed.

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