DOMAIN VARIATIONS OF THE FIRST EIGENVALUE VIA A STRICT FABER-KRAHN TYPE INEQUALITY

T. V. ANOOP AND K. ASHOK KUMAR

Abstract. For $d \geq 2$ and $\frac{2d+2}{d+2} < p < \infty$, we prove a strict Faber-Krahn type inequality for the first eigenvalue $\lambda_1(\Omega)$ of the p-Laplace operator on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ (with mixed boundary conditions) under the polarizations. We apply this inequality to the obstacle problems on the domains of the form $\Omega \setminus \mathcal{O}$, where $\mathcal{O} \subset\subset \Omega$ is an obstacle. Under some geometric assumptions on Ω and $\mathcal{O}$, we prove the strict monotonicity of $\lambda_1(\Omega \setminus \mathcal{O})$ with respect to certain translations and rotations of $\mathcal{O}$ in Ω.

1. Introduction

In 1877, Lord Rayleigh [26] conjectured that '*the disk is the only planar domain that minimizes the first Dirichlet eigenvalue of the Laplace operator among all planar domains of fixed area.*' Nearly after 45 years, this conjecture was proved by Faber [18] and Krahn [23] for the planar domains (in 1923), and it is extended for higher dimensional domains by Krahn [24] (in 1925). This result is known as the Faber-Krahn inequality which is also available for the first Dirichlet eigenvalue of the p-Laplace operator Δ_p, defined by $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with $p \in (1, \infty)$, see for example [25, page 191] and [20, II.4]. For a domain $\Omega \subset \mathbb{R}^d$, the Faber-Krahn inequality states that

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega), \tag{1.1}$$

where $\lambda_1(D)$ denotes the first Dirichlet eigenvalue of the p-Laplace operator on a domain D and Ω^* is the open ball centred at the origin in $\mathbb{R}^d$ with the same Lebesgue measure as that of Ω. If Ω is a ball, then the equality holds in (1.1). The question *'for which domains the strict inequality holds in* (1.1)*?'* is addressed in [2, 9, 14, 16, 21].

Noting that Ω^* is the Schwarz symmetrization of Ω, the inequality (1.1) asserts that the first Dirichlet eigenvalue decreases under the Schwarz symmetrization. Next, we see that a similar result easily holds under the polarization as well. The polarization is one of the simplest rearrangements on $\mathbb{R}^d$ that was first introduced for sets by Wolontis [32], and for functions by Ahlfors [1] (for $d = 2$) and Baernstein and Taylor [6] (for $d \geq 2$). We refer to [5, 10, 12, 28, 31] for further reading on polarizations and their applications. Now, we define the polarization of measurable sets and functions with respect to an open affine-halfspace in $\mathbb{R}^d$. Let H be an open affine-halfspace in $\mathbb{R}^d$ (called a *polarizer*), and let σ_H be the reflection with respect to the boundary ∂H in $\mathbb{R}^d$. We denote the set of all polarizers in $\mathbb{R}^d$ by $\mathcal{H}$.

Definition 1.1 (Polarization)**.** *Let $H \in \mathcal{H}$ and $\Omega \subseteq \mathbb{R}^d$. The polarization $P_H(\Omega)$ and the dual-polarization $P^H(\Omega)$ of Ω with respect to H are defined as:*

$$P_H(\Omega) = [(\Omega \cup \sigma_H(\Omega)) \cap H] \cup [\Omega \cap \sigma_H(\Omega)],$$
$$P^H(\Omega) = [(\Omega \cup \sigma_H(\Omega)) \cap H^{\mathsf{c}}] \cup [\Omega \cap \sigma_H(\Omega)].$$

For a measurable function $u : \mathbb{R}^d \longrightarrow \mathbb{R}$, the polarization $P_H(u)$ with respect to H is defined as

$$P_H(u)(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & \text{for } x \in H, \\ \min\{u(x), u(\sigma_H(x))\}, & \text{for } x \in \mathbb{R}^d \setminus H. \end{cases}$$

Now, for $u : \Omega \longrightarrow \mathbb{R}$ let $\widetilde{u}$ be the zero extension of u to $\mathbb{R}^d$. The polarization $P_H(u)$ is defined as the restriction of $P_H(\widetilde{u})$ to $P_H(\Omega)$.

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Remark 1.2. The polarization of the sets and the functions satisfy the following relation:

$$P_H(\mathbb{1}_\Omega) = \mathbb{1}_{P_H(\Omega)}, \text{ for any } \Omega \subseteq \mathbb{R}^d,$$

where $\mathbb{1}_\Omega$ denotes the characteristic function of Ω.

In Figure 1, the dark shaded regions on the right side represent the polarization $P_H(\Omega)$ of Ω with respect to H.

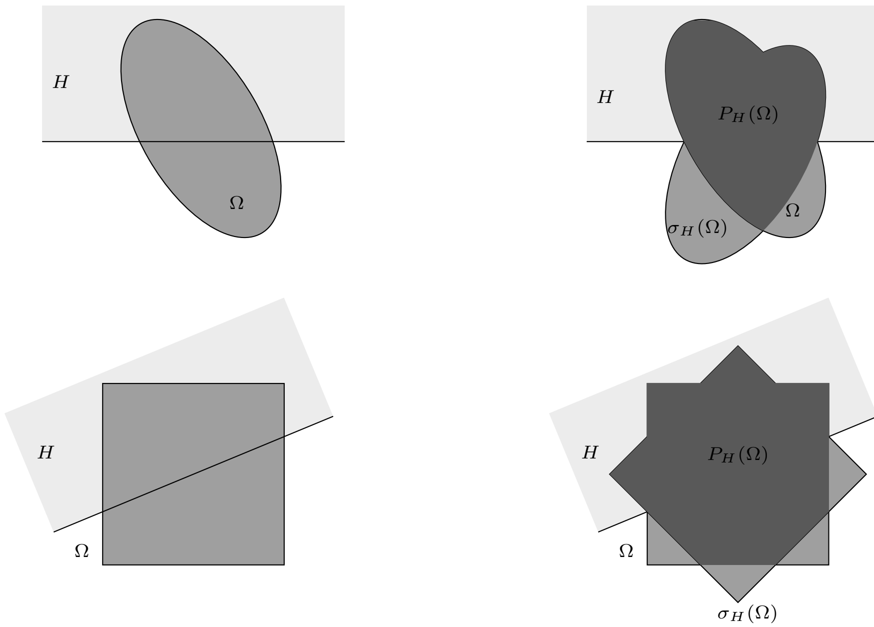


Figure 1. Polarization of an ellipse and a square.

For $H \in \mathcal{H}$, the polarization P_H is a rearrangement (preserves the inclusion order and the measure) on $\mathbb{R}^d$, see [13, Section 3]. Further, P_H takes an open set to an open set and a closed set to a closed set in $\mathbb{R}^d$. Throughout this article, we consider $p \in (1, \infty)$, unless otherwise specified. For a non-negative function $u \in W_0^{1,p}(\Omega)$ the polarization $P_H(u) \in W_0^{1,p}(P_H(\Omega))$ and the norms are preserved, see [13, Corollary 5.1]:

$$\|u\|_{p,\Omega} = \|P_H(u)\|_{p,P_H\Omega} \text{ and } \|\nabla u\|_{p,\Omega} = \|\nabla P_H(u)\|_{p,P_H\Omega}.$$

Therefore, we have the equality in the Pólya-Szëgo type inequality for the polarizations on $\mathbb{R}^d$. As an immediate consequence, the variational characterization of $\lambda_1(\Omega)$ yields the following Faber-Krahn type inequality:

$$\lambda_1(P_H(\Omega)) \leq \lambda_1(\Omega). \tag{1.2}$$

Clearly, if $P_H(\Omega) = \Omega$ or $P_H(\Omega) = \sigma_H(\Omega)$ then the equality holds in (1.2). In this article, we identify the domains for which the strict inequality holds in (1.2) for certain values of p. More precisely, we show that, if $p > \frac{2d+2}{d+2}$ and the equality holds in (1.2) then $P_H(\Omega) = \Omega$ or $P_H(\Omega) = \sigma_H(\Omega)$. We prove this result for the first eigenvalue of the p-Laplace operator with mixed boundary conditions on the multiply connected domains of the following form:

($\mathbf{A_0}$) $\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \subset \mathbb{R}^d$ is a bounded Lipschitz domain with $\Omega_{\text{in}} \subset\subset \Omega_{\text{out}}$, and $\Omega_{\text{in}} = \bigcup_{j=1}^{m} \Omega_j$, where Ω_j is simply connected and $\overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset$ for $i, j \in \{1, 2, \dots, m\}$ with $i \neq j$.

For $\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}$ as in ($\mathbf{A_0}$), we consider the following family of admissible polarizers

$$\mathcal{H}_{\text{ad}} := \Big\{ H \in \mathcal{H} : \sigma_H(\Omega_{\text{in}}) \subset\subset \Omega_{\text{out}} \Big\}.$$

Since $\Omega_{\text{in}} \subset\subset \Omega_{\text{out}}$, the set $\mathcal{H}_{\text{ad}}$ is always non-empty, and for $H \in \mathcal{H}_{\text{ad}}$ we have (see Proposition 2.12):

$$P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}) = P_H(\Omega_{\text{out}}) \setminus P^H(\overline{\Omega_{\text{in}}}) \text{ and } \partial P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}) = \partial P_H(\Omega_{\text{out}}) \sqcup \partial P^H(\Omega_{\text{in}}).$$

For $H \in \mathcal{H}_{\text{ad}}$, we consider the following eigenvalue problems for the p-Laplace operator on both Ω and $P_H(\Omega)$ with mixed boundary conditions:

Neumann condition on $\partial\Omega_{\text{in}}$:

$$\left.\begin{aligned} -\Delta_p u &= \nu|u|^{p-2}u \text{ in } \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}, \\ u &= 0 \text{ on } \partial\Omega_{\text{out}}, \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \partial\Omega_{\text{in}}; \end{aligned}\right\} \tag{1.3}$$

$$\left.\begin{aligned} -\Delta_p v &= \nu|v|^{p-2}v \text{ in } P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}), \\ v &= 0 \text{ on } \partial P_H(\Omega_{\text{out}}), \\ \frac{\partial v}{\partial n} &= 0 \text{ on } \partial P^H(\Omega_{\text{in}}); \end{aligned}\right\} \tag{1.4}$$

Neumann condition on $\partial\Omega_{\text{out}}$:

$$\left.\begin{aligned} -\Delta_p u &= \tau|u|^{p-2}u \text{ in } \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}, \\ u &= 0 \text{ on } \partial\Omega_{\text{in}}, \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \partial\Omega_{\text{out}}; \end{aligned}\right\} \tag{1.5}$$

$$\left.\begin{aligned} -\Delta_p v &= \tau|v|^{p-2}v \text{ in } P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}), \\ v &= 0 \text{ on } \partial P^H(\Omega_{\text{in}}), \\ \frac{\partial v}{\partial n} &= 0 \text{ on } \partial P_H(\Omega_{\text{out}}), \end{aligned}\right\} \tag{1.6}$$

where $\nu, \tau \in \mathbb{R}$.

The above eigenvalue problems can be collectively expressed as the following problem:

$$\left.\begin{aligned} -\Delta_p u &= \gamma|u|^{p-2}u \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_N, \end{aligned}\right\} \tag{$\mathcal{E}$}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain with $\partial\Omega = \Gamma_N \sqcup \Gamma_D$, and $\gamma \in \mathbb{R}$. Let

$$\mathcal{C}^{0,1}_{\Gamma_D}(\Omega) := \Big\{ v \text{ is a Lipschitz continuous function on } \Omega \text{ with } \operatorname{supp}(v) \cap \Gamma_D = \emptyset \Big\},$$

and we consider the following Sobolev space:

$$W^{1,p}_{\Gamma_D}(\Omega) = \text{ the closure of } \mathcal{C}^{0,1}_{\Gamma_D}(\Omega) \text{ in } W^{1,p}(\Omega).$$

If $\Gamma_N = \emptyset$ (equivalently $\Gamma_D = \partial\Omega$) then $W^{1,p}_{\Gamma_D}(\Omega) = W^{1,p}_0(\Omega)$. A real number γ is said to be an eigenvalue of ($\mathcal{E}$) if there exists a non-zero function $u \in W^{1,p}_{\Gamma_D}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x - \gamma \int_\Omega |u|^{p-2}uv \,\mathrm{d}x = 0 \text{ for every } v \in W^{1,p}_{\Gamma_D}(\Omega),$$

and the function u is called as an eigenfunction corresponding to the eigenvalue γ. The standard variational arguments establish the existence of an infinite subset of eigenvalues tending to infinity (see [3, Proposition A.1]). The first eigenvalue $\gamma_1(\Omega)$ of ($\mathcal{E}$) is simple (the dimension of the eigenspace is one) and the corresponding eignfunctions have constant sign in Ω (see [3, Proposition A.2]). Moreover, the first eigenvalue $\gamma_1(\Omega)$ of ($\mathcal{E}$) has the following variational characterization:

$$\gamma_1(\Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x : u \in W^{1,p}_{\Gamma_D}(\Omega) \text{ with } \int_\Omega |u|^p \,\mathrm{d}x = 1 \right\}.$$

Now, we state a Faber-Krahn type inequality for the first eigenvalues of the eigenvalue problems (1.3) and (1.4), and similarly for the first eigenvalues of the eigenvalue problems (1.5) and (1.6).

Theorem 1.3. *Let $p \in (1,\infty)$, $\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \subset \mathbb{R}^d$ be a domain as given in* ($\mathbf{A_0}$)*, and $H \in \mathcal{H}_{\text{ad}}$.*

(i) *If $\sigma_H(\Omega_{\text{in}}) = \Omega_{\text{in}}$, then*

$$\nu_1(P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}})) \leq \nu_1(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}). \tag{1.7}$$

(ii) *If* $\Omega_{\text{in}} \neq \emptyset$ *and* $\sigma_H(\Omega_{\text{out}}) = \Omega_{\text{out}}$*, then*

$$\tau_1(P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}})) \leq \tau_1(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}). \tag{1.8}$$

(iii) *If* $\frac{2d+2}{d+2} < p < \infty$*, and the equality holds in* (1.7) *or* (1.8) *then*

$$P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}) = \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \text{ or } P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}) = \sigma_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}).$$

Remark 1.4.

(i) If $\Omega_{\text{in}} = \emptyset$, in (i) then ν_1 corresponds to the first Dirichlet eigenvalue λ_1 and thus (1.7) gives: for every $H \in \mathcal{H}_{\text{ad}}$,

$$\lambda_1(P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}})) \leq \lambda_1(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}).$$

(ii) If $\Omega_{\text{in}} = \emptyset$, then $\tau_1(\Omega_{\text{out}}) = \tau_1(P_H(\Omega_{\text{out}})) = 0$, for every $H \in \mathcal{H}$. This is the reason why we impose the condition $\Omega_{\text{in}} \neq \emptyset$ in (ii).

(iii) The symmetry assumptions in (i) and (ii) of Theorem 1.3 ensure that $\Gamma_N \subseteq \partial P_H(\Omega)$ and hence the Neumann boundary is unaltered under such polarizations. This fact is crucially used in our proof. Obtaining the same conclusions of Theorem 1.3 without these additional symmetry assumptions seems to be a challenging problem.

Application to the domain variations: Next, we apply Theorem 1.3 for the domains of the form $\Omega \setminus \mathcal{O} \subset \mathbb{R}^d$ to study the monotonicity of the first eigenvalue of ($\mathcal{E}$) on $\Omega \setminus \mathcal{O}$ under certain translations and rotations of $\mathcal{O}$ within Ω. We assume the following:

($\mathbf{A_1}$) $\mathcal{O} \subset \Omega$ is a closed set with nonempty interior such that $\Omega \setminus \mathcal{O}$ is a bounded Lipschitz domain.

The set $\mathcal{O}$ in ($\mathbf{A_1}$) is called as *an obstacle*. The main idea is to express the translations and the rotations of $\mathcal{O}$ in terms of polarizations of punctured domain $\Omega \setminus \mathcal{O}$. Then we apply Theorem 1.3 and get the monotonicity of the eigenvalue.

The monotonicity along a straight line: In this case, we set $\Omega_{\text{in}} = \emptyset$ and $\Omega_{\text{out}} = \Omega \setminus \mathcal{O}$ is a bounded Lipschitz domain in $\mathbb{R}^d$. For a given $h \in \mathbb{S}^{d-1}$, we study the monotonicity of the first Dirichlet eigenvalue of the p-Laplace operator with respect to the translations of the obstacle $\mathcal{O}$ in the h-direction within Ω. Without loss of generality, we may assume that the origin $0 \in \mathcal{O}$. We consider the following family of polarizers:

$$H_s = \left\{x \in \mathbb{R}^d : x \cdot h < s\right\}, \text{ for } s \in \mathbb{R}. \tag{1.9}$$

We make the following geometric assumption on Ω and $\mathcal{O}$:

($\mathbf{A_2}$) $P_{H_0}(\Omega) = \Omega$, and $\mathcal{O}$ is Steiner symmetric with respect to the hyperplane ∂H_0 (see Definition 2.7).

The translations of $\mathcal{O}$ in the directions of h are given by

$$\mathcal{O}_s = sh + \mathcal{O} \text{ for } s \in \mathbb{R}. \tag{1.10}$$

For Ω and $\mathcal{O}$ as given in ($\mathbf{A_2}$), define $\mathrm{L}_{\mathcal{O}} = \left\{s \in \mathbb{R} : P_{H_s}(\Omega) = \Omega \text{ and } \mathcal{O}_s \subset \Omega\right\}$. Let $\lambda_1(s)$ be the first eigenvalue of (1.3) with $\Omega_{\text{in}} = \emptyset$ and $\Omega_{\text{out}} = \Omega \setminus \mathcal{O}_s$ for $s \in \mathrm{L}_{\mathcal{O}}$. For $s \in \mathbb{R}$, let $\Sigma_s := \left\{x \in \Omega : x \cdot h > s\right\}$. A set $A \subseteq \mathbb{R}$ is said to be convex in the h-direction, if any line segment parallel to the $\mathbb{R}h$-axis with endpoints in A completely lies in A. Now, we have the following strict monotonicity result.

Theorem 1.5. *Let* $\frac{2d+2}{d+2} < p < \infty$ *and* $h \in \mathbb{S}^{d-1}$*. Assume that* $\mathcal{O}, \Omega \subset \mathbb{R}^d$ *satisfy* ($\mathbf{A_1}$) *and* ($\mathbf{A_2}$)*. If the set* $\Sigma_{s_0} \bigcup \sigma_{H_{s_0}}(\Sigma_{s_0})$ *is convex in the* h*-direction for some* $s_0 \in \mathrm{L}_{\mathcal{O}}$*, then the set* $\{s \in \mathrm{L}_{\mathcal{O}} : s \geq s_0\}$ *is an interval and* $\lambda_1(\cdot)$ *is strictly decreasing on this interval.*

Throughout this article, for given $a \in \mathbb{R}^d$ and $r \geq 0$, we denote $B_r(a) = \left\{x \in \mathbb{R}^d : |x - a| < r\right\}$, the open ball centered at a with the radius r, and the closure of $B_r(a)$ by $\overline{B}_r(a)$.

Remark 1.6. In Theorem 1.5, if Ω itself is convex in the h-direction, then $\mathrm{L}_{\mathcal{O}}$ is an interval containing 0. In particular, if $\Omega = B_R(0)$, $\mathcal{O} = \overline{B}_r(0)$ for $0 < R < r < \infty$ and $h = e_1 = (1, 0, \dots, 0) \in \mathbb{S}^{d-1}$. Then, both Ω and $\mathcal{O}$ are Steiner symmetric with respect to ∂H_0, and $\mathrm{L}_{\mathcal{O}} = [0, R - r)$. Therefore, by Theorem 1.5, the first Dirichlet eigenvalue $\lambda_1(B_R(0) \setminus \overline{B}_r(se_1))$ is strictly decreasing for $s \in [0, R - r)$. Thus, Theorem 1.5 gives an alternate proof for many existing strict monotonicity results that were proved using the shape derivative (Hadamard perturbation) formula. For example, Kesavan [22] and Harrell-Kröger-Kurata [19], and Anoop-Bobkov-Sasi [4] for $p \in \left(\frac{2d+2}{d+2}, \infty\right)$.

Remark 1.7. Due to the symmetry restrictions on the Neumann boundary in Theorem 1.3, the monotonicity results (similar to that of Dirichlet eigenvalue in Remark 1.6), when the Neumann boundary condition is specified on $\partial B_R(0)$ can not be deduced from Theorem 1.5. However, such a monotonicity result is proved for $p = 2$, by Anoop-Ashok-Kesavan [5] using the Hadamard perturbation formula and some geometric properties of the first eigenfunctions. This result is open for general $p \neq 2$.

The monotonicity with respect to the rotations about a point: Next, we study the monotonicity of the first eigenvalue of ($\mathcal{E}$) on $\Omega \setminus \mathcal{O}$ with respect to the rotations of the obstacle $\mathcal{O}$ in Ω about a point $a \in \mathbb{R}^d$. We set $\mathbb{R}^+ = [0, \infty)$, and make the following geometric assumptions on both Ω and $\mathcal{O}$:

- ($\mathbf{A_3}$) The domain Ω and the obstacle $\mathcal{O}$ are foliated Schwarz symmetric with respect to the ray $a + \mathbb{R}^+\eta$, for some $\eta \in \mathbb{S}^{d-1}$ (see Definition 2.7).

For $s \in [-1, 1]$, let $\theta_s := \arccos(s) \in [0, \pi]$. For $\xi \in \mathbb{S}^{d-1} \setminus \{\eta\}$, let $R_{s,\xi}$ be the simple rotation on $\mathbb{R}^d$ with the plane of rotation is $X_\xi := \operatorname{span}\{\eta, \xi\}$ and the angle of rotation is θ_s from the ray $\mathbb{R}^+\eta$ in the counter-clockwise direction. The rotation of the obstacle $\mathcal{O}$ by $R_{s,\xi}$ about the point a is given by

$$\mathcal{O}_{s,\xi} = a + R_{s,\xi}(-a + \mathcal{O}). \tag{1.11}$$

Now, we observe the following facts (see Proposition 2.9 and Lemma 4.3):

(a) The rotated obstacle $\mathcal{O}_{s,\xi}$ is foliated Schwarz symmetric with respect to the ray $a + \mathbb{R}^+ R_{s,\xi}(\eta)$.
(b) For any rotation R that fixes η, $\mathcal{O} = a + R(-a + \mathcal{O})$ and $\Omega = a + R(-a + \Omega)$.
(c) For any distinct $\xi_1, \xi_2 \in \mathbb{S}^{d-1} \setminus \{\eta\}$, there exists R that fixes η such that

$$R(-a + \Omega \setminus \mathcal{O}_{s,\xi_1}) = -a + \Omega \setminus \mathcal{O}_{s,\xi_2}.$$

From the above observations, it is evident that we only need to consider the rotations of the obstacle by $R_{s,\xi}$ with respect to a in a X_ξ-plane for a fixed $\xi \in \mathbb{S}^{d-1} \setminus \{\eta\}$. Thus for $s \in [-1, 1]$, we set $\mathcal{O}_s = \mathcal{O}_{s,\xi}$ and consider

$$\begin{aligned} \mathrm{C}_{\mathcal{O}} &:= \Big\{ s \in [-1, 1] : \mathcal{O}_s \subset \Omega \Big\}, \\ \gamma_1(s) &:= \gamma_1(\Omega \setminus \mathcal{O}_s), \ \text{the first eigenvalue of } (\mathcal{E}) \text{ on } \Omega \setminus \mathcal{O}_s \text{ for } s \in \mathrm{C}_{\mathcal{O}}. \end{aligned} \tag{1.12}$$

In this article, we consider the following types of Ω and $\Gamma_N \subseteq \partial\Omega$:

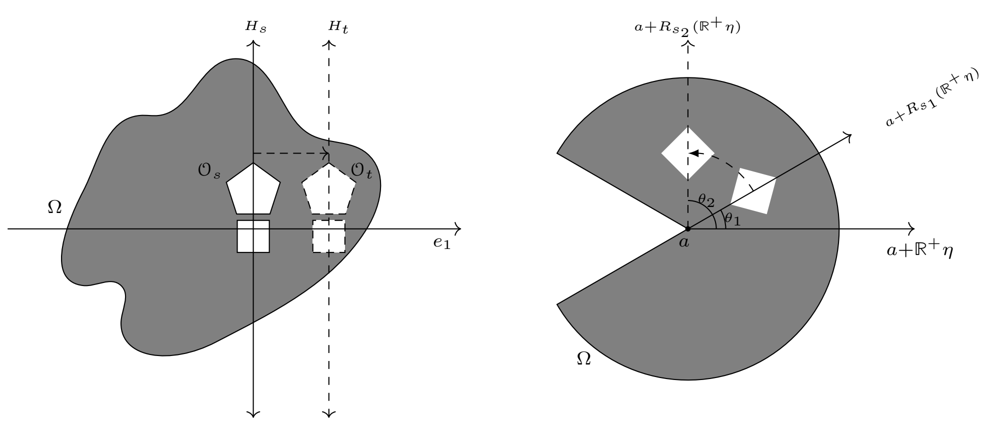


FIGURE 2. The translations of $\mathcal{O}$ along the e_1-axis; and rotations of $\mathcal{O}$ about the point $a \in \mathbb{R}^d$, here $\theta_i = \arccos(s_i)$ for $i = 1, 2$ with $s_1 > s_2$.

$(\mathbf{A_4})$ $\Omega = \Omega_0 \setminus \overline{B}_{\rho_0}(a)$, where $\overline{B}_{\rho_0}(a) \subsetneq \Omega_0 \subset \mathbb{R}^d$, $\rho_0 \geq 0$; and $\Gamma_N = \partial B_{\rho_0}(a)$.

$(\mathbf{A_5})$ $\Omega = B_R(a) \setminus \overline{\Omega_1}$, where $\overline{\Omega_1} \subset B_R(a)$, and $\Gamma_N = \partial B_R(a)$.

Now, we state our monotonicity result for $\gamma_1(.)$ on $\mathrm{C}_{\mathcal{O}}$.

Theorem 1.8. *Let $\frac{2d+2}{d+2} < p < \infty$ and $\Omega \subset \mathbb{R}^d$ be a domain. Assume that the pair Ω and Γ_N satisfy either $(\mathbf{A_4})$ or $(\mathbf{A_5})$. If Ω and $\mathcal{O}$ satisfy $(\mathbf{A_1})$ and $(\mathbf{A_3})$ for some $a \in \mathbb{R}^d$ and $\eta \in \mathbb{S}^{d-1}$, then $\mathrm{C}_{\mathcal{O}}$ is an interval. In addition, if Ω is not radial with respect to a, then $\gamma_1(\cdot)$ is strictly increasing on $\mathrm{C}_{\mathcal{O}}$.*

Remark 1.9. If Ω is radial with respect to the point a (see Corollary 2.11), then the first eigenvalue $\gamma_1(\cdot)$ remains as a constant on $\mathrm{C}_{\mathcal{O}}$.

The rest of this article is organized as follows. In Section 2, the polarization of measurable sets and functions are introduced, and some of their important properties are discussed. Further, the characterizations of Steiner and foliated Schwarz symmetries using polarizations are given in Section 2. Also, we include a strong comparison principle and a few interior and boundary regularity results that are essential for the development of this article. A proof of Faber-Krahn inequality (Theorem 1.3) is given in Section 3. The proofs of strict monotonicity results (Theorem 1.5 and 1.8) are given in Section 4. Many important remarks and explicit examples are included in Section 5.

2. Preliminaries

In this section, we discuss some of the important properties of the polarization of the sets and functions. Further, we give the definitions of Steiner and foliated Schwarz symmetries, and their characterizations in terms of polarizations. Lastly, we give some regularity results and strong comparison principles for the solutions of the p-Laplace operator.

2.1. Polarization of sets.

We discuss a few simple properties of the polarization of sets.

Proposition 2.1. *Let $H \in \mathcal{H}$ and $A, C \subseteq \mathbb{R}^d$. Then,*

(i) *$P_H(A)$ is open, if A is open; and $P_H(A)$ is closed if A is closed;*
(ii) *$P_H(A) \subseteq P_H(C)$, if $A \subseteq C$;*
(iii) *$P_H(A \cap C) \subseteq P_H(A) \cap P_H(C)$ and $P_H(A) \cup P_H(C) \subseteq P_H(A \cup C)$;*
(iv) *$P_H(\sigma_H(A)) = P_H(A)$, $\sigma_H(P_H(A)) = P^H(A)$, and $\sigma_H(P^H(A)) = P_H(A)$;*
(v) *$P_H(A^{\mathsf{c}}) = \left(P^H(A)\right)^{\mathsf{c}}$.*

Proof. Recall that, for $H \in \mathcal{H}$, the polarizations of a set $A \subseteq \mathbb{R}^d$ are given by

$$P_H(A) = [(A \cup \sigma_H(A)) \cap H] \cup [A \cap \sigma_H(A)], \text{ and } P^H(A) = [(A \cup \sigma_H(A)) \cap H^{\mathsf{c}}] \cup [A \cap \sigma_H(A)].$$

Since $A \cap \partial H = \sigma_H(A) \cap \partial H$, we can also write

$$P_H(A) = \left[(A \cup \sigma_H(A)) \cap \overline{H}\right] \cup [A \cap \sigma_H(A)] \text{ and } P^H(A) = \left[(A \cup \sigma_H(A)) \cap \overline{H}^{\mathsf{c}}\right] \cup [A \cap \sigma_H(A)].$$

Now, (i)-(iii) follow easily from the above observations.
(iv) This follows from the fact $\sigma_H(H) = \overline{H}^{\mathsf{c}}$ and the above observations.
(v) By the definition, we have $P_H(A^{\mathsf{c}}) = [(A^{\mathsf{c}} \cup \sigma_H(A^{\mathsf{c}})) \cap H] \bigcup [A^{\mathsf{c}} \cap \sigma_H(A^{\mathsf{c}})]$, and hence

$$\begin{aligned}(P_H(A^{\mathsf{c}}))^{\mathsf{c}} &= [(A \cap \sigma_H(A)) \cup H^{\mathsf{c}}] \cap [A \cup \sigma_H(A)] \\ &= [A \cap \sigma_H(A)] \cup [(A \cup \sigma_H(A)) \cap H^{\mathsf{c}}] = P^H(A).\end{aligned}$$ ∎

The following proposition characterizes the invariance of a set under polarizations.

Proposition 2.2. *Let $H \in \mathcal{H}$ and $A \subseteq \mathbb{R}^d$. Then*

(i) *$P_H(A) = A$ if and only if $\sigma_H(A) \cap H \subseteq A$;*
(ii) *$P^H(A) = A$ if and only if $\sigma_H(A) \cap H^c \subseteq A$;*
(iii) *$P_H(A) = P^H(A)$ if and only if $\sigma_H(A) = A$.*

Proof. (i) From the definition of $P_H(A)$, it is clear that

$$P_H(A) = [(A \cup \sigma_H(A)) \cap H] \cup [A \cap \sigma_H(A) \cap H^{\mathsf{c}}] \,. \tag{2.1}$$

If $P_H(A) = A$, then $P_H(A) \cap H = A \cap H$. Thus the above equation yields $(A \cup \sigma_H(A)) \cap H \subseteq A$, and hence we must have $\sigma_H(A) \cap H \subseteq A$. Conversely, assume that $\sigma_H(A) \cap H \subseteq A$. Then, by applying σ_H on both sides, and using the fact that $A \cap \partial H = \sigma_H(A) \cap \partial H$, we obtain $A \cap H^{\mathsf{c}} \subseteq \sigma_H(A)$, Therefore, $A \cap H^{\mathsf{c}} = A \cap \sigma_H(A) \cap H^{\mathsf{c}}$. From the assumption, we also have $A \cap H = [(A \cup \sigma_H(A)) \cap H]$. Now, using (2.1), we easily conclude that $P_H(A) = A$.
(ii) From Proposition 2.1-(iv), we have $\sigma_H(P^H(A)) = P_H(A)$ and $P_H(\sigma_H(A)) = P_H(A)$. Therefore, we get $P^H(A) = A$ if and only if $P_H(\sigma_H(A)) = \sigma_H(A)$. Now, from (i) we obtain

$$P_H(\sigma_H(A)) = \sigma_H(A) \text{ if and only if } A \cap H \subseteq \sigma_H(A).$$

Now applying σ_H on both sides of last inclusion and using the fact that $\sigma_H(A) \cap \partial H = A \cap \partial H$, we get

$$P^H(A) = A \text{ if and only if } \sigma_H(A) \cap H^c \subseteq A.$$

(iii) From the definitions of $P_H(A)$ and $P^H(A)$, it is clear that

$$P_H(A) = [(A \cup \sigma_H(A)) \cap H] \cup [A \cap \sigma_H(A) \cap H^{\mathsf{c}}] \,,$$
$$P^H(A) = [(A \cup \sigma_H(A)) \cap H^{\mathsf{c}}] \cup [A \cap \sigma_H(A) \cap H] \,.$$

If $P_H(A) = P^H(A)$, then $(A \cup \sigma_H(A)) \cap H = A \cap \sigma_H(A) \cap H$ and $A \cap \sigma_H(A) \cap H^{\mathsf{c}} = (A \cup \sigma_H(A)) \cap H^{\mathsf{c}}$. Therefore $A \cup \sigma_H(A) = A \cap \sigma_H(A)$, and hence $\sigma_H(A) = A$. Conversely, assume that $\sigma_H(A) = A$. Then, from above equations, we get $P_H(A) = P^H(A) = A$. $\square$

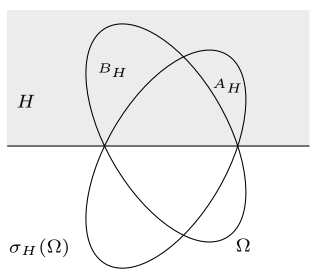


FIGURE 3. The sets A_H and B_H of $P_H(\Omega) \cap H$.

Proposition 2.3. *Let $H \in \mathcal{H}$ and $\Omega \subseteq \mathbb{R}^d$ be an open set. Then,*

(i) *$P_H(\Omega) \neq \Omega$ if and only if $A_H := \sigma_H(\Omega) \cap \Omega^{\mathsf{c}} \cap H$ has non-empty interior;*
(ii) *$P_H(\Omega) \neq \Omega$ if and only if $B_H := \Omega \cap \sigma_H(\Omega^{\mathsf{c}}) \cap H$ has non-empty interior.*

Proof. (i) First, we observe that the interior of A_H is $\sigma_H(\Omega) \cap \overline{\Omega}^{\mathsf{c}} \cap H$. Since $\sigma_H(\Omega) \cap H$ is open, from Proposition 2.2, we get $P_H(\Omega) \neq \Omega$ if and only if $\sigma_H(\Omega) \cap H \nsubseteq \overline{\Omega}$. Clearly, $\sigma_H(\Omega) \cap H \nsubseteq \overline{\Omega}$ if and only if $\overline{\Omega}^{\mathsf{c}} \cap \sigma_H(\Omega) \cap H \neq \emptyset$.
(ii) For $H \in \mathcal{H}$, we have $\sigma_H(H) \in \mathcal{H}$. Then from Proposition 2.1, $P_H(\Omega) = \sigma_H(\Omega)$ if and only if $P_{\sigma_H(H)}(\Omega) = \Omega$. The proof follows from (i) by replacing H with $\sigma_H(H)$. $\square$

Remark 2.4. For an open set $\Omega \subset \mathbb{R}^d$, if $P_H(\Omega) \neq \Omega$ then the interior of $P_H(\Omega) \setminus \Omega$ is non-empty. Therefore, if $P_H(\Omega) \neq \Omega$ then $P_H(\Omega)$ can not be equal to Ω up to a set of measure zero (or up to a set of p-capacity zero).

Now, we prove that the set $P_H(\Omega) \cap H$ is a domain when Ω is a domain. For this, we need the following lemma.

Lemma 2.5. *Let $H \in \mathcal{H}$ and $\Omega \subseteq \mathbb{R}^d$ be a domain. If $\sigma_H(\Omega) = \Omega$, then both $\Omega \cap H$ and $\Omega \cap \overline{H}^{\mathsf{c}}$ are connected.*

Proof. Let $f : \Omega \cap \overline{H} \longrightarrow \{0,1\}$ be a continuous function. Using the symmetry of Ω, we define

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{for } x \in \Omega \cap \overline{H}, \\ f \circ \sigma_H(x), & \text{for } x \in \Omega \cap H^{\mathsf{c}}. \end{cases}$$

Then $\widetilde{f}$ is a continuous function on Ω, since $\sigma_H(x) = x$ for $x \in \partial H$. By the connectedness of Ω, $\widetilde{f}$ is constant on Ω. In particular, f is constant on $\Omega \cap \overline{H}$, and hence $\Omega \cap \overline{H}$ is connected. Therefore $\operatorname{int}\big(\Omega \cap \overline{H}\big) = \Omega \cap H$ is connected, and hence $\sigma_H(\Omega \cap H) = \Omega \cap \overline{H}^{\mathsf{c}}$ is also connected. ☐

Proposition 2.6. *Let $H \in \mathcal{H}$ and $\Omega \subseteq \mathbb{R}^d$ be a domain. Then $P_H(\Omega) \cap H$ is a domain.*

Proof. First, we observe that $P_H(\Omega) \cap H = (\Omega \cup \sigma_H(\Omega)) \cap H$ is open. For proving the connectedness, we consider the following two cases: **(a)** $\Omega \cap \sigma_H(\Omega) = \emptyset$, and **(b)** $\Omega \cap \sigma_H(\Omega) \neq \emptyset$.

(a) $\Omega \cap \sigma_H(\Omega) = \emptyset$: In this case, we have $\Omega \cap \partial H = \emptyset$, since $\Omega \cap \partial H \subset \Omega \cap \sigma_H(\Omega)$. Therefore, Ω is the union of two open sets $\Omega \cap H$ and $\Omega \cap \overline{H}^{\mathsf{c}}$. By the connectedness of Ω, one of them is equal to Ω. If $\Omega \cap H = \Omega$ then $P_H(\Omega) = \Omega$, and hence $P_H(\Omega) \cap H = \Omega$. If $\Omega \cap \overline{H}^{\mathsf{c}} = \Omega$ then $P_H(\Omega) = \sigma_H(\Omega)$, and hence $P_H(\Omega) \cap H = \sigma_H(\Omega)$.

(b) $\Omega \cap \sigma_H(\Omega) \neq \emptyset$: In this case, we have $\Omega \cup \sigma_H(\Omega)$ is connected, and it is symmetric with respect to ∂H. Thus, by Lemma 2.5, $(\Omega \cup \sigma_H(\Omega)) \cap H = P_H(\Omega) \cap H$ is connected.
Therefore, in both of the cases, $P_H(\Omega) \cap H$ is domain. ☐

The Steiner, axial and foliated Schwarz symmetries: A set in $\mathbb{R}^d$ is said to have certain symmetry, if it is invariant under corresponding symmetrization or rearrangement on $\mathbb{R}^d$. Here, we directly give the definitions of the Steiner and the foliated Schwarz symmetries without defining the associated symmetrizations (see [30, Definition 3.1 and Definition 3.2]). The foliated Schwarz symmetrization with respect to a ray $a + \mathbb{R}^+\eta$ is the cap symmetrization with respect to $a + \mathbb{R}^+\eta$, see [30, Definition 3.2].

Definition 2.7. *Let $A \subseteq \mathbb{R}^d$ be a measurable set.*

(1) ***Steiner symmetry.*** *Let S be an affine-hyperplane in $\mathbb{R}^d$. For each $x \in S$, let L_x be the line passing through x and orthogonal to S. Then A is said to be Steiner symmetric with respect to S, if*
$$\text{for each } x \in S, \quad A \cap L_x = B_\rho(x) \cap L_x \text{ for some } \rho \geq 0.$$
(2) ***Axial symmetry.*** *Let L be a line in $\mathbb{R}^d$. For each $x \in L$, let S_x be the affine hyperplane passing through x and orthogonal to L. Then A is said to be axially symmetric with respect to L, if*
$$\text{for each } x \in L, \quad A \cap \partial B_\rho(x) \cap S_x = \partial B_\rho(x) \cap S_x \text{ for some } \rho \geq 0.$$
(3) ***Foliated Schwarz symmetry.*** *Let $a + \mathbb{R}^+\eta$ be a ray starting for some $a \in \mathbb{R}^d$ and $\eta \in \mathbb{S}^{d-1}$. Then A said to be foliated Schwarz symmetric with respect to $a + \mathbb{R}^+\eta$, if*
$$\text{for every } r > 0, \quad A \cap \partial B_r(a) = B_\rho(a + r\eta) \cap \partial B_r(a) \text{ for some } \rho \geq 0.$$

Remark 2.8. We observe that:

(i) a set $A \subseteq \mathbb{R}^d$ is Steiner symmetric with respect to an affine-hyperplane S, if and only if A is invariant under the reflection with respect to S and convex in the orthogonal direction to S;
(ii) a set $A \subseteq \mathbb{R}^d$ is axially symmetric with respect to a line L, if and only if A is invariant under the reflection with respect to every affine hyperplane containing L. In particular, if $L = \mathbb{R}\eta$ and R is any rotation on $\mathbb{R}^d$ such that $R(\eta) = \eta$, then $R(A) = A$. This follows from the definition, since the planes of rotation of such R can not contain η, and hence those planes must be orthogonal to η;
(iii) let $A \subseteq \mathbb{R}^d$ be foliated Schwarz symmetric with respect to $a + \mathbb{R}^+\eta$. Let $I_A := \big\{r > 0 : A \cap \partial B_r(a) \neq \emptyset\big\}$. For $r \in I_A$, let $\rho(r) > 0$ be such that $A \cap \partial B_r(a) = B_{\rho(r)}(a + r\eta) \cap \partial B_r(a)$. Then,
$$A = \bigcup_{r \in I_A} B_{\rho(r)}(a + r\eta) \cap \partial B_r(a). \tag{2.2}$$

The following proposition provides some properties of the foliated Schwartz symmetric sets.

Proposition 2.9. *If $A \subseteq \mathbb{R}^d$ is foliated Schwarz symmetric with respect to a ray $a + \mathbb{R}^+\eta$ then*

(i) *A is axially symmetric with respect to $a + \mathbb{R}\eta$,*

(ii) *for any linear map T and $b \in \mathbb{R}^d$, the set $b + T(A)$ is foliated Schwarz symmetric with respect to $b + T(a) + \mathbb{R}^+ T(\eta)$,*

(iii) *$R(-a + A) = -a + A$, for any rotation R on $\mathbb{R}^d$ that fixes η.*

Proof. (i) Observe that, for every $r \in I_A$, the set $B_{\rho(r)}(a + r\eta) \cap \partial B_r(a)$ is axially symmetric with respect to $a + \mathbb{R}\eta$. Now, using (2.2), we conclude that A is axially symmetric with respect to $a + \mathbb{R}\eta$.
(ii) Let $r > 0$, then

$$(b + T(A)) \cap \partial B_r(b + T(a)) = b + T(A \cap \partial B_r(a)) = b + T(B_\rho(a + r\eta) \cap \partial B_r(a)), \text{ for some } \rho \geq 0,$$

where the last equality follows from the definition foliated Schwarz symmetry. Thus

$$\begin{aligned}(b + T(A)) \cap \partial B_r(b + T(a)) &= b + B_\rho(T(a) + rT(\eta)) \cap \partial B_r(T(a)) \\ &= B_\rho(b + T(a) + rT(\eta)) \cap \partial B_r(b + T(a)).\end{aligned}$$

Now, we obtain the required conclusion by the definition of foliated Schwarz symmetry.
(iii) By taking $T = I$ and $b = -a$ in (ii), we get $-a + A$ is foliated Schwarz symmetric with respect to $\mathbb{R}^+\eta$. Thus by (i), $-a + A$ is axially symmetric with respect to $\mathbb{R}\eta$. Since R fixes η, from (ii) of Remark 2.8, we conclude $R(-a + A) = -a + A$. ☐

Next, we characterize the foliated Schwarz and Steiner symmetric sets using the polarizations. First, we consider the following polarizers: for given $a \in \mathbb{R}^d, \eta \in \mathbb{S}^{d-1}$, let

$$\mathcal{H}_{a,\eta} := \Big\{ H \in \mathcal{H} : a + \mathbb{R}^+\eta \subset H \text{ and } a \in \partial H \Big\}.$$

Some useful characterizations of the Steiner symmetry (from [11, Lemma 2.2]), foliated Schwarz symmetry (from [30, Section 3]) are given in the following proposition.

Proposition 2.10. *Let $A \subseteq \mathbb{R}^d$ be any set.*

(i) *Let $H_s \in \mathcal{H}$ be as given in (1.9). Then, for $s_0 \in \mathbb{R}$, the following statements are equivalent:*
  (a) *the set A is Steiner symmetric with respect to the affine-hyperplane ∂H_{s_0},*
  (b) *$P_{H_s}(A) = A$, for every $s \geq s_0$; and $P^{H_s}(A) = A$, for every $s \leq s_0$.*

(ii) *Let $a \in \mathbb{R}^d$ and $\eta \in \mathbb{S}^{d-1}$. Then the following are equivalent:*
  (a) *the set A is foliated Schwarz symmetric with respect to the ray $a + \mathbb{R}^+\eta$,*
  (b) *$P^H(A) = \sigma_H(A)$, for every $H \in \mathcal{H}_{a,\eta}$.*

We have the following corollary.

Corollary 2.11. Let $A \subseteq \mathbb{R}^d$ be any set. If A is foliated Schwarz symmetric with respect to both the rays $a + \mathbb{R}^+\eta$ and $a - \mathbb{R}^+\eta$ for some $a \in \mathbb{R}^d$ and $\eta \in \mathbb{S}^{d-1}$. Then A is radial with respect to the point a.

Proof. Notice that, A is radial with respect to $a \in \mathbb{R}^d$ provided $A \cap \partial B_r(a) = \partial B_r(a)$ for every $r \in I_A$, where $I_A = \{r \in \mathbb{R} : A \cap \partial B_r(a) \neq \emptyset\}$. Since A is foliated Schwarz symmetric with respect to both the rays $a + \mathbb{R}^+\eta$ and $a - \mathbb{R}^+\eta$, for each $r \in I_A$ we get:

$$A \cap \partial B_r(a) = B_{\rho_1}(a + r\eta) \cap \partial B_r(a) = B_{\rho_2}(a - r\eta) \cap \partial B_r(a) \text{ for some } \rho_1, \rho_2 \geq 0. \tag{2.3}$$

Since $|a - (a - r\eta)| = r$, from (2.3) we obtain $a - r\eta \in B_{\rho_1}(a + r\eta)$. Thus $\rho_1 \geq |a - r\eta - (a + r\eta)| = 2r$, and hence $B_{\rho_1}(a + r\eta) \cap \partial B_r(a) = \partial B_r(a)$. Now, from (2.3) we conclude that $A \cap \partial B_r(a) = \partial B_r(a)$. ☐

2.2. **Polarization of punctured domains.** We consider the polarization of the punctured domains of the form $A \setminus C$, where $A \subseteq \mathbb{R}^d$ is open, and $C \subset A$ is closed. Clearly $\partial(A \setminus C) = \partial A \sqcup \partial C$.

Proposition 2.12. *Let $A \subseteq \mathbb{R}^d$ be open and $C \subset A$ be closed. If $H \in \mathcal{H}$ is such that $\sigma_H(C) \subset A$, then*

(i) *$P_H(A \setminus C) = P_H(A) \setminus P^H(C)$,*

(ii) *$P^H(C) \subset P_H(A)$, in particular $\partial P_H(A \setminus C) = \partial P_H(A) \sqcup \partial P^H(C)$.*

Proof. (i) For $A \subseteq \mathbb{R}^d$, denote $P_H^+(A) = P_H(A) \cap H$ and $P_H^-(A) = P_H(A) \cap H^{\mathsf{c}}$. Thus $P_H(A) = P_H^+(A) \sqcup P_H^-(A)$, and

$$P_H(A) \cap P_H(C^{\mathsf{c}}) = \left[P_H^+(A) \cap P_H^+(C^{\mathsf{c}})\right] \sqcup \left[P_H^-(A) \cap P_H^-(C^{\mathsf{c}})\right]. \tag{2.4}$$

On the other hand, we have $P_H^-(A\cap C^{\mathsf c}) = P_H^-(A)\cap P_H^-(C^{\mathsf c})$. Since $C, \sigma_H(C)\subseteq A$, we get $\sigma_H(A)\cup C^{\mathsf c} = A\cup\sigma_H(C^{\mathsf c}) = \mathbb{R}^d$. Thus, $(A\cap C^{\mathsf c})\cup\sigma_H(A\cap C^{\mathsf c}) = (A\cup\sigma_H(A))\cap(C^{\mathsf c}\cup\sigma_H(C^{\mathsf c}))$, and hence $P_H^+(A\cap C^{\mathsf c}) = P_H^+(A)\cap P_H^+(C^{\mathsf c})$. Therefore, from (2.4) and using $P_H(C^{\mathsf c}) = (P^H(C))^{\mathsf c}$ (Proposition 2.1-(v)), we obtain

$$P_H(A\setminus C) = P_H(A)\cap P_H(C^{\mathsf c}) = P_H(A)\setminus P^H(C).$$

(ii) Since $C\cup\sigma_H(C)$ is a symmetric set in A, by the definitions of P^H and P_H, we get

$$P^H(C)\subseteq C\cup\sigma_H(C) = P_H(C\cup\sigma_H(C))\subset P_H(A).$$

Moreover, $P^H(C)$ is closed and $P_H(A)$ is open in $\mathbb{R}^d$. Thus,

$$\partial P_H(A\setminus C) = \partial\left(P_H(A)\setminus P^H(C)\right) = \partial P_H(A)\sqcup\partial P^H(C). \qquad \square$$

Remark 2.13. The assumption $\sigma_H(C)\subset A$ is essential for the conclusions of the above proposition. To see this, consider $A = B_R(0)$, $C = \overline{B}_r(0)$, and the polarizers $H_s := \{x\in\mathbb{R}^d : x_1 < s\}$ for $s\in\mathbb{R}$. For $s > \frac{R-r}{2}$, we have $|\sigma_{H_t}(C)\cap A^{\mathsf c}| = |\overline{B}_r(2te_1)\cap B_R(0)^{\mathsf c}| > 0$, where $|A|$ is the Lebesgue measure of $A\subseteq\mathbb{R}^d$. Then, $|P_{H_t}(A)\setminus P^{H_t}(C)| = |B_R(0)\setminus\overline{B}_r(2te_1)| > |B_R(0)\setminus\overline{B}_r(0)|$. Since P_H is measure preserving, we conclude that $P_H(A)\setminus P^H(C)\neq P_H(A\setminus C)$.

2.3. **Polarization of functions.** Now, we consider the polarization of functions defined on a domain $\Omega\subseteq\mathbb{R}^d$ and discuss some important properties of polarization of functions, such as Lipschitz continuity, non-expansivity, norm preserving property. Recall the definition of polarization of functions (from Definition 1.1).

Proposition 2.14. *Let $H\in\mathcal{H}$, and $u\in\mathcal{C}(\mathbb{R}^d)$ be a non-negative function. Then*

$$\operatorname{supp}(P_H(u)) = P_H(\operatorname{supp}(u)).$$

Proof. Let $F = \operatorname{supp}(u)$. Clearly $u = u\circ\sigma_H = 0$ on $F^{\mathsf c}\cap\sigma_H(F^{\mathsf c})$, and $u = 0$ or $u\circ\sigma_H = 0$ on $F^{\mathsf c}\cup\sigma_H(F^{\mathsf c})$. Since $u\geq 0$, by the definition, we get $P_H(u) = 0$ on $\left[(F^{\mathsf c}\cup\sigma_H(F^{\mathsf c}))\cap H^{\mathsf c}\right]\cup\left[F^{\mathsf c}\cap\sigma_H(F^{\mathsf c})\right] = P^H(F^{\mathsf c})$. Now, since $P^H(F^{\mathsf c}) = (P_H(F))^{\mathsf c}$ (from Proposition 2.1-(v)), we get $\operatorname{supp}(P_H(u))\subseteq P_H(F)$. The other way inclusion is easy to see from the definition. Therefore, $\operatorname{supp}(P_H(u)) = P_H(\operatorname{supp}(u))$. $\square$

Remark 2.15. Similarly, for non-positive function $u\in\mathcal{C}(\mathbb{R}^d)$, $\operatorname{supp}(P_H(u))\subseteq P^H(\operatorname{supp}(u))$. More generally, for any function $u\in\mathcal{C}(\mathbb{R}^d)$ we have $\operatorname{supp}(P_H(u)) = P_H(\operatorname{supp}(u^+))\cup P^H(\operatorname{supp}(u^-))$ (see [11, Section-2]), where $u^+ = \max\{0,u\}$ and $u^- = \min\{0,u\}$.

The Hölder continuity of polarizations of Hölder continuous functions defined on $\mathbb{R}^d$ is given in [13, Corollary 3.1]. The same result holds for the functions defined on a symmetric domain.

Proposition 2.16. *Let $\Omega_0\subseteq\mathbb{R}^d$ be a domain and $H\in\mathcal{H}$ such that $\sigma_H(\Omega_0) = \Omega_0$. If $u\in\mathcal{C}^{0,\alpha}(\Omega_0)$ for some $\alpha\in(0,1]$, then $P_H u\in\mathcal{C}^{0,\alpha}(\Omega_0)$.*

Proof. For $u\in\mathcal{C}^{0,\alpha}(\Omega_0)$, there exists $L>0$ such that $|u(x)-u(y)|\leq L|x-y|^\alpha$ for any $x,y\in\Omega_0$. For simplicity of notation, we denote the reflection $\sigma_H(z)$ of $z\in\mathbb{R}^d$ with respect to ∂H by z^*. Let $x,y\in\Omega_0$. Since $\sigma_H(\Omega_0) = \Omega_0$, both $x^*, y^*\in\Omega_0$, and $\operatorname{supp}(P_H(u))\subseteq\Omega_0$ (from Remark 2.15). If both $x,y\in H$, then

$$\begin{aligned}|P_Hu(x)-P_Hu(y)| &\leq \left|\max\{u(x),u(x^*)\}-\max\{u(y),u(y^*)\}\right|\\ &\leq \max\Big\{|u(x)-u(y)|, |u(x^*)-u(y^*)|\Big\}\leq L|x-y|^\alpha.\end{aligned}$$

Similarly, if $x,y\in H^{\mathsf c}$ then $|P_Hu(x)-P_Hu(y)|\leq L|x-y|^\alpha$. Now, if $x\in H$ and $y\in H^{\mathsf c}$ then $|x-y^*| = |x^*-y|\leq|x-y|$. Therefore

$$\begin{aligned}|P_Hu(x)-P_Hu(y)| &\leq \left|\max\{u(x),u(x^*)\}-\min\{u(y),u(y^*)\}\right|\\ &\leq \max\Big\{|u(x)-u(y)|, |u(x)-u(y^*)|, |u(y)-u(x^*)|, |u(x^*)-u(y^*)|\Big\}\\ &\leq L|x-y|^\alpha. \qquad \square\end{aligned}$$

Now, we state the following non-expansive property of polarization, see [13, Theorem 3.1] and [15, Theorem 3, Corollary 1].

Proposition 2.17. *Let $\Omega_0 \subset \mathbb{R}^d$, and j be any Young function. Then, for any $H \in \mathcal{H}$ and any non-negative measurable functions u, v on Ω_0,*

$$\int_{P_H\Omega_0} j\left(|P_H u - P_H v|\right) \mathrm{d}x \leq \int_{\Omega_0} j(|u-v|)\,\mathrm{d}x.$$

In particular, for $j(t) = t^p,\ 1 \leq p < \infty$,

$$\|P_H u - P_H v\|_{p, P_H\Omega_0} \leq \|u - v\|_{p,\Omega_0} \text{ for any non-negative } u, v \in L^p(\Omega_0).$$

We state the following invariance property of polarizations, see [29, Proposition 2.3.] and [31, Lemma 3.1].

Proposition 2.18. *Let $\Omega_0 \subseteq \mathbb{R}^d$ be an open set and $H \in \mathcal{H}$ such that $\sigma_H(\Omega_0) = \Omega_0$. If $u \in W^{1,p}(\Omega_0)$ then $P_H(u) \in W^{1,p}(\Omega_0)$, and*

$$\|u\|_p = \|P_H u\|_p \text{ and } \|\nabla u\|_p = \|\nabla P_H u\|_p . \tag{2.5}$$

Proof. Let $u \in W^{1,p}(\Omega_0)$. Since Ω_0 is symmetric with respect to ∂H, we have $v := u \circ \sigma_H \in W^{1,p}(\Omega_0)$. Moreover, using the standard arguments we can easily show that, $|u-v|$, $f := |u-v|\mathbb{1}_{\Omega_0 \cap H}$, and $g := -|u-v|\mathbb{1}_{\Omega_0 \cap H^c}$ are in $W^{1,p}(\Omega_0)$. Thus $P_H(u) = \frac{1}{2}(u+v+f+g)$ is also in $W^{1,p}(\Omega_0)$. To prove that the norms are preserved, first observe that

$$P_H u = \begin{cases} u & \text{a.e., in } [(\Omega_0 \cap H) \cap \{u > v\}] \cup [(\Omega_0 \cap H^{\mathsf{c}}) \cap \{u < v\}], \\ v & \text{a.e., in } [(\Omega_0 \cap H^{\mathsf{c}}) \cap \{u > v\}] \cup [(\Omega_0 \cap H) \cap \{u < v\}]; \end{cases}$$

$$\nabla P_H u = \begin{cases} \nabla u & \text{a.e., in } [(\Omega_0 \cap H) \cap \{u > v\}] \cup [(\Omega_0 \cap H^{\mathsf{c}}) \cap \{u < v\}], \\ \nabla v & \text{a.e., in } [(\Omega_0 \cap H^{\mathsf{c}}) \cap \{u > v\}] \cup [(\Omega_0 \cap H) \cap \{u < v\}]. \end{cases}$$

Now, by integrating $|P_H(u)|^p$ and $|\nabla P_H(u)|^p$ over Ω_0, and using $\sigma_H\left((\Omega_0 \cap H) \cap \{u > v\}\right) = \left(\Omega_0 \cap \overline{H}^{\mathsf{c}}\right) \cap \{u < v\}$ we get (2.5). □

Recall that, for a domain $\Omega \subseteq \mathbb{R}^d$ and $\Gamma_D \subseteq \partial\Omega$, the Sobolev space $W^{1,p}_{\Gamma_D}(\Omega)$ is defined by

$$W^{1,p}_{\Gamma_D}(\Omega) = \text{the closure of } \mathcal{C}^{0,1}_{\Gamma_D}(\Omega) \text{ in } W^{1,p}(\Omega),$$

where $\mathcal{C}^{0,1}_{\Gamma_D}(\Omega) = \left\{\varphi \in \mathcal{C}^{0,1}(\Omega) : \operatorname{supp}(\phi) \cap \Gamma_D = \emptyset\right\}$. We give the analogous result of Proposition 2.18 for the functions in $W^{1,p}_{\Gamma_D}(\Omega)$ in the following proposition.

Proposition 2.19. *Let $\Omega = \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \subset \mathbb{R}^d$ be as given in $(\mathbf{A_0})$, $\Gamma_D \subseteq \partial\Omega$ and $H \in \mathcal{H}_{\text{ad}}$. Let $\varphi \in \mathcal{C}^{0,1}_{\Gamma_D}(\Omega)$ be any non-negative function.*

(i) *If $\Gamma_D = \partial\Omega_{\text{out}}$ and $\sigma_H(\Omega_{\text{in}}) = \Omega_{\text{in}}$, then $P_H(\varphi) \in \mathcal{C}^{0,1}_{\partial P_H(\Omega_{\text{out}})}(P_H(\Omega))$.*
(ii) *If $\Gamma_D = \partial\Omega_{\text{in}}$ and $\sigma_H(\Omega_{\text{out}}) = \Omega_{\text{out}}$, then $P_H(\varphi) \in \mathcal{C}^{0,1}_{\partial P^H(\Omega_{\text{in}})}(P_H(\Omega))$.*

In both of the cases (2.5) holds.

Proof. (i) Let $\Omega_0 = \mathbb{R}^d \setminus \overline{\Omega_{\text{in}}}$. Then $\Omega \subset \Omega_0$, and $\sigma_H(\Omega_0) = \Omega_0$. Let $\varphi \in \mathcal{C}^{0,1}_{\Gamma_D}(\Omega)$ be a non-negative function, and let $\widetilde{\varphi}$ be its zero extension to Ω_0. Then $\widetilde{\varphi} \in \mathcal{C}^{0,1}(\Omega_0)$, and hence by Proposition 2.16, $P_H(\widetilde{\varphi}) \in \mathcal{C}^{0,1}(\Omega_0)$. Therefore, $P_H(\varphi) = P_H(\widetilde{\varphi})\mathbb{1}_{P_H(\Omega)} \in \mathcal{C}^{0,1}(P_H\Omega)$. Next, we show that $P_H(\varphi) = 0$ on $\partial P_H(\Omega_{\text{out}})$. Let $M = \operatorname{supp}(\varphi) \subsetneq \Omega_{\text{out}}$. Since $\operatorname{supp}(P_H(\varphi)) \subseteq P_H(M)$ is closed, $P_H(\Omega_{\text{out}})$ is open and $P_H(M) \subset P_H(\Omega_{\text{out}})$, we obtain $\operatorname{supp}(P_H(\varphi)) \cap \partial P_H(\Omega_{\text{out}}) = \emptyset$ as required.
(ii) In this case, let $\Omega_0 = \Omega_{\text{out}}$. For a non-negative function $\varphi \in W^{1,p}_{\Gamma_D}(\Omega)$, as before we get $P_H(\varphi) = P_H(\widetilde{\varphi})\mathbb{1}_{P_H(\Omega)} \in \mathcal{C}^{0,1}(P_H(\Omega))$. Let $M = \operatorname{supp}(\varphi)$. Then $M \cap \overline{\Omega_{\text{in}}} = \emptyset$ and $M \subset \overline{\Omega_{\text{in}}}^{\mathsf{c}}$. Now, using Proposition 2.1 we obtain

$$P_H(M) \subset P_H(\overline{\Omega_{\text{in}}}^{\mathsf{c}}) \subseteq P_H(\Omega_{\text{in}}^{\mathsf{c}}) = \left(P^H(\Omega_{\text{in}})\right)^{\mathsf{c}}.$$

Since $\operatorname{supp}(P_H(\varphi)) \subseteq P_H(M)$ is closed, and $P^H(\Omega_{\text{in}})$ is open, we get $\operatorname{supp}(P_H(\varphi)) \cap \partial P^H(\Omega_{\text{in}}) = \emptyset$. Therefore, $P_H(\varphi) = 0$ on $\partial P^H(\Omega_{\text{in}})$. □

Using the standard approximation techniques and Proposition 2.17 (the non-expansivity of polarizations), we can prove the following analogous result of Proposition 2.18, for the functions in $W^{1,p}_{\Gamma_D}(\Omega)$.

Proposition 2.20. *Let Ω, $\Gamma_D \subseteq \partial\Omega$ and H be as given in Proposition 2.19. Let $u \in W^{1,p}_{\Gamma_D}(\Omega)$ be any non-negative function.*

(i) *If $\Gamma_D = \partial\Omega_{\text{out}}$ and $\sigma_H(\Omega_{\text{in}}) = \Omega_{\text{in}}$, then $P_H(u) \in W^{1,p}_{\partial P_H(\Omega_{\text{out}})}(P_H(\Omega))$.*
(ii) *If $\Gamma_D = \partial\Omega_{\text{in}}$ and $\sigma_H(\Omega_{\text{out}}) = \Omega_{\text{out}}$, then $P_H(u) \in W^{1,p}_{\partial P^H(\Omega_{\text{in}})}(P_H(\Omega))$.*

In both of the cases (2.5) holds.

2.4. **Regularity results and Strong comparison principles.** Next, we recall a few regularity results for the eigenfunctions. Using Moser type iteration arguments [7, Proposition 1.2] and the arguments from [8, Remark 2.8], we get that the eigenfunctions are in L^q for any $q \in [1, \infty]$. Now, the local $\mathcal{C}^{1,\alpha}$-regularity results of [17, Theorem 1 and 2] give the following boundary regularity of the eigenfunctions.

Proposition 2.21. *Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain and $1 < p < \infty$. Let $u \in W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ be a weak solution of $-\Delta_p u = \lambda |u|^{p-2} u$ for some $\lambda \in \mathbb{R}$. Then there exists $\alpha \in (0, 1)$ such that $u \in \mathcal{C}^{1,\alpha}_{\text{loc}}(\Omega) \cap \mathcal{C}^{0,\alpha}(\overline{\Omega})$.*

The following strong comparison principle for the distributional solutions of the p-Laplace operator is given in [27, Theorem 1.4].

Proposition 2.22. *Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain and $\frac{2d+2}{d+2} < p < \infty$. Let $u, v \in C^1(\overline{\Omega})$ be positive distributional solutions of $-\Delta_p u - g(u) = 0$ in Ω, for a non-negative Lipschitz function g on $[0, \infty)$ with $g(s) > 0$ for $s > 0$. If $u \le v$ in Ω, then either $u < v$ in Ω or $u \equiv v$ in Ω.*

3. Strict Faber-Krahn type inequality under polarization

In this section, we give A proof for Theorem 1.3. Recall the following two subsets of $P_H(\Omega) \cap H$:

$$A_H = \Omega^{\mathsf{c}} \cap \sigma_H(\Omega) \cap H \text{ and } B_H = \Omega \cap \sigma_H(\Omega^{\mathsf{c}}) \cap H.$$

We need the following lemma.

Lemma 3.1. *Let $\Omega = \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \subset \mathbb{R}^d$ be a domain as given in ($\mathbf{A_0}$), and $H \in \mathcal{H}_{\text{ad}}$. Then $\overline{\Omega \cap H} \cap \overline{A_H} \subseteq \partial\Omega$. Furthermore,*

(i) *if $\sigma_H(\Omega_{\text{in}}) = \Omega_{\text{in}}$ then $\overline{\Omega \cap H} \cap \overline{A_H} \subseteq \partial\Omega_{\text{out}}$;*
(ii) *if $\sigma_H(\Omega_{\text{out}}) = \Omega_{\text{out}}$ then $\overline{\Omega \cap H} \cap \overline{A_H} \subseteq \partial\Omega_{\text{in}}$.*

Proof. If $A_H = \emptyset$, then trivially $\emptyset = \overline{\Omega \cap H} \cap \overline{A_H} \subset \partial\Omega$. Let $A_H \neq \emptyset$, then from Proposition 2.3, we obtain $P_H(\Omega) \neq \Omega$, and A_H has non-empty interior. Since $\Omega \cap H$ and A_H are disjoint and $P_H(\Omega) \cap H = (\Omega \cap H) \cup A_H$, using the connectedness of $P_H(\Omega) \cap H$ we conclude that $\overline{\Omega \cap H} \cap \overline{A_H} = \partial(\Omega \cap H) \cap \partial A_H \neq \emptyset$. Clearly $\Omega \cap \partial(\Omega \cap H) \cap \partial A_H = \emptyset$ and hence $\partial(\Omega \cap H) \cap \partial A_H \subseteq \partial\Omega$.
(i) If $\sigma_H(\Omega_{\text{in}}) = \Omega_{\text{in}}$, then we can write

$$\begin{aligned} A_H = \Omega^{\mathsf{c}} \cap \sigma_H(\Omega) \cap H &= (\Omega^{\mathsf{c}}_{\text{out}} \cup \Omega_{\text{in}}) \cap \sigma_H(\Omega_{\text{out}}) \cap \Omega^{\mathsf{c}}_{\text{in}} \cap H \\ &= \Omega^{\mathsf{c}}_{\text{out}} \cap \sigma_H(\Omega_{\text{out}}) \cap H. \end{aligned}$$

Since $\Omega_{\text{in}} \subset\subset \Omega_{\text{out}}$, we have $\partial\Omega = \partial\Omega_{\text{out}} \sqcup \partial\Omega_{\text{in}}$ and $\partial\Omega_{\text{in}} \cap \partial A_H = \emptyset$. Therefore, $\overline{\Omega \cap H} \cap \overline{A_H} \subset \partial\Omega_{\text{out}}$.
(ii) Similarly, for $\sigma_H(\Omega_{\text{out}}) = \Omega_{\text{out}}$ we have $A_H = \Omega_{\text{in}} \cap \sigma_H(\Omega_{\text{in}}) \cap H$. Therefore, we obtain $\overline{\Omega \cap H} \cap \overline{A_H} \subset \partial\Omega_{\text{in}}$. □

For any non-negative function $u \in \mathcal{C}(\overline{\Omega})$, let $\widetilde{u}$ be its zero extension to $\mathbb{R}^d$ and let

$$M_u = \Big\{ x \in P_H(\Omega) \cap H : P_H(\widetilde{u})(x) > \widetilde{u}(x) \Big\}.$$

Next, we prove a lemma that plays a significant role in our results.

Lemma 3.2. *Let $\Omega = \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \subset \mathbb{R}^d$ be a domain as given in ($\mathbf{A_0}$), $\Gamma_D \subseteq \partial\Omega$, and $H \in \mathcal{H}_{\text{ad}}$. Let $u \in \mathcal{C}(\overline{\Omega})$ be a non-negative function with $u = 0$ on Γ_D. If Γ_D satisfies one of the following assumptions:*

$$(a)\ \Gamma_D = \partial\Omega, \quad (b)\ \Gamma_D = \partial\Omega_{\text{out}} \text{ and } \sigma_H(\Omega_{\text{in}}) = \Omega_{\text{in}}, \quad (c)\ \Gamma_D = \partial\Omega_{\text{in}} \text{ and } \sigma_H(\Omega_{\text{out}}) = \Omega_{\text{out}},$$

then $\widetilde{u}$ is continuous on $P_H(\Omega) \cap H$. Moreover, if $\Omega \neq P_H(\Omega) \neq \sigma_H(\Omega)$ then there exists a ball $B_0 \subset \Omega \cap H$ such that

$$P_H(u) > u \text{ in } B_0 \cap M_u \text{ and } P_H(u) \equiv u \text{ in } B_0 \cap M_u^{\mathsf{c}}.$$

Proof. If $P_H(\Omega) = \Omega$ then $\widetilde{u} = u$ in $P_H(\Omega) \cap H$, and hence it is continuous. If $P_H(\Omega) \neq \Omega$, then from Proposition 2.3 we have $A_H \neq \emptyset$ and $\Omega \cap H \subsetneq P_H(\Omega) \cap H$. Clearly $\widetilde{u} = u$ on $\Omega \cap H$ and $\widetilde{u} = 0$ on A_H, and hence $\widetilde{u}$ is continuous on both $\Omega \cap H$ and A_H. If Γ_D satisfies one of the assumptions (a)-(c), then by Lemma 3.1 we get $\emptyset \neq \overline{\Omega \cap H} \cap \overline{A_H} \subseteq \Gamma_D$. Therefore, $\widetilde{u} = u = 0$ on $\overline{\Omega \cap H} \cap \overline{A_H}$ and hence $\widetilde{u}$ is continuous on $(\Omega \cap H) \cup A_H = P_H(\Omega) \cap H$.
Now assume that $\Omega \neq P_H(\Omega) \neq \sigma_H(\Omega)$. Then from Proposition 2.3, both A_H and B_H have non-empty interiors. By the definition of $P_H(\widetilde{u})$, we get $P_H(\widetilde{u}) \geq \widetilde{u}$ in $P_H(\Omega) \cap H$, and

$$\begin{aligned} &\text{in } A_H : \widetilde{u} = 0, \widetilde{u} \circ \sigma_H = u \circ \sigma_H > 0 \text{ and hence } P_H(\widetilde{u}) = u \circ \sigma_H > \widetilde{u}; \\ &\text{in } B_H : \widetilde{u} = u > 0, \widetilde{u} \circ \sigma_H = 0 \text{ and hence } P_H(\widetilde{u}) = \widetilde{u}. \end{aligned} \tag{3.1}$$

Let $N = \{x \in P_H(\Omega) \cap H : P_H(\widetilde{u})(x) = \widetilde{u}(x)\}$. Since $P_H(\widetilde{u})$ is also continuous on $P_H(\Omega) \cap H$ (Proposition 2.16), from (3.1) we get $N \subsetneq \Omega \cap H$ is a non-empty closed set and $M_u = (P_H(\Omega) \cap H) \setminus N$ is a non-empty open set in $P_H(\Omega) \cap H$. Now, by the connectedness of $P_H(\Omega) \cap H$ we must have $\partial M_u \cap N \neq \emptyset$. For $x_0 \in \partial M_u \cap N$, let $B_0 = B_r(x_0) \subset \Omega \cap H$. Then B_0 has all the the desired properties. ☐

Now, we prove Theorem 1.3.

Proof of Theorem 1.3. Let $1 < p < \infty$, $\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \subset \mathbb{R}^d$ be as given in $(\mathbf{A_0})$ and $H \in \mathcal{H}_{\text{ad}}$. Denote $\Omega = \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}$.
(i) Assume that $\sigma_H(\Omega_{\text{in}}) = \Omega_{\text{in}}$. Let $0 < u \in \mathcal{C}^{0,\alpha}_{\partial P_H(\Omega_{\text{out}})}(\overline{\Omega})$ be an eigenfunction corresponding to $\nu_1(\Omega)$. Define $v = P_H(u)$ in $P_H(\Omega)$ then, from Proposition 2.19, we get $v \in \mathcal{C}^{0,\alpha}_{\partial P_H(\Omega_{\text{out}})}(P_H(\Omega))$, and

$$\|u\|_{p,\Omega} = \|v\|_{p,P_H(\Omega)} \text{ and } \|\nabla u\|_{p,\Omega} = \|\nabla v\|_{p,P_H(\Omega)}.$$

From the variational characterization of $\nu_1(P_H(\Omega))$, we obtain:

$$\nu_1(P_H(\Omega)) \leq \nu_1(\Omega). \tag{3.2}$$

(ii) Assume that $\sigma_H(\Omega_{\text{out}}) = \Omega_{\text{out}}$. Let $0 < u \in \mathcal{C}^{0,\alpha}_{\partial P^H(\Omega_{\text{in}})}(\overline{\Omega})$ be an eigenfunction corresponding to $\tau_1(\Omega)$. Define $v = P_H(u)$ in $P_H(\Omega)$ then, from Proposition 2.19, we obtain $v \in \mathcal{C}^{0,\alpha}_{\partial P^H(\Omega_{\text{in}})}(P_H(\Omega))$, and

$$\|u\|_{p,\Omega} = \|v\|_{p,P_H(\Omega)} \text{ and } \|\nabla u\|_{p,\Omega} = \|\nabla v\|_{p,P_H(\Omega)}.$$

From the variational characterization of $\tau_1(P_H(\Omega))$, we get

$$\tau_1(P_H(\Omega)) \leq \tau_1(\Omega). \tag{3.3}$$

(iii) Let $\frac{2d+2}{d+2} < p < \infty$. Assume that, the equality holds in (3.2). Let $0 \leq u \in \mathcal{C}^{0,\alpha}_{\partial P^H(\Omega_{\text{out}})}(\overline{\Omega})$ be an eigenfunction corresponding to the eigenvalue $\nu_1(\Omega)$. On the contrary, assume that $\Omega \neq P_H(\Omega) \neq \sigma_H(\Omega)$. Then by Lemma 3.2, there exists a ball $B_0 \subset \Omega \cap H$ such that

$$v > u \text{ in } B_0 \cap M_u \text{ and } v \equiv u \text{ in } B_0 \cap M_u^{\mathsf{c}}, \tag{3.4}$$

where $M_u = \{x \in P_H(\Omega) \cap H : v(x) > u(x)\}$ is a non-empty open set. Then, both $u, v \in \mathcal{C}^1(\overline{B_0})$ are positive distributional solutions for the following problem in B_0:

$$-\Delta_p u - \lambda |u|^{p-2} u = 0 \text{ in } B_0.$$

Now, for $\frac{2d+2}{d+2} < p < \infty$, the strong comparison principle (Proposition 2.22) implies that

$$\text{either } u < v \text{ or } u \equiv v \text{ in } B_0.$$

This is a contradiction to (3.4), and hence we must have $P_H(\Omega) = \Omega$ or $P_H(\Omega) = \sigma_H(\Omega)$. If the equality holds in (3.3), the proof will follow using a similar set of arguments as given above. ☐

4. Strict monotonicity of first eigenvalues via polarization

In this section, we prove Theorem 1.5 and Theorem 1.8. The main idea is to express the translations and the rotations of the obstacle $\mathcal{O}$ in terms of polarizations and apply the Faber-Krahn type inequality to get the desired monotonicity.

4.1. **Monotonicity along a straight line.** Now, we give a proof for Theorem 1.5. First, we recall that:

$$\text{for given } h \in \mathbb{S}^{d-1}, \quad H_s := \left\{x \in \mathbb{R}^d : x \cdot h < s\right\}, \quad \Sigma_s := \{x \in \Omega : x \cdot h \geq s\}, \text{for } s \in \mathbb{R},$$

$P_{H_0}(\Omega) = \Omega$, the obstacle $\mathcal{O}$ is Steiner symmetric with respect to ∂H_0, and the translations of $\mathcal{O}$ in the h-direction are given by $\mathcal{O}_s = sh + \mathcal{O}$ for $s \in \mathbb{R}$, and

$$\mathrm{L}_{\mathcal{O}} := \Big\{s \in \mathbb{R} : P_{H_s}(\Omega) = \Omega \text{ and } \mathcal{O}_s \subset \Omega\Big\}.$$

We observe the following facts:

$$\text{for } x \in \mathbb{R}^d, \quad \sigma_{H_0}(x) = x - 2(x \cdot h)h, \text{ and } \sigma_{H_s}(x) = 2sh + \sigma_{H_0}(x) \text{ for } s \in \mathbb{R}. \tag{4.1}$$

Lemma 4.1. *If $\mathcal{O} \subseteq \mathbb{R}^d$ satisfies $\sigma_{H_0}(\mathcal{O}) = \mathcal{O}$, then for any $s, t \in \mathbb{R}$, $th + \mathcal{O}_s = \mathcal{O}_{s+t}$ and $\sigma_{H_t}(\mathcal{O}_s) = \mathcal{O}_{2t-s}$.*

Proof. It is easy to verify that $th+\mathcal{O}_s = (s+t)h+\mathcal{O} = \mathcal{O}_{s+t}$ and $\sigma_{H_s}(\mathcal{O}_s) = \mathcal{O}_s$. Since $\sigma_{H_t}(x) = 2th+\sigma_{H_0}(x) = 2(t-s)h+2sh+\sigma_{H_0}(x) = 2(t-s)h+\sigma_{H_s}(x)$ any $x \in \mathbb{R}^d$, we get $\sigma_{H_t}(\mathcal{O}_s) = 2(t-s)h+\sigma_{H_s}(\mathcal{O}_s) = 2(t-s)h+\mathcal{O}_s = \mathcal{O}_{2t-s}$. ∎

Proof of Theorem 1.5. Let the set $\Sigma_{s_0} \bigcup \sigma_{H_{s_0}}(\Sigma_{s_0})$ is convex in the h-direction for some $s_0 \in \mathrm{L}_{\mathcal{O}}$. Let $\mathrm{R}_{\mathcal{O}} := \sup \mathrm{L}_{\mathcal{O}}$.

<u>**The interval** $[s_0, \mathrm{R}_{\mathcal{O}}) \subseteq \mathrm{L}_{\mathcal{O}}$</u>**:** Let $s \in (s_0, \mathrm{R}_{\mathcal{O}})$. Clearly, $\Sigma_s \subset \Sigma_{s_0}$ and the convexity of the set $\Sigma_{s_0} \bigcup \sigma_{H_{s_0}}(\Sigma_{s_0})$ in the h-direction implies that $\sigma_{H_s}(\Sigma_s) \subset \Sigma_{s_0} \bigcup \sigma_{H_{s_0}}(\Sigma_{s_0}) \subset \Omega$ (see Proposition 2.10-(i)). Therefore, by Proposition 2.2, we get $P_{H_s}(\Omega) = \Omega$. Next, we show $\mathcal{O}_s \subset \Omega$. By the definition of $\mathrm{R}_{\mathcal{O}}$, there exists $s_1 \in \mathrm{L}_{\mathcal{O}}$ such that $s < s_1 < \mathrm{R}_{\mathcal{O}}$. Observe that $\mathcal{O}_{s_0}, \mathcal{O}_{s_1} \subset \Sigma_{s_0} \bigcup \sigma_{H_{s_0}}(\Sigma_{s_0})$, and $s = ts_0 + (1-t)s_1$, for some $t \in (0,1)$. Thus $\mathcal{O}_s = t\mathcal{O}_{s_0} + (1-t)\mathcal{O}_{s_1}$, and hence the convexity of $\Sigma_{s_0} \bigcup \sigma_{H_{s_0}}(\Sigma_{s_0})$ in the h-direction implies that $\mathcal{O}_s \subset \Omega$. Therefore $s \in \mathrm{L}_{\mathcal{O}}$.

<u>**Monotonicity of** $\lambda_1(\cdot)$ **on** $[s_0, \mathrm{R}_{\mathcal{O}})$</u>**:** Let $s < t$ in $[s_0, \mathrm{R}_{\mathcal{O}})$. Then $\overline{s} = \frac{s+t}{2} \in \mathrm{L}_{\mathcal{O}}$ and hence $P_{H_{\overline{s}}}(\Omega) = \Omega$. Since $\mathcal{O}_s$ is Steiner symmetric with respect to ∂H_s and $\overline{s} > s$, from Proposition 2.10-(i), we get $P^{H_{\overline{s}}}(\mathcal{O}_s) = \sigma_{H_{\overline{s}}}(\mathcal{O}_s)$. From Lemma 4.1 (since $\sigma_{H_0}(\mathcal{O}) = \mathcal{O}$) we also have $\sigma_{H_{\overline{s}}}(\mathcal{O}_s) = \mathcal{O}_{2\overline{s}-s} = \mathcal{O}_t$. Therefore, from Proposition 2.12 we obtain

$$P_{H_{\overline{s}}}(\Omega \setminus \mathcal{O}_s) = P_{H_{\overline{s}}}(\Omega) \setminus P^{H_{\overline{s}}}(\mathcal{O}_s) = \Omega \setminus \mathcal{O}_t.$$

For $\frac{2d+2}{d+2} < p < \infty$, the Faber-Krahn type inequality (Theorem 1.3) implies that $\lambda_1(t) < \lambda_1(s)$. Therefore, the first Dirichlet eigenvalue $\lambda_1(\cdot)$ is strictly decreasing on $[s_0, \mathrm{R}_{\mathcal{O}})$. ∎

Remark 4.2. If we drop the convexity assumption from Theorem 1.5, then L_0 might not be an interval. However, for any $s, t \in L_0$ with $\frac{s+t}{2} \in L_0$ and $s < t$, the above proof still yields $\lambda_1(t) < \lambda_1(s)$.

4.2. **Monotonicity with respect to the rotations about a point.** Now, we prove Theorem 1.8. First recall that, for $\xi \in \mathbb{S}^{d-1} \setminus \{\eta\}$ the rotations of the obstacle $\mathcal{O}$ with the plane of rotation is $X_\xi = \operatorname{span}\{\eta, \xi\}$ about the point $a \in \mathbb{R}^d$ are given by: for $s \in [-1,1]$,

$$\mathcal{O}_{s,\xi} := a + R_{s,\xi}(-a + \mathcal{O}),$$

where $R_{s,\xi}$ is the simple rotation in $\mathbb{R}^d$ with X_ξ as the plane of rotation and $\theta_s = \arccos(s) \in [0, \pi]$ as the angle of rotation from the ray $\mathbb{R}^+\eta$ in the counter-clockwise direction. We prove the following lemmas.

Lemma 4.3. *For any distinct $\xi_1, \xi_2 \in \mathbb{S}^{d-1} \setminus \{\eta\}$, there exists a simple rotation R such that*

$$R(-a + \Omega \setminus \mathcal{O}_{s,\xi_1}) = -a + \Omega \setminus \mathcal{O}_{s,\xi_2}.$$

Proof. Let $\xi_1, \xi_2 \in \mathbb{S}^{d-1} \setminus \{\eta\}$, define

$$\widetilde{\xi_i} = \frac{\xi_i - (\xi_i \cdot \eta)\eta}{\|\xi_i - (\xi_i \cdot \eta)\eta\|} \text{ for } i = 1, 2. \tag{4.2}$$

Observe that, the rotation of η under R_{s,ξ_i} is given by $R_{s,\xi_i}(\eta) = s\eta + \sqrt{1-s^2}\,\widetilde{\xi_i}$ for $i = 1, 2$. Consider the plane $X = \operatorname{span}\left\{\widetilde{\xi_1}, \widetilde{\xi_2}\right\}$, that is orthogonal to η. Let R be the simple rotation such that $R(\widetilde{\xi_1}) = \widetilde{\xi_2}$. Thus R must fix η, and

$$R \circ R_{s,\xi_1}(\eta) = R(s\eta + \sqrt{1-s^2}\,\widetilde{\xi_1}) = s\eta + \sqrt{1-s^2}\,\widetilde{\xi_2} = R_{s,\xi_2}(\eta).$$

Therefore, for $r>0$, $\rho>0$,

$$R\left(B_\rho(rR_{s,\xi_1}(\eta))\cap\partial B_r(0)\right)=B_\rho(rR\circ R_{s,\xi_1}(\eta))\cap\partial B_r(0)=B_\rho(rR_{s,\xi_2}(\eta))\cap\partial B_r(0),$$

and from (2.2) we obtain $R\left(R_{s,\xi_1}(-a+\mathcal{O})\right)=R_{s,\xi_2}(-a+\mathcal{O})$, and hence $R\left(-a+\mathcal{O}_{s,\xi_1}\right)=-a+\mathcal{O}_{s,\xi_2}$. Since R fixes η, and Ω is foliated Schwarz symmetric with respect to $a+\mathbb{R}^+\eta$, we get $R(-a+\Omega)=-a+\Omega$. Thus we obtain

$$\begin{aligned}R(-a+\Omega\setminus\mathcal{O}_{s,\xi_1})&=R(-a+\Omega)\setminus R(-a+\mathcal{O}_{s,\xi_1})\\&=(-a+\Omega)\setminus(-a+\mathcal{O}_{s,\xi_2})=-a+\Omega\setminus\mathcal{O}_{s,\xi_2}.\end{aligned}$$

□

From Lemma 4.3, we only need to consider the rotations of the obstacle by $R_{s,\xi}$ about the point a in a X_ξ-plane for a fixed $\xi\in\mathbb{S}^{d-1}\setminus\{\eta\}$. Thus for $s\in[-1,1]$, we set $\mathcal{O}_s=\mathcal{O}_{s,\xi}$. Recall that:

$$\text{for } a\in\mathbb{R}^d \text{ and } \eta\in\mathbb{S}^{d-1},\quad \mathcal{H}_{a,\eta}=\Big\{H\in\mathcal{H}: a\in\partial H \text{ and } a+\mathbb{R}^+\eta\subset H\Big\}.$$

Lemma 4.4. *Let $a\in\mathbb{R}^d$, $\eta\in\mathbb{S}^{d-1}$, and $\mathcal{O}\subset\mathbb{R}^d$ is foliated Schwarz symmetric with respect to the ray $a+\mathbb{R}^+\eta$. Let $\xi\in\mathbb{S}^{d-1}\setminus\{\eta\}$ and the rotations of $\mathcal{O}$ be as given in (1.11). Then, for any $s<t$ in $[-1,1]$ there exists $H\in\mathcal{H}_{a,\eta}$ such that*

$$\text{(a) } \sigma_H(\mathcal{O}_t)=\mathcal{O}_s,\quad \text{(b) } P^H(\mathcal{O}_t)=\mathcal{O}_s\quad \textit{and} \text{ (c) } P_H(\mathcal{O}_s)=\mathcal{O}_t.$$

Proof. Let $h=R_s(\eta)-R_t(\eta)$ and consider the polarizer $H:=\left\{z\in\mathbb{R}^d:(z-a)\cdot h<0\right\}$. Observe that $a\in\partial H$, and for $r>0$,

$$\begin{aligned}(a+r\eta-a)\cdot h&=r\eta\cdot[R_s(\eta)-R_t(\eta)]=r(s-t)<0,\\(a+rR_t(\eta)-a)\cdot h&=rR_t(\eta)\cdot[R_s(\eta)-R_t(\eta)]=r[R_s(\eta)\cdot R_t(\eta)-1]<0.\end{aligned}$$

Therefore $H\in\mathcal{H}_{a,\eta}\cap\mathcal{H}_{a,R_t(\eta)}$.
(a) Notice that, $\|h\|=2[1-R_s(\eta)\cdot R_t(\eta)]$. Now, for $x=a+rR_t(\eta)$, $r>0$ we get

$$\sigma_H(x)=x-\frac{2(x-a)\cdot h}{\|h\|^2}h=a+rR_t(\eta)-\frac{2r[R_s(\eta)\cdot R_t(\eta)-1]}{2[1-R_s(\eta)\cdot R_t(\eta)]}(R_s(\eta)-R_t(\eta))=a+rR_s(\eta).$$

Therefore, $\sigma_H(a+\mathbb{R}^+R_t(\eta))=a+\mathbb{R}^+R_s(\eta)$, and hence from (2.2) we obtain

$$\sigma_H(\mathcal{O}_t)=\bigcup_{r\in I_{\mathcal{O}}}B_{\rho(r)}(\sigma_H(a+rR_t(\eta)))\cap\partial B_r(\sigma_H(a))=\bigcup_{r\in I_{\mathcal{O}}}B_{\rho(r)}(a+rR_s(\eta))\cap\partial B_r(a)=\mathcal{O}_s.$$

(b) Since $H\in\mathcal{H}_{a,R_t(\eta)}$ and $\mathcal{O}_t$ is foliated Schwarz symmetric with respect to $a+\mathbb{R}^+R_t(\eta)$, Proposition 2.10-(ii) implies that $P^H(\mathcal{O}_t)=\sigma_H(\mathcal{O}_t)=\mathcal{O}_s$.
(c) Since $P_H(\sigma_H(\mathcal{O}_t))=P_H(\mathcal{O}_t)$ (Proposition 2.1-(iv)), we get $P_H(\mathcal{O}_s)=P_H(\sigma_H(\mathcal{O}_t))=P_H(\mathcal{O}_t)=\mathcal{O}_t$. □

Proof of Theorem 1.8. Given $\frac{2d+2}{d+2}<p<\infty$, Ω and $\mathcal{O}$ are foliated Schwarz symmetric with respect to $a+\mathbb{R}^+\eta$.

The set $C_{\mathcal{O}}$ is interval: We show that, for any $s\in C_{\mathcal{O}}$, the interval $[s,1]\subseteq C_{\mathcal{O}}$. Let $t\in(s,1]$. From Lemma 4.4-(c) there exists $H\in\mathcal{H}_{a,\eta}$ such that $P_H(\mathcal{O}_s)=\mathcal{O}_t$. Since $H\in\mathcal{H}_{a,\eta}$ and Ω is foliated Schwarz symmetric with respect to $a+\mathbb{R}^+\eta$, from Proposition 2.10-(ii), we get $P_H(\Omega)=\Omega$. Now, $\mathcal{O}_s\subset\Omega$ implies that $\mathcal{O}_t=P_H(\mathcal{O}_s)\subset P_H(\Omega)=\Omega$. Therefore, $t\in C_{\mathcal{O}}$ and hence $[s,1]\subseteq C_{\mathcal{O}}$.

Monotonicity of $\gamma_1(\cdot)$: Let $s<t$ in $C_{\mathcal{O}}$. From Lemma 4.4, there exists $H\in\mathcal{H}_{a,\eta}$ such that $P^H(\mathcal{O}_t)=\mathcal{O}_s$. Since Ω is foliated Schwarz symmetric with respect to $a+\mathbb{R}^+\eta$, from Proposition 2.10-(ii), we have $P_H(\Omega)=\Omega$, and from Proposition 2.12 we get

$$P_H(\Omega\setminus\mathcal{O}_t)=P_H(\Omega)\setminus P^H(\mathcal{O}_t)=\Omega\setminus\mathcal{O}_s.$$

If Ω satisfies $(\mathbf{A_4})$ then $\Omega=\Omega_0\setminus\overline{B}_\rho(a)$ and $\Gamma_N=\partial B_\rho(a)$. In this case, we set $\Omega_{\text{out}}=\Omega_0\setminus\mathcal{O}_t$ and $\Omega_{\text{in}}=B_{\rho_0}(a)$ (in Theorem 1.3) so that $\Omega_{\text{out}}\setminus\overline{\Omega_{\text{in}}}=\Omega\setminus\mathcal{O}_t$, $\Gamma_N=\partial\Omega_{\text{in}}$ and $\Gamma_D=\partial\Omega_{\text{out}}$. Therefore, we have $\nu_1(\Omega_{\text{out}}\setminus\overline{\Omega_{\text{in}}})=\gamma_1(\Omega\setminus\mathcal{O}_t)$ and $\nu_1(P_H(\Omega_{\text{out}}\setminus\overline{\Omega_{\text{in}}}))=\gamma_1(\Omega\setminus\mathcal{O}_s)$. Since $\sigma_H(\Omega_{\text{in}})=\Omega_{\text{in}}$, from Theorem 1.3-(i) we get

$$\gamma_1(\Omega\setminus\mathcal{O}_s)\le\gamma_1(\Omega\setminus\mathcal{O}_t).$$

Similarly, if Ω satisfies $(\mathbf{A_5})$ then $\Omega = B_R(a) \setminus \overline{\Omega_1}$ and $\Gamma_N = \partial B_R(a)$. In this case, we set $\Omega_{\text{out}} = B_R(a)$ and $\Omega_{\text{in}} = \Omega_1 \cup \mathcal{O}_t$ (in Theorem 1.3) so that $\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} = \Omega \setminus \mathcal{O}_t$, $\Gamma_N = \partial\Omega_{\text{out}}$ and $\Gamma_D = \partial\Omega_{\text{in}}$. Therefore, we have $\tau_1(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}) = \gamma_1(\Omega \setminus \mathcal{O}_t)$ and $\tau_1(P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}})) = \gamma_1(\Omega \setminus \mathcal{O}_s)$. Since $\sigma_H(\Omega_{\text{out}}) = \Omega_{\text{out}}$, from Theorem 1.3-(ii) we get

$$\gamma_1(\Omega \setminus \mathcal{O}_s) \leq \gamma_1(\Omega \setminus \mathcal{O}_t).$$

Since Ω is not radially symmetric with the center a, in both cases, we have $\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \neq P_H\left(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}\right) \neq \sigma_H\left(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}\right)$. Thus, the strict Faber-Krahn type inequality (Theorem 1.3-(iii)) implies

$$\gamma_1(s) < \gamma_1(t).$$

Therefore, $\gamma_1(\cdot)$ is strictly increasing on $\mathrm{C}_{\mathcal{O}}$. □

5. Some remarks and examples

Example 1. Let $\Omega \subset \mathbb{R}^2$ is given by $\Omega = \left\{(x, y) : x^2 + y^2 < R^2, x \leq 0\right\} \cup \left\{(x, y) : |x| + |y| < 2R, x \geq 0\right\}$ for $R > 0$, and $\mathcal{O}$ is the rhombus given by $|x| + |y| \leq 2\ell$ with $\ell < R$ (see Figure 4). Since $\mathcal{O}$ is Steiner symmetric with respect to the hyperplanes $S_1 := \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and $S_2 := \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$, we can consider the translations $\mathcal{O}$ along the x-axis, as well as along the straight line $y = x$.

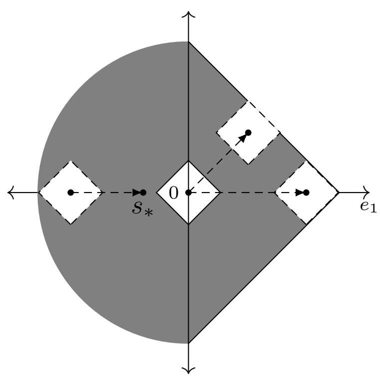


Figure 4. Example 1

Along the x-axis, the translations of $\mathcal{O}$ are $\mathcal{O}_s = (s, 0) + \mathcal{O} \subset \Omega$ for $|s| < R - \ell$. For $|s| < R - \ell$, let $\lambda_1(s) = \lambda_1(\Omega \setminus \mathcal{O}_s)$, the first Dirichlet eigenvalue of the p-Laplacian, for $\frac{2d+2}{d+2} < p < \infty$. Let

$$s_* = \sup\left\{s \in (-R + \ell, 0) : \sigma_{H_s}(\{(x, y) \in \Omega : x < s\}) \subset \Omega\right\}.$$

Now, applying Theorem 1.5

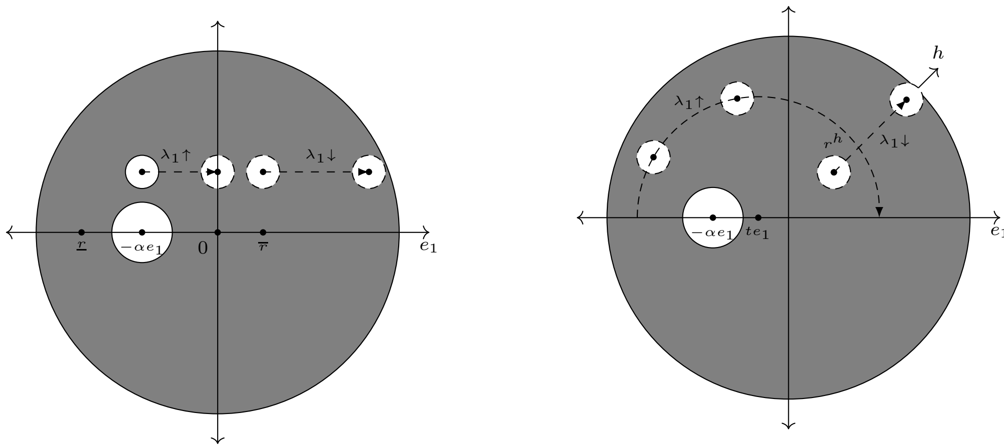
(i) with $h = (1, 0)$, we get $\lambda_1(s)$ is strictly decreasing for $s \in [0, R - \ell)$,
(ii) with $h = (-1, 0)$, we get $\lambda_1(s)$ is strictly increasing for $s \in (-R + \ell, s_*)$.

We get similar monotonicity results for the translations of $\mathcal{O}$ along the straight line $y = x$.

Remark 5.1. In Theorem 1.8, we can consider the obstacle $\mathcal{O}$ of the form $\mathcal{O} = \bigcup_{j=1}^{k} \overline{B}_{\rho_j}(z_j) \subset \Omega$, a finite union of closed balls, such that the centers z_j's lie on the ray $a + \mathbb{R}^+\eta$. Now, the rotations of the obstacle $\mathcal{O}$ about the point a are given by

$$\mathcal{O}_s := \bigcup_{j=1}^{k} \overline{B}_{\rho_j}(a + R_s(-a + z_j)) \text{ for } s \in [-1, 1], \text{ and } \mathrm{C}_{\mathcal{O}} := \left\{s \in [-1, 1] : \mathcal{O}_s \subset \Omega\right\}.$$

In this case, also, we have the same conclusions as Theorem 1.8.



(A) Translations in the e_1-direction

(B) Rotations with respect to te_1, $t \in [-\alpha, 0]$; and translations in the h-direction

FIGURE 5. An eccentric annular domain with a hole.

5.1. **An eccentric annular domain with a spherical hole.** Let $\frac{2d+2}{d+2} < p < \infty$. For given $0 < r < R$ and $0 \leq \alpha < R - r$, we consider an eccentric annular domain $\Omega = B_R(0) \setminus \overline{B_r}(-\alpha e_1) \subset \mathbb{R}^d$ (see Figure 5). Let $\rho > 0$ be such that $\overline{B}_\rho(y) \subset \Omega$ for some $y \in \mathbb{R}^d$. Now define

$$\Omega_\rho := \left\{ y \in \Omega : \overline{B}_\rho(y) \subset \Omega \right\}, \text{ and } \lambda_1(y) := \lambda_1\left(\Omega \setminus \overline{B}_\rho(y)\right) \text{ for } y \in \Omega_\rho.$$

We want to study the behaviour of $\lambda_1(\cdot)$ on Ω_ρ. It is easy to observe that

(a) $\overline{B}_\rho(y)$ is Steiner symmetric with respect to any affine-hyperplane through x;
(b) $\overline{B}_\rho(y)$ is foliated Schwarz symmetric with respect to $a + \mathbb{R}^+(y - a)$ for any $a \in \mathbb{R}^d$;
(c) Ω is foliated Schwarz symmetric with respect to $te_1 + \mathbb{R}^+ e_1$ for $t \in [-\alpha, 0]$;
(d) the sets $\{x \in \Omega : x_1 < \underline{r}\} \bigcup \sigma_{H_{\underline{r}}}(\{x \in \Omega : x_1 < \underline{r}\})$ and $\{x \in \Omega : x_1 > \overline{r}\} \bigcup \sigma_{H_{\overline{r}}}(\{x \in \Omega : x_1 > \overline{r}\})$ are convex in the e_1-direction, where $\overline{r} = \frac{R+r-\alpha}{2}$, $\underline{r} = -\frac{R+r+\alpha}{2}$.

For $y \in \mathbb{R}^d$, we write $y = (s, z) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Now for a given $z \in \mathbb{R}^{d-1}, \beta > 0$, we consider the sets

$$\mathrm{L}_z := \left\{ s \in (-R, R) : (s, z) \in \Omega_\rho \right\};$$
$$\mathrm{S}_\beta(te_1) := \Omega_\rho \cap \partial B_\beta(te_1), \quad t \in [-\alpha, 0].$$

Remark 5.2. Let $(s, z_1) \in \Omega_\rho$. Then we have the following:

(i) Using the axial symmetry of Ω, we obtain $\lambda_1(s, z) = \lambda_1(s, z_1)$, for $z \in \mathbb{R}^{d-1}$ such that $(s, z) \in \Omega_\rho$ and $|z| = |z_1|$.
(ii) From (a), (d) and Theorem 1.5 with $h = -e_1$ (and $h = e_1$) , we get

$$\lambda_1(\cdot, z_1) \text{ is strictly increasing on } \mathrm{L}_{z_1} \bigcap \left(-\sqrt{(R-\rho)^2 - |z_1|^2}, \underline{r} \right],$$

and

$$\lambda_1(\cdot, z_1) \text{ is strictly decreasing on } \mathrm{L}_{z_1} \bigcap \left[\overline{r}, \sqrt{(R-\rho)^2 - |z_1|^2} \right).$$

(iii) If (s_1, z_1) and $(s_2, z_2) \in \Omega_\rho$ such that $s_1 < s_2$ and $(s_1 - t)^2 + |z_1|^2 = (s_2 - t)^2 + |z_2|^2$ for some $t \in [-\alpha, 0]$, then by Theorem 1.8, we get

$$\lambda_1(s_1, z_1) < \lambda_1(s_2, z_2).$$

In particular, for $s_1, s_2 \in [-\alpha, 0]$ with $s_1 < s_2$, by taking $t = \frac{s_1+s_2}{2}$ we obtain

$$\lambda_1(\cdot, z) \text{ is strictly increasing on } \mathrm{L}_z \cap [-\alpha, 0].$$

Remark 5.3. In general, for any $h = (h_1, h') \in \mathbb{S}^{d-1}$, let

$$r^h := \frac{\sqrt{R^2 - \alpha^2 |h'|^2} + r - \alpha h_1}{2}.$$

Then, the set $\{x \in \Omega : x \cdot h > r^h\} \bigcup \sigma_{H_{r_h}}(\{x \in \Omega : x \cdot h > r^h\})$ is convex in the h-direction. For $y \in \Omega_\rho$, define $\mathrm{L}_y = \left\{ s \in [0, R) : y + sh \in \Omega_\rho \right\}$. Now, from (a) and Theorem 1.5 we get

$$\lambda_1(y + sh) \text{ is strictly decreasing for } s \in \mathrm{L}_y \cap [r^h, R).$$

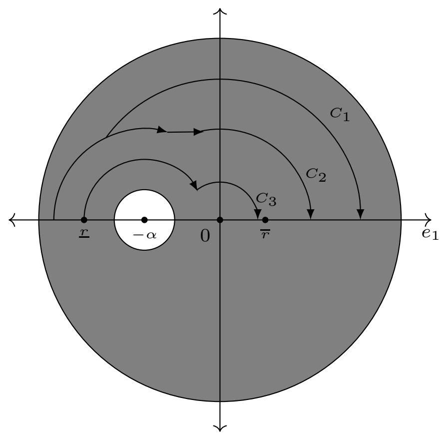


FIGURE 6. Monotonicity along certain paths in Ω.

Remark 5.4. Let $C : (-R, R) \longrightarrow \Omega$ be a continuous path in Ω such that (see Figure 6):

- on $(-R, -\alpha)$, C is a circular arc centered at $t_1 e_1$ with $t_1 \in [-\alpha, 0]$;
- on $[-\alpha, 0]$, C is either a circular arc centered at $t_2 e_1$ with $t_2 \in [-\alpha, 0]$ or a line segment parallel to the e_1-axis;
- on $(0, R)$, C is a circular arc centered at $t_3 e_1$ with $t_3 \in [-\alpha, 0]$.

Now, from Remark 5.2, we see that $\lambda_1(\cdot)$ is strictly increasing along the path C; i.e., for any $(s_1, z_1), (s_2, z_2) \in C$ with $s_1 < s_2$ we have $\lambda_1(s_1, z_1) < \lambda_1(s_2, z_2)$.

Remark 5.5 (**Optimal placement of the obstacle**)**.** For $y \in \Omega_\rho$, we have $y, |y|e_1 \in \mathrm{S}_{|y|}(0)$. If $y_1 < |y|$, then using Theorem 1.8 we obtain $\lambda_1(y) < \lambda(|y|e_1)$. Thus

$$\sup \{\lambda_1(y) : y \in \Omega_\rho\} = \sup \{\lambda_1(s, 0) : s \in \mathrm{L}_0 \cap [0, \overline{r}]\}.$$

If $0 < \rho < \alpha - r$, then $0, \overline{r} \in \mathrm{L}_0$, and hence by (iii) of Remark 5.2 we get

$$\sup \{\lambda_1(y) : y \in \Omega_\rho\} = \max \{\lambda_1(s, 0) : s \in [0, \overline{r}]\}.$$

On the other hand, if $\alpha \le r$ or $\rho > \alpha - r$, then $0 \notin \mathrm{L}_0$. Thus, the above arguments fail to conclude that the supremum is attained in Ω_ρ. However, from a *Mathematica 12* plot of $\lambda_1(\cdot)$ on $[0, R) \cap L_0$ (for the various values of α, r, and ρ), we observed that the maximum is attained at a unique point in $(0, \overline{r}) \cap L_0$. Giving an analytic explanation of this behaviour of $\lambda_1(\cdot, 0)$ in $[0, \overline{r}] \cap \mathrm{L}_0$ seems to be an interesting problem to explore.

5.2. **The symmetries of the first eigenfunctions:** In this subsection, we take $p \in (1, \infty)$ and $\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \subset \mathbb{R}^d$ is a domain as given in $(\mathbf{A_0})$. We establish that the first eigenfunctions of (1.3) and (1.6) inherit some of the symmetries of the underlying domains.

Remark 5.6. Let $H \in \mathcal{H}_{\text{ad}}$ be such that $P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}) = \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}$.

(i) Let u be an eigenfunction corresponding to the first eigenvalue $\nu_1(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}})$ of (1.3). Assume that $\sigma_H(\Omega_{\text{in}}) = \Omega_{\text{in}}$. If u is positive, then from Proposition 2.19 and (2.5), we see that $P_H(u)$ is also an eigenfunction corresponding to $\nu_1(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}})$. Since the norms of u and $P_H(u)$ are same, by the simplicity of ν_1, we get $P_H(u) = u$. If u is negative then, we get $P^H(u) = -P_H(-u) = u$.

(ii) Similarly, if $\sigma_H(\Omega_{\text{out}}) = \Omega_{\text{out}}$ and v is a positive eigenfunction corresponding to the first eigenvalue $\tau_1(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}})$ of (1.5), then $P_H(v) = v$ and $P^H(-v) = -v$.

Definition 5.7. *Let Ω be foliated Schwarz symmetric with respect to the ray $a + \mathbb{R}^+\eta$. Then a function $u : \Omega \longrightarrow \mathbb{R}$ is said to be foliated Schwarz symmetric with respect to the same ray, if $P_H(u) = u$ for every $H \in \mathcal{H}_{a,\eta}$ (see [13, Lemma 6.3] and [30, Section 3]).*

Remark 5.8. Let $\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}$ be foliated Schwarz symmetric with respect to the ray $a + \mathbb{R}^+\eta$ for some $a \in \mathbb{R}^d$ and $\eta \in \mathbb{S}^{d-1}$. Then, we have $P_H(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}) = \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}$ for every $H \in \mathcal{H}_{a,\eta}$.

(i) Assume that $\Omega_{\text{in}} = B_r(a)$ for some $r \geq 0$. Then $\sigma_H(\Omega_{\text{in}}) = \Omega_{\text{in}}$ for every $H \in \mathcal{H}_{a,\eta}$. Hence, for any positive eigenfunction u corresponding to $\nu_1(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}})$, from Remark 5.6, we obtain $P_H(u) = u$ for every $H \in \mathcal{H}_{a,\eta}$. Thus, u is foliated Schwarz symmetric with respect to the ray $a + \mathbb{R}^+\eta$.

(ii) Similarly, Assume that $\Omega_{\text{out}} = B_R(a)$ for some $R > 0$. Then, any positive eigenfunction u corresponding to the first eigenvalue $\tau_1(\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}})$ of (1.5) is foliated Schwarz symmetric with respect to $a + \mathbb{R}^+\eta$.

Remark 5.9. Let $\Omega = B_R(0) \setminus \overline{B}_r(-\alpha e_1) \subset \mathbb{R}^d$ be the eccentric annular domain as given in Subsection 5.1. Then Ω is foliated Schwarz symmetric with respect to $te_1 + \mathbb{R}^+e_1$ for $t \in [-\alpha, 0]$.

(i) If, we take $\Omega_{\text{out}} = \Omega$ and $\Omega_{\text{in}} = \emptyset$, then any positive eigenfunction corresponding to the first Dirichlet eigenvalue $\lambda_1(\Omega)$ is foliated Schwarz symmetric with respect to $te_1 + \mathbb{R}^+e_1$ for every $t \in [-\alpha, 0]$.

(ii) If, we take $\Omega_{\text{out}} = B_R(0)$ and $\Omega_{\text{in}} = B_r(-\alpha e_1)$, then
- any positive eigenfunction corresponding to the first eigenvalue $\nu_1(\Omega)$ of (1.3) is foliated Schwarz symmetric with respect to the ray $-\alpha e_1 + \mathbb{R}^+e_1$;
- any positive eigenfunction corresponding to the first eigenvalue $\tau_1(\Omega)$ of (1.5) is foliated Schwarz symmetric with respect to the ray $\mathbb{R}^+e_1$.

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T. V. Anoop, Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India
Email address: anoop@iitm.ac.in

K. Ashok Kumar, Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India
Email address: s.r.asoku@gmail.com