The relaxed energy of fractional Sobolev maps with values into the circle

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Abstract. We deal with the weak sequential density of smooth maps in the fractional Sobolev classes of $W^{s,p}$ maps in high dimension domains and with values into the circle. When s is lower than one, using interpolation theory we introduce a natural energy in terms of optimal extensions on suitable weighted Sobolev spaces. The relaxation problem is then discussed in terms of Cartesian currents. When $sp = 1$, the energy gap in the relaxed functional is always finite and is given by the minimal connection of the singularities times an energy weight, obtained through a minimum problem for one dimensional $W^{1/p,p}$ maps with degree one. When $sp > 1$, instead, concentration on codimension one sets needs unbounded energy. We finally treat the case where s is greater than one, obtaining an almost complete picture.

Keywords: fractional Sobolev spaces; weighted Sobolev spaces; relaxation; singularities; minimal connections; Cartesian currents.

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Introduction

In this paper we deal with the weak sequential density property of smooth maps in the fractional Sobolev class $W^{s,p}(B^n, \mathbb{S}^1)$, where B^n is the unit ball, $\mathbb{S}^1$ the unit circle, $s > 0$, and $1 < p < \infty$.

When $0 < s < 1$, the fractional Sobolev space $W^{s,p}(B^n)$ is given by the L^p-functions $u : B^n \to \mathbb{R}$ with finite fractional Gagliardo energy

$$|u|^p_{s,p} := \int_{B^n} \int_{B^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \, .$$

It is Banach space when equipped with the norm $\|u\|_{s,p} := \|u\|_{L^p} + |u|_{s,p}$. When $s > 1$ is not integer, denoting by m and σ the integer and fractional part of s, respectively, the space $W^{s,p}(B^n)$ is given by the Sobolev functions $u \in W^{m,p}(B^n)$ such that the Gagliardo seminorm $|D^m u|_{\sigma,p}$ is finite, where $D^m u$ is the tensor of the m-th order weak derivatives of u. It is again a Banach space when equipped with the norm $\|u\|_{s,p} := \|u\|_{W^{m,p}} + |D^m u|_{\sigma,p}$. Denoting by

$$\mathbb{S}^1 := \{y \in \mathbb{R}^2 \, : \, |y| = 1\}$$

the unit circle centered at the origin in the target space, we thus let

$$W^{s,p}(B^n, \mathbb{S}^1) := \{u \in W^{s,p}(B^n, \mathbb{R}^2) \, : \, |u(x)| = 1 \quad \text{for a.e. } x \in B^n\}$$

where $W^{s,p}(B^n, \mathbb{R}^2)$ is the space of functions $u : B^n \to \mathbb{R}^2$ with components in $W^{s,p}(B^n)$.

The problem of *strong density of smooth maps* $u : \bar{B}^n \to \mathbb{S}^1$ in $W^{s,p}(B^n, \mathbb{S}^1)$ is completely settled:

Theorem 0.1 *The class $C^\infty(\bar{B}^n, \mathbb{S}^1)$ is dense in $W^{s,p}(B^n, \mathbb{S}^1)$ in low dimension $n = 1$ for any s and p, and, when $n \geq 2$, if and only if $sp < 1$ or $sp \geq 2$.*

When $sp < 1$, the statement readily follows from the existence of a lifting in $W^{s,p}(B^n)$, see [5], through a density argument, see [6].

When $s \geq 1$ and $sp \geq 2$, a similar argument based on existence of suitable liftings, see [11], and on a convolution argument, see [10], applies.

When $sp > n$, the strong density of smooth maps follows from the embedding of $W^{s,p}(B^n)$ in the class $C^0(B^n)$, and when $sp = n$ (or in low dimension $n = 1$ for any s and p) from the embedding in the class VMO of functions with vanishing mean oscillation. In these cases, in fact, one may apply an argument that goes back to [29] and is based on an approximation by convolutions with a smooth kernel, followed by a projection onto $\mathbb{S}^1$. Therefore, such an argument works for more general target manifolds $\mathcal{Y}$ through the Nash embedding theorem.

When $0 < s < 1$ and $2 \le sp < n$, the strong density theorem is proved by Brezis-Mironescu in [12], and it is essentially based on two facts. Firstly, the authors introduce the class $\mathcal{R}_{s,p}(B^n, \mathbb{S}^1)$ of maps u in $W^{s,p}(B^n, \mathbb{S}^1)$ which are smooth (continuous) outside a singular set Σ_u given by a finite union of $(n - [sp] - 1)$-manifolds of B^n, where $[sp]$ denotes the integer part of sp. For example, Σ_u is a finite set of points when $n = 2$ and $sp = 1$. Extending results that go back to [4, 3] for Sobolev maps in $W^{1,p}(B^n, \mathcal{Y})$, they in fact prove:

Theorem 0.2 *Every map in $W^{s,p}(B^n, \mathbb{S}^1)$ is the strong limit of a sequence in $\mathcal{R}_{s,p}(B^n, \mathbb{S}^1)$.*

Secondly, since the high order homotopy groups of $\mathbb{S}^1$ are all trivial, $\pi_i(\mathbb{S}^1) \simeq 0$ for $i \ge 2$ integer, they are able to remove the singular set Σ_u of maps in $\mathcal{R}_{s,p}$, when $[sp] \ge 2$.

The first homotopy group $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$ being non-trivial, counterexamples to the strong density of smooth maps exist when $n \ge 2$ and the integer part of sp is equal to one.

Example 0.3 If $n = 2$, the map $u(x) = x/|x|$ belongs to $W^{s,p}(B^2, \mathbb{S}^1)$ if $1 \le sp < 2$, and u has degree one around the origin. On the other hand, smooth maps in $C^\infty(B^2, \mathbb{S}^1)$ have degree zero around the origin, and the degree is continuous w.r.t. the strong convergence in $W^{s,p}$, when $1 \le sp < 2$. Therefore, the map u cannot be the strong $W^{s,p}$ limit of a sequence $\{u_h\} \subset C^\infty(B^2, \mathbb{S}^1)$. If $n \ge 3$, a similar counterexample is given by the map $u(x) = (x_1, x_2)/|(x_1, x_2)|$, see e.g. [12].

Denoting by $W^{s,p}_S(B^n, \mathbb{S}^1)$ the *strong closure of $C^\infty(B^n, \mathbb{S}^1)$ in the $W^{s,p}$-norm*, and assuming $n \ge 2$, we thus have:

$$W^{s,p}_S(B^n, \mathbb{S}^1) = W^{s,p}(B^n, \mathbb{S}^1) \quad \iff \quad sp < 1 \quad \text{or} \quad sp \ge 2 \tag{0.1}$$

whereas

$$W^{s,p}_S(B^n, \mathbb{S}^1) \subsetneq W^{s,p}(B^n, \mathbb{S}^1) \quad \iff \quad 1 \le sp < 2 \quad \text{and} \quad n \ge 2\,.$$

However, in [13] it is defined a *distributional Jacobian* $\mathbb{J}u$ of maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ that characterizes the obstruction to the strong approximation by smooth maps: namely, the strong closure of $C^\infty(B^n, \mathbb{S}^1)$ in the $W^{s,p}$-norm agrees with the class of maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ such that $\mathbb{J}u = 0$.

For $1 \le sp < 2$ and $n \ge 2$, one may ask whether the *weak sequential density of smooth maps* holds true in the whole class $W^{s,p}(B^n, \mathbb{S}^1)$. This is false when $1 < sp < 2$. In that case, in fact, there exist maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ with non-zero Jacobian $\mathbb{J}u$, such that for any smooth sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ with $u_h \to u$ a.e. (or in $L^p(B^n, \mathbb{R}^2)$) one has $\sup_h \|u_h\|_{s,p} = \infty$, see [8].

In the relevant case $sp = 1$, the latter problem is completely settled for $s = 1/2$ and $p = 2$, i.e. in the class $W^{1/2,2}(B^n, \mathbb{S}^1)$. In that framework, in fact, using that functions in $W^{1/2,2}(B^n)$ identify the traces of the Sobolev space $W^{1,2}(B^n \times (0, +\infty))$, the distributional Jacobian $\mathbb{J}u$ was defined by Hang-Lin [25], and actually in any dimension $n \ge 2$ it can be written by means of homological arguments, see [23].

In this paper, we deal with the problem of weak sequential density of smooth maps in any dimension $n \ge 2$. When $0 < s < 1$, we introduce on the class $W^{s,p}(B^n, \mathbb{S}^1)$ a natural energy $u \mapsto \mathcal{E}_{s,p}(u)$, see (0.5) below. In the case $s = 1/2$ and $p = 2$ already considered in [18, 23], the energy $\mathcal{E}_{1/2,2}(u)$ of a map $u \in W^{1/2,2}(B^n, \mathbb{S}^1)$ is given by the Dirichlet integral $\int |DU|^2\, dx\, dt$ of the harmonic extension $U : B^n \times (0, 1) \to \mathbb{R}^2$ of u. We then analyze the corresponding *relaxed energy* (0.6). Energy concentration only occurs on codimension one sets and with a finite amount of energy, when $sp = 1$. When $sp > 1$, instead, energy blows up. Finally, when $s > 1$, the problem of weak sequential density of smooth maps is partially solved.

In order to state our results, we introduce some more notation.

THE ENERGY. For $\gamma \in \mathbb{R}$ and $p > 1$, we denote by $W^{1,p}_\gamma(B^n \times (0, +\infty))$ the weighted Sobolev space given by the functions $U \in L^p((B^n \times (0, +\infty))$ which are approximately differentiable a.e. and with approximate gradient DU a measurable function satisfying

$$\int_{\Omega\times(0,+\infty)} t^\gamma |DU(x,t)|^p\,dx\,dt<\infty\,,\qquad \Omega=B^n\,. \tag{0.2}$$

By interpolation theory, see e.g. [27], it turns out that when $0<s<1$, the fractional Sobolev space $W^{s,p}(B^n)$ agrees with the Besov space $B^s_{p,p}(B^n)$, for any $p>1$, and hence with the class of traces $u(x)=U(x,0)$ on $t=0$ of functions U in $W^{1,p}_\gamma(B^n\times(0,+\infty))$, where

$$\gamma=\gamma(s,p):=p(1-s)-1\,,\quad p>1\,,\quad 0<s<1\,. \tag{0.3}$$

Notice that when $s=1-1/p$, one has $\gamma=0$ and $W^{1-1/p,p}(B^n)$ agrees with the class of traces on $t=0$ of the Sobolev space $W^{1,p}(B^n\times(0,+\infty))$. By the previous discussion, a particular case of our interest is when $sp=1$, so that $\gamma=p-2$. In that case, since the fractional Gagliardo energy becomes

$$|u|^p_{1/p,p}=\int_{B^n}\int_{B^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+1}}\,dx\,dy$$

it turns out that in any dimension n the class of bounded functions in $W^{1/p,p}(B^n)$ is continuously embedded in $W^{1/q,q}(B^n)$ for each $1<p<q$. Moreover, in low dimension $n=2$, when $\Omega=\mathbb{R}^2$ and $sp=1$, so that $\gamma=p-2$, the energy (0.2) is scale invariant for any $p>1$.

Denote now by $\mathbf{C}^{n+1}$ the $(n+1)$-dimensional cylinder

$$\mathbf{C}^{n+1}:=B^n\times(0,1)$$

by $W^{1,p}_\gamma(\mathbf{C}^{n+1},\mathbb{R}^2)$ the class of functions $U=(U^1,U^2):\mathbf{C}^{n+1}\to\mathbb{R}^2$ with components U^j in $W^{1,p}_\gamma(\mathbf{C}^{n+1})$, and consider for $0<s<1$ and $p>1$ the *energy*

$$\mathcal{E}^p_{\gamma(s,p)}(U):=\int_{\mathbf{C}^{n+1}} t^{\gamma(s,p)}|DU(x,t)|^p\,dx\,dt\,,\qquad \gamma(s,p):=p(1-s)-1\,. \tag{0.4}$$

For any bounded function $u\in W^{s,p}(B^n,\mathbb{R}^2)\cap L^\infty$, where $0<s<1$ and $p>1$, we let

$$U:=\mathrm{Ext}(u)$$

denote a bounded function that *minimizes the energy* $\mathcal{E}^p_{\gamma(s,p)}(U)$ *among all* $U\in W^{1,p}_{\gamma(s,p)}(\mathbf{C}^{n+1},\mathbb{R}^2)\cap L^\infty$ *such that* $U(x,0)=u(x)$ *on* $B^n\times\{0\}$ *in the sense of the traces.*

Such a minimizer exists and is smooth inside $\mathbf{C}^{n+1}$, by the convexity of the functional $U\mapsto\mathcal{E}^p_{\gamma(s,p)}(U)$. Moreover, if $u\in W^{s,p}(B^n,\mathbb{S}^1)$, by a projection argument we may assume $\mathrm{Ext}(u):\mathbf{C}^{n+1}\to\mathbb{D}^2$, where

$$\mathbb{D}^2:=\{y\in\mathbb{R}^2\ :\ |y|\le 1\}$$

is the unit disk in the target space. In addition, see [27], if $\{u_h\}\subset W^{s,p}(B^n,\mathbb{R}^2)\cap L^\infty$ is a sequence converging a.e. in B^n to a function $u\in W^{s,p}(B^n,\mathbb{R}^2)\cap L^\infty$, it turns out that *the strong convergence* $u_h\to u$ *in* $W^{s,p}(B^n,\mathbb{R}^2)$ *is equivalent to the convergence* $u_h\to u$ *in* $L^p(B^n,\mathbb{R}^2)$ *joined with the energy convergence*

$$\lim_{h\to\infty}\mathcal{E}^p_{\gamma(s,p)}(\mathrm{Ext}(u_h))=\mathcal{E}^p_{\gamma(s,p)}(\mathrm{Ext}(u))\,.$$

When $0<s<1$, it is then natural to introduce on the class of maps $u\in W^{s,p}(B^n,\mathbb{S}^1)$ the $W^{s,p}$*-energy*

$$\mathcal{E}_{s,p}(u):=\mathcal{E}^p_{\gamma(s,p)}(U)\,,\quad U=\mathrm{Ext}\,u:\mathbf{C}^{n+1}\to\mathbb{D}^2 \tag{0.5}$$

where $\mathcal{E}^p_{\gamma(s,p)}(U)$ is given by (0.4). In the same spirit as for Lebesgue's relaxed area, we correspondingly introduce the *relaxed energy*

$$\widetilde{\mathcal{E}}_{s,p}(u):=\inf\Big\{\liminf_{h\to\infty}\mathcal{E}_{s,p}(u_h)\mid\{u_h\}\subset C^\infty(B^n,\mathbb{S}^1),\ u_h\to u\text{ strongly in }L^p(B^n,\mathbb{R}^2)\Big\}\,. \tag{0.6}$$

On account of (0.1), the *energy gap*

$$\mathcal{G}_{s,p}(u) := \widetilde{\mathcal{E}}_{s,p}(u) - \mathcal{E}_{s,p}(u) \tag{0.7}$$

can be non-zero only if $1 \leq sp < 2$ and $n \geq 2$. In the particular case where $sp > 1$, moreover, the cited results from [13] yield the existence of maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ with non-zero distributional Jacobian $\mathbb{J}u$ and for which $\mathcal{G}_{s,p}(u) = +\infty$.

Main new results. We first deal with the case where $sp = 1$. For maps in $W^{1/2,2}(B^n, \mathbb{S}^1)$, the explicit formula for the relaxed energy was obtained in [23], using tools from Geometric Measure Theory. In this paper, we show that *the energy gap* $\mathcal{G}_{1/p,p}(u)$ *in* (0.7) *is always finite* in the class $W^{1/p,p}(B^n, \mathbb{S}^1)$, for any $p > 1$.

In order to find the explicit formula for the energy gap, we rewrite the distributional Jacobian $\mathbb{J}u$ from [13], that carries the relevant information on the *singularities* of u, in terms of an $(n-2)$-dimensional current $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$, see Definition 3.1. For any $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$, we in fact obtain:

$$\mathcal{G}_{1/p,p}(u) = 0 \qquad \Longleftrightarrow \qquad \mathbb{P}(u) = 0\,.$$

Notice that in low dimension $n = 2$, the distribution $\mathbb{P}(u)$ agrees with the extension of the distributional determinant $T : W^{1/p,p}(B^2, \mathbb{S}^1) \to W^{1,\infty}(B^2, \mathbb{R})^*$ by Bourgain-Brezis-Mironescu [7].

In order to prove the equivalence in the previous centered formula, we first show that for every u in $W^{1/p,p}(B^n, \mathbb{S}^1)$ the current $\mathbb{P}(u)$ is an *integral flat chain*, i.e., that there exists an integer multiplicity (say i.m.) rectifiable current $L \in \mathcal{R}_{n-1}(B^n)$ satisfying $(\partial L) \llcorner B^n = \mathbb{P}(u)$.

Such a property holds true when $p = 2$ as a consequence of the *coarea formula* first considered by Almgren-Browder-Lieb [1], see [23]. In fact, extensions of maps in $W^{1/2,2}(B^n, \mathbb{S}^1)$ belong to the Sobolev space $W^{1,2}(\mathbf{C}^{n+1}, \mathbb{R}^2)$.

Therefore, when $1 < p < 2$, due to the continuous embedding $W^{1/p,p}(B^n, \mathbb{S}^1) \subset W^{1/2,2}(B^n, \mathbb{S}^1)$ it turns out that the integral flat chain $\mathbb{P}(u)$ is automatically defined for all maps $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$.

This direct argument fails to hold if $p > 2$, since in that case there are maps $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ such that $\|u\|_{1/2,2} = \infty$. Notwithstanding, in this paper we obtain the validity of the coarea formula by making use of some relevant estimates due to Bourgain-Brezis-Mironescu [7].

In addition, when $\mathbb{P}(u) \neq 0$, we are able to express the energy gap (0.7) through the *minimal connection* $m_{i,B^n}(\mathbb{P}(u))$ of the singularity $\mathbb{P}(u)$, times a positive weight E_p, namely:

$$\mathcal{G}_{1/p,p}(u) = E_p \cdot m_{i,B^n}(\mathbb{P}(u)) < \infty \qquad \forall\, u \in W^{1/p,p}(B^n, \mathbb{S}^1)\,, \quad \forall\, n \geq 2\,,\ p > 1\,. \tag{0.8}$$

In the latter formula, the term $m_{i,B^n}(\mathbb{P}(u))$ denotes the mass $\mathbf{M}(L)$ of the minimal i.m. rectifiable current $L \in \mathcal{R}_{n-1}(B^n)$ satisfying $(\partial L) \llcorner B^n = \mathbb{P}(u)$. If e.g. $n = 2$ and $u \in R_{1/p,p}(B^2, \mathbb{S}^1)$, letting $\Sigma_u = \{x_1, \ldots, x_m\}$ and denoting by $d_i \in \mathbb{Z}$ the degree of u at x_i, the minimal connection of the singular set Σ_u is the mass-minimizing current $L \in \mathcal{R}_1(B^2)$ satisfying $(\partial L) \llcorner B^2 = \sum_{i=1}^m d_i \cdot \delta_{x_i}$, where δ_x is the unit Dirac mass at the point $x \in B^2$.

Moreover, the positive constant E_p in formula (0.8) is given by the *energy minimum of degree one maps* in $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$. When $p = 2$, in [23, 28] we in fact obtained that $E_2 = 2\pi$, that is equal to the Dirichlet energy $\int |DU|^2$ of the harmonic map from the half-space to $\mathbb{R}^2$ whose trace agrees with the inverse of the stereographic map from $\mathbb{S}^1$ onto $\mathbb{R}$, see Appendix A.

Content of the paper. In Sec. 1, we use some estimates taken from [7] to find a suitable extension of maps in $W^{1/p,p}(B^n, \mathbb{S}^1)$ with finite mapping area, Proposition 1.1, which yields to the validity of the coarea formula for any $p > 1$, Theorem 1.2.

In Sec. 2, we recall some notation from Geometric Measure Theory and introduce the class of Cartesian currents $\mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, for which we establish a closure-compactness property, Theorem 2.5. Roughly speaking, a current T in $\mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ takes the form:

$$T = G_u + L \times [\![\mathbb{S}^1]\!] \tag{0.9}$$

where G_u is the *current carried by the graph* of some map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ and $L \in \mathcal{R}_{n-1}(B^n)$ is an i.m. rectifiable current in such a way that $\partial T = 0$ on compactly supported smooth $(n-1)$-forms in

$B^n \times \mathbb{S}^1$. In low dimension $n = 1$, we then discuss a notion of degree in our context, by extending the definition given in [7] for maps $g \in W^{1/p,p}(\mathbb{S}^1, \mathbb{S}^1)$.

In Sec. 3, we introduce the $(n-2)$-current of the singularities $\mathbb{P}(u)$, showing that it is an integral flat chain, Proposition 3.7. For this purpose, we take advantage of Theorem 0.2 by Brezis-Mironescu [12]. Notice that when $n \geq 2$, for a current T in $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ as in (0.9), the null boundary condition of T is equivalent to the equality $(\partial L) \llcorner B^n = -\mathbb{P}(u)$, whereas it is automatically verified when $n = 1$.

In Sec. 4, we deal with the factor E_p in the energy gap formula (0.8), showing that for any $p > 1$ and any fixed integer $d \in \mathbb{Z}$, the energy minimum among all maps in $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ with degree d is equal to $|d| \cdot E_p$, Proposition 4.3.

In Sec. 5, we introduce a suitable functional $T \mapsto E_{1/p,p}(T)$ on Cartesian currents, see (5.1), that agrees with the energy $\mathcal{E}_{1/p,p}(u)$ in case of graphs of "smooth" maps $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$, see (0.5). Our functional turns out to be lower semicontinuous along weakly converging sequences of smooth graphs, Theorem 5.1.

In Sec. 6, we provide in low dimension $n = 2$ the approximation of dipoles for $W^{1/p,p}$-maps with values in $\mathbb{S}^1$, Theorem 6.4. Our Proposition 6.5 is in accordance with the case $N = 1$ of [7, Thm. 2.4], where the authors analyzed the dipole problem for maps in $W^{N/p,p}(\mathbb{S}^{N+1}, \mathbb{S}^N)$. We also show how to remove *homologically trivial* point singularities, Proposition 6.1.

In Sec. 7, we prove for any $p > 1$ and $n \geq 2$ a strong density result for our class of Cartesian currents. Namely, in Theorem 7.1 we show that *for every T in $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ we can find a smooth sequence $\{u_h\}$ in $C^\infty(B^n, \mathbb{S}^1)$ such that $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and $\mathcal{E}_{1/p,p}(u_h) \to E_{1/p,p}(T)$ as $h \to \infty$.*

We briefly sketch here its proof. On account of Proposition 3.7 and of Federer's strong polyhedral approximation theorem [14], we are able to reduce to the case where the current T takes the form (0.9) for some map $u \in \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$, where the current $L \in \mathcal{R}_{n-1}(B^n)$ is a finite sum of pairwise disjoint oriented polyhedral $(n-1)$-chains. In order to approximate the $(n-1)$-dimensional "dipoles", in high dimension $n \geq 3$ we apply Proposition 7.3, that is proved in Appendix B. Finally, in order to remove the $(n-2)$-dimensional singular set, in high dimension $n \geq 3$ we apply Proposition 7.4, that is proved in Appendix C. In the easier case where $n = 2$, we directly apply Theorem 6.4 and Proposition 6.1.

In Sec. 8, we first we collect the closure-compactness properties for the class of Cartesian currents $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, Theorem 8.1, extending results proved in [23] when $p = 2$.

We then prove the explicit formula (0.8) for the energy gap (0.7). The proof of Theorem 8.2 is based on our main results previously stated: Proposition 3.7 and Theorems 2.5, 5.1, and 7.1. Notice that formula (0.8) implies that *every map in $W^{1/p,p}(B^n, \mathbb{S}^1)$ belongs to the $W^{1/p,p}$-weak sequential closure of smooth maps in $C^\infty(B^n, \mathbb{S}^1)$.*

When $n \geq 2$, $0 < s < 1$, $p > 1$, and $1 < sp < 2$, in Theorem 8.4 we then obtain:

$$\widetilde{\mathcal{E}}_{s,p}(u) = \begin{cases} \mathcal{E}_{s,p}(u) & \text{if} \quad \mathbb{P}(u) = 0 \\ +\infty & \text{if} \quad \mathbb{P}(u) \neq 0 \end{cases} \qquad \forall\, u \in W^{s,p}(B^n, \mathbb{S}^1)\,.$$

It remains to consider the ranges of s and p for which the strong density of smooth maps fails to hold, see Theorem 0.1, but $s > 1$. In Corollary 8.6, we give a partial result: if $1 < p < 2$, $1 < s < 2$, $1 < sp < 2$, $n \geq 2$, and $u \in W^{s,p}(B^n, \mathbb{S}^1)$ is such that there exists a sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ converging a.e. to u and with equibounded $W^{s,p}$-norms, $\sup_h \|u_h\|_{s,p} < \infty$, then one has $\mathbb{P}(u) = 0$.

Notice that when $s > 1$, our definition of energy (0.5) does not make sense. A possible way to prove the converse implication to Corollary 8.6 is proposed in Remark 8.7 as an Open Question.

For the sake of brevity, the case of maps with *prescribed boundary values* is not treated in this paper: it can be readily discussed by making straightforward modifications as e.g. in [23]. Finally, in the case $s = 1$, the relaxation problem of Sobolev maps in $W^{1,p}(B^n, \mathbb{S}^1)$ is treated in [24].

1 Coarea formula

In this section, we find a coarea formula for a suitable extension of maps in $W^{1/p,p}(B^n, \mathbb{S}^1)$. To this purpose, we make use of some relevant estimates obtained by Bourgain-Brezis-Mironescu in [7].

A RELEVANT ESTIMATE. Let $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ and $U \in W^{1,p}_{p-2}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ the harmonic extension of u to $\mathbf{C}^{n+1} := B^n \times (0,1)$, where $n \geq 1$ is integer and $p > 1$ real. Following [7, Lemma 1.2], we denote

$$G := \{(x,t) \in \mathbf{C}^{n+1} \,:\, |U(x,t)| \leq 1/2\}$$

and we let $d : B^n \to]0,1/2]$ the function such that $d(x) := 1/2$ if $|U(x,t)| \geq 1/2$ for each $t \in (0,1/2)$, and

$$d(x) := \min\{t \in (0,1/2) \,:\, |U(x,t)| \leq 1/2\}$$

otherwise. Using that $|DU(x,t)| \leq c/t$ for some absolute constant c, for any exponent $\alpha > 1$ one has

$$\int_G |DU(x,t)|^\alpha \, dx\, dt \leq c \int_{B^n} \Big(\int_{d(x)}^1 t^{-\alpha}\, dt \Big)\, dx \leq C \int_{B^n} \frac{1}{d(x)^{\alpha-1}}\, dx\,.$$

In a similar way to the case $\alpha = 2$, using that $t > d(x)$ if $(x,t) \in G$, for each $p > 1$ we estimate

$$\int_G t^{p-2} |DU(x,t)|^p \, dx\, dt \leq \int_G \frac{C}{t^2}\, dx\, dt \leq C \int_{B^n} \frac{1}{d(x)}\, dx \tag{1.1}$$

where $C = C(n,p)$. Moreover, as in [7, Lemma 1.3], since $U \in W^{2/p,p}(B^n \times I, \mathbb{R}^2)$, where $I = (0,1/2)$, using the embedding of $W^{2/p,p}(I)$ in the Hölder class $C^{0,1/p}(I)$, it turns out that for a.e. $x \in B^n$ the function $\varphi_x(t) := U(x,t)$ belongs to $W^{2/p,p}(I, \mathbb{R}^2)$, whence to $C^{0,1/p}(I, \mathbb{R}^2)$, so that we have:

$$\frac{1}{2} \leq |\varphi_x(d(x)) - \varphi_x(0)| \leq C\, d(x)^{1/p} \|\varphi_x\|_{C^{0,1/p}(I)} \leq C\, d(x)^{1/p}\, \|\varphi_x\|_{W^{2/p,p}(I,\mathbb{R}^2)}$$

and hence

$$\frac{1}{d(x)} \leq C\, \|\varphi_x\|^p_{W^{2/p,p}(I)}\,.$$

Therefore, using the inequality on Besov-type spaces

$$\int_{B^n} \|\varphi_x\|^p_{W^{2/p,p}(I,\mathbb{R}^2)}\, dx = \int_{B^n} \|U(x,\cdot)\|^p_{W^{2/p,p}(I,\mathbb{R}^2)}\, dx \leq C\, \|U\|^p_{W^{2/p,p}(\mathbf{C}^{n+1},\mathbb{R}^2)} \leq C\, \|u\|^p_{W^{1/p,p}(B^n,\mathbb{R}^2)}$$

by (1.1) one gets the estimate

$$\int_G t^{p-2} |DU(x,t)|^p \, dx\, dt \leq C_1 \int_{B^n} \frac{1}{d(x)}\, dx \leq C_2 \int_{\mathbf{C}^{n+1}} t^{p-2} |DU(x,t)|^p \, dx\, dt \tag{1.2}$$

for some positive constants C_1, C_2 only depending on n and p.

In the sequel, we choose a smooth function $\Phi : \mathbb{R}^2 \to \mathbb{D}^2$ such that $\Phi(y) = y/|y|$ if $|y| \geq 1/2$, where $y = (y_1, y_2)$, and Φ is a bi-Lipschitz map from $\{y \,:\, |y| \leq 1/2\}$ to $\mathbb{D}^2$.

Setting $V := \Phi \circ U$, we clearly have:

$$|DV(x,t)| \leq C_1\, |DU(x,t)| \quad \forall\, (x,t) \in \mathbf{C}^{n+1}\,, \quad |DU(x,t)| \leq C_2\, |DV(x,t)| \qquad \forall\, (x,t) \in G\,. \tag{1.3}$$

Denote now by $V^\#(dy^1 \wedge dy^2)$ the 2-form in $\mathbf{C}^{n+1}$ given by the pull-back by V of the 2-form $dy^1 \wedge dy^2$. One has

$$|V^\#(dy^1 \wedge dy^2)| = J_V \tag{1.4}$$

where J_V is the Jacobian of the map V, so that $J_V(x,t)^2$ is the sum of all the 2×2 minors of the gradient matrix $DV(x,t)$. Therefore, by the area formula one has $J_V(x,t) = 0$ if $(x,t) \in G$ whereas by the parallelogram inequality one gets the general estimate $J_V(x,t) \leq C_n\, |DV(x,t)|^2$.

These are the main facts that led Bourgain-Brezis-Mironescu [7] to obtain the estimate

$$|\deg g| \leq C_p\, \|g\|^p_{1/p,p} \qquad \forall\, p > 1$$

on the degree $\deg g$ of maps $g \in W^{1/p,p}(\mathbb{S}^1, \mathbb{S}^1)$. In dimension $n = 2$, they also build up for any $p > 1$ the unique extension $T : W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1) \to W^{1,\infty}(\mathbb{S}^2, \mathbb{R})^*$ of the *distributional determinant* $T(g) = \mathrm{Det}(\nabla g)$ of maps $g \in W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1) \cap W^{1,2}$, obtaining the estimate

$$|\langle T(g), \zeta\rangle| \leq C_p\, \|g\|^p_{1/p,p} \cdot \|\nabla\zeta\|_{L^\infty} \qquad \forall\, \zeta \in W^{1,\infty}(\mathbb{S}^2)$$

for any $g \in W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1)$. Extending previous facts from the easier case $p = 2$, they also prove for every $p > 1$ and $g \in W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1)$ the existence of two sequences $(P_i), (N_i) \subset \mathbb{S}^2$ such that

$$T(g) = \pi \cdot \sum_i (\delta_{P_i} - \delta_{N_i}) \,, \quad \sum_i |P_i - N_i| \le C_p \, \|g\|^p_{1/p,p} \,.$$

COAREA FORMULA. With the previous notation, we similarly obtain in any dimension $n \ge 1$ the following estimate:

Proposition 1.1 *Let $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ for some $p > 1$. Then we have:*

$$\int_{\mathbf{C}^{n+1}} |V^{\#}(dy^1 \wedge dy^2)| \, dx \, dt \le C \int_{\mathbf{C}^{n+1}} t^{p-2} \, |DU(x,t)|^p \, dx \, dt \tag{1.5}$$

for some real constant $C > 0$ only depending on n, p.

PROOF: By the previous facts, inequality (1.5) readily follows when $p = 2$, and hence for $1 < p < 2$, by the continuous embedding $W^{1/p,p}(B^n, \mathbb{S}^1) \subset W^{1/2,2}(B^n, \mathbb{S}^1)$. When $p > 2$, letting $\alpha = \alpha(p) = 2(p-2)/p$, by the Hölder inequality with exponents $q = p/2$ and $q' = p/(p-2)$ we get:

$$\begin{aligned} \int_G |DV(x,t)|^2 \, dx \, dt \le & \quad C \int_G (t^{\alpha} \, |DU(x,t)|^2) \, t^{-\alpha} \, dx \, dt \\ \le & \quad C \Big(\int_G t^{p-2} |DU(x,t)|^p \, dx \, dt \Big)^{2/p} \cdot \Big(\int_G t^{-2} \, dx \, dt \Big)^{(p-2)/p} \end{aligned}$$

where by (1.1) and (1.2) we can estimate

$$\Big(\int_G t^{-2} \, dx \, dt \Big)^{(p-2)/p} \le C_2 \Big(\int_{\mathbf{C}^{n+1}} t^{p-2} |DU(x,t)|^p \, dx \, dt \Big)^{(p-2)/p}$$

Since by (1.3) and (1.4)

$$\int_{\mathbf{C}^{n+1}} |V^{\#}(dy^1 \wedge dy^2)| \, dx \, dt = \int_G J_V(x,t) \, dx \, dt \le C \int_G |DV(x,t)|^2 \, dx \, dt$$

the assertion readily follows. □

We thus obtain the validity of the *coarea formula* in the sense of Almgren-Browder-Lieb [1]:

Theorem 1.2 *Let $n \ge 1$ and $p > 1$. For every map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ there exists a smooth extension $V \in W^{1,p}_{p-2}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ and a regular value $y \in \mathbb{D}^2$ for V such that*

$$\mathcal{H}^{n-1}(V^{-1}(\{y\})) \le C \int_{\mathbf{C}^{n+1}} t^{p-2} |DU(x,t)|^p \, dx \, dt \tag{1.6}$$

for some real constant C only depending on n and p.

PROOF: Choose $V := \Phi \circ U$, where $U \in W^{1,p}_{p-2}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ is the harmonic extension of u. We have

$$\int_{\mathbb{D}^2} \mathcal{H}^{n-1}(V^{-1}(\{y\})) \, d\mathcal{H}^2(y) = \int_{\mathbf{C}^{n+1}} J_V(x,t) \, dx \, dt = \int_{\mathbf{C}^{n+1}} |V^{\#}(dy^1 \wedge dy^2)| \, dx \, dt$$

and hence we can find a regular value $y \in \mathbb{D}^2$ such that

$$\mathcal{H}^{n-1}(V^{-1}(\{y\})) \le \frac{1}{\pi} \cdot \int_{\mathbf{C}^{n+1}} |V^{\#}(dy^1 \wedge dy^2)| \, dx \, dt \,.$$

The assertion follows from Proposition 1.1. □

2 Cartesian currents and degree

In this section, we recall some notation from Geometric Measure Theory and introduce the class of Cartesian currents $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, for which we establish a closure-compactness property, Theorem 2.5. In low dimension $n = 1$, we then discuss a notion of degree in our context, by extending the definition given in [7] for maps $g \in W^{1/p,p}(\mathbb{S}^1, \mathbb{S}^1)$.

RECTIFIABLE CURRENTS. Let $0 \leq k \leq N$ integer and $\Omega \in \mathbb{R}^N$ an open set. The space $\mathcal{D}_k(\Omega)$ of k-currents in Ω is the strong dual of the space $\mathcal{D}^k(\Omega)$ of compactly supported smooth k-forms. The weak convergence $T_h \rightharpoonup T$ in $\mathcal{D}_k(\Omega)$ is defined by duality through the formula

$$T_h(\omega) \to T(\omega) \qquad \forall\, \omega \in \mathcal{D}^k(\Omega)\,.$$

The *mass* of a current $T \in \mathcal{D}_k(\Omega)$ is defined by

$$\mathbf{M}(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^k(\Omega),\ \|\omega\| \leq 1\}$$

where $\|\omega\|$ is the *comass* norm of ω. Therefore, the mass functional is lower semicontinuous w.r.t. the weak convergence. The *boundary* of a current T in $\mathcal{D}_k(\Omega)$, when $k \geq 1$, is defined by duality as

$$\partial T(\eta) := T(d\eta)\,, \qquad \eta \in \mathcal{D}^{k-1}(\Omega)$$

yielding to a current ∂T in $\mathcal{D}_{k-1}(\Omega)$.

A current $T \in \mathcal{D}_k(\Omega)$ is called *integer multiplicity* (say i.m.) *rectifiable* in $\mathcal{R}_k(\Omega)$ if one has

$$T(\omega) = \int_{\mathcal{M}} \theta\, \langle \omega, \xi \rangle\, d\mathcal{H}^k \qquad \forall\, \omega \in \mathcal{D}^k(\Omega)$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure, $\mathcal{M}$ is a countably k-rectifiable set of Ω, with $\mathcal{H}^k(\mathcal{M}) < \infty$, the $\mathcal{H}^k \llcorner \mathcal{M}$-measurable function $\xi : \mathcal{M} \to \Lambda^k \mathbb{R}^N$ gives for $\mathcal{H}^k$-a.e. $z \in \mathcal{M}$ a unit simple k-vector $\xi(z)$ that provides an orientation to the approximate tangent k-space to $\mathcal{M}$ at z, and θ is an integer valued, $\mathcal{H}^k \llcorner \mathcal{M}$-summable, and non-negative multiplicity function. Therefore, one has $\mathbf{M}(T) = \int_{\mathcal{M}} \theta\, d\mathcal{H}^k < \infty$.

If e.g. $\mathcal{M}$ is an oriented k-submanifold of Ω with finite k-volume, the linear functional $\omega \mapsto \int_{\mathcal{M}} \omega$ on $\mathcal{D}^k(\Omega)$ defines a current $[\![\, \mathcal{M} \,]\!]$ in $\mathcal{R}_k(\Omega)$ with finite mass equal to $\mathcal{H}^k(\mathcal{M})$. We address to [30] or [20, Vol. I] for further details on GMT tools.

In particular, when $\Omega = A \times \mathbb{R}^m$, where $A \subset \mathbb{R}^k$ is a bounded domain, and $\mathcal{M} = \mathcal{G}_v$ is the graph of a Lipschitz function $v : A \to \mathbb{R}^m$, the k-current $G_v = [\![\, \mathcal{G}_v \,]\!]$ carried by the graph of v acts on k-forms $\omega \in \mathcal{D}^k(A \times \mathbb{R}^m)$ as

$$G_v(\omega) = ((Id \bowtie v)_\# [\![\, A \,]\!], \omega) := \int_A (Id \bowtie v)^\# \omega$$

where $(Id \bowtie v)^\# \omega$ is the pull-back of ω through the graph map $(Id \bowtie v)(x) := (x, v(x))$. For example, if $\omega = \gamma \wedge \psi \in \mathcal{D}^k(A \times \mathbb{R}^m)$, where $\gamma \in \mathcal{D}^{k-h}(A)$, $\psi \in \mathcal{D}^h(\mathbb{R}^m)$, and $0 \leq h \leq \min\{k, m\}$, then

$$G_v(\gamma \wedge \psi) = \int_A (Id \bowtie v)^\# (\gamma \wedge \psi) = \int_A \gamma \wedge v^\# \psi\,. \tag{2.1}$$

By the area formula one then computes

$$\mathbf{M}(G_v) = \int_A J_{Id \bowtie v}(x)\, dx = \mathcal{H}^k(\mathcal{G}_v)$$

where $J_{Id \bowtie v}$ is the Jacobian of the graph map. If e.g. $k \geq m = 2$, one has

$$J_{Id \bowtie v} = \sqrt{1 + |Dv|^2 + |M_2(Dv)|^2}$$

where $|M_2(Dv)|^2$ is the sum of the square of the 2×2 minors of the gradient matrix Dv, so that $|M_2(Dv)| = J_v$ and in particular $|M_2(Dv)| = |\det Dv|$ if $k = 2$, see e.g. [20].

Example 2.1 If U is a Sobolev map in $W^{1,2}(\mathbf{C}^{n+1},\mathbb{R}^2)$, the $(n+1)$-current G_U in $\mathbf{C}^{n+1}\times\mathbb{R}^2$ carried by its graph is defined (in an approximate sense) by $G_U := (Id \bowtie U)_\# [\![\mathbf{C}^{n+1}]\!]$, compare [20]. Actually, G_U has finite mass and is an i.m. rectifiable current in $\mathcal{R}_{n+1}(\mathbf{C}^{n+1}\times\mathbb{R}^2)$. In fact, by the area formula and the parallelogram inequality we get the bound

$$\mathbf{M}(G_U) = \int_{\mathbf{C}^{n+1}} J_{Id\bowtie U}\, dz \le c\Big(1 + \int_{\mathbf{C}^{n+1}} |DU|^2\, dz\Big) < \infty$$

for some absolute constant $c>0$, not depending on U. We also have

$$\partial G_U(\eta) = 0 \qquad \forall\, \eta\in\mathcal{D}^n(\mathbf{C}^{n+1}\times\mathbb{R}^2),$$

a property that reads as the null-boundary condition

$$(\partial G_U) \llcorner \mathbf{C}^{n+1}\times\mathbb{R}^2 = 0\,. \tag{2.2}$$

In fact, equation (2.2) is readily checked if U is smooth, by Stoke's theorem, and it is preserved by the weak convergence $G_{U_h}\rightharpoonup G_U$ as currents, that holds true by dominated convergence if $U_h\to U$ strongly in $W^{1,2}(\mathbf{C}^{n+1},\mathbb{R}^2)$. Then, a standard density argument applies to infer (2.2).

By Proposition 1.1, we are able to extend the previous features to our setting, as follows.

Proposition 2.2 *Let $u\in W^{1/p,p}(B^n,\mathbb{S}^1)$, where $p>1$, and let $V:=\Phi\circ U$, where $U\in W^{1,p}_{p-2}(\mathbf{C}^{n+1},\mathbb{D}^2)$ is the harmonic extension of u. Then the current G_V is i.m. rectifiable in $\mathcal{R}_{n+1}(\mathbf{C}^{n+1}\times\mathbb{R}^2)$, with finite mass bounded by*

$$\mathbf{M}(G_V) = \int_{\mathbf{C}^{n+1}} J_{Id\bowtie V}\, dz \le c\Big(1 + \int_{\mathbf{C}^{n+1}} t^{p-2}|DU(x,t)|^p\, dz\Big) \tag{2.3}$$

for some constant $c>0$, not depending on u. Moreover, the null-boundary condition (2.2) *holds true.*

PROOF: The continuous embedding $W^{1,p}_{p-2}(\mathbf{C}^{n+1},\mathbb{R}^2)\subset W^{1,1}(\mathbf{C}^{n+1},\mathbb{R}^2)$ holds for any $p>1$. In fact, letting $\alpha=(p-2)/p$, by the Hölder inequality with exponents p and $p'=p/(p-1)$ we get:

$$\begin{aligned}\int_{\mathbf{C}^{n+1}} |DU(z)|\, dz &= \int_G (t^\alpha\, |DU(x,t)|)\, t^{-\alpha}\, dx\, dt \\ &\le \Big(\int_{\mathbf{C}^{n+1}} t^{p-2}|DU(x,t)|^p\, dx\, dt\Big)^{1/p}\cdot\Big(\int_{\mathbf{C}^{n+1}} t^{-\alpha\cdot(p-1)/p}\, dx\, dt\Big)^{(p-1)/p} < \infty\end{aligned}$$

as $-\alpha\cdot(p-1)/p = (2-p)(p-1)/p^2 > -1$ if $p>1$. Recalling that $|V^\#(dy^1\wedge dy^2)| = |M_2(DV)|$ and that $|DV|\le C\,|DU|$, the mass estimate (2.3) follows from (1.5). Finally, similarly to the $W^{1,2}$ case mentioned in Example 2.1, the null boundary condition (2.2) is readily checked by a standard density argument, on account of the mass estimate (2.3) and of the dominated convergence theorem. □

Due to the previous facts, similarly to the case $p=2$ analyzed in [18, 22], we are able to introduce a good notion of current G_u carried by the graph of a map $u\in W^{1/p,p}(B^n,\mathbb{S}^1)$.

Definition 2.3 *To any map $u\in W^{1/p,p}(B^n,\mathbb{S}^1)$ we associate an n-current G_u in $\mathcal{D}_n(B^n\times\mathbb{S}^1)$ by setting*

$$G_u := (-1)^{n-1}(\partial G_V)\llcorner((B^n\times\{0\})\times\mathbb{R}^2) \qquad on \quad \mathcal{D}^n(B^n\times\mathbb{S}^1)\,, \tag{2.4}$$

where $V:=\Phi\circ U$ and $U\in W^{1,p}_{p-2}(\mathbf{C}^{n+1},\mathbb{D}^2)$ is the harmonic extension of u.

In formula (2.4), the boundary ∂G_V is seen by extending the action of the current G_V to forms in $\mathcal{D}^{n+1}(\mathbb{R}^{n+1}\times\mathbb{R}^2)$. By Federer's support theorem [14], we in fact infer that the current G_u actually belongs to the class $\mathcal{D}_n(B^n\times\mathbb{S}^1)$. Notice however that in general G_u is not i.m. rectifiable, even in low dimension $n=2$, and fails to satisfy the null-boundary condition $(\partial G_u)\llcorner B^n\times\mathbb{S}^1=0$.

Remark 2.4 In low dimension $n=1$, the null-boundary condition $(\partial G_u)\llcorner B^1\times\mathbb{S}^1=0$ holds true as a consequence of the strong density of smooth maps, see Theorem 0.1.

Cartesian currents. Following [20, 23], we introduce for any $p > 1$ and $n \geq 1$ the class of Cartesian currents $\mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$. They are given by the class of currents $T \in \mathcal{D}_n(B^n \times \mathbb{S}^1)$ that decompose as

$$T = G_{u_T} + L_T \times [\![\mathbb{S}^1]\!] \tag{2.5}$$

for some $u_T \in W^{1/p,p}(B^n, \mathbb{S}^1)$ and $L_T \in \mathcal{R}_{n-1}(B^n)$ and that satisfy the null-boundary condition

$$(\partial T) \llcorner B^n \times \mathbb{S}^1 = 0\,. \tag{2.6}$$

Notice that condition (2.6) is automatically satisfied when $n = 1$, see Remark 2.4. Moreover, in general a current T in $\mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ fails to have bounded mass and to be i.m. rectifiable current in $\mathcal{R}_n(B^n \times \mathbb{S}^1)$, if e.g. $u_T \notin W^{1,1}(B^n, \mathbb{R}^2)$. However, the following compactness property holds:

Theorem 2.5 *Let $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $\sup_h \mathcal{E}_{1/p,p}(u_h) < \infty$ for some $p > 1$. Then, there exists a Cartesian current $T \in \mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ as in (2.5) such that, possibly passing to a not relabeled subsequence, $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and $u_h \to u_T$ in $L^p(B^n, \mathbb{R}^2)$.*

Proof: According to Definition 2.3, let $V_h := \Phi \circ U_h$, where $U_h \in W^{1,p}_{p-2}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ is the harmonic extension of u_h, so that for each h

$$G_{u_h} = (-1)^{n-1} (\partial G_{V_h}) \llcorner ((B^n \times \{0\}) \times \mathbb{R}^2) \qquad \text{on} \quad \mathcal{D}^n(B^n \times \mathbb{S}^1)\,. \tag{2.7}$$

Following [23], we now define a suitable map $W_h : \widetilde{\mathbf{C}}^{n+1} \to \mathbb{S}^2$, where $\widetilde{\mathbf{C}}^{n+1} := B^n \times (-1,1)$ and $\mathbb{S}^2 := \{y \in \mathbb{R}^3 \,:\, |y| = 1\}$ is the unit sphere. Denoting by $\mathbb{S}^2_\pm := \{y \in \mathbb{S}^2 \,:\, \pm y_3 \geq 0\}$ the upper and lower half-spheres, we consider a couple of bi-Lipschitz maps $\Phi^\pm : \mathbb{D}^2 \to \mathbb{S}^2_\pm$ such that ${\Phi^\pm}_{|\mathbb{S}^1}(z) = (z, 0)$, define

$$W_h(x,t) := \begin{cases} \Phi^+ \circ V_h(x,t) & \text{if } t \geq 0 \\ \Phi^- \circ V_h(x,-t) & \text{if } t \leq 0 \end{cases} \qquad x \in B^n$$

and denote $G_{W_h} \in \mathcal{D}_{n+1}(\widetilde{\mathbf{C}}^{n+1} \times \mathbb{S}^2)$ the current carried by the graph of W_h. According to (0.4), we shall thus work with the energy $W \mapsto \mathcal{E}^p_{p-2}(W) := \int_{\widetilde{\mathbf{C}}^{n+1}} |t|^{p-2} |DW|^p \, dx\, dt$.

Since $\sup_h \mathcal{E}_{1/p,p}(u_h) < \infty$, by the mass estimate (2.3) we infer that

$$\sup_h \mathbf{M}(G_{W_h}) \leq C \cdot \sup_h \mathcal{E}^p_{p-2}(W_h) < \infty$$

so that $\{G_{W_h}\}$ is a sequence of i.m. rectifiable currents in $\mathcal{R}_{n+1}(\widetilde{\mathbf{C}}^{n+1} \times \mathbb{S}^2)$ with equibounded masses.

Furthermore, the following null-boundary condition holds:

$$(\partial G_{W_h}) \llcorner \widetilde{\mathbf{C}}^{n+1} \times \mathbb{S}^2 = 0 \qquad \forall\, h\,.$$

In fact, each function W_h is the strong $W^{1,p}_{p-2}$-limit of a smooth sequence $\{W^{(h)}_k\} \subset C^\infty(\widetilde{\mathbf{C}}^{n+1}, \mathbb{R}^2)$. By the mass estimate (2.3), on account of the dominated convergence theorem one has $G_{W^{(h)}_k} \rightharpoonup G_{W_h}$ weakly in $\mathcal{D}_{n+1}(\widetilde{\mathbf{C}}^{n+1} \times \mathbb{R}^2)$ as $k \to \infty$, whereas each graph current $G_{W^{(h)}_k}$ satisfies the previous null-boundary condition, by Stokes theorem (see [20, Vol. I, Sec. 3.2.5] for a similar argument).

Therefore, by Federer-Fleming's closure theorem [16], a subsequence of $\{G_{W_h}\}$ weakly converges in $\mathcal{D}_{n+1}(\widetilde{\mathbf{C}}^{n+1} \times \mathbb{S}^2)$ to an i.m. rectifiable current $\widetilde{T} \in \mathcal{R}_{n+1}(\widetilde{\mathbf{C}}^{n+1} \times \mathbb{S}^2)$ satisfying $(\partial \widetilde{T}) \llcorner \widetilde{\mathbf{C}}^{n+1} \times \mathbb{S}^2 = 0$.

Arguing as in [17], by the rectifiable slices theorem [31] it turns out that $\widetilde{T} = G_W + L \times [\![\mathbb{S}^2]\!]$ for some function $W \in W^{1,1}(\widetilde{\mathbf{C}}^{n+1}, \mathbb{S}^2)$, with Jacobian $J_W \in L^1(\widetilde{\mathbf{C}}^{n+1})$, and some i.m. rectifiable current $L \in \mathcal{R}_{n-1}(\widetilde{\mathbf{C}}^{n+1})$. Therefore, $\widetilde{T}$ is a Cartesian current in the class $\mathrm{cart}(\widetilde{\mathbf{C}}^{n+1} \times \mathbb{S}^2)$, see [20].

Since moreover $W_h \to W$ in $L^p(\widetilde{\mathbf{C}}^{n+1}, \mathbb{R}^2)$, by lower semicontinuity of the energy $W \mapsto \mathcal{E}^p_{p-2}(W)$ it turns out that, in an obvious sense, $W \in W^{1,p}_{p-2}(\widetilde{\mathbf{C}}^{n+1}, \mathbb{S}^2)$. Also, using that $W_h(x,0) = u_h(x)$, a further subsequence of $\{u_h\}$ strongly converges in $L^p(B^n, \mathbb{R}^2)$ to some map $u \in L^p(B^n, \mathbb{S}^1)$, whence we get $W(x,0) = u(x)$ in the sense of the traces and therefore $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$.

Furthermore, by the definition of W_h, it turns out that the current L is supported in the closure of $B^n \times \{0\}$. Therefore, on account of definition (2.7), by a slicing argument we infer that the current $T = G_u + L \times [\![\mathbb{S}^1]\!]$ satisfies the null-boundary condition (2.6).

In conclusion, T belongs to the class $\mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ and actually $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, as required. $\square$

DEGREE. For $p > 1$, denote now by $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ the class of locally summable maps $u : \mathbb{R} \to \mathbb{S}^1$ such that $u(x) - P_u \in L^p(\mathbb{R}, \mathbb{R}^2)$ for some point $P_u \in \mathbb{S}^1$, and $|u|_{1/p,p} < \infty$, where

$$|u|^p_{1/p,p} := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^p}{|x - y|^2} \, dx \, dy < \infty \, .$$

The class $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ is equipped with the norm $\|u - P_u\|_{L^p} + |u|_{1/p,p}$.

We define the *degree* of a map u in $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ through the formula

$$\deg u := \frac{1}{\pi} \int_{\mathbb{R}^2_+} V^{\#}(dy^1 \wedge dy^2) \tag{2.8}$$

where $\mathbb{R}^2_+ := \{(x, t) \in \mathbb{R}^2 \mid t > 0\}$ denotes the upper half plane, $U \in W^{1,p}_{p-2}(\mathbb{R}^2_+, \mathbb{D}^2)$ is the harmonic extension of u, and $V := \Phi \circ U$, as before. We have:

Proposition 2.6 *The degree of maps in* $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ *is strongly continuous. Moreover,* $\deg u \in \mathbb{Z}$ *for each* $u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$.

PROOF: Let $u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$. Arguing as in the proof of Proposition 1.1, we have:

$$\int_{\mathbb{R}^2_+} |V^{\#}(dy^1 \wedge dy^2)| \, dx \, dt \le C_p \int_{\mathbb{R}^2_+} t^{p-2} \, |DU(x, t)|^p \, dx \, dt$$

for some real constant $C_p > 0$ depending on p. Let $\{u_h\} \subset W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ such that $u_h \to u$ strongly in $W^{1/p,p}$. For each h, denote $V_h := \Phi \circ U_h$, where $U_h \in W^{1,p}_{p-2}(\mathbb{R}^2_+, \mathbb{D}^2)$ is the harmonic extension of u_h. The strong convergence $u_h \to u$ in $W^{1/p,p}(\mathbb{R}, \mathbb{R}^2)$ implies the strong convergence $V_h \to V$ in $W^{1,p}_{p-2}(\mathbb{R}^2_+, \mathbb{R}^2)$. Therefore, by the above estimate, the dominated convergence theorem yields

$$\lim_{h\to\infty} \int_{\mathbb{R}^2_+} V_h^{\#}(dy^1 \wedge dy^2) = \int_{\mathbb{R}^2_+} V^{\#}(dy^1 \wedge dy^2)$$

whence $\deg u_h \to \deg u$. Since moreover $n = 1$, there exists a sequence $\{u_h\} \subset C^1(\mathbb{R}, \mathbb{S}^1)$ such that $u_h \to u$ strongly in $W^{1/p,p}$. By means of a cut-off argument, for each h we readily find a smooth map $W_h : \mathbb{R}^2 \to \mathbb{D}^2$ and a point $P_h \in \mathbb{S}^1$ such that $W_h(x, t) - P_h$ has compact support contained in $\mathbb{R}^2_+$, and

$$\int_{\mathbb{R}^2_+} |W_h^{\#}(dy^1 \wedge dy^2) - V_h^{\#}(dy^1 \wedge dy^2)| < \frac{1}{2} \, .$$

It is then readily checked that

$$\int_{\mathbb{R}^2_+} W_h^{\#}(dy^1 \wedge dy^2) = d_h \cdot \pi$$

for some $d_h \in \mathbb{Z}$. Therefore, we get $\deg u_h = d_h$ for each h, whence $\deg u \in \mathbb{Z}$, as $\deg u_h \to \deg u$. $\square$

Remark 2.7 Of course, due to the bubbling phenomenon, the degree fails to be continuous w.r.t. the weak sequential convergence in $W^{1/p,p}$. It suffices to consider a sequence $\{u_h\} \subset C^1(\mathbb{R}, \mathbb{S}^1)$ with $\sup_h \|u_h\|_{1/p,p} < \infty$, $\deg u_h = 1$ for each h, and such that $u_h \to P$ a.e., but $G_{u_h} \rightharpoonup G_P + \delta_0 \times [\![\mathbb{S}^1]\!]$ in $\mathcal{D}_1(\mathbb{R} \times \mathbb{S}^1)$, where G_P is the graph current of the constant map equal to some $P \in \mathbb{S}^1$.

Moreover, as in Definition 2.3, with $n = 1$, if $u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ we can define the 1-current G_u in $\mathcal{D}_1(\mathbb{R} \times \mathbb{S}^1)$ by setting

$$G_u := (\partial G_V) \llcorner ((\mathbb{R} \times \{0\}) \times \mathbb{R}^2) \qquad \text{on} \quad \mathcal{D}^1(\mathbb{R} \times \mathbb{S}^1) \, .$$

Actually, using a cut-off argument on the function V, and arguing essentially as in the proof of Proposition 3.2 below, it can be checked that

$$\deg u = \frac{1}{\pi} \, G_u(\pi_2^{\#} \omega_{\mathbb{S}^1}) \, . \tag{2.9}$$

3 Singularities and minimal connections

In this section, we describe in any dimension $n \geq 2$ the singular set of maps $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ in terms of homological tools. Namely, by means of the coarea formula, Theorem 1.2, we build up an $(n-2)$-dimensional *integral flat chain* $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$ that in dimension $n = 2$ agrees with the extension $T : W^{1/p,p}(B^2, \mathbb{S}^1) \to W^{1,\infty}(B^2, \mathbb{R})^*$ of the distributional determinant by Bourgain-Brezis-Mironescu [7].

SINGULARITIES. Assume now $n \geq 2$, and let $\omega_{\mathbb{S}^1}$ denote the *normalized volume 1-form* in $\mathbb{S}^1$

$$\omega_{\mathbb{S}^1} := \frac{1}{2\pi}\left(y^1 dy^2 - y^2 dy^1\right) \tag{3.1}$$

so that $[\![\mathbb{S}^1]\!](\omega_{\mathbb{S}^1}) = \int_{\mathbb{S}^1} \omega_{\mathbb{S}^1} = 1$. Moreover, let $\pi_1 : A \times \mathbb{R}^2 \to A$ and $\pi_2 : A \times \mathbb{R}^2 \to \mathbb{R}^2$ denote the orthogonal projections onto the two factors, where $A = B^n$ or $A = \mathbf{C}^{n+1}$.

Definition 3.1 *The* singularities *of a map u in $W^{1/p,p}(B^n, \mathbb{S}^1)$, where $p > 1$, are represented by the $(n-2)$-dimensional current $\mathbb{P}(u)$ in $\mathcal{D}_{n-2}(B^n)$ defined by*

$$\mathbb{P}(u)(\phi) := \partial G_u(\pi_1^\# \phi \wedge \pi_2^\# \omega_{\mathbb{S}^1}), \qquad \phi \in \mathcal{D}^{n-2}(B^n)$$

where G_u is given by Definition 2.3.

We write explicitly the action of $\mathbb{P}(u)$, recovering in the case $p = 2$ the definition of singularities introduced by Hang-Lin [25]. For this purpose, following [23], we choose a smooth decreasing cut-off function $\eta : [0,1] \to [0,1]$ such that $\eta(t) = 1$ for $t \in [0, 1/4]$ and $\eta(t) = 0$ for $t \in [3/4, 1]$, and for any form $\phi \in \mathcal{D}^k(B^n)$ we denote by $\widetilde{\phi}$ the k-form in $\mathbf{C}^{n+1}$ given by $\widetilde{\phi} := \phi \wedge \eta$.

Proposition 3.2 *For every $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ and $\phi \in \mathcal{D}^{n-2}(B^n)$ we have*

$$\mathbb{P}(u)(\phi) = \frac{1}{\pi} \int_{\mathbf{C}^{n+1}} d\widetilde{\phi} \wedge V^\#(dy^1 \wedge dy^2)$$

where the extension V is chosen as in Definition 2.3.

PROOF: Since $d\pi_2^\# \omega_{\mathbb{S}^1} = \pi_2^\# d\omega_{\mathbb{S}^1} = 0$, as $\omega_{\mathbb{S}^1}$ is a closed 1-form in $\mathbb{S}^1$, we compute

$$\mathbb{P}(u)(\phi) = G_u(d\pi_1^\# \phi \wedge \pi_2^\# \omega_{\mathbb{S}^1}) = G_u(\pi_1^\# d\phi \wedge \pi_2^\# \omega_{\mathbb{S}^1}).$$

Denote by $\widehat{\omega}_{\mathbb{S}^1}$ a 1-form in $\mathcal{D}^1(\mathbb{R}^2)$ that agrees with the right-hand side of (3.1) on $\mathbb{D}^2$. By the definition (2.4), using that V satisfies the null-boundary condition (2.2) we have

$$\begin{aligned}
\partial G_V(\pi_1^\# d\widetilde{\phi} \wedge \pi_2^\# \widehat{\omega}_{\mathbb{S}^1}) &= (-1)^{n-1} G_u(\pi_1^\# (d_x \widetilde{\phi} + d_t \widetilde{\phi})_{|t=0} \wedge \pi_2^\# \omega_{\mathbb{S}^1}) \\
&= (-1)^{n-1} G_u(\pi_1^\# d\phi \wedge \pi_2^\# \omega_{\mathbb{S}^1}).
\end{aligned}$$

We thus obtain:

$$\mathbb{P}(u)(\phi) = (-1)^{n-1} \partial G_V(\pi_1^\# d\widetilde{\phi} \wedge \pi_2^\# \widehat{\omega}_{\mathbb{S}^1}) = G_V(\pi_1^\# d\widetilde{\phi} \wedge d\pi_2^\# \widehat{\omega}_{\mathbb{S}^1}) = G_V(\pi_1^\# d\widetilde{\phi} \wedge \pi_2^\# d\widehat{\omega}_{\mathbb{S}^1}). \tag{3.2}$$

Therefore, it suffices to observe that since $V(\mathbf{C}^{n+1}) \subset \mathbb{D}^2$, then

$$V^\# \, d\widehat{\omega}_{\mathbb{S}^1} = \frac{1}{\pi} V^\# (dy^1 \wedge dy^2)$$

and recall the action (2.1) of the current G_V, on account of Proposition 2.2. □

CARTESIAN MAPS. By the previous notation, it turns out that a map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ has zero homological singularities, i.e., satisfies $\mathbb{P}(u) = 0$, if and only if the current G_u associated to its graph has no inner boundary, i.e.,

$$\partial G_u = 0 \quad \text{on} \quad \mathcal{D}^{n-1}(B^n \times \mathbb{S}^1). \tag{3.3}$$

For this reason, we give the following

Definition 3.3 *Let $n \geq 1$ and $p > 1$. A map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ is said to be a* Cartesian map *in* $\text{cart}^{1/p,p}(B^n, \mathbb{S}^1)$ *if the current G_u satisfies the null-boundary condition* (3.3).

Notice that the strong convergence $u_h \to u$ in $W^{1/p,p}(B^n, \mathbb{R}^2)$ yields the weak convergence $G_{u_h} \rightharpoonup G_u$ in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$. Therefore, in low dimension $n = 1$ we get the equality

$$W^{1/p,p}(B^1, \mathbb{S}^1) = \text{cart}^{1/p,p}(B^1, \mathbb{S}^1)$$

by the strong density of smooth maps, whereas in high dimension we clearly have:

$$W^{1/p,p}(B^n, \mathbb{S}^1) \subsetneq \text{cart}^{1/p,p}(B^n, \mathbb{S}^1) \qquad \forall\, n \geq 2\,.$$

When $n \geq 2$, condition $\mathbb{P}(u_h) = 0$ clearly holds true if $u_h : B^n \to \mathbb{S}^1$ is smooth, say Lipschitz, and the null-boundary condition (3.3) is preserved by the weak convergence $G_{u_h} \rightharpoonup G_u$ in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, which implies the weak convergence $\mathbb{P}(u_h) \rightharpoonup \mathbb{P}(u)$ in $\mathcal{D}_{n-2}(B^n)$. Therefore, we immediately obtain that

$$W_S^{1/p,p}(B^n, \mathbb{S}^1) \subset \text{cart}^{1/p,p}(B^n, \mathbb{S}^1)$$

where, we recall, $W_S^{1/p,p}(B^n, \mathbb{S}^1)$ denotes the strong closure of smooth maps $u \in C^\infty(B^n, \mathbb{S}^1)$ in the $W^{1/p,p}$-norm.

Example 3.4 Coming back to Example 0.3, the map $u(x) = x/|x|$ belongs to $W^{s,p}(B^2, \mathbb{S}^1)$ if $1 \leq sp < 2$. Following [20, Vol. I, Sec. 4.2.5], we have:

$$(\partial G_u) \llcorner (B^2 \times \mathbb{S}^1) = -\delta_{\mathbf{0}} \times [\![\, \mathbb{S}^1 \,]\!]$$

where $\mathbf{0}$ is the origin in $\mathbb{R}^2$. Therefore, on account of Definition 3.1 we infer that $\mathbb{P}(u) = -\delta_{\mathbf{0}}$.

If $n \geq 3$, the map $u(x) = (x_1, x_2)/|(x_1, x_2)|$ belongs to $W^{s,p}(B^n, \mathbb{S}^1)$ if $1 \leq sp < 2$, and this time

$$(\partial G_u) \llcorner (B^n \times \mathbb{S}^1) = -[\![\, \Delta \,]\!] \times [\![\, \mathbb{S}^1 \,]\!]$$

where the $(n-2)$-disk $\Delta := \{x \in B^n \mid (x_1, x_2) = (0, 0)\}$ is oriented by $e_3 \wedge \cdots \wedge e_n$, $\{e_i\}_{i=1}^n$ being the canonical basis in $\mathbb{R}^n$. As a consequence, we get $\mathbb{P}(u) = -[\![\, \Delta \,]\!]$.

Instead, an example of Cartesian map according to Definition 3.3 is given e.g. by the content of [12, Lemma 5]. Taking in fact

$$u(x) := (\cos\psi(x), \sin\psi(x))\,, \quad \psi : B^n \setminus \{0\} \to \mathbb{R}\,, \quad \psi(x) := \frac{1}{|x|^\alpha}$$

it turns out that $u \in W^{s,p}(B^n, \mathbb{S}^1)$ for every $0 < s < 1$ and $p > 1$ with $1 \leq sp < n$, provided that $0 < \alpha < (n - sp)/sp$. Therefore, if $0 < \alpha < n - 1$, where $n \geq 2$, we have $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ for every $p > 1$. Now, letting $u_h = (\cos\psi_h, \sin\psi_h)$, where $\psi_h(x) := \max\{\psi(x), h\}$ and $h \in \mathbb{N}^+$, we infer that $\{u_h\} \subset W^{1/p,p}(B^n, \mathbb{S}^1)$ is a sequence of Lipschitz maps strongly converging to u in $W^{1/p,p}(B^n, \mathbb{R}^2)$. Using that $\mathbb{P}(u_h) = 0$ for each h, we infer that $\mathbb{P}(u) = 0$, whence $u \in \text{cart}^{1/p,p}(B^n, \mathbb{S}^1)$ for any $p > 1$.

In Corollary 8.3, we shall prove the equality

$$W_S^{1/p,p}(B^n, \mathbb{S}^1) = \text{cart}^{1/p,p}(B^n, \mathbb{S}^1) \quad \forall\, n \geq 2\,, \quad \forall\, p > 1$$

so that for every map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$,

$$u \in W_S^{1/p,p}(B^n, \mathbb{S}^1) \iff \mathbb{P}(u) = 0\,.$$

Moreover, we shall see that when $u \notin W_S^{1/p,p}(B^n, \mathbb{S}^1)$, the energy gap $\mathcal{G}_{1/p,p}(u)$ in the relaxation process, see (0.7), can be described in terms of the *minimal integral connection* of the singularity $\mathbb{P}(u)$ times a constant weight E_p only depending on the exponent $p > 1$, with $E_2 = 2\pi$ in the easier case $p = 2$.

Real and integral mass. Let $0 \leq k \leq n - 2$ integers. Recall from [20]:

Definition 3.5 *For any current* $\Gamma \in \mathcal{D}_k(B^n)$*, we denote by*

$$m_{r,B^n}(\Gamma) := \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{k+1}(B^n)\,, \quad (\partial D) \llcorner B^n = \Gamma\}$$
$$m_{i,B^n}(\Gamma) := \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{k+1}(B^n)\,, \quad (\partial L) \llcorner B^n = \Gamma\}$$

the real mass *and* integral mass *of* Γ relative to B^n*, respectively.*

In case $m_{i,B^n}(\Gamma) < \infty$, an i.m. rectifiable current $L \in \mathcal{R}_{k+1}(B^n)$ is an *integral minimal connection for the mass* of Γ *allowing connections to the boundary* of B^n, if $(\partial L) \llcorner B^n = \Gamma$ and $\mathbf{M}(L) = m_{i,B^n}(\Gamma)$. In general, one has $m_{r,B^n}(\Gamma) \leq m_{i,B^n}(\Gamma)$. However, by Federer's theorem [14], for $k = 0$, or by Hardt-Pitts' result [26], when $k = n - 2$, if $m_{i,B^n}(\Gamma) < \infty$ one has

$$m_{r,B^n}(\Gamma) = m_{i,B^n}(\Gamma)\,.$$

Following [23], we now introduce for every $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ the current $\mathbb{D}(u) \in \mathcal{D}_{n-1}(B^n)$ given by

$$\mathbb{D}(u)(\gamma) := G_V(\pi_1^{\#}\widetilde{\gamma} \wedge \pi_2^{\#} d\widehat{\omega}_{\mathbb{S}^1})\,, \qquad \gamma \in \mathcal{D}^{n-1}(B^n) \tag{3.4}$$

where the extension V is chosen as in Definition 2.3 and the $(n-1)$-form $\widetilde{\gamma}$ in $\mathbf{C}^{n+1}$ is given as above by $\widetilde{\gamma} := \gamma \wedge \eta$. Since we have

$$\mathbf{M}(\mathbb{D}(u)) \leq \int_{\mathbf{C}^{n+1}} |V^{\#}(dy^1 \wedge dy^2)|\, dx\, dt$$

by Proposition 1.1 we infer that $\mathbb{D}(u)$ has finite mass, namely:

$$\mathbf{M}(\mathbb{D}(u)) \leq C \int_{\mathbf{C}^{n+1}} t^{p-2}\, |DU(x,t)|^p\, dx\, dt \tag{3.5}$$

for some real constant $C > 0$ only depending on n and p.

Furthermore, on account of formula (3.2), in Proposition 3.2 we have actually obtained that

$$\mathbb{P}(u) = (\partial \mathbb{D}(u)) \llcorner B^n\,.$$

Therefore, it turns out that $\mathbb{P}(u)$ is an $(n-2)$-dimensional *real flat chain*, and

$$m_{r,B^n}(\mathbb{P}(u)) \leq \mathbf{M}(\mathbb{D}(u)) < \infty\,.$$

Minimal integral connection. By means of the coarea formula, Theorem 1.2, we now obtain that if $u \in \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$ the current $\mathbb{P}(u)$ of the singularities is an *integral flat chain.* If e.g. $n = 2$, this implies that $\mathbb{P}(u)$ is a finite sum of Dirac masses.

Proposition 3.6 *Let* $u \in \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$*, where* $n \geq 2$ *and* $p > 1$*. Then there exists* $L \in \mathcal{R}_{n-1}(B^n)$ *with* $\mathbf{M}(L) < \infty$ *such that* $\mathbb{P}(u) = (\partial L) \llcorner B^n$*.*

Proof: Choose a regular value $y \in \mathbb{D}^2$ of the extension $V \in W^{1,p}_{p-2}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ such that (1.6) holds, so that $\mathcal{M} := V^{-1}(\{y\})$ is a countably $(n-1)$-rectifiable set of $\mathbf{C}^{n+1}$. Consider the current $\widetilde{L} \in \mathcal{D}_{n-1}(\mathbf{C}^{n+1})$ given by

$$\widetilde{L}(\omega) := \int_{\mathcal{M}} \langle \omega, \xi \rangle\, d\mathcal{H}^{n-1}, \qquad \omega \in \mathcal{D}^{n-1}(\mathbf{C}^{n+1})$$

where $\xi := \eta/|\eta|$ and η is the $(n-1)$-vector $\eta := *V^{\#}(dy^1 \wedge dy^2)$, where $*$ is the *Hodge operator* in $\mathbb{R}^{n+1}$. Therefore, when $n = 2$ the 1-vector field η agrees with the D-field introduced by Brezis-Coron-Lieb [9].

Since ξ is an orienting unit $(n-1)$-vector field of the approximate tangent $(n-1)$-space to $\mathcal{M}$ at $\mathcal{H}^{n-1} \llcorner \mathcal{M}$-a.e. $z \in \mathbf{C}^{n+1}$, it turns out that $\widetilde{L}$ is i.m. rectifiable in $\mathcal{R}_{n-1}(\mathbf{C}^{n+1})$, with finite mass bounded by the right-hand side of formula (1.6). In addition, by (2.4) and Definition 3.1 it turns out that

$$(\partial \widetilde{L}) \llcorner (B^n \times \{0\}) = \mathbb{P}(u)\,.$$

It then suffices to take L equal to the push forward of $\widetilde{L}$ through the orthogonal projection of $\mathbf{C}^{n+1}$ onto $B^n \times \{0\}$. □

We now recall that any map u in $W^{1/p,p}(B^n, \mathbb{S}^1)$ is the strong limit of a sequence $\{u_h\}$ in the class $\mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$, see Theorem 0.2. Arguing as e.g. in [20, Vol. II, Sec. 4.2.5], we thus prove the following:

Proposition 3.7 *For any $n \geq 2$ and $p > 1$, let $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ and $\{u_h\} \subset \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$ such that $u_h \to u$ strongly in $W^{1/p,p}(B^n, \mathbb{R}^2)$. Then*

(i) *$\mathbf{M}(\mathbb{D}(u) - \mathbb{D}(u_h)) \to 0$ as $h \to \infty$;*

(ii) *there exists $L \in \mathcal{R}_{n-1}(B^n)$ such that $\mathbb{P}(u) = (\partial L) \llcorner B^n$;*

(iii) *if $L_{u_h,u}$ denotes an i.m. rectifiable current of least mass in $\mathcal{R}_{n-1}(B^n)$ such that*

$$(\partial L_{u_h,u}) \llcorner B^n = \mathbb{P}(u) - \mathbb{P}(u_h)\,, \tag{3.6}$$

then $\mathbf{M}(L_{u_h,u}) \to 0$ as $h \to \infty$.

Proof: Since $m_{i,B^n}(\mathbb{P}(u_h)) < \infty$, for every $h \in \mathbb{N}$ there exists an i.m. rectifiable current $L_h \in \mathcal{R}_{n-1}(B^n)$ such that $\mathbb{P}(u_h) = (\partial L_h) \llcorner B^n$ and $m_{i,B^n}(\mathbb{P}(u_h)) = \mathbf{M}(L_h)$. By the bound (3.5), since the strong convergence of $u_h \to u$ is equivalent to the strong convergence of $U_h \to U$ in $W^{1,p}_{p-2}(\mathbf{C}^{n+1}, \mathbb{R}^2)$, using the dominated convergence theorem we obtain property (i). Therefore, possibly passing to a (not relabeled) subsequence we may and do assume that

$$m_{r,B^n}(\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h)) \leq 2^{-h} \qquad \forall\, h \in \mathbb{N}$$

and again that

$$\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h) = (\partial \widetilde{L}_h) \llcorner B^n\,,$$

where $\widetilde{L}_h$ is an integral minimal connection of $\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h)$. Therefore, by Hardt-Pitts' result [26]

$$\mathbf{M}(\widetilde{L}_h) = m_{i,B^n}(\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h)) = m_{r,B^n}(\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h)) \leq 2^{-h}\,.$$

Therefore, the current $L := L_0 + \sum_{h=0}^{\infty} \widetilde{L}_h$ is i.m. rectifiable in $\mathcal{R}_{n-1}(B^n)$, whereas due to the weak convergence $\mathbb{P}(u_h) \rightharpoonup \mathbb{P}(u)$ in $\mathcal{D}_{n-2}(B^n)$ we get

$$(\partial L) \llcorner B^n = (\partial L_0) \llcorner B^n + \sum_{h=0}^{\infty} (\partial \widetilde{L}_h) \llcorner B^n = \mathbb{P}(u_0) + \sum_{h=0}^{\infty} \big(\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h)\big) = \mathbb{P}(u)\,.$$

We thus obtain property (ii), whereas property (iii) readily follows. □

Remark 3.8 If $n = 2$, and replacing B^2 with $\mathbb{S}^2$, our definition of singular set $\mathbb{P}(g) \in \mathcal{D}_0(\mathbb{S}^2)$ of maps $g \in W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1)$ agrees essentially with the extension $T : W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1) \to W^{1,\infty}(\mathbb{S}^2, \mathbb{R})^*$ of the distributional determinant $\mathrm{Det}(\nabla g)$ by Bourgain-Brezis-Mironescu [7]. Therefore, if e.g. $u \in W^{1/p,p}(B^2, \mathbb{S}^1)$ is constant in a neighborhood of ∂B^2, arguing as in [7] for the case of maps in $W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1)$, we obtain the existence of two sequences $(P_i), (N_i) \subset B^2$ such that

$$\mathbb{P}(u) = \sum_{i=1}^{\infty} (\delta_{P_i} - \delta_{N_i})\,, \qquad m_i(\mathbb{P}(u)) = \sum_{i=1}^{\infty} |P_i - N_i| < \infty$$

where the integral mass of the integral flat chain $\mathbb{P}(u) \in \mathcal{D}_0(B^2)$ is given by

$$m_i(\mathbb{P}(u)) := \min\{\mathbf{M}(L) \mid L \in \mathcal{R}_1(B^2),\ \partial L = \mathbb{P}(u)\}\,.$$

For our purposes, we finally point out the following:

Remark 3.9 Let $T \in \mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, so that (2.5) holds. It is readily checked that

$$(2.6) \text{ holds} \iff \mathbb{P}(u_T) = -(\partial L_T) \llcorner B^n\,. \tag{3.7}$$

Therefore, on account of property (ii) in Proposition 3.7, it turns out that the class of Cartesian currents with underlying map u_T equal to u

$$\mathcal{T}_u^{1/p,p} := \{T \in \mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1) \mid u_T = u \text{ in } (2.5)\} \tag{3.8}$$

is non-empty for every $u \in W^{1/p,p}(B^n \times \mathbb{S}^1)$ and $p > 1$.

4 Energy concentration

In this section, we discuss a minimum problem that turns out to be strictly related to the energy concentration phenomenon in the relaxation process on the class $W^{1/p,p}(B^n,\mathbb{S}^1)$.

A MINIMUM PROBLEM. For any $p>1$, denote by

$$\mathcal{F}_p := \{u \in W^{1/p,p}(\mathbb{R},\mathbb{S}^1) \mid \deg u = 1\}$$

see (2.8) and Proposition 2.6, and let

$$E_p := \inf\{\mathcal{E}_{1/p,p}(u,\mathbb{R}) \mid u \in \mathcal{F}_p\} \tag{4.1}$$

where, similarly as before, for any $u \in W^{1/p,p}(\mathbb{R},\mathbb{S}^1)$ we let

$$\mathcal{E}_{1/p,p}(u,\mathbb{R}) := \mathcal{E}^p_{p-2}(U,\mathbb{R}^2_+) = \int_{\mathbb{R}^2_+} t^{p-2}\,|DU(x,t)|^p\,dx\,dt < \infty\,, \quad U := \mathrm{Ext}(u)$$

$\mathrm{Ext}(u)$ being the energy minimizer among all functions $U \in W^{1,p}_{p-2}(\mathbb{R}^2_+,\mathbb{D}^2)$ such that $U(x,0)=u(x)$ on $\mathbb{R}\times\{0\}$ in the sense of the traces.

Remark 4.1 The energy functional $U \mapsto \mathcal{E}^p_{p-2}(U,\mathbb{R}^2_+)$ is scale-invariant for each $p>1$. More precisely, if $U_{(r)}(x,t) := U(rx,rt)$, where $U=\mathrm{Ext}(u)$ for some map $u \in W^{1/p,p}(\mathbb{R},\mathbb{S}^1)$, the trace $u_{(r)}$ of $U_{(r)}$ on $t=0$ belongs to $W^{1/p,p}(\mathbb{R},\mathbb{S}^1)$ and $\mathcal{E}^p_{p-2}(U_{(r)},\mathbb{R}^2_+) = \mathcal{E}^p_{p-2}(U,\mathbb{R}^2_+)$ for each $r>0$, whence $U_{(r)} = \mathrm{Ext}(u_{(r)})$.

By convexity of the integrand, the functional $u \mapsto \mathcal{E}_{1/p,p}(u,\mathbb{R})$ is sequentially lower semicontinuous in $\mathcal{F}_p$. However, the class $\mathcal{F}_p$ is not closed w.r.t. the weak convergence, see Remark 2.7, so that (a part from the case $p=2$, see Appendix A) we expect that the minimum in (4.1) fails to be attained. Notwithstanding, we have:

Proposition 4.2 *For every $p>1$, the energy minimum in* (4.1) *is a real positive constant $E_p>0$.*

PROOF: Arguing as in the proof of Proposition 1.1 in the case $n=1$, but this time with $\mathbb{R}^2_+$ instead of $\mathbf{C}^2$, for every $p>1$ and $u \in W^{1/p,p}(\mathbb{R},\mathbb{S}^1)$ we get the estimate

$$\int_{\mathbb{R}^2_+} |V^\#(dy^1\wedge dy^2)|\,dx\,dt \le C_p \int_{\mathbb{R}^2_+} t^{p-2}\,|DU(x,t)|^p\,dx\,dt$$

for some real constant $C_p>0$ only depending on p. On account of definition (2.8), this yields that

$$\pi \le \Big|\int_{\mathbb{R}^2_+} V^\#(dy^1\wedge dy^2)\Big| \le C_p\cdot \mathcal{E}_{1/p,p}(u,\mathbb{R})$$

for each $u \in \mathcal{F}_p$, whence $E_p \ge \pi/C_p > 0$, as required. □

When $p=2$, in Example A.1 from the first appendix we compute the weight $E_2 = 2\pi$. Therefore, extending the result obtained in [23] in the case $p=2$, the explicit formula for the relaxed energy (0.6) is expected to be:

$$\widetilde{\mathcal{E}}_{1/p,p}(u) = \mathcal{E}_{1/p,p}(u) + E_p\cdot m_{i,B^n}(\mathbb{P}(u)) < \infty$$

for every $u \in W^{1/p,p}(B^n,\mathbb{S}^1)$, $n\ge 2$, and $p>1$. This property will be proved in Theorem 8.2 below.

ENERGY MINIMUM WITH FIXED DEGREE. In the sequel, for each map $u \in W^{1/p,p}(\mathbb{R},\mathbb{S}^1)$ and each open set $A\subset\mathbb{R}$ we define the energy of u on A by means of the restriction $u_{|A}$, i.e.,

$$\mathcal{E}_{1/p,p}(u,A) := \mathcal{E}^p_{p-2}(U,A\times(0,1)) = \int_{A\times(0,1)} t^{p-2}|DU(x,t)|^p\,dx\,dt\,, \quad U = \mathrm{Ext}(u_{|A}) \tag{4.2}$$

where $\mathrm{Ext}(u_{|A})$ is the energy minimizer among all maps $U \in W^{1,p}_{p-2}(A\times(0,1),\mathbb{R}^2)\cap L^\infty$ such that $U(x,0)=u(x)$ on $A\times\{0\}$. Notice that in general $\mathcal{E}_{1/p,p}(u,A) \le \mathcal{E}^p_{p-2}(\mathrm{Ext}(u),A\times(0,1))$, where $\mathrm{Ext}(u)$ is the energy minimizer in $W^{1,p}_{p-2}(\mathbb{R}^2_+,\mathbb{D}^2)$, so that $\mathcal{E}_{1/p,p}(u,\mathbb{R}) := \mathcal{E}^p_{p-2}(\mathrm{Ext}(u),\mathbb{R}^2_+)$.

Recalling the definition (2.8), Proposition 2.6, and (4.1), we now prove:

Proposition 4.3 *For any $p > 1$ and $d \in \mathbb{Z}$, we have*

$$\inf\{\mathcal{E}_{1/p,p}(u,\mathbb{R}) \mid u \in \mathcal{F}_p(d)\} = |d| \cdot E_p \tag{4.3}$$

where $\mathcal{F}_p(d) := \{u \in W^{1/p,p}(\mathbb{R},\mathbb{S}^1) \mid \deg u = d\}$.

PROOF: We prove in two steps the inequalities "$\leq$" and "$\geq$" in (4.3), where the case $d \in \{0,1,-1\}$ trivially follows, whence it clearly suffices to consider the case $d \in \mathbb{N}^+$, with $d \geq 2$.

STEP 1: THE INEQUALITY "$\leq$". Let $P_S := (0,-1)$, the "south pole" in $\mathbb{S}^1$. We make use of the following

Lemma 4.4 *For each $\varepsilon > 0$ we can find a map $u_\varepsilon \in W^{1/p,p}(\mathbb{R},\mathbb{S}^1)$ such that $u_\varepsilon(x) = P_S$ if $|x| \geq 1$, $\deg u_\varepsilon = 1$, and $\mathcal{E}_{1/p,p}(u_\varepsilon,\mathbb{R}) \leq E_p + \varepsilon$.*

PROOF: Let $u \in \mathcal{F}_p = \mathcal{F}_p(1)$ such that $\mathcal{E}_{1/p,p}(u,\mathbb{R}) < E_p + \varepsilon/2$. Since $u - P_u \in L^p(\mathbb{R},\mathbb{S}^1)$ for some $P_u \in \mathbb{S}^1$, by left composition with a rotation in the target space we can choose $P_u = P_S$. Let now $U = \mathrm{Ext}(u)$, and for each $h \in \mathbb{N}^+$ define

$$U_h(x,t) = \phi(|(x,t)| - h)\, U(x,t) + (1 - \phi(|(x,t)| - h)) \cdot P_S$$

for some smooth decreasing cut-off function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(\rho) = 1$ for $\rho \leq 0$ and $\phi(\rho) = 0$ for $\rho \geq 1$. We have that $U_h \to U$ strongly in $W^{1,p}_{p-2}(\mathbb{R}^2_+,\mathbb{R}^2)$, whence for h sufficiently large we get

$$|\mathcal{E}^p_{p-2}(U_h,\mathbb{R}^2_+) - \mathcal{E}^p_{p-2}(U,\mathbb{R}^2_+)| < \varepsilon/2\,.$$

By scale invariance of the energy, Remark 4.1, it suffices to define $u_\varepsilon(x) = u_h(x/(h+1))$ for h large. □

Now, by gluing together d "copies" of the map u_ε from Lemma 4.4, i.e., by letting $u_{\varepsilon,d}(x) = u_\varepsilon(x+2i)$ if $x \in (2i, 2i+1)$ for $i = 0,\ldots,d-1$, and $u_{\varepsilon,d}(x) = P_S$ elsewhere in $\mathbb{R}$, we find a map $u \in \mathcal{F}_p(d)$ satisfying $\mathcal{E}_{1/p,p}(u_{\varepsilon,d},\mathbb{R}) \leq d \cdot E_p + d \cdot \varepsilon$. Therefore, inequality "$\leq$" in (4.3) follows by letting $\varepsilon \to 0$.

STEP 2: THE INEQUALITY "$\geq$". Assume by contradiction that there exists $u \in \mathcal{F}_p(d)$ such that

$$\mathcal{E}_{1/p,p}(u,\mathbb{R}) = (E_p - \eta)\, d \tag{4.4}$$

for some constant $\eta > 0$. By a density argument, a truncation procedure as in the previous lemma, by scale invariance of the energy, and by using a left composition with a rotation, we may and do assume that $u \in C^\infty(\mathbb{R},\mathbb{S}^1)$ with $u(x) \equiv P_S$ for $|x| \geq 1$. Let $C = u^{-1}(\{P_S\})$ and $A = \mathbb{R} \setminus C$, so that A is a bounded open subset in B^1.

Let $\{A_j\}$ denote the (at most countable) family of connected components (open intervals) of A. For each j, we let u_j denote the function equal to u on A_j and to P_S on $\mathbb{R} \setminus A_j$. We have $u_j \in W^{1/p,p}(\mathbb{R},\mathbb{S}^1)$, whence the degree $d_j \in \mathbb{Z}$ of u_j is well-defined, and by (2.9) we readily infer that $\sum_j d_j = d$.

We now see that $d_j \in \{0,-1,+1\}$ for each j. Since in fact $u_j \equiv P_S$ in $\mathbb{R} \setminus A_j$ and $u_j(x) \neq P_S$ for each $x \in A_j$, it cannot happen that $|d_j| \geq 2$, otherwise for topological reasons the continuous map u_j should cover the whole circle $\mathbb{S}^1$ at least once outside A_j, a contradiction.

We thus may assume (after a relabeling) that $d_j = 1$ and hence $u_j \in \mathcal{F}_p$ for each for $j = 1,\ldots,d$. Using that $\mathcal{E}_{1/p,p}(u,\mathbb{R}) \geq \sum_{j=1}^d \mathcal{E}_{1/p,p}(u_j,\mathbb{R})$, by (4.4) we find for some $j = 1,\ldots,d$

$$\mathcal{E}_{1/p,p}(u_j,\mathbb{R}) \leq \frac{1}{d}\, \mathcal{E}_{1/p,p}(u,\mathbb{R}) = E_p - \eta\,, \qquad \eta > 0$$

so that by (4.1) we get contradiction in the formula (4.4), as required. □

5 A lower semicontinuous energy on currents

In this section, we introduce a suitable functional $T \mapsto E_{1/p,p}(T)$ on Cartesian currents that agrees with the energy $\mathcal{E}_{1/p,p}(u)$ if $T = G_u$ for some map $u \in \mathrm{cart}^{1/p,p}(B^n,\mathbb{S}^1)$, see (5.1). Our functional turns out to be lower semicontinuous along weakly converging sequences of smooth graphs, Theorem 5.1.

For any $p>1$, if $T\in \text{cart}^{1/p,p}(B^n\times\mathbb{S}^1)$ as in (2.5), we define

$$E_{1/p,p}(T):=\mathcal{E}_{1/p,p}(u_T)+E_p\cdot \mathbf{M}(L_T)\quad\text{if}\quad T=G_{u_T}+L_T\times[\![\,\mathbb{S}^1\,]\!] \tag{5.1}$$

where $E_p>0$ is the real constant given by (4.1). The localized energy on open sets $A\subset B^n$ is

$$E_{1/p,p}(T,A):=\mathcal{E}_{1/p,p}(u,A)+E_p\cdot\mathbf{M}(L\llcorner A)\,,\qquad u=u_T\,,\ L=L_T$$

where $\mathcal{E}_{1/p,p}(u,A)$ is defined again by (4.2), the function $\text{Ext}(u_{|A})$ being the energy minimizer among all maps $U\in W^{1,p}_{p-2}(A\times(0,1),\mathbb{R}^2)\cap L^\infty$ such that $U(x,0)=u(x)$ on $A\times\{0\}$. We recall that in general $\mathcal{E}^p_{p-2}(\text{Ext}(u),A\times(0,1))\geq \mathcal{E}_{1/p,p}(u,A)$, and that if A_1, A_2 are pairwise disjoint open sets in B^n

$$\mathcal{E}_{1/p,p}(u,A_1\cup A_2)\geq \mathcal{E}_{1/p,p}(u,A_1)+\mathcal{E}_{1/p,p}(u,A_2)\,,\qquad A_1\cap A_2=\emptyset\,. \tag{5.2}$$

The following lower semicontinuity property holds true for any $p>1$ and in any dimension n.

Theorem 5.1 *Let $T\in\text{cart}^{1/p,p}(B^n\times\mathbb{S}^1)$. Let $\{u_h\}\subset C^\infty(B^n,\mathbb{S}^1)$ such that $\sup_h\mathcal{E}_{1/p,p}(u_h)<\infty$, $G_{u_h}\rightharpoonup T$ weakly in $\mathcal{D}_n(B^n\times\mathbb{S}^1)$, and $u_h\to u_T$ in $L^p(B^n,\mathbb{R}^2)$. Then we have:*

$$E_{1/p,p}(T)\leq \liminf_{h\to\infty}\mathcal{E}_{1/p,p}(u_h)\,. \tag{5.3}$$

PROOF: We divide the proof in three steps. In Step 1, we first consider the case of low dimension $n=1$. In Step 2, we deal with the case $n=2$, using a dimension reduction argument and Step 1. In Step 3, we treat the case of high dimension $n\geq 3$ by induction.

STEP 1: THE CASE $n=1$. If $T\in\text{cart}^{1/p,p}(B^1\times\mathbb{S}^1)$, in formula (2.5) we have $L=L_T\in\mathcal{R}_0(B^1)$, hence L is a finite sum of Dirac masses with integer weight, namely

$$L=\sum_{i=1}^m d_i\,\delta_{x_i}\,,\quad \mathbf{M}(L)=\sum_{i=1}^m|d_i| \tag{5.4}$$

where the x_i's are distinct points in B^1 and $d_i\in\mathbb{Z}\setminus\{0\}$ for each i. Choose a family of open intervals $\{A_i\mid i=1,\dots,m\}$, each A_i centered at the point x_i, with pairwise disjoint closures $\overline{A}_i$ which are also well contained in B^1. Due to the continuous embedding of $W^{1/p,p}(B^1,\mathbb{S}^1)$ in the class VMO, see e.g. [7], for each $\varepsilon>0$ we can select each A_i small enough in such a way that:

i) the map u_T has small oscillation on each A_i and small energy, $\mathcal{E}_{1/p,p}(u_T,A_i)<\varepsilon/m$;

ii) for $1=1,\dots,m$, we can find a map $v_i\in W^{1/p,p}(\mathbb{R},\mathbb{S}^1)$ such that

$$\|v_i\|_{W^{1/p,p}(\mathbb{R}\setminus\overline{A}_i,\mathbb{R}^2)}+\|u_T-v_i\|_{W^{1/p,p}(A_i,\mathbb{R}^2)}<\frac{\varepsilon}{m}\,;$$

iii) setting $\Omega=B^1\setminus\bigcup\{\overline{A}_i\mid i=1,\dots,m\}$, we have

$$\mathcal{E}_{1/p,p}(u_T)\leq\mathcal{E}_{1/p,p}(u_T,\Omega)+\sum_{i=1}^m\mathcal{E}_{1/p,p}(u_T,A_i)+\varepsilon\,;$$

iv) on account of Proposition 4.3, if $\varepsilon>0$ is small enough we have $\deg v_i=0$ for each i.

By lower semicontinuity of the energy $u\mapsto\mathcal{E}_{1/p,p}(u,\Omega)$, we thus get

$$\liminf_{h\to\infty}\mathcal{E}_{1/p,p}(u_h,\Omega)\geq\mathcal{E}_{1/p,p}(u_T,\Omega)\geq\mathcal{E}_{1/p,p}(u_T)-\sum_{i=1}^m\mathcal{E}_{1/p,p}(u,A_i)-\varepsilon\geq\mathcal{E}_{1/p,p}(u_T)-2\varepsilon\,. \tag{5.5}$$

Now, for each i we have that $G_{u_h}\llcorner(A_i\times\mathbb{S}^1)\rightharpoonup G_{u_T}\llcorner(A_i\times\mathbb{S}^1)+d_i\,\delta_{x_i}$ in $\mathcal{D}_1(A_i\times\mathbb{S}^1)$. Moreover, using that $\{u_h\}$ has equibounded energy, for h large enough we can find a smooth function $v_h(i)\in W^{1/p,p}(\mathbb{R},\mathbb{S}^1)$ such that

$$\|u_h-v_h(i)\|_{W^{1/p,p}(A_i,\mathbb{R}^2)}\leq a_h\,,\quad \|v_i-v_h(i)\|_{W^{1/p,p}(\mathbb{R}\setminus\overline{A}_i,\mathbb{R}^2)}\leq a_h$$

where $a_h \to 0$. This yields that $G_{v_h(i)} \rightharpoonup G_{v_i} + d_i\,\delta_{x_i}$ in $\mathcal{D}_1(\mathbb{R}\times\mathbb{S}^1)$.

On account of (2.9), and using that $\deg v_i = 0$, we thus infer that $\deg(v_h(i)) \to d_i$ as $h \to \infty$, whence Proposition 4.3 yields that $\mathcal{E}_{1/p,p}(v_h(i),\mathbb{R}) \geq E_p \cdot |d_i|$ for h large. As a consequence, for h large we get

$$\mathcal{E}_{1/p,p}(u_h, A_i) \geq E_p \cdot |d_i| - \varepsilon/m \qquad \forall\, i = 1,\dots,m\,. \tag{5.6}$$

By the estimates (5.5) and (5.6), using that by (5.2)

$$\mathcal{E}_{1/p,p}(u_h) \geq \mathcal{E}_{1/p,p}(u_h,\Omega) + \sum_{i=1}^m \mathcal{E}_{1/p,p}(u_h, A_i) \qquad \forall\, h$$

we thus get

$$\liminf_{h\to\infty} \mathcal{E}_{1/p,p}(u_h) \geq \mathcal{E}_{1/p,p}(u_T) + E_p \cdot \sum_{i=1}^m |d_i| - 3\varepsilon$$

and hence, letting $\varepsilon \to 0$, the lower semicontinuity inequality (5.3) follows on account of (5.1) and (5.4).

Finally, the inequality

$$\liminf_{h\to\infty} \mathcal{E}_{1/p,p}(u_h, A) \geq E_{1/p,p}(T, A)$$

is similarly obtained for any open subset $A \subset B^1$.

STEP 2: THE CASE $n = 2$. Denote for simplicity $u_T = u_\infty$, $T_h := G_{u_h}$, $T_\infty := G_{u_\infty} + L_T \times \mathbb{S}^1$.

Choose a direction $\nu \in S^1_+ := \{x \in \mathbb{R}^2 \,:\, |x| = 1,\ x_1 \geq 0\}$, denote by π_ν the 1-D space spanned by ν, and fix an orienting unit vector $\tau(\nu)$ of the 1-D subspace of $\mathbb{R}^2$ orthogonal to π_ν. For any non-empty and open subset A of B^2, denote by A_ν the orthogonal projection of A onto π_ν, and for any $y \in A_\nu$

$$A^\nu_y := \{z \in \mathbb{R} \mid y\,\nu + z\,\tau(\nu) \in A\}$$

the (non-empty) section of A corresponding to y. Accordingly, for any function $u_h : A \to \mathbb{S}^1$ and any $y \in A_\nu$, the sliced function $(u_h)^\nu_y : A^\nu_y \to \mathbb{S}^1$ is defined by

$$(u_h)^\nu_y(z) := u_h(y\,\nu + z\,\tau(\nu))\,, \quad h \in \overline{\mathbb{N}}.$$

Taking $\Omega := B^2$, for any $h \in \overline{\mathbb{N}}$ the 1-dimensional slice (cf. [20, Vol. I, Sec. 2.2.5])

$$(T_h)^\nu_y := T_h \llcorner \Omega^\nu_y \times \mathbb{S}^1$$

defines a Cartesian current in $\mathrm{cart}^{1/p,p}(\Omega^\nu_y \times \mathbb{S}^1)$ for $\mathcal{H}^1$-a.e. $y \in \Omega_\nu$, and actually

$$(T_h)^\nu_y = G_{(u_h)^\nu_y}\,, \quad (T_\infty)^\nu_y = G_{(u_\infty)^\nu_y} + (L_T \llcorner \Omega^\nu_y) \times [\![\, \mathbb{S}^1 \,]\!]\,.$$

Moreover, by the definition (5.1), the energy of the sliced current $(T_h)^\nu_y$ is given for $\mathcal{L}^1$-a.e. $y \in A_\nu$ by

$$\begin{aligned} E_{1/p,p}((T_h)^\nu_y, A^\nu_y) &:= \mathcal{E}_{1/p,p}((u_h)^\nu_y, A^\nu_y) \qquad \forall\, h \in \mathbb{N} \\ E_{1/p,p}((T_\infty)^\nu_y, A^\nu_y) &:= \mathcal{E}_{1/p,p}((u_\infty)^\nu_y, A^\nu_y) + E_p \cdot \mathbf{M}(L_T \llcorner A^\nu_y)\,. \end{aligned}$$

Therefore, setting

$$E_{1/p,p}(T_h, A;\nu) := \int_{\pi_\nu} E_{1/p,p}((T_h)^\nu_y, A^\nu_y)\,dy\,, \qquad h \in \overline{\mathbb{N}} \tag{5.7}$$

by the inequalities

$$\mathcal{E}_{1/p,p}(u_h, A) \geq \int_{A_\nu} \mathcal{E}_{1/p,p}((u_h)^\nu_y, A^\nu_y)\,dy \quad \forall\, h \in \overline{\mathbb{N}}\,, \qquad \mathbf{M}(L_T \llcorner A) \geq \int_{A_\nu} \mathbf{M}(L_T \llcorner A^n_y)\,dy$$

we infer that

$$E_{1/p,p}(T_h, A) \geq E_{1/p,p}(T_h, A;\nu) \qquad \forall\, h \in \overline{\mathbb{N}}\,. \tag{5.8}$$

Moreover, using that

$$\lim_{h\to\infty}\int_{\pi_\nu}\int_{A^\nu_y}|(u_h)^\nu_y-(u_\infty)^\nu_y|^p\,dz\,dy=\lim_{h\to\infty}\int_A|u_h-u_\infty|^p\,dx=0$$

we can find a strictly increasing sequence $\{h(k)\}\subset\mathbb{N}$ such that

$$\liminf_{h\to\infty}E_{1/p,p}(T_h,A;\nu)=\lim_{k\to\infty}E_{1/p,p}(T_{h(k)},A;\nu)$$

and the sliced currents $(T_{h(k)})^\nu_y$ converge to $(T_\infty)^\nu_y$ weakly in $\mathcal{D}_1(A^\nu_y\times\mathbb{S}^1)$ as $k\to\infty$ for $\mathcal{H}^1$-a.e. $y\in\pi_\nu$. By Step 1, we thus have for any such y

$$\liminf_{k\to\infty}E_{1/p,p}((T_{h(k)})^\nu_y,A^\nu_y)\geq E_{1/p,p}((T_\infty)^\nu_y,A^\nu_y)\,. \tag{5.9}$$

Integrating both sides of (5.9) on π_ν, using Fatou's lemma, (5.7), and (5.8) we thus get for any $\nu\in S^1_+$

$$\liminf_{h\to\infty}E_{1/p,p}(T_h,A)\geq\liminf_{k\to\infty}E_{1/p,p}(T_{h(k)},A;\nu)\geq E_{1/p,p}(T_\infty,A;\nu)\,. \tag{5.10}$$

Consider now the Radon measure

$$\lambda:=\mathcal{L}^3\llcorner\mathbf{C}^3+\theta\,\mathcal{H}^1\llcorner(\operatorname{set}L_T\times\{0\})$$

where θ is the density function of the i.m. rectifiable current $L_T\in\mathcal{R}_1(B^2)$ corresponding to the weak limit current T, and $\operatorname{set}L_T$ is the 1-rectifiable subset of B^2 given by the *points with positivity density* θ.

Let $\{\nu^{(i)}\}_i\subset S^1_+$ be a countable dense sequence. Setting for $(x,t)\in B^n\times J$, where $J:=[0,1]$,

$$\varphi_i(x,t):=\begin{cases}t^{p-2}|D\operatorname{Ext}(u_\infty)^{\nu^{(i)}}_y(x,t)|^p & \text{if}\quad (x,t)\in\mathbf{C}^3\setminus(\operatorname{set}L_T\times\{0\})\,,\quad x=y\,\nu^{(i)}+z\,\tau(\nu^{(i)})\\ E_p & \text{if}\quad x\in\operatorname{set}L_T\,,\quad t=0\end{cases}$$

we obtain for each i and each open set $A\subset B^2$:

$$E_{1/p,p}(T_\infty,A;\nu^{(i)})=\int_{A\times J}\varphi_i\,d\lambda\,.$$

By (5.2) and by the superadditivity of the lim inf operator, using (5.10) we thus get

$$\liminf_{h\to\infty}E_{1/p,p}(T_h)\geq\sum_i\int_{A_i\times J}\varphi_i\,d\lambda \tag{5.11}$$

for any finite family of pairwise disjoint open sets $A_i\subset B^2$. We now recall that by [2, Lemma 2.35]

$$\int_{B^2\times J}\sup_{i\in\mathbb{N}}\varphi_i\,d\lambda=\sup\Big\{\sum_{i\in I}\int_{A_i\times J}\varphi_i\,d\lambda\Big\}$$

where the supremum is taken over all finite sets of indices $I\subset\mathbb{N}$ and all families $\{A_i\}_{i\in I}$ of pairwise disjoint open sets with compact closure in B^2. By (5.11), we then conclude that

$$\liminf_{h\to\infty}\mathcal{E}_{1/p,p}(u_h)=\liminf_{h\to\infty}E_{1/p,p}(T_h)\geq\int_{B^2\times J}\sup_{i\in\mathbb{N}}\varphi_i\,d\lambda=E_{1/p,p}(T_\infty)\,.$$

Finally, the inequality

$$\liminf_{h\to\infty}\mathcal{E}_{1/p,p}(u_h,A)\geq E_{1/p,p}(T_\infty,A)$$

is similarly obtained for any open subset $A\subset B^2$, as required.

Step 3: the case $n\geq 3$. Arguing as e.g. in the proof of [2, Thm. 5.4], we apply a reduction argument to the case $n-1$ in a similar way to Step 2, and an induction argument on the dimension n. We omit any further detail. □

6 Approximate dipoles

In this section, we provide in low dimension $n=2$ the approximation of dipoles for $W^{1/p,p}$-maps with values in $\mathbb{S}^1$, see [9, 19], and [20, Vol. II, Sec. 4.2.3]. Using an argument similar to [23], we first show how to remove *homologically trivial* point singularities in dimension $n=2$.

Removing point singularities. If $p>1$, $0<s<1$, and $1\le sp<2$, by the continuous embedding $W^{s,p}(B^n,\mathbb{S}^1)\subset W^{1/p,p}(B^n,\mathbb{S}^1)$, it turns out that Definitions 2.3 and 3.1 concerning the graph current G_u and the singularity $\mathbb{P}(u)$ extend to maps $u\in W^{s,p}(B^n,\mathbb{S}^1)$. We have:

Proposition 6.1 *Let $p>1$, $0<s<1$, and $1\le sp<2$. Let $u\in\mathcal{R}_{s,p}(B^2,\mathbb{S}^1)$ be such that $\mathbb{P}(u)=0$, so that* (3.3) *holds, with $n=2$. Then there exists a sequence of smooth maps $\{u_h\}\subset C^\infty(B^2,\mathbb{S}^1)$ which converges to u strongly in $W^{s,p}$.*

Proof: Since we use a local argument, we may assume that u has only one singularity at the origin, i.e., $u\in C^\infty(B^2\setminus\{0\},\mathbb{S}^1)$. For $0<r<1$, we denote

$$Q_r:=B^3_r\cap\mathbf{C}^3\,,\qquad F_r:=Q_r\cap(B^2\times\{0\})\,,$$
$$\partial^+Q_r:=\partial B^3_r\cap\{z=(x,t)\in\mathbf{C}^3\mid t>0\}\,.$$

Let $U=\mathrm{Ext}(u)\in W^{1,p}_{\gamma(s,p)}(\mathbf{C}^3,\mathbb{R}^2)$, where $\gamma(s,p)$ is given by (0.3). According to (0.4), for any Borel set $\Omega\subset\mathbf{C}^3$ we let

$$\mathcal{E}^p_{\gamma(s,p)}(U,\Omega):=\int_\Omega t^{\gamma(s,p)}|DU(x,t)|^p\,dx\,dt\,,\qquad \gamma=\gamma(s,p):=p(1-s)-1\,.$$

Given $\varepsilon>0$, let $0<R=R(\varepsilon)\ll1$ be such that $\mathcal{E}^p_{\gamma(s,p)}(U,Q_R)\le\varepsilon$. Since

$$\mathcal{E}^p_{\gamma(s,p)}(U,Q_R\setminus Q_{R/2})=\int_{R/2}^R dr\int_{\partial^+Q_r}t^{\gamma(s,p)}|DU|^p\,d\mathcal{H}^2$$

there exists $r=r_\varepsilon\in[R/2,R]$ such that

$$\mathcal{E}^p_{\gamma(s,p)}(U,\partial^+Q_r):=\int_{\partial^+Q_r}t^{\gamma(s,p)}|DU|^p\,d\mathcal{H}^2\le\frac{4}{R}\mathcal{E}^p_{\gamma(s,p)}(U,Q_R\setminus Q_{R/2})\le\frac{4\varepsilon}{R}\,. \tag{6.1}$$

In order to remove the singularity of u, we have to show that

$$\{w\in W^{s,p}(B^2_r,\mathbb{R}^2)\cap C^0(\bar B^2_r,\mathbb{S}^1)\mid w_{|\partial B^2_r}=u_{|\partial B^2_r}\}\neq\emptyset\,, \tag{6.2}$$

i.e., that $u_{|\partial B^2_r}$ is homotopic to a constant map in $\mathbb{S}^1$. Since the first homotopy group $\pi_1(\mathbb{S}^1)\simeq\mathbb{Z}$ is commutative, it suffices to show that $d\,u_{|\partial B^2_r}{}^{\#}\omega_{\mathbb{S}^1}=0$. This property holds true since by condition (3.3) we get:

$$\begin{aligned}\int_{\partial B^2_r}u_{|\partial B^2_r}{}^{\#}\omega_{\mathbb{S}^1}&=G_{u_{|\partial B^2_r}}(\pi_2^{\#}\omega_{\mathbb{S}^1})=\partial G_{u_{|B^2_r}}(\pi_2^{\#}\omega_{\mathbb{S}^1})\\&=G_{u_{|B^2_r}}(d\pi_2^{\#}\omega_{\mathbb{S}^1})=G_{u_{|B^2_r}}(\pi_2^{\#}d\omega_{\mathbb{S}^1})=0\,.\end{aligned} \tag{6.3}$$

As a consequence, there exists a smooth extension $u_r:B^2_r\to\mathbb{S}^1$ of $u_{|\partial B^2_r}$ with finite $W^{s,p}$-norm.

Let $V_r:Q_r\to\mathbb{D}^2$ a minimizer of the energy $V\mapsto\mathcal{E}^p_{\gamma(s,p)}(V,Q_r)$ among the maps in $W^{1,p}_{\gamma(s,p)}(Q_r,\mathbb{R}^2)$ satisfying the boundary conditions

$$\begin{cases}V=U & \text{on}\quad \partial^+Q_r\\ V=u_r & \text{on}\quad F_r\,.\end{cases}$$

Let $0<\delta<r$ to be fixed later. Define $U_r:\mathbf{C}^3\to\mathbb{D}^2$ by

$$U_r(z):=\begin{cases}V_r\left(\dfrac{r}{\delta}z\right) & \text{if}\quad |z|\le\delta\\[2mm] U\left(r\dfrac{z}{|z|}\right) & \text{if}\quad \delta\le|z|\le r\\[2mm] U(z) & \text{if}\quad |z|\ge r\end{cases}$$

so that $U_r \in W^{1,p}_{\gamma(s,p)}(\mathbf{C}^3, \mathbb{R}^2)$ is continuous and with trace $u_r(x) := U_r(x,0)$ in $W^{s,p}(B^2, \mathbb{S}^1)$. We have

$$\mathcal{E}^p_{\gamma(s,p)}(U_r, Q_\delta) = \Big(\frac{\delta}{r}\Big)^{2-sp} \mathcal{E}^p_{\gamma(s,p)}(V_r, Q_r)\,.$$

Moreover, as in the case $s = 1/2$ and $p = 2$, we can estimate

$$\mathcal{E}^p_{\gamma(s,p)}(U_r, Q_r \setminus Q_\delta) \le c\, r\, \mathcal{E}^p_{\gamma(s,p)}(U, \partial^+ Q_r) \tag{6.4}$$

for some absolute constant $c > 0$. Therefore, by (6.1), using that $r < R$, and taking $\delta = \delta(\varepsilon)$ sufficiently small, since $2 - sp > 0$ we get:

$$\mathcal{E}^p_{\gamma(s,p)}(U_r) \le \mathcal{E}^p_{\gamma(s,p)}(U) + 4c\varepsilon + \Big(\frac{\delta}{r}\Big)^{2-sp} \mathcal{E}^p_{\gamma(s,p)}(V_r, Q_r) \le \mathcal{E}^p_{\gamma(s,p)}(U) + (4c+1)\,\varepsilon\,.$$

Letting $\varepsilon \to 0$ we infer that $U_{r_\varepsilon} \to U$ in $W^{1,p}_{\gamma(s,p)}(\mathcal{C}^3, \mathbb{R}^2)$ and finally that $u_{r_\varepsilon} \to u$ in $W^{s,p}(B^2, \mathbb{R}^2)$, with $u_{r_\varepsilon} \in W^{s,p}(B^2, \mathbb{S}^1)$ continuous. By a standard argument as e.g. in [29], we finally approximate u_{r_ε} by smooth functions in $C^\infty(B^2, \mathbb{S}^1)$, as required. □

THE DIPOLE CONSTRUCTION. We now restrict to the case $sp = 1$, where we adapt some results from [21], to which we refer for further details. To fix the notation, let $a_+, a_- \in B^2$ and $L \in \mathcal{R}_1(B^2)$ the 1-current integration over the segment joining a_- to a_+, with mass $\mathbf{M}(L) = l := |a_+ - a_-| \in (0,1)$, oriented so that the boundary $\partial L = \delta_{a_+} - \delta_{a_-}$. We assume for simplicity

$$a_+ := (l, 0)\,, \qquad a_- := (0,0)\,.$$

Also, if $P \in \mathbb{S}^1$, we let G_P denote the current carried by the graph of the map equal to P on B^2.

In the sequel we also denote $D^2 := \{(x,t) \in \mathbb{R}^2 \mid x^2 + t^2 < r^2\}$ and

$$B^+ := \bar{D}^2 \cap \mathcal{C}^2\,, \qquad \partial^+ B := \partial D^2 \cap \{(x,t) \in \mathcal{C}^2 \mid t > 0\}\,,$$
$$J := \partial B^+ \setminus \partial^+ B = [-1,1] \times \{0\}\,.$$

We first notice that by Lemma 4.4, taking a left composition with a rotation, we readily obtain:

Proposition 6.2 *For every $P \in \mathbb{S}^1$ there exists a family $\{f^P_\varepsilon\}_{\varepsilon>0}$ of Lipschitz functions $f^P_\varepsilon : B^+ \to \mathbb{D}^2$ such that $f^P_{\varepsilon|\partial^+ B} \equiv P$, $f^P_\varepsilon(J) \subset \mathbb{S}^1$, $f^P_{\varepsilon\#}[\![B^+]\!] = [\![\mathbb{D}^2]\!]$, $f^P_{\varepsilon\#}[\![J]\!] = [\![\mathbb{S}^1]\!]$, and*

$$\mathcal{E}_{1/p,p}(f^P_\varepsilon, B^+) \le E_p + \frac{\varepsilon}{2}\,.$$

In Lemma 6.3, we extend a result proved in [21] when $p = 2$. For $0 < \delta < 1$ and $0 < m \le 1$, let

$$\varphi^m_\delta(y) := \min\{my, m(l-y), \delta\}\,, \qquad 0 \le y \le l\,.$$

Consider the map $\phi^m_\delta : (0,l) \times B^+ \to \mathbf{C}^3$

$$\phi^m_\delta(z) := \big(x_1, \varphi^m_\delta(x_1)x_2, \varphi^m_\delta(x_1)t\big)\,, \qquad z = (x_1, x_2, t)$$

and define

$$\begin{aligned} \Omega^m_\delta &:= \phi^m_\delta((0,l) \times B^+) \\ &= \{(x_1,x_2,t) \in \mathbf{C}^3 \mid (x_2)^2 + t^2 < \varphi^m_\delta(x_1)^2\,, \quad 0 < x_1 < l\}\,. \end{aligned}$$

Lemma 6.3 *Let $V : (0,l) \times B^+ \to \mathbb{R}^2$ be a $W^{1,p}_{p-2}$ function and let*

$$V^m_\delta(z) := V \circ (\phi^m_\delta)^{-1}(z)\,, \qquad z \in \Omega^m_\delta\,.$$

Then there exists an absolute constant $c > 0$ such that

$$\begin{aligned} \int_{\Omega^m_\delta} t^{p-2} |DV^m_\delta|^p\, dz &\le \int_{(0,l)\times B^+} t^{p-2} |D_{(x_2,t)} V|^p\, dz + c\,\delta^p \int_{(0,l)\times B^+} t^{p-2} |D_{x_1} V|^2\, dz \\ &+ c\, m^p \int_{((0,\delta/m)\cup(l-\delta/m,l))\times B^+} t^{p-2} |D_{(x_2,t)} V|^p\, dz\,. \end{aligned} \tag{6.5}$$

PROOF: It suffices to notice that with $s = \varphi_\delta^m(x_1)t$, then $s^{p-2}|\det D\phi_\delta^m(\phi_\delta^m(x,t))| = t^{p-2}\varphi_\delta^m(x_1)^p$. We then argue by a change of variables as in the case $p=2$ considered in [20, Vol. II, Sec. 4.2.3]. □

APPROXIMATE DIPOLES. We then obtain the following

Theorem 6.4 *For every $P \in \mathbb{S}^1$, there exists a sequence of maps $\{u_\varepsilon\} \subset C^1(B^2 \setminus \{a_-, a_+\}, \mathbb{S}^1)$ such that $G_{u_h} \rightharpoonup G_P + L \times \mathbb{S}^1$ weakly in $\mathcal{D}_2(B^2 \times \mathbb{S}^1)$ and*

$$\mathcal{E}_{1/p,p}(u_h) \to l \cdot E_p \,, \qquad l := |a_+ - a_-| \,.$$

PROOF: For $\varepsilon > 0$ small, first define $W_\varepsilon : (0,l) \times B^+ \to \mathbb{R}^2$ by $W_\varepsilon(x_1, x_2, t) := f_\varepsilon^P(x_2, t)$, where f_ε^P is given by Proposition 6.2, so that W_ε is a smooth function. Then apply Lemma 6.3 with $V := W_\varepsilon$ and $m = 1$, to obtain a map $W_{\varepsilon,\delta} := W_\varepsilon \circ (\phi_\delta^1)^{-1}$. Finally set

$$U_\varepsilon(z) := \begin{cases} W_{\varepsilon,\delta}(z) & \text{if} \quad z \in \Omega_\delta^1 \\ P & \text{otherwise}\,, \end{cases}$$

so that U_ε belongs to $C^1(\mathbf{C}^3 \setminus \{a_-, a_+\}, \mathbb{R}^2)$, the trace $u_\varepsilon(x) := U_\varepsilon(x, 0)$ belongs to $W^{1/p,p}(B^2, \mathbb{S}^1)$, and by (6.5)

$$l \cdot E_p \le \mathcal{E}_{p-2}^p(U_\varepsilon) \le l \cdot \mathcal{E}_{p-2}^p(f_\varepsilon^P, B^+) + \frac{\varepsilon}{2}$$

if δ is sufficiently small in dependence of ε and of the Lipschitz constant of W_ε. The assertion then follows from Proposition 6.2, letting $u_h = u_{\varepsilon_h}$ with $\varepsilon_h \searrow 0$. □

THE DIPOLE PROBLEM. Notice that the function u_ε in Theorem 6.4 has degree $\deg(u_\varepsilon, a\pm) = \pm 1$, where the degree is defined in the classical sense by taking the restriction of u_ε to small circles around the singularities $a_\pm$. Equivalently, according to Example 3.4 we have: $\mathbb{P}(u_\varepsilon) = -\delta_{a_+} + \delta_{a_-}$. As a consequence, we readily obtain:

Proposition 6.5 *Let $p > 1$ and $\mathcal{G}_p$ denote the class of maps $u \in W^{1/p,p}(\mathbb{R}^2, \mathbb{S}^1)$ which are smooth outside the points $a_\pm$ and such that $\deg(u, a_\pm) = \pm 1$. Then*

$$\inf\{\mathcal{E}_{1/p,p}(u, \mathbb{R}^2) \mid u \in \mathcal{G}_p\} = l \cdot E_p \,, \qquad l = |a_+ - a_-| \,.$$

PROOF: Inequality "$\le$" follows from Theorem 6.4, extending u_ε as the constant function P on $\mathbb{R}^2 \setminus B^2$. On the other hand, for every $u \in \mathcal{G}_p$ and for a.e. $x_1 \in (0,l)$, the restriction $u_{x_1}(y) := u(x_1, y)$ is a map in $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ with degree one, whence $\mathcal{E}_{1/p,p}(u_{x_1}, \mathbb{R}) \ge E_p$, by (4.1). Integrating on $y \in (0,l)$, we get

$$\mathcal{E}_{1/p,p}(u, \mathbb{R}^2) \ge \int_0^l \mathcal{E}_{1/p,p}(u_{x_1}, \mathbb{R})\, dx_1 \ge l \cdot E_p$$

which yields the inequality "$\ge$". □

Proposition 6.5 is in accordance with the case $N = 1$ of [7, Thm. 2.4], where the authors analyzed the dipole problem for maps in $W^{N/p,p}(\mathbb{S}^{N+1}, \mathbb{S}^N)$.

7 Approximation by smooth graphs

In this section, and in the appendices B and C, we prove the following strong density result:

Theorem 7.1 *Let $n \ge 2$ and $p > 1$. For every $T \in \mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, there exists a sequence of smooth maps $\{u_h\}$ in $C^\infty(B^n, \mathbb{S}^1)$ such that $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and*

$$\lim_{h\to\infty} \mathcal{E}_{1/p,p}(u_h) = E_{1/p,p}(T) \,.$$

PROOF: We make use of a readaptation of the proof for the case $p = 2$ taken from [23], in the simpler case where the target space is the circle $\mathbb{S}^1$. We divide the proof in four steps.

STEP 1: REDUCTION TO FINITE MASS SINGULARITIES. Let $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, so that (2.5) holds and hence $\mathbb{P}(u_T) = -(\partial L_T) \llcorner B^n$, see (3.7). By Proposition 3.7, we readily infer:

Lemma 7.2 *There exists a sequence* $\{u_h\}$ *in* $\mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$ *strongly converging to* $u = u_T$ *in* $W^{1/p,p}$ *such that if* $L_{u_h,u}$ *is given by* (3.6), *then*

$$T_h := G_{u_h} + (L_{u_h,u} + L_T) \times \mathbb{S}^1$$

belongs to $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, *all the boundary masses* $\mathbf{M}(\partial(L_{u_h,u} + L_T))$ *are finite,* $T_h \rightharpoonup T$ *weakly in* $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, *and* $E_{1/p,p}(T_h) \to E_{1/p,p}(T)$ *as* $h \to \infty$.

As a consequence, we can assume that $u_T \in \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$ and that $\mathbb{P}(u_T)$ has finite mass, whence $T = G_{u_T} + L \times [\![\mathbb{S}^1]\!]$, where $L \in \mathcal{R}_{n-1}(B^n)$ satisfies $(\partial L) \llcorner B^n = -\mathbb{P}(u_T)$. Therefore, by Federer's boundary rectifiability theorem [14], we infer that the boundary current ∂L is i.m. rectifiable in $\mathcal{R}_{n-2}(B^n)$.

STEP 2: APPROXIMATION BY POLYHEDRAL CHAINS. We can thus write $L = \sum_{q=1}^m L_q$ where the L_q's are integral $(n-1)$-currents in B^n with pairwise disjoint supports. Using Federer's strong polyhedral approximation theorem [14], for every $\varepsilon > 0$ and $q = 1, \dots, m$ we find an integral polyhedral $(n-1)$-chain P_q^ε with support contained in a small neighborhood of radius $c\,\varepsilon$ of the support of the L_q's, and a function $U_\varepsilon \in C^1(\mathbf{C}^{n+1}, \mathbb{R}^2)$, with trace $u_\varepsilon(x) := U_\varepsilon(x, 0) \in \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$, such that if

$$T_\varepsilon := G_{u_\varepsilon} + \sum_{q=1}^m P_q^\varepsilon \times \mathbb{S}^1 \,,$$

then $T_\varepsilon \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, T_ε converges weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ to T as $\varepsilon \to 0$, and

$$E_{1/p,p}(T_\varepsilon) = \mathcal{E}_{1/p,p}(u_\varepsilon) + E_p \cdot \sum_{q=1}^m \mathbf{M}(P_q^\varepsilon) \to \mathcal{E}_{1/p,p}(u_T) + E_p \cdot \mathbf{M}(L) = E_{1/p,p}(T) \,.$$

Moreover, since the L_q's have disjoint supports, we may and do choose the polyhedral chains P_q^ε in such a way that for every small $\varepsilon > 0$ they have pairwise disjoint supports contained in B^n, and u_ε is locally Lipschitz on $B^n \setminus \bigcup_{q=1}^m \text{spt}\, \partial P_q^\varepsilon$, i.e., outside the $(n-2)$-skeleton of each P_q^ε. Also, possibly dividing the simplices of a triangulation of P_q^ε, we may and do assume that every polyhedral $(n-1)$-chain P_q^ε is the union of a finite number of oriented $(n-1)$-simplices Δ which only intersect at the boundary points.

STEP 3: APPROXIMATING DIPOLES. In dimension $n = 2$, we apply Proposition 6.2. In high dimension $n \geq 3$, we apply Proposition 7.3, that is proved in Appendix B. For this purpose, we fix some notation.

For $n \geq 3$, we let Δ denote the $(n-1)$-simplex in B^n given by the convex hull

$$\Delta := \text{coh}(\{0_{\mathbb{R}^n}, l\, e_1, l\, e_2, \dots, l\, e_{n-1}\}) \,, \qquad 0 < l < 1 \,, \tag{7.1}$$

$(e_1, \dots, e_n)$ being the standard basis in $\mathbb{R}^n$. Denote by

$$z = (x, t) = (\widetilde{x}, x_n, t) \,, \qquad \widetilde{x} = (x_1, \dots, x_{n-1})$$

a generic point z in $\mathbf{C}^{n+1}$. Also, for $\delta > 0$ and $0 < m \ll 1$, we let

$$\varphi_\delta^m(y) := \min\{my, \delta\} \,, \qquad y \geq 0 \,,$$

we denote by

$$y(\widetilde{x}) := \text{dist}(\widetilde{x}, \partial\Delta)$$

the distance of $\widetilde{x}$ from the boundary of the $(n-1)$-simplex Δ, and we set

$$\phi_\delta^m(z) := \big(\widetilde{x},\, \varphi_\delta^m(y(\widetilde{x}))x_n,\, \varphi_\delta^m(y(\widetilde{x}))t\big)$$

so that if

$$\Omega_\delta^m := \phi_\delta^m(\Delta \times B^+) \,, \qquad B^+ := \{(x_n, t) \in B^2 \mid t > 0\} \,,$$

then Ω_δ^m is a small "neighbor" of the $(n-1)$-simplex $\Delta \times \{0\}$ in $\mathbf{C}^{n+1}$.

Proposition 7.3 *Let $U \in W^{1,p}_{p-2}(\mathbf{C}^{n+1}, \mathbb{R}^2)$ be a map which is smooth in the interior of $\Omega^{m_0}_{\delta_0}$, for some fixed small $m_0, \delta_0 > 0$, and such that the trace $u(x) := U(x, 0)$ belongs to $W^{1/p,p}(B^n, \mathbb{S}^1)$. Then for every $\varepsilon > 0$, $0 < \delta < \delta_0$, and $0 < m < m_0$, there exists a map $U_\varepsilon : \mathbf{C}^{n+1} \to \mathbb{R}^2$ with trace $u_\varepsilon(x) := U_\varepsilon(x, 0)$ in $W^{1/p,p}(B^n, \mathbb{S}^1)$ such that U_ε is smooth in the closure of Ω^m_δ, except for the $(n-2)$-skeleton of a triangulation of Δ. Moreover, $G_{u_\varepsilon} \rightharpoonup G_u + [\![\Delta]\!] \times [\![\mathbb{S}^1]\!]$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ as $\varepsilon \to 0$ and*

$$\mathcal{E}_{1/p,p}(u_\varepsilon) \leq \mathcal{E}_{1/p,p}(u) + E_p \cdot \mathcal{H}^{n-1}(\Delta) + \varepsilon\,. \tag{7.2}$$

Once we have applied Proposition 6.2, when $n = 2$, or Proposition 7.3, when $n \geq 3$, in order to approximate the dipoles $P_q \times \mathbb{S}^1$, by a diagonal argument we find a sequence $\{u_\varepsilon\}$ in $R_{1/p,p}(B^n \times \mathbb{S}^1)$ whose graphs G_{u_ε} weakly converge to T in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, with $E_{1/p,p}(G_{u_\varepsilon}) \to E_{1/p,p}(T)$. Moreover, by the construction it turns out that u_ε is smooth except on a singular set Σ_ε of B^n given by the $(n-2)$-skeleton of a triangulation of the union of the polyhedral $(n-1)$-chains P_q, and that $\mathbb{P}(u_\varepsilon) = 0$, i.e., u_ε is a Cartesian map in $\text{cart}^{1/p,p}(B^n, \mathbb{S}^1)$.

Step 4: Removing the singularities. In order to remove the $(n-2)$-dimensional singular set Σ_ε, in low dimension $n = 2$ we apply Proposition 6.1. In high dimension $n \geq 3$, we make use of the following variant of a result from [23], that is proved in Appendix C.

Proposition 7.4 *Under the previous hypotheses, for $\varepsilon > 0$ small enough there exists a sequence of smooth maps $\{u^{(\varepsilon)}_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ which converges to u_ε strongly in $W^{1/p,p}(B^n, \mathbb{R}^2)$ as $h \to \infty$.*

The proof of Theorem 7.1 is then completed by a diagonal argument. □

8 Relaxed energy

In this section, we provide in any dimension $n \geq 2$ the explicit formula for the relaxed energy (0.6) in the class $W^{s,p}(B^n, \mathbb{S}^1)$ for any $0 < s < 1$ and $p > 1$. We then give a partial result concerning for the case $1 < s < 2$. Recalling that in dimension $n \geq 2$ we have $W^{s,p}_S(B^n, \mathbb{S}^1) = W^{s,p}(B^n, \mathbb{S}^1)$ if and only if $sp < 1$ or $sp \geq 2$, see Theorem 0.1, in the sequel we assume $1 \leq sp < 2$ and $p > 1$, by firstly considering the case $s = 1/p$, Theorem 8.2, where we apply previous results proved in this paper.

For the sake of completeness, we first collect some properties concerning the class $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ from Definition 3.3, which where already proved in [23] when $p = 2$

Cartesian currents. Using Theorems 2.5, 5.1, and 7.1, we obtain:

Theorem 8.1 *Let $n \geq 2$ and $p > 1$. Then:*

i) *for every $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ there exists a smooth sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $u_h \to u_T$ strongly in $L^p(B^n, \mathbb{R}^2)$, $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, and $\mathcal{E}_{1/p,p}(u_h) \to E_{1/p,p}(T)$;*

ii) *the class $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ is closed along weakly converging sequences of currents with equibounded energies;*

iii) *the functional $T \mapsto E_{1/p,p}(T)$ is sequentially lower semicontinuous in the class $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$.*

Proof: Property i) is Theorem 7.1. As to property ii), assume that $\{T_h\} \subset \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ is such that $\sup_h E_{1/p,p}(T_h) < \infty$. By applying Theorem 7.1, for each h we find a sequence $\{v^{(h)}_k\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $G_{v^{(h)}_k} \rightharpoonup T_h$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and $\mathcal{E}_{1/p,p}(v^{(h)}_k) \to E_{1/p,p}(T_h)$ as $k \to \infty$. Letting $v_h := v^{(h)}_h$, we have $\sup_h \mathcal{E}_{1/p,p}(v_h) < \infty$, whence by Theorem 2.5 a (not relabeled) subsequence is such that $G_{v_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ for some $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$. We now recall that the weak convergence restricted to currents in $\mathcal{R}_n(B^n \times \mathbb{S}^1)$ with no inner boundary is metrizable, being equivalent to the flat metric convergence, see [30, Thm. 31.2]. Therefore, by a diagonal argument we find that a subsequence of $\{T_h\}$ weakly converges to T.

As to property iii), assume now that $\{T_h\} \subset \mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ satisfies $\sup_h E_{1/p,p}(T_h) < \infty$ and $T_h \rightharpoonup T$ for some $T \in \mathrm{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$. As before, we find a sequence $\{v_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $G_{v_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and

$$\liminf_{h\to\infty} \mathcal{E}_{1/p,p}(v_h) \le \liminf_{h\to\infty} E_{1/p,p}(T_h)\,.$$

On account of Theorem 5.1, we thus get

$$E_{1/p,p}(T) \le \liminf_{h\to\infty} E_{1/p,p}(T_h)\,.$$

We omit any further detail. □

An explicit formula. Assume now $sp = 1$ and $p > 1$. The following theorem implies that *every map in $W^{1/p,p}(B^n, \mathbb{S}^1)$ belongs to the $W^{1/p,p}$-weak sequential closure of smooth maps in $C^\infty(B^n, \mathbb{S}^1)$.*

Theorem 8.2 *Let $n \ge 2$ and $p > 1$. For every $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ the relaxed energy $\widetilde{\mathcal{E}}_{1/p,p}(u)$ is finite, and we have:*

$$\widetilde{\mathcal{E}}_{1/p,p}(u) = \mathcal{E}_{1/p,p}(u) + E_p \cdot m_{i,B^n}(\mathbb{P}(u)) < \infty$$

where $E_p > 0$ is given by the minimum problem (4.1), and $m_{i,B^n}(\mathbb{P}(u))$ is the integral mass relative to B^n of the current $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$ of the singularities of u, see Definitions 3.1 *and* 3.5*.*

Proof: We claim that for every $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$

$$\widetilde{\mathcal{E}}_{1/p,p}(u) = \inf\{E_{1/p,p}(T) \mid T \in \mathcal{T}_u^{1/p,p}\} < \infty \tag{8.1}$$

where the class $\mathcal{T}_u^{1/p,p}$ of Cartesian currents with underlying function u is defined in (3.8).

In fact, in Remark 3.9 we have seen that the class $\mathcal{T}_u^{1/p,p}$ is non-empty, whereas by Theorem 7.1, for any $T \in \mathcal{T}_u^{1/p,p}$ we find a smooth sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and $\mathcal{E}_{1/p,p}(u_h) \to E_{1/p,p}(T)$. Since $u_h \to u_T$ in $L^p(B^n, \mathbb{R}^2)$ and $u_T = u$, we infer that the inequality "$\le$" holds in (8.1), and hence that $\widetilde{\mathcal{E}}_{1/p,p}(u) < \infty$.

On the other hand, if $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ satisfies $\sup_h \mathcal{E}_{1/p,p}(u_h) < \infty$, and $u_h \to u$ in $L^p(B^n, \mathbb{R}^2)$, by Theorem 2.5 we find a (not relabeled) subsequence such that $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ to some $T \in \mathcal{T}_u^{1/p,p}$. By Theorem 5.1, we have that $E_{1/p,p}(T) \le \liminf_h \mathcal{E}_{1/p,p}(u_h)$, whence the inequality "$\ge$" holds too in (8.1).

Now, using again Proposition 3.7, we know that $\mathbb{P}(u)$ is an integral flat chain in $\mathcal{D}_{n-2}(B^n)$, whence there exists $L_u \in \mathcal{R}_{n-2}(B^n)$ such that

$$(\partial L_u) \llcorner B^n = -\mathbb{P}(u) \quad \text{and} \quad \mathbf{M}(L_u) = m_{i,B^n}(\mathbb{P}(u))\,.$$

Setting then $T_u := G_u + L_u \times [\![\mathbb{S}^1]\!]$, by (3.7) we infer that the null-boundary condition (2.6) holds, whence by (2.5) it turns out that $T_u \in \mathcal{T}_u^{1/p,p}$. By the definition in (5.1) we thus get

$$\inf\{E_{1/p,p}(T) \mid T \in \mathcal{T}_u^{1/p,p}\} = E_{1/p,p}(T_u) = \mathcal{E}_{1/p,p}(u) + E_p \cdot \mathbf{M}(L_u)$$

which implies the explicit formula for the relaxed energy, on account of (8.1). □

Coming back to Definition 3.3, we thus readily obtain:

Corollary 8.3 *For any $p > 1$, we have:*

$$W_S^{1/p,p}(B^n, \mathbb{S}^1) = \mathrm{cart}^{1/p,p}(B^n, \mathbb{S}^1) \quad \forall\, n \ge 2\,.$$

The case $0 < s < 1$, $1 < sp < 2$. In these ranges of s and p, the strong density of smooth maps fails to hold, see Theorem 0.1. Since $sp > 1$, by the continuous embedding $W^{s,p}(B^n, \mathbb{S}^1) \subset W^{s,1/s}(B^n, \mathbb{S}^1)$ the graph current G_u in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and the singular set $\mathbb{P}(u)$, an integral flat chain in $\mathcal{D}_{n-2}(B^n)$, are well defined for each map $u \in W^{s,p}(B^n, \mathbb{S}^1)$.

However, in dimension $n = 2$, differently to the scale invariance property for the case $sp = 1$, see Remark 4.1, letting $\gamma(s,p)$ as in (0.3), if $U \in W^{1,p}_{\gamma(s,p)}(\mathbb{R}^2_+, \mathbb{R}^2)$ and $U_{(r)}(x,t) := U(rx, rt)$ we get

$$\mathcal{E}^p_{\gamma(s,p)}(U_{(r)}, \mathbb{R}^2_+) = r^{sp-1} \mathcal{E}^p_{\gamma(s,p)}(U, \mathbb{R}^2_+) \qquad \forall\, r > 0\,.$$

Therefore, since $sp > 1$ we have $\mathcal{E}^p_{\gamma(s,p)}(U_{(r)}, \mathbb{R}^2_+) \to 0$ as $r \to 0^+$.

On account of Theorem 8.2, by the above facts we infer that the relaxed energy $\widetilde{\mathcal{E}}_{s,p}(u)$ is finite if and only if $\mathbb{P}(u) = 0$, in which case it agrees with the energy $\mathcal{E}_{s,p}(u)$. More precisely, we have:

Theorem 8.4 *Let $n \geq 2$, $0 < s < 1$, and $p > 1$, with $1 < sp < 2$. Then for every $u \in W^{s,p}(B^n, \mathbb{S}^1)$*

$$\widetilde{\mathcal{E}}_{s,p}(u) = \begin{cases} \mathcal{E}_{s,p}(u) & \text{if } \ \mathbb{P}(u) = 0 \\ +\infty & \text{if } \ \mathbb{P}(u) \neq 0\,. \end{cases}$$

Proof: The implication

$$\mathbb{P}(u) = 0 \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{s,p}(u) = \mathcal{E}_{s,p}(u) < \infty$$

is a consequence of the following approximation result:

Proposition 8.5 *Under the previous values of s, p, and n, if $u \in W^{s,p}(B^n, \mathbb{S}^1)$ satisfies $\mathbb{P}(u) = 0$, there exists a sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $u_h \to u$ in $L^p(B^n, \mathbb{R}^2)$ and $\mathcal{E}_{s,p}(u_h) \to \mathcal{E}_{s,p}(u)$ as $h \to \infty$.*

Proof: Since $G_u \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, arguing as in Steps 1–2 of the proof of Theorem 7.1 we find a sequence of maps u_ε satisfying $\mathbb{P}(u_\varepsilon) = 0$, which are smooth except on a singular set Σ_ε of B^n given by the $(n-2)$-skeleton of a finite triangulation in B^n, and with $u_\varepsilon \to u$ strongly in $W^{s,p}(B^n, \mathbb{R}^2)$ as $\varepsilon \to 0$. We then apply Proposition 6.2, when $n = 2$, or Proposition C.1 from Appendix C, in high dimension $n \geq 3$, in order to remove the homologically trivial singularities. Further details are omitted. □

Conversely, we now show that

$$\widetilde{\mathcal{E}}_{s,p}(u) < \infty \quad \Longrightarrow \quad \mathbb{P}(u) = 0\,.$$

Let $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $\sup_h \mathcal{E}_{s,p}(u_h) < \infty$ and $u_h \to u$ in $L^p(B^n, \mathbb{R}^2)$. Since by the continuous embedding $\sup_h \mathcal{E}_{1/p,p}(u_h) < \infty$ and $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$, by Theorem 2.5 we find a (not relabeled) subsequence such that $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ to some $T \in \mathcal{T}^{1/p,p}_u$.

Assume by contradiction that $\mathbb{P}(u) \neq 0$. Then, $T = G_u + L \times [\![\mathbb{S}^1]\!]$ for some $L \in \mathcal{R}_{n-1}(B^n)$ with positive mass, $\mathbf{M}(L) > 0$. Therefore, if $\mathcal{L} := \text{set}\, L$ is the set of points of positive density for L, we have $\mathcal{H}^{n-1}(\mathcal{L}) > 0$.

For $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{L}$, we denote by I_x the line segment given by the intersection of B^n with the straight line of $\mathbb{R}^n$ containing x and orthogonal to the approximate tangent $(n-1)$-space to $\mathcal{L}$ at x. Then, by a slicing argument, the 1-dimensional restriction of G_{u_h} to $I_x \times \mathbb{S}^1$ yields (possibly passing to a subsequence) a sequence of graphs of smooth maps $u_{h|I_x} : I_x \to \mathbb{S}^1$ with equibounded $\mathcal{E}_{s,p}$-energies.

Moreover, denoting by

$$U^{(x)}_h := \text{Ext}(u_{h|I_x}) : I_x \times (0,1) \to \mathbb{R}^2$$

the extension of $u_{h|I_x}$ in $W^{1,p}_{\gamma(s,p)}(I_x \times (0,1), \mathbb{R}^2)$, where $\gamma(s,p)$ is given by (0.3), we have

$$\sup_h \int_{I_x \times (0,1)} t^{\gamma(s,p)} |DU^{(x)}_h(y,t)|^p \, dy \, dt = C < \infty\,.$$

Therefore, using that $t^{p-2} = t^{sp-1} \cdot t^{\gamma(s,p)}$, for $0 < r < 1$ we estimate:

$$\int_{I_x \times (0,r)} t^{p-2} |DU^{(x)}_h(y,t)|^p \, dy \, dt \leq r^{sp-1} \int_{I_x \times (0,r)} t^{\gamma(s,p)} |DU^{(x)}_h(y,t)|^p \, dy \, dt \leq C\, r^{sp-1} \quad \forall\, h \tag{8.2}$$

where $C\, r^{sp-1} \to 0$ as $r \to 0^+$, since $sp > 1$.

On the other hand, by a slicing argument, the 1-currents $G_{u_{h|I_x}}$ have to converge near the point x to the current $G_{u_{|I_x}} + \sigma\,\delta_x \times \mathbb{S}^1$, where $\sigma \in \mathbb{Z} \setminus \{0\}$ agrees (up to the sign) with the density of the current L at x. Therefore, by lower semicontinuity, Theorem 5.1, we have

$$\liminf_{h\to\infty} \mathcal{E}_{1/p,p}(u_{h|I_x}, I_x) \geq E_p > 0 \tag{8.3}$$

where E_p is given by (4.1). Since (8.3) is in contradiction with (8.2), we must have $\mathbf{M}(L) = 0$, which yields $\mathbb{P}(u) = 0$, by (3.7), as required. □

Notice that by Theorems 8.2 and 8.4, when $n \geq 2$, $0 < s < 1$, and $p > 1$, with $sp \geq 1$, one has:

$$W^{s,p}_S(B^n, \mathbb{S}^1) = \{u \in W^{s,p}_S(B^n, \mathbb{S}^1) \mid \mathbb{P}(u) = 0\}\,. \tag{8.4}$$

THE CASE $1 < s < 2$, $1 < sp < 2$. These are the ranges of s and p for which the strong density of smooth maps fails to hold, see Theorem 0.1, but $s > 1$. Therefore, our definition of energy (0.5), with $\mathcal{E}^p_{\gamma(s,p)}(U)$ given by (0.4), does not make sense, see Remark 8.7. However, we obtain a partial result:

Corollary 8.6 *Let $u \in W^{s,p}(B^n, \mathbb{S}^1)$, where $1 < p < 2$, $1 < s < 2$, $1 < sp < 2$, and $n \geq 2$. Assume that there exists a sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ converging a.e. to u and such that $\sup_h \|u_h\|_{s,p} < \infty$. Then, $\mathbb{P}(u) = 0$.*

PROOF: Since $sp > 1$, we have already seen that the current G_u and the singular set $\mathbb{P}(u)$ are well defined. This time, by means of a Gagliardo-Niremberg type inequality, see [5, Appendix D], the continuous embedding $W^{s,p}(B^n, \mathbb{S}^1) \subset W^{r,q}(B^n, \mathbb{S}^1)$ holds for any $0 < r < 1$ and $q > p$ such that $rq = sp$, with $\|u\|_{r,q} \leq C\, \|u\|^{r/s}_{s,p}$. This yields that $\sup_h \mathcal{E}_{r,q}(u_h) < \infty$, whence u has finite relaxed energy, $\widetilde{\mathcal{E}}_{r,q}(u) < \infty$. Since $rq > 1$ and $0 < r < 1$, by Theorem 8.4 we conclude that $\mathbb{P}(u) = 0$. □

Remark 8.7 In order to prove the converse statement in Corollary 8.6, coming back to [27], one may observe that when $1 < s < 2$ the class $W^{s,p}(B^n)$ is given by the traces on $t = 0$ of Sobolev functions U in $W^{1,p}(B^n \times (0, +\infty))$ with approximate second gradient D^2U a measurable function satisfying

$$\int_{B^n \times (0,+\infty)} t^\gamma |D^2U(x,t)|^p \, dx\, dt < \infty\,, \qquad \gamma = p(2-s) - 1\,.$$

Similarly as before, one may thus introduce on maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ the energy $\mathcal{E}_{s,p}(u)$ as e.g.

$$\mathcal{E}_{s,p}(u) := \int_{\mathbf{C}^{n+1}} t^{p(2-s)-1} |D^2U(x,t)|^p \, dx\, dt < \infty\,, \quad U = \mathrm{Ext}(u)\,. \tag{8.5}$$

A positive answer to the following Open Question would imply the validity of formula (8.4) for any $p > 1$ and $s > 0$, with $1 \leq sp < 2$.

Open Question: Let $1 < p < 2$, $1 < sp < 2$, $1 < s < 2$, and $n \geq 2$. If $u \in W^{s,p}(B^n, \mathbb{S}^1)$ satisfies condition $\mathbb{P}(u) = 0$, then there exists a sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ converging to u a.e. and such that $\mathcal{E}_{s,p}(u_h) \to \mathcal{E}_{s,p}(u)$, where the energy is given by (8.5) and $\mathrm{Ext}(u)$ is an energy minimizer among the smooth maps U from $\mathbf{C}^{n+1}$ to $\mathbb{D}^2$ with trace on $t = 0$ equal to u.

Appendix A Harmonic maps

The Euler-Lagrange system associated to the minimum problem (4.1) reads as

$$\mathrm{div}(|t\, DU(x,t)|^{p-2}\, DU) = 0\,, \qquad (x,t) \in \mathbb{R}^2_+$$

and finding the energy minimum E_p in (4.1) is a difficult task. However, when $p = 2$ we reduce to the harmonic map equation $\Delta U(x,t) = 0$ and one has $E_2 = 2\pi$, an energy minimizer $u_2 \in W^{1/2,2}(\mathbb{R}, \mathbb{S}^1)$ being given by the trace on $t = 0$ of the function $U_2 : \mathbb{R}^2_+ \to \mathbb{D}^2$

$$U_2(x,t) := \Big(\frac{2}{x^2 + (t+1)^2}\, x,\ \frac{1 - (x^2 + t^2)}{x^2 + (t+1)^2} \Big)$$

see (A.1), that is by the inverse u_2 of the stereographic map $\sigma : \mathbb{S}^1 \to \mathbb{R}$ from the south pole $P_S = (0, -1)$.

Example A.1 When $p = 2$, by the parallelogram inequality one has $2|\det G| \leq |G|^2$ for each $G \in \mathbb{R}^{2\times 2}$ and hence, using that $U(\Omega) = \mathbb{D}^2$ if $u \in \mathcal{F}_2$ is smooth, by the area formula we get the energy lower bound

$$\mathcal{E}_2(U, \mathbb{R}^2_+) = \int_{\mathbb{R}^2_+} |DU(x,t)|^2 \, dx \, dt \geq 2 \int_{\mathbb{R}^2_+} |\det DU(x,t)|^2 \, dx \, dt \geq 2\pi \,.$$

Therefore, by a density argument we infer that $E_2 \geq 2\pi$. Following [28], consider now the complex map

$$h(z) := \frac{1 - i\bar{z}}{\bar{z} - i} \,, \quad z \in \mathbb{C} \setminus \{-i\} \,.$$

It is readily checked that h is a biholomorphic map between the half-space $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and the unit disc $\mathbb{E} := \{z \in \mathbb{C} : |z| < 1\}$, and that $h(z) \to -i$ as $|z| \to +\infty$. Setting $z = x + it$ for $x \in \mathbb{R}$ and $t > 0$, we have $h(z) = f(x,t) + i\, g(x,t)$ where

$$f(x,t) := \frac{2x}{\Delta(x,t)} \,, \quad g(x,t) := \frac{1 - (x^2 + t^2)}{\Delta(x,t)} \,, \quad \Delta = \Delta(x,t) := x^2 + (t+1)^2 \,.$$

Therefore, the following Cauchy-Riemann equations are satisfied:

$$f_{,x} = -g_{,t} = \frac{2}{\Delta(x,t)^2} \big((t+1)^2 - x^2 \big) \,, \quad f_{,t} = g_{,x} = \frac{4x(t+1)}{\Delta(x,t)^2} \,.$$

In particular, the mapping $U_2(x,t) := (f,g)(x,t)$ is conformal, i.e.,

$$\langle D_i U_2, D_j U_2 \rangle_{\mathbb{R}^2} = \delta_{ij} \, \frac{4}{\Delta^2} \quad \forall \, (x,t) \in \mathbb{R}^2_+ \,, \quad 1 \leq i \leq j \leq 2 \,.$$

Therefore, one has $|DU_2|^2 = 2|\det DU_2|$ for each $(x,t) \in \mathbb{R}^2_+$, and since $U_2 : \mathbb{R}^2_+ \to \mathbb{D}$ is bijective one has

$$\mathcal{E}^2_0(U_2, \mathbb{R}^2_+) = \int_{\mathbb{R}^2_+} |DU_2(x,t)|^2 \, dx \, dt = 2 \int_{\mathbb{R}^2_+} |\det DU_2(x,t)| \, dx \, dt = 2\pi \,.$$

Finally, the trace $u_2 : \mathbb{R} \to \mathbb{S}^1$

$$u_2(x) := U_2(x,0) = \Big(\frac{2x}{1+x^2}, \frac{1-x^2}{1+x^2} \Big) \,, \quad x \in \mathbb{R} \tag{A.1}$$

satisfies $\deg u_2 = 1$, whence $u_2 \in \mathcal{F}_2$ and definitely $E_2 = 2\pi$.

Appendix B Approximate dipoles

In this appendix we give the proof of Proposition 7.3. It is a readaptation of the proof of [23, Prop. 7.3] for the case $p = 2$, in the simpler case where the target manifold is the unit circle $\mathbb{S}^1$.

Proof of Proposition 7.3: Recall that $n \geq 3$. Let ψ be a bi-Lipschitz homeomorphism of B^n which takes the $(n-1)$-simplex Δ given by (7.1) onto the $(n-1)$-disk $D \times \{0\}$ of diameter l, where

$$D := \{\widetilde{x} \in \mathbb{R}^{n-1} \,:\, |\widetilde{x}| \leq l/2\}$$

with Lipschitz constant

$$\operatorname{Lip} \psi \leq K \,, \qquad \operatorname{Lip} \psi^{-1} \leq K \tag{B.1}$$

where $K = K(n)$ does not depend on l, but possibly on the distance of Δ from ∂B^n. Also, let $V : \mathbf{C}^{n+1} \to \mathbb{R}^2$ be given by

$$V(z) := U \circ \Psi^{-1}(z) \,, \qquad \Psi(z) = \Psi(x,t) := (\psi(x), t) \,.$$

Denoting $\mathbf{0} = (0,0) \in \mathbb{R}^2$, we finally set

$$\begin{aligned} \Omega_\rho &:= \{z \in \mathbf{C}^{n+1} \mid \operatorname{dist}(z, \partial D \times \{\mathbf{0}\}) < \rho\} \\ \partial^+\Omega_\rho &:= \{z \in \mathbf{C}^{n+1} \mid \operatorname{dist}(z, \partial D \times \{\mathbf{0}\}) = \rho\} \end{aligned}$$

fix $0 < R < l/2$, and let $p : \Omega_R \to \partial D \times \{\mathbf{0}\}$ denote the nearest point projection, so that for every $z \in \Omega_R$

$$|z - p(z)| = \operatorname{dist}(z, \partial D \times \{\mathbf{0}\}) .$$

By applying the coarea formula w.r.t. "cylindrical type" coordinates defined around the $(n-2)$-sphere $\partial D \times \{\mathbf{0}\}$, since property

$$\int_{\Omega_R} t^{p-2} |DV|^p \, dz = \int_0^R d\rho \int_{\partial^+\Omega_\rho} t^{p-2} |DV|^p \, d\mathcal{H}^n < \infty$$

yields

$$\liminf_{\rho \to 0^+} \rho \int_{\partial^+\Omega_\rho} t^{p-2} |DV|^2 \, d\mathcal{H}^n = 0$$

we can choose a small radius $r > 0$ and replace V on Ω_r by the map

$$V_r(z) := V\Big(p(z) + r \frac{z - p(z)}{|z - p(z)|}\Big) \tag{B.2}$$

so that as in (6.4) we estimate

$$\mathcal{E}^p_{p-2}(V_r, \Omega_r) \le c_n \cdot r \int_{\partial^+\Omega_r} t^{p-2} |DV|^p \, d\mathcal{H}^n = O(r) \tag{B.3}$$

where $O(r_j) \to 0$ along a sequence $r_j \searrow 0$.

We now define $\widetilde{\Omega}^m_\delta := \widetilde{\phi}^m_\delta(D \times B^+)$, where for $z = (\widetilde{x}, x_n, t)$ we let

$$\widetilde{\phi}^m_\delta(z) := (\widetilde{x}, \varphi^m_\delta(\widetilde{y}(\widetilde{x})) x_n, \varphi^m_\delta(\widetilde{y}(\widetilde{x})) t) , \qquad \widetilde{y}(\widetilde{x}) := \operatorname{dist}(\widetilde{x}, \partial D) .$$

Moreover, setting $r_{\delta,m} := \delta \, \dfrac{\sqrt{1+m^2}}{m}$ we define

$$K^m_\delta := \Big\{ z \in \mathbf{C}^{n+1} \mid 0 < \operatorname{dist}(z, \partial D \times \{\mathbf{0}\}) < r_{\delta,m} \, , \ 0 < \widetilde{y}(\widetilde{x}) < \frac{\delta}{m} \, , \ \sqrt{{x_n}^2 + t^2} < m \cdot \widetilde{y}(\widetilde{x}) \Big\}$$

and we notice that if $r_{\delta,m} < r$, by (B.2) it turns out that the restriction of V to K^m_δ does not depend on the distance of z from $\partial D \times \{0\}$.

We now wish that the following conditions hold true:

(i) V maps K^m_δ into a set of diameter ε;

(ii) V maps $\widetilde{\Omega}^m_\delta$ into a set of diameter ε.

If it is not the case, we let $\{\Delta_i\}_{i=1}^{c(n)}$ be a barycentric-type subdivision of Δ into smaller simplices of side $l/2$, whose number $c(n)$ only depends on n, starting from the 1-faces of Δ. Moreover, possibly slightly moving the centers of the 1-faces of Δ, without loss of generality we can assume that the restriction of V to each k-face of Δ_i has finite $\mathcal{E}_{1/p,p}$-energy, for every $k = 2, \ldots, n$ and every i. We then apply the previous construction to each Δ_i, where K, see (B.1), is an upper bound for the Lipschitz constants of the homeomorphisms of B^n which map Δ_i onto D_i, the $(n-1)$-disk of diameter $l/2$, for every i.

If V does not satisfy conditions (i) and (ii) on the sets $K^m_{\delta,i}$ and $\Omega^m_{\delta,i}$ corresponding to D_i, we start again with the previous procedure, by taking a barycentric subdivision of Δ_i as above.

Notice that V is smooth on the interior of Ω^m_δ, for δ and m sufficiently small, and, by paying a small amount of energy, we can assume that V does not depend on the distance of z from ∂D_i on $K^m_{\delta,i}$. We then infer that the conditions (i) and (ii) above are obtained after a finite number of barycentric subdivisions, by first taking $0 < m = m(\varepsilon) \ll 1$ and then $\delta = \delta(m, r) > 0$ small. Therefore, in the sequel we omit to write the index i corresponding to the simplex Δ_i of the given (finite) subdivision of Δ.

Let now $W_\varepsilon : D \times B^+ \to \mathbb{R}^N$ be given by

$$W_\varepsilon(\widetilde{x}, x_n, t) := f_\varepsilon^P(x_n, t) \tag{B.4}$$

where f_ε^P is given by Proposition 6.2 in correspondence to the point $P := U(p)$ for some given $p \in \operatorname{int}(\Delta)$. Setting

$$\Phi_\varepsilon(z) := W_\varepsilon \circ (\widetilde{\phi}_\delta^m)^{-1}(z)\,, \qquad z \in \widetilde{\Omega}_\delta^m$$

arguing as in Theorem 6.4 and (B.3) we estimate

$$\mathcal{E}_{p-2}^p(\Phi_\varepsilon, \widetilde{\Omega}_\delta^m) \le \mathcal{H}^{n-1}(\Delta) \cdot \mathcal{E}_{p-2}^p(f_\varepsilon^P, B^+) + \frac{\varepsilon}{2\mu K^2}$$

if we choose $\delta = \delta(W_\varepsilon, \varepsilon, m, K, \mu)$ sufficiently small. Here, μ is the finite number of the Δ_i's obtained in the previous subdivision of Δ. We now introduce the cylindrical coordinates

$$z = (\widetilde{x}, x_n, t) = F(\rho, \theta, \widetilde{x}) := (\widetilde{x}, \rho\cos\theta, \rho\sin\theta)\,, \qquad \rho > 0, \quad \theta \in [0, \pi]$$

so that $\rho = \sqrt{{x_n}^2 + t^2}$, denote

$$\widehat{W}(\rho, \theta, \widetilde{x}) := W(F(\rho, \theta, \widetilde{x}))$$

and define $V_\varepsilon : \widetilde{\Omega}_\delta^m \to \mathbb{R}^2$ by

$$\widehat{V}_\varepsilon(\rho, \theta, \widetilde{x}) := \begin{cases} \widehat{\Phi}_\varepsilon(2\rho, \theta, \widetilde{y}) & \text{if} \quad 0 \le \rho < \varphi_\delta^m(\widetilde{y})/2 \\ \widehat{\Psi}_\delta^m(\rho, \theta, \widetilde{y}) & \text{if} \quad \varphi_\delta^m(\widetilde{y})/2 \le \rho < \varphi_\delta^m(\widetilde{y}) \end{cases}$$

for all $\theta \in [0, \pi]$ and $\widetilde{x} \in \operatorname{int}(\Delta)$, where $\widetilde{y} = \widetilde{y}(\widetilde{x}) := \operatorname{dist}(\widetilde{x}, \partial D)$ and

$$\widehat{\Psi}_\delta^m(\rho, \theta, \widetilde{y}) := \left(\frac{2\rho}{\varphi_\delta^m(\widetilde{y})} - 1 \right) \cdot \widehat{V}(\varphi_\delta^m(\widetilde{y}), \theta, \widetilde{y}) + \left(2 - \frac{2\rho}{\varphi_\delta^m(\widetilde{y})} \right) \cdot P\,.$$

We also extend $V_\varepsilon \equiv V$ outside $\widetilde{\Omega}_\delta^m$. By conditions (i) and (ii) above we thus estimate

$$\mathcal{E}_{p-2}^p(V_\varepsilon, \widetilde{\Omega}_\delta^m) \le \mathcal{H}^{n-1}(\Delta) \cdot (E_p + 4\varepsilon^2) + \frac{\varepsilon}{2\mu K^2}\,. \tag{B.5}$$

We finally define

$$U_\varepsilon(z) := V_\varepsilon \circ \Psi(z)\,.$$

Possibly repeating the argument for each simplex Δ_i of the given subdivision of Δ, by (B.5) and (B.1) we estimate

$$\mathcal{E}_{p-2}^p(U_\varepsilon) \le \mathcal{E}_{p-2}^p(U) + \mathcal{H}^{n-1}(\Delta) \cdot (E_p + 4K^2\varepsilon^2) + \frac{\varepsilon}{2}$$

so that (7.2) follows for $\varepsilon > 0$ small. □

Appendix C Removing homologically trivial singularities

In this appendix, we give the proof of Proposition 7.4 by taking $s = 1/p$ in Proposition C.1 below. As before, it is a readaptation of the proof of [23, Prop. 7.4] for the case $s = 1/2$ and $p = 2$, in the simpler case where the target manifold is the unit circle $\mathbb{S}^1$.

Proposition C.1 *Let $n \ge 3$, $p > 1$, $0 < s < 1$ such that $1 \le sp < 2$, and let $u_\varepsilon \in \mathcal{R}_{s,p}(B^n \times \mathbb{S}^1)$ which is smooth except on a singular set Σ_ε of B^n given by the $(n-2)$-skeleton of a triangulation of the union of polyhedral $(n-1)$-chains P_q, $q = 1, \ldots, m$. If $\mathbb{P}(u_\varepsilon) = 0$, there exists a sequence of smooth maps $\{u_h^{(\varepsilon)}\} \subset C^\infty(B^n, \mathbb{S}^1)$ which converges to u_ε strongly in $W^{s,p}(B^n, \mathbb{R}^2)$ as $h \to \infty$.*

PROOF: Let $U_\varepsilon : B^n\times]-1,1[\to \mathbb{D}^2$ given by $U_\varepsilon(x,t) := (\operatorname{Ext} u_\varepsilon)(x,t)$ if $t>0$, and $U_\varepsilon(x,t) := (\operatorname{Ext} u_\varepsilon)(x,-t)$ if $t<0$. In the proof, we shall then work with the energy

$$U \mapsto \mathcal{E}^p_{\gamma(s,p)}(U) := \int_{B^n\times]-1,1[} |t|^{\gamma(s,p)} |DU(x,t)|^p\,dx\,dt$$

where $\gamma(s,p)$ is given by (0.3). For $m\in\mathbb{N}^*$ and $a\in[1/4m,3/4m]^{n+1}$, we denote by $\mathcal{L}_m=\mathcal{L}_m(a)$ the grid of $\mathbb{R}^{n+1}$

$$\mathcal{L}_m := a + \bigcup_{z\in\mathbb{Z}^{n+1}} \frac{1}{m}\cdot z$$

and by $\mathcal{L}_m^{(k+1)}$ the family of all the $(k+1)$-faces Q of the $(n+1)$-cubes of $\mathcal{L}_m$ which intersect the n-disk $B^n\times\{0\}$, for $k=1,\dots,n$. Moreover, we let $\mathcal{F}_m^{(k)}$ denote the set of k-faces F obtained by intersecting the $(k+1)$-faces Q of $\mathcal{L}_m^{(k+1)}$ with the n-disk $B^n\times\{0\}$, i.e.,

$$F = Q\cap (B^n\times\{0\})\,. \tag{C.1}$$

We finally set

$$G_m := B^n\times]-10m^{-1},10m^{-1}[\,.$$

Similarly to [22], we may and do choose $a=a(m,U_\varepsilon)$ so that the following conditions hold:

(i) for every $k=1,\dots n-1$, the restriction of U_ε to every $(k+1)$-face of $\mathcal{L}_m^{(k+1)}$ is a $W^{1,p}_{\gamma(s,p)}$ function;

(ii) there exists an absolute constant $c>0$ such that

$$\mathcal{E}^p_{\gamma(s,p)}(U_\varepsilon,\cup\mathcal{L}_m^{(k+1)}) \le c\,m^{n-k}\,\mathcal{E}^p_{\gamma(s,p)}(U_\varepsilon,G_m) \qquad \forall\, k=1,\dots,n-1\,. \tag{C.2}$$

Moreover, recalling that $n\ge 3$, since the singular set Σ_ε is given by the $(n-2)$-skeleton of some fixed triangulation of the P_q's, by a slicing argument, for m sufficiently large we can also require that

(iii) Σ_ε does not intersect the 1-faces of $\mathcal{F}_m^{(1)}$;

(iv) every 2-face F of $\mathcal{F}_m^{(2)}$ intersects Σ_ε at almost one interior point $p_F\in\operatorname{int}(F)$, which does not belong to the $(n-3)$-skeleton of the triangulation of the polyhedral $(n-1)$-chains P_q ;

(v) the restriction $u_{\varepsilon|F}$ of u_ε to any 2-face F of $\mathcal{F}_m^{(2)}$ is continuous, possibly except at the point p_F;

(vi) in this case, if $p_F\in\operatorname{spt} P_q$, we have

$$\partial G_{u_{\varepsilon|F}} \llcorner F\times\mathbb{S}^1 = 0 \qquad \text{on} \quad \mathcal{D}^1(F\times\mathbb{S}^1)\,. \tag{C.3}$$

As a consequence, arguing as in (6.3), by (C.3) we infer that

$$\{w\in W^{s,p}(F,\mathbb{R}^2)\cap C^0(F,\mathbb{S}^1) \mid w_{|\partial F} = u_{\varepsilon|\partial F}\}\neq\emptyset \tag{C.4}$$

holds true for every 2-face F of $\mathcal{F}_m^{(2)}$.

In order to remove the singular set Σ_ε of u_ε, we make use of an argument taken from [22]. To this aim, at the 1^{st} step we set $U_m^{(\varepsilon)}\equiv U_\varepsilon$ on $\cup\mathcal{L}_m^{(2)}$. We then argue by induction on the dimension $k=2,\dots,n$ and, at the k^{th} step, we set $U_m^{(\varepsilon)}\equiv U_\varepsilon$ on every $Q\in\mathcal{L}_m^{(k+1)}$ which does not intersect the n-disk $B^n\times\{0\}$. Moreover, we define $U_m^{(\varepsilon)}$ on every $Q\in\mathcal{L}_m^{(k+1)}$ which intersects $B^n\times\{0\}$ by means of a "cone" construction starting from the restriction $U^{(\varepsilon)}_{m|\partial Q}$ of $U_m^{(\varepsilon)}$ to the boundary ∂Q. To do this, if $F\in\mathcal{F}_m^{(k)}$ is given by (C.1), it suffices to require that the trace φ_F of $U^{(\varepsilon)}_{m|\partial Q}$ on the boundary of F has a continuous extension $\Phi_F\in W^{s,p}(F,\mathbb{S}^1)$.

Notice that at the 2^{nd} step, this last condition is given by (C.4). In order to extend this condition to the case $k \geq 3$, for every $k \geq 2$ at the k^{th} step we first modify the definition of $u_m^{(\varepsilon)}(x) := U_m^{(\varepsilon)}(x,0)$ on $\mathcal{F}_m^{(k)}$ in a suitable way (see the k^{th} step of the proof of Theorem 2 on p. 457 of [22] for further details).

We secondly extend $U_m^{(\varepsilon)}$ to every $Q \in \mathcal{L}_m^{(k+1)}$ in a continuous way, so that its trace $u_m^{(\varepsilon)}$ belongs to $W^{s,p}(F,\mathbb{S}^1)$ and

$$\mathcal{E}^p_{\gamma(s,p)}(U_m^{(\varepsilon)},Q) \leq \frac{c}{m}\,\mathcal{E}^p_{\gamma(s,p)}(U_\varepsilon,\partial Q)\,. \tag{C.5}$$

More precisely, let $v_Q : Q \to \mathbb{R}^2$ be defined by $v_Q(z) = v_Q^{\pm}(z)$ if $z \in Q^{\pm}$, where

$$Q^{\pm} := \{z = (x,t) \in Q \mid \pm t \geq 0\}$$

and $v_Q^{\pm} : Q^{\pm} \to \mathbb{R}^2$ is the solution of the minimum problem for the energy $\int_{Q^{\pm}} |t|^{\gamma(s,p)}|DV(x,t)|^p\,dx\,dt$ with boundary condition

$$\begin{cases} v_Q^{\pm} = U_m^{(\varepsilon)} & \text{on} \quad \partial Q^{\pm} \cap \{(x,t) \mid \pm t > 0\} \\ v_Q^{\pm} = \Phi_F & \text{on} \quad F \end{cases}$$

where $\Phi_F : F \to \mathbb{S}^1$ is a continuous $W^{s,p}$-extension of the boundary datum $\varphi_F(x) := U^{(\varepsilon)}_{m|\partial Q}(x,0)$. Assuming e.g. that the center of Q is the origin $0_{\mathbb{R}^{n+1}}$, we define $U_m^{(\varepsilon)}$ on Q by setting, for $0 < \delta \ll 1/2m$,

$$U_m^{(\varepsilon)}(z) := \begin{cases} v_Q\Big(\dfrac{z}{2m\delta}\Big) & \text{if} \;\; \|z\| \leq \delta \\ U_m^{(\varepsilon)}\Big(\dfrac{z}{2m\|z\|}\Big) & \text{if} \;\; \delta \leq \|z\| \leq \dfrac{1}{2m} \end{cases} \qquad z \in Q$$

where $\|z\| := \sup_i |z_i|$ if $z = (z_1,\dots,z_{n+1})$, so that $\|z\| = 1/2m$ if $z \in \partial Q$. Therefore, we have:

$$\mathcal{E}^p_{\gamma(s,p)}(U_m^{(\varepsilon)},\{\|z\| < \delta\}) \leq (2m\delta)^{n-sp}\mathcal{E}^p_{\gamma(s,p)}(v_Q,\{\|z\| < 1/2m\})$$

where $n - sp > 0$. A similar definition works in the general case, so that (C.5) holds true and $u_m^{(\varepsilon)}(x) := U_m^{(\varepsilon)}(x,0)$ belongs to $W^{s,p}(F,\mathbb{S}^1)$.

Repeating the argument for $k = 2,\dots,n$, from (C.5) we estimate

$$\mathcal{E}^p_{\gamma(s,p)}(U_m^{(\varepsilon)},\cup\mathcal{L}_m^{(n+1)}) \leq C(n)\sum_{k=1}^{n-1}\frac{1}{m^{n-k}}\mathcal{E}^p_{\gamma(s,p)}(U_\varepsilon,\mathcal{L}_m^{(k+1)})$$

and hence, by (C.2), we obtain

$$\mathcal{E}^p_{\gamma(s,p)}(U_m^{(\varepsilon)},\cup\mathcal{L}_m^{(n+1)}) \leq C(n)\,\mathcal{E}^p_{\gamma(s,p)}(U_\varepsilon,G_m) \to 0$$

as $m \to +\infty$, since $|G_m| \to 0$. We finally set $U_m^{(\varepsilon)} = U_\varepsilon$ on $\mathbf{C}^{n+1} \setminus \cup\mathcal{L}_m^{(n+1)}$, as required. □

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