Geometric Flows on Planar Lattices

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Presentation

The scope of these notes is to present a model case – but already complex enough – of motion in heterogeneous media. Even though this analysis will be performed in the relatively simplified setting of a periodic lattice, where the heterogeneous structure is somewhat built-in in the environment itself, this must be thought of as a case study for a large class of inhomogeneous media. In such a lattice setting we consider the simplest order parameter –obtained by labelling the nodes of the lattice with zeros or ones–, and define an energy that favours constant values of such an order parameter and penalizes the creation of 'discrete interfaces'. In such a way we expect an overall geometric motion driven by surface minimization such as mean-curvature flow. The very discrete structure of the environment provides an obstruction to this motion, with energy barriers that contrast this evolution and in our mind are a prototype of the effect of local minima in a general energy-driven motion in a heterogeneous structure. The seemingly unsolvable contrast between an overall tendency towards motion and a microscopic pinning by local minima can be overcome by resorting to a notion of 'homogenized motion' obtained with a balance between minimization of a scaled energy and a scaled dissipation at proper time and space scales, adapting the minimizing-movement approach that has led to a general approach to gradient flow type evolutions in the last twenty years.

The content of these notes stem from a course given by the first author at the Institute for Advanced Study, *Technische Universität München*, as part of the Summer School "Multiscale Phenomena in Geometry and Dynamics" organized by M. Cicalese and C. Kühn from July 22nd to 26th, 2019. The course was aimed at treating a variational approach to geometric flow on lattices on one hand as an example where we can tackle a simplified version of the general problem of evolution in heterogeneous media, referred to by De Giorgi as a "hard nut to crack", and on the other hand as a subject where the audience might get in touch with various advanced topics in modern Applied Analysis, such as homogenization, gradient flows on metric spaces, geometric evolution, Γ-convergence tools, applications of Geometric Measure Theory, properties of interfacial energies, etc. The present notes are a substantial enlargement and a completion of the content of the course, including more theoretical issues on minimizing movements, the detailed treatment of examples only hinted at in the course and of new ones, and some additional results on the convergence of lattice energies.

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