A new space of generalised functions with bounded variation motivated by fracture mechanics

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Abstract. We introduce a new space of generalised functions with bounded variation to prove the existence of a solution to a minimum problem that arises in the variational approach to fracture mechanics in elastoplastic materials. We study the fine properties of the functions belonging to this space and prove a compactness result. In order to use the Direct Method of the Calculus of Variations we prove a lower semicontinuity result for the functional occurring in this minimum problem. Moreover, we adapt a nontrivial argument introduced by Friedrich to show that every minimizing sequence can be modified to obtain a new minimizing sequence that satisfies the hypotheses of our compactness result.

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1. Introduction

The variational approach to rate-independent evolution problems developed in [10] and [11] is based on a time discretization scheme, where the approximate solution at a given time is obtained by solving an incremental minimum problem which involves the solution at the previous time. The same approach was introduced independently in fracture mechanics in [8] (we refer also to [3] for further developments in this field).

In this framework, the study of crack growth in linearly elastic-perfectly plastic materials in the small strain regime leads to incremental minimization problems that involve the crack Γ as well as the elastic part e and the plastic part p of the strain. In the (generalised) antiplane case, the reference configuration is a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, the crack is a Borel set $\Gamma \subset \overline{\Omega}$,

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with $\mathcal{H}^{d-1}(\Gamma) < +\infty$, and the displacement is a function $u\colon \Omega\setminus\Gamma \to \mathbb{R}$, whose gradient is additively decomposed as $Du = e + p$, where e, the elastic part, is an L^2-function defined in $\Omega \setminus \Gamma$ and p, the plastic part, is a bounded Radon measure defined on $\overline{\Omega} \setminus \Gamma$.

Given a Borel set $\Gamma_0 \subset \overline{\Omega}$ (the crack at the previous time), with $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$, and a bounded Radon measure p_0 in $\overline{\Omega} \setminus \Gamma_0$ (the plastic strain at the previous time), the incremental minimum problem takes the form

$$\min \Big\{ \frac{1}{2} \int_\Omega |e|^2 dx + |p - p_0|(\overline{\Omega} \setminus \Gamma) + \mathcal{H}^{d-1}(\Gamma \setminus \Gamma_0) \Big\}, \tag{1.1}$$

where the minimum is taken over all competing cracks $\Gamma \supset \Gamma_0$ (irreversibility condition) and all pairs (e, p) such that e is an L^2- function, p is a Radon measure, and $e+p = Du$ in $\Omega\setminus\Gamma$ for some displacement u satisfying prescribed boundary conditions (see Section 4 for a precise formulation).

The purpose of this paper is to prove the existence of a solution to problem (1.1). In [6] we considered the same problem only in the case $d = 2$, with the additional constraint that Γ and Γ_0 are compact and satisfy an a priori bound on the number of their connected components. In this case, given a minimizing sequence $(\Gamma_k, e_k, p_k)_k$, we can extract a subsequence (not relabelled) such that $\Gamma_k \to \Gamma$ in the Hausdorff metric, $e_k \rightharpoonup e$ weakly in L^2, and $p_k \overset{*}{\rightharpoonup} p$ locally weakly* as measures on $\overline{\Omega}\setminus\Gamma$. Therefore the existence of a solution to (1.1) follows from the Direct Method of the Calculus of Variations, since all terms in (1.1) are lower semicontinuous.

This simplified approach cannot be applied when $d > 2$, nor when $d = 2$ without bounds on the number of connected components of Γ. For this reason we prefer to rewrite the minimum problem (1.1) in terms of the displacement u, considered as a function defined $\mathcal{L}^d$-a.e. in Ω.

Let u_0 be the displacement at the previous time, let e_0 be its elastic strain, so that $Du_0 = e_0 + p_0$, and let $f\colon \mathbb{R}^d \to \mathbb{R}$ be defined by $f(\xi) = \frac{1}{2}|\xi|^2$, if $|\xi| \le 1$, and $f(\xi) = |\xi| - \frac{1}{2}$, if $|\xi| \ge 1$. Setting $v = u - u_0$, if $v \in BV(\Omega)$, the space of functions with bounded variation in Ω, then (Γ, u) is a solution to (1.1) if and only if v is a solution of

$$\min \Big\{ \int_\Omega f(\nabla v + e_0) dx + |D^c v|(\Omega) + \int_{J_v\setminus\Gamma_0} |[v]| \wedge 1 d\mathcal{H}^{d-1} \Big\} \tag{1.2}$$

with suitable boundary conditions, and $\Gamma = \Gamma_0 \cup \{x \in J_v : |[v](x)| \ge 1\}$, see Lemmas 4.1 and 4.2. Here and in the rest of the paper ∇v is the approximate gradient of v (see (2.6)), J_v and $[v]$ are the jump set and the jump of v (see Section 2), $D^c v$ is the Cantor part of the distributional gradient Dv of v (see (2.5)), and $a \wedge b := \min\{a, b\}$ for every $a, b \in \mathbb{R}$.

Unfortunately, there are boundary conditions for which the minimum problem (1.2) has no solution in the space of functions of bounded variation, as shown by the example provided in Proposition 6.5. The reason is that, while the first term in (1.2) controls ∇v and the second one controls $D^c v$, the third term does not control the whole jump part of Dv, which is given by the integral of $|[v]|$ on J_v.

Therefore, in order to prove the existence of a solution to the minimum problem (1.2), and hence (1.1), we consider a larger functional space for the admissible displacements, which we denote by $GBV_\star(\Omega)$. This is a subset of the space $GBV(\Omega)$ of generalised functions of bounded variation introduced in [1, Section 1] (see also [2, Definition 4.26]). In Section 3 we study the fine properties of functions in $GBV_\star(\Omega)$, as well as some structure properties of this space. In particular, we prove in Theorem 3.11 that, if $(v_k)_k$ is a minimizing sequence of (1.2) in $GBV_\star(\Omega)$ and

$$\sup_k \int_\Omega \psi(|v_k|)dx < +\infty\,, \tag{1.3}$$

for some continuous function ψ with $\psi(t) \to +\infty$ as $t \to +\infty$, then a subsequence of $(v_k)_k$ converges pointwise $\mathcal{L}^d$-a.e. to a function $v \in GBV_\star(\Omega)$.

In Section 5 we prove that every minimizing sequence of problem (1.2) can be modified in order to obtain a new minimizing sequence which satisfies (1.3) for a suitable function ψ, depending on the sequence. The construction of ψ is not trivial and is achieved by adapting to $GBV_\star(\Omega)$ the arguments introduced in [9] for $GSBV^p(\Omega)$.

To use the Direct Method of the Calculus of Variations we prove in Theorem 6.1 that the functional considered in (1.2) is lower semicontinuous. If e_0 is constant we can easily reduce the problem to the case $e_0 = 0$, which was studied in [4]. The result can be easily extended to the piecewise constant case by a localization argument. The general case is obtained by approximation.

The existence of a minimizing sequence satisfying (1.3) and the semicontinuity result imply that there exists a solution to (1.2) in $GBV_\star(\Omega)$, see Theorem 6.2. Since problem (1.1) is equivalent to problem (1.2) in $GBV_\star(\Omega)$, see Lemmas 4.1 and 4.2, we conclude that problem (1.1) has a solution, see Corollary 6.3.

2. Preliminaries on BV-spaces

In this section we fix the general notation used in the paper and we recall the fine properties of BV and GBV functions that will be used in the sequel.

For every topological space X, for every Borel set $Y \subset X$, and for every finite-dimensional Hilbert space Ξ, the space of Ξ-valued bounded Borel measures on Y is denoted by $\mathcal{M}_b(Y;\Xi)$.

Throughout this section U is an open set in $\mathbb{R}^d$. For every set $E \subset U$ the characteristic function $\chi_E\colon U \to \mathbb{R}$ is defined by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. For every Borel measure μ on U and every Borel set $E \subset U$ the Borel measure $\mu \llcorner E$ on U is defined by $\mu \llcorner E(B) = \mu(B \cap E)$ for every Borel set $B \subset U$. If $f \in L^1(U;\mu)$ the Borel measure $f\mu$ on U is defined by $(f\mu)(B) := \int_B f d\mu$ for every Borel set $B \subset U$.

The Lebesgue measure is denoted by $\mathcal{L}^d$ and the $(d-1)$-dimensional Haussdorff measure by $\mathcal{H}^{d-1}$. For every $\mathcal{L}^d$-measurable set $E \subset U$ and every

$\alpha \in [0,1]$ the set $E^{(\alpha)}$ of points in U of $\mathcal{L}^d$-density α for E is defined by

$$E^{(\alpha)} := \{x \in U : \lim_{\rho\to 0+} \frac{\mathcal{L}^d(E\cap B_\rho(x))}{\mathcal{L}^d(B_\rho(x))} = \alpha\}\,,$$

where $B_\rho(x)$ is the ball in $\mathbb{R}^d$ of radius ρ and centre x.

Given an $\mathcal{L}^d$-measurable set $E \subset U$ and an $\mathcal{L}^d$-measurable function $u\colon E \to \overline{\mathbb{R}}$, we say that $a \in \overline{\mathbb{R}}$ is the approximate limit of $u(y)$ as y tends to a point $x \in E^{(\alpha)}$ for some $\alpha > 0$, in symbols

$$\operatorname*{ap\,lim}_{y\to x} u(y) = a\,,$$

if for every neighbourhood A of a in $\overline{\mathbb{R}}$ we have

$$\lim_{\rho\to 0+} \rho^{-d}\mathcal{L}^d(\{y \in E\cap B_\rho(x) : u(y) \notin A\}) = 0\,. \tag{2.1}$$

It follows from the definition that if $f\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a continuous function and $\operatorname{ap\,lim}_{y\to x} u(y)$ exists, then

$$\operatorname*{ap\,lim}_{y\to x} f(u(y)) = f(\operatorname*{ap\,lim}_{y\to x} u(y))\,. \tag{2.2}$$

Moreover, if $u, v\colon E \to \overline{\mathbb{R}}$ are $\mathcal{L}^d$-measurable functions, F is an $\mathcal{L}^d$-measurable subset of E, $u = v$ in F, and $x \in F^{(1)}$, then $\operatorname{ap\,lim}_{y\to x} u(y)$ exists if and only if $\operatorname{ap\,lim}_{y\to x} v(y)$ exists, and in this case

$$\operatorname*{ap\,lim}_{y\to x} v(y) = \operatorname*{ap\,lim}_{y\to x} u(y)\,. \tag{2.3}$$

Let $u\colon U \to \overline{\mathbb{R}}$ be an $\mathcal{L}^d$-measurable function. We define the jump set J_u as the set of all points $x \in U$ such that there exist $u^+(x), u^-(x) \in \overline{\mathbb{R}}$ with $u^+(x) \neq u^-(x)$ and a unit vector $\nu_u(x) \in \mathbb{R}^d$ such that, setting

$$U^+ := \{y \in U : (y-x)\cdot\nu_u(x) > 0\} \quad \text{and} \quad U^- := \{y \in U : (y-x)\cdot\nu_u(x) < 0\}\,,$$

we have

$$u^+(x) = \operatorname*{ap\,lim}_{y\to x} u|_{U^+}(y) \quad \text{and} \quad u^-(x) = \operatorname*{ap\,lim}_{y\to x} u|_{U^-}(y)\,.$$

It is easy to see that the triple $(u^+(x), u^-(x), \nu_u(x))$ is uniquely defined up to a swap of the first two terms and a change of sign in the third one. For every $x \in J_u$ we set $[u](x) := u^+(x) - u^-(x)$. It can be proved that J_u is a Borel set and $[u]\colon J_u \to \overline{\mathbb{R}}$ is a Borel function.

For the general properties of the space $BV(U)$ of functions of bounded variation on U we refer to [7, Chapter 5] and [2, Chapter 3]. Let us fix $u \in BV(U)$. As a consequence of [2, Theorem 3.78], we have that for $\mathcal{H}^{d-1}$-a.e. $x \in U \setminus J_u$ there exists

$$\tilde{u}(x) := \operatorname*{ap\,lim}_{y\to x} u(y)\,.$$

Moreover, the function $\tilde{u}$ defined by this formula is finite for $\mathcal{H}^{d-1}$-a.e. $x \in U \setminus J_u$. For every $m \in \mathbb{R}_+$ we set

$$\{|\tilde{u}| \le m\} := \{x \in U \setminus J_u : \tilde{u}(x) \text{ exists and } |\tilde{u}(x)| \le m\}\,. \tag{2.4}$$

The set $\{|\tilde{u}| > m\}$ is defined in a similar way.

By the definition of $BV(U)$ the distributional gradient of u, denoted by Du, belongs to $\mathcal{M}_b(U;\mathbb{R}^d)$. It can be decomposed as

$$Du = \nabla u \mathcal{L}^d + D^c u + D^j u\,, \tag{2.5}$$

where $\nabla u \in L^1(U;\mathbb{R}^d)$, $D^j u := Du \llcorner J_u$ is the jump part of Du, while $D^c u$, called the Cantor part of Du, is a bounded Radon measure with values in $\mathbb{R}^d$, singular with respect to $\mathcal{L}^d$ and such that $D^c u(B) = 0$ on every Borel set $B \subset U$ with $\mathcal{H}^{d-1}(B) < +\infty$. Moreover, for $\mathcal{L}^d$- a.e. $x \in U$ the vector $\nabla u(x)$ is the approximate gradient of u at x, i.e.,

$$\operatorname*{ap\,lim}_{y\to x} \frac{u(y) - \tilde{u}(x) - \nabla u(x)\cdot(y-x)}{|y-x|} = 0\,. \tag{2.6}$$

Finally, we have

$$D^j u = [u]\nu_u \mathcal{H}^{d-1} \llcorner J_u. \tag{2.7}$$

For every $t \in \mathbb{R}$ and $m \in \mathbb{R}_+$ we set $t^{(m)} := (t \wedge m) \vee (-m)$ and note that the function $t \mapsto t^{(m)}$ is Lipschitz continuous with constant 1. If u is an $\overline{\mathbb{R}}$-valued function defined in U, $u^{(m)}$ is the function defined by $u^{(m)}(x) = u(x)^{(m)}$ for every $x \in U$.

The following theorem gives a formula for the distributional gradient of the truncation of a BV function.

Theorem 2.1. *Let $u \in BV(U)$ and let $m \in \mathbb{R}_+$. Then $u^{(m)} \in BV(U)$ and*

$$Du^{(m)} = \chi_{\{|u|\le m\}} \nabla u \mathcal{L}^d + \chi_{\{|\tilde{u}|\le m\}} D^c u + [u^{(m)}]\nu_u \mathcal{H}^{d-1} \llcorner J_u\,.$$

Proof. It is enough to apply [2, Theorem 3.99] to $f(t) = t^{(m)}$. □

Remark 2.2. Let $u \in BV(U)$ and let $m \in \mathbb{R}_+$. It follows from Theorem 2.1 that

$$\nabla u^{(m)} = 0 \quad \mathcal{L}^d\text{-a.e. in } \{|u| > m\}\,, \tag{2.8}$$

$$D^c u^{(m)}(B) = 0 \quad \text{for every Borel set } B \subset \{|\tilde{u}| > m\}\,. \tag{2.9}$$

The following lemma provides a strong localization property for Du.

Lemma 2.3. *Let $u, v \in BV(U)$ and let E be a Borel set contained in $U \setminus (J_u \cup J_v)$. Suppose that $\tilde{u} = \tilde{v}$ $\mathcal{H}^{d-1}$-a.e. in E. Then $\nabla u = \nabla v$ $\mathcal{L}^d$-a.e. in E and $D^c u(B) = D^c v(B)$ for every Borel set $B \subset E$.*

Proof. It is enough to apply [2, Proposition 3.92 and Remark 3.93]. □

We refer to [2, Chapter 3] for the definition and the main properties of the sets of finite perimeter. If $E \subset U$ is a $\mathcal{L}^d$-measurable set, its perimeter in U is denoted by $P(E,U)$. When $P(E,U) < +\infty$, the reduced boundary of E in U is denoted by $\partial^* E$ and for every $x \in \partial^* E$ the approximate inner unit normal vector is denoted by $\nu_E(x)$.

The following lemma provides a precise formula for the gradient of the product between a bounded BV function and the characteristic function of a set with finite perimeter.

Lemma 2.4. *Let $u \in BV(U) \cap L^\infty(U)$ and let $E \subset U$ be a $\mathcal{L}^d$-measurable set with $P(E,U) < +\infty$. Then there exists $\gamma_E u \in L^\infty(\partial^* E, \mathcal{H}^{d-1})$ such that for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E$ we have*

$$\lim_{\rho \to 0+} \frac{1}{\rho^d} \int_{B^+_\rho(x)} |u(y) - (\gamma_E u)(x)| dy = 0 \,,$$

where $B^+_\rho(x) = \{y \in B_\rho(x) : (y-x)\cdot \nu_E(x) > 0\}$. Moreover, setting $v := u\chi_E$ we have $v \in BV(U) \cap L^\infty(U)$ and

$$\nabla v = \chi_E \nabla u \tag{2.10}$$

$$D^c v = \chi_{E^{(1)}} D^c u \tag{2.11}$$

$$D^j v = \chi_{E^{(1)}} D^j u + (\gamma_E u) \nu_E \mathcal{H}^{d-1} \llcorner \partial^* E \,. \tag{2.12}$$

Proof. The statement about $\gamma_E u$ is proved in [2, Theorem 3.77]. The properties concerning v can be easily deduced from [2, Theorem 3.84]. ☐

We shall use the space $GBV(U)$ of generalised functions of bounded variation introduced in [1, Section 1] (see also [2, Definition 4.26]).

Remark 2.5. Since $u^{(m)} = (u^{(n)})^{(m)}$ for $0 < m < n$, Remark 2.2 implies that (2.8) and (2.9) hold also for every $u \in GBV(U)$ and every $m \in \mathbb{R}_+$.

In the following proposition we summarize the fine properties of functions in $GBV(U)$.

Proposition 2.6. *Let $u \in GBV(U)$. Then the following properties hold:*

(a) *(precise values) for $\mathcal{H}^{d-1}$-a.e. $x \in U \setminus J_u$ there exists*

$$\tilde{u}(x) := \operatorname*{ap\,lim}_{y \to x} u(y) \in \overline{\mathbb{R}} \,; \tag{2.13}$$

moreover, if $x \in \{|u| \leq m\}^{(1)} \setminus J_u$ for some $m \in \mathbb{R}_+$ and $\tilde{u}(x)$ exists, then $\tilde{u}(x) = \widetilde{u^{(m)}}(x) \in \mathbb{R}$; in particular, $\tilde{u} = \widetilde{u^{(m)}}$ $\mathcal{H}^{d-1}$-a.e. on $\{|u| \leq m\}^{(1)} \setminus J_u$;

(b) *(approximate differentiability) there exists a Borel function, denoted by $\nabla u \colon U \to \mathbb{R}^d$, such that for $\mathcal{L}^d$-a.e. $x \in U$ we have $\tilde{u}(x) \in \mathbb{R}$ and* (2.6) *holds; moreover, for every $m \in \mathbb{R}_+$ we have*

$$\nabla u(x) = \nabla u^{(m)}(x) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \{|u| \leq m\} \,; \tag{2.14}$$

(c) *(jumps) the set J_u is countably $\mathcal{H}^{d-1}$-rectifiable and for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$ the vector $\nu_u(x)$ is orthogonal to the approximate tangent space to J_u at x (according to* [2, Definition 2.86]*); moreover, for every $m \in \mathbb{R}_+$ we have $J_{u^{(m)}} \subset J_u$ up to a set of $\mathcal{H}^{d-1}$-measure zero and $|[u^{(m)}]| \leq |[u]|$ $\mathcal{H}^{d-1}$-a.e. on $J_{u^{(m)}} \cap J_u$; finally, for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$, there exists $m_x \in \mathbb{N}$ such that $x \in J_{u^{(m)}}$ for every $m \in \mathbb{N}$ with $m \geq m_x$ and $[u^{(m)}](x) \to [u](x)$ as $m \to \infty$ with $m \in \mathbb{N}$;*

(d) *(Cantor part) for every $m, n \in \mathbb{R}_+$, with $m \leq n$, we have $D^c u^{(m)}(B) = D^c u^{(n)}(B)$ for every Borel set $B \subset \{|\tilde{u}| \leq m\}$ and $|D^c u^{(m)}|(E) \leq |D^c u^{(n)}|(E)$ for every Borel set $E \subset U$.*

Proof. Properties (a), (b), and (c) can be deduced from [2, Theorem 4.34]. Equality $\tilde{u}(x) = \widetilde{u^{(m)}}(x)$ in (a) follows from (2.3).

Let us prove (d). For every $x \in \{|\tilde{u}| \le m\}$ we have $\tilde{u}(x)^{(m)} = \tilde{u}(x)$, therefore the Lipschitz continuity of $t \mapsto t^{(m)}$ implies that $|u^{(m)}(y) - \tilde{u}(x)| \le |u(y) - \tilde{u}(x)|$ for every $y \in U$. The definition of ap lim then gives that $\operatorname{ap\,lim}_{y\to x} u^{(m)}(y) = \tilde{u}(x)$. Hence $\widetilde{u^{(m)}}(x) = \tilde{u}(x)$, and the same result holds for $u^{(n)}$. The conclusion follows from Lemma 2.3 and Remark 2.5. □

In the following theorem we show that, if $u \in GBV(U)$ satisfies condition (2.15) below, then we can define an $\mathbb{R}^d$-valued Radon measure that plays the role of the Cantor part of Du, even if the measure Du cannot be defined.

Theorem 2.7. *Let $u \in GBV(U)$ be such that*

$$\sup_{m>0} |D^c u^{(m)}|(U) < +\infty\,. \tag{2.15}$$

Then there exists a unique measure $\mu \in \mathcal{M}_b(U;\mathbb{R}^d)$ such that for every $m \in \mathbb{R}_+$

$$\mu(B) = D^c u^{(m)}(B) \quad \textit{for every Borel set } B \subset \{|\tilde{u}| \le m\}\,, \tag{2.16}$$

and

$$\mu(B) = 0 \quad \textit{for every Borel set } B \subset \{|\tilde{u}| = +\infty\} \cup J_u\,. \tag{2.17}$$

Proof. Let $E_0 := \emptyset$ and for every $k \in \mathbb{N}$ let $E_k := \{|\tilde{u}| \le k\}$. By Proposition 2.6(d) for every $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} |D^c u^{(k)}|(E_k \setminus E_{k-1}) = \sum_{k=1}^{n} |D^c u^{(n)}|(E_k \setminus E_{k-1}) \le |D^c u^{(n)}|(U)\,, \tag{2.18}$$

hence, by (2.15),

$$\sum_{k=1}^{+\infty} |D^c u^{(k)}|(E_k \setminus E_{k-1}) < +\infty\,. \tag{2.19}$$

If $B \subset U$ is a Borel set we define

$$\mu(B) := \sum_{k=1}^{+\infty} D^c u^{(k)}(B \cap E_k \setminus E_{k-1})\,. \tag{2.20}$$

Recalling (2.4) it follows immediately from the definition that (2.17) holds. By (2.19) the series converges absolutely and its sum is finitely additive with respect to B. To prove the countable additivity of μ it is enough to show that

$$\mu(B^j) \to 0 \tag{2.21}$$

whenever $(B^j)_j$ is a decreasing sequence of Borel sets with empty intersection. In this case by definition we have

$$\mu(B^j) = \sum_{k=1}^{+\infty} D^c u^{(k)}(B^j \cap E_k \setminus E_{k-1})\,,$$

and for every $k \in \mathbb{N}$ we have $D^c u^{(k)}(B^j \cap E_k \setminus E_{k-1}) \to 0$ as $j \to \infty$. This implies (2.21) by the dominated convergence theorem for series, which can be applied thanks to (2.19).

To prove (2.16) we fix $m \in \mathbb{R}_+$ and a Borel set $B \subset \{|\tilde{u}| \le m\}$. Let n be the smallest integer larger than or equal to m. Since $\{|\tilde{u}| \le m\} \subset E_n$, we have $B \subset \bigcup_{k=1}^n (E_k \setminus E_{k-1})$ and $B \cap E_k \setminus E_{k-1} = \emptyset$ for $k > n$. Therefore (2.20) and Proposition 2.6(d) give

$$\mu(B) = \sum_{k=1}^n D^c u^{(k)}(B \cap E_k \setminus E_{k-1}) = \sum_{k=1}^n D^c u^{(n)}(B \cap E_k \setminus E_{k-1})$$
$$= D^c u^{(n)}(B) = D^c u^{(m)}(B)\,.$$

This concludes the proof. □

Definition 2.8. Assume that $u \in GBV(U)$ satisfies (2.15). The measure μ introduced in Theorem 2.7 is denoted by $D^c u$.

Lemma 2.3 ensures that this definition is consistent with (2.5) whenever $u \in BV(U)$. The following proposition shows that the total variation $|D^c u|$ of $D^c u$ coincides with the measure introduced in [2, Definition 4.33].

Proposition 2.9. *Assume that $u \in GBV(U)$ satisfies* (2.15). *Then*

$$D^c u^{(m)}(B) \to D^c u(B) \text{ as } m \to +\infty\,, \tag{2.22}$$
$$\lim_{m\to+\infty} |D^c u^{(m)}|(B) = \sup_{m>0} |D^c u^{(m)}|(B) = |D^c u|(B)\,, \tag{2.23}$$

for every Borel set $B \subset U$.

Proof. Using the notation in the proof of Theorem 2.7, by (2.16) and (2.17) we have

$$D^c u^{(m)}(B \cap \{|\tilde{u}| \le m\}) = D^c u(B \cap \{|\tilde{u}| \le m\}) \to D^c u(B)\,. \tag{2.24}$$

From Remark 2.5 it follows that $D^c u^{(m)}(B \setminus (\{|\tilde{u}| \le m\} \cup J_u)) = 0$. Since $D^c u^{(m)}(J_u) = 0$, we conclude that $D^c u^{(m)}(B) = D^c u^{(m)}(B \cap \{|\tilde{u}| \le m\})$. Together with (2.24) this implies (2.22).

The first equality in (2.23) follows from Proposition 2.6(d). To prove the other equality, for every Borel set $B \subset U$ we define

$$\nu(B) := \lim_{m\to+\infty} |D^c u^{(m)}|(B) = \sup_{m>0} |D^c u^{(m)}|(B)\,. \tag{2.25}$$

Using the monotonicity with respect to m stated in Proposition 2.6(d) we can prove that ν is a Borel measure. Therefore, it is enough to prove (2.23) for every Borel set $B \subset \{|\tilde{u}| \le m\}$ and for every Borel set $B \subset \{|\tilde{u}| = +\infty\} \cup J_u$. The former follows from (2.16), while the latter follows from (2.17) and Remark 2.5, taking into account the fact that $|D^c u^{(m)}|(J_u) = 0$ by the general properties of the Cantor part of the gradient of a BV function. □

3. The function space used in our problem

We now introduce the function space that will be used to formulate and solve problem (1.1). Throughout this section U is a bounded open set in $\mathbb{R}^d$.

Definition 3.1. The space $GBV_\star(U)$ is defined as the space of functions $u\colon U \to \mathbb{R}$ such that $u^{(m)} \in BV(U)$ for every $m \in \mathbb{R}_+$ and

$$\sup_{m\in\mathbb{R}_+} \Big(\int_U |\nabla u^{(m)}|dx + |D^c u^{(m)}|(U) + \int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \Big) < +\infty\,. \quad (3.1)$$

It follows from the definition of $GBV(U)$ that $GBV_\star(U) \subset GBV(U)$. Moreover, using Remark 2.5 and Proposition 2.6 it is easy to see that the supremum in (3.1) can be taken over $m \in \mathbb{N}$.

Remark 3.2. If $d = 1$ and $u \in GBV_\star(U)$, then (3.1) implies that there exist at most a finite number of jump points x of u with $|[u](x)| \geq 1$. From this property and from (3.1) we can deduce that $u \in BV(U)$. Hence $GBV_\star(U) = BV(U)$ if $d = 1$.

On the contrary, when $d \geq 2$ we have $GBV_\star(U) \neq BV(U)$. Indeed, let $x_0 \in U$ and let $R > 0$ be such that $B_R(x_0) \subset U$. For every $k \in \mathbb{N}$ let $R_k := 2^{-k}R$ and let $u\colon U \to \mathbb{R}$ be defined by $u(x) := 1/R_k^{d-1}$, if $x \in B_{R_k}(x_0) \setminus B_{R_{k+1}}(x_0)$ for $k \in \mathbb{N}$, and $u(x) := 0$, if $x \in U \setminus B_R(x_0)$. For every $m \in \mathbb{R}_+$ we have $\nabla u^{(m)} = 0$, $D^c u^{(m)} = 0$, and

$$\int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \leq \sigma_{d-1} \sum_k R_k^{d-1} < +\infty\,,$$

where $\sigma_{d-1} := \mathcal{H}^{d-1}(B_1(0))$. This shows that $u \in GBV_\star(U)$. Since

$$\int_{J_u} |[u]| d\mathcal{H}^{d-1} = \sigma_{d-1} \sum_{k=1}^{\infty} \Big(\frac{1}{R_k^{d-1}} - \frac{1}{R_{k-1}^{d-1}} \Big) R_k^{d-1} = \sigma_{d-1} \sum_{k=1}^{\infty} \Big(1 - \frac{1}{2^{d-1}} \Big) = +\infty\,,$$

we conclude that $u \notin BV(U)$.

For every $u \in GBV(U)$ we define

$$J_u^1 := \{x \in J_u : |[u](x)| \geq 1\}\,. \quad (3.2)$$

Proposition 3.3. *The space $GBV_\star(U)$ coincides with the set of functions $u \in GBV(U)$ such that*

$$\nabla u \in L^1(U;\mathbb{R}^d)\,, \quad (3.3)$$

$$\sup_{m\in\mathbb{R}_+} |D^c u^{(m)}|(U) < +\infty\,, \quad (3.4)$$

$$\int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} < +\infty\,. \quad (3.5)$$

Remark 3.4. Inequality (3.5) is equivalent to

$$\int_{J_u \setminus J_u^1} |[u]| d\mathcal{H}^{d-1} < +\infty \quad \text{and} \quad \mathcal{H}^{d-1}(J_u^1) < +\infty\,. \quad (3.6)$$

Therefore, every $u \in GBV_\star(U)$ satisfies (3.6).

Proof of Proposition 3.3. Assume $u \in GBV_\star(U)$. Since U is the union of the sets $\{|u| \le m\}$ for $m \in \mathbb{N}$ (recall that u is finite valued), from (2.14) and (3.1) we deduce (3.3). Inequality (3.4) follows from (3.1), while (3.5) can be deduced from (3.1) and Proposition 2.6(c).

Conversely, assume $u \in GBV(U)$ and that (3.3)-(3.5) are satisfied. From the definition of $GBV(U)$, see [2, Definition 4.26], for every $m \in \mathbb{R}_+$ we have $u^{(m)} \in BV_{loc}(U)$. To prove that $u^{(m)} \in BV(U)$ it is enough to show that $|Du^{(m)}|(U) < +\infty$. By the extension of (2.5) to $BV_{loc}(U)$ we have

$$Du^{(m)} = \nabla u^{(m)} \mathcal{L}^d + D^c u^{(m)} + [u^{(m)}] \nu_{u^{(m)}} \mathcal{H}^{d-1} \llcorner J_{u^{(m)}} \,. \tag{3.7}$$

By (2.14) we have that $\nabla u^{(m)} = \nabla u$ $\mathcal{L}^d$-a.e. in $\{|u| \le m\}$, while by Remark 2.5 $\nabla u^{(m)} = 0$ $\mathcal{L}^d$-a.e. in $\{|u| > m\}$. Therefore (3.3) ensures that $\nabla u^{(m)} \in L^1(U;\mathbb{R}^d)$ and

$$\sup_{m \in \mathbb{R}_+} \int_U |\nabla u^{(m)}| dx < +\infty \,. \tag{3.8}$$

By Proposition 2.6(c) we have $J_{u^{(m)}} \subset J_u$ up to a set of $\mathcal{H}^{d-1}$ measure zero and $|[u^{(m)}]| \le |[u]|$ $\mathcal{H}^{d-1}$-a.e. on $J_{u^{(m)}}$. Therefore

$$\begin{aligned} \int_{J_{u^{(m)}}} |[u^{(m)}]| d\mathcal{H}^{d-1} &\le \int_{J_{u^{(m)}} \setminus J_u^1} |[u^{(m)}]| d\mathcal{H}^{d-1} + 2m \mathcal{H}^{d-1}(J_u^1) \\ &\le \int_{J_u \setminus J_u^1} |[u]| d\mathcal{H}^{d-1} + 2m \mathcal{H}^{d-1}(J_u^1) < +\infty \,, \end{aligned} \tag{3.9}$$

where the last inequality follows from (3.6). By (3.4), (3.7), (3.8), and (3.9) we conclude that $|Du^{(m)}|(U) < +\infty$, which implies $u^{(m)} \in BV(U)$.

Finally, recalling again Proposition 2.6(c), by (3.5) we have

$$\int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \le \int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} < +\infty \,. \tag{3.10}$$

Together with (3.4) and (3.8), this implies (3.1), hence $u \in GBV_\star(U)$. □

Remark 3.5. Let $\lambda > 0$ and $u \in GBV(U)$. Then

$$\int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} < +\infty \quad \Longleftrightarrow \quad \int_{J_u} |[u]| \wedge \lambda d\mathcal{H}^{d-1} < +\infty \,.$$

Indeed, if $\int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} < +\infty$ and $\lambda \le 1$ we have $\int_{J_u} |[u]| \wedge \lambda d\mathcal{H}^{d-1} \le \int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} < +\infty$. If $\int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} < +\infty$ and $\lambda > 1$ we have $\int_{J_u} |[u]| \wedge \lambda d\mathcal{H}^{d-1} \le \int_{J_u \setminus J_u^1} |[u]| d\mathcal{H}^{d-1} + \lambda \mathcal{H}^{d-1}(J_u^1) \le \lambda \int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} < +\infty$. The converse implication can be proved in the same way.

Given a function $u \in GBV_\star(U)$, by Theorem 2.7 and Proposition 3.3 the measures $\nabla u \mathcal{L}^d$ and $D^c u$ are well-defined and belong to $\mathcal{M}_b(U;\mathbb{R}^d)$. Since in general $|[u]| \notin L^1(J_u, \mathcal{H}^{d-1})$, we cannot use (2.5) and (2.7) to define a measure which plays the role of Du. However, this is possible on a suitable subset of U and leads to a measure which will be crucial in the sequel.

Definition 3.6. Let $u \in GBV_\star(U)$ and let $\Gamma \subset U$ be a Borel set with

$$\int_{J_u \setminus \Gamma} |[u]| d\mathcal{H}^{d-1} < +\infty \,. \tag{3.11}$$

The measure $D^\Gamma u \in \mathcal{M}_b(U \setminus \Gamma; \mathbb{R}^d)$ is defined by

$$D^\Gamma u := \nabla u \mathcal{L}^d + D^c u + [u]\nu_u \mathcal{H}^{d-1} \llcorner (J_u \setminus \Gamma) \,. \tag{3.12}$$

Remark 3.7. If $u \in BV(U)$, using (2.5) and (2.7) we see that $D^\Gamma u$ coincides with the restriction of Du to $U \setminus \Gamma$.

It is known that, if $u \in BV(U)$, then the approximate limit $\tilde{u}(x)$ is finite for $\mathcal{H}^{d-1}$-a.e. $x \in U \setminus J_u$, while $u^+(x)$ and $u^-(x)$ are finite for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$ (see [7, Theorem 5.9.3]). These properties do not hold for an arbitrary function in $GBV(U)$. The following theorem shows that they hold for functions in $GBV_\star(U)$.

Theorem 3.8. *Let $u \in GBV_\star(U)$. Then $\tilde{u}(x)$ is finite for $\mathcal{H}^{d-1}$-a.e. $x \in U \setminus J_u$, while $u^+(x)$ and $u^-(x)$ are finite for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$.*

Proof. It is enough to repeat the proof of [2, Theorem 4.40], replacing the hypothesis $\mathcal{H}^{d-1}(S_u^*) < +\infty$ with (3.5). □

It is well-known that $GBV(U)$ is not a vector space (see [2, Remark 4.27]). The additional properties considered in the definition of $GBV_\star(U)$ lead to the following result.

Theorem 3.9. *$GBV_\star(U)$ is a vector space.*

Proof. It is obvious that, if $u \in GBV_\star(U)$ and $\lambda \in \mathbb{R}$, then $\lambda u \in GBV_\star(U)$. Given $u, v \in GBV_\star(U)$ we want to prove that $u + v \in GBV_\star(U)$. The first step is to show that, given $m \in \mathbb{R}_+$, we have $(u+v)^{(m)} \in BV(U)$. For every $r \in \mathbb{R}_+$, by the definition of $GBV_\star(U)$ the functions $u^{(r)}$ and $v^{(r)}$ belong to $BV(U)$ hence

$$w_r^m := (u^{(r)} + v^{(r)})^{(m)} \in BV(U) \,. \tag{3.13}$$

By Theorem 2.1 we have

$$\nabla w_r^m = \chi_{\{|u^{(r)}+v^{(r)}| \le m\}} (\nabla u^{(r)} + \nabla v^{(r)}) \,, \tag{3.14}$$

$$D^c w_r^m = \chi_{\{|\tilde{u}^{(r)}+\tilde{v}^{(r)}| \le m\}} (D^c u^{(r)} + D^c v^{(r)}) \,. \tag{3.15}$$

To estimate $|Dw_r^m|(U)$ we write

$$|Dw_r^m|(U) = \int_U |\nabla w_r^m| dx + |D^c w_r^m|(U) + \int_{J_{w_r^m}} |[w_r^m]| d\mathcal{H}^{d-1} \,.$$

Taking into account the definition of $GBV_\star(U)$, by (3.14), (3.15) there exists a constant $C > 0$, independent of m, such that

$$\int_U |\nabla w_r^m| dx \le \int_U |\nabla u^{(r)}| dx + \int_U |\nabla v^{(r)}| dx \le C \,, \tag{3.16}$$

$$|D^c w_r^m|(U) \le |D^c u^{(r)}|(U) + |D^c v^{(r)}|(U) \le C \,, \tag{3.17}$$

for every $r \in \mathbb{R}_+$. Since by (3.13), $|[w_r^m]| \le |[u^{(r)}]| + |[v^{(r)}]|$ $\mathcal{H}^{d-1}$-a.e. on $J_{w_r^m}$ and $J_{w_r^m} \subset J_{u^{(r)}} \cup J_{v^{(r)}}$ up to a set of $\mathcal{H}^{d-1}$-measure zero, recalling also that $|w_r^m| \le m$, we have

$$\begin{aligned}\int_{J_{w_r^m}} |[w_r^m]| d\mathcal{H}^{d-1} &\le \int_{J_{u^{(r)}} \setminus J^1_{u^{(r)}}} |[u^{(r)}]| d\mathcal{H}^{d-1} + \int_{J_{v^{(r)}} \setminus J^1_{v^{(r)}}} |[v^{(r)}]| d\mathcal{H}^{d-1} \\ &\quad + 2m\mathcal{H}^{d-1}(J^1_{u^{(r)}}) + 2m\mathcal{H}^{d-1}(J^1_{v^{(r)}}) \\ &\le (2m+1) \int_{J_{u^{(r)}}} |[u^{(r)}]| \wedge 1 d\mathcal{H}^{d-1} + (2m+1) \int_{J_{v^{(r)}}} |[v^{(r)}]| \wedge 1 d\mathcal{H}^{d-1} .\end{aligned}$$

Using the definition of $GBV_\star(U)$ we see that there exists a constant $C_m > 0$ such that

$$\int_{J_{w_r^m}} |[w_r^m]| d\mathcal{H}^{d-1} \le C_m \tag{3.18}$$

for every $r \in \mathbb{R}_+$. Since $w_r^m \to (u+v)^{(m)} \in L^1(U)$ as $r \to \infty$, by (2.5) and (3.16)-(3.18) we deduce that $(u+v)^{(m)} \in BV(U)$.

To conclude the proof we have to show that

$$\sup_{m \in \mathbb{R}_+} \int_U |\nabla (u+v)^{(m)}| dx < +\infty \,, \tag{3.19}$$

$$\sup_{m \in \mathbb{R}_+} |D^c (u+v)^{(m)}|(U) < +\infty \,, \tag{3.20}$$

$$\sup_{m \in \mathbb{R}_+} \int_{J_{(u+v)^{(m)}}} |[(u+v)^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} < +\infty \,. \tag{3.21}$$

To prove (3.19) for every $r \ge 0$ we set $A_r := \{|u| \le r\} \cap \{|v| \le r\}$. Since $(u^{(r)} + v^{(r)})^{(m)} = (u+v)^{(m)}$ in A_r, by Lemma 2.3 we have $\nabla (u^{(r)} + v^{(r)})^{(m)} = \nabla (u+v)^{(m)}$ $\mathcal{L}^d$-a.e. in A_r. Recalling (3.16) we obtain

$$\begin{aligned}\int_{A_r} |\nabla (u+v)^{(m)}| dx &= \int_{A_r} |\nabla (u^{(r)} + v^{(r)})^{(m)}| dx \\ &\le \int_U |\nabla (u^{(r)} + v^{(r)})^{(m)}| dx \le C \,,\end{aligned}$$

and since $A_r \nearrow U$ as $r \to +\infty$ (recall that u and v have finite values) we get (3.19).

To prove (3.20) for every $r \ge 0$ we set, according to (2.4),

$$\tilde{A}_r := \{|\tilde{u}| \le r)\} \cap \{|\tilde{v}| \le r\} \,. \tag{3.22}$$

By (2.2) applied to $f(t) := t^{(m)}$ we have $\tilde{u}(x) = \operatorname{ap\,lim}_{y \to x} u^{(r)}(y)$ for every $x \in \tilde{A}_r$. Hence $(\widetilde{u^{(r)}} + \widetilde{v^{(r)}})^{(m)} = (\tilde{u} + \tilde{v})^{(m)}$ in $\tilde{A}_r$. By Lemma 2.3 we have $D^c (u^{(r)} + v^{(r)})^{(m)} = D^c (u+v)^{(m)}$ as measures on $\tilde{A}_r$ and from (3.17) we get

$$|D^c (u+v)^{(m)}|(\tilde{A}_r) = |D^c (u^{(r)} + v^{(r)})^{(m)}|(\tilde{A}_r) \le |D^c (u^{(r)} + v^{(r)})^{(m)}|(U) \le C \,.$$

Recalling that $\tilde{A}_r \nearrow U \setminus (\{|\tilde{u}| = +\infty\} \cup \{|\tilde{v}| = +\infty\})$ and that $\mathcal{H}^{d-1}(\{|\tilde{u}| = +\infty\} \cup \{|\tilde{v}| = +\infty\}) = 0$ by Theorem 3.8, we can pass to the limit as $r \to +\infty$ and we obtain that (3.20) holds.

It remains to prove (3.21). To this end we observe that by Proposition 3.3 there exists a constant $C > 0$ such that

$$\int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} \le C \quad \text{and} \quad \int_{J_v} |[v]| \wedge 1 d\mathcal{H}^{d-1} \le C\,.$$

Since $|[(u+v)^{(m)}]| \wedge 1 \le |[u+v]| \wedge 1 \le |[u]| \wedge 1 + |[v]| \wedge 1$, we have that

$$\int_{J_{(u+v)^{(m)}}} |[(u+v)^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \le 2C$$

for every m. This concludes the proof. □

Proposition 3.10. *Let $u, v \in GBV_\star(U)$, $\lambda \in \mathbb{R}$, and let $\Gamma \subset U$ be a Borel set. Then*

$$\nabla(u+v) = \nabla u + \nabla v \quad \textit{and} \quad \nabla(\lambda u) = \lambda \nabla u \quad \mathcal{L}^d\text{-a.e. in } U\,, \tag{3.23}$$

$$D^c(u+v) = D^c u + D^c v \quad \textit{and} \quad D^c(\lambda u) = \lambda D^c u \quad \textit{on } U\,, \tag{3.24}$$

$$D^\Gamma(u+v) = D^\Gamma u + D^\Gamma v \quad \textit{and} \quad D^\Gamma(\lambda u) = \lambda D^\Gamma u \quad \textit{on } U \setminus \Gamma\,. \tag{3.25}$$

Proof. Equalities (3.23) follow immediately from the definition of the approximate limit. The second equality in (3.24) follows from the definition of $D^c u$. To prove the first equality, we set $w := u + v$ and we fix m and s with $0 \le 2s \le m$. For every $r \ge s$ we have $\tilde{w}^{(m)} = \tilde{w} = \tilde{u} + \tilde{v} = \tilde{u}^{(r)} + \tilde{v}^{(r)}$ $\mathcal{H}^{d-1}$-a.e. in $\tilde{A}_s$, where $\tilde{A}_s$, is defined in (3.22). Since $w^{(m)} \in BV(U)$ by Theorem 3.9 and $u^{(r)}$, $v^{(r)} \in BV(U)$ by the definition of $GBV_\star(U)$, using (2.2) and Lemma 2.3 for every Borel set B in U we obtain

$$D^c w^{(m)}(B \cap \tilde{A}_s) = D^c(u^{(r)} + v^{(r)})(B \cap \tilde{A}_s) = D^c u^{(r)}(B \cap \tilde{A}_s) + D^c v^{(r)}(B \cap \tilde{A}_s)\,.$$

By Proposition 2.9 we can pass to the limit as $r \to +\infty$ and we get

$$D^c(u+v)^{(m)}(B \cap \tilde{A}_s) = D^c u(B \cap \tilde{A}_s) + D^c v(B \cap \tilde{A}_s)\,.$$

Taking the limit as $m \to +\infty$ and using Proposition 2.9 again we obtain

$$D^c(u+v)(B \cap \tilde{A}_s) = D^c u(B \cap \tilde{A}_s) + D^c v(B \cap \tilde{A}_s)\,.$$

Finally, arguing as in the proof of Theorem 3.9 we can pass to the limit as $s \to +\infty$ and obtain the first equality in (3.24).

By (3.23) and (3.24) to prove the first equality in (3.25) it is enough to show that

$$\int_{J_{u+v} \cap B} [u+v] \nu_{u+v} d\mathcal{H}^{d-1} = \int_{J_u \cap B} [u] \nu_u d\mathcal{H}^{d-1} + \int_{J_v \cap B} [v] \nu_v d\mathcal{H}^{d-1}$$

for every Borel set $B \subset U \setminus \Gamma$. This follows easily from the linearity of the jump and the locality property of approximate tangent spaces (see, e.g., [2, (2.65)]), which gives (up to a sign) $\nu_{u+v} = \nu_u$ $\mathcal{H}^{d-1}$-a.e. on $J_{u+v} \cap J_u$ and $\nu_{u+v} = \nu_v$ $\mathcal{H}^{d-1}$-a.e. on $J_{u+v} \cap J_v$. The second equality in (3.25) is trivial. □

Theorem 3.11 (Compactness). *Let $(u_k)_k$ be a sequence in $GBV_\star(U)$. Assume that there exist a constant $M > 0$ and a continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(t) \to +\infty$ as $t \to +\infty$, such that*

$$\int_U |\nabla u_k| dx + |D^c u_k|(U) + \int_{J_{u_k}} |[u_k]| \wedge 1 d\mathcal{H}^{d-1} \leq M \,, \tag{3.26}$$

$$\sup_k \int_U \psi(|u_k|) dx < +\infty \,. \tag{3.27}$$

Then there exist a subsequence, not relabelled, and a function $u \in GBV_\star(U)$ such that $u_k \to u$ $\mathcal{L}^d$-a.e. in U.

Proof. We claim that for every $m \in \mathbb{N}$ the truncated functions $u_k^{(m)}$ are bounded in $BV(U)$. Indeed, by Proposition 2.6(b) we have that $\nabla u_k^{(m)} = \nabla u_k$ $\mathcal{L}^d$-a.e. on $\{|u_k| \leq m\}$, while by Remark 2.5 we have $\nabla u_k^{(m)} = 0$ $\mathcal{L}^d$-a.e. on $\{|u_k| > m\}$. By (3.26) this implies that

$$\int_U |\nabla u_k^{(m)}| dx \leq \int_U |\nabla u_k| dx \leq M \,. \tag{3.28}$$

By (2.23) we have also

$$|D^c u_k^{(m)}|(U) \leq |D^c u_k|(U) \leq M \,. \tag{3.29}$$

As for the estimate on the jump part, we observe that, by Proposition 2.6(c), we have that $J_{u_k^{(m)}} \subset J_{u_k}$ up to a set of $\mathcal{H}^{d-1}$-measure zero, and $|[u_k^{(m)}]| \leq |[u_k]|$ $\mathcal{H}^{d-1}$-a.e. on $J_{u_k^{(m)}} \cap J_{u_k}$. Then

$$\int_{J_{u_k^{(m)}}} |[u_k^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \leq \int_{J_{u_k}} |[u_k]| \wedge 1 d\mathcal{H}^{d-1} \,, \tag{3.30}$$

$$\begin{aligned} \int_{J_{u_k^{(m)}}} |[u_k^{(m)}]| d\mathcal{H}^{d-1} &\leq \int_{J_{u_k^{(m)}} \setminus J^1_{u_k^{(m)}}} |[u_k^{(m)}]| d\mathcal{H}^{d-1} + 2m \mathcal{H}^{d-1}(J^1_{u_k^{(m)}}) \\ &\leq (1 + 2m) \int_{J_{u_k}} |[u_k]| \wedge 1 d\mathcal{H}^{d-1} \leq (1+2m) M \,. \end{aligned} \tag{3.31}$$

Therefore, by (2.5), (3.28), (3.29), and (3.31), the functions $u_k^{(m)}$ are bounded in $BV(U)$ uniformly with respect to k.

By the compactness of the embedding of $BV(U)$ into $L^1_{loc}(U)$, using a diagonal argument we can extract a subsequence of $(u_k)_k$, not relabelled, such that for every $m \in \mathbb{N}$, the sequence $(u_k^{(m)})_k$ converges $\mathcal{L}^d$-a.e. in U to a function $v^m \in L^\infty(U)$. Since the BV-norm is lower semicontinuous with respect to L^1-convergence, we obtain that $v^m \in BV(U)$.

We observe that

$$m < n \quad \Longrightarrow \quad (v^n)^{(m)} = v^m \,. \tag{3.32}$$

This is an obvious consequence of the fact that $(u_k^{(n)})^{(m)} = u_k^{(m)}$. From (3.32) we have that

$$m < n \qquad \Longrightarrow \qquad \{|v^n| = n\} \subset \{|v^m| = m\}\,.$$

Let E_∞ be the intersection of the sets $\{|v^m| = m\}$ for $m \in \mathbb{N}$. We claim that

$$\mathcal{L}^d(E_\infty) = 0\,. \tag{3.33}$$

To prove this property we observe that it is not restrictive to assume that the function ψ in (3.27) is increasing. For every $m \in \mathbb{N}$ by the Fatou Lemma we have

$$\psi(m)\mathcal{L}^d(E_\infty) = \int_{E_\infty} \psi(|v^m|)dx \le \liminf_k \int_{E_\infty} \psi(|u_k^{(m)}|)dx \le \sup_k \int_U \psi(|u_k|)dx\,,$$

where in the last inequality we used the monotonicity of ψ. Since, by assumption, $\psi(m) \to +\infty$ as $m \to +\infty$, from (3.27) we obtain (3.33).

If $x \in U \setminus E_\infty$ there exists $m \in \mathbb{N}$ such that $|v^m(x)| < m$. We set

$$u(x) := v^m(x)$$

and we observe that, by (3.32), the function u is well-defined on $U \setminus E_\infty$ and $u^{(m)} = v^m$ in $U \setminus E_\infty$ for every $m \in \mathbb{N}$. We also set $u(x) = 0$ for $x \in E_\infty$. Since $u^{(m)} = v^m$ $\mathcal{L}^d$-a.e. on U we conclude that $u^{(m)} \in BV(U)$ for every $m \in \mathbb{N}$ and $u_k^{(m)} \to u^{(m)}$ strongly in $L^1(U)$ as $k \to +\infty$. By (3.26), (3.28), (3.29), and (3.30) we obtain that

$$\int_U |\nabla u_k^{(m)}|dx + |D^c u_k^{(m)}|(U) + \int_{J_{u_k^{(m)}}} |[u_k^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \le M\,. \tag{3.34}$$

By [4, Theorem 2.1] we deduce that

$$\int_U |\nabla u^{(m)}|dx + |D^c u^{(m)}|(U) + \int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \le M\,, \tag{3.35}$$

hence $u \in GBV_\star(U)$. □

4. The incremental minimum problem

In this section we present a precise formulation of the incremental minimum problem (1.1), which appears in the variational approach to the quasistatic crack growth in elastic–perfectly plastic materials. The reference configuration is a bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. The crack in the reference configuration is represented by a Borel set $\Gamma \subset \overline{\Omega}$, with $\mathcal{H}^{d-1}(\Gamma) < +\infty$. The set $\overline{\Omega} \setminus \Gamma$ represents the elasto-plastic part of the body.

Since we are dealing with the antiplane case, the displacement of each material point is described by a function $u\colon \overline{\Omega} \setminus \Gamma \to \mathbb{R}$. Regarding u as a function defined $\mathcal{L}^d$-a.e. in Ω, we assume that

$$u \in GBV_\star(\Omega)\,, \int_{J_u \setminus \Gamma} |[u]| d\mathcal{H}^{d-1} < +\infty\,, \text{ and } \int_{\partial\Omega \setminus \Gamma} |u| d\mathcal{H}^{d-1} < +\infty\,. \tag{4.1}$$

Here and in the rest of the paper the trace on $\partial\Omega$ of a function $v \in GBV(\Omega)$ is still denoted by v. The strain corresponding to the displacement u is given by the measure $D^\Gamma u \in \mathcal{M}_b(\Omega \setminus \Gamma; \mathbb{R}^d)$ introduced in Definition 3.6 with U replaced by Ω and Γ replaced by $\Gamma \cap \Omega$.

The Dirichlet boundary condition is assigned using the trace on $\partial\Omega$ of a function $w \in H^1(\Omega)$. The elastic part of the strain is denoted by e and the plastic part by p. We assume that

$$e \in L^2(\Omega; \mathbb{R}^d) \quad \text{and} \quad p \in \mathcal{M}_b(\overline{\Omega} \setminus \Gamma; \mathbb{R}^d)\,, \tag{4.2}$$

$$D^\Gamma u = e + p \text{ as measures on } \Omega \setminus \Gamma\,, \tag{4.3}$$

$$p = (w-u)\nu_\Omega \mathcal{H}^{d-1} \text{ as measures on } \partial\Omega \setminus \Gamma\,, \tag{4.4}$$

where ν_Ω is the outer unit normal to $\partial\Omega$. Here and in the rest of the paper we identify an L^1-function φ and the measure $\varphi\mathcal{L}^d$. To simplify the exposition, for every Borel set $\Gamma \subset \overline{\Omega}$ and every $w \in H^1(\Omega)$, it is convenient to introduce the set $\mathcal{A}(\Gamma, w)$ of all triples (u, e, p) which satisfy (4.1)-(4.4).

From the definition of $D^\Gamma u$ it follows that if $\mathcal{H}^{d-1}(\Gamma) < +\infty$, the absolutely continuous part p^a of p with respect to $\mathcal{L}^d$ satisfies

$$\nabla u = e + p^a \quad \mathcal{L}^d\text{-a.e. in } \Omega\,, \tag{4.5}$$

while the singular part p^s of p with respect to $\mathcal{L}^d$ satisfies

$$p^s(B) = D^c u(B) + \int_{J_u \cap B} [u]\nu_u d\mathcal{H}^{d-1} \text{ for every Borel set } B \subset \Omega \setminus \Gamma, \tag{4.6}$$

$$p^s(B) = p(B) = \int_B (w-u)\nu_\Omega d\mathcal{H}^{d-1} \text{ for every Borel set } B \subset \partial\Omega \setminus \Gamma. \tag{4.7}$$

In our incremental minimum problem the data at the previous time are

$$\text{a Borel set } \Gamma_0 \subset \overline{\Omega} \quad \text{with} \quad \mathcal{H}^{d-1}(\Gamma_0) < +\infty\,, \tag{4.8}$$

$$w_0 \in H^1(\Omega) \quad \text{and} \quad (u_0, e_0, p_0) \in \mathcal{A}(\Gamma_0, w_0)\,. \tag{4.9}$$

Given

$$w \in H^1(\Omega) \tag{4.10}$$

the precise formulation of the incremental minimum problem (1.1) is

$$\min_{\substack{\Gamma \text{ Borel}, \Gamma_0 \subset \Gamma \subset \overline{\Omega} \\ (u,e,p) \in \mathcal{A}(\Gamma, w)}} \Big\{ \frac{1}{2} \int_\Omega |e|^2 dx + |p - p_0|(\overline{\Omega} \setminus \Gamma) + \mathcal{H}^{d-1}(\Gamma \setminus \Gamma_0) \Big\}\,. \tag{4.11}$$

To solve this problem we introduce the function $f \colon \mathbb{R}^d \to \mathbb{R}$ defined by

$$f(\xi) := \min_{\eta \in \mathbb{R}^d} \Big\{ \tfrac{1}{2}|\eta|^2 + |\xi - \eta| \Big\} = \begin{cases} \frac{1}{2}|\xi|^2 & \text{if } |\xi| \le 1\,, \\ |\xi| - \frac{1}{2} & \text{if } |\xi| \ge 1\,. \end{cases} \tag{4.12}$$

The minimum in the definition of $f(\xi)$ is attained for $\eta = \begin{cases} \xi & \text{if } |\xi| \le 1\,, \\ \xi/|\xi| & \text{if } |\xi| \ge 1\,. \end{cases}$

It is convenient to introduce the maps $\pi_1, \pi_2 : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\pi_1(\xi) = \begin{cases} \xi & \text{if } |\xi| \le 1 \\ \xi/|\xi| & \text{if } |\xi| \ge 1 \end{cases} \qquad \pi_2(\xi) = \begin{cases} 0 & \text{if } |\xi| \le 1 \\ \xi - \xi/|\xi| & \text{if } |\xi| \ge 1\,. \end{cases}$$

We note that

$$\xi = \pi_1(\xi) + \pi_2(\xi)\,, \quad |\pi_1(\xi)| \le 1\,, \quad \text{and} \quad f(\xi) = \tfrac{1}{2}|\pi_1(\xi)|^2 + |\pi_2(\xi)|\,. \tag{4.13}$$

For later use we observe that the definition (4.12) of f implies

$$|\xi| - \tfrac{1}{2} \le f(\xi) \le |\xi| \quad \text{for every } \xi \in \mathbb{R}^d\,. \tag{4.14}$$

To deal with the boundary condition (4.4) in (4.11) it is convenient to introduce a bounded open set Ω' with

$$\overline{\Omega} \subset \Omega' \tag{4.15}$$

and to extend w, w_0, e_0 in such a way that

$$w, w_0 \in H^1(\Omega') \quad \text{and} \quad e_0 \in L^2(\Omega'; \mathbb{R}^d)\,. \tag{4.16}$$

We now prove that problem (4.11) is equivalent to the following minimum problem

$$\min_{\substack{\Gamma \text{ Borel}, \Gamma_0 \subset \Gamma \subset \overline{\Omega} \\ v \in GBV_\star(\Omega') \\ v = w - w_0 \text{ a.e. in } \Omega' \setminus \Omega}} \Big\{ \int_{\Omega'} f(\nabla v + e_0) dx + |D^c v|(\Omega') + \int_{J_v \setminus \Gamma} |[v]| d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(\Gamma) \Big\}. \tag{4.17}$$

Lemma 4.1. *Assume* (4.8), (4.9), (4.15), *and* (4.16). *Let* Γ *and* (u, e, p) *be a solution of* (4.11) *and let* $v := u - u_0$ *in* Ω *and* $v := w - w_0$ *in* $\Omega' \setminus \Omega$. *Then* Γ *and* v *solve* (4.17).

Conversely, assume that Γ *and* v *solve* (4.17) *and let* $u := v|_\Omega + u_0$, $e := \pi_1(\nabla v|_\Omega + e_0|_\Omega)$, $p := D^\Gamma u - e$ *in* $\Omega \setminus \Gamma$, *and* $p := (w - u)\nu_\Omega \mathcal{H}^{d-1}$ *on* $\partial\Omega \setminus \Gamma$. *Then* Γ *and* (u, e, p) *solve* (4.11).

Proof. Let Γ and (u, e, p) be a solution of (4.11). It is clear that $\mathcal{H}^{d-1}(\Gamma \setminus \Gamma_0) < +\infty$, hence (4.8) implies that $\mathcal{H}^{d-1}(\Gamma) < +\infty$. Let v be as in the statement of the lemma. To prove that Γ and v solve (4.17) we fix a Borel set $\hat\Gamma$, with $\Gamma_0 \subset \hat\Gamma \subset \overline{\Omega}$, and $\hat v \in GBV_\star(\Omega')$, with

$$\hat v = w - w_0 \quad \mathcal{L}^d\text{-a.e. in } \Omega' \setminus \Omega\,. \tag{4.18}$$

We want to show that

$$\begin{aligned} &\int_{\Omega'} f(\nabla v + e_0) dx + |D^c v|(\Omega') + \int_{J_v \setminus \Gamma} |[v]| d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(\Gamma) \\ &\le \int_{\Omega'} f(\nabla \hat v + e_0) dx + |D^c \hat v|(\Omega') + \int_{J_{\hat v} \setminus \hat\Gamma} |[\hat v]| d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(\hat\Gamma)\,. \end{aligned} \tag{4.19}$$

It is not restrictive to assume that

$$\int_{J_{\hat v} \setminus \hat\Gamma} |[\hat v]| d\mathcal{H}^{d-1} < +\infty \quad \text{and} \quad \mathcal{H}^{d-1}(\hat\Gamma) < +\infty\,. \tag{4.20}$$

We set

$$\hat e := \pi_1(\nabla \hat v|_\Omega + e_0|_\Omega) \text{ and } \hat g := \nabla \hat v|_\Omega + \nabla u_0 - \hat e = \nabla \hat v|_\Omega + e_0|_\Omega + p_0^a - \hat e\,, \tag{4.21}$$

where the last equality follows from (4.5) and (4.9). Then $\hat e \in L^2(\Omega;\mathbb{R}^d)$ and $\hat g \in L^1(\Omega;\mathbb{R}^d)$. We now define $\hat u := \hat v|_\Omega + u_0$ and note that $\hat u \in GBV_\star(\Omega)$ by Theorem 3.9. Moreover we define $\hat p \in \mathcal{M}_b(\overline{\Omega}\setminus\hat\Gamma;\mathbb{R}^d)$ by

$$\hat p := \hat g + D^c\hat u + [\hat u]\nu_{\hat u}\mathcal{H}^{d-1}\llcorner(J_{\hat u}\setminus\hat\Gamma) + (w-\hat u)\nu_\Omega\mathcal{H}^{d-1}\llcorner(\partial\Omega\setminus\hat\Gamma)\,.$$

We remark that $\int_{J_{\hat u}\setminus\hat\Gamma}|[\hat u]|d\mathcal{H}^{d-1}<+\infty$ and $\int_{\partial\Omega\setminus\hat\Gamma}|\hat u|d\mathcal{H}^{d-1}<+\infty$ by (4.1), (4.9), (4.18), and (4.20). This shows that the definition of $\hat p$ makes sense. We note that $D^{\hat\Gamma}\hat u = \hat e + \hat p$ in $\Omega\setminus\hat\Gamma$ and $\hat p = (w-\hat u)\nu_\Omega\mathcal{H}^{d-1}$ on $\partial\Omega\setminus\hat\Gamma$, hence $(\hat u,\hat e,\hat p)\in\mathcal{A}(\hat\Gamma,w)$. Consequently, the minimality of Γ and (u,e,p) gives

$$\begin{aligned}&\frac12\int_\Omega|e|^2dx + |p-p_0|(\overline{\Omega}\setminus\Gamma) + \mathcal{H}^{d-1}(\Gamma\setminus\Gamma_0)\\ &\le \frac12\int_\Omega|\hat e|^2dx + |\hat p-p_0|(\overline{\Omega}\setminus\hat\Gamma) + \mathcal{H}^{d-1}(\hat\Gamma\setminus\Gamma_0)\,.\end{aligned}\tag{4.22}$$

Since $\hat v = w - w_0$ $\mathcal{L}^d$-a.e. in $\Omega'\setminus\Omega$ and $\hat u = \hat v|_\Omega + u_0$ we have that

$$\int_{J_{\hat v}\setminus\hat\Gamma}|[\hat v]|d\mathcal{H}^{d-1} = \int_{J_{\hat u-u_0}\setminus\hat\Gamma}|[\hat u-u_0]|d\mathcal{H}^{d-1} + \int_{\partial\Omega\setminus\hat\Gamma}|w-\hat u-w_0+u_0|d\mathcal{H}^{d-1}\,.$$

On the other hand by the definition of $\hat p$ we have

$$\begin{aligned}|\hat p-p_0|(\overline{\Omega}\setminus\hat\Gamma) &= \int_\Omega|\hat g-p_0^a|dx + |D^c(\hat u-u_0)|(\Omega)\\ &+\int_{J_{\hat u-u_0}\setminus\hat\Gamma}|[\hat u-u_0]|d\mathcal{H}^{d-1} + \int_{\partial\Omega\setminus\hat\Gamma}|w-\hat u-w_0+u_0|d\mathcal{H}^{d-1}\\ &= \int_\Omega|\hat g-p_0^a|dx + |D^c\hat v|(\Omega) + \int_{J_{\hat v}\setminus\hat\Gamma}|[\hat v]|d\mathcal{H}^{d-1}\,,\end{aligned}$$

hence, by (4.13) and (4.21),

$$\begin{aligned}&\int_{\Omega'}f(\nabla\hat v+e_0)dx + |D^c\hat v|(\Omega') + \int_{J_{\hat v}\setminus\hat\Gamma}|[\hat v]|d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(\hat\Gamma)\\ &= \gamma + \frac12\int_\Omega|\hat e|^2dx + |\hat p-p_0|(\overline{\Omega}\setminus\hat\Gamma) + \mathcal{H}^{d-1}(\hat\Gamma)\,,\end{aligned}\tag{4.23}$$

where

$$\gamma := \int_{\Omega'\setminus\Omega}f(\nabla w-\nabla w_0+e_0)dx\,.\tag{4.24}$$

Similarly, using the definition (4.12) of f instead of (4.13), we obtain

$$\begin{aligned}&\int_{\Omega'}f(\nabla v+e_0)dx + |D^cv|(\Omega') + \int_{J_v\setminus\Gamma}|[v]|d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(\Gamma)\\ &\le \gamma + \frac12\int_\Omega|e|^2dx + |p-p_0|(\overline{\Omega}\setminus\Gamma) + \mathcal{H}^{d-1}(\Gamma)\,.\end{aligned}\tag{4.25}$$

Then from (4.22), (4.23), and (4.25) we obtain

$$\int_{\Omega'} f(\nabla v + e_0)dx + |D^c v|(\Omega') + \int_{J_v\setminus\Gamma} |[v]|d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(\Gamma)$$
$$\leq \int_{\Omega'} f(\nabla \hat{v} + e_0)dx + |D^c \hat{v}|(\Omega') + \int_{J_{\hat{v}}\setminus\hat{\Gamma}} |[\hat{v}]|d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(\hat{\Gamma})\,, \tag{4.26}$$

which shows that Γ and v solve (4.17).

Conversely, assume that Γ and v solve (4.17). By (4.8) it is clear that $\mathcal{H}^{d-1}(\Gamma) < +\infty$. We observe that the triple (u, e, p) defined in the second part of the statement of the lemma belongs to $\mathcal{A}(\Gamma, w)$ and that $p = p^a + D^c u + [u]\nu_u \mathcal{H}^{d-1} \llcorner (J_u \setminus \Gamma)$ in $\Omega \setminus \Gamma$ while $p = (w - u)\nu_\Omega \mathcal{H}^{d-1}$ on $\partial\Omega \setminus \Gamma$. To prove that Γ and (u, e, p) solve (4.11) we fix a Borel set $\hat{\Gamma}$ with $\Gamma_0 \subset \hat{\Gamma} \subset \overline{\Omega}$ and a triple $(\hat{u}, \hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w)$. We want to show that

$$\frac{1}{2}\int_\Omega |e|^2 dx + |p - p_0|(\overline{\Omega}\setminus\Gamma) + \mathcal{H}^{d-1}(\Gamma\setminus\Gamma_0)$$
$$\leq \frac{1}{2}\int_\Omega |\hat{e}|^2 dx + |\hat{p} - p_0|(\overline{\Omega}\setminus\hat{\Gamma}) + \mathcal{H}^{d-1}(\hat{\Gamma}\setminus\Gamma_0)\,. \tag{4.27}$$

Let $\hat{v} := \hat{u} - u_0$ in Ω and $\hat{v} := w - w_0$ in $\Omega' \setminus \Omega$. Then, arguing as in the first part of the proof we obtain in this case (4.25) with an equality and (4.23) with the inequality $\leq$. Then (4.27) follows from (4.22) and (4.26), which is an obvious consequence of (4.17). □

We now prove that (4.17) is equivalent to the minimum problem

$$\min_{\substack{v\in GBV_\star(\Omega') \\ v=w-w_0 \text{ a.e. in } \Omega'\setminus\Omega}} \Big\{ \int_{\Omega'} f(\nabla v + e_0)dx + |D^c v|(\Omega') + \int_{J_v\setminus\Gamma_0} |[v]|\wedge 1 d\mathcal{H}^{d-1} \Big\}. \tag{4.28}$$

We recall that in (3.2) we defined $J_v^1 := \{x \in J_v : |[v](x)| \geq 1\}$.

Lemma 4.2. *If* Γ *and* v *solve* (4.17)*, then* v *is a solution of* (4.28)*. Conversely, if* v *is a solution of* (4.28) *and* $\Gamma := (J_v^1 \cup \Gamma_0) \cap \overline{\Omega}$*, then* Γ *and* v *solve* (4.17)*.*

Proof. Assume that Γ and v solve (4.17). Let $\hat{v} \in GBV_\star(\Omega')$ be such that $\hat{v} = w - w_0$ $\mathcal{L}^d$-a.e. in $\Omega' \setminus \Omega$ and let $\hat{\Gamma} = (J_{\hat{v}}^1 \cup \Gamma_0) \cap \overline{\Omega}$. Then

$$\int_{J_v\setminus\Gamma_0} |[v]|\wedge 1 d\mathcal{H}^{d-1} \leq \int_{J_v\setminus\Gamma} |[v]|d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(\Gamma\setminus\Gamma_0)$$

and

$$\int_{J_{\hat{v}}\setminus\hat{\Gamma}} |[\hat{v}]|d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(\hat{\Gamma}\setminus\Gamma_0) = \int_{J_{\hat{v}}\setminus\Gamma_0} |[\hat{v}]|\wedge 1 d\mathcal{H}^{d-1}\,,$$

where we used the fact that $\mathcal{H}^{d-1}(J_{\hat v}\backslash\overline{\Omega})=0$ since $\hat v\in H^1(\Omega'\backslash\overline{\Omega})$. Therefore, by the minimality of Γ and v we have

$$\begin{aligned}
&\int_{\Omega'} f(\nabla v+e_0)dx+|D^cv|(\Omega')+\int_{J_v\backslash\Gamma_0}|[v]|\wedge 1d\mathcal{H}^{d-1}\\
&\leq \int_{\Omega'} f(\nabla v+e_0)dx+|D^cv|(\Omega')+\int_{J_v\backslash\Gamma}|[v]|d\mathcal{H}^{d-1}+\mathcal{H}^{d-1}(\Gamma\setminus\Gamma_0)\\
&\leq \int_{\Omega'} f(\nabla \hat v+e_0)dx+|D^c\hat v|(\Omega')+\int_{J_{\hat v}\backslash\hat\Gamma}|[\hat v]|d\mathcal{H}^{d-1}+\mathcal{H}^{d-1}(\hat\Gamma\setminus\Gamma_0)\\
&= \int_{\Omega'} f(\nabla \hat v+e_0)dx+|D^c\hat v|(\Omega')+\int_{J_{\hat v}\backslash\Gamma_0}|[\hat v]|\wedge 1d\mathcal{H}^{d-1}\,,
\end{aligned}$$

which proves that v solves (4.28).

Conversely, assume now that v solves (4.28) and let $\Gamma:=(J_v^1\cup\Gamma_0)\cap\overline{\Omega}$. Let $\hat\Gamma$ be a Borel set with $\Gamma_0\subset\hat\Gamma\subset\overline{\Omega}$, let $\hat v\in GBV_\star(\Omega')$ with $\hat v=w-w_0$ $\mathcal{L}^d$-a.e. in $\Omega'\setminus\Omega$, and let $\hat\Gamma_1:=J_{\hat v}^1\cup\Gamma_0$. By the minimality of v we have

$$\begin{aligned}
&\int_{\Omega'} f(\nabla v+e_0)dx+|D^cv|(\Omega')+\int_{J_v\backslash\Gamma}|[v]|d\mathcal{H}^{d-1}+\mathcal{H}^{d-1}(\Gamma\setminus\Gamma_0)\\
&= \int_{\Omega'} f(\nabla v+e_0)dx+|D^cv|(\Omega')+\int_{J_v\backslash\Gamma_0}|[v]|\wedge 1d\mathcal{H}^{d-1}\\
&\leq \int_{\Omega'} f(\nabla \hat v+e_0)dx+|D^c\hat v|(\Omega')+\int_{J_{\hat v}\backslash\Gamma_0}|[\hat v]|\wedge 1d\mathcal{H}^{d-1}\,.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{J_{\hat v}\backslash\Gamma_0}|[\hat v]|\wedge 1d\mathcal{H}^{d-1}=\int_{J_{\hat v}\backslash\hat\Gamma_1}|[\hat v]|d\mathcal{H}^{d-1}+\mathcal{H}^{d-1}(\hat\Gamma_1\setminus\Gamma_0)\\
&=\int_{(J_{\hat v}\backslash\hat\Gamma_1)\backslash\hat\Gamma}|[\hat v]|d\mathcal{H}^{d-1}+\int_{(J_{\hat v}\backslash\hat\Gamma_1)\cap\hat\Gamma}|[\hat v]|d\mathcal{H}^{d-1}+\mathcal{H}^{d-1}(\hat\Gamma_1\setminus\Gamma_0)\\
&\leq \int_{(J_{\hat v}\backslash\hat\Gamma_1)\backslash\hat\Gamma}|[\hat v]|d\mathcal{H}^{d-1}+\mathcal{H}^{d-1}(\hat\Gamma\setminus\hat\Gamma_1)+\mathcal{H}^{d-1}(\hat\Gamma_1\setminus\Gamma_0)\\
&\leq \int_{J_{\hat v}\backslash\hat\Gamma}|[\hat v]|d\mathcal{H}^{d-1}+\mathcal{H}^{d-1}(\hat\Gamma\setminus\Gamma_0)\,,
\end{aligned}$$

we obtain that

$$\begin{aligned}
&\int_{\Omega'} f(\nabla v+e_0)dx+|D^cv|(\Omega')+\int_{J_v\backslash\Gamma}|[v]|d\mathcal{H}^{d-1}+\mathcal{H}^{d-1}(\Gamma\setminus\Gamma_0)\\
&\leq \int_{\Omega'} f(\nabla \hat v+e_0)dx+|D^c\hat v|(\Omega')+\int_{J_{\hat v}\backslash\hat\Gamma}|[\hat v]|d\mathcal{H}^{d-1}+\mathcal{H}^{d-1}(\hat\Gamma\setminus\Gamma_0)\,,
\end{aligned}$$

which shows that Γ and v solve (4.17). □

The results of this section show that the existence of a solution to the minimum problem (4.11) can be obtained by proving that the minimum problem (4.28) has a solution. To this aim we shall use the Direct Method of the Calculus of Variations. Unfortunately, not every energy-bounded sequence for

(4.28) is relatively compact. For instance, if $w = w_0 = 0$, $\Gamma_0 = \emptyset$, $e_0 = 0$, and $v_k = k\chi_E$, where E is a set of finite perimeter with $\mathcal{L}^d(E) > 0$, then $(v_k)_k$ is energy-bounded for (4.28), but it has no subsequence which converges $\mathcal{L}^d$-a.e. to a finite-valued function.

The origin of this problem is the fact that, in general, an energy-bounded sequence does not satisfy (3.27). In the next section we shall construct a relatively compact minimizing sequence for problem (4.28), while in Section 6 we shall prove a lower semicontinuity result, which will allow us to obtain the existence of a minimizer.

5. Construction of a relatively compact minimizing sequence

In this section Ω is a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary, c_1, c_2 are constants with $0 < c_1 \leq c_2$, and $a_1, a_2 \in L^1(\Omega)$. Given a Borel set $\Gamma_0 \subset \overline{\Omega}$, with $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$, and a Borel measurable function $g\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$, with

$$c_1|\xi| - a_1(x) \leq g(x,\xi) \leq c_2|\xi| + a_2(x) \tag{5.1}$$

for $\mathcal{L}^d$-a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^d$, we consider the functional $\mathcal{G}^g_{\Gamma_0}$ defined by

$$\mathcal{G}^g_{\Gamma_0}(u) := \int_\Omega g(x,\nabla u)dx + |D^c u|(\Omega) + \int_{J_u\setminus\Gamma_0} |[u]| \wedge 1 d\mathcal{H}^{d-1} \tag{5.2}$$

for every $u \in GBV_\star(\Omega)$. The aim of this section is to show that, if $(u_k)_k$ is a minimizing sequence for $\mathcal{G}^g_{\Gamma_0}$, then we can modify it by means of piecewise constant translations obtaining a new minimizing sequence which satisfies the hypotheses of the compactness Theorem 3.11. The construction of the modified sequence follows the lines of [9] and requires several steps. We begin by constructing a suitable Caccioppoli partition (see [2, Definition 4.16]).

Lemma 5.1 (L^∞-approximation with piecewise constant functions). *For every $M > 0$ and for every $u \in GBV_\star(\Omega)$, with*

$$\|\nabla u\|_{L^1(\Omega;\mathbb{R}^d)} + |D^c u|(\Omega) + \int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} \leq M\,, \tag{5.3}$$

there exist a Caccioppoli partition $(P_j)_j$ of Ω and corresponding translations $(b_j)_j \subset \mathbb{R}$ such that the function

$$v := u - \sum_{j=1}^\infty b_j \chi_{P_j}$$

belongs to $BV(\Omega) \cap L^\infty(\Omega)$ and the following estimates hold:

$$\sum_{j=1}^\infty \mathcal{H}^{d-1}(\partial^* P_j) \leq 2 + 2M + \mathcal{H}^{d-1}(\partial\Omega) \tag{5.4}$$

$$\|v\|_{L^\infty(\Omega)} \leq 2M\,. \tag{5.5}$$

Proof. We may assume that

$$A := \int_\Omega |\nabla u| dx + |D^c u|(\Omega) + \int_{J_u \setminus J_u^1} |[u]| d\mathcal{H}^{d-1} > 0 \,.$$

Indeed, if this is not the case, by (3.1) for every m we can apply [2, Theorem 4.23] to the truncated function $u^{(m)}$, obtaining that $u^{(m)}$ is piecewise constant (see [2, Definition 4.21]). This implies that the function u itself is piecewise constant and there is nothing to prove.

By the coarea formula in $GBV(\Omega)$ (see [2, Theorem 4.34(d)] applied with $B = \Omega \setminus J_u^1$) for a.e. $t \in \mathbb{R}$ the set $\{u > t\}$ has finite perimeter in Ω and

$$\int_{-\infty}^{+\infty} \mathcal{H}^{d-1}(\Omega \cap \partial^*\{u > t\} \setminus J_u^1) dt = A \,.$$

For every $i \in \mathbb{Z}$ there exists $t_i \in (iA, (i+1)A)$ such that the set $\{u > t_i\}$ has finite perimeter in Ω and

$$\mathcal{H}^{d-1}(\Omega \cap \partial^*\{u > t_i\} \setminus J_u^1) \le \frac{1}{A} \int_{iA}^{(i+1)A} \mathcal{H}^{d-1}(\Omega \cap \partial^*\{u > t\} \setminus J_u^1) dt \,. \quad (5.6)$$

For every $i \in \mathbb{Z}$ let $E_i := \{u > t_i\} \setminus \{u > t_{i+1}\}$. It is clear that $(E_i)_i$ is a partition of Ω, that each E_i has finite perimeter in $\mathbb{R}^d$ (recall that Ω has Lipschitz boundary, hence we can apply [2, Remark 4.20]), and that

$$\sum_{i \in \mathbb{Z}} \mathcal{H}^{d-1}(\Omega \cap \partial^* E_i \setminus J_u^1) \le \frac{2}{A} \int_{-\infty}^{+\infty} \mathcal{H}^{d-1}(\Omega \cap \partial^*\{u > t\} \setminus J_u^1) dt = 2 \,. \quad (5.7)$$

Let us prove now that

$$\sum_{i \in \mathbb{Z}} \mathcal{H}^{d-1}(\partial^* E_i) \le 2 + 2M + \mathcal{H}^{d-1}(\partial\Omega) \,. \quad (5.8)$$

First of all, we claim that every $x \in J_u^1$ belongs at most to two sets $\partial^* E_i$. Indeed, it is known (see, e.g., [2, Theorem 3.61]) that for every $x \in \partial^* E_i$ we have $\lim_{\rho \to 0} \mathcal{L}^d(E_i \cap B_\rho(x))/\mathcal{L}^d(B_\rho(x)) = \frac{1}{2}$. Therefore, if $x \in \partial^* E_i \cap \partial^* E_j \cap \partial^* E_k$ for some $i < j < k$, we would have $\lim_{\rho \to 0} \mathcal{L}^d((E_i \cup E_j \cup E_k) \cap B_\rho(x))/\mathcal{L}^d(B_\rho(x)) = \frac{3}{2}$, which is clearly impossible. This proves our claim, which implies that

$$\sum_{i \in \mathbb{Z}} \mathcal{H}^{d-1}(\partial^* E_i \cap J_u^1) \le 2\mathcal{H}^{d-1}(J_u^1) \le 2M \,. \quad (5.9)$$

A similar argument shows that every $x \in \partial\Omega$ belongs to at most one set $\partial^* E_i$, hence

$$\sum_{i \in \mathbb{Z}} \mathcal{H}^{d-1}(\partial^* E_i \cap \partial\Omega) \le \mathcal{H}^{d-1}(\partial\Omega) \,. \quad (5.10)$$

Therefore (5.7), (5.9), and (5.10) give (5.8), which shows that $(E_i)_i$ is a Caccioppoli partition of Ω.

Let us define $v := u - \sum_{i \in \mathbb{Z}} t_i \chi_{E_i}$. For every $x \in E_i$ we have

$$0 \le v(x) = u(x) - t_i \le t_{i+1} - t_i \le 2A \le 2M \,, \quad (5.11)$$

which shows that

$$\|v\|_{L^\infty(\Omega)} \le 2M\,. \tag{5.12}$$

We show that $v \in BV(\Omega)$. To this end let us consider $v_k := \sum_{|i|\le k} z_i$, with $z_i := (u - t_i)\chi_{E_i}$. By (5.11) we have $z_i = (u^{(m_i)} - t_i)\chi_{E_i}$, with $m_i = 2M + |t_i|$. Since both $u^{(m_i)} - t_i$ and χ_{E_i} belong to $BV(\Omega) \cap L^\infty(\Omega)$, by Lemma 2.4 we have $z_i \in BV(\Omega)$. Recalling (5.11) we have $[u^{(m_i)}] = [u]$ and $0 \le [u^{(m_i)} - t_i] \le 2M$ on $E_i^{(1)} \cap J_u$ by the definition of $E^{(1)}$, while the trace operator γ_{E_i} defined in Lemma 2.4 satisfies $|\gamma_{E_i}(u^{m_i} - t_i)| \le 2M$ $\mathcal{H}^{d-1}$-a.e. on $\partial^* E_i$. Using Lemma 2.4 again, from these properties we obtain

$$\begin{aligned}
|Dz_i|(\Omega) &= \int_\Omega |\nabla z_i| dx + |D^c z_i|(\Omega) + |D^j z_i|(\Omega) \\
&\le \int_{E_i} |\nabla u^{(m_i)}|\, dx + |D^c u^{(m_i)}|(E_i^{(1)}) \\
&\quad + \int_{E_i^{(1)} \cap J_{u^{(m_i)}}} |[u^{(m_i)}]| d\mathcal{H}^{d-1} + 2M\mathcal{H}^{d-1}(\partial^* E_i) \\
&\le \int_{E_i} |\nabla u|\, dx + |D^c u|(E_i^{(1)}) + \int_{E_i^{(1)} \cap (J_u \setminus J_u^1)} |[u]| d\mathcal{H}^{d-1} \\
&\quad + 2M\mathcal{H}^{d-1}(E_i^{(1)} \cap J_u^1) + 2M\mathcal{H}^{d-1}(\partial^* E_i)\,,
\end{aligned}$$

where the last inequality follows from Proposition 2.6(c) and (2.23). Since the sets $E_i^{(1)}$ are pairwise disjoint, by (5.8)

$$|Dv_k|(\Omega) \le \int_\Omega |\nabla u| dx + |D^c u|(\Omega) + \int_{J_u \setminus J_u^1} |[u]| d\mathcal{H}^{d-1} + \hat{M}\,,$$

where $\hat{M} := 2M\big(2 + 3M + \mathcal{H}^{d-1}(\partial\Omega)\big)$. Since the right-hand side is finite, we obtain that $|Dv_k|(\Omega)$ is bounded uniformly with respect to k. On the other hand, since $(E_i)_i$ is a partition, inequality (5.12) implies that the sequence $(v_k)_k$ is bounded in $L^\infty(\Omega)$ and that $v_k \to v$ strongly in $L^1(\Omega)$. Therefore $v \in BV(\Omega)$.

To conclude the proof it is enough to take $P_j = E_{\sigma(j)}$ and $b_j = t_{\sigma(j)}$ where $\sigma\colon \mathbb{N} \to \mathbb{Z}$ is bijective. □

In the following lemma the Caccioppoli partition is finite and we provide a precise estimate on the translations.

Lemma 5.2 (Piecewise Poincaré inequality). *Let* $\alpha \ge 1$ *and let* $0 < \theta < 1$. *Then there exist positive constants* C_Ω *and* $C_{\theta,\alpha,d}$ *such that for every* $u \in GBV_\star(\Omega)$ *there exist a finite Caccioppoli partition* $\Omega = \bigcup_{j=1}^J P_j \cup R_1 \cup R_2$, *a finite family of translations* $(b_j)_{j=1}^J \subset \mathbb{R}$, *and a constant* $\lambda \in [1, C_{\theta,\alpha,d}]$, *depending on u, satisfying the following estimates:*

$$\mathcal{L}^d(R_1 \cup R_2) \le \theta C_\Omega \mathcal{H}^{d-1}(J_u^1 \cup \partial\Omega)\,, \tag{5.13}$$

$$\mathcal{H}^{d-1}(\partial^* R_1) \le \theta C_\Omega \mathcal{H}^{d-1}(J_u^1 \cup \partial\Omega)\,, \tag{5.14}$$

$$\sum_{j=1}^J \mathcal{H}^{d-1}(\partial^* P_j) + \mathcal{H}^{d-1}(\partial^* R_2) \le C_\Omega \mathcal{H}^{d-1}(J_u^1 \cup \partial\Omega)\,, \tag{5.15}$$

$$\max_{1\le j\le J} \|u - b_j\|_{L^\infty(P_j)} \le \lambda\Big(\int_\Omega |\nabla u| dx + |D^c u|(\Omega) + \int_{J_u \setminus J_u^1} |[u]| d\mathcal{H}^{d-1}\Big), \tag{5.16}$$

$$\min_{1\le j\le J} \operatorname*{ess\,inf}_{R_2} |u - b_j| \ge \alpha\lambda\Big(\int_\Omega |\nabla u| dx + |D^c u|(\Omega) + \int_{J_u \setminus J_u^1} |[u]| d\mathcal{H}^{d-1}\Big), \tag{5.17}$$

$$\min_{1\le i<j\le J} |b_i - b_j| \ge \alpha\lambda\Big(\int_\Omega |\nabla u| dx + |D^c u|(\Omega) + \int_{J_u \setminus J_u^1} |[u]| d\mathcal{H}^{d-1}\Big). \tag{5.18}$$

Proof. It is enough to repeat the proof of [9, Lemma 3.5] replacing the space $GSBV^p(\Omega;\mathbb{R}^m)$ by $GBV_\star(\Omega)$, $\mathcal{H}^{d-1}(J_u \cup \partial\Omega)$ by $\mathcal{H}^{d-1}(J_u^1 \cup \partial\Omega)$, $\|\nabla u\|_{L^1(\Omega)}$ by $\|\nabla u\|_{L^1(\Omega;\mathbb{R}^d)} + |D^c u|(\Omega) + \int_{J_u\setminus J_u^1} |[u]| d\mathcal{H}^{d-1}$, and [9, Theorem 2.5] by Corollary 5.1. □

The following theorem shows that we can modify a function u by means of piecewise constant translations, with a precise control on the value taken by the functional $\mathcal{G}^g_{\Gamma_0}$ defined in (5.2) on the modified function.

Theorem 5.3 (Piecewise translated functions). *Let $M > 0$ and $0 < \theta < 1$. Then there exist positive constants $C_{M,\Omega}$ and $C_{M,\theta,\Omega}$ with the following property: for every $u \in GBV_\star(\Omega)$ with*

$$\|\nabla u\|_{L^1(\Omega;\mathbb{R}^d)} + |D^c u|(\Omega) + \int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} \le M \tag{5.19}$$

there exist a finite Caccioppoli partition $\Omega = \bigcup_{j=1}^J P_j \cup R$ and a finite family of translations $(t_j)_{j=1}^J \subset \mathbb{R}$ such that the function

$$v := \sum_{j=1}^J (u - t_j)\chi_{P_j}\,, \tag{5.20}$$

belongs to $BV(\Omega) \cap L^\infty(\Omega)$ and the following estimates hold:

$$\mathcal{L}^d(R) \le \theta C_{M,\Omega}\,, \tag{5.21}$$

$$\sum_{j=1}^J \mathcal{H}^{d-1}(\partial^* P_j) + \mathcal{H}^{d-1}(\partial^* R) \le C_{M,\Omega}\,, \tag{5.22}$$

$$\|v\|_{L^\infty(\Omega)} \le C_{M,\theta,\Omega}\,, \tag{5.23}$$

$$\mathcal{G}^g_{\Gamma_0}(v) \le \mathcal{G}^g_{\Gamma_0}(u) + \theta C_{M,\Omega} + \|a\|_{L^1(R)}\,, \tag{5.24}$$

for every g satisfying (5.1) *and for every Borel set $\Gamma_0 \subset \Omega$ with $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$, where $a := |a_1| + |a_2|$. Moreover, $\{v = 0\} \supset \{u = 0\}$ (up to a set* *of negligible $\mathcal{L}^d$ measure). Finally, we can choose $(P_j)_{j=1}^J$ and $(t_j)_{j=1}^J$ so that the following additional property holds: for every collection $(t'_j)_{j=1}^J$, with $|t_j - t'_j| \leq \theta^{-1}\|v\|_{L^\infty(\Omega)}$, the function $v' := \sum_{j=1}^J (u - t'_j)\chi_{P_j}$ belongs to $BV(\Omega) \cap L^\infty(\Omega)$ and satisfies* (5.24).

Proof. It is enough to repeat the proof of [9, Theorem 3.2] replacing c_3 by 1, $GSBV^p_M(\Omega;\mathbb{R}^m)$ by $\{u \in GBV_\star(\Omega) : (5.19) \text{ holds}\}$, $\mathcal{H}^{d-1}(J_u)$ by $\mathcal{H}^{d-1}(J^1_u)$, $\mathcal{H}^{d-1}(J_u \cup \partial\Omega)$ by $\mathcal{H}^{d-1}(J^1_u \cup \partial\Omega)$, $\|\nabla u\|_{L^1(\Omega)}$ by $\|\nabla u\|_{L^1(\Omega;\mathbb{R}^d)} + |D^c u|(\Omega) + \int_{J_u\setminus J^1_u} |[u]| d\mathcal{H}^{d-1}$, and [9, Lemma 3.5] by Lemma 5.2 above, obtaining that (5.21)-(5.23) hold.

To prove that $v \in BV(\Omega)$ it is enough to show that $(u-t_j)\chi_{P_j} \in BV(\Omega)$. We observe that for $m \geq C_{M,\theta,\Omega} + |t_j|$ we have $(u - t_j)\chi_{P_j} = (u^{(m)} - t_j)\chi_{P_j}$ by (5.23). Since $u^{(m)} - t_j$ and χ_{P_j} belong to $BV(\Omega) \cap L^\infty(\Omega)$, we conclude that $(u - t_j)\chi_{P_j} \in BV(\Omega)$.

It remains to prove (5.24). More precisely, we shall prove that

$$\int_\Omega g(x, \nabla v)dx \leq \int_\Omega g(x, \nabla u)dx + \int_R a dx\,, \tag{5.25}$$

$$|D^c v|(\Omega) \leq |D^c u|(\Omega)\,, \tag{5.26}$$

$$\int_{J_v\setminus\Gamma_0} |[v]| \wedge 1 d\mathcal{H}^{d-1} \leq \int_{J_u\setminus\Gamma_0} |[u]| \wedge 1 d\mathcal{H}^{d-1} + \theta C_{M,\Omega}\,, \tag{5.27}$$

which gives (5.24). Inequality (5.25) can be proved as in the proof of [9, formula (14)], while (5.27) can be proved as in the last part of [9, Theorem 3.2]. As for (5.26) we begin by observing that by (5.23) there exists a constant $m > 0$ such that $v = \sum_{j=1}^J (u^{(m)} - t_j)\chi_{P_j}$. By (2.11) and (2.23) we obtain

$$|D^c v|(\Omega) = \sum_{j=1}^J |D^c(u^{(m)} - t_j)|(P_j^{(1)}) \leq \sum_{j=1}^J |D^c u|(P_j^{(1)}) \leq |D^c u|(\Omega)\,,$$

which gives (5.26). □

The previous result can be extended to the case of functions satisfying prescribed boundary conditions in the usual BV sense considered in (4.17) and (4.28). To this aim we introduce a bounded open set $\Omega' \subset \mathbb{R}^d$ with Lipschitz boundary and containing $\overline{\Omega}$.

Corollary 5.4 (The case of boundary conditions). *Let $M > 0$ and $0 < \theta < 1$. Then there exist positive constants $C_{M,\Omega'}$ and $C_{M,\theta,\Omega'}$ with the following property: for each $h \in W^{1,1}(\Omega')$ with $\|\nabla h\|_{L^1(\Omega';\mathbb{R}^d)} \leq M$ and each $u \in GBV_\star(\Omega')$ with $u = h$ $\mathcal{L}^d$-a.e. on $\Omega' \setminus \overline{\Omega}$ and*

$$\int_{\Omega'} |\nabla u| dx + |D^c u|(\Omega') + \int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} \leq M \tag{5.28}$$

there exist a finite Caccioppoli partition $\Omega' = \bigcup_{j=1}^J P_j \cup R$ *and a finite family of translations* $(t_j)_{j=1}^J$ *such that the function*

$$v := h\chi_R + \sum_{j=1}^J (u - t_j)\chi_{P_j} \tag{5.29}$$

satisfies

$$v \in BV(\Omega')\,, \;\; v = h \;\mathcal{L}^d\text{-a.e. on } \Omega' \setminus \overline{\Omega}\,, \;\; v - h \in BV(\Omega') \cap L^\infty(\Omega')\,, \tag{5.30}$$

and the following estimates hold:

$$\|v - h\|_{L^\infty(\Omega')} \le C_{M,\theta,\Omega'}\,, \tag{5.31}$$

$$\mathcal{L}^d(R) \le \theta C_{M,\Omega'}\,, \tag{5.32}$$

$$\sum_{j=1}^J \mathcal{H}^{d-1}(\partial^* P_j) + \mathcal{H}^{d-1}(\partial^* R) \le C_{M,\Omega'}\,, \tag{5.33}$$

$$\mathcal{L}^d(P_j \cap (\Omega' \setminus \overline{\Omega})) > 0 \quad \text{for at most one index } j\,, \tag{5.34}$$

$$\mathcal{G}^g_{\Gamma_0}(v, \Omega') \le \mathcal{G}^g_{\Gamma_0}(u, \Omega') + \theta C_{M,\Omega'} + \|a\|_{L^1(R)} + c_2\|\nabla h\|_{L^1(R;\mathbb{R}^d)}, \tag{5.35}$$

for every Borel measurable $g\colon \Omega' \times \mathbb{R}^d \to \mathbb{R}$ *satisfying* (5.1) *on* $\Omega' \times \mathbb{R}^d$*, and for every Borel set* $\Gamma_0 \subset \Omega'$*, with* $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$*, where* $\mathcal{G}^g_{\Gamma_0}(u, \Omega')$ *is defined as in* (5.2)*, with* Ω *replaced by* Ω'*, and* $a = |a_1| + |a_2|$*. Moreover, we can choose* $(P_j)_{j=1}^J$ *and* $(t_j)_{j=1}^J$ *so that the following additional property holds: for every collection* $(t'_j)_{j=1}^J$*, with* $|t_j - t'_j| \le \theta^{-1}\|v - h\|_{L^\infty(\Omega')}$*, the function* $v' := h\chi_R + \sum_{j=1}^J (u - t'_j)\chi_{P_j}$ *belongs to* $BV(\Omega')$ *and satisfies* (5.35).

Proof. We apply Theorem 5.3 to the function $u - h$ on Ω', which belongs to $GBV_\star(\Omega')$ by Theorem 3.9. Arguing as in the proof of [9, Corollary 3.3] we obtain a finite Caccioppoli partition $\Omega' = \bigcup_{j=1}^J P_j \cup R$ and a finite family of translations $(t_j)_{j=1}^J \subset \mathbb{R}$ such that (5.32)-(5.34) hold, the function

$$z := \sum_{j=1}^J (u - h - t_j)\chi_{P_j}\,, \tag{5.36}$$

belongs to $BV(\Omega') \cap L^\infty(\Omega')$, and

$$\|z\|_{L^\infty(\Omega')} \le C_{M,\theta,\Omega'}\,. \tag{5.37}$$

Moreover, by (5.26) and (5.27), z satisfies

$$|D^c z|(\Omega') \le |D^c(u - h)|(\Omega') = |D^c u|(\Omega')\,, \tag{5.38}$$

$$\int_{J_z \setminus \Gamma_0} |[z]| \wedge 1 d\mathcal{H}^{d-1} \le \int_{J_{u-h} \setminus \Gamma_0} |[u - h]| \wedge 1 d\mathcal{H}^{d-1} + \theta C_{M,\Omega'}$$

$$= \int_{J_u \setminus \Gamma_0} |[u]| \wedge 1 d\mathcal{H}^{d-1} + \theta C_{M,\Omega'}\,. \tag{5.39}$$

Let $v := z + h$. Then $v \in BV(\Omega')$ and satisfies (5.29), (5.30), and (5.31).

It remains to prove (5.35). More precisely, we shall prove that

$$\int_{\Omega'} g(x,\nabla v)dx \leq \int_{\Omega'} g(x,\nabla u)dx + \int_R a dx + c_2 \int_R |\nabla h| dx\,, \tag{5.40}$$

$$|D^c v|(\Omega') \leq |D^c u|(\Omega')\,, \tag{5.41}$$

$$\int_{J_v\setminus\Gamma_0} |[v]| \wedge 1 d\mathcal{H}^{d-1} \leq \int_{J_u\setminus\Gamma_0} |[u]| \wedge 1 d\mathcal{H}^{d-1} + \theta C_{M,\Omega'}\,, \tag{5.42}$$

which gives (5.35).

Inequalities (5.41) and (5.42) follow from (5.38) and (5.39), respectively, since $D^c v = D^c z$, $J_v = J_z$, and $[z] = [v]$ (recall that $h \in W^{1,1}(\Omega')$).

To prove (5.40) we observe that $\nabla v = \nabla u$ $\mathcal{L}^d$-a.e. on $\Omega' \setminus R$, while $\nabla v = \nabla h$ $\mathcal{L}^d$-a.e. on R, so that

$$\begin{aligned}\int_{\Omega'} g(x,\nabla v)dx &= \int_{\Omega'\setminus R} g(x,\nabla u)dx + \int_R g(x,\nabla h)dx \\ &\leq \int_{\Omega'} g(x,\nabla u)dx + \int_R a dx + c_2 \int_R |\nabla h| dx\,,\end{aligned}$$

which concludes the proof. □

We are now in a position to prove the main result of this section.

Theorem 5.5 (Existence of modifications satisfying (3.27)). *Let $h \in W^{1,1}(\Omega')$, let $g\colon \Omega' \times \mathbb{R}^d \to \mathbb{R}$ be a Borel measurable function satisfying (5.1) on $\Omega' \times \mathbb{R}^d$, and let $\Gamma_0 \subset \Omega'$ be a Borel set with $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$. Let $(u_k)_k$ be a sequence in $GBV_\star(\Omega')$ with $u_k = h$ on $\Omega' \setminus \overline{\Omega}$. Assume that there exists $C > 0$ such that*

$$\mathcal{G}^g_{\Gamma_0}(u_k, \Omega') \leq C \quad \text{for every } k. \tag{5.43}$$

Then there exist a subsequence of $(u_k)_k$, not relabelled, modifications $y_k \in GBV_\star(\Omega')$ of u_k, with $y_k = h$ on $\Omega' \setminus \overline{\Omega}$, and a continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(t) \to +\infty$ as $t \to \infty$ such that

$$\mathcal{G}^g_{\Gamma_0}(y_k, \Omega') \leq \mathcal{G}^g_{\Gamma_0}(u_k, \Omega') + \tfrac{1}{k}\,, \tag{5.44}$$

$$\sup_k \int_{\Omega'} \psi(|y_k|) dx < +\infty\,. \tag{5.45}$$

Remark 5.6. By (5.44), if $(u_k)_k$ is a minimizing sequence for the functional $\mathcal{G}^g_{\Gamma_0}$ with $u_k = h$ $\mathcal{L}^d$-a.e. in $\Omega' \setminus \overline{\Omega}$, then the same is true for $(y_k)_k$. Inequalities (5.43) and (5.44), together with (5.1), imply that $(y_k)_k$ satisfies (3.26), while (5.45) guarantees that (3.27) also holds. Hence, by Theorem 3.11 there exists a subsequence of $(y_k)_k$, not relabelled, and a function $u \in GBV_\star(\Omega')$ such that $y_k \to u$ $\mathcal{L}^d$-a.e. in Ω'.

Proof of Theorem 5.5. We repeat the proof of [9, Theorem 3.8] with some modifications. By (5.43) we have

$$c_1 \int_{\Omega'} |\nabla u_k| dx + |D^c u_k|(\Omega') + \int_{J_{u_k}} |[u_k]| \wedge 1 d\mathcal{H}^{d-1} \leq M_1 \tag{5.46}$$

with $M_1 := C + \|a\|_{L^1(\Omega')} + \mathcal{H}^{d-1}(\Gamma_0)$, where $a = |a_1| + |a_2|$. We define $\theta_\ell := 2^{-\ell}$ and apply Corollary 5.4. Let us remark that, since we will pass to subsequences (not relabelled), we will eventually have only the inequality

$$\theta_\ell \leq 2^{-\ell}\,. \tag{5.47}$$

Step 1 (Application of Corollary 5.4*)* We apply Corollary 5.4 to the functions u_k and the boundary data h with parameters θ_ℓ and $M := \frac{M_1}{\min\{c_1,1\}}$. We find finite Caccioppoli partitions $\Omega' = \cup_{j\geq 1} P_j^{k,\ell} \cup R_k^\ell$ and piecewise translated functions $v_k^\ell \in BV(\Omega')$ defined by

$$v_k^\ell := h + \sum_{j\geq 1} (u_k - t_j^{k,\ell} - h)\chi_{P_j^{k,\ell}} = h\chi_{R_k^\ell} + \sum_{j\geq 1}(u_k - t_j^{k,\ell})\chi_{P_j^{k,\ell}}\,, \tag{5.48}$$

where $(t_j^{k,\ell})_{j\geq 1}$ are suitable finite families of translations. For notational convenience we shall also use the notation $P_0^{k,\ell} = R_k^\ell$ so that $(P_j^{k,\ell})_{j\geq 0}$ is a partition of Ω'. By Corollary 5.4 we have

$$v_k^\ell \in BV(\Omega')\,,\ v_k^\ell = h\ \mathcal{L}^d\text{-a.e. on } \Omega' \setminus \overline{\Omega}\,,\ v_k^\ell - h \in L^\infty(\Omega')\,, \tag{5.49}$$

$$\|v_k^\ell - h\|_{L^\infty(\Omega')} \leq C_{M,\theta_\ell,\Omega'}\,, \tag{5.50}$$

$$\mathcal{L}^d(R_k^\ell) \leq \theta_\ell C_{M,\Omega'}\,, \tag{5.51}$$

$$\textstyle\sum_{j\geq 0} \mathcal{H}^{d-1}(\partial^* P_j^{k,\ell}) \leq C_{M,\Omega'}\,, \tag{5.52}$$

$$\text{for every } k,\ell \text{ there is at most one } j \text{ with } \mathcal{L}^d(P_j^{k,\ell} \cap \Omega' \setminus \overline{\Omega}) > 0, \tag{5.53}$$

$$\mathcal{G}^g_{\Gamma_0}(v_k^\ell, \Omega') \leq \mathcal{G}^g_{\Gamma_0}(u_k, \Omega') + \theta_\ell C_{M,\Omega'} + \|a\|_{L^1(R_k^\ell)} + c_2\|\nabla h\|_{L^1(R_k^\ell;\mathbb{R}^d)}. \tag{5.54}$$

By (5.51) there exists a decreasing sequence η_ℓ converging to zero such that

$$\|a\|_{L^1(R_k^\ell)} + c_2\|\nabla h\|_{L^1(R_k^\ell;\mathbb{R}^d)} \leq \eta_\ell C_{M,\Omega'}\,,$$

which together with (5.54) gives

$$\mathcal{G}^g_{\Gamma_0}(v_k^\ell, \Omega') \leq \mathcal{G}^g_{\Gamma_0}(u_k, \Omega') + (\theta_\ell + \eta_\ell) C_{M,\Omega'} \quad \text{for every } k \text{ and } \ell. \tag{5.55}$$

For later use we recall that for every family $(\hat t_j^{k,\ell})_{j\geq 1}$, with $|t_j^{k,\ell} - \hat t_j^{k,\ell}| \leq \theta_\ell^{-1}\|v_k^\ell - h\|_{L^\infty(\Omega')}$, we have that the functions $\hat v_k^\ell := h\chi_{R_k^\ell} + \sum_{j\geq 1}(u_k - \hat t_j^{k,\ell})\chi_{P_j^{k,\ell}}$ belong to $BV(\Omega')$ and satisfy

$$\mathcal{G}^g_{\Gamma_0}(\hat v_k^\ell, \Omega') \leq \mathcal{G}^g_{\Gamma_0}(u_k, \Omega) + (\theta_\ell + \eta_\ell) C_{M,\Omega'}\,, \tag{5.56}$$

see (5.35).

Step 2 (Limiting objects for each ℓ) By (5.43), (5.50), and (5.55) we obtain that for every ℓ the sequence $(v_k^\ell)_k$ is bounded in $BV(\Omega')$. Indeed, arguing as in the proof of (5.46), by (5.43) and (5.55) we have that

$$c_1 \int_{\Omega'} |\nabla v_k^\ell| dx + |D^c v_k^\ell|(\Omega') + \int_{J_{v_k^\ell}} |[v_k^\ell]| \wedge 1 d\mathcal{H}^{d-1} \leq M_1 + (\theta_\ell + \eta_\ell) C_{M,\Omega'}\,.$$

This implies that $\|\nabla v_k^\ell\|_{L^1(\Omega';\mathbb{R}^d)}$, $|D^c v_k^\ell|(\Omega')$, and $\mathcal{H}^{d-1}(J^1_{v_k^\ell})$ are bounded. Since $[h] = 0$ we have $\int_{J_{v_k^\ell}} |[v_k^\ell]| d\mathcal{H}^{d-1} = \int_{J_{v_k^\ell}\setminus J^1_{v_k^\ell}} |[v_k^\ell]| \wedge 1 d\mathcal{H}^{d-1} + \int_{J^1_{v_k^\ell}} |[v_k^\ell -$

$h]|d\mathcal{H}^{d-1} \leq M_1 + (\theta_\ell + \eta_\ell)C_{M,\Omega'} + 2\mathcal{H}^{d-1}(J^1_{v^\ell_k})\|v^\ell_k - h\|_{L^\infty(\Omega')}$. By (5.50) we obtain that $\int_{J_{v^\ell_k}} |[v^\ell_k]| d\mathcal{H}^{d-1}$ is bounded with respect to k. Together with the previous bounds this implies that $|Dv^\ell_k|(\Omega')$ is bounded uniformly with respect to k. Since $v^\ell_k = h$ on $\Omega' \setminus \overline{\Omega}$, by the Poincaré inequality we deduce that $(v^\ell_k)_k$ is bounded in $BV(\Omega')$.

Using a diagonal argument we obtain a subsequence of $(k)_k$ (not relabelled) such that for every ℓ there exist a function $v^\ell \in BV(\Omega')$ and a constant $L_\ell \in [0, C_{M,\theta_\ell,\Omega'}]$ (see (5.50)) such that

$$v^\ell_k \to v^\ell \quad \text{in } L^1(\Omega') \quad \text{and} \quad \|v^\ell_k - h\|_{L^\infty(\Omega')} \to L_\ell\,. \tag{5.57}$$

By the semicontinuity of the L^∞-norm we obtain

$$\|v^\ell - h\|_{L^\infty(\Omega')} \leq L_\ell\,. \tag{5.58}$$

Arguing as in Step 2 of the proof of [9, Theorem 3.8] we find Caccioppoli partitions $(P^\ell_j)_{j\geq 0}$ and $(P_j)_{j\geq 0}$ such that after extracting (not relabelled) subsequences in ℓ and k, we get

$$\sum_{j\geq 0} \mathcal{L}^d(P^\ell_j \triangle P_j) \leq 2^{-\ell} \quad \text{and} \quad \sum_{j\geq 0} \mathcal{L}^d(P^{k,\ell}_j \triangle P^k_j) \leq 2^{-\ell} \quad \text{for all } k \geq \ell\,. \tag{5.59}$$

Step 3 (Conclusion of the proof) If $(L_\ell)_\ell$ does not tend to $+\infty$ as $\ell \to +\infty$, by (5.58) there exists a subsequence, not relabelled, such that $(v^\ell - h)_\ell$ is bounded in $L^\infty(\Omega')$. Then $(v^\ell)_\ell$ is bounded in $L^1(\Omega')$ and we can take $\psi(t) = t$ to obtain

$$\sup_\ell \int_{\Omega'} \psi(|v^\ell|)dx < +\infty\,. \tag{5.60}$$

The conclusion can now be obtained by repeating Step 5 of the proof of [9, Theorem 3.8] replacing $\hat{v}^\ell$, $\hat{v}^\ell_k$, and E_k by v^ℓ, v^ℓ_k, and $\mathcal{G}^g_{\Gamma_0}(\cdot, \Omega')$, respectively.

If $L_\ell \to +\infty$, passing to a subsequence, not relabelled, we may assume that $L_\ell < L_{\ell+1}$. By the definition of L_ℓ, for every ℓ we can find an increasing sequence $(k_\ell)_\ell$ such that $\|v^\ell_k - h\|_{L^\infty(\Omega')} < \|v^{\ell+1}_k - h\|_{L^\infty(\Omega')}$ for every ℓ and for every $k \geq k_\ell$. This allows us to follow the lines of Step 3 of the proof of [9, Theorem 3.8]. Namely we replace the translations $t^{k,\ell}_j$ by the translations $\hat{t}^{k,\ell}_j$ introduced in that paper and we consider the corresponding functions $\hat{v}^\ell_k$ defined as in (5.48) with $t^{k,\ell}_j$ replaced by $\hat{t}^{k,\ell}_j$. This construction leads to the fact that $\hat{v}^\ell_k$ satisfies (5.56) and

$$\|\hat{v}^\ell_k - h\|_{L^\infty(\Omega')} \leq 2\sum_{m=1}^{\ell} C_{M,\theta_m,\Omega'} \quad \text{for every } k \geq k_\ell\,. \tag{5.61}$$

Hence we can repeat the argument leading to (5.57) and we obtain a subsequence of $(k)_k$ (not relabelled) and, for every ℓ, a function $\hat{v}^\ell \in BV(\Omega')$ such that

$$\hat{v}^\ell_k \to \hat{v}^\ell \quad \text{in } L^1(\Omega')\,. \tag{5.62}$$

The conclusion can now be obtained by repeating Steps 4 and 5 in the proof of [9, Theorem 3.8] with E_k replaced by $\mathcal{G}^g_{\Gamma_0}(\cdot, \Omega')$. □

6. Existence result

In this section we shall prove that the minimum problem (4.28) has a solution. As observed at the end of Section 4, this will lead to the proof of the existence of a solution to problem (4.11).

Let Ω and Ω' be bounded open sets in $\mathbb{R}^d$ with Lipschitz boundary and with $\overline{\Omega} \subset \Omega'$, and let $f\colon \mathbb{R}^d \to \mathbb{R}$ be the function defined by (4.12). We begin by proving a lower semicontinuity result.

Theorem 6.1. *Let $\Phi \in L^1(\Omega';\mathbb{R}^d)$ and let $\Gamma_0 \subset \Omega'$ be a Borel set with $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$. Then the functional $\mathcal{F}^{\Phi}_{\Gamma_0}\colon GBV_\star(\Omega') \to [0,+\infty]$ defined by*

$$\mathcal{F}^{\Phi}_{\Gamma_0}(v) := \int_{\Omega'} f(\nabla v + \Phi)dx + |D^c v|(\Omega') + \int_{J_v \setminus \Gamma_0} |[v]| \wedge 1 d\mathcal{H}^{d-1}$$

is lower semicontinuous with respect to the convergence in measure on Ω'.

Proof. Let us fix a bounded open set $U \subset \mathbb{R}^d$. The first step in the proof is to show that the functional

$$\mathcal{F}_{0,U}(v) := \int_U f(\nabla v)dx + |D^c v|(U) + \int_{J_v \cap U} |[v]| \wedge 1 d\mathcal{H}^{d-1}$$

is lower semicontinuous on $GBV_\star(U)$ with respect to the convergence in measure on U. Let $(u_k)_k \subset GBV_\star(U)$ be a sequence converging in measure to some $u \in GBV_\star(U)$ such that $(\mathcal{F}_{0,U}(u_k))_k$ is bounded. For every $m > 0$ the sequence of truncations $(u_k^{(m)})_k$ converges to $u^{(m)}$ in measure on U and $\mathcal{F}_{0,U}(u_k^{(m)}) \le \mathcal{F}_{0,U}(u_k)$ by Proposition 2.6.

Let us show that $(u_k^{(m)})_k$ is bounded in $BV(U)$ by a constant depending on m. The first inequality in (4.14) implies that

$$\int_U |\nabla u_k^{(m)}|dx + |D^c u_k^{(m)}|(U) + \int_{J_{u_k^{(m)}}} |[u_k^{(m)}]| d\mathcal{H}^{d-1} \le c_m \mathcal{F}_{0,U}(u_k) + \tfrac{1}{2}\mathcal{L}^d(U)\,,$$

where $c_m := 1 + 2m$, hence the boundedness of $(u_k^{(m)})_k$ in $BV(U)$ follows from (2.5).

By [4, Theorem 2.1] we have

$$\mathcal{F}_{0,U}(u^{(m)}) \le \liminf_{k\to\infty} \mathcal{F}_{0,U}(u_k^{(m)}) \le \liminf_{k\to\infty} \mathcal{F}_{0,U}(u_k)\,. \tag{6.1}$$

Passing to the limit as $m \to +\infty$ and using Propositions 2.6 and 2.9 we obtain

$$\mathcal{F}_{0,U}(u) \le \liminf_{k\to\infty} \mathcal{F}_{0,U}(u_k)\,,$$

which concludes the proof of the lower semicontinuity of $\mathcal{F}_{0,U}$ on $GBV_\star(U)$ with respect to the convergence in measure on U.

Given $\xi \in \mathbb{R}^d$, for every $v \in GBV_\star(U)$ let

$$\mathcal{F}_{\xi,U}(v) := \int_U f(\nabla v + \xi)dx + |D^c v|(U) + \int_{J_v} |[v]| \wedge 1 d\mathcal{H}^{d-1}\,.$$

Since $\mathcal{F}_{\xi,U}(v) = \mathcal{F}_{0,U}(v+\ell_\xi)$, where $\ell_\xi(x) := \xi\cdot x$, we deduce that $\mathcal{F}_{\xi,U}$ is lower semicontinuous on $GBV_\star(U)$ with respect to the convergence in measure on U.

To prove a similar result for $\mathcal{F}^\Phi_{\Gamma_0}$ we fix a sequence $(v_k)_k \subset GBV_\star(\Omega')$ which converges in measure on Ω' to a function $v \in GBV_\star(\Omega')$, and an increasing sequence $(K_j)_j$ of compact subsets of Γ_0 such that $\mathcal{H}^{d-1}(\Gamma_0 \setminus K_j) \to 0$. It is easy to construct a sequence $(\Phi_j)_j$ of piecewise constant functions converging to Φ in $L^1(\Omega';\mathbb{R}^d)$ such that for every j there exists a partition $U^1_j, \dots, U^{i_j}_j, N_j$ of $\Omega' \setminus K_j$, with U^i_j open and $\mathcal{H}^{d-1}(N_j) < +\infty$, such that $\Phi_j = \xi^i_j$ in U^i_j for suitable constant vectors $\xi^i_j \in \mathbb{R}^d$. It is not restrictive to assume also that $\mathcal{H}^{d-1}(J_v \cap N_j) = 0$. By the previous step of the proof, for every j we have

$$\mathcal{F}^{\Phi_j}_{K_j}(v) = \sum_{i=1}^{i_j} \mathcal{F}_{\xi^i_j,U^i_j}(v) \le \sum_{i=1}^{i_j} \liminf_{k\to\infty} \mathcal{F}_{\xi^i_j,U^i_j}(v_k) \le \liminf_{k\to\infty} \mathcal{F}^{\Phi_j}_{K_j}(v_k)\,. \tag{6.2}$$

Since f is Lipschitz continuous with constant 1, for every $u \in GBV_\star(\Omega')$ we have

$$|\mathcal{F}^{\Phi_j}_{K_j}(u) - \mathcal{F}^{\Phi}_{\Gamma_0}(u)| \le \|\Phi_j - \Phi\|_{L^1(\Omega';\mathbb{R}^d)} + \mathcal{H}^{d-1}(\Gamma_0 \setminus K_j)\,,$$

hence

$$\mathcal{F}^{\Phi}_{\Gamma_0}(v) \le \liminf_{k\to\infty} \mathcal{F}^{\Phi}_{\Gamma_0}(v_k) + 2\big(\|\Phi_j - \Phi\|_{L^1(\Omega';\mathbb{R}^d)} + \mathcal{H}^{d-1}(\Gamma_0 \setminus K_j)\big)\,.$$

Passing to the limit as $j \to \infty$ we obtain the lower semicontinuity inequality along the sequence $(v_k)_k$. □

We are now ready to prove the existence of a solution to the minimum problem (4.28).

Theorem 6.2. *Let $w \in H^1(\Omega')$, let $\Phi \in L^1(\Omega';\mathbb{R}^d)$, and let $\Gamma_0 \subset \Omega'$ be a Borel set with $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$. Then the minimum problem*

$$\min_{\substack{v\in GBV_\star(\Omega')\\ v=w \text{ a.e. in } \Omega'\setminus\Omega}} \Big\{ \int_{\Omega'} f(\nabla v + \Phi)dx + |D^c v|(\Omega') + \int_{J_v\setminus\Gamma_0} |[v]| \wedge 1 d\mathcal{H}^{d-1} \Big\} \tag{6.3}$$

has a solution.

Proof. Since $\mathcal{F}^\Phi_{\Gamma_0}$ coincides with the functional $\mathcal{G}^g_{\Gamma_0}$ introduced in (5.2), with $g(x,\xi) := f(\xi + \Phi(x))$, and by (4.14) g satisfies (5.1), we can apply Theorem 5.5 and obtain that there exist a minimizing sequence $(u_k)_k \subset GBV_\star(\Omega')$, with $u_k = w$ $\mathcal{L}^d$-a.e. in $\Omega' \setminus \Omega$, and a continuous function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$, with $\psi(t) \to +\infty$ as $t \to +\infty$, such that (3.26) and (3.27) hold. Then by the Compactness Theorem 3.11 there exist a subsequence, not relabelled, and a function $u \in GBV_\star(\Omega')$ such that $u_k \to u$ $\mathcal{L}^d$-a.e. in Ω'. By the Semicontinuity

Theorem 6.1 we obtain that

$$\int_{\Omega'} f(\nabla u + \Phi)dx + |D^c u|(\Omega') + \int_{J_u\setminus\Gamma_0} |[u]| \wedge 1 d\mathcal{H}^{d-1}$$
$$\leq \liminf_k \Big(\int_{\Omega'} f(\nabla u_k + \Phi)dx + |D^c u_k|(\Omega') + \int_{J_{u_k}\setminus\Gamma_0} |[u_k]| \wedge 1 d\mathcal{H}^{d-1} \Big) .$$

Since $(u_k)_k$ is a minimizing sequence and $u = w$ $\mathcal{L}^d$- a.e. in $\Omega'\setminus\Omega$ we conclude that u is a solution of the minimum problem (6.3). □

We now show that the minimum problem (4.11) has a solution.

Corollary 6.3. *Let $\Gamma_0 \subset \overline{\Omega}$ be a Borel set with $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$, let w_0, $w \in H^1(\Omega)$, and let $(u_0, e_0, p_0) \in \mathcal{A}(\Gamma_0, w_0)$. Then the minimum problem*

$$\min_{\substack{\Gamma \text{ Borel}, \Gamma_0\subset\Gamma\subset\overline{\Omega} \\ (u,e,p)\in\mathcal{A}(\Gamma,w)}} \Big\{ \frac{1}{2} \int_\Omega |e|^2 dx + |p - p_0|(\overline{\Omega} \setminus \Gamma) + \mathcal{H}^{d-1}(\Gamma \setminus \Gamma_0) \Big\} \tag{6.4}$$

has a solution.

Proof. By the equivalence results proved in Lemmas 4.1 and 4.2, the conclusion follows from Theorem 6.2. □

We conclude the paper with two results which show that in general we cannot find a solution v to the minimum problem (6.3) with $v \in BV(\Omega')$. In the following proposition we show that this may happen even if $w = 0$.

Proposition 6.4. *Assume that $d \geq 2$. Then there exist a function $\Phi \in L^1(\Omega'; \mathbb{R}^d)$ and a Borel set $\Gamma_0 \subset \overline{\Omega}$, with $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$, such that the solution of the minimum problem* (6.3) *with $w = 0$ is unique, belongs to $L^1(\Omega')$, but does not belong to $BV(\Omega')$.*

Proof. It is not restrictive to assume that $0 \in \Omega$ and that Ω' is connected. Let $0 < R < 1$ be such that $[-R, R]^d \subset \Omega$, let $\psi, h : (0, R] \to [0, +\infty)$ be defined by

$$\psi(r) := \frac{1}{r} - \frac{1}{R} \quad \text{and} \quad h(r) := r^2 \quad \text{for every } r \in (0, R] , \tag{6.5}$$

and let $\Omega_{R,h} := \{(x_1, \dots, x_d) \in (-R, R)^d : 0 < x_1 < R,\ |x_2| < h(x_1)\}$.

For every $x = (x_1, \dots, x_d) \in \Omega'$ we define

$$v_0(x) := \begin{cases} \psi(x_1) & \text{if } x \in \Omega_{R,h} , \\ 0 & \text{otherwise} . \end{cases}$$

By (6.5) it is easy to see that $v_0 \in GBV_\star(\Omega') \cap L^1(\Omega')$, $v_0 = 0$ in $\Omega' \setminus \Omega$, $D^c v_0 = 0$, $\mathcal{H}^{d-1}(J_{v_0}) < +\infty$, and $J_{v_0} \supset \{x \in (-R, R)^d : 0 < x_1 < R,\ x_2 = \pm h(x_1)\}$. Since on this set $|[v_0](x)| = \psi(x_1)$, from (6.5) we deduce

$$\int_{J_{v_0}} |[v_0]| d\mathcal{H}^{d-1} = +\infty ,$$

which shows that $v_0 \notin BV(\Omega')$.

Let us define $\Phi := -\nabla v_0$ and $\Gamma_0 := J_{v_0}$. Then it is clear that v_0 is a solution of (6.3) since the value of the functional in v_0 is equal to zero. If v is another solution we must have

$$\int_{\Omega'} f(\nabla v - \nabla v_0)dx + |D^c v|(\Omega') + \int_{J_v \setminus J_{v_0}} |[v]| \wedge 1 d\mathcal{H}^{d-1} = 0\,.$$

This implies $\nabla v = \nabla v_0$ $\mathcal{L}^d$-a.e. in Ω', $D^c v = 0$ in Ω', and $\mathcal{H}^{d-1}(J_v \setminus J_{v_0}) = 0$. Therefore $v - v_0 \in W^{1,1}(\Omega' \setminus \overline{J}_{v_0})$, $\nabla(v - v_0) = 0$ $\mathcal{L}^d$-a.e. in Ω', and $v - v_0 = 0$ $\mathcal{L}^d$-a.e. in $\Omega' \setminus \Omega$. Since $\Omega' \setminus \overline{J}_{v_0}$ is connected, we conclude that $v = v_0$ $\mathcal{L}^d$-a.e. in Ω'. □

In the following proposition we consider the case $\Phi = 0$.

Proposition 6.5. *Assume $d \geq 2$. Then there exist a function $w \in H^1(\Omega')$ and a Borel set $\Gamma_0 \subset \overline{\Omega}$, with $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$, such that the solution of the minimum problem* (6.3) *with $\Phi = 0$ is unique, belongs to $L^1(\Omega')$, but does not belong to $BV(\Omega')$.*

Proof. It is not restrictive to assume that Ω' is connected. Since Ω has Lipschitz boundary, up to a change in the coordinate system, we may assume that there exist an open set $A \subset \mathbb{R}^{d-1}$, an interval $I \subset \mathbb{R}$, and a Lipschitz function $g\colon A \to I$ such that $\Omega \cap (A \times I) = \{(y, z) \in A \times I : z < g(y)\}$. It is not restrictive to assume that $\overline{A} \times \overline{I} \subset \Omega'$. We fix a nonempty open set $A' \subset\subset A$ and a sequence $(y_k)_{k \geq k_0} \subset A'$ such that the balls in $\mathbb{R}^{d-1}$ of centre y_k and radius $1/k^2$ are pairwise disjoint and contained in A'.

For every $y_0 \in \mathbb{R}^{d-1}$, $h > 0$, and $r > 0$ let

$$C_r^h(y_0) := \{(y, z) \in \mathbb{R}^{d-1} \times \mathbb{R} : |y - y_0| < r\,,\ |z - g(y_0)| < h\}$$

be the cylinder of centre $(y_0, g(y_0))$, height $2h$, and radius r. For every pair of open sets U and V, with $U \subset V \subset \mathbb{R}^d$, let

$$\mathrm{cap}(U, V) = \min\{\int_V |Du|^2 dx : u \in H_0^1(V), u = 1\ \mathcal{L}^d\text{-a.e. in } U\} \tag{6.6}$$

be the relative capacity of U in V.

Let $L > 0$ be the Lipschitz constant of g. We may assume that $C_{1/k^2}^{L/k^2}(y_k) \subset A' \times I$ for every $k \geq k_0$. Since we have $\mathrm{cap}(C_r^{Lr}(y_k), C_{1/k^2}^{L/k^2}(y_k)) \to 0+$ as $r \to 0+$, there exists r_k such that

$$0 < r_k < 1/k^2 \quad \text{and} \quad \mathrm{cap}(C_{r_k}^{Lr_k}(y_k), C_{1/k^2}^{L/k^2}(y_k)) < \frac{1}{k^{4d-2}}\,. \tag{6.7}$$

Let w_k be the solution of the minimum problem (6.6) with $U = C_{r_k}^{Lr_k}(y_k)$ and $V = C_{1/k^2}^{L/k^2}(y_k)$, extended to zero out of $C_{1/k^2}^{L/k^2}(y_k)$, and let

$$w := \sum_{k=k_0}^{\infty} k^{2d-3} w_k\,.$$

By (6.7) the series converges in $H^1_0(\Omega')$, hence $w \in H^1_0(\Omega')$. Moreover

$$w = k^{2d-3} \quad \mathcal{L}^d\text{-a.e. in } C^{Lr_k}_{r_k}(y_k)\,, \tag{6.8}$$

$$w = 0 \quad \mathcal{L}^d\text{-a.e. in } \Omega' \setminus (A' \times I)\,. \tag{6.9}$$

Let $\Gamma_1 := \partial\Omega \cap (\bar{A}' \times I) \setminus \bigcup_k C^{Lr_k}_{r_k}(y_k)$, $\Gamma_2 := \bigcup_k \partial C^{L/k^2}_{1/k^2}(y_k) \cap \Omega$, and $\Gamma_0 := \Gamma_1 \cup \Gamma_2$. Then

$$\mathcal{H}^{d-1}(\Gamma_0) \le \mathcal{H}^{d-1}(\partial\Omega) + \sum_k \mathcal{H}^{d-1}(\partial C^{L/k^2}_{1/k^2}(y_k))$$
$$\le \mathcal{H}^{d-1}(\partial\Omega) + \sum_k \omega_{d-1} 1/k^{2(d-1)} + 2\sum_k \sigma_{d-2} L/k^{2(d-2)+2} < +\infty\,,$$

where ω_{d-1} is the $(d-1)$-measure of the unit ball in $\mathbb{R}^{d-1}$ and σ_{d-2} is the $(d-2)$-measure of its boundary.

Let $v_0 \colon \Omega' \to \mathbb{R}$ be defined by $v_0 := w$ in $\Omega' \setminus \Omega$, $v_0 := k^{2d-3}$ in $C^{L/k^2}_{1/k^2}(y_k) \cap \Omega$ and $v_0 := 0$ in $\Omega \setminus \bigcup_k C^{L/k^2}_{1/k^2}(y_k)$. Since $\nabla v_0 = 0$ $\mathcal{L}^d$-a.e. in Ω, $\nabla v_0 = \nabla w$ $\mathcal{L}^d$-a.e. in $\Omega' \setminus \Omega$, $D^c v_0 = 0$ in Ω', $J_{v_0} \subset \Gamma_0$, and $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$, we have that $v_0 \in GBV_\star(\Omega')$. We observe that the functional in the minimum problem (6.3) with $\Phi = 0$ attains the value $\int_{\Omega' \setminus \Omega} f(\nabla w)dx$ at v_0, hence v_0 is a solution to this problem.

If v is another solution of (6.3) we must have $v = w$ in $\Omega' \setminus \Omega$ and

$$\int_\Omega f(\nabla v)dx + |D^c v|(\Omega') + \int_{J_v \setminus \Gamma_0} |[v]| \wedge 1 d\mathcal{H}^{d-1} = 0\,.$$

This implies $\nabla v = 0$ $\mathcal{L}^d$-a.e. in Ω, $D^c v = 0$ in Ω', and $\mathcal{H}^{d-1}(J_v \setminus \Gamma_0) = 0$. Hence $v \in W^{1,1}(\Omega' \setminus \Gamma_0)$. Since $\nabla v = \nabla v_0$ $\mathcal{L}^d$-a.e. in $\Omega' \setminus \Gamma_0$ and $\Omega' \setminus \Gamma_0$ is connected, from the equality $v = v_0$ $\mathcal{L}^d$-a.e. in $\Omega' \setminus \Omega$ we conclude that $v = v_0$ $\mathcal{L}^d$-a.e. in Ω'.

Since

$$\int_{\Omega'} |v_0|dx \le \int_{\Omega' \setminus \Omega} |w|dx + \sum_{k=k_0}^\infty k^{2d-3} \mathcal{L}^d(C^{L/k^2}_{1/k^2}(y_k))$$
$$\le \int_{\Omega' \setminus \Omega} |w|dx + 2\omega_{d-1} L \sum_{k=k_0}^\infty \left(\frac{1}{k^2}\right)^{d-1} \frac{1}{k^2} k^{2d-3} < +\infty\,,$$

we have that $v_0 \in L^1(\Omega')$.

To prove that $v_0 \notin BV(\Omega)$ we estimate the integral of the jump of v_0. Since $|[v_0]| = k^{2d-3}$ $\mathcal{H}^{d-1}$-a.e. on $\partial C^{L/k^2}_{1/k^2}(y_k) \cap \Omega$, and the base of this cylinder is contained in Ω, we have

$$\int_{J_{v_0}} |[v_0]| d\mathcal{H}^{d-1} \ge \omega_{d-1} \sum_{k=k_0}^\infty \left(\frac{1}{k^2}\right)^{d-1} k^{2d-3} = +\infty\,.$$

This shows that $v_0 \notin BV(\Omega')$, since for every $v \in BV(\Omega')$ we have $[v] \in L^1(J_v, \mathcal{H}^{d-1})$. $\square$

Remark 6.6. By the equivalence results proved in Lemmas 4.1 and 4.2, if Γ_0 and w are as in Proposition 6.5, then the minimum problem (6.4) corresponding to $w_0 = 0$ and $(u_0, e_0, p_0) = (0, 0, 0)$ has a unique solution (u, e, p) with $u \notin BV(\Omega)$.

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