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MEAN-FIELD LIMIT OF A HYBRID SYSTEM FOR MULTI-LANE CAR-TRUCK TRAFFIC

Maria Teresa Chiri

Department of Mathematics
Penn State University
University Park, PA 16802, USA

Xiaoqian Gong

School of Mathematical and Statistical Sciences
Arizona State University
Tempe, AZ, USA

Benedetto Piccoli

Department of Mathematical Sciences and Center for Computational and Integrative Biology
Rutgers University
Camden, NJ, USA

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Abstract. In the present work we model multi-lane traffic flow in presence of two population of vehicles: cars and trucks. We first develop a finite-dimensional hybrid system which rely on continuous Bando-Follow-the-Leader dynamics coupled with discrete events motivated by the lane-change maneuvers. Then we rigorously prove that the mean-field limit is given by a system of Vlasov-type PDE with source terms generated by the lane-change maneuvers of the human-driven vehicles.

1. **Introduction.** Mathematical models for traffic flow are mainly classified into microscopic, mesoscopic, macroscopic, and cellular, depending on the scale at which they represent vehicular traffic [1, 4, 36]. Generally, the scale is chosen according to the type of traffic characteristics to be captured. In this paper, we are interested in microscopic models and mesoscopic descriptions.

Microscopic models are developed with the idea of explicitly reproducing the individual behaviors of drivers, such as reactions to traffic changes and interactions with other vehicles, therefore the dynamic is expressed in terms of trajectories of the single vehicles, by means of ODEs. Two of the most successful microscopic models are the Optimal Velocity model, also known as the Bando model [3], and

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*Corresponding author.

the Follow-the-Leader model [17, 24, 40], in which the acceleration of the single vehicle is controlled according to the velocity of the leading vehicle.
Mesoscopic traffic flow models were derived as bridge between the family of microscopic models and the family of macroscopic models which interpret traffic as a continuum flow. Usually mesoscopic models describe vehicle flow in aggregate terms such as in probability distributions. Mean-field equations fall into this category and aim to provide an aggregate and statistical view of traffic by capturing and predicting the main phenomenology of microscopic dynamics. Among the relevant literature, in this context, we can mention classic works like [32, 37, 38] and more recent results, e.g. [9, 13, 24, 31, 33, 25]. The passagge from microscopic to mesoscopic description can be also rigorously performed by using the generalized version of the classical Wasserstain distance [2]. The analysis, through the progressive change of scale, is not only a peculiarity of traffic flow models, but extends to other research areas such as biology [10, 12], economics [45] and social sciences [11].

Heterogeneous and multi-lane traffic flow modeling is fundamental to understand the dynamics and control of complex traffic systems. Specifically in this work we consider two populations of vehicles: cars and trucks. For other relevant contribution in multipopulation traffic models, see [5, 46, 39, 29]. We model the multi-lane traffic by hybrid systems because of its hybrid nature: the continuous dynamics on each lane and the discrete events due to lane-changing maneuvers. The lane-change is one of the most common maneuvers, which generates interaction and risk [26] among vehicles on motorways. Current models for multi-lane traffic include two-dimensional models [23, 43], in which lane changing rules are not explicitly prescribed and models treating lanes as discrete entities [27, 41].

Our main contribution regards the the formalization of the passage from microscopic dynamics to mesoscopic in the case of the two before mentioned populations of vehicles (cars and trucks). The model used is the combined Bando-Follow-the-Leader one for both the populations. In particular, we reformulate it by replacing the interaction with the closest vehicle ahead with a short-range interaction kernel which allows to write the system of ODEs in a convolution framework. Continuous dynamics are combined with discrete dynamics generated by the lane changing rules, which are designed following [30]. This leads to an hybrid system (see [7, 34, 16, 18, 44]) whose mean field limit is given by a system of two Vlasov-type PDEs with source terms [14, 22, 28]. These source terms are generated by the discrete lane changing rules and induce the measure solutions to change mass in time, thus the limit is obtained using the generalized Wasserstein distance [35].

This complete representation of multi-lane heterogenous traffic at microscopic and mesoscopic scales together connected by a rigorous limiting procedure has also been extendended by the same authors to the case of two populations of human-driven vehicles and autonomous vehicles [8]. The main difference is that a control is introduced in the acceleration of autonomous vehicles with the idea that they can influence the general dynamics of the other two populations. Moreover the number of autonomous vehicles remains finite in the limiting procedure.
The paper is organized as follows. In Section 2 we introduce the notation used to capture the heterogeneous traffic and the main notions necessary for what we are going to prove. We describe in detail the combined Bando-Follow-the-leader model and how to reformulate it in convolution form. This is propaedeutic to the derivation of the mean-field limit. The lane change rules, together with the key ideas behind, are explored in the Subsection 2.2. Soon after we present an

overview on the generalized Wasserstein distance and a revised version of Ascoli-Arzelá theorem which play an essential role in the process of derivation of the Vlasov-Poisson type equations with source term. In Section 3 we define the "cool-down" time model assumption, which is critical to describe the frequencies of the vehicles' lane-changing behavior and to prove the well-posedness of our heterogeneus multi-lane traffic model. Then we introduce the finite dimensional hybrid system which captures the continuous dynamics on each lane and discrete events for lane-changes. Section 4 is finally devoted to the rigorous derivation of the mean-field limit for the hybrid system which leads to two coupled Vlasov Poisson type equation with source term. In Section 5 we discuss future research directions opened by this work.

2. Preliminaries.

In this section, we recall the original and convolutional form of a car-following model, Bando-Follow-the-leader model, and design lane-changing rules based on distance headway and acceleration for multi-lane traffic in both homogeneous and heterogeneous traffic conditions. We also give an overview on the generalized Wasserstein distance [35] and a revised version of Ascoli-Arzela theorem introduced in [20]. In the end, we formally derive the mean-field limit of a finite dimensional system and listed the results on partial differential equations of Vlasov-type with and without source terms.

2.1. Car-following model: Bando-Follow-the-Leader model.

The Bando-Follow-the-Leade (Bando-FtL) model is a first order car-following model introduced in [42]. The main idea of the Bando-FtL model is that the ego vehicle adjusts its own acceleration based on its space headway, optimal velocity (determined by its space headway) and its leader's velocity. We refer the readers to [19] for the well-posedness of the Bando-FtL model.

We assume that the vehicles move from left to right and vehicle $n+1 \in \mathbb{N}_{\geq 1}$ is the leader of vehicle $n \in \mathbb{N}_{\geq 1}$. Let $(x_n, v_n)\colon [0,T] \to \mathbb{R} \times \mathbb{R}_{\geq 0}$ be the position-velocity vector of vehicle n, where $T > 0$ is fixed, $l_n \in \mathbb{R}_{>0}$ be the length of vehicle n, $h_n = x_{n+1} - x_n - l_n$ be the space headway of vehicle n, and $V\colon (0,+\infty) \mapsto [0,+\infty); h \to V(h)$ be the optimal velocity function which describes the desired velocity corresponding to space headway. Usually, the optimal velocity function V is increasing with respect to the headway. For example, one may choose the following optimal velocity function as in [42], for any $h \in (0,+\infty)$,

$$V(h) = v_{\max} \frac{\tanh(h - d_s) + \tanh(l_v + d_s)}{1 + \tanh(l_v + d_s)}, \tag{1}$$

We recall the Bando-FtL model in two traffic conditions: homogeneous and heterogeneous. In the case of homogeneous traffic where the vehicles' physical dimensions do not vary much, the governing equation of the Bando-FtL model is as follows: for vehicle $n \in \mathbb{N}_{\geq 1}$

$$\begin{cases} \dot{x}_n = v_n, \\ \dot{v}_n = \alpha(V(h_n) - v_n) + \beta \frac{v_{n+1} - v_n}{(h_n)^2}, \end{cases} \tag{2}$$

where α, β are positive with proper dimensions. We develop the homogeneous Bando-FtL model to its heterogeneous form by giving subscripts to the model parameters as follows: for vehicle $n \in \mathbb{N}_{\geq 1}$

$$\begin{cases} \dot{x}_n = v_n, \\ \dot{v}_n = \alpha_n (V_n(h_n) - v_n) + \beta_n \frac{v_{n+1} - v_n}{(h_n)^2}, \end{cases} \tag{3}$$

where α_n, β_n are again positive parameters with proper dimensions, V_n is an optimal velocity function depending on the headway $h_n = x_{n+1} - x_n - l_n$. In the case of heterogeneous traffic with cars and trucks, due to the four different car-truck car-following combinations (C-C, C-T, T-C, T-T), all parameters α_n, β_n and the optimal velocity function V_n have four different alternatives. For example, α_n can be α_{cc},α_{ct}, α_{tc}, or α_{tt}. Here α_{ct} represents the weight of the Bando term in the Bando-FtL model in the case of car-following-truck. The vehicle length l_n has two alternatives the car length $l_c > 0$ and the truck length $l_t > 0$,

Now we want to rewrite the heterogeneous Bando-FtL model (3) into its convolutional form. Instead of only considering one leading vehicle ahead, the drivers adjust their accelerations and decelerations according to the types and velocities of vehicles in nearby front, their own velocities and the optimal velocities depending on their space headways. Of course, we cannot expect that the strength of the interaction in C-C case is the same as in T-T case, and also the order of two different type of vehicles plays an important role. Hence the strength of the interaction in the configuration car-truck must be different based on the car-truck car-following combinations. Therefore, we assume that the ego vehicle only interact with other front nearby vehicles that is at most $\varepsilon_n > 0$, $n \in \{cc, tc, ct, tt\}$, away. We call ε_n the strength of interaction.

For convenience of notation, for the heterogeneous traffic containing cars and trucks, we introduce $\mathcal{I} = \{1, \dots, P+S\}$ be the set of index for all the vehicles, $\mathcal{I}_P = \{1, \dots, P\}$ the set of index for cars and $\mathcal{I}_S = \{P+1, \dots, P+S\}$ for trucks. Note that $\mathcal{I} = \mathcal{I}_P \cup \mathcal{I}_S$. We define the following two time dependent atomic probability measures

$$\mu_P(t) = \frac{1}{P} \sum_{i \in \mathcal{I}_P} \delta_{(x_i(t), v_i(t))}, \qquad \mu_S(t) = \frac{1}{S} \sum_{i \in \mathcal{I}_S} \delta_{(x_i(t), v_i(t))} \tag{4}$$

tracking the position-velocity of cars and trucks at time $t \in [0, T]$.

Consider convolution kernels of the form

$$H_1^n : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R} \quad \text{ with } n \in \{cc, tc, ct, tt\}$$

$$(x, v) \mapsto \alpha_n h_n(x)(V_n(-x) - v)$$

where $h_n : \mathbb{R} \mapsto \mathbb{R}_{\geq 0}$ is a smooth function supported compactly on $[-\varepsilon_n, 0]$. Here h_n measures the interaction of two vehicles depending on their distance and types. The Bando-Term in (3) can be rewritten as

$$\begin{aligned} &\left(H_1^{cc} *_1 \mu_P + H_1^{tc} *_1 \mu_S\right)(x_i, v_i) \qquad (5) \\ &= \tfrac{\alpha_{cc}}{P} \textstyle\sum_{k \in \mathcal{I}_P} h_{cc}(x_i - x_k)\left(V_{cc}(x_k - x_i) - v_i\right) + \tfrac{\alpha_{tc}}{S} \textstyle\sum_{k \in \mathcal{I}_S} h_{tc}(x_i - x_k)\left(V_{tc}(x_k - x_i) - v_i\right) \end{aligned}$$

for cars ($i \in \mathcal{I}_P$) and

$$\begin{aligned} &\left(H_1^{ct} *_1 \mu_P + H_1^{tt} *_1 \mu_S\right)(x_i, v_i) \qquad (6) \\ &= \tfrac{\alpha_{ct}}{P} \textstyle\sum_{k \in \mathcal{I}_P} h_{ct}(x_i - x_k)\left(V_{ct}(x_k - x_i) - v_i\right) + \tfrac{\alpha_{tt}}{S} \textstyle\sum_{k \in \mathcal{I}_S} h_{tt}(x_i - x_k)\left(V_{tt}(x_k - x_i) - v_i\right) \end{aligned}$$

for trucks ($i \in \mathcal{I}_S$). Here $*_1$ is the convolution with respect to the first variable.

In analogous way we introduce four kernels

$$H_2^n : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R} \quad \text{ with } n \in \{cc, tc, ct, tt\}$$

$$(x, v) \mapsto \beta_n h_n(x) \frac{-v}{x^2}$$

and rewrite the FtL-term in (3) as

$$\begin{aligned}&\left(H_2^{cc} * \mu_P + H_2^{tc} * \mu_S\right)(x_i, v_i)\\&\quad= \frac{\beta_{cc}}{P} \sum_{k \in \mathcal{I}_P} h_{cc}(x_i - x_k) \frac{v_k - v_i}{(x_i - x_k)^2} + \frac{\beta_{tc}}{S} \sum_{k \in \mathcal{I}_S} h_{tc}(x_i - x_k) \frac{v_k - v_i}{(x_i - x_k)^2}\,,\end{aligned} \tag{7}$$

for $i \in \mathcal{I}_P$ and

$$\begin{aligned}&\left(H_2^{ct} * \mu_P + H_2^{tt} * \mu_S\right)(x_i, v_i)\\&\quad= \frac{\beta_{ct}}{P} \sum_{k \in \mathcal{I}_P} h_{ct}(x_i - x_k) \frac{v_k - v_i}{(x_i - x_k)^2} + \frac{\beta_{tt}}{S} \sum_{k \in \mathcal{I}_S} h_{tt}(x_i - x_k) \frac{v_k - v_i}{(x_i - x_k)^2}\,,\end{aligned} \tag{8}$$

for $i \in \mathcal{I}_S$. In this case $*$ is the convolution with respect to both space and speed.

This leads to the following convolutional formulation of the Bando-FtL model with two distinct dynamics for cars and trucks:

$$\begin{aligned}\dot{x}_i &= v_i \quad i \in \mathcal{I}\\ \dot{v}_i &= \begin{cases} \left(H_1^{cc} *_1 \mu_P + H_1^{tc} *_1 \mu_S\right)(x_i, v_i) + \left(H_2^{cc} * \mu_P + H_2^{tc} * \mu_S\right)(x_i, v_i) & i \in \mathcal{I}_P \\ \left(H_1^{ct} *_1 \mu_P + H_1^{tt} *_1 \mu_S\right)(x_i, v_i) + \left(H_2^{ct} * \mu_P + H_2^{tt} * \mu_S\right)(x_i, v_i) & i \in \mathcal{I}_S. \end{cases}\end{aligned} \tag{9}$$

2.2. **Lane-changing rules based on acceleration.** Inspired by [30], we design a lane-changing rule based on a trade-off between the expected own advantage and the disadvantage imposed on other drivers. In particular, the follower in the target lane is involved in the decision process. The subjective utility of a change of lane increases with the gap to the new leader in the target lane. However, if the velocity of this leader is lower, it may be convenient to stay in the present lane. A criterion for the utility including the above mentioned situations is the difference in the accelerations after and before the lane change. The formulation in terms of accelerations has several advantages. Indeed the evaluation of the traffic situation is transferred to the acceleration function of the Bando-FtL model with the result that the lane change rules are compact and depend only on a small number of additional parameters.

Now we consider P cars and S trucks on an open stretch road of $L \in \mathbb{N}_{>0}$ lanes. Let $\mathcal{K} = \{1, \ldots, L\}$. First of all, we consider the lane-changing condition in a homogeneous traffic flow. Let $\Delta > 0$ be fixed. The choice of Δ depends on the vehicle type in the homogeneous traffic flow. Let $a_i^k(t)$ be the acceleration of vehicle i on the current lane $k \in \mathcal{K}$ at time $t \in [0, T]$, and $\bar{a}_i^{k'}(t)$ the expected acceleration of vehicle i on the adjacent lane $k' = k + 1$ or $k - 1$ at time t. In addition, We assume that the accelerations and expected accelerations (if there is lane-changing) of vehicles in the homogeneous traffic flow are bounded from above by $M \in \mathbb{R}_{\geq \Delta}$. Denote $i_L^{k'}$, $i_F^{k'}$ the label of the leading and following vehicle of i-th vehicle on the adjacent lane k', respectively, if vehicle i performs lane-changing from lane k to lane k' and $\bar{a}^{k'}_{i^{k'}_F}(t)$ the expected acceleration of vehicle $i^{k'}_F$ at time $t \in [0,T]$.

In a homogeneous traffic flow, vehicle i will perform lane changing at time $t \in [0,T]$ from lane k to lane k' with probability $p([\bar{a}^{k'}_i(t) - a^k_i(t) - \Delta]_+, [\bar{a}^{k'}_i(t) + \Delta]_+, [\bar{a}^{k'}_{i^{k'}_F}(t) + \Delta]_+)$ if the following conditions are satisfied

$$\text{Incentive: } \bar{a}^{k'}_i(t) \geq a^k_i(t) + \Delta$$
$$\text{Safety: } \bar{a}^{k'}_i(t) \geq -\Delta \text{ and } \bar{a}^{k'}_{i^{k'}_F}(t) \geq -\Delta.$$

In particular, if the expected acceleration of vehicle i on its neighbor lane k' is sufficiently bigger than its actual acceleration on its current lane k, then vehicle i has the incentive to perform lane-changing from lane k to lane k'. The safety condition guarantees that there is no excessive breaking for both vehicle i and its new follower $i^{k'}_F$ on the adjacient lane k' if vehicle i changing from lane k to lane k'.

Furthermore, a possible choice of the probability function is

$$p\colon (\mathbb{R}^+)^3 \to [0,1] \text{ with } p(b_1,b_2,b_3) = \frac{1}{C}\big(1 - e^{-\gamma b_1 b_2 b_3}\big) \in [0,1], \gamma > 0, \tag{10}$$

where C is a renormalization constant defined as

$$C = \max_{[0,2M+\Delta]^3} \big(1 - e^{-\gamma b_1 b_2 b_3}\big) = 1 - e^{-\gamma(2M+\Delta)^3}.$$

But our result will be valid for any function having the following properties:

- It strictly increases with respect to each one of its input;
- If one of its input is zero, then the probability function value is zero.

Now we will consider the lane-changing condition in a heterogeneous traffic flow encompassing cars and trucks. Specifically, we need to modify the "incentive" and "safety" conditions according to the different car-following combinations and vehicle types.

Let $\Delta^n > 0$, with $n \in \{cc, ct, tc, tt, c, t\}$, be fixed. Vehicle $i \in \mathcal{I}$ will change from lane k to lane k' at time $t \in [0,T]$ with a given certain probability (to be defined later) if the following conditions are satisfied:

$$\text{Incentive: } \bar{a}^{k'}_i(t) \geq \begin{cases} a^k_i(t) + \Delta^{cc} \text{ if } i, i^{k'}_F \in \mathcal{I}_P, \\ a^k_i(t) + \Delta^{tc} \text{ if } i \in \mathcal{I}_P, i^{k'}_F \in \mathcal{I}_S, \\ a^k_i(t) + \Delta^{ct} \text{ if } i \in \mathcal{I}_S, i^{k'}_F \in \mathcal{I}_P, \\ a^k_i(t) + \Delta^{tt} \text{ if } i, i^{k'}_F \in \mathcal{I}_S; \end{cases} \tag{11}$$

$$\text{Safety: } \bar{a}^{k'}_i(t) \geq \begin{cases} -\Delta^c \text{ and } \bar{a}^{k'}_{i^{k'}_F}(t) \geq -\Delta^c \text{ if } i, i^{k'}_F \in \mathcal{I}_P \\ -\Delta^c \text{ and } \bar{a}^{k'}_{i^{k'}_F}(t) \geq -\Delta^t \text{ if } i \in \mathcal{I}_P, i^{k'}_F \in \mathcal{I}_S, \\ -\Delta^t \text{ and } \bar{a}^{k'}_{i^{k'}_F}(t) \geq -\Delta^c \text{ if } i \in \mathcal{I}_S, i^{k'}_F \in \mathcal{I}_P, \\ -\Delta^t \text{ and } \bar{a}^{k'}_{i^{k'}_F}(t) \geq -\Delta^t \text{ if } i, i^{k'}_F \in \mathcal{I}_S. \end{cases} \tag{12}$$

For instance, the probability of vehicle $i \in \mathcal{I}_P$ performing lane-changing from lane k to lane k' with its new follower on lane k' being a truck, i.e., $i^{k'}_F \in \mathcal{I}_S$, at time $t \in [0,T]$ is

$$p([\bar{a}^{k'}_i(t) - a^k_i(t) - \Delta^{tc}]_+,\, [\bar{a}^{k'}_i(t) + \Delta^c]_+,\, [\bar{a}^{k'}_{i^{k'}_F}(t) + \Delta^t]_+)$$

and the probability of vehicle $i \in \mathcal{I}_S$ performing lane-changing from lane k to lane k' with its new follower on lane k' being a car, i.e., $i_F^{k'} \in \mathcal{I}_P$, at time $t \in [0,T]$ is

$$p([\bar{a}_i^{k'}(t) - a_i^k(t) - \Delta^{ct}]_+,\, [\bar{a}_i^{k'}(t) + \Delta^t]_+,\, [\bar{a}_{i_F^{k'}}^{k'}(t) + \Delta^c]_+).$$

Here the probability function p is defined in Eq. (10) with renormalization constant

$$C = \max_{[0,2M^*+\Delta^*]^3} \left(1 - e^{-\gamma b_1 b_2 b_3}\right) = 1 - e^{-\gamma (2M^*+\Delta^*)^3},$$

where $\Delta^* = \min\{\Delta^{cc}, \Delta^{ct}, \Delta^{tc}, \Delta^{tt}, \Delta^c, \Delta^t\}$ and $M^* \in \mathbb{R}_{\geq \Delta}$ is a common upper bound for the acceleration of both cars and trucks, i.e., for every $t \in [0,T]$ and $i \in \mathcal{I}$, $|a_i(t)| < M^*$.

Note that in the heterogeneous traffic condition, each acceleration a_i^k, $\bar{a}_i^{k'}$, and $\bar{a}_{i_L^{k'}}^{k'}$ has four different alternatives based on the four different car-truck car-following combinations. Furthermore, by equation (3), each acceleration a_i^k, $\bar{a}_i^{k'}$, and $\bar{a}_{i_L^{k'}}^{k'}$ depends on the space gap, the velocity of the reference vehicle and the velocity of the leader of the reference vehicle. The incentive condition defined in equation (11) implies that, before changing lane, vehicle i needs to check its space gap, velocity and velocity difference with its leading vehicle on the current and adjacent lane. The safety condition defined in equation (11) implies that there is no excessive breaking for vehicle i and its follower $i_L^{j'}$ on the adjacent lane k' after lane-changing.

2.3. Overview on Generalized Wasserstein Distance and a revised version of Ascoli-Arzelá theorem.

In this subsection, we recall some notions and properties related to the generalized Wasserstein distance ([35]) and give the statement of a revised version of Ascoli-Arzelá theorem ([20]).

The Generalized Wasserstein Distance.

In the following we denote with

- d the dimension of the space;
- $\mathcal{M}$ the space of Borel measures with finite mass on $\mathbb{R}^d$;
- $\operatorname{supp}\mu$ the support of measure $\mu \in \mathcal{M}$;
- $\mathcal{P}$ be the space of probability measures (the measures in $\mathcal{M}$ with unit mass) on $\mathbb{R}^d$;
- $\mathcal{M}^p$ be the space of Borel measures with finite p-th moment on $\mathbb{R}^d$;
- $\mathcal{M}_0^{ac} \subset \mathcal{M}$ the subset of measures that are absolutely continuous with respect to the Lebesgue measure with bounded support.

Given a measure $\mu \in \mathcal{M}$, we denote its mass by $|\mu|\colon = \mu(\mathbb{R}^d)$. Given a Borel map $\gamma\colon \mathbb{R}^d \mapsto \mathbb{R}^d$, the push-forward of $\mu \in \mathcal{M}$ by γ, $\gamma\#\mu$, is defined as for every Borel set $A \subset \mathbb{R}^d$, $\gamma\#\mu(A)\colon = \mu(\gamma^{-1}(A))$. One can see that the mass of $\gamma\#\mu$ is identical to the mass of μ, i.e., $|\mu| = |\gamma\#\mu|$.

We use the notation $\mu_1 \leq \mu$ when μ_1 is absolutely continuous with repsect to $\mu \in \mathcal{M}$ and for every Borel set $A \subset \mathbb{R}^d$, $\mu_1(A) \leq \mu(A)$.

Now we recall the definition of the generalized Wasserstein distance on $\mathcal{M}$.

Definition 2.1. Given $a, b \in (0,\infty)$ and $p \geq 1$, the generalized Wasserstein distance between two measures $\mu, \nu \in \mathcal{M}^p$ is

$$W_p^{a,b}(\mu,\nu)\colon = \inf_{\substack{\tilde{\mu},\tilde{\nu} \in \mathcal{M}^p \\ |\tilde{\mu}| = |\tilde{\nu}|}} \left(a\left(|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}|\right) + b W_p(\tilde{\mu},\tilde{\nu})\right), \tag{13}$$

where $W_p(\tilde{\mu},\tilde{\nu})$ is the Wasserstein distance between the measures $\tilde{\mu},\tilde{\nu}\in\mathcal{M}^p$ with $|\tilde{\mu}|=|\tilde{\nu}|$.

We recall that the standard Wasserstein distance is defined only for Borel measures with the same mass and combining it with the L^1 distance we get the generalized Wasserstein distance which can be applied instead to measures with different masses.

Remark 2.1. Note that the infimum on the right hand side of equation (13) is actually a minimum if one takes $\tilde{\mu}\le\mu$ and $\tilde{\nu}\le\nu$.

We recall the following key result (Proposition 2 in [35]).

Proposition 2.2. The following properties of the generalized Wasserstein distance $W_1^{1,1}$ hold for measures $\mu,\nu,\mu_1,\mu_2,\nu_1,\nu_2\in\mathcal{M}^p$

$$\begin{aligned}&W_1^{1,1}(k\mu,k\nu)\le kW_1^{1,1}(\mu,\nu)\text{ for }k\ge 0;\\&W_1^{1,1}(\mu_1+\mu_2,\nu_1+\nu_2)\le W_1^{1,1}(\mu_1,\nu_1)+W_1^{1,1}(\mu_2,\nu_2).\end{aligned}$$

To be self-contained, we list the following two lemmata from [20].

Lemma 2.3. For every $f,g\colon\mathbb{R}^n\to\mathbb{R}^n$ bounded Borel measureable functions and $\mu\in\mathcal{M}^1(\mathbb{R}^n)$, one has

$$W_1^{1,1}(f\#\mu,g\#\mu)\le\|f-g\|_{L^\infty(\operatorname{supp}\mu)}.$$

Moreover if f is a locally Lipschitz continuous Borel measurable function with Lipschitz constant L on the ball B of $\mathbb{R}^n$, then for $\mu,\nu\in\mathcal{M}^1(\mathbb{R}^n)$ compactly supported on B,

$$W_1^{1,1}(f\#\mu,f\#\nu)\le\max\{L,1\}W_1^{1,1}(\mu,\nu).$$

Lemma 2.4. Let H be a locally Lipschitz continuous map with sub-linear growth. Let $R>0$ be fixed, and d be the dimension of the space, and $\mu,\nu\colon[0,T]\mapsto\mathcal{M}^1(\mathbb{R}^d)$ be continuous maps with respect to the generalized Wasserstein distance $W_1^{1,1}$ such that for every $t\in[0,T]$

$$\operatorname{supp}\mu(t)\subset B(0,R)\text{ and }\operatorname{supp}\nu(t)\subset B(0,R).$$

For every $\rho>0$, there exists a constant $L_{\rho,R}$ such that

$$\|H*\mu(t)-H*\nu(t)\|_{L^\infty(B(0,\rho))}\le L_{\rho,R}W_1^{1,1}(\mu(t),\nu(t)).\tag{14}$$

An extended version of Ascoli-Arzelá theorem

In the following we recall an extended version of Ascoli-Arzelá theorem from [20].

Theorem 2.5. Let K be a compact subset of $\mathbb{R}$ and let E be a complete and totally bounded metric space with metric d. Consider a sequence of functions $\{f_n\}_{n=1}^\infty$ in $C(K;E)$. If there exists $L>0$, such that the following is true: for any $\varepsilon>0$, there exists $N>0$, such that, whenever $n\ge N$,

$$d(f_n(t),f_n(s))\le L|t-s|+\min\{\varepsilon,|t-s|\},\forall s,t\in K$$

then the sequence $\{f_n\}_{n=1}^\infty$ has a uniformly convergent sub-sequence.

2.4. **Formal derivation of a mean-field limit of a finite dimensional ODE system.** In this subsection, we start with a finite dimensional ODE system and derive its mean-field limit formally.

Let $R > 0$ be fixed. Let $M, N \in \mathbb{Z}_{>0}$. Denote with D the domain $\mathbb{R} \times \mathbb{R}_{\geq 0}$, with $D^M = \mathbb{R}^M \times \mathbb{R}^M_{\geq 0}$ and $D^N = \mathbb{R}^N \times \mathbb{R}^N_{\geq 0}$. Let $H_i \colon D \to \mathbb{R}$ with $i = 1, \dots, 8$ be locally Lipschitz continuous maps with sub-linear growth. Then given an initial datum $I_0 \colon = (x_0, v_0, y_0, w_0) \in D^M \times D^N$, there exists a unique solution $I(t) \colon = (x(t), v(t), y(t), w(t)) \in D^M \times D^N$ on the whole time interval $[0, T]$ to the following system of ODEs on $D^M \times D^N$

$$\begin{aligned}
&\dot{x}_i(t) = v_i(t), \quad i = 1, \dots, M \\
&\dot{v}_i(t) = \left(H_1 *_1 \mu_M + H_2 *_1 \nu_N + H_3 * \mu_M + H_4 * \nu_N\right)(x_i, v_i) \\
&\dot{y}_j(t) = w_j(t), \quad j = 1, \dots, N \\
&\dot{w}_j(t) = \left(H_5 *_1 \mu_M + H_6 *_1 \nu_N + H_7 * \mu_M + H_8 * \nu_N\right)(y_j, w_j)
\end{aligned} \tag{15}$$

where $\mu_M, \nu_N \colon [0, T] \mapsto \mathcal{P}(D) \cap \mathcal{M}^1(D)$ are defined as follows

$$\mu_M(t) = \frac{1}{M} \sum_{i=1}^{M} \delta_{(x_i(t), v_i(t))}, \qquad \nu_N(t) = \frac{1}{N} \sum_{j=1}^{N} \delta_{(y_j(t), w_j(t))}. \tag{16}$$

For more details, we refer the readers to [15].

Let us further assume that for each $t \in [0, T]$, the empirical measures $\mu_M(t), \nu_N(t)$ in $\mathcal{P}(D) \cap \mathcal{M}^1(D)$ are with uniform support in both M and N. By Prohorov's theorem (see [6]) it follows that the sequences $(\mu_M)_M$ and $(\nu_N)_N$ are weakly* relatively compact. Therefore, there exist subsequences $(\mu_{M_k})_k$, $(\nu_{N_k})_k$ and $\mu, \nu \in \mathcal{P}(D) \cap \mathcal{M}^1(D)$ such that

$$\begin{aligned}
&\mu_{M_k} \to \mu \text{ as } k \to \infty \\
&\nu_{N_k} \to \nu \text{ as } k \to \infty
\end{aligned} \tag{17}$$

with weak* convergence in $\mathcal{P}(D) \cap \mathcal{M}^1(D)$ pointwise in time.

Now we take $M, N \to \infty$ in Eq. (15) and derive the mean-field limit of the finite-dimensional ODE system formally. Let us consider a test function $\varphi \in C^1_0(D)$ and we compute

$$\begin{aligned}
&\frac{d}{dt} \langle \mu_M(t) \,,\, \varphi \rangle = \frac{1}{M} \sum_{i=1}^{M} \frac{d}{dt} \varphi(x_i(t), v_i(t)) \\
=&\frac{1}{M} \sum_{i=1}^{M} \big(\partial_x \varphi(x_i(t), v_i(t)) v_i(t) + \partial_v \varphi(x_i(t), v_i(t)) \dot{v}_i(t) \big) \\
=&\frac{1}{M} \sum_{i=1}^{M} \partial_x \varphi(x_i(t), v_i(t)) v_i(t) + \\
&+ \frac{1}{M} \sum_{i=1}^{M} \big(\partial_v \varphi(x_i(t), v_i(t)) \left(H_1 *_1 \mu_M + H_2 *_1 \nu_N + H_3 * \mu_M + H_4 * \nu_N\right)(x_i, v_i) \big) \\
=&\langle \mu_M(t), \partial_x \varphi(x, v) v \rangle \\
&+ \langle \mu_M(t), \partial_v \varphi(x, v) \left(H_1 *_1 \mu_M + H_2 *_1 \nu_N + H_3 * \mu_M + H_4 * \nu_N\right)(x, v) \rangle \\
=&- \langle \partial_x \mu_M(t) v, \varphi \rangle \\
&- \langle \partial_v \left(H_1 *_1 \mu_M + H_2 *_1 \nu_N + H_3 * \mu_M + H_4 * \nu_N\right)(x, v) \mu_M(t), \varphi \rangle
\end{aligned}$$

which implies

$$\partial_t\mu_M(t) + v\partial_x\mu_M(t) + \partial_v[(H_1 *_1 \mu_M + H_2 *_1 \nu_N + H_3 * \mu_M + H_4 * \nu_N)(x,v)\mu_M(t)] = 0. \tag{18}$$

In addition, we have

$$\begin{aligned}
&\frac{d}{dt}\left\langle \nu_N(t)\,,\,\varphi\right\rangle = \frac{1}{N}\sum_{j=1}^{N}\frac{d}{dt}\varphi(y_j(t),w_j(t))\\
=&\sum_{j=1}^{N}\partial_x\varphi(y_j(t),w_j(t))w_j(t) + \sum_{j=1}^{N}\partial_v\varphi(y_i(t),w_i(t))\dot{w}_j(t)\\
=&\frac{1}{N}\sum_{j=1}^{N}\partial_x\varphi(y_j(t),w_j(t)) +\\
&+\frac{1}{N}\sum_{j=1}^{N}\big(\partial_v\varphi(y_j(t),w_j(t))\,(H_5 *_1 \mu_M + H_6 *_1 \nu_N + H_7 * \mu_M + H_8 * \nu_N)\,(y_j,w_j)\big)\\
=&\left\langle \nu_N(t), \partial_x\varphi(x,v)v\right\rangle\\
&+\left\langle \nu_N(t), \partial_v\varphi(x,v)\,(H_5 *_1 \mu_M + H_6 *_1 \nu_N + H_7 * \mu_M + H_8 * \nu_N)\,(x,v)\right\rangle\\
=&-\left\langle \partial_x\nu_N(t)v, \varphi(x,v)\right\rangle\\
&-\Big\langle \partial_v\Big(\nu_N(t)\,(H_5 *_1 \nu_N + H_6 *_1 \nu_N + 7 * \mu_M + H_8 * \nu_N)\,(x,v)\Big), \varphi\Big\rangle
\end{aligned}$$

from which we conclude

$$\partial_t\nu_N(t) + v\partial_x\nu_N(t) + \partial_v\left[(H_5 *_1 \mu_M + H_6 *_1 \nu_N + H_7 * \mu_M + H_8 * \nu_N)(x,v)\nu_N(t)\right] = 0. \tag{19}$$

Combine with equations (17), (18) and (19), for the limit of $k \to \infty$ of the subsequences $(\mu_{M_k})_k$ and $(\nu_{N_k})_k$, formally we have

$$\begin{aligned}
&\partial_t\mu(t) + v\partial_x\mu(t) + \partial_v[(H_1 *_1 \mu + H_2 *_1 \nu + H_3 * \mu + H_4 * \nu)\,(x,v)\mu(t)] = 0,\\
&\partial_t\nu(t) + v\partial_x\nu(t) + \partial_v\left[(H_5 *_1 \mu + H_6 *_1 \nu + H_7 * \mu + H_8 * \nu)(x,v)\nu(t)\right] = 0.
\end{aligned}$$

2.5. **Partial Differential Equations of Vlasov-type.** In the last section, we recall some related results on partial differential equations of Vlasov-type with and without source terms.

2.5.1. *Partial Differential Equations of Vlasov-type without Source Term.* A family of Lipschitz continuous flow maps is associated to the system (15)

$$\mathcal{T}_t^{\mu,\nu}\colon I_0 \in D^M \times D^N \mapsto I(t) \in D^M \times D^N. \tag{20}$$

indexed by $t \in [0,T]$. For more details we refer to [15].

Given the initial conditions $(\mu_0,\nu_0) \in \Big(\mathcal{P}(D)\cap\mathcal{M}^1(D)\Big)^2$ with bounded support, we say that the couple of measures $(\mu(t),\nu(t))$ is a weak equi-compactly supported solution of the following Vlasov-type PDE system with the initial datum (μ_0,ν_0),

$$\begin{aligned}
&\partial_t\mu + v\cdot\partial_x\mu + \partial_v\cdot[(H_1 *_1 \mu + H_2 *_1 \nu + H_3 * \mu + H_4 * \nu)\mu] = 0,\\
&\partial_t\nu + v\cdot\partial_x\nu + \partial_v\cdot[(H_5 *_1 \mu + H_6 *_1 \nu + H_7 * \mu + H_8 * \nu)\nu] = 0,
\end{aligned} \tag{21}$$

if (i) $\mu(0)=\mu_0$ and $\nu(0)=\nu_0$;
(ii) $\operatorname{supp}\mu(t), \operatorname{supp}\nu(t) \subset B^D(0,R)$ for all $t\in[0,T]$;
(iii) for every $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D \varphi(x,v)\,\mathrm{d}\mu(t)(x,v) = \int_D \nabla\varphi(x,v)\cdot\tilde{\omega}(t,x,v)\,\mathrm{d}\mu(t)(x,v),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D \varphi(y,w)\,\mathrm{d}\nu(t)(y,w) = \int_D \nabla\varphi(y,w)\cdot\hat{\omega}(t,y,w)\,\mathrm{d}\nu(t)(y,w)$$

where $\tilde{\omega}(t,x,v)=\tilde{\omega}_{H_1,H_2,H_3,H_4,\mu,\nu}(t,x,v)\colon [0,T]\times D\mapsto\mathbb{R}^2$ is defined as

$$\tilde{\omega}_{H_1,H_2,H_3,H_4,\mu,\nu}(t,x,v)\colon = \left(v, \left(H_1 *_1 \mu + H_2 *_1 \nu + H_3 * \mu + H_4 * \nu\right)(x,v)\right), \quad (22)$$

and $\hat{\omega}(t,y,w)=\hat{\omega}_{H_5,H_6,H_7,H_8,\mu,\nu}(t,y,w)\colon [0,T]\times D\mapsto\mathbb{R}^2$ is defined as

$$\hat{\omega}_{H_5,H_6,H_7,H_8,\mu,\nu}(t,y,w)\colon = \left(v, \left(H_5 *_1 \mu + H_6 *_1 \nu + H_7 * \mu + H_8 * \nu\right)(y,w)\right). \quad (23)$$

Furthermore, following from Section 8.1 in [2], the couple of measures $(\mu(t),\nu(t))$ is a weak equi-compactly supported solution of the system (21) if and only if it satisfies condition (ii) and the measure-theoretical fixed point equation $(\mu(t),\nu(t)) = (\mathcal{T}_t^{\mu,\nu})\#(\mu_0,\nu_0)$ where the flow function $\mathcal{T}_t^{\mu,\nu}$ is defined in equation (20).

2.5.2. *Partial Differential Equations of Vlasov-type with Source Term.* Now we consider solutions to the following Vlasov-type PDE system with initial datum $(\mu_0,\nu_0)\in\left(\mathcal{M}_0^{ac}(D)\cap\mathcal{M}^1(D)\right)^2$, and source terms G_1 and G_2

$$\begin{aligned}&\partial_t\mu + v\partial_x\mu + \partial_v\left[(H_1 *_1 \mu + H_2 *_1 \nu + H_3 * \mu + H_4 * \nu)\mu\right] = G_1(\mu,\nu)\\ &\partial_t\nu + v\partial_x\nu + \partial_y\left[(H_5 *_1 \mu + H_6 *_1 \nu + H_7 * \mu + H_8 * \nu)\nu\right] = G_2(\mu,\nu)\end{aligned} \quad (24)$$

under the following hypotheses:

(**A1**) $G_1(\mu,\nu), G_2(\mu,\nu)$ have uniformly bounded mass and support, that is, there exist $Q, R,$ such that $|G_1(\mu,\nu)|(D), |G_2(\mu,\nu)|(D)\le Q$, and $\operatorname{supp}(G_1(\mu,\nu)), \operatorname{supp}(G_2(\mu,\nu))\subset B^D(0,R)$;

(**A2**) G_1 and G_2 are Lipschitz, that is, there exists $L,$ such that, for any μ,μ', $\nu,\nu'\in\mathcal{M}^1(D), W_1^{1,1}(G_i(\mu,\nu),G_i(\mu',\nu'))\le L\Big(W_1^{1,1}(\mu,\mu')+W_1^{1,1}(\nu,\nu')\Big)$, $i=1,2$.

A coupled of measures $(\mu(t),\nu(t))$ are weak solutions of equation (24) with a given initial datum $(\mu_0,\nu_0)\in\left(\mathcal{M}_0^{ac}(D)\cap\mathcal{M}^1(D)\right)^2$, if $\mu(0)=\mu_0, \nu(0)=\nu_0$ and if for every $\varphi\in C_c^\infty(\mathbb{R}^2)$, it holds

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_D \varphi(x,v)\,\mathrm{d}\mu(t)(x,v) =\\ &= \int_D \varphi(x,v)\,\mathrm{d}G_1(\mu,\nu)(x,v) + \int_{D_1}\nabla\varphi(x,v)\cdot\tilde{\omega}_{H_1,H_2,H_3,H_4,\mu,\nu}(t,x,v)\,\mathrm{d}\mu(t)(x,v),\end{aligned}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_D \varphi(y,w)\,\mathrm{d}\mu(t)(y,w) =\\ &= \int_D \varphi_2(y,w)\,\mathrm{d}G_2(\mu,\nu)(y,w) + \int_D\nabla\varphi(y,w)\cdot\hat{\omega}_{H_5,H_6,H_7,H_8,\mu,\nu}(t,y,w)\,\mathrm{d}\nu(t)(y,w),\end{aligned}$$

where $\tilde{\omega}, \hat{\omega}$ is as defined in equations (22) and (23). We have the following:

Theorem 2.6. Given an initial datum $(\mu_0, \nu_0) \in \Big(\mathcal{M}_0^{ac}(D) \cap \mathcal{M}^1(D)\Big)^2$, under the hypotheses (**A1**)-(**A2**), there exists a unique weak solution $(\mu(t), \nu(t))$ to the system (24) in $\Big(\mathcal{M}_0^{ac}(D) \cap \mathcal{M}^1(D)\Big)^2$.

A weak solution $(\mu(t), \nu(t))$ to equation (24) can be constructed using a sample-and-hold Lagrangian scheme. For a fixed $j \in \mathbb{N}^+$, define $\Delta t\colon = \frac{T}{2^j}$ and decompose the time interval $[0, T]$ in $[0, \Delta t), [\Delta t, 2\Delta t), \dots, [(2^j - 1)\Delta t, 2^j \Delta t)$, define

Initial step $(\mu_j(0), \nu_j(0)) \coloneqq (\mu_0, \nu_0)$;

Recursive step 1 $(\mu_j((n+1)\Delta t), \nu_j((n+1)\Delta t)) \coloneqq \mathcal{T}_{\Delta t}^{\mu_j(n\Delta t), \nu(n\Delta t)} \#(\mu_j(n\Delta t),$
$\nu_j(n\Delta t)) + \Delta t\big(G_1(\mu_j(n\Delta t), \nu_j(n\Delta t)), G_2(\mu_j(n\Delta t), \nu_j(n\Delta t))\big)$;

Recursive step 2 $(\mu_j(t), \nu_j(t)) \coloneqq \mathcal{T}_{\tau}^{\mu_j(n\Delta t), \nu(n\Delta t)} \#(\mu_j(n\Delta t), \nu_j(n\Delta t)) +$
$+\tau\big(G_1(\mu_j(n\Delta t), \nu_j(n\Delta t)), G_2(\mu_j(n\Delta t), \nu_j(n\Delta t))\big)$;

where n is the maximum integer such that $t - n\Delta t \geq 0$ and $\tau\colon = t - n\Delta t$. Then $(\mu(t), \nu(t)) = \lim_{j\to\infty} (\mu_j(t), \nu_j(t))$ is the unique weak solution to equation (24). For more detail, please see [35].

3. **Finite-dimensional hybrid system.** In order to describe the frequencies of the vehicles' lane change behavior and prove the well-posedness of our heterogeneous multi-lane traffic model, it is critical to introduce the model assumption "cool-down" time. Indeed, empirical observations showed that the lane-changing frequency of vehicles on the highway is low. A key example is a study done on the German highway which shows that only 15% of the vehicles performes lane-change while traveling the recorded road segment [26]. For this reason, the chance of two vehicles performing lane-change at exactly the same time is even lower and it is reasonable to assume that this does not happen at all. In the next we state mathematically what just explained.

Each vehicle $i \in \mathcal{I}$ is associated to a timer τ_i and the initial timer differs from vehicle to vehicle. We introduce a "cool-down" time $\bar{\tau} = \frac{T}{N_\tau}$, with $N_\tau \in \mathbb{N}_{\geq 0}$ large. Every vehicle checks the lane-changing conditions only when its timer reaches $\bar{\tau}$. When this happens, the vehicle's timer is then set to 0. More explicitely, for each vehicle $i \in \mathcal{I}$, its timer τ_i satisfies the differential equation

$$\dot{\tau}_i(t) = 1,\ \tau_i(0) = \tau_{i,0},\ t \in [0, \bar{\tau})$$

with the following assumption on the initial data:

$$i \neq j \in \mathcal{I} \quad \Longrightarrow \quad \tau_{i,0} \neq \tau_{j,0}. \tag{25}$$

When $t = \bar{\tau}$ we set $\tau_i(t) = 0$. We can also model a large lane-change frequency by simply choosing a small cool-down time $\bar{\tau}$.

In the case of finitely many vehicles, the presence of the cool-down time, $\bar{\tau}$ allows us to consider a small time interval $[0, t_1]$ during which there is no vehicle changing lane, with

$$t_1 = \min_{i\in\mathcal{I}} \{\bar{\tau} - \tau_{i,0}\}$$

Condider the space $X = \mathbb{R} \times \mathbb{R}_{\geq 0} \times [0, \bar{\tau})$ and the set $\mathcal{L} = \big\{\ell = (\ell_i)_{i\in\mathcal{I}} \in \mathcal{K}^{P+S}\big\}$ of symbols that represent all possible lane labels of all vehicles among cars and trucks.

Let $A_\ell \subset X$ be the set of triples position-velocity-timer of all vehicles among which there are at least two vehicles occupying the same lane and position at certain time, i.e.

$$A_\ell = \big\{ \big(x_i, v_i, \tau_i\big)_{i\in\mathcal{I}} \in X \colon \exists t \in [0,T], i_1, i_2 \in \mathcal{I}, \\ s.t., x_{i_1}(t) = x_{i_2}(t) \wedge \ell_{i_1}(t) = \ell_{i_2}(t), \text{ with } \ell_{i_1}, \ell_{i_2} \in \mathcal{K} \big\}, \tag{26}$$

As in Section 2, let $\mathcal{I} = \{1, \dots, P+S\}$ be the set of index for all the vehicles, $\mathcal{I}_P = \{1, \dots, P\}$ the set of index for cars and $\mathcal{I}_S = \{P+1, \dots, P+S\}$ for trucks. Denote respectively with with $\mathcal{I}_P^k(t)$ and $\mathcal{I}_S^k(t)$ the set of indices for cars and trucks on lane k at time $t \in [0,T]$, with $P_k(t)$ and $S_k(t)$ the number of cars and trucks on lane k at time $t \in [0,T]$.
The time dependent atomic probability measures on the k lane are given by

$$\mu_P^k(t) = \frac{1}{P_k(t)} \sum_{i\in\mathcal{I}_P^k(t)} \delta_{(x_i(t), v_i(t))}, \qquad \mu_S^k(t) = \frac{1}{S_k(t)} \sum_{i\in\mathcal{I}_S^k(t)} \delta_{(x_i(t), v_i(t))}. \tag{27}$$

where $(x_i(t), v_i(t))$ are solutions of the following first order system:

$$\begin{aligned} &\dot{x}_i = v_i \quad i \in \mathcal{I}\,, \\ &\dot{v}_i = \begin{cases} \big(H_1^{cc} *_1 \mu_P^k + H_1^{tc} *_1 \mu_S^k\big)(x_i, v_i) + \big(H_2^{cc} * \mu_P^k + H_2^{tc} * \mu_S^k\big)(x_i, v_i) & i \in \mathcal{I}_P^k \\ \big(H_1^{ct} *_1 \mu_P^k + H_1^{tt} *_1 \mu_S^k\big)(x_i, v_i) + \big(H_2^{ct} * \mu_P^k + H_2^{tt} * \mu_S^k\big)(x_i, v_i) & i \in \mathcal{I}_S^k \end{cases}. \end{aligned} \tag{28}$$

And finally consider the switching set $LC(\Sigma)$ describing the lane-changing mechanism of the finitely many vehicles:

$$\begin{aligned} LC(\Sigma) = \Big\{ &(\ell, (x_i, v_i, \tau_i), \ell', (x_i', v_i', \tau_i'))_{i\in\mathcal{I}} \in (\mathcal{L} \times X)^2 \colon \\ &\exists\, i_0 \in \mathcal{I}, \exists\, t_0 \in [0, \bar{\tau}), s.t., j \neq i_0, (\ell_j(t_0), x_j(t_0), v_j(t_0), \tau_j(t_0)) \\ &=(\ell_j'(t_0), x_j'(t_0), v_j'(t_0), \tau_j'(t_0)) \wedge (x_{i_0}(t_0), v_{i_0}(t_0)) \\ &=(x_{i_0}'(t_0), v_{i_0}'(t_0)), \ell_{i_0}'(t_0) = \ell_{i_0}(t_0) \pm 1, \tau_{i_0}'(t_0) = 0 \Big\}. \end{aligned} \tag{29}$$

Now we are ready to give the definition of hybrid system.

Definition 3.1. A hybrid system is a 4-tuple $\Sigma = (\mathcal{L}, \mathcal{M}, g, SW)$ where:

(1) $\mathcal{L} = \big\{ \ell = (\ell_i)_{i\in\mathcal{I}} \in \mathcal{K}^{P+S} \big\}$ is a finite set of symbols that represent all possible lane labels of all vehicles;
(2) $\mathcal{M} = \{\mathcal{M}_\ell\}_{\ell\in\mathcal{L}}$, where $\mathcal{M}_\ell = (X \setminus A_\ell)^{P+S}$, with A_ℓ defined in (26).
(3) $g = \{g_\ell\}_{\ell\in\mathcal{L}}$, $g_\ell \colon \mathcal{M}_\ell \mapsto \mathbb{R}^{3(P+S)}$, $g_{(\ell_i)} = (v_i, a_i, 1)$, where $a_i = \dot{v}_i$ as defined in systems (28);
(4) SW is a subset of $LC(\Sigma)$, where $LC(\Sigma)$ is the set of states for which a lane-changing can occur, that is (29).

We need two further definitions before stating and proving the result of existence of solutions for the hybrid system.

Definition 3.2. A hybrid state of the hybrid system Σ is a 4-tuple $(\ell, x, v, \tau) \in \mathcal{L} \times \mathcal{M}_\ell$. The set of all the hybrid states of the hybrid system Σ will be called $\mathcal{HS}$.

Definition 3.3. Let $(\ell_0, x_0, v_0, \tau_0) \in (\mathcal{K} \times X)^{P+S}$ be an initial condition to the hybrid system Σ and assume that τ_0 satisfies (25). A trajectory of the hybrid system Σ with initial condition $(\ell_0, x_0, v_0, \tau_0)$ is a map $\varphi\colon [0,T] \to \mathcal{HS}$, $\varphi(t) = (\ell(t), x(t), v(t), \tau(t))$, such that

(1) $(\ell(0), x(0), v(0), \tau(0)) = (\ell_0, x_0, v_0, \tau_0)$;
(2) $\ell_i[0, \bar\tau - \tau_{i,0}) = \ell_{i,0}, \quad i \in \mathcal{I}$;
$\ell_i[n\bar\tau - \tau_{i,0}, (n+1)\bar\tau - \tau_{i,0}) = \ell_{i,n} \in \mathcal{L}, \quad i \in \mathcal{I}$;
(3) $\tau_i(n\bar\tau - \tau_{i,0}) = 0 \quad i \in \mathcal{I}$;
(4) $\lim\limits_{t \to (n\bar\tau - \tau_{i,0})^-} x_i(t) = x_i(n\bar\tau - \tau_{i,0})$;
(5) For almost every $t \in [0,T]$
$$\frac{d}{dt}(x_i, v_i, \tau_i) = g_{\ell_i(t)}(x_i(t), v_i(t), \tau_i(t)) \quad i \in \mathcal{I}. \tag{30}$$

Theorem 3.1 (Existence and uniqueness of trajectories to the hybrid system Σ)**.** Let $H_1^n : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, and $H_2^n : \mathbb{R} \times \mathbb{R}_{\geq} 0 \to \mathbb{R}$ with $n \in \{cc, ct, tc, tt\}$ be locally Lipschitz convolution kernels with sub-linear growth and let $(\ell_0, x_0, v_0, \tau_0) \in (\mathcal{K} \times X)^{P+S}$ be a given initial datum. Then there exists a unique trajectory $\varphi : [0,T] \to \mathcal{HS}$ to hybrid system Σ, which is also Lipschitz continuous in time over the time interval in which no lane change occurs.

Proof. Let $t_0 = \min_{i \in \mathcal{I}}\{\bar\tau - \tau_{i,0}\}$ where $\tau_{i,0}$ is the i-th component of the vector $\tau_0 \in [0, \bar\tau)^{P+S}$. By definition no lane change is performed in the time interval $[0, t_0)$ and the dynamic of each vehicle in the lane k is given by (28). More compactly, we can call $\varphi^k(t) = (x^k(t), v^k(t)) \in (\mathbb{R} \times \mathbb{R}_{\geq 0})^{P_k + S_k}$ the trajectory of vehicles on the lane k over the time interval $[0, t_0)$ and re-write the system (28) as
$$\dot\varphi^k(t) = g^k(t, \varphi^k(t)). \tag{31}$$
Here $g^k\colon [0, t_0) \times (\mathbb{R} \times \mathbb{R}_{\geq 0})^{P_k+S_k} \mapsto (\mathbb{R} \times \mathbb{R}_{\geq 0})^{P_k+S_k}$ is defined as
$$g^k(t, \varphi^k(t)) = (v^k(t), a^k(t)),$$
with
$$a_i^k(t) = \begin{cases} \left(H_1^{cc} *_1 \mu_P^k + H_1^{tc} *_1 \mu_S^k\right)(x_i, v_i) + \left(H_2^{cc} * \mu_P^k + H_2^{tc} * \mu_S^k\right)(x_i, v_i) & i \in \mathcal{I}_P^k \\ \left(H_1^{ct} *_1 \mu_P^k + H_1^{tt} *_1 \mu_S^k\right)(x_i, v_i) + \left(H_2^{ct} * \mu_P^k + H_2^{tt} * \mu_S^k\right)(x_i, v_i) & i \in \mathcal{I}_S^k. \end{cases}$$
By the regularity and growth assumptions on the convolution kernels, it is immediate to check that
$$\|g^k(t, \varphi^k(t))\| \leq C(1 + \|\varphi^k(t)\|) \tag{32}$$
for a constant C which does not depend on the number of vehicles (see Lemma 3.4 in [20] for details). Therefore the Caratheodory Theorem [21] yields the existence of solution φ^k to the linear system (31) on the time interval $[0, t_0)$ with initial data $\varphi_0^k = (x_0^k, v_0^k) \in (\mathbb{R} \times \mathbb{R}_{\geq 0})^{P_k+S_k}$. Moreover the solution satisfies the following growth condition
$$\|\varphi^k(t)\| \leq (\|\varphi_0^k\| + Ct_0)e^{Ct_0} \tag{33}$$
which implies also the Lipschitzianity. Indeed for any times $t, t' \in [0, t_0)$ we have
$$\begin{aligned} \|\varphi^k(t') - \varphi^k(t)\| &\leq \int_t^{t'} \left\|g^k(s, \varphi^k(s))\right\| ds \\ &\leq \int_t^{t'} C(1 + \|\varphi^k(s)\|)\, ds| \leq C(1 + (\|\varphi_0^k\| + Ct_0)e^{Ct_0})|t' - t|. \end{aligned}$$

Analogously, on all the finitely time intervals in which there is no lane change, the Caratheodory Theorem still yields the existence of a unique Lipschitz trajectory for vehicles in the same lane. □

4. The mean-field limit of the finite-dimensional hybrid system.

In the next, we let the number of cars and trucks approach infinity. The emerging equations do not describe anymore the trajectories of the single vehicle but the evolution of density of each class of vehicles in space and velocity.

4.1. A system of coupled PDEs with source term.

For convenience, we introduce the following compact notation for equation (9):

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \Big((H_1^{n_1} *_1 \mu_P + H_1^{n_2} *_1 \mu_S + H_2^{n_1} * \mu_P + H_2^{n_2} * \mu_S \Big)(x_i, v_i) \end{aligned} \tag{34}$$

with $(n_1, n_2) = (cc, tc)$ if $i \in \mathcal{I}_P$ and $(n_1, n_2) = (ct, tt)$ if $i \in \mathcal{I}_S$.

What we are going to prove is that the mean field limit of the hybrid system in Definition 3.1 is a system of two Vlasov-type equations with source terms. These source terms are generated by the lane-change behaviour in the four different car-truck car-following combinations and induce the measure solutions to change mass in time, therefore the limit is obtained by using the generalized Wasserstein distance. In detail we will derive the following limit system

$$\partial_t \nu_c^k + v \partial_x \nu_c^k + \partial_v \Big[\big(H_1^{cc} *_1 \nu_c^k + H_1^{tc} *_1 \nu_t^k + H_2^{cc} * \nu_c^k + H_2^{tc} * \nu_t^k \big) \nu_c^k \Big] = G_1(\nu_c^k, \nu_t^k, \nu_c^{k'}, \nu_t^{k'}) , \tag{35}$$

$$\partial_t \nu_t^k + v \partial_x \nu_t^k + \partial_v \Big[\big(H_1^{ct} *_1 \nu_c^k + H_1^{tt} *_1 \nu_t^k + H_2^{ct} * \nu_c^k + H_2^{tt} * \nu_t^k \big) \nu_t^k \Big] = G_2(\nu_c^k, \nu_t^k, \nu_c^{k'}, \nu_t^{k'}) , \tag{36}$$

where ν_c^k and ν_t^k represent respectively the density of cars and trucks on the lane k. To describe the derivation process of the source terms G_1 and G_2, we need to introduce the average accelerations A_P^k and A_S^k defined as

$$\begin{aligned} A_P^k &= H_1^{cc} *_1 \mu_P^k + H_1^{tc} *_1 \mu_S^k + H_2^{cc} * \mu_P^k + H_2^{tc} * \mu_S^k, \\ A_S^k &= H_1^{ct} *_1 \mu_P^k + H_1^{tt} *_1 \mu_S^k + H_2^{ct} * \mu_P^k + H_2^{tt} * \mu_S^k, \end{aligned}$$

with μ_P^k and μ_S^k the probability measures given in (4) on lane k. Since μ_P^k, μ_S^k are both compactly supported and the convolution kernels are, by assumption, locally Lipschitz and with sub-linear growth, it follows that both the average accelerations are bounded.

We define the map p_1 as

$$p_1(b_1, b_2, b_3, b_4, b_5) = \frac{1}{C}\big(1 - e^{-\gamma_2 b_1 b_2 b_3 b_4 b_5}\big) ,$$

$$p_1([A_P^{k'} - A_P^k - \Delta^{cc}]_+, \, [A_S^{k'} - A_P^k - \Delta^{tc}]_+, \, [A_P^{k'} + \Delta^c]_+, \, [A_S^{k'} + \Delta^t]_+, \, [A_S^{k'} + \Delta^t]_+) ,$$

representing the probability of cars performing lane change from lane k to lane k' and analogously the map p_2 as

$$p_2(b_1, b_2, b_3, b_4, b_5) = \frac{1}{C}\big(1 - e^{-\gamma_2 b_1 b_2 b_3 b_4 b_5}\big)$$

$$p_2([A_P^{k'} - A_S^k - \Delta^{ct}]_+, \, [A_S^{k'} - A_S^k - \Delta^{tt}]_+, \, [A_P^{k'} + \Delta^c]_+, \, [A_S^{k'} + \Delta^t]_+, \, [A_S^{k'} + \Delta^t]_+)$$

which is instead the probability of trucks performing lane change from lane k to lane k'.
Looking closely at the probability p_1 (the same considerations also apply to p_2), we can observe that it is strictly positive only if the safety and incentive conditions are strictly satisfied in an average sense, i.e. if

$$A_P^{k'} > A_P^k + \Delta^{cc}\,, \quad A_S^{k'} > A_P^k + \Delta^{tc}\,, \quad A_P^{k'} > -\Delta^c\,, \quad \text{and } A_S^{k'} > -\Delta^t.$$

Thanks to this notation we can define the source terms as

$$G_1(\nu_c^k,\nu_t^k,\nu_c^{k'},\nu_t^{k'}) \tag{37}$$
$$= \Big[G_1^{k-1,k}(\nu_c^{k-1},\nu_t^{k-1},\nu_c^k,\nu_t^k) - G_1^{k,k-1}(\nu_c^{k-1},\nu_t^{k-1},\nu_c^k,\nu_t^k)\Big](1-\delta_1(k)) \tag{38}$$
$$+ \Big[G_1^{k+1,k}(\nu_c^{k+1},\nu_t^{k+1},\nu_c^k,\nu_t^k) - G_1^{k,k+1}(\nu_c^{k+1},\nu_t^{k+1},\nu_c^k,\nu_t^k)\Big](1-\delta_N(k)) \tag{39}$$

$$G_2(\nu_c^k,\nu_t^k,\nu_c^{k'},\nu_t^{k'}) \tag{40}$$
$$= \Big[G_2^{k-1,k}(\nu_c^{k-1},\nu_t^{k-1},\nu_c^k,\nu_t^k) - G_2^{k,k-1}(\nu_c^{k-1},\nu_t^{k-1},\nu_c^k,\nu_t^k)\Big](1-\delta_1(k)) \tag{41}$$
$$+ \Big[G_2^{k+1,k}(\nu_c^{k+1},\nu_t^{k+1},\nu_c^k,\nu_t^k) - G_2^{k,k+1}(\nu_c^{k+1},\nu_t^{k+1},\nu_c^k,\nu_t^k)\Big](1-\delta_N(k)) \tag{42}$$

where $k' \in \{k-1, k+1\}$ and $G_1^{j,j'}$,$G_2^{j,j'}$ are respectively given by

$$G_1^{j,j'}(\nu_c^j,\nu_t^j,\nu_c^{j'},\nu_t^{j'})$$
$$= p_1([A_P^{j'} - A_P^j - \Delta^{cc}]_+,\, [A_S^{j'} - A_P^j - \Delta^{tc}]_+,\, [A_P^{j'} + \Delta^c]_+,\, [A_P^{j'} + \Delta^c]_+,\, [A_S^{j'} + \Delta^t]_+)\nu_c^j\,,$$
$$G_2^{j,j'}(\nu_c^j,\nu_t^j,\nu_c^{j'},\nu_t^{j'})$$
$$= p_2([A_P^{j'} - A_S^j - \Delta^{ct}]_+,\, [A_S^{j'} - A_S^j - \Delta^{tt}]_+,\, [A_S^{j'} + \Delta^t]_+,\, [A_P^{j'} + \Delta^c]_+,\, [A_S^{j'} + \Delta^t]_+)\nu_t^j\,,$$

with $j, j' \in \{k-1, k, k+1\}$.

4.2. **The weak solution to the coupled PDEs.**

Definition 4.1 (Weak solution to the coupled PDEs)**.** Given initial datum $(\vec{\nu}_{c,0}, \vec{\nu}_{t,0}) \in \Big(\mathcal{M}_0^{ac}(D) \cap \mathcal{M}^1(D)\Big)^{2L}$, we say that $(\vec{\nu}_c, \vec{\nu}_t) : [0,T] \to \Big(\mathcal{M}_0^{ac}(D) \cap \mathcal{M}^1(D)\Big)^{2L}$ is a solution to the coupled PDEs (35)-(36), if for every test function $\varphi \in C_c^\infty(D)$ and for all $k \in \{1, \ldots, L\}$, ν_c^k and ν_t^k are compactly supported in $B(0,R)$ for some $R > 0$, and for almost every $t \in [0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} \varphi(x,v)\,\mathrm{d}\nu_c^k(t)(x,v) =$$
$$= \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} \varphi(x,v)\,\mathrm{d}G_1(\nu_c^k,\nu_t^k,\nu_c^{k'},\nu_t^{k'})(t)(x,v)$$
$$+ \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} \big(\nabla\varphi(x,v)\cdot\omega_c^k(t,x,v)\big)\,\mathrm{d}\nu_c^k(t)(x,v) \tag{43}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} \varphi(x,v)\,\mathrm{d}\nu_t^k(t)(x,v) = \\
&= \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} \varphi(x,v)\,\mathrm{d}G_2(\nu_c^k,\nu_t^k,\nu_c^{k'},\nu_t^{k'})(t)(x,v) \\
&+ \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} \left(\nabla\varphi(x,v)\cdot\omega_t^k(t,x,v)\right)\,\mathrm{d}\nu_t^k(t)(x,v) \qquad (44)
\end{aligned}$$

where

$$\omega_c^k(t,x,v) = \left(v, \left(H_1^{cc} *_1 \nu_c^k + H_1^{tc} *_1 \nu_t^k + H_2^{cc} * \nu_c^k + H_2^{tc} * \nu_t^k\right)(x,v)\right),$$

and

$$\omega_t^k(t,x,v) = \left(v, \left(H_1^{ct} *_1 \nu_c^k + H_1^{tt} *_1 \nu_t^k + H_2^{ct} * \nu_c^k + H_2^{tt} * \nu_t^k\right)(x,v)\right).$$

4.3. **Existence of solutions to the coupled PDEs.**

Theorem 4.1 (Existence of weak solutions to the couple PDEs)**.** Let the initial datum $(\vec{\nu}_{c,0},\vec{\nu}_{t,0}) \in \Big(\mathcal{M}_0^{ac}(D)\cap\mathcal{M}^1(D)\Big)^{2L}$ be given. Assume that convolutional kernels H_q^n with $q=1,2$ and $n\in\{cc,ct,tc,tt\}$ are locally Lipschitz and with sub-linear growth. Then there exists a solution $(\vec{\nu}_c,\vec{\nu}_t) : [0.T] \to \Big(\mathcal{M}_0^{ac}(D)\cap\mathcal{M}^1(D)\Big)^{2N}$ to the coupled PDEs (35)-(36) as in Definition 4.1.

Proof. As first step we construct a sequence of discrete measures converging to the initial datum in the generalized Wasserstein distance. Indeed on each lane $k\in\{1,\dots,L\}$, there exists a infinite set of couples $(x_{i,0}^k, v_{i,0}^k)\in\mathbb{R}\times\mathbb{R}_{\geq 0}$, such that

$$\nu_{c,0}^k = \lim_{p^k\to\infty} m_c \sum_{i\in\mathcal{I}_P^k} \delta_{(x_{i,0}^k, v_{i,0}^k)} \tag{45}$$

and

$$\nu_{t,0}^k = \lim_{s^k\to\infty} m_t \sum_{i\in\mathcal{I}_S^k} \delta_{(x_{i,0}^k, v_{i,0}^k)}, \tag{46}$$

where $\vec{\nu}_{c,0} = (\nu_{c,0}^k)_{k=1}^L$ and $\vec{\nu}_{t,0} = (\nu_{t,0}^k)_{k=1}^L$, $\mathcal{I}_P^k$ is a finite subset of indices for cars and $\mathcal{I}_S^k$ is a finite subset of indices for trucks on lane k, while $p^k = \#\mathcal{I}_P^k$ and $s^k = \#\mathcal{I}_S^k$ represent the number of cars and trucks on lane k respectively. The constants m_c and m_t are the average masses for cars and trucks defined as

$$m_c = \frac{\sum_{j=1}^L \|\nu_{c,0}^j\|}{\sum_{j=1}^L p^j}, \qquad m_t = \frac{\sum_{j=1}^L \|\nu_{t,0}^j\|}{\sum_{j=1}^L s^j}.$$

Let P be the set of cars and S the set of trucks on the open stretch road. The couples $(x_0,v_0) = (x_{i,0},v_{i,0})$, with $i\in\mathcal{I}$ (set of indices for both vehicles) introduced for the approximation represent the initial positions and velocities of cars and trucks. We also define the following multi-valued functions:

$$\mathcal{I}_P^k(\cdot) : [0,T] \to \mathcal{P}(\mathcal{I}) \qquad \text{and} \qquad \mathcal{I}_S^k(\cdot) : [0,T] \to \mathcal{P}(\mathcal{I})$$

where $\mathcal{P}(\cdot)$ stands for power set. These keep trace of the set of the indices for cars and trucks on each lane on the time interval $[0,T]$. To each one of these multi-functions it is naturally associated a map counting the number of cars and number of trucks present on the lane k at any time, i.e.

$$P^k(\cdot):[0,T]\to\mathbb{N}_{\geq 0}\,, \qquad\qquad P^k(t)=\#\,\mathcal{I}_P^k(t)\,,$$
$$S^k(\cdot):[0,T]\to\mathbb{N}_{\geq 0}\,, \qquad\qquad S^k(t)=\#\,\mathcal{I}_S^k(t).$$

with $P^k(0)=p^k$ and $S^k(0)=s^k$. By theorem 3.1, we know that for $t\in[0,T]$ and $i\in\mathcal{I}_P^{\ell_i(t)}(t)$, there exists a unique map $(x_i,v_i)\colon[0,T]\to \mathrm{proj}_{1,2}(\mathcal{M}_{\ell_i(t)})$ (where $\mathrm{proj}_{1,2}:\mathbb{R}^3\to\mathbb{R}^2; (x,y,z)\mapsto(x,y)$) which represents the positions and velocities of cars on lane $\ell_i(t)$ during the time interval $[0,T]$ with $(x_i(0),v_i(0))=(x_{i,0},v_{i,0})$. Define a discrete measure

$$\nu^{p^k}(t)=m_c\sum_{i\in\mathcal{I}_P^k(t)}\delta_{(x_i(t),v_i(t))}. \tag{47}$$

Similarly, for $t\in[0,T]$, $i\in\mathcal{I}_S^{\ell_i(t)}(t)$, there exists a unique map $(x_i,v_i)\colon[0,T]\to \mathrm{proj}_{1,2}(\mathcal{M}_{\ell_i(t)})$ representing positions and velocities of trucks on lane $\ell_i(t)$ during the time interval $[0,T]$ with $(x_i(0),v_i(0))=(x_{i,0},v_{i,0})$. We can define a discrete measure

$$\nu^{s^k}(t)=m_t\sum_{i\in\mathcal{I}_S^k(t)}\delta_{(x_i(t),v_i(t))}. \tag{48}$$

Note that $m_c\to 0$ as $p^k\to\infty$. Therefore there exists a constant $L>0$ which satisfies the following condition: for every $\varepsilon>0$, we can find $N_1>0$, such that whenever $p^k>N_1$,

$$W_1^{1,1}(\nu^{p^k}(s),\nu^{p^k}(t))<L|s-t|+\min\{\varepsilon,|s-t|\}\,\forall s,t\in[0,T].$$

By Theorem 2.5, there exist a convergent subsequence (ν^{p^k}) (for simplicity, we use the some notation for the subsequence as the notation for the original sequence) and $\nu_c^k\in\mathcal{M}(D)$ such that

$$\nu^{p^k}\to\nu_c^k \text{ as } p^k\to\infty.$$

Analogously we can conclude that there exists a subsequence of ν^{s^k} converging to $\nu_t^k\in\mathcal{M}(D)$ as $s^k\to\infty$. Next we will show that $(\nu_c^k,\nu_t^k)\in\big(\mathcal{M}(D)\big)^2$ with $k\in\{1,\dots,N\}$, is a weak solution to the coupled PDEs (35) and (36) as in Definition 4.1. Let $\mathcal{I}_{P_1}^k$ be the set of indices of cars on lane k not performing lane-change over the whole time interval $[0,T]$ and set $p_1^k=|\mathcal{I}_{P_1}^k|$. Consider the following discrete measure to track positions and velocities for cars in this set:

$$\nu^{p_1^k}(t)=m_c\sum_{i\in\mathcal{I}_{P_1}^k}\delta_{(x_i(t),v_i(t))}.$$

then, for any test function $\varphi\in\mathcal{C}^\infty(D)$ we have

$$\begin{aligned}
&\tfrac{d}{dt}\langle\varphi\,,\,\nu^{p_1^k}\rangle=\tfrac{d}{dt}m_c\sum_{i\in\mathcal{I}_{P_1}^k}\varphi(x_i(t),v_i(t))\\
&=m_c\sum_{i\in\mathcal{I}_{P_1}^k}\big(\partial_x\varphi(x_i(t),v_i(t))v_i(t)+\partial_v\varphi(x_i(t),v_i(t))\dot v_i(t)\big)\\
&=m_c\sum_{i\in\mathcal{I}_{P_1}^k}\partial_x\varphi(x_i(t),v_i(t))v_i(t)
\end{aligned}$$

$$
\begin{aligned}
&+ m_c \sum_{i\in\mathcal{I}^k_{P_1}} \partial_v\varphi(x_i(t),v_i(t))\big(H_1^{cc} *_1 \nu^{p^k}(t) + H_1^{tc} *_1 \nu^{s^k}(t) \\
&\qquad + H_2^{cc} * \nu^{p^k}(t) + H_2^{tc} * \nu^{s^k}(t)\big)(x_i(t),v_i(t)) \\
&= \langle \partial_x\varphi(x,v)v\,,\, \nu^{p_1^k}\rangle \\
&\quad + \langle \partial_v\varphi(x,v)\big(H_1^{cc} *_1 \nu^{p^k}(t) + H_1^{tc} *_1 \nu^{s^k}(t) \\
&\qquad + H_2^{cc} * \nu^{p^k}(t) + H_2^{tc} * \nu^{s^k}(t)\big)(x,v)\,,\, \nu^{p_1^k}\rangle\,.
\end{aligned} \tag{49}
$$

For all $t \in [0,T]$ and $s \in [0,t]$, by integrating both sides of (49) we get

$$
\begin{aligned}
\langle \varphi, \nu^{p_1^k}(s) - \nu^{p_1^k}(0)\rangle = \int_0^s \Big[\int_{\mathbb{R}\times\mathbb{R}^+} \partial_x\varphi(x,v)v + \partial_v\varphi(x,v)\big(H_1^{cc} *_1 \nu^{p^k}(t) \\
+ H_1^{tc} *_1 \nu^{s^k}(t) + H_2^{cc} * \nu^{p^k}(t) + H_2^{tc} * \nu^{s^k}(t)\big)(x,v)\, \mathrm{d}\nu^{p_1^k}(t)(x,v)\Big]\, \mathrm{d}t.
\end{aligned} \tag{50}
$$

Analogously, let $\mathcal{I}^k_{S_1}$ be the set of indices of trucks on lane k not performing lane-change over the whole time interval $[0,T]$ and set $s_1^k = |\mathcal{I}^k_{S_1}|$. Consider the following discrete measure which keeps trace of positions and velocities for these trucks:

$$
\nu^{s_1^k}(t) = m_t \sum_{i\in\mathcal{I}^k_{S_1}} \delta_{(x_i(t),v_i(t))}\,.
$$

For any test function $\varphi \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}_{\geq 0})$ we have

$$
\begin{aligned}
&\tfrac{d}{dt}\langle \varphi\,,\, \nu^{s_1^k}\rangle = \tfrac{d}{dt} m_t \sum_{i\in\mathcal{I}^k_{S_1}} \varphi(x_i(t),v_i(t)) \\
&= m_t \sum_{i\in\mathcal{I}^k_{S_1}} \big(\partial_x\varphi(x_i(t),v_i(t))v_i(t) + \partial_v\varphi(x_i(t),v_i(t))\dot v_i(t)\big) \\
&= m_t \sum_{i\in\mathcal{I}^k_{S_1}} \partial_x\varphi(x_i(t),v_i(t))v_i(t) \\
&\quad + m_t \sum_{i\in\mathcal{I}^k_{S_1}} \partial_v\varphi(x_i(t),v_i(t))\big(H_1^{ct} *_1 \nu^{p^k}(t) + H_1^{tt} *_1 \nu^{s^k}(t) \\
&\qquad + H_2^{ct} * \nu^{p^k}(t) + H_2^{tt} * \nu^{s^k}(t)\big)(x_i(t),v_i(t)) \\
&= \langle \partial_x\varphi(x,v)v\,,\, \nu^{s_1^k}\rangle \\
&\quad + \langle \partial_v\varphi(x,v)\big(H_1^{ct} *_1 \nu^{p^k}(t) + H_1^{tt} *_1 \nu^{s^k}(t) \\
&\qquad + H_2^{ct} * \nu^{p^k}(t) + H_2^{tt} * \nu^{s^k}(t)\big)(x,v)\,,\, \nu^{s_1^k}\rangle
\end{aligned} \tag{51}
$$

For all $t \in [0,T]$ and $s \in [0,t]$, by integrating both sides of (51)

$$
\begin{aligned}
\langle \varphi, \nu^{s_1^k}(s) - \nu^{s_1^k}(0)\rangle = \int_0^s \Big[\int_{\mathbb{R}\times\mathbb{R}^+} \partial_x\varphi(x,v)v + \partial_v\varphi(x,v)\big(H_1^{ct} *_1 \nu^{p^k}(t) \\
+ H_1^{tt} *_1 \nu^{s^k}(t) + H_2^{ct} * \nu^{p^k}(t) + H_2^{tt} * \nu^{s^k}(t)\big)(x,v)\, \mathrm{d}\nu^{s_1^k}(t)(x,v)\Big]\, \mathrm{d}t.
\end{aligned} \tag{52}
$$

Let p_1^k and s_1^k (respectively the number of cars and trucks not performing lane change on lane k) go to infinity, then on the left hand side of equations (50)-(52)

we have

$$\lim_{p_1^k\to\infty}\langle\varphi,\nu^{p_1^k}(s)-\nu^{p_1^k}(0)\rangle=\langle\varphi,\nu_c^k-\nu_{c,0}^k\rangle, \tag{53}$$

$$\lim_{s_1^k\to\infty}\langle\varphi,\nu^{s_1^k}(s)-\nu^{s_k^1}(0)\rangle=\langle\varphi,\nu_t^k-\nu_{t,0}^k\rangle. \tag{54}$$

By the dominated convergence theorem, on the right hand side of (50)-(52) we have that for all test function $\varphi\in C_c^\infty(D)$,

$$\begin{aligned}\lim_{p_1^k\to\infty}\int_0^s\left(\int_{\mathbb{R}\times\mathbb{R}^+}\partial_x\varphi(x,v)v\right)\mathrm{d}\nu^{p_1^k}(t)(x,v)\,\mathrm{d}t\\ =\int_0^s\left(\int_{\mathbb{R}\times\mathbb{R}^+}\partial_x\varphi(x,v)v\right)\,\mathrm{d}\nu_c^k(t)(x,v)\,\mathrm{d}t\end{aligned} \tag{55}$$

$$\begin{aligned}\lim_{s_1^k\to\infty}\int_0^s\left(\int_{\mathbb{R}\times\mathbb{R}^+}\partial_x\varphi(x,v)v\right)\mathrm{d}\nu^{s_1^k}(t)(x,v)\,\mathrm{d}t\\ =\int_0^s\left(\int_{\mathbb{R}\times\mathbb{R}^+}\partial_x\varphi(x,v)v\right)\,\mathrm{d}\nu_t^k(t)(x,v)\,\mathrm{d}t\end{aligned} \tag{56}$$

and

$$\begin{aligned}&\lim_{s_1^k\to\infty}\lim_{p_1^k\to\infty}\int_0^s\Big(\partial_v\varphi(x,v)\big(H_1^{cc}*_1\nu^{p^k}(t)+H_1^{tc}*_1\nu^{s^k}(t)\\ &\qquad+H_2^{cc}*\nu^{p^k}(t)+H_2^{tc}*\nu^{s^k}(t)\big)(x,v)\Big)\,\mathrm{d}\nu^{p_1^k}(t)(x,v)\,\mathrm{d}t\\ &=\int_0^s\big(\partial_v\varphi(x,v)\big(H_1^{cc}*_1\nu_c^k(t)+H_1^{tc}*_1\nu_t^k(t)\\ &\qquad+H_2^{cc}*\nu_c^k(t)+H_2^{tc}*\nu_t^k(t)\big)(x,v)\big)\,\mathrm{d}\nu_c^k(t)(x,v)\,\mathrm{d}t,\end{aligned} \tag{57}$$

$$\begin{aligned}&\lim_{s_1^k\to\infty}\lim_{p_1^k\to\infty}\int_0^s\Big(\partial_v\varphi(x,v)\big(H_1^{ct}*_1\nu^{p^k}(t)+H_1^{tt}*_1\nu^{s^k}(t)\\ &\qquad+H_2^{ct}*\nu^{p^k}(t)+H_2^{tt}*\nu^{s^k}(t)\big)(x,v)\Big)\,\mathrm{d}\nu^{s_1^k}(t)(x,v)\,\mathrm{d}t\\ &=\int_0^s\big(\partial_v\varphi(x,v)\big(H_1^{ct}*_1\nu_c^k(t)+H_1^{tt}*_1\nu_t^k(t)\\ &\qquad+H_2^{ct}*\nu_c^k(t)+H_2^{tt}*\nu_t^k(t)\big)(x,v)\big)\,\mathrm{d}\nu_t^k(t)(x,v)\,\mathrm{d}t.\end{aligned} \tag{58}$$

Indeed,for every $r>0$, Lemma 2.4 yields

$$\begin{aligned}&\lim_{s_1^k\to\infty}\lim_{p_1^k\to\infty}\Big\|\big(H_1^{cc}*_1\nu^{p^k}(t)+H_1^{tc}*_1\nu^{s^k}(t)+H_2^{cc}*\nu^{p^k}(t)+H_2^{tc}*\nu^{s^k}(t)\big)(x,v)\\ &-\big(H_1^{cc}*_1\nu_c^k(t)+H_1^{tc}*_1\nu_t^k(t)+H_2^{cc}*\nu_c^k(t)+H_2^{tc}*\nu_t^k(t)\big)(x,v)\Big\|_{L^\infty(B(0,r))}=0,\end{aligned} \tag{59}$$

$$\begin{aligned}&\lim_{s_1^k\to\infty}\lim_{p_1^k\to\infty}\Big\|\big(H_1^{cc}*_1\nu^{p^k}(t)+H_1^{ct}*_1\nu^{s^k}(t)+H_2^{ct}*\nu^{p^k}(t)+H_2^{tt}*\nu^{s^k}(t)\big)(x,v)\\ &-\big(H_1^{ct}*_1\nu_c^k(t)+H_1^{tt}*_1\nu_t^k(t)+H_2^{ct}*\nu_c^k(t)+H_2^{tt}*\nu_t^k(t)\big)(x,v)\Big\|_{L^\infty(B(0,r))}=0.\end{aligned} \tag{60}$$

Since $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_{\geq 0})$,

$$\lim_{s_1^k \to \infty} \lim_{p_1^k \to \infty} \Big\| \partial_v \varphi(x,v) \Big[\big(H_1^{cc} *_1 \nu^{p^k}(t) + H_1^{tc} *_1 \nu^{s^k}(t) + H_2^{cc} * \nu^{p^k}(t) + H_2^{tc} * \nu^{s^k}(t) \big)(x,v)$$
$$- \big(H_1^{cc} *_1 \nu_P^k(t) + H_1^{tc} *_1 \nu_t^k(t) + H_2^{cc} * \nu_c^k(t) + H_2^{tc} * \nu_t^k(t) \big)(x,v) \Big] \Big\|_{L^\infty(B(0,r))} = 0, \quad (61)$$

$$\lim_{s_1^k \to \infty} \lim_{p_1^k \to \infty} \Big\| \partial_v \varphi(x,v) \Big[\big(H_1^{ct} *_1 \nu^{p^k}(t) + H_1^{tt} *_1 \nu^{s^k}(t) + H_2^{ct} * \nu^{p^k}(t) + H_2^{tt} * \nu^{s^k}(t) \big)(x,v)$$
$$- \big(H_1^{ct} *_1 \nu_c^k(t) + H_1^{tt} *_1 \nu_t^k(t) + H_2^{ct} * \nu_c^k(t) + H_2^{tt} * \nu_t^k(t) \big)(x,v) \Big] \Big\|_{L^\infty(B(0,r))} = 0, \qquad (62)$$

which implies equations (57) and (58). Now let $\mathcal{I}_{P_2}^k$ be the set of indices of cars on lane k performing lane-change at least once during the time interval $[0,T]$. Assume that $|\mathcal{I}_{P_2}^k| = p_2^k$. By the lane-changing conditions (11) and (12), we consider the following discrete measure to track the positions and velocities of these cars:

$$\nu^{p_2^k}(t) =$$
$$= \sum_{i \in \mathcal{I}_{P_2}^{k-1}} m_c\, \delta_{(x_i(t),v_i(t))} p_1([\bar{a}_{c,i}^k - a_i^{k-1} - \Delta^{cc}]_+,\, [\bar{a}_{t,i}^k - a_i^{k-1} - \Delta^{tc}]_+,$$
$$[\bar{a}_i^k + \Delta^c]_+,\, [\bar{a}_{i_F^k}^k + \Delta^c]_+, [\bar{a}_{i_F^k}^k + \Delta^t]_+)(1 - \delta_1(k))$$
$$- \sum_{i \in \mathcal{I}_{P_2}^{k}} m_c\, \delta_{(x_i(t),v_i(t))} p_1([\bar{a}_{c,i}^{k-1} - a_i^{k} - \Delta^{cc}]_+,\, [\bar{a}_{t,i}^{k-1} - a_i^{k} - \Delta^{tc}]_+,$$
$$[\bar{a}_i^{k-1} + \Delta^c]_+,\, [\bar{a}_{i_F^{k-1}}^{k-1} + \Delta^c]_+, [\bar{a}_{i_F^{k-1}}^{k-1} + \Delta^t]_+)(1 - \delta_1(k))$$
$$+ \sum_{i \in \mathcal{I}_{P_2}^{k+1}} m_c\, \delta_{(x_i(t),v_i(t))} p_1([\bar{a}_{c,i}^k - a_i^{k+1} - \Delta^{cc}]_+,\, [\bar{a}_{t,i}^k - a_i^{k+1} - \Delta^{tc}]_+,$$
$$[\bar{a}_i^k + \Delta^c]_+,\, [\bar{a}_{i_F^k}^k + \Delta^c]_+, [\bar{a}_{i_F^k}^k + \Delta^t]_+)(1 - \delta_N(k))$$
$$- \sum_{i \in \mathcal{I}_{P_2}^{k}} m_c\, \delta_{(x_i(t),v_i(t))} p_1([\bar{a}_{c,i}^{k+1} - a_i^{k} - \Delta^{cc}]_+,\, [\bar{a}_{t,i}^{k+1} - a_i^{k} - \Delta^{tc}]_+,$$
$$[\bar{a}_i^{k+1} + \Delta^c]_+,\, [\bar{a}_{i_F^{k+1}}^{k+1} + \Delta^c]_+, [\bar{a}_{i_F^{k+1}}^{k+1} + \Delta^t]_+)(1 - \delta_N(k))$$

where for $j \in \{1, \ldots, N\}$ and $j' = j+1$ or $j-1$. The accelerations a_i^j, $\bar{a}_{c,i}^{j'}$ and $\bar{a}_{t,i}^{j'}$ are respectively given by

$$a_i^j = \dot{v}_i^j = \big(H_1^{cc} *_1 \nu^{p^j} + H_1^{tc} *_1 \nu^{s^j} + H_2^{cc} * \nu^{p^j} + H_2^{tc} * \nu^{s^j} \big)(x_i, v_i) \quad i \in \mathcal{I}_P\,,$$
$$\bar{a}_{c,i}^{j'} = \left(H_1^{cc} *_1 \nu^{p^{j'}} + H_1^{tc} *_1 \nu^{s^{j'}} \right)(x_i, v_i) + \left(H_2^{cc} * \nu^{p^{j'}} + H_2^{tc} * \nu^{s^{j'}} \right)(x_i, v_i) \quad i \in \mathcal{I}_P^j\,,$$
$$\bar{a}_{t,i}^{j'} = \left(H_1^{ct} *_1 \nu^{p^{j'}} + H_1^{tt} *_1 \nu^{s^{j'}} \right)(x_i, v_i) + \left(H_2^{ct} * \nu^{p^{j'}} + H_2^{tt} * \nu^{s^{j'}} \right)(x_i, v_i) \quad i \in \mathcal{I}_P^j.$$

Similarly, for trucks, let $\mathcal{I}_{S_2}^k$ be the set of indices of trucks on lane k that perform lane-changing at least once during the time interval $[0,T]$. Denote the number of these trucks by s_2^k, i.e. $|\mathcal{I}_{S_2}^k| = s_2^k$. Again, by the lane-changing conditions (11) and (12), we consider the following discrete measure to track the positions and velocities of these trucks:

$$\nu^{s_2^k}(t) =$$
$$= \sum_{i \in \mathcal{I}_{S_2}^{k-1}} m_t\, \delta_{(x_i(t),v_i(t))} p_2([\bar{a}_{c,i}^k - a_i^{k-1} - \Delta^{ct}]_+,\, [\bar{a}_{t,i}^k - a_i^{k-1} - \Delta^{tt}]_+,$$
$$[\bar{a}_i^k + \Delta^t]_+,\, [\bar{a}_{i_F^k}^k + \Delta^c]_+, [\bar{a}_{i_F^k}^k + \Delta^t]_+)(1 - \delta_1(k))$$

$$
\begin{aligned}
&- \sum_{i\in\mathcal{I}^k_{S_2}} m_t\, \delta_{(x_i(t),v_i(t))} p_2([\bar a^{k-1}_{c,i} - a^k_i - \Delta^{ct}]_+,\, [\bar a^{k-1}_{t,i} - a^k_i - \Delta^{tt}]_+,\\
&\qquad\qquad [\bar a^{k-1}_i + \Delta^t]_+,\, [\bar a^{k-1}_{i^{k-1}_F} + \Delta^c]_+, [\bar a^{k-1}_{i^{k-1}_F} + \Delta^t]_+)(1-\delta_1(k))\\
&+ \sum_{i\in\mathcal{I}^{k+1}_{S_2}} m_t\, \delta_{(x_i(t),v_i(t))} p_2([\bar a^{k}_{c,i} - a^{k+1}_i - \Delta^{ct}]_+,\, [\bar a^{k}_{t,i} - a^{k+1}_i - \Delta^{tt}]_+,\\
&\qquad\qquad [\bar a^{k}_i + \Delta^t]_+,\, [\bar a^{k}_{i^{k}_F} + \Delta^c]_+, [\bar a^{k}_{i^{k}_F} + \Delta^t]_+)(1-\delta_N(k))\\
&- \sum_{i\in\mathcal{I}^k_{S_2}} m_t\, \delta_{(x_i(t),v_i(t))} p_2([\bar a^{k+1}_{c,i} - a^k_i - \Delta^{ct}]_+,\, [\bar a^{k+1}_{t,i} - a^k_i - \Delta^{tt}]_+,\\
&\qquad\qquad [\bar a^{k+1}_i + \Delta^t]_+,\, [\bar a^{k+1}_{i^{k+1}_F} + \Delta^c]_+, [\bar a^{k+1}_{i^{k+1}_F} + \Delta^t]_+)(1-\delta_N(k))
\end{aligned}
$$

with $j \in \{1,\dots,N\}$, $j' = j+1$ or $j-1$ and again the accelerations are given by

$$
\begin{aligned}
a^j_i &= \dot v^j_i = \big(H^{ct}_1 *_1 \nu^{p^j} + H^{tt}_1 *_1 \nu^{s^j} + H^{ct}_2 * \nu^{p^j} + H^{tt}_2 * \nu^{s^j}\big)(x_i,v_i) \quad i\in\mathcal{I}_S,\\
\bar a^{j'}_{c,i} &= \left(H^{cc}_1 *_1 \nu^{p^{j'}} + H^{tc}_1 *_1 \nu^{s^{j'}}\right)(x_i,v_i) + \left(H^{cc}_2 * \nu^{p^{j'}} + H^{tc}_2 * \nu^{s^{j'}}\right)(x_i,v_i) \quad i\in\mathcal{I}^j_P\\
\bar a^{j'}_{t,i} &= \left(H^{ct}_1 *_1 \nu^{p^{j'}} + H^{tt}_1 *_1 \nu^{s^{j'}}\right)(x_i,v_i) + \left(H^{ct}_2 * \nu^{p^{j'}} + H^{tt}_2 * \nu^{s^{j'}}\right)(x_i,v_i) \quad i\in\mathcal{I}^j_S.
\end{aligned}
$$

Now on each lane k we let the number of cars and trucks performing lane-chance (p^k_2, s^k_2) go to infinity. Then we have

$$
\begin{aligned}
&\lim_{p^k_2\to\infty}\lim_{s^k_2\to\infty} \nu^{p^k_2}(t) =\\
&\quad= \nu^{k-1}_c p_1([A^k_P - A^{k-1}_P - \Delta^{cc}]_+, [A^k_S - A^{k-1}_P - \Delta^{tc}]_+,\\
&\qquad\qquad [A^k_P + \Delta^c]_+, [A^k_P + \Delta^c]_+, [A^k_S + \Delta^t]_+)(1-\delta_1(k))\\
&\qquad - \nu^k_c p_1([A^{k-1}_P - A^k_P - \Delta^{cc}]_+, [A^{k-1}_S - A^k_P - \Delta^{tc}]_+,\\
&\qquad\qquad [A^{k-1}_P + \Delta^c]_+, [A^{k-1}_P + \Delta^c]_+, [A^{k-1}_S + \Delta^t]_+)(1-\delta_1(k))\\
&\qquad + \nu^{k+1}_c p_1([A^k_P - A^{k+1}_P - \Delta^{cc}]_+, [A^k_S - A^{k+1}_P - \Delta^{tc}]_+,\\
&\qquad\qquad [A^k_P + \Delta^c]_+, [A^k_P + \Delta^c]_+, [A^k_S + \Delta^t]_+)(1-\delta_N(k))\\
&\qquad - \nu^k_c p_1([A^{k+1}_P - A^k_P - \Delta^{cc}]_+, [A^{k+1}_S - A^k_P - \Delta^{tc}]_+,\\
&\qquad\qquad [A^{k+1}_P + \Delta^c]_+, [A^{k+1}_P + \Delta^c]_+, [A^{k+1}_S + \Delta^t]_+)(1-\delta_N(k))\\
&\quad= \Big[G^{k-1,k}_1(\nu^{k-1}_c, \nu^{k-1}_t, \nu^k_c, \nu^k_t) - G^{k,k-1}_1(\nu^k_c, \nu^k_t, \nu^{k-1}_c, \nu^{k-1}_t)\Big](1-\delta_1(k))\\
&\qquad + \Big[G^{k+1,k}_1(\nu^{k+1}_c, \nu^{k+1}_t, \nu^{k-1}_c, \nu^{k-1}_t) - G^{k,k+1}_1(\nu^k_c, \nu^k_t, \nu^{k+1}_c, \nu^{k+1}_t)\Big](1-\delta_N(k))\\
&\quad= G_1(\nu^k_c, \nu^k_t, \nu^{k'}_c, \nu^{k'}_t)
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
&\lim_{s^k_2\to\infty}\lim_{p^k_2\to\infty} \nu^{s^k_2}(t) =\\
&\quad= \nu^{k-1}_t p_2([A^k_P - A^{k-1}_S - \Delta^{ct}]_+, [A^k_S - A^{k-1}_S - \Delta^{tt}]_+,\\
&\qquad\qquad [A^k_S + \Delta^t]_+, [A^k_P + \Delta^c]_+, [A^k_P + \Delta^t]_+)(1-\delta_1(k))\\
&\qquad - \nu^k_t\, p_2([A^{k-1}_P - A^k_S - \Delta^{ct}]_+, [A^{k-1}_S - A^k_S - \Delta^{tt}]_+,\\
&\qquad\qquad [A^{k-1}_S + \Delta^t]_+, [A^{k-1}_P + \Delta^c]_+, [A^{k-1}_S + \Delta^t]_+)(1-\delta_1(k))\\
&\qquad + \nu^{k+1}_t\, p_2([A^k_P - A^{k+1}_S - \Delta^{ct}]_+, [A^k_S - A^{k+1}_S - \Delta^{tt}]_+,\\
&\qquad\qquad [A^k_S + \Delta^t]_+, [A^k_P + \Delta^c]_+, [A^k_S + \Delta^t]_+)(1-\delta_N(k))
\end{aligned}
$$

$$
\begin{aligned}
&- \nu_t^k\, p_2([A_P^{k+1} - A_S^k - \Delta^{ct}]_+, [A_S^{k+1} - A_S^k - \Delta^{tt}]_+,\\
&\qquad\qquad [A_S^{k+1} + \Delta^t]_+, [A_P^{k+1} + \Delta^c]_+, [A_S^{k+1} + \Delta^t]_+)(1 - \delta_N(k))\\
&= \Big[G_2^{k-1,k}(\mu_P^{k-1}, \nu_t^{k-1}, \nu_c^k, \nu_t^k) - G_2^{k,k-1}(\nu_c^k, \nu_t^k, \mu_P^{k-1}, \nu_t^{k-1})\Big](1 - \delta_1(k))\\
&\quad + \Big[G_2^{k+1,k}(\nu_c^{k+1}, \nu_t^{k+1}, \nu_c^{k-1}, \nu_t^{k-1}) - G_2^{k,k+1}(\nu_c^k, \nu_t^k, \nu_c^{k+1}, \nu_t^{k+1})\Big](1 - \delta_N(k))\\
&= G_2(\nu_c^k, \nu_t^k, \nu_c^{k'}, \nu_t^{k'}) \qquad (64)
\end{aligned}
$$

Set $\nu_c^k = \lim\limits_{p_1^k\to\infty} \lim\limits_{p_2^k\to\infty} (\nu^{p_1^k} + \nu^{p_2^k})$ and $\nu_t^k = \lim\limits_{s_1^k\to\infty} \lim\limits_{s_2^k\to\infty} (\nu^{s_1^k} + \nu^{s_2^k})$. By combining equations (55),(57),(63) and (56),(58),(64) we can observe thath the constructed couple (ν_c^k, ν_t^k), $k \in \{1, \dots, N\} \in \mathcal{M}_0^{ac}(D) \cap \mathcal{M}^1(D)$ is a weak solution to the coupled PDEs (4)-(36). $\square$

4.4. **Uniqueness of solutions to the coupled PDEs.**

Theorem 4.2 (Continuity with respect to the initial conditions)**.** For $q = 1, 2$, let $(\nu_c^q, \nu_t^q) \in \Big(\mathcal{M}_0^{ac}(D) \cap \mathcal{M}^1(D)\Big)^{2L}$ be two weak solutions for the coupled equations (35),(36) over the time interval $[0, T]$ associated to the initial data $(\nu_{0,c}^q, \nu_{0,t}^q) \in \Big(\mathcal{M}_0^{ac}(D) \cap \mathcal{M}^1(D)\Big)^{2L}$. Then for all $k \in \{1, \dots, L\}$, there exists a positive constant C_0 such that

$$
\begin{aligned}
&W_1^{1,1}(\nu_c^{k,1}(t), \nu_c^{k,2}(t)) + W_1^{1,1}(\nu_t^{k,1}(t), \nu_t^{k,2}(t))\\
&\qquad \le C_0 \sum_{i=1}^{L} \Big(W_1^{1,1}(\nu_{0,c}^{i,1}, \nu_{0,c}^{i,2}) + W_1^{1,1}(\nu_{0,t}^{i,1}, \nu_{0,t}^{i,2})\Big), \quad t \in [0, T]. \qquad (65)
\end{aligned}
$$

Here we have assumed for each $q = 1, 2$, $\nu_{0,c}^q = (\nu_{0,c}^{k,q})_{k=1}^L$ and the same for ν_c^q, ν_t^q and $\nu_{0,t}^q$.

Proof. Let $(\nu_c^{k,q}, \nu_t^{k,q}) : [0, T] \to \Big(\mathcal{M}_0^{ac}(D) \cap \mathcal{M}^1(D)\Big)^2$ be two solutions to system (35),(36) over the time interval $[0, T]$ associated to the initial data $(\nu_{0,c}^{k,q}, \nu_{0,t}^{k,q}) \in \Big(\mathcal{M}_0^{ac}(D) \cap \mathcal{M}^1(D)\Big)^2$ with $q = 1, 2$ and $k \in \{1, \dots, L\}$. Let $t \in [0, T]$ be fixed and denote $\Delta t = \frac{T}{2^j}$ for a fixed $j \in \mathbb{N}^+$. Consider the partition of $[0, T]$ into sub-intervals $[0, \Delta t)$, $[\Delta t, 2\Delta t), \dots, [(2^j - 1)\Delta t, 2^j \Delta t)$ and let n be the maximum integer such that $t - n\Delta t \ge 0$, then $t \in [n\Delta t, (n + 1)\Delta t)$. As mentioned in Section 2.5, $(\nu_c^{k,q}, \nu_t^{k,q}) = \lim\limits_{j\to\infty} (\nu_{j,c}^{k,q}, \nu_{j,t}^{k,q})$, where $(\nu_{j,c}^{k,q}, \nu_{j,t}^{k,q})$ is constructed according to the following scheme:

$$
\begin{aligned}
&(\nu_{j,c}^{k,q}(0), \nu_{j,t}^{k,q}(0)) := (\nu_{0,c}^{k,q}, \nu_{0,t}^{k,q});\\
&(\nu_{j,c}^{k,q}((n+1)\Delta t), \nu_{j,t}^{k,q}((n+1)\Delta t)) := \mathcal{T}_{\Delta t}^{\nu_{j,c}^{k,q}(n\Delta t), \nu_{j,t}^{k,q}(n\Delta t)} \#(\nu_{j,c}^{k,q}(n\Delta t), \nu_{j,t}^{k,q}(n\Delta t)) +\\
&\qquad + \Delta t\big(G_1(\nu_{j,c}^{k,q}(n\Delta t), \nu_{j,t}^{k,q}(n\Delta t), \nu_{j,c}^{k',q}(n\Delta t), \nu_{j,t}^{k',q}(n\Delta t)),\\
&\qquad\qquad G_2(\nu_{j,c}^{k,q}(n\Delta t), \nu_{j,t}^{k,q}(n\Delta t), \nu_{j,c}^{k',q}(n\Delta t), \nu_{j,t}^{k',q}(n\Delta t))\big);\\
&(\nu_{j,c}^{k,q}(t), \nu_{j,t}^{k,q}(t)) := \mathcal{T}_{\tau}^{\nu_{j,c}^{k,q}(n\Delta t), \nu_{j,t}^{k,q}(n\Delta t)} \#(\nu_{j,c}^{k,q}(n\Delta t), \nu_{j,t}^{k,q}(n\Delta t)) +\\
&\qquad + \tau\big(G_1(\nu_{j,c}^{k,q}(n\Delta t), \nu_{j,t}^{k,q}(n\Delta t), \nu_{j,c}^{k',q}(n\Delta t), \nu_{j,t}^{k',q}(n\Delta t)),\\
&\qquad\qquad G_2(\nu_{j,c}^{k,q}(n\Delta t), \nu_{j,t}^{k,q}(n\Delta t), \nu_{j,c}^{k',q}(n\Delta t), \nu_{j,t}^{k',q}(n\Delta t))\big).
\end{aligned}
$$

with $k' \in \{k-1, k+1\}$ and $\tau = t - n\Delta t$. Observe that for $t = (n+1)\Delta t$ we have $\tau = \Delta t$.
The key observation is that in this approximation procedure the equations in (9) for the evolution of different vehicles, are decoupled. The consequence is that the flow $\mathcal{T}_\tau^{\nu_{j,c}^{k,q}(n\Delta t), \nu_{j,t}^{k,q}(n\Delta t)}$ has two components $(\mathcal{T}_c^{k,q}, \mathcal{T}_t^{k,q})$ representing respectively the evolution of cars and trucks.

For every $t \in [0, T]$,

$$\begin{aligned}
W_1^{1,1}&(\nu_{j,c}^{k,1}(t), \nu_{j,c}^{k,2}(t)) = \\
&= W_1^{1,1}\Big(\mathcal{T}_c^{k,1}\#(\nu_{j,c}^{k,1}(n\Delta t)) + \tau G_1(\nu_{j,c}^{k,1}(n\Delta t), \nu_{j,t}^{k,1}(n\Delta t), \nu_{j,c}^{k',1}(n\Delta t), \nu_{j,t}^{k',1}(n\Delta t))\ , \\
&\qquad \mathcal{T}_c^{k,2}\#(\nu_{j,c}^{k,2}(n\Delta t)) + \tau G_1(\nu_{j,c}^{k,2}(n\Delta t), \nu_{j,t}^{k,2}(n\Delta t), \nu_{j,c}^{k',2}(n\Delta t), \nu_{j,t}^{k',2}(n\Delta t))\Big) \\
&\le W_1^{1,1}\Big(\mathcal{T}_c^{k,1}\#(\nu_{j,c}^{k,1}(n\Delta t))\ , \mathcal{T}_c^{k,2}\#(\nu_{j,c}^{k,2}(n\Delta t))\Big) \\
&\qquad + \tau W_1^{1,1}\Big(G_1(\nu_{j,c}^{k,1}(n\Delta t), \nu_{j,t}^{k,1}(n\Delta t), \nu_{j,c}^{k',1}(n\Delta t), \nu_{j,t}^{k',1}(n\Delta t)), \\
&\qquad\qquad G_1(\nu_{j,c}^{k,2}(n\Delta t), \nu_{j,t}^{k,2}(n\Delta t), \nu_{j,c}^{k',2}(n\Delta t), \nu_{j,t}^{k',2}(n\Delta t))\Big)
\end{aligned} \tag{66}$$

Lemma 2.3 together with the estimate (6.14) in [15] yield the existence of a radius $\rho > 0$ and three constants $L_1, L_2, L_3 > 0$ such that

$$\begin{aligned}
W_1^{1,1}&\Big(\mathcal{T}_c^{k,1}\#(\nu_{j,c}^{k,1}(n\Delta t))\ , \mathcal{T}_c^{k,2}\#(\nu_{j,c}^{k,2}(n\Delta t))\Big) \\
&\le W_1^{1,1}\Big(\mathcal{T}_c^{k,1}\#(\nu_{j,c}^{k,1}(n\Delta t))\ , \mathcal{T}_c^{k,1}\#(\nu_{j,c}^{k,2}(n\Delta t))\Big) \\
&\qquad + W_1^{1,1}\Big(\mathcal{T}_c^{k,2}\#(\nu_{j,c}^{k,1}(n\Delta t))\ , \mathcal{T}_c^{k,2}\#(\nu_{j,c}^{k,2}(n\Delta t))\Big) \\
&\le L_1 W_1^{1,1}(\nu_{j,c}^{k,1}(n\Delta t), \nu_{j,c}^{k,2}(n\Delta t)) \\
&\qquad + L_3 \int_{n\Delta t}^{t} e^{L_2(t-s)}\Big[W_1^{1,1}(\nu_{j,c}^{k,1}(s)), \nu_{j,c}^{k,2}(s)) + W_1^{1,1}(\nu_{j,t}^{k,1}(s)), \nu_{j,t}^{k,2}(s))\Big] ds
\end{aligned} \tag{67}$$

where in the first passage we applied the triangular inequality and then the Lipschitz continuity of the flow map with an application of the Gronwall's Lemma.
On the other side the source G_1 is Lipschitz continuous in all the input with constant L_{G_1}, therefore

$$\begin{aligned}
W_1^{1,1}&\Big(G_1(\nu_{j,c}^{k,1}(n\Delta t), \nu_{j,t}^{k,1}(n\Delta t), \nu_{j,c}^{k',1}(n\Delta t), \nu_{j,t}^{k',1}(n\Delta t)), \\
&\qquad G_1(\nu_{j,c}^{k,2}(n\Delta t), \nu_{j,t}^{k,2}(n\Delta t), \nu_{j,c}^{k',2}(n\Delta t), \nu_{j,t}^{k',2}(n\Delta t))\Big) \\
&\le L_{G_1}\Big(W_1^{1,1}(\nu_{j,c}^{k,1}(n\Delta t), \nu_{j,c}^{k,2}(n\Delta t)) + W_1^{1,1}(\nu_{j,t}^{k,1}(n\Delta t), \nu_{j,t}^{k,2}(n\Delta t)) \\
&\qquad + W_1^{1,1}(\nu_{j,c}^{k',1}(n\Delta t), \nu_{j,c}^{k',2}(n\Delta t)) + W_1^{1,1}(\nu_{j,t}^{k',1}(n\Delta t), \nu_{j,t}^{k',2}(n\Delta t))\Big)
\end{aligned} \tag{68}$$

Combining recursively (66)-(67)-(68) we find the following estimate:

$$W_1^{1,1}(\nu_{j,c}^{k,1}(t), \nu_{j,c}^{k,2}(t)) \le C_1 \sum_{i=1}^{L} \Big(W_1^{1,1}(\nu_{0,c}^{i,1}, \nu_{0,c}^{i,2}) + W_1^{1,1}(\nu_{0,t}^{i,1}, \nu_{0,t}^{i,2}) \Big) \tag{69}$$

with C_1 positive constant. In analogous way we can derive

$$W_1^{1,1}(\nu_{j,t}^{k,1}(t), \nu_{j,t}^{k,2}(t)) \le C_2 \sum_{i=1}^{L} \Big(W_1^{1,1}(\nu_{0,c}^{i,1}, \nu_{0,c}^{i,2}) + W_1^{1,1}(\nu_{0,t}^{i,1}, \nu_{0,t}^{i,2}) \Big) \tag{70}$$

for $C_2 > 0$. By adding (69) to (70) and taking the limit for $j \to \infty$ we obtain (65) with $C_0 = \max\{C_1, C_2\}$.

□

5. **Future Work.** In the future, one may study the dynamics of finitely many vehicles including cars and trucks on a multi-lane in an appropriate numerical scheme. In particular, the parameters for the Bando-Follow-the-leader model and the lane-changing probability functions are needed to be trained. Furthermore, the convergence of the finite-dimensional hybrid system to the Vlasov type PDE with a source term can also be studied numerically. In addition, one may also add the dynamics of finitely many controlled autonomous vehicles to the study and focus on an optimal control problem to minimize, for instance, energy cost, and so on. In that case, we expect to have the convergence of a finite-dimensional hybrid optimal control problem to an infinite-dimensional hybrid optimal control problem.

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E-mail address: mxc6028@psu.edu
E-mail address: xiaoqian.gong@asu.edu
E-mail address: piccoli@camden.rutgers.edu