Existence of minimizers for a generalized liquid drop model with fractional perimeter

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Abstract

We consider the minimization problem of the functional given by the sum of the fractional perimeter and a general Riesz potential, which is one generalization of Gamow's liquid drop model. We first show the existence of minimizers for any volumes if the kernel of the Riesz potential decays faster than that of the fractional perimeter. We also prove the existence of generalized minimizers for any volumes if the kernel of the Riesz potential just vanishes at infinity. Finally, we study the asymptotic behavior of minimizers when the volume goes to infinity and we prove that a sequence of minimizers converges to the Euclidean ball up to translations if the kernel of the Riesz potential decays sufficiently fast.

Keywords. liquid drop model, fractional perimeter, generalized minimizers.

Contents

1 Introduction

We study existence and asymptotic behavior of minimizers for the minimization problem

$$E_{s,g}[m] \coloneqq \inf\left\{\mathcal{E}_{s,g}(E) \mid E \subset \mathbb{R}^N : \text{measurable}, \ |E| = m\right\} \tag{1.1}$$

for any $m > 0$, where we define the functional $\mathcal{E}_{s,g}$ as

$$\mathcal{E}_{s,g}(E) \coloneqq P_s(E) + V_g(E) \tag{1.2}$$

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for any measurable set $E \subset \mathbb{R}^N$. Note that the first term P_s of (1.2) is the *fractional s-perimeter* with $s \in (0, 1)$ defined by

$$P_s(E) \coloneqq \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}}\,dx\,dy$$

for any measurable set $E \subset \mathbb{R}^N$, and the second term V_g of (1.2) is the generalized Riesz potential, defined by

$$V_g(E) \coloneqq \int_E \int_E g(x-y)\,dx\,dy$$

for any measurable set $E \subset \mathbb{R}^N$, where $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is a non-negative measurable function (see Section 2 for the precise assumptions on g).

Problem (1.1) can be regarded as a nonlocal counterpart of the minimization problem

$$E_g[m] \coloneqq \inf\left\{\mathcal{E}_g(E) \coloneqq P(E) + V_g(E) \mid E \subset \mathbb{R}^N,\, |E| = m\right\} \tag{1.3}$$

where we let $P(E)$ be the classical perimeter of a set E.

A relevant physical case of Problem (1.3) is when $N = 3$ and $g(x) = |x|^{-1}$ for $x \in \mathbb{R}^N \setminus \{0\}$. In this case, this problem is referred to as Gamow's liquid drop model and was first investigated by George Gamow in [19] to explain the behavior of atoms and provide a simple model of the nuclear fission. In this model, an atomic nucleus can be regarded as nucleons (protons and neutrons) contained in a set $E \subset \mathbb{R}^N$. The nucleons are assumed to be concentrated with constant density which implies that the number of nucleons is proportional to $|E|$. From a physical point of view, the classical perimeter term corresponds to surface tension, which is minimised by spherical nuclei. On the other hand, the Riesz potential corresponds to a Coulomb repulsion, which tends to drive nuclei away from each other. Due to these properties, the competition between the perimeter term and Riesz potential occurs. By rescaling, one can easily observe this phenomenon. Indeed, using the dilation $\lambda \mapsto \lambda E$ for a set E, we have that

$$P(\lambda E) + V_g(\lambda E) = \lambda^{N-1}\, P(E) + \int_E \int_E \lambda^{2N}\, g(\lambda(x-y))\,dx\,dy$$

for any $\lambda > 1$ and measurable set $E \subset \mathbb{R}^N$. Then, if the kernel g satisfies $g(x) \approx |x|^{-(N+\delta)}$ as $|x| \to \infty$ for $\delta < 1$, we have that $\lambda^{2N}\, g(\lambda\, x) \approx \lambda^{N-\delta}$. Thus, the Riesz potential dominates the perimeter as $\lambda \to \infty$ since $\delta < 1$. On the other hand, if $\lambda \to 0$, then the perimeter dominates the Riesz potential. In Problem (1.1), the fractional perimeter P_s with $s \in (0, 1)$ behaves like the classical perimeter when s approaches to 1 (see the asymptotic behavior of the s-fractional perimeter and more general results in [2, 4, 8, 12, 24, 25, 34]). Choksi, Muratov, and Topaloglu in [11] published a survey on this model and further discussions on this model are written there.

Now let us review the previous works on the classical liquid drop model. Recently, Frank and Nam in [16] revisited this model and some references are also therein. The main interest from the mathematical point of view is to investigate the following three topics: the existence of minimizer, the non-existence of minimizer, and the minimality of the ball. Knüpfer and Muratov in [22, 23] considered when g is equal to $|x|^{-\alpha}$ for $\alpha \in (0, N)$ with $N \geq 2$ and proved that there exists constants $0 < m_0 \leq m_1 \leq m_2 < \infty$ such that the following three things hold: if $N \geq 2$, $\alpha \in (0, N)$, and $m \leq m_1$, then Problem (1.3) admits a minimizer; if $N \geq 2$, $\alpha \in (0, 2)$, and $m > m_2$, then Problem (1.3) does not admit a minimizer; finally, if $m \leq m_0$, then the ball is the unique minimizer whenever either $N = 2$ and $\alpha \in (0, 2)$, or $3 \leq N \leq 7$ and $\alpha \in (0, N-1)$. Later, Julin in [21] proved that, if $N \geq 3$ and $g(x) = |x|^{-(N-2)}$, the ball is the unique minimizer of $\mathcal{E}_g$ whenever m is sufficiently small. Bonacini and Cristoferi in [3] studied the case of the full parameter range $N \geq 2$ and $\alpha \in (0, N-1)$ when $g(x) = |x|^{-\alpha}$. Moreover, for a small parameter α, Bonacini and Cristoferi in [3] gave a complete characterization of the ground state. Namely, they showed that, if α is sufficiently small, there exists a constant m_c such that the ball is the unique minimizer of $\mathcal{E}_g$ for $m \leq m_c$ and $\mathcal{E}_g$ does not have minimizers for $m > m_c$. In a slightly different context, Lu and Otto in [29] showed the non-existence of minimizers for large volumes and that the ball is the unique minimizer for small volumes when $N = 3$ and $g(x) = |x|^{-1}$. Lu and Otto were motivated by the ionization conjecture and the energy studied by them includes a background potential, which behaves like an attractive term. In the similar context to [29], Frank, Nam, and Van Den Bosch in [17] showed the non-existence of minimizers for large volumes. In contrast, Alama, Bronsard, Choksi, and Topaloglu in [1] proved that a variant of Gamow's model including the background potential admits minimizers for any volume, due to the effects from the background potential against the Riesz potential. Very recently, Novaga and Pratelli in [31] showed the existence of generalized minimizers for the energy associated with $\mathcal{E}_g$ for any volume. After this work, Carazzato, Fusco, and Pratelli in [9] showed that the ball is the unique minimizer for small volumes in any dimensions and for a general function g. Concerning the behavior of (generalized) minimizers for large volumes, Pegon in [33] showed that, if the kernel g decays sufficiently fast at infinity and if the volume is sufficiently large, then minimizers exist and converge to a ball, up to rescaling, when the volume goes to infinity. Shortly after, Merlet and Pegon in [30] proved that, in dimension $N = 2$, minimizers are actually balls for large enough volumes.

A remarkable feature of our results is that, if the kernel g of the Riesz potential decays faster than that of the fractional perimeter (and is not necessarily compactly supported), then there always exists a minimizer of $\mathcal{E}_{s,g}$ for any volumes. This phenomena is not well-understood in the classical case. Indeed, Rigot in [35] proved the existence of minimizers of $\mathcal{E}_g$ for any volumes under the assumption that the kernel g has a compact support. For the case that the kernel g does not have a compact support, Pegon in [33] recently showed the existence of minimizers of $\mathcal{E}_g$ only for sufficiently large volumes whenever g decays sufficiently fast. In contrast, we show that, even if the kernel g does not have a compact support, there exists a minimizer of $\mathcal{E}_{s,g}$ for any volumes whenever the kernel g decays sufficiently fast. Thus, we can say that, unlike the classical cases in [35] and [33], the fractional perimeter, which can be understood as an interpolation between the classical perimeter and volume measure, would play an important role of ensuring the existence of minimizers for any volumes.

By a heuristic argument, one can observe that, if g decays sufficiently fast, the fractional perimeter dominates the Riesz potential even if the volume is sufficiently large. Indeed, if $g(x) \lesssim |x|^{-(N+s')}$ and $s' > s$, then we obtain that

$$\begin{aligned}\mathcal{E}_{s,g}(\lambda E) &= \lambda^{N-s}\, P_s(E) + \lambda^{2N} \int_E \int_E g(\lambda(x-y))\, dx\, dy \\ &= \lambda^{N-s} \left(P_s(E) + \lambda^{s-s'} \int_E \int_E \lambda^{N+s'} g(\lambda(x-y))\, dx\, dy \right)\end{aligned}$$

for any set $E \subset \mathbb{R}^N$ and $\lambda > 0$. Since we assume that $s' > s$, the Riesz potential could be dominated by the fractional perimeter term as $\lambda \to \infty$. Thus, one natural question is what would be the behavior of the energy like in the case that the kernel g behaves like or is controlled by the kernel $|x|^{-(N+s)}$ of the fractional perimeter P_s.

In this paper, we answer this question. More precisely, we obtain the existence of minimizers for any volume and we characterize the asymptotic behavior of minimizers as the volume goes to infinity, assuming that the kernel g decays faster than the kernel of the fractional perimeter P_s. More precisely, we first prove the existence of minimizers of $\mathcal{E}_{s,g}$ for any volume if g is non-negative, radially non-increasing, symmetric with respect to the origin, and decays faster than the kernel of P_s. For the precise assumptions on g, we refer to Section 3. The strategy of the proof is inspired by the concentration-compactness lemma by Lions [27, 28], and we will give some intuitive explanation of the strategy before proving the claim in Section 4.

We then prove the existence of generalized minimizers of a generalized functional $\widetilde{\mathcal{E}}_{s,g}$, which we will define later, under the assumption that the kernel g vanishes at infinity. It is easy to see that this assumption is weaker than the assumption that g decays faster than the kernel of the fractional perimeter, which we imposed to prove the first result. For convenience, we here give the definitions of the generalized functional and generalized minimizers. For any $m > 0$, we define a *generalized functional* of $\mathcal{E}_{s,g}$ over the family of sequences of the sets $\{E^k\}_{k\in\mathbb{N}}$ with $\sum_{k=1}^{\infty} |E^k| = m$ as

$$\widetilde{\mathcal{E}}_{s,g}\left(\{E^k\}_{k\in\mathbb{N}}\right) := \sum_{k=1}^{\infty} \mathcal{E}_{s,g}(E^k). \tag{1.4}$$

Then we consider the minimization problem

$$\inf\left\{\widetilde{\mathcal{E}}_{s,g}\left(\{E^k\}_{k\in\mathbb{N}}\right) \mid E^k\text{: measurable for any } k,\ \sum_k |E^k| = m\right\} \tag{1.5}$$

and show the existence of a minimizer of Problem (1.5) for any $m > 0$. We call such a minimizer a *generalized minimizer* of $\mathcal{E}_{s,g}$. The precise statement will be given Theorem 2.4 in Section 2. The idea to prove our second result is to show the identity

$$\inf\{\mathcal{E}_{s,g}(E) \mid |E| = m\} = \inf\left\{\widetilde{\mathcal{E}}_{s,g}(\{E^k\}_k) \mid \sum_{k=1}^{\infty} |E^k| = m\right\}$$

for any $m > 0$ and apply the same method which we use in the proof of our first result.

Finally, we investigate the asymptotic behavior of minimizers as the volume goes to infinity, under the assumption that g decays faster at infinity than the kernel $|x|^{-(N+s)}$ of the fractional perimeter P_s. Here we require an assumption on g which is stronger than the one we assume in the existence result. To study the asymptotic behavior, we consider an equivalent minimization problem. More precisely, one can have two problems equivalent to $E_{s,g}[m]$ for $m > 0$ under a proper decay assumption on g. Indeed, if the kernel g is integrable in $\mathbb{R}^N$, one can rewrite the Riesz potential as

$$\int_E \int_E g(x-y)\,dx\,dy = |E|\,\|g\|_{L^1(\mathbb{R}^N)} - \int_E \int_{E^c} g(x-y)\,dx\,dy$$

for any measurable set $E \subset \mathbb{R}^N$ with $|E| < \infty$. Hence, the minimization problem (1.1) becomes

$$\widehat{E}_{s,g}[m] := \inf\left\{P_s(E) - \int_E \int_{E^c} g(x-y)\,dx\,dy \mid |E| = m\right\} \tag{1.6}$$

for any $m > 0$. Moreover, by rescaling, one can further modify the minimization problem (1.6) into the equivalent problem

$$\widehat{E}^{\lambda}_{s,g}[|B_1|] := \inf\left\{\widehat{\mathcal{E}}^{\lambda}_{s,g}(F) := P_s(F) - \int_E \int_{E^c} \lambda^{N+s} g(\lambda(x-y))\,dx\,dy \mid |F| = |B_1|\right\} \tag{1.7}$$

for any $\lambda > 0$. Note that we will revisit more precisely the notations in (1.6) and (1.7) in Section 2. With this notation, our last theorem is as follows; suppose that $\{F_n\}_n$ is any sequence of the minimizers of $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ such that $\lambda_n \to \infty$ and $|F_n| = |B_1|$ for any n. Then we have that the full sequence satisfies

$$|F_n \Delta B_1| \xrightarrow[n\to\infty]{} 0$$

up to translations.

The organization of this paper is as follows: in Section 2, we will state our main results, namely, the existence of minimizers, the existence of generalized minimizers, and the convergence of any sequence of rescaled minimizers to the ball. In Section 3, we will give several preliminary properties of minimizers of our energy. In Section 4, we will prove the existence of minimizers for any volumes and, in Section 5, we will prove the existence of generalized minimizers for any volumes. In Section 6, we will study the asymptotic behavior of rescaled minimizers as the volume goes to infinity. We will also give the Γ-convergence result for our energy.

2 Main results

We start with the assumptions on the kernel g of the Riesz potential in the energy $\mathcal{E}_{s,g}$. Throughout this paper, we assume that $s \in (0,\,1)$ and $g : \mathbb{R}^N\setminus\{0\} \to \mathbb{R}$ is in $L^1_{loc}(\mathbb{R}^N)$ and not identically equal to zero. We consider the following conditions on g:

(g1) g is non-negative and radially non-increasing, namely,

$$g(\lambda\, x) \leq g(x) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\} \text{ and } \lambda \geq 1.$$

(g2) g is symmetric with respect to the origin, namely, $g(-x) = g(x)$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

When we prove the existence of minimizers of $\mathcal{E}_{s,g}$ in Section 4, we further assume the following condition on g:

(g3) There exist constants $R_0 > 1$ and $\beta \in (0,\,1)$ such that

$$g(x) \leq \frac{\beta}{|x|^{N+s}} \quad \text{for any } |x| \geq R_0.$$

(g decays faster than the kernel of P_s far away from the origin.)

On the other hand, when we prove the existence of generalized minimizers of $\widetilde{\mathcal{E}}_{s,g}$ in Section 5, we assume the following condition:

(g4) g vanishes at infinity, namely, $g(x) \to 0$ as $|x| \to \infty$.

Note that (g4) is weaker than (g3). Moreover, when we study the asymptotic behavior of rescaled minimizers with large volumes in Section 6, we further impose the following assumption on g:

(g5) There exists a constant $\gamma \in (0,\,1)$ such that

$$g(x) \leq \frac{\gamma}{|x|^{N+s}} \quad \text{for any } x \in \mathbb{R}^N \setminus \{0\}, \quad g(x) = o\left(\frac{1}{|x|^{N+s}}\right) \quad \text{as } |x| \to \infty.$$

Note that (g5) is stronger than (g3). Thus, we have the following implication of the conditions on g:

$$\text{(g5)} \Rightarrow \text{(g3)} \Rightarrow \text{(g4)}$$

where $p \Rightarrow q$ means that p implies q.

Remark 2.1. From the assumption that $g \in L^1_{loc}(\mathbb{R}^N)$, we can easily show that $V_g(B) < \infty$ for any ball $B \subset \mathbb{R}^N$. Indeed, one may compute

$$V_g(B) \leq \int_B \int_{2B(y)} g(x-y)\, dx\, dy = |B| \int_{2B(0)} g(x)\, dx < \infty.$$

Moreover, with (g3), we actually have that g is integrable in $\mathbb{R}^N$. Indeed, if $g \in L^1_{loc}(\mathbb{R}^N)$, we have that $\|g\|_{L^1(B_{R_0})} < \infty$. On the other hand, from (g3) and the integrability of $|x|^{-(N+s)}$ in $B^c_{R_0}$, we also have that $\|g\|_{L^1(B^c_{R_0})} < \infty$. Hence, the claim holds true.

Remark 2.2. A condition ensuring the assumption (g5) is the existence of constants $R_0 > 1$, $\gamma \in (0,\,1)$, and $t > s$ such that

$$g(x) \leq \begin{cases} \dfrac{\gamma}{|x|^{N+s}} & \text{if } 0 < |x| < R_0 \\ \dfrac{1}{|x|^{N+t}} & \text{if } |x| \geq R_0. \end{cases} \tag{2.1}$$

We can easily observe that (2.1) is stronger than assumption (g3).

Now we can state the main results of this paper. In the first result, we show the existence of minimizers of $\mathcal{E}_{s,g}$ for any volume under the assumption that g decays faster than the kernel of P_s at infinity.

Theorem 2.3. *Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions* (g1), (g2), *and* (g3). *Then, there exists a minimizer of $\mathcal{E}_{s,g}$ with volume m for any $m > 0$.*

Moreover, the boundary of every minimizer has the regularity of class $C^{1,\alpha}$ with $\alpha \in (0, 1)$ except a closed set of Hausdorff dimension $N - 3$.

The proof is inspired by so-called the "concentration-compactness" lemma by Lions in [27, 28] and we apply the same idea shown in [13]. We will roughly explain the idea of the proof in Section 4.

In the second theorem, we show the existence of generalized minimizer of $\widetilde{\mathcal{E}}_{s,g}$ for any volume, under the assumption that g vanishes at infinity. Notice that this assumption is weaker than the one we impose in Theorem 2.3.

Theorem 2.4. *Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions* (g1), (g2), *and* (g4). *Then, there exists a generalized minimizer of $\widetilde{\mathcal{E}}_{s,g}$ for any $m > 0$, namely, there exist a number $M \in \mathbb{N}$ and a sequence of sets $\{E^k\}_{k=1}^M$ such that*

$$\sum_{k=1}^{M} \mathcal{E}_{s,g}(E^k) = \inf \left\{ \widetilde{\mathcal{E}}_{s,g}(\{F^k\}_k) \mid \sum_{k=1}^{\infty} |F^k| = m \right\},$$

and E^k is also a minimizer of $\mathcal{E}_{s,g}$ among sets of volume $|E^k|$ for every $k \in \mathbb{N}$.

As we mentioned in Section 1, the idea of the proof is based on the observation that Problem (1.5) can be reduced into Problem (1.1).

Finally, we study the asymptotic behavior of minimizers of $\mathcal{E}_{s,g}$ when the volume goes to infinity, under the assumption that g decays much faster than the kernel $|x|^{-(N+s)}$ of P_s far away from the origin.

Before stating the theorem, in order to study the behavior of the minimizers of the minimization problem $E_{s,g}[m]$ for any $m > 0$, it is convenient to lift the volume constraint onto the functional itself and work with fixed volume $|B_1|$. To see this, we first define a rescaled kernel by

$$g_\lambda(x) := \lambda^{N+s}\, g(\lambda\, x) \tag{2.2}$$

for any $x \neq 0$ and $\lambda > 0$. Then we show the equivalence of the rescaled problem in the following proposition.

Proposition 2.5 (Equivalent problem)**.** *Let $m > 0$. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is in $L^1_{loc}(\mathbb{R}^N)$. Then, setting $\lambda^N := m\,|B_1|^{-1}$, we have that the problem $E_{s,g}[m]$ is equivalent to*

$$E^{\lambda}_{s,g}(B_1) := \inf \left\{ P_s(F) + V_{g_\lambda}(F) \mid F \subset \mathbb{R}^N : \textit{measurable},\ |F| = |B_1| \right\}$$

where g_λ is given in (2.2).

Moreover, under the assumption that g is integrable on $\mathbb{R}^N$, the minimization problem $E_{s,g}[m]$ is also equivalent to Problem (1.7).

Proof. Given any E with $|E| = m$ and setting $F := \lambda^{-1} E$ where $\lambda^N = m\,|B_1|^{-1}$, we have that $|F| = |B_1|$ and

$$\begin{aligned}
\mathcal{E}_{s,g}(E) &= \lambda^{N-s}\, P_s(F) + \lambda^{2N} \int_F \int_F g(\lambda(x-y))\, dx\, dy \\
&= \lambda^{N-s} \left(P_s(F) + \int_F \int_F \lambda^{N+s} g(\lambda(x-y))\, dx\, dy \right) \\
&= \lambda^{N-s} \left(P_s(F) + V_{g_\lambda}(F) \right)
\end{aligned} \tag{2.3}$$

where $g_\lambda(x) := \lambda^{N+s} g(\lambda x)$ as in (2.2). For the latter part of the claim, we first recall the equivalent minimization problem

$$\widehat{E}_{s,g}[m] := \inf \left\{ P_s(E) - \int_E \int_{E^c} g(x-y)\, dx\, dy \right\},$$

which is equivalent to the problem $E_{s,g}[m]$ for any $m > 0$. Thus, from (2.3), we obtain that

$$\mathcal{E}_{s,g}(E) = \lambda^{N-s} \left(P_s(F) - \int_F \int_{F^c} g_\lambda(x-y)\, dx\, dy + m\, \|g\|_{L^1(\mathbb{R}^N)} \right).$$

Hence, we conclude that the claim is valid. □

Now we are ready to state the last theorem of this present paper.

Theorem 2.6. *Let $s \in (0,\, 1)$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset (1,\, \infty)$ with $\lambda_n \to \infty$ as $n \to \infty$. Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of minimizers for $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ with $|F_n| = |B_1|$ for each $n \in \mathbb{N}$. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions* (g1), (g2), *and* (g5). *Then, the sequence $\{F_n\}_{n\in\mathbb{N}}$ converges to the unit ball B_1, up to translations, in the sense of L^1-topology, namely,*

$$|F_n \Delta B_1| \xrightarrow[n\to\infty]{} 0.$$

The idea of the proof is based on the same argument of Theorem 2.3, and we will give the precise description of the proof in Section 6. Moreover, we show the Γ-convergence of the energy $\widehat{\mathcal{E}}^{\lambda}_{s,g}$ as $\lambda \to \infty$. For the detail, the readers should refer to Proposition 6.1 in Section 6.

3 Preliminary results for minimizers of $\mathcal{E}_{s,g}$

In this section, we collect several properties for minimizers of $\mathcal{E}_{s,g}$ under the assumptions on g in Section 2

First of all, we recall an important property of the fractional perimeter P_s with $0 < s < 1$.

Proposition 3.1. *For any $s \in (0,\, 1)$ and measurable set $E \subset \mathbb{R}^N$ with $|E| < \infty$, it holds that $P_s(E \cap K) \leq P_s(E)$ for every convex set $K \subset \mathbb{R}^N$.*

The proof can be found in [15, Lemma B.1] and we omit the proof of this proposition here. We also refer to [7, Corollary 5.3] and [2] for related properties to Proposition 3.1.

We next prove the boundedness of minimizers of $\mathcal{E}_{s,g}$ among sets of volume m.

Lemma 3.2 (Boundedness of minimizers)**.** *Let $m > 0$. Assume that the kernel g satisfies the conditions* (g1) *and* (g2). *If $E \subset \mathbb{R}^N$ is a minimizer of $\mathcal{E}_{s,g}$ with $|E| = m$, then E is essentially bounded, namely, there exists a constant $R > 0$ such that $|E \setminus B_R(0)| = 0$.*

Proof. Let E be a minimizer of $\mathcal{E}_{s,g}$ with $|E| = m$. By setting $\phi(r) := |E \setminus B_r(0)|$ for any $r > 0$, we have that $\phi'(r) = -\mathcal{H}^{N-1}(E \cap \partial B_r(0))$ for a.e. $r > 0$. In order to prove the claim, we suppose by contradiction that $\phi(r) > 0$ for any $r > 0$. Setting $E_r := E \cap B_r(0)$ and $(\lambda_r)^N := \frac{m}{m-\phi(r)}$ for any $r > 0$, then we choose $\lambda_r E_r$ as the competitor of E for each $r > 0$ with $\phi(r) < m$ and thus, we have that

$$\begin{aligned} \mathcal{E}_{s,g}(E) \leq \mathcal{E}_{s,g}(\lambda_r E_r) &\leq (\lambda_r)^{N-s} P_s(E_r) + (\lambda_r)^{2N} V_{g_{\lambda_r}}(E_r) \\ &\leq \mathcal{E}_{s,g}(E_r) + \left((\lambda_r)^{N-s} - 1\right) P_s(E_r) + \left((\lambda_r)^{2N} - 1\right) V_g(E_r). \end{aligned} \tag{3.1}$$

Since $\phi(r) \to 0$ as $r \to \infty$, we can choose a constant $R_0 > 0$, depending on E, such that $\phi(r) \leq m/2$ for any $r \geq R_0$ and thus, we may assume that

$$(\lambda_r)^{N-s} - 1 \leq c_0\, \phi(r), \quad (\lambda_r)^{2N} - 1 \leq c_0'\, \phi(r) \tag{3.2}$$

for any $r \geq R_0$ where c_0 and c_0' are some positive constants depending only on N, s, and m. We recall the following identities of the fractional perimeter and the Riesz potential: for any measurable sets A, $B \subset \mathbb{R}^N$ with $|A \cap B| = 0$,

$$P_s(A \cup B) = P_s(A) + P_s(B) - 2 \int_A \int_B \frac{dx\, dy}{|x-y|^{N+s}} \tag{3.3}$$

$$V_g(A \cup B) = V_g(A) + V_g(B) + 2 \int_A \int_B g(x-y)\, dx\, dy. \tag{3.4}$$

The proof follows from the direct computations. Then, from (3.3) and (3.4) and by combining (3.2) with (3.1), we have that

$$\begin{aligned} P_s(E \setminus B_r(0)) &\leq P_s(E \setminus B_r(0)) + V_g(E \setminus B_r(0)) \\ &\leq 2 \int_{E \cap B_r(0)} \int_{E \setminus B_r(0)} \frac{dx\, dy}{|x-y|^{N+s}} + c_0\, \phi(r)\, P_s(E_r) + c_0'\, \phi(r)\, V_g(E_r) \end{aligned} \tag{3.5}$$

for any $r \geq R_0$. From Proposition 3.1 and the definition of V_g, we have that, for any $r > 0$,

$$P_s(E_r) + V_g(E_r) \leq P_s(E) + V_g(E) = E_{s,g}[m].$$

Thus, from (3.5), we obtain that

$$P_s(E \setminus B_r(0)) \leq 2 \int_{E \cap B_r(0)} \int_{E \setminus B_r(0)} \frac{dx\, dy}{|x-y|^{N+s}} + (c_0 + c_0') E_{s,g}[m]\, \phi(r)$$

for any $r \geq R_0$. Now using the isoperimetric inequality of P_s and the fact that $E \cap B_r(0) \subset B^c_{|y|-r}(y)$ for any $y \in E \setminus B_r(0)$, we obtain

$$\begin{aligned} \frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}}\, \phi(r)^{\frac{N-s}{N}} &\leq 2 \int_{E \setminus B_r(0)} \int_{B^c_{r-|y|}(y)} \frac{dx\, dy}{|x-y|^{N+s}} + (c_0 + c_0') E_{s,g}[m]\, \phi(r) \\ &= 2|\partial B_1| \int_{E \setminus B_r(0)} \int_{|y|-r}^{\infty} \frac{1}{t^{1+s}}\, dt\, dy + (c_0 + c_0') E_{s,g}[m]\, \phi(r) \\ &= \frac{2|\partial B_1|}{s} \int_{E \setminus B_r(0)} \frac{1}{(|y|-r)^s}\, dy + (c_0 + c_0') E_{s,g}[m]\, \phi(r) \\ &= \frac{2|\partial B_1|}{s} \int_r^{\infty} \frac{-\phi'(\sigma)}{(\sigma - r)^s}\, d\sigma + (c_0 + c_0') E_{s,g}[m]\, \phi(r) \end{aligned} \tag{3.6}$$

for any $r > 0$. Here we have used the co-area formula in the last equality. Since ϕ is non-increasing, there exists a constant $R_0' = R_0'(N, s, m) > 0$ such that

$$(c_0 + c_0') E_{s,g}[m]\, \phi(r) \leq \frac{P_s(B_1)}{2|B_1|^{\frac{N-s}{N}}}\, \phi(r)^{\frac{N-s}{N}} \tag{3.7}$$

for any $r \geq \max\{R_0, R_0'\}$. From (3.6) and (3.7), we obtain

$$c_1\, \phi(r)^{\frac{N-s}{N}} \leq c_2 \int_r^{\infty} \frac{-\phi'(\sigma)}{(\sigma - r)^s}\, d\sigma \tag{3.8}$$

for any $r \geq \max\{R_0, R_0'\}$ where we set $c_1 := (2|B_1|^{\frac{N-s}{N}})^{-1} P_s(B_1)$ and $c_2 := 2s^{-1}|\partial B_1|$. By integrating the both sides in (3.8) over $r \in [R, \infty)$ for any fixed constant $R \geq \max\{R_0, R_0'\}$ and changing the order of the integration, we obtain

$$\begin{aligned} c_1 \int_R^{\infty} \phi(r)^{\frac{N-s}{N}}\, dr \leq c_2 \int_R^{\infty} \int_r^{\infty} \frac{-\phi'(\sigma)}{(\sigma - r)^s}\, d\sigma\, dr &= c_2 \int_R^{\infty} \int_R^{\sigma} \frac{-\phi'(\sigma)}{(\sigma - r)^s}\, dr\, d\sigma \\ &= -\frac{c_2}{1-s} \int_R^{\infty} \phi'(\sigma)\, (\sigma - R)^{1-s}\, d\sigma. \end{aligned} \tag{3.9}$$

Hence, by employing the same argument shown in [13, Lemma 4.1] and [10, Proposition 3.2] together with (3.9), we obtain that $\phi(R) = 0$, which contradicts the assumption that $\phi(r) > 0$ for any $r > 0$. Therefore, we obtain that there exists $\hat{R} > 0$ such that $|E \setminus B_{\hat{R}}| = 0$. $\square$

Next, by using assumption (g4), we show the sub-additivity result of the function $m \mapsto E_{s,g}[m]$. We recall that $E_{s,g}[m]$ is defined by

$$\inf\left\{\mathcal{E}_{s,g}(E) \mid E \subset \mathbb{R}^N\text{: measurable, } |E| = m\right\}.$$

for any $m > 0$.

Lemma 3.3 (Sub-additivity of $E_{s,g}$)**.** *Let $m > 0$ be any number. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies* (g1), (g2), *and* (g4). *Then, for any $m_1 \in (0, m]$, it holds*

$$E_{s,g}[m] \leq E_{s,g}[m_1] + E_{s,g}[m - m_1].$$

Proof. The idea is in the same spirit as the one shown in [29, Lemma 3] (see also [32]).

Let $m > 0$ be any constant and we take any $m_1 \in (0, m)$. By definition, for any $\eta > 0$, there exist measurable sets E_1, $E_2 \subset \mathbb{R}^N$ with the volume constraints $|E_1| = m_1$ and $|E_2| = m - m_1$ such that

$$\mathcal{E}_{s,g}(E_1) + \mathcal{E}_{s,g}(E_2) \leq E_{s,g}[m_1] + E_{s,g}[m_2] + \eta. \tag{3.10}$$

Now we may assume that E_1 and E_2 are bounded. Indeed, we can observe that the minimum of $\mathcal{E}_{s,g}$ among unbounded sets of volume m is not smaller than the minimum of $\mathcal{E}_{s,g}$ among bounded sets of volume m. To see this, for any unbounded set E with $|E| = m$, we can choose sufficiently large $R > 1$ in such a way that $|E \setminus B_R(0)|$ is as small as possible. Then, setting $\widehat{E} \coloneqq \lambda(R)\,(E \cap B_R(0))$ where $\lambda(R)^N \coloneqq \frac{m}{m - |E \setminus B_R(0)|} \geq 1$, we obtain that

$$|\widehat{E}| = \lambda(R)^N\,(m - |E \setminus B_R(0)|) = m$$

and

$$\begin{aligned}
\mathcal{E}_{s,g}(\widehat{E}) &\leq \lambda(R)^{2N}\mathcal{E}_{s,g}(E \cap B_R(0)) \\
&\leq \lambda(R)^{2N}\mathcal{E}_{s,g}(E) - P_s(E \setminus B_R(0)) + 2\int_{E \cap B_R(0)}\int_{E \setminus B_R(0)} \frac{dx\,dy}{|x-y|^{N+s}}.
\end{aligned} \tag{3.11}$$

Here we have used (3.3). From the isoperimetric inequality and the computation in (3.6) in Lemma 3.2, we have that

$$\mathcal{E}_{s,g}(\widehat{E}) \leq \lambda(R)^{2N}\mathcal{E}_{s,g}(E) - C_1\,|E \setminus B_R(0)|^{\frac{N-s}{N}} + C_2\int_R^\infty \frac{\mathcal{H}^{N-1}(E \cap \partial B_\sigma(0))}{(\sigma - R)^s}\,d\sigma \tag{3.12}$$

where we set $C_1 \coloneqq P_s(B_1)\,|B_1|^{-\frac{N-s}{N}}$ and $C_2 \coloneqq 2s^{-1}|\partial B_1|$. Since E is unbounded, we have that the function $R \mapsto |E \setminus B_R(0)|$ is non-increasing and not equal to zero for any $R > 0$. Thus, by applying the same argument in Lemma 3.2, we can find that there exists a sequence $\{R_i\}_{i\in\mathbb{N}}$ such that $R_i \to \infty$ as $i \to \infty$ and

$$-C_1\,|E \setminus B_{R_i}(0)|^{\frac{N-s}{N}} + C_2\int_{R_i}^\infty \frac{\mathcal{H}^{N-1}(E \cap \partial B_\sigma(0))}{(\sigma - R_i)^s}\,d\sigma < 0 \tag{3.13}$$

for any $i \in \mathbb{N}$. Hence, from (3.12) and (3.13), it follows that

$$\inf\{\mathcal{E}_{s,g}(E) \mid E\text{: measurable \& bounded, } |E| = m\} \leq \mathcal{E}_{s,g}(\widehat{E}) < \lambda(R_i)^{2N}\mathcal{E}_{s,g}(E)$$

for any $i \in \mathbb{N}$. From the fact that $\lambda(R_i) \to 1$ as $i \to \infty$, the arbitrariness of E and by letting $i \to \infty$, we finally obtain that

$$\inf\{\mathcal{E}_{s,g}(E) \mid E\text{: bounded, } |E| = m\} \leq \inf\{\mathcal{E}_{s,g}(E) \mid E\text{: unbounded, } |E| = m\},$$

as we desired.

Now we focus on the case that both E_1 and E_2 are bounded. Since E_1, E_2 are bounded, we can find a vector $e \in \mathbb{S}^{N-1}$ such that it follows that

$$\operatorname{dist}(E_1, (E_2 + d\,e)) \xrightarrow[d\to\infty]{} \infty.$$

Then we may compute the energy as follows:

$$\begin{aligned}\mathcal{E}_{s,g}(E_1 \cup (E_2 + d\,e)) &= P_s(E_1 \cup (E_2 + d\,e)) + V_g(E_1 \cup (E_2 + d\,e))\\ &\le P_s(E_1) + P_s(E_2 + d\,e)\\ &\qquad + V_g(E_1) + V_g(E_2 + d\,e) + 2\int_{E_1}\int_{E_2 + d\,e} g(x-y)\,dx\,dy\\ &\le \mathcal{E}_{s,g}(E_1) + \mathcal{E}_{s,g}(E_2) + 2\int_{E_1}\int_{E_2+d\,e} g(x-y)\,dx\,dy.\end{aligned}$$

Here we have used the translation invariance of P_s and V_g. From assumption (g4), which says that g vanishes at infinity, we can show that

$$\int_{E_1}\int_{E_2+d\,e} g(x-y)\,dx\,dy \xrightarrow[d\to\infty]{} 0.$$

Since $|E_1 \cup (E_2 + d\,e)| = |E_1| + |E_2| = m$ for sufficiently large $d > 0$ and from (3.10), we obtain

$$E_{s,g}[m_1 + m - m_1] \le E_{s,g}[m_1] + E_{s,g}[m - m_1] + \eta + o(1).$$

Letting $d \to \infty$ and then $\eta \to 0$, we conclude that the lemma is valid. □

4 Existence of minimizers for $\mathcal{E}_{s,g}$ for any volumes

In this section we prove Theorem 2.3, giving the existence of minimizers of the functional $\mathcal{E}_{s,g}$ for any volume $m > 0$, under the assumption that g decays faster than the kernel of the fractional perimeter P_s.

As we mentioned in the introduction, the proof is inspired by the so-called "concentration-compactness" principle introduced by P. L. Lions in [27, 28]. When one studies the variational problems in unbounded domain, the possible loss of compactness may occur from the vanishing or splitting into many pieces of minimizing sequences (see [13, 20, 10]), so that one may not apply the direct method of calculus of variations to our problem.

Proof of Theorem 2.3. Let $m > 0$ be any number and let $\{E_n\}_{n\in\mathbb{N}}$ be any minimizing sequence for the functional $\mathcal{E}_{s,g}$ with $|E_n| = m$ for all n. We divide the proof into 4 steps.

Step 1. We first show that, under (g1) and (g2), there exist sets $\{G^j\}_j$ such that

$$\sum_j \mathcal{E}_{s,g}(G^j) \le \liminf_{n\to\infty} \mathcal{E}_{s,g}(E_n), \quad \sum_j |G^j| = m.$$

We first decompose $\mathbb{R}^N$ into infinitely many disjoint cubes of side 1 (denoted by $\{Q^i\}_{i=1}^{\infty}$). From Lemma 3.2, each minimizer E_n is bounded and thus, we can choose a number $I_n \in \mathbb{N}$ in such a way that $|E_n \cap Q^i| > 0$ for any $i \in \{1, \cdots, I_n\}$ and $|E_n \cap Q^i| = 0$ for any $i > I_n$. We set $x_n^i \coloneqq |E_n \cap Q^i|$ and we have that

$$\sum_{i=1}^{I_n} x_n^i = |E_n| = m \tag{4.1}$$

for any $n \in \mathbb{N}$. Since E_n is a minimizer with $|E_n| = m$ for any n, we can choose a ball with volume m as a competitor and then, from the local integrability of g, have

$$\sup_{n\in\mathbb{N}} P_s(E_n) \leq P_s(B_m) + V_g(B_m) \leq \left(\frac{m}{|B_1|}\right)^{\frac{N-s}{N}} P_s(B_1) + m\,\|g\|_{L^1(2B_m)} < \infty \tag{4.2}$$

where B_m is an open ball with volume m for each $m > 0$. From (4.2) and the isoperimetric inequality shown in [13, Lemma 2.5], we obtain

$$\sum_{i=1}^{I_n} (x_n^i)^{\frac{N-s}{N}} \leq C \sum_{i=1}^{I_n} P_s(E_n; Q^i) \leq 2C P_s(E_n) \leq C_1 < \infty \tag{4.3}$$

for any $n \in \mathbb{N}$, where C and C_1 are the positive constants independent of n. Here we define the localised fractional perimeter $P_s(A; B)$ by

$$P_s(A; B) \coloneqq \int_{A\cap B}\int_{A^c} \frac{dx\,dy}{|x-y|^{N+s}} + \int_{A\cap B^c}\int_{A^c\cap B} \frac{dx\,dy}{|x-y|^{N+s}} \tag{4.4}$$

for any measurable sets A, $B \subset \mathbb{R}^N$. Up to reordering the cubes $\{Q^i\}_i$, we may assume that $\{x_n^i\}_i$ is a non-increasing sequence for any $n \in \mathbb{N}$. Thus, applying the technical result shown in [20, Lemma 4.2] or [13, Lemma 7.4] with (4.1) and (4.3), we obtain that

$$\sum_{i=k+1}^{\infty} x_n^i \leq \frac{C_2}{k^{\frac{s}{N}}} \tag{4.5}$$

for any $k \in \mathbb{N}$, where we set $x_n^i \coloneqq 0$ for any $i > I_n$ and C_2 is the positive constant independent of n and k. Hence, by a diagonal argument we have that, up to extracting a subsequence, $x_n^i \to \alpha^i \in [0,\, m]$ as $n \to \infty$ for every $i \in \mathbb{N}$ and, from (4.1) and (4.5),

$$\sum_{i=1}^{\infty} \alpha^i = m. \tag{4.6}$$

Now we choose a point $z_n^i \in E_n \cap Q^i$ for each i and n whenever $|E_n \cap Q^i| > 0$. Up to extracting a further subsequence, we may assume that $|z_n^i - z_n^j| \to c^{ij} \in [0,\, \infty]$ as $n \to \infty$ for each i, $j \in \mathbb{N}$ and, since we have, from (4.2), the uniform bound of the sequence $\{P_s(E_n - z_n^i)\}_{n\in\mathbb{N}}$ and its upper-bound is independent of i, there exists a measurable set $G^i \subset \mathbb{R}^N$ such that, up to a subsequence,

$$\chi_{E_n - z_n^i} \xrightarrow[n\to\infty]{} \chi_{G^i} \quad \text{in } L^1_{loc}\text{-topology.}$$

We define the relation $i \sim j$ for every i, $j \in \mathbb{N}$ as $c^{ij} < \infty$ and we denote by $[i]$ the equivalent class of i. Moreover, we define the set of the equivalent class by $\mathcal{I}$. Then we show the following:

$$\sum_{[i]\in\mathcal{I}} P_s(G^i) \leq \liminf_{n\to\infty} P_s(E_n), \qquad \sum_{[i]\in\mathcal{I}} V_g(G^i) \leq \liminf_{n\to\infty} V_g(E_n). \tag{4.7}$$

Indeed, we first fix $M \in \mathbb{N}$ and $R > 0$ and we take the equivalent classes $i_1, \cdots, i_M$. Notice that, if $p \neq q$, then $|z_n^{i_p} - z_n^{i_q}| \to \infty$ as $n \to \infty$ and thus we have that $\{z_n^{i_p} + Q_R\}_p$ are disjoint sets for large n and

$$\int_{z_n^{i_p}+Q_R}\int_{z_n^{i_q}+Q_R} \frac{1}{|x-y|^{N+s}}\,dx\,dy \xrightarrow[n\to\infty]{} 0$$

where Q_R is the cube of side R. Now we recall the following inequality for the localized fractional perimeter:

$$P_s(E; A) + P_s(E; B) \leq P_s(E; A\cup B) + 2\int_A\int_B \frac{dx\,dy}{|x-y|^{N+s}} \tag{4.8}$$

for any measurable disjoint sets A, $B \subset \mathbb{R}^N$. As a consequence, from the lower semi-continuity of P_s, we obtain

$$\begin{aligned}\sum_{p=1}^{M} P_s(G^{i_p}; Q_R) &\leq \liminf_{n\to\infty} \sum_{p=1}^{M} P_s(E_n - z_n^{i_p}; Q_R) \\ &= \liminf_{n\to\infty} \sum_{p=1}^{M} P_s(E_n; z_n^{i_p} + Q_R) \\ &\leq \liminf_{n\to\infty} P_s\left(E_n; \bigcup_{p=1}\left(z_n^{i_p} + Q_R\right)\right) \\ &\qquad + \liminf_{n\to\infty} 2\sum_{p\neq q} \int_{z_n^{i_p}+Q_R}\int_{z_n^{i_q}+Q_R} \frac{dx\,dy}{|x-y|^{N+s}} \\ &\leq \liminf_{n\to\infty} P_s(E_n).\end{aligned}$$

Letting $R \to \infty$ and then $M \to \infty$, we obtain the first inequality in (4.7).

For the second inequality in (4.7), we again take any $M \in \mathbb{N}$ and $R > 0$. Then, in the same way as we have observed in the first claim, we have, from (3.4), Fatou's lemma, and the non-negativity of g, that

$$\begin{aligned}\sum_{p=1}^{M} V_g(G^{i_p} \cap Q_R) &\leq \liminf_{n\to\infty} \sum_{p=1}^{M} V_g\left(\left(E_n - z_n^{i_p}\right) \cap Q_R\right) \\ &= \liminf_{n\to\infty} \sum_{p=1}^{M} V_g\left(E_n \cap \left(z_n^{i_p} + Q_R\right)\right) \\ &\leq \liminf_{n\to\infty} V_g\left(E_n \cap \bigcup_{p=1}^{M}\left(z_n^{i_p} + Q_R\right)\right) \\ &\leq \liminf_{n\to\infty} V_g(E_n).\end{aligned}$$

Here we have used the fact that the sets $\{z_n^{i_p} + Q_R\}_{p=1}^M$ are disjoint if n is sufficiently large from the choice of the points $\{z_n^{i_p}\}_{p=1}^M$. Thus, letting $R \to \infty$ and then $M \to \infty$, we obtain the second claim.

Now we show that

$$\sum_{[i]\in\mathcal{I}} |G^i| = m.$$

Indeed, from the L^1_{loc}-convergence of $\{\chi_{E_n - z_n^i}\}_{n\in\mathbb{N}}$ for any i, we have that, for any $R > 0$ sufficiently large,

$$|G^i| \geq |G^i \cap Q_R| = \lim_{n\to\infty} |(E_n - z_n^i) \cap Q_R|. \tag{4.9}$$

If $j \in \mathbb{N}$ is such that $j \sim i$ and $c^{ij} < \frac{R}{100}$, then we have that $Q^i - z_n^i \subset Q_R$ for large $R > 0$ and all n. Thus, from (4.9), it follows

$$\begin{aligned}|(E_n - z_n^i) \cap Q_R| &= \sum_{i=1}^{I_n} |(E_n - z_n^i) \cap Q_R \cap \left(Q^i - z_n^i\right)| \\ &\geq \sum_{j:\, c^{ij} < \frac{R}{100}} |(E_n - z_n^i) \cap Q_R \cap \left(Q^i - z_n^i\right)| \\ &= \sum_{j:\, c^{ij} < \frac{R}{100}} |E_n \cap Q^i| \end{aligned}\tag{4.10}$$

for all n and large $R > 0$. Therefore, combining (4.10) with (4.9), we obtain

$$|G^i| \geq \sum_{j:\, c^{ij} < \frac{R}{100}} \alpha^i$$

and, letting $R \to \infty$, we have

$$|G^i| \geq \sum_{j:\, c^{ij} < \infty} \alpha^i = \sum_{j \in [i]} \alpha^i.$$

Hence, recalling (4.6), we have

$$\sum_{[i] \in \mathcal{I}} |G^i| \geq \sum_{[i] \in \mathcal{I}} \sum_{j \in [i]} \alpha^i = m. \tag{4.11}$$

Now we show the other inequality in the following manner; for any $M \in \mathbb{N}$ and $R > 0$, we take the equivalent classes $i_1, \cdots, i_M$. Then, from the choice of $\{G^i\}_i$, we have that

$$\begin{aligned} \sum_{p=1}^{M} |G^{i_p} \cap Q_R| &= \lim_{n \to \infty} \sum_{p=1}^{M} \left| \left(E_n - z_n^{i_p} \right) \cap Q_R \right| \\ &= \lim_{n \to \infty} \sum_{p=1}^{M} \left| E_n \cap \left(z_n^{i_p} + Q_R \right) \right|. \end{aligned} \tag{4.12}$$

Recalling the condition that $|z_n^{i_p} - z_n^{i_q}| \to \infty$ as $n \to \infty$ if $p \neq q$, we have that, for sufficiently large $n \in \mathbb{N}$, $\left(z_n^{i_p} + Q_R \right) \cap \left(z_n^{i_q} + Q_R \right) = \emptyset$ for any $p \neq q$. From (4.12), we have that

$$\sum_{p=1}^{M} |G^{i_p} \cap Q_R| = \lim_{n \to \infty} \left| E_n \cap \bigcup_{p=1}^{M} \left(z_n^{i_p} + Q_R \right) \right| \leq m$$

and thus, letting $R \to \infty$ and then $M \to \infty$, we obtain that

$$\sum_{[i] \in \mathcal{I}} |G^i| = \sum_{p=1}^{\infty} |G^{i_p}| \leq m.$$

This completes the proof of the claim. Taking into account all the above arguments, we obtain the existence of sets $\{G^i\}_{[i] \in \mathcal{I}}$ satisfying the properties that

$$\sum_{[i] \in \mathcal{I}} \mathcal{E}_{s,g}(G^i) \leq \liminf_{n \to \infty} \mathcal{E}_{s,g}(E_n), \qquad \sum_{[i] \in \mathcal{I}} |G^i| = m. \tag{4.13}$$

Step 2. We now claim that, under (g1), (g2), and (g3), each component G^i for $[i] \in \mathcal{I}$ is a minimizer of $\mathcal{E}_{s,g}$ among sets with volume $|G^i|$. Moreover, we show that G^i is bounded for each $[i] \in \mathcal{I}$. Note that this claim is actually valid under (g4), which is weaker than (g3).

Indeed, we first recall the definition of $E_{s,g}$, which says that

$$E_{s,g}[m] \coloneqq \inf \{ \mathcal{E}_{s,g}(E) \mid |E| = m \}$$

for any $m > 0$, and the sub-additivity result of the function $m \mapsto E_{s,g}[m]$ as in Lemma 3.3. Notice that we have imposed (g3) in Section 3, which is stronger than (g4). Thus, we can also apply Lemma 3.3 to this case. From (4.13), we have that

$$
\begin{aligned}
\sum_{p=1}^{M}\left(\mathcal{E}_{s,g}(G^{i_p}) - E_{s,g}[|G^{i_p}|]\right) &\le E_{s,g}[m] - \sum_{p=1}^{M} E_{s,g}[|G^{i_p}|] \\
&\le E_{s,g}\left[\sum_{p=M+1}^{\infty}|G^{i_p}|\right] + E_{s,g}\left[\sum_{p=1}^{M}|G^{i_p}|\right] - \sum_{p=1}^{M} E_{s,g}[|G^{i_p}|] \\
&\le E_{s,g}\left[\sum_{p=M+1}^{\infty}|G^{i_p}|\right] \qquad (4.14)
\end{aligned}
$$

for any $M \in \mathbb{N}$. We can observe that $E_{s,g}[m] \to E_{s,g}[0] = 0$ as $m \to 0$ because $E_{s,g}[m]$ can be bounded by the quantity $C_1\, m^{\frac{N-s}{N}} + C_2\, m$ for small $m > 0$, where C_1 and C_2 are the constants depending only on N, s, and g. Hence, letting $M \to \infty$ in (4.14), we obtain that

$$
\sum_{[i]\in\mathcal{I}}\left(\mathcal{E}_{s,g}(G^{i}) - E_{s,g}[|G^{i}|]\right) = \sum_{p=1}^{\infty}\left(\mathcal{E}_{s,g}(G^{i_p}) - E_{s,g}[|G^{i_p}|]\right) \le 0
$$

and, from the fact that each term of the series is non-negative, we conclude that each term of the series is equal to zero. This implies that, for every $[i] \in \mathcal{I}$, G^i is a minimizer of $\mathcal{E}_{s,g}$ among sets with volume $|G^i|$. To see the boundedness of $\{G^i\}_{[i]\in\mathcal{I}}$, it is sufficient to apply Lemma 3.2 to G^i for each $[i] \in \mathcal{I}$. This completes the proof of Step 2.

Step 3. We now show that, under (g1) and (g2), there exist a number $H \in \mathbb{N}$ and a family of bounded sets $\{\widetilde{G}^p\}_{p=1}^{H}$ such that

$$
\sum_{p=1}^{H}\mathcal{E}_{s,g}\left(\widetilde{G}^p\right) \le \sum_{[i]\in\mathcal{I}}\mathcal{E}_{s,g}(G^i) \le \liminf_{n\to\infty}\mathcal{E}_{s,g}(E_n), \quad \sum_{p=1}^{H}\left|\widetilde{G}^p\right| = m. \qquad (4.15)
$$

Indeed, letting $\{G^{i_p}\}_{p=1}^{\infty}$ be as in Step 2, we first set $m^p := |G^{i_p}|$ for any $p \in \mathbb{N}$ and, since $\sum_{p=1}^{\infty} m^p = m$, we can observe that $m^p \to 0$ as $p \to \infty$ and, moreover, $\mu_\ell := \sum_{p=\ell+1}^{\infty} m^p \to 0$ as $\ell \to \infty$. Then, we can choose $\widetilde{p} \in \mathbb{N}$ such that $m^{\widetilde{p}} \ge \frac{m}{2^{\widetilde{p}+1}}$. By using the family $\{G^{i_p}\}_{p=1}^{\infty}$, we construct a family of sets $\{\widetilde{G}^p\}_{p=1}^{H}$ for some $H \in \mathbb{N}$, depending only on N, s, and m, in the following manner; we choose $H \in \mathbb{N}$ so large that $H \ge \widetilde{p}$ and set $\widetilde{G}^p := G^{i_p}$ for any $p \in \{1, \cdots, H\}$ with $p \neq \widetilde{p}$ and $\widetilde{G}^{\widetilde{p}} := \lambda\, G^{\widetilde{p}}$ where $\lambda^N := \frac{m^{\widetilde{p}}+\mu_H}{m^{\widetilde{p}}}$. From the definition of $\widetilde{G}^p$, we have the volume identity that

$$
\sum_{p=1}^{H}\left|\widetilde{G}^p\right| = \sum_{p=1,\, p\neq\widetilde{p}}^{H}\left|G^{i_p}\right| + \lambda^N\,|G^{i_{\widetilde{p}}}| = \sum_{p=1,\, p\neq\widetilde{p}}^{H} m^p + m^{\widetilde{p}} + \mu_H = m. \qquad (4.16)
$$

Now we compute the energy of $\{\widetilde{G}^p\}_{p=1}^{H}$ to show that the total energy of each elements of $\{\widetilde{G}^p\}_{p=1}^{H}$ is lower than that of $\{G^{i_p}\}_{p=1}^{\infty}$; from the definition of $\lambda \ge 1$, we have that

$$
\begin{aligned}
\sum_{p=1}^{H}\mathcal{E}_{s,g}(\widetilde{G}^p) &\le \sum_{p=1,\, p\neq\widetilde{p}}^{H}\mathcal{E}_{s,g}(G^{i_p}) + \lambda^{2N}\,\mathcal{E}_{s,g}(G^{i_{\widetilde{p}}}) \\
&= \sum_{p=1}^{\infty}\mathcal{E}_{s,g}(G^{i_p}) + \left(\lambda^{2N} - 1\right)\,\mathcal{E}_{s,g}(G^{i_{\widetilde{p}}}) - \sum_{p=H+1}^{\infty}\mathcal{E}_{s,g}(G^{i_p}) \\
&\le \sum_{[i]\in\mathcal{I}}\mathcal{E}_{s,g}(G^{i_p}) + \frac{2^{\widetilde{p}+1}\,E_{s,g}[m^{\widetilde{p}}]}{m}\,\mu_H - \sum_{p=H+1}^{\infty} P_s(G^{i_p}). \qquad (4.17)
\end{aligned}
$$

Here, in the last inequality, we have also used (4.13). From the isoperimetric inequality of P_s and (4.17), we further obtain that

$$\begin{aligned}\sum_{p=1}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) &\le \sum_{[i]\in\mathcal{I}} \mathcal{E}_{s,g}(G^i) + \frac{2^{\widetilde{p}+1}\, E_{s,g}[m^{\widetilde{p}}]}{m}\, \mu_H - C \sum_{p=H+1}^{\infty} (m^p)^{\frac{N-s}{N}} \\ &\le \sum_{[i]\in\mathcal{I}} \mathcal{E}_{s,g}(G^i) + \frac{2^{\widetilde{p}+1}\, E_{s,g}[m^{\widetilde{p}}]}{m}\, \mu_H - C \left(\sum_{p=H+1}^{\infty} m^p \right)^{\frac{N-s}{N}} \\ &= \sum_{[i]\in\mathcal{I}} \mathcal{E}_{s,g}(G^i) + \frac{2^{\widetilde{p}+1}\, E_{s,g}[m^{\widetilde{p}}]}{m}\, \mu_H - C\,(\mu_H)^{\frac{N-s}{N}}.\end{aligned}$$

Taking the number H so large that $H \ge \widetilde{p}$ and

$$\frac{2^{\widetilde{p}+1}\, E_{s,g}[m^{\widetilde{p}}]}{m}\, \mu_H - C\,(\mu_H)^{\frac{N-s}{N}} \le 0,$$

we obtain (4.15), and thus this completes Step 3.

Step 4. We finally show that, under (g1), (g2), and (g3), there exists $p' \in \{1, 2, \cdots, H\}$ such that

$$\mathcal{E}_{s,g}\left(\widetilde{G}^{p'}\right) \le \liminf_{n\to\infty} \mathcal{E}_{s,g}(E_n) = E_{s,g}[m], \qquad \left|\widetilde{G}^{p'}\right| = m,$$

where $H \in \mathbb{N}$ and $\{\widetilde{G}^p\}_{p=1}^H$ are given in the previous step.

Let $\{\widetilde{G}^p\}_{p=1}^H$ be a family given in the previous step. From the proof in the previous step, we can choose $p' \in \{1, 2, \cdots, H\}$ such that $|\widetilde{G}^{p'}| > 0$. Moreover, since $\widetilde{G}^p$ is bounded for every $p \in \{1, 2, \cdots, H\}$, we can choose points $\{z^p\}_{p=1,\, p\neq p'}^H$ such that each set $\widetilde{G}^p + R\, z^p$ is disjoint with the others for large $R > 1$. We thus can compute the energy as follows; from the translation invariance of $\mathcal{E}_{s,g}$, it holds that

$$\begin{aligned}\sum_{p=1}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) &= \sum_{p=1\, p\neq p',q}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) + \mathcal{E}_{s,g}(\widetilde{G}^{p'}) + \mathcal{E}_{s,g}(\widetilde{G}^q) \\ &= \sum_{p=1\, p\neq p',q}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) + \mathcal{E}_{s,g}(\widetilde{G}^{p'}) + \mathcal{E}_{s,g}(\widetilde{G}^q + R\, z^q) \\ &= \sum_{p=1\, p\neq p',q}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) + \mathcal{E}_{s,g}(\widetilde{G}^{p'} \cup (\widetilde{G}^q + R\, z^q)) \\ &\quad + 2\int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^q + R\, z^q} \frac{dx\, dy}{|x-y|^{N+s}} - 2\int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^q + R\, z^q} g(x-y)\, dx\, dy\end{aligned}$$

for any $q \in \{1, \cdots, H\}$ with $q \neq p'$ and sufficiently large $R > 1$. Recalling the assumption (g3) that $g(x) \le \beta |x|^{-(N+s)}$ for any $|x| \ge R_0$ and some $\beta \in (0, 1)$, and choosing $R > 1$ in such a way that the set $\widetilde{G}^q + R\, z^q$ has the distance of more than R_0 from $\widetilde{G}^{p'}$, we obtain that

$$\begin{aligned}\sum_{p=1}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) \ge &\sum_{p=1,\, p\neq p',\, q}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) + \mathcal{E}_{s,g}(\widetilde{G}^{p'} \cup (\widetilde{G}^q + R\, z^q)) \\ &+ 2(1-\beta) \int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^q + R\, z^q} \frac{dx\, dy}{|x-y|^{N+s}}. \end{aligned} \tag{4.18}$$

By repeating the same argument finite times for the rest of the sets $\{\widetilde{G}^p\}_{p=1,\, p\neq p', q}^H$ with sufficiently large $R > 1$, we obtain the similar inequalities to (4.18) and, finally, we can derive the inequality that

$$\sum_{p=1}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) \geq \mathcal{E}_{s,g}\left(\widetilde{G}^{p'} \cup \bigcup_{p=1,\, p\neq p'}^{H} \left(\widetilde{G}^p + R\, z^p\right)\right)$$
$$+\, 2(1-\beta) \sum_{p=1,\, p\neq p'}^{H} \int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^p + R\, z^p} \frac{dx\, dy}{|x-y|^{N+s}}. \tag{4.19}$$

Since $\widetilde{G}^{p'} \cup \bigcup_{p=1,\, p\neq p'}^{H} \left(\widetilde{G}^p + R\, z^p\right)$ are the union of disjoint sets, we have, from (4.16), that

$$\left| \widetilde{G}^{p'} \cup \bigcup_{p=1,\, p\neq p'}^{H} \left(\widetilde{G}^p + R\, z^p\right) \right| = \sum_{p=1}^{H} |\widetilde{G}^p| = m.$$

Thus, from (4.19), we obtain

$$\begin{aligned} &2(1-\beta) \sum_{p=1,\, p\neq p'}^{H} \int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^p + R\, z^p} \frac{dx\, dy}{|x-y|^{N+s}} + E_{s,g}[m] \\ &\leq \sum_{p=1,\, p\neq p'}^{H} \int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^p + R\, z^p} \frac{dx\, dy}{|x-y|^{N+s}} + \mathcal{E}_{s,g}\left(\widetilde{G}^{p'} \cup \bigcup_{p=1,\, p\neq p'}^{H} \left(\widetilde{G}^p + R\, z^p\right)\right) \\ &\leq \sum_{p=1}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) \leq E_{s,g}[m] \end{aligned}$$

and it follows that

$$2(1-\beta) \sum_{p=1,\, p\neq p'}^{H} \int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^p + R\, z^p} \frac{dx\, dy}{|x-y|^{N+s}} \leq 0$$

for large $R > 1$. Since each term of the sum is non-negative, $\beta < 1$, and $|\widetilde{G}^{p'}| > 0$, we conclude that $|\widetilde{G}^p| = 0$ for all $p \neq p'$. Therefore, the final claim is valid and this completes the proof of Theorem 2.3. □

4.1 Regularity of the boundaries of minimizers

In this subsection, we consider the regularity of the boundary of a minimizer of $\mathcal{E}_{s,g}$ under suitable assumptions on the kernel g.

Given $\Lambda > 0$, we say that a measurable set $E \subset \mathbb{R}^N$ is an *almost-minimizer* with constant Λ if

$$P_s(E) \leq P_s(F) + \Lambda\, |E \Delta F| \tag{4.20}$$

for any measurable set $F \subset \mathbb{R}^N$.

We recall the result proved by Figalli, Fusco, Maggi, Millot, and Morini in [15, Corollary 3.5] on the regularity of almost-minimizers, combining the result proved by Savin and Valdinoci in [36, Corollary 2]. The result in [36] is that the singular set has Hausdorff dimension at most $N-3$.

Theorem 4.1. *Let $s_0 \in (0, 1)$ and $\Lambda > 0$. Then there exist positive constants $\varepsilon_0 \in (0, 1)$, $C_0 > 0$, and $\alpha \in (0, 1)$, depending on N, s_0, and Λ, with the following property. If $s \in [s_0, 1)$ and E is an almost-minimizer with constant Λ, then ∂E is of class $C^{1,\alpha}$, out of a closed singular set of $\mathcal{H}^{N-3}$-dimension.*

As a consequence of Theorem 4.1 we can obtain the regularity of the minimizers of $\mathcal{E}_{s,g}$. To see this, we reduce the minimization problem $E_{s,g}[m]$ for any $m > 0$ to another minimization problem. More precisely, we show that any solutions of the minimization problem $E_{s,g}[m]$ are also the solutions of the unconstrained minimization problem

$$\inf \left\{ \mathcal{E}_{s,g,\mu_0}(E) \mid E \subset \mathbb{R}^N\text{: measurable} \right\}$$

for some constant $\mu_0 > 0$ and any $m > 0$, where we define $\mathcal{E}_{s,g,\mu_0}$ as

$$\mathcal{E}_{s,g,\mu}(F) \coloneqq \mathcal{E}_{s,g}(F) + \mu \, \left| |F| - m \right|$$

for any $F \subset \mathbb{R}^N$ and $\mu > 0$.

Proposition 4.2 (Reduction to an unconstrained problem)**.** *Let $m > 0$. Assume that the kernel g satisfies the assumptions* (g1) *and* (g2)*. Then there exists a constant $\mu_0 = \mu_0(N, s, g, m) > 0$ such that, if E is a minimizer of $\mathcal{E}_{s,g}$ with $|E| = m$, then E is also a minimizer of $\mathcal{E}_{s,g,\mu}$ among sets in $\mathbb{R}^N$ for any $\mu \geq \mu_0$.*

Proof. Suppose by contradiction that, for any $n \in \mathbb{N}$, there exist a minimizer E_n of $\mathcal{E}_{s,g}$ with $|E_n| = m$ and a constant $\mu(n) \geq n$ such that E_n is not a minimizer of $\mathcal{E}_{s,g,\mu(n)}$. Then, by assumption, we can choose a sequence $\{F_n\}_{n\in\mathbb{N}}$ such that

$$\mathcal{E}_{s,g,\mu(n)}(F_n) < \mathcal{E}_{s,g,\mu(n)}(E_n) \tag{4.21}$$

for any $n \in \mathbb{N}$. First of all, we show that $|F_n| \xrightarrow[n\to\infty]{} m$. Indeed, we set B_m as a open ball in $\mathbb{R}^N$ whose volume is equal to m. Then from (4.21) and the minimality of E_n with $|E_n| = m$ for any $n \in \mathbb{N}$, we have that

$$\mathcal{E}_{s,g,\mu(n)}(F_n) < \mathcal{E}_{s,g,\mu(n)}(E_n) = \mathcal{E}_{s,g}(E_n) = E_{s,g}[m]. \tag{4.22}$$

Thus, denoting r_m by the radius of the ball B_m and using the change of variables, we obtain

$$\mu(n) \left| |F_n| - m \right| < E_{s,g}[m] < \infty \tag{4.23}$$

for any $n \in \mathbb{N}$. From the definition of r_m, the right-hand side in (4.23) is finite and independent of n. Hence, letting $n \to \infty$ in (4.23), we obtain the claim that $|F_n| \to m$ as $n \to \infty$. Finally, we derive a contradiction in the following manner. We first define $\widetilde{F}_n$ as $\widetilde{F}_n \coloneqq \lambda_n F_n$ where $\lambda_n^N \coloneqq m \, |F_n|^{-1}$ and, by definition, we can observe that $|\widetilde{F}_n| = m$. In the sequel, we may assume that, up to extracting a subsequence, $|F_n| \leq m$ for $n \in \mathbb{N}$. Indeed, we suppose by contradiction that, for any subsequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ of $\{F_n\}_{n\in\mathbb{N}}$, we always have that $|F_{n_k}| > m$ for any $k \in \mathbb{N}$. From the continuity of the Lebesgue measure, for each $k \in \mathbb{N}$, there exists a constant $R_k > 0$ such that $|F_{n_k} \cap B_{R_{n_k}}(0)| = m$ for every $k \in \mathbb{N}$. Thus, from the minimality of E_n for any $n \in \mathbb{N}$ and Proposition 3.1, we have the estimate that

$$\mathcal{E}_{s,g,\mu(n)}(E_{n_k}) = \mathcal{E}_{s,g}(E_{n_k}) \leq \mathcal{E}_{s,g}(F_{n_k} \cap B_{R_{n_k}}(0)) \leq P_s(F_{n_k}) + V_g(F_{n_k}) = \mathcal{E}_{s,g}(F_{n_k})$$

for any $k \in \mathbb{N}$, which contradicts the estimate (4.21) since $\mathcal{E}_{s,g}(F_{n_k}) \leq \mathcal{E}_{s,g,\mu(n)}(F_{n_k})$ for any $k \in \mathbb{N}$. Hence, from (4.21), the minimality of E_n, and the assumption that $\lambda_n \geq 1$ for any $n \in \mathbb{N}$, we have

$$\mathcal{E}_{s,g,\mu(n)}(F_n) < \mathcal{E}_{s,g}(E_n) \leq \mathcal{E}_{s,g}(\widetilde{F}_n) \leq \lambda_n^{N-s} \, P_s(F_n) + \lambda_n^{2N} \, V_g(F_n). \tag{4.24}$$

From the definition, we notice that $\left| |F_n| - m \right| = |\lambda_n^{-N} \, m - m| = m \, \lambda_n^{-N} \, |\lambda_n^N - 1|$ for any n. Hence, from (4.24) and dividing the both side of (4.24) by $\left| |F_n| - m \right|$, we obtain

$$\mu(n) \leq m^{-1} \, \lambda_n^N \frac{|\lambda_n^{N-s} - 1|}{|\lambda_n^N - 1|} P_s(F_n) + m^{-1} \, \lambda_n^N \frac{|\lambda_n^{2N} - 1|}{|\lambda_n^N - 1|} V_g(F_n) \tag{4.25}$$

for any $n \in \mathbb{N}$. Recalling (4.21) and (4.22), we have that $P_s(F_n) + V_g(F_n) < E_{s,g}[m] < \infty$. Moreover, we can observe that $\frac{|\lambda_n^{N-s}-1|}{|\lambda_n^N-1|} \le 2$ and $\frac{|\lambda_n^{2N}-1|}{|\lambda_n^N-1|} \le 2$ for sufficiently large $n \in \mathbb{N}$. Therefore, from (4.25), we obtain

$$\mu(n) \le 8m^{-1}\, E_{s,g}[m] \tag{4.26}$$

for sufficiently large $n \in \mathbb{N}$ and thus obtain a contradiction. $\square$

Now we are ready to show the regularity of minimizers for $\mathcal{E}_{s,g}$

Lemma 4.3 (Regularity of minimizers)**.** *Let $s \in (0, 1)$ and let $m > 0$. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies* (g1)*,* (g2)*, and* (g3)*. If $E \subset \mathbb{R}^N$ is a minimizer of $\mathcal{E}_{s,g}$ among sets with volume m, then ∂E is of class $C^{1,\alpha}$, for some $0 < \alpha < 1$, out of a closed singular set of Hausdorff dimension $N - 3$.*

Proof. Let $E \subset \mathbb{R}^N$ be a minimizer of $\mathcal{E}_{s,g}$ with $|E| = m$. In order to apply the regularity result of Theorem 4.1 to our case, it is sufficient to show that the set E is an almost-minimizer in the sense of (4.20) for some constant $\Lambda > 0$. From Proposition 4.2, we know that E with $|E| = m$ is also a solution to the minimization problem

$$\min\{\mathcal{E}_{s,g,\mu_0}(E) \mid E \subset \mathbb{R}^N\text{: measurable}\}.$$

where $\mu_0 > 0$ is as in Proposition 4.2 and is independent of E. Hence, from the minimality of E, we have that

$$\mathcal{E}_{s,g,\mu_0}(E) \le \mathcal{E}_{s,g,\mu_0}(F) \tag{4.27}$$

for any bounded measurable set $F \subset \mathbb{R}^n$. We may assume that F is finite with respect to the s-fractional perimeter; otherwise the inequality (4.27) is obviously valid. Then from the fact that $|E| = m$, we have

$$\begin{aligned} P_s(E) &\le P_s(F) + V_g(F) - V_g(E) + \mu_0\, ||F| - |E|| \\ &\le P_s(F) + V_g(F) - V_g(E) + \mu_0\, |F \Delta E|. \end{aligned} \tag{4.28}$$

Regarding the Riesz potential, we can compute the difference $V_g(F) - V_g(E)$ as follows:

$$\begin{aligned} |V_g(F) - V_g(E)| &\le \left| \int_F \int_{F\cup E} g(x-y)\, dx\, dy - \int_E \int_{F\cup E} g(x-y)\, dx\, dy \right| \\ &\le 2 \int_{F\Delta E} \int_{F\cup E} g(x-y)\, dx\, dy \\ &\le 2|F\Delta E| \int_{\mathbb{R}^N} g(x)\, dx. \end{aligned} \tag{4.29}$$

Note that, from the local integrability of g and assumption (g3), the kernel g is integrable in $\mathbb{R}^N$ and thus, the right-hand side in (4.29) is finite. Hence, by combining (4.29) with (4.28), we obtain that

$$P_s(E) \le P_s(F) + \Big(2\|g\|_{L^1(\mathbb{R}^N)} + \mu_0\Big)\ |F\Delta E|$$

for any measurable set $F \subset \mathbb{R}^N$. Therefore, by employing Theorem 4.1, we can prove the claim. $\square$

5 Existence of generalized minimizers for $\widetilde{\mathcal{E}}_{s,g}$ for any volumes

In this section, we prove Theorem 2.4, namely, the existence of generalized minimizers for the generalized functional $\widetilde{\mathcal{E}}_{s,g}$ given as (1.4) for any volumes. To see this, we impose slightly more general assumptions on g than we do to prove the existence of minimizers of $\mathcal{E}_{s,g}$ for any volumes in Section 4. More precisely, we assume that the kernel g satisfies the assumptions (g1), (g2), and (g4) in Section 3.

Before proving the main theorem, we show one lemma, saying that one can modify a "generalized" minimizing sequence for the generalized functional $\widetilde{\mathcal{E}}_{s,g}$ into a "usual" minimizing sequence for the functional $\mathcal{E}_{s,g}$. More precisely, we prove

Lemma 5.1. *Let $s \in (0, 1)$. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions* (g1)*,* (g2)*, and* (g4)*. Then, for any $m > 0$, it follows that*

$$E_{s,g}[m] = \inf\{\mathcal{E}_{s,g}(E) \mid |E| = m\} = \inf\left\{\widetilde{\mathcal{E}}_{s,g}(\{E^k\}_k) \mid \sum_{k=1}^{\infty}|E^k| = m\right\} = \widetilde{E}_{s,g}[m].$$

Proof. The idea of the proof is based on the concentration-compactness lemma that we apply in the proof of Theorem 2.3.

First of all, we can readily observe that the inequality

$$\inf\{\mathcal{E}_{s,g}(E) \mid |E| = m\} \geq \inf\left\{\widetilde{\mathcal{E}}_{s,g}(\{E^k\}_k) \mid \sum_{k=1}^{\infty}|E^k| = m\right\}$$

holds true. Hence, it remains for us to prove the other inequality. To see this, we take any minimizing sequence $\{\{E_n^k\}_k\}_n$ for the generalized functional $\widetilde{\mathcal{E}}_{s,g}$. Then it follows that, for any $\varepsilon > 0$, there exists a number $n_0 \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty}\mathcal{E}_{s,g}(E_n^k) = \widetilde{\mathcal{E}}_{s,g}(\{E_n^k\}_k) \leq \widetilde{E}_{s,g}[m] + \varepsilon, \quad \sum_{k=1}^{\infty}|E_n^k| = m \tag{5.1}$$

for any $n \geq n_0$. Since the minimum is attained with a minimizing sequence of which each element is bounded, we may assume that E_n^k is bounded for each k, $n \in \mathbb{N}$. In the sequel, we fix one $n \in \mathbb{N}$ with $n \geq n_0$ until we give another remark.

We first show that there exist a number $K_n \in \mathbb{N}$ and a sequence $\{\widetilde{E}_n^k\}_{k=1}^{K_n}$, constructed from $\{E_n^k\}_k$, such that

$$\sum_{k=1}^{K_n}\mathcal{E}_{s,g}(\widetilde{E}_n^k) \leq \sum_{k=1}^{\infty}\mathcal{E}_{s,g}(E_n^k), \quad \sum_{k=1}^{K_n}|\widetilde{E}_n^k| = m. \tag{5.2}$$

The proof of this claim is the same as Step 3 in the proof of Theorem 2.3, since we assume that the kernel g only satisfies (g1) and (g2), thus we will not repeat the proof here.

Let now $\{\widetilde{E}_n^k\}_{k=1}^{K_n}$ be as in the previous claim. Since we have that $\sum_{k=1}^{K_n}|E_n^k| = m$, we can choose one $k' \in \mathbb{N}$ with $|E_n^{k'}| > 0$. Since we have assumed that the sets $\{E_n^k\}_{k=1}^{K_n}$ are bounded, we can choose the points $\{z_n^k\}_{k=1,\,k\neq k'}^{K_n}$ such that each set $E_n^k + R\,z_n^k$ is far away from the others for sufficiently large $R > 1$. We can thus compute the energy as follows; from the translation invariance of $\mathcal{E}_{s,g}$, it holds that

$$\begin{aligned}\sum_{k=1}^{K_n}\mathcal{E}_{s,g}(E_n^k) &= \sum_{k=1,\,k\neq k',\ell}^{K_n}\mathcal{E}_{s,g}(E_n^k)+\mathcal{E}_{s,g}(E_n^{k'})+\mathcal{E}_{s,g}(E_n^{\ell})\\ &= \sum_{k=1,\,k\neq k',\ell}^{K_n}\mathcal{E}_{s,g}(E_n^k)+\mathcal{E}_{s,g}(E_n^{k'})+\mathcal{E}_{s,g}(E_n^{\ell}+R\,z_n^{\ell})\\ &= \sum_{k=1,\,k\neq k',\ell}^{K_n}\mathcal{E}_{s,g}(E_n^k)+\mathcal{E}_{s,g}(E_n^{k'}\cup(E_n^{\ell}+R\,z_n^{\ell}))\\ &\qquad +2\int_{E_n^{k'}}\int_{E_n^{\ell}+R\,z_n^{\ell}}\frac{dx\,dy}{|x-y|^{N+s}}-2\int_{E_n^{k'}}\int_{E_n^{\ell}+R\,z_n^{\ell}}g(x-y)\,dx\,dy\end{aligned}$$

for any $\ell\in\{1,\cdots,K_n\}$ with $\ell\neq k'$ and sufficiently large $R>1$. Thus, we obtain that

$$\begin{aligned}\sum_{k=1,\,k\neq k',\ell}^{K_n}\mathcal{E}_{s,g}(E_n^k)+\mathcal{E}_{s,g}(E_n^{k'}\cup(E_n^{\ell}+R\,z_n^{\ell})) \leq \sum_{k=1}^{K_n}\mathcal{E}_{s,g}(E_n^k)&\\ +2\int_{E_n^{k'}}\int_{E_n^{\ell}+R\,z_n^{\ell}}g(x-y)\,dx\,dy&\end{aligned}\tag{5.3}$$

for any $\ell\in\{1,\cdots,K_n\}$ with $\ell\neq k'$ and sufficiently large $R>1$. By repeating the same argument finite times for the rest of the sets $\{E_n^k\}_{k=1,\,k\neq k',\ell}^{K_n}$ with sufficiently large $R>1$ and from the translation invariance of $\mathcal{E}_{s,g}$, we can derive the inequality

$$\begin{aligned}\mathcal{E}_{s,g}\left(E_n^{k'}\cup\bigcup_{k=1,\,k\neq k'}^{K_n}\left(E_n^k+R\,z_n^k\right)\right)\leq\sum_{k=1}^{K_n}\mathcal{E}_{s,g}(E_n^k)&\\ +2\sum_{1\leq k\neq\ell\leq K_n}\int_{F_n^k(R)}\int_{F_n^{\ell}(R)}g(x-y)\,dx\,dy&\end{aligned}\tag{5.4}$$

where we define the sets $\{F_n^k(R)\}_{k=1}^{K_n}$ in such a way that $F_n^k(R)\coloneqq E_n^k+R\,z_n^k$ if $k\neq k'$ and $F_n^{k'}(R)\coloneqq E_n^{k'}$. Note that the sets $\{F_n^k(R)\}_{k=1}^{K_n}$ satisfy

$$\operatorname{dist}\left(F_n^k(R),\,F_n^{\ell}(R)\right)\xrightarrow[R\to\infty]{}\infty\tag{5.5}$$

for any $k,\,\ell\in\{1,\cdots,K_n\}$ with $k\neq\ell$. Since $\sum_{k=1}^{K_n}|E_n^k|=m$ and $E_n^{k'}\cup\bigcup_{k=1,\,k\neq k'}^{K_n}\left(E_n^k+R\,z^{i_p}\right)$ are the union of disjoint sets, we have that

$$\left|E_n^{k'}\cup\bigcup_{k=1,\,k\neq k'}^{K_n}\left(E_n^k+R\,z^{i_p}\right)\right|=\sum_{k=1}^{K_n}|E_n^k|=m.$$

Thus, from (5.1) and (5.4), we obtain

$$\begin{aligned}E_{s,g}[m]&\leq\mathcal{E}_{s,g}\left(E_n^{k'}\cup\bigcup_{k=1,\,k\neq k'}^{K_n}\left(E_n^k+R\,z_n^k\right)\right)\\ &\leq\sum_{k=1}^{K_n}\mathcal{E}_{s,g}(E_n^k)\\ &\qquad+2\sum_{1\leq k\neq\ell\leq K_n}\int_{F_n^k(R)}\int_{F_n^{\ell}(R)}g(x-y)\,dx\,dy\end{aligned}$$

$$
\begin{aligned}
&\le \widetilde{E}_{s,g}[m] + \varepsilon \\
&\qquad + 2 \sum_{1\le k\neq \ell \le K_n} \int_{F_n^k(R)} \int_{F_n^\ell(R)} g(x-y)\,dx\,dy
\end{aligned} \tag{5.6}
$$

Hence, if we show that the last term of the right-hand side in (5.6) converges to zero as $R \to \infty$ for each $n \ge n_0$, then we conclude, from the arbitrariness of ε, that the inequality

$$
\inf\{\mathcal{E}_{s,g}(E) \mid |E| = m\} = E_{s,g}[m] \le \widetilde{E}_{s,g}[m] = \inf\left\{ \widetilde{\mathcal{E}}_{s,g}(\{E^k\}_k) \mid \sum_{k=1}^{\infty} |E^k| = m \right\}
$$

holds and this is what we want. To conclude the proof of the lemma, it is sufficient to show that, under assumption (g4), the convergence

$$
\sum_{1\le k\neq \ell \le K_n} \int_{F_n^k(R)} \int_{F_n^\ell(R)} g(x-y)\,dx\,dy \xrightarrow[R\to\infty]{} 0
$$

holds for each $n \ge n_0$. We fix $n \ge n_0$. From assumption (g4), we have that, for any $\varepsilon > 0$, there exists a constant $R(\varepsilon) > 0$ such that $g(z) < \varepsilon$ for any $|z| \ge R(\varepsilon)$. On the other hand, from (5.5), we can also choose a constant $R'(\varepsilon) > 0$ such that $|x-y| \ge R(\varepsilon)$ for any $R > R'(\varepsilon)$, $x \in F_n^k(R)$, $y \in F_n^\ell(R)$, and $k,\, \ell \in \{1, \cdots, K_n\}$ with $k \neq \ell$. Thus, taking these into account, we obtain that, for any $R > R'(\varepsilon)$,

$$
\sum_{1\le k\neq \ell \le K_n} \int_{F_n^k(R)} \int_{F_n^\ell(R)} g(x-y)\,dx\,dy < \varepsilon \sum_{k=1}^{K_n} |F_n^k(R)| \sum_{\ell=1}^{K_n} |F_n^\ell(R)|.
$$

Recalling the definition of the sets $\{F_n^k(R)\}_k$, we have that $\sum_{k=1}^{K_n} |F_n^k(R)| \le m$. Therefore, we obtain that

$$
\sum_{1\le k\neq \ell \le K_n} \int_{F_n^k(R)} \int_{F_n^\ell(R)} g(x-y)\,dx\,dy < m^2\,\varepsilon
$$

for any $R > R'(\varepsilon)$ and this completes the proof of the claim.

□

Now we prove Theorem 2.4, namely, the existence of generalized minimizers of $\widetilde{\mathcal{E}}_{s,g}$ under the assumptions (g1), (g2), and (g4) in Section 3.

Proof of Theorem 2.4. Let $m > 0$. Thanks to Lemma 5.1, it is sufficient to take any sequence $\{E_n\}_{n\in\mathbb{N}}$ such that $|E_n| = m$ for any $n \in \mathbb{N}$ and

$$
\lim_{n\to\infty} \mathcal{E}_{s,g}(E_n) = \widetilde{E}_{s,g}[m] \tag{5.7}
$$

instead of taking a minimizing sequence for $\widetilde{E}_{s,g}[m]$.

We now apply the same argument as we conducted in Step 1, 2, and 3 in the proof of Theorem 2.3, because we only need the assumptions (g1), (g2), and (g4) for the arguments in Step 1, 2, and 3 to work. Therefore, this enables us to choose a finite number of measurable sets $\{G^p\}_{p=1}^H$ with $H \in \mathbb{N}$ such that

$$
\sum_{p=1}^{H} \mathcal{E}_{s,g}(G^p) \le \liminf_{n\to\infty} \mathcal{E}_{s,g}(E_n), \quad \sum_{i=1}^{H} |G^p| = m. \tag{5.8}
$$

Moreover, each set G^p is a minimizer of $\mathcal{E}_{s,g}$ among sets of volume $|G^p|$. Therefore, from (5.7) and (5.8), we conclude that the sequence $\{G^p\}_{p=1}^H$ is a generalized minimizer of $\widetilde{\mathcal{E}}_{s,g}$ with $\sum_{p=1}^{H} |G^p| = m$ as we desired. □

6 Asymptotic behavior of minimizers for large volumes

In this section, we study the asymptotic behavior of minimizers of $\mathcal{E}_{s,g}$ with large volumes under the assumption that the kernel g decays sufficiently fast. To see this, we first prove the Γ-convergence in L^1-topology of the functional associated with Problem (1.7) to the fractional perimeter P_s as m goes to infinity. Since it is well-known that a sequence of minimizers for a functional converges to a minimizer of its Γ-limit, we can derive the convergence of a sequence of the minimizers to the unit ball, by rescaling, up to translations.

6.1 Γ-convergence of $\widehat{\mathcal{E}}^{\lambda}_{s,g}$ to the fractional perimeter as $\lambda \to \infty$

Before proving Theorem 2.6, we establish the Γ-convergence result for the energy $\mathcal{E}^{\lambda}_{s,g}$ under the assumption that the kernel g decays sufficiently fast. To see this, we give several notations and the definition of the functional $\mathcal{F}^{\lambda}_{s,g}$ on $L^1(\mathbb{R}^N)$. First, we recall the definition of the s-fractional Sobolev semi-norm $[f]_{W^{s,1}}(\mathbb{R}^N)$ as follows:

$$[f]_{W^{s,1}}(\mathbb{R}^N) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|}{|x-y|^{N+s}}\,dx\,dy$$

for $f \in L^1$. Note that $[\chi_E]_{W^{s,1}}(\mathbb{R}^N) = P_s(E)$ for any measurable set $E \subset \mathbb{R}^N$. As is shown in [5, Proposition 4.2 and Corollary 4.4], it follows that any integrable function of bounded variation is also finite with respect to the fractional semi-norm $[\cdot]_{W^{s,1}}$. In order to study the Γ-convergence of the sequence $\{\widehat{\mathcal{E}}^{\lambda}_{s,g}\}_{\lambda>1}$ given in Proposition 2.5, we define the functional $\widehat{\mathcal{F}}^{\lambda}_{s,g}$ as

$$\widehat{\mathcal{F}}^{\lambda}_{s,g}(f) := \begin{cases} [f]_{W^{s,1}}(\mathbb{R}^N) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x)-f(y)|\, g_{\lambda}(x-y)\,dx\,dy \\ \qquad\qquad \text{if } f=\chi_F \text{ for some bounded set } F\subset\mathbb{R}^N \text{ with } P_s(F)<\infty, \\ \infty \qquad\qquad \text{otherwise.} \end{cases} \tag{6.1}$$

Note that the functional $\widehat{\mathcal{F}}^{\lambda}_{s,g}(f)$ for any $\lambda > 0$ is well-defined. Moreover, if $f = \chi_E$ for some bounded set E with $P_s(E) < \infty$, then we can easily see that $\widehat{\mathcal{F}}^{\lambda}_{s,g}(f) = \widehat{\mathcal{E}}^{\lambda}_{s,g}(E)$.

Now we prove the Γ-convergence of the functional $\widehat{\mathcal{F}}^{\lambda_n}_{s,g}$ to $\widehat{\mathcal{F}}^{\infty}_{s}$ (we give the definition of $\widehat{\mathcal{F}}^{\infty}_{s}$ in the following proposition) as $n \to \infty$ in the L^1-topology.

Proposition 6.1 (Γ-convergence to the s-fractional semi-norm)**.** *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be any sequence of positive real numbers going to infinity as $n \to \infty$. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions* (g1)*,* (g2)*, and* (g5) *in Section 3. Then the sequence $\{\widehat{\mathcal{F}}^{\lambda_n}_{s,g}\}_{n\in\mathbb{N}}$ Γ-converges, with respect to L^1-topology, to the functional $\widehat{\mathcal{F}}^{\infty}_{s}$ defined by*

$$\widehat{\mathcal{F}}^{\infty}_{s}(f) := \begin{cases} [f]_{W^{s,1}}(\mathbb{R}^N) & \textit{if } f=\chi_F \textit{ for some bounded } F\subset\mathbb{R}^N \textit{ with } P_s(F)<\infty, \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. We recall the definition of the Γ-convergence. We say that $\{\widehat{\mathcal{F}}^{\lambda_n}_{s,g}\}_{n\in\mathbb{N}}$ Γ-converges to $\widehat{\mathcal{F}}^{\infty}_{s}$ with respect to L^1-topology if the two estimates hold

$$\Gamma_{L^1}\text{-}\limsup_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f) \leq \widehat{\mathcal{F}}^{\infty}_{s}(f), \quad \widehat{\mathcal{F}}^{\infty}_{s}(f) \leq \Gamma_{L^1}\text{-}\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f)$$

for any $f \in L^1(\mathbb{R}^N)$, where we set

$$\Gamma_{L^1}\text{-}\limsup_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f) := \min\left\{ \limsup_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) \mid f_n \xrightarrow[n\to\infty]{} f \text{ in } L^1(\mathbb{R}^N) \right\} \tag{6.2}$$

and

$$\Gamma_{L^1}\text{-}\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f) := \min\left\{\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) \mid f_n \xrightarrow[n\to\infty]{} f \text{ in } L^1(\mathbb{R}^N)\right\}. \tag{6.3}$$

Note that the minimum in (6.2) and (6.3) is attained by the diagonal argument.

First of all, we prove that $\Gamma_{L^1}\text{-}\limsup_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f) \le \widehat{\mathcal{F}}^{\infty}_{s}(f)$ for any $f \in L^1(\mathbb{R}^N)$. In the case that f is not a characteristic function of some bounded set of finite fractional perimeter, we obviously have that $\widehat{\mathcal{F}}^{\infty}_{s}(f) = \infty$ and the inequality holds. Thus, we may assume that $f = \chi_F$ for a bounded set $F \subset \mathbb{R}^N$ with $P_s(F) < \infty$. Setting a sequence $\{f_n\}_{n\in\mathbb{N}}$ as $f_n = f = \chi_F$ for any $n \in \mathbb{N}$, we obtain, from the non-negativity of g, that

$$\widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) \le \widehat{\mathcal{F}}^{\infty}_{s}(f)$$

for any $n \in \mathbb{N}$ and thus, it follows that $\Gamma_{L^1}\text{-}\limsup_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f) \le \widehat{\mathcal{F}}^{\infty}_{s}(f)$.

Next we prove that $\widehat{\mathcal{F}}^{\infty}_{s}(f) \le \Gamma_{L^1}\text{-}\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f)$ for any $f \in L^1(\mathbb{R}^N)$. We take any sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R}^N)$ such that $f_n \to f$ in L^1 as $n \to \infty$. In the case that f is not a characteristic function of some bounded set of finite fractional perimeter, we claim that there exists a number $n_0 \in \mathbb{N}$ such that f_n is also not a characteristic function of a measurable set for any $n \ge n_0$. Indeed, we suppose by contradiction that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} = \chi_{F_{n_k}}$ for some measurable set $F_{n_k} \subset \mathbb{R}^N$ for any $k \in \mathbb{N}$. Since $f_{n_k} \to f$ in L^1 as $k \to \infty$ and $f_{n_k} \in \{0, 1\}$ for any $k \in \mathbb{N}$, we can show that $f \in \{0, 1\}$ a.e. in $\mathbb{R}^N$. Indeed, suppose that there exists a point $x_0 \in \mathbb{R}^N$ such that $f(x_0) \neq 0, 1$. Then, from the convergence $f_{n_k} \to f$ as $k \to \infty$ in L^1, we have that $f_{n_k} \to f$ a.e. in $\mathbb{R}^N$. Thus we have that, for sufficiently large k,

$$f_{n_k}(x_0) - \frac{||f(x_0)| - 1|}{2} < f(x_0) < f_{n_k}(x_0) + \frac{||f(x_0)| - 1|}{2}.$$

Since $f_{n_k}(x_0) \in \{0, 1\}$ for all k, we obtain a contradiction. Hence, f can be written as $f = \chi_F$ for some measurable $F \subset \mathbb{R}^N$. This contradicts the assumption that f is not a characteristic function. Hence, we conclude that, for large $n \in \mathbb{N}$, $\widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) = \infty$ and the claim holds true. Thus, in the sequel, we may assume that $f = \chi_F$ for some bounded set $F \subset \mathbb{R}^N$. In addition, we may assume that $P_s(F) < \infty$ because of the lower semi-continuity of P_s and the assumption (g5), namely, the condition that $g(x) \le \frac{\gamma}{|x|^{N+s}}$ for any $x \neq 0$ and some $\gamma \in (0, 1)$.

Under the above assumption, we first compute the second term of the functional $\widehat{\mathcal{F}}^{\lambda_n}_{s,g}$ in (6.1). Let $\varepsilon \in (0, 1)$. From assumption (g5), we can choose a constant $R_\varepsilon > 1$ such that $g(x) \le \frac{\varepsilon}{|x|^{N+s}}$ for $|x| \ge R_\varepsilon$. Then, from the definition of g_{λ_n} for any $n \in \mathbb{N}$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f_n(x) - f_n(y)|\, g_{\lambda_n}(x-y)\, dx\, dy \\
&= \iint_{\{(x,y) \mid \lambda_n|x-y| < R_\varepsilon\}} |f_n(x) - f_n(y)|\, g_{\lambda_n}(x-y)\, dx\, dy \\
&\qquad + \iint_{\{(x,y) \mid \lambda_n|x-y| \ge R_\varepsilon\}} |f_n(x) - f_n(y)|\, g_{\lambda_n}(x-y)\, dx\, dy \\
&\le \iint_{\{(x,y) \mid \lambda_n|x-y| < R_\varepsilon\}} \frac{|f_n(x) - f_n(y)|}{|x-y|^{N+s}}\, dx\, dy \\
&\qquad + \varepsilon \iint_{\{(x,y) \mid \lambda_n|x-y| \ge R_\varepsilon\}} \frac{|f_n(x) - f_n(y)|}{|x-y|^{N+s}}\, dx\, dy
\end{aligned} \tag{6.4}$$

for any $n \in \mathbb{N}$. Thus, from the definition of $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$, the assumption (g5), and (6.4), we can obtain

$$\begin{aligned}\widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) &\geq [f_n]_{W^{s,1}(\mathbb{R}^N)} - \frac{1}{2}\iint_{\{(x,y)\,|\,\lambda_n|x-y|<R_\varepsilon\}} \frac{|f_n(x)-f_n(y)|}{|x-y|^{N+s}}\,dx\,dy \\ &\quad - \frac{\varepsilon}{2}\iint_{\{(x,y)\,|\,\lambda_n|x-y|\geq R_\varepsilon\}} \frac{|f_n(x)-f_n(y)|}{|x-y|^{N+s}}\,dx\,dy \\ &\geq \frac{1-\varepsilon}{2}\iint_{\{(x,y)\,|\,\lambda_n|x-y|\geq R_\varepsilon\}} \frac{|f_n(x)-f_n(y)|}{|x-y|^{N+s}}\,dx\,dy \end{aligned} \tag{6.5}$$

for any $\varepsilon \in (0,\,1)$ and $n \in \mathbb{N}$. Thus, letting first $n \to \infty$ and then $\varepsilon \to 0$ with Fatou's lemma, we finally obtain

$$\begin{aligned}\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) &\geq \limsup_{\varepsilon\to 0} \frac{1-\varepsilon}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|f(x)-f(y)|}{|x-y|^{N+s}}\,dx\,dy \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|}{|x-y|^{N+s}}\,dx\,dy = [f]_{W^{s,1}(\mathbb{R}^N)}.\end{aligned}$$

Therefore, from the above arguments, we complete the proof. □

6.2 Convergence of minimizers of $\widehat{\mathcal{E}}^{\lambda}_{s,g}$ to the ball as $\lambda \to \infty$

Now we prove Theorem 2.6. To observe the convergence, we consider Problem (1.7) and finally we take the limit $\lambda \to \infty$ instead of Problem 1.1 with the limit $m \to \infty$.

The strategy for the proof of the asymptotic behavior of the minimizers is as follows; in contrast to the idea for studying the asymptotic behavior of minimizers, for instance, in [15, 33], we may not be able to employ naively the quantitative isoperimetric inequality for the fractional perimeter P_s and a Fuglede-type argument. The reason would be that the volume of the symmetric difference between a minimizer and a ball can be bounded from above only by the volume and perimeter of that difference. Since we deal with minimizers with large volumes, it is not obvious whether the bound of the symmetric difference can give us the L^1-convergence of minimizers to the ball. Therefore, we adopt another strategy in the following way; we first take any sequence $\{F_n\}_n$ of the minimizers for $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ with $|F_n| = |B_1|$. Then we apply the "concentration-compactness" lemma that we used to show the existence of minimizers in Section 4. As a consequence, we can choose a sequence of sets $\{G^i\}_i$ and points $\{z^i_n\}_{i,n}$ such that, up to extracting a subsequence,

$$\sum_i P_s(G^i) \leq \liminf_{n\to\infty} \widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n), \quad F_n - z^i_n \xrightarrow[n\to\infty]{} G^i \quad \text{in } L^1_{loc}, \quad \sum_i |G^i| = |B_1| \tag{6.6}$$

thanks to the assumptions on g. Then, from the isoperimetric inequality of P_s and the minimality of F_n, we can actually obtain that each G^i coincides with the ball, up to translations and negligible sets, whenever $|G^i| > 0$. Finally, from (6.6) and the isoperimetric inequality of the fractional perimeter P_s, we can show that the only one element in $\{G^i\}_i$ coincides with the ball B_1, up to translations and negligible sets. Therefore, from Brezis-Lieb lemma, the convergence in (6.6) is improved to the L^1-convergence and this concludes the proof.

Proof of Theorem 2.6. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be any sequence going to infinity as $n \in \mathbb{N}$ and we take any sequence $\{F_n\}_{n\in\mathbb{N}}$ of the minimizers for $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ with $|F_n| = |B_1|$ for any $n \in \mathbb{N}$. From the assumption (g5), we have that $g_{\lambda_n}(x) \leq \gamma|x|^{-(N+s)}$ for any $|x| \neq 0$ and some $\gamma \in (0,\,1)$. From the minimality of F_n for each $n \in \mathbb{N}$, we have that

$$P_s(F_n) \leq P_s(B_1) + P_{g_{\lambda_n}}(F_n) = P_s(B_1) + \gamma\, P_s(F_n)$$

for any $n \in \mathbb{N}$ and thus, we obtain that $\{P_s(F_n)\}_n$ is uniformly bounded with respect to n, namely, $\sup_{n\in\mathbb{N}} P_s(F_n) \le (1-\gamma)^{-1} P_s(B_1) < \infty$. As a consequence of the boundedness of $\{P_s(F_n)\}_n$, we can now apply the same method as in the proof of Theorem 2.3 (see also [13]) to the sequence $\{F_n\}_n$. Although we already discuss the method in the proof of Theorem 2.3, we rewrite the argument in the sequel for convenience.

Step 1. We first show that, under (g1), (g2), and (g5), there exists a sequence (not necessarily finite) of sets $\{G^j\}_j$ such that

$$\sum_j P_s(G^j) \le \liminf_{n\to\infty} \widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n), \quad \sum_j |G^j| = |B_1|. \tag{6.7}$$

To see this, we apply the same strategy as in Step 1 of the proof of Theorem 2.3 and thus we can show the existence of a sequence $\{G^j\}_j$ such that

$$\sum_j |G^j| = |B_1|.$$

The only thing that we need to do is to show the inequality

$$\sum_j P_s(G^j) \le \liminf_{n\to\infty} \widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n). \tag{6.8}$$

Indeed, we can prove this inequality in the same way as in Step 1 in the proof of Theorem 2.3, combining the assumption (g5) with the argument in the proof of Proposition 6.1. For convenience, we do not omit the proof.

As we show in Step 1 in the proof of Theorem 2.3, we can choose not only sets $\{G^j\}_j$ but also points $\{z^j_n\}_{j,\,n}$ such that

$$\chi_{F_n - z^j_n} \xrightarrow[n\to\infty]{} \chi_{G^j} \quad \text{in } L^1_{loc}, \quad |z^j_n - z^k_n| \xrightarrow[n\to\infty]{} c^{jk} \in [0,\,\infty]$$

for any $j \neq k$. Then we define the equivalent class $[j]$ in such a way that $c^{jk} < \infty$ if $k \in [j]$.

We now fix $M \in \mathbb{N}$ and $R > 0$ and we take the equivalent classes $j_1, \cdots, j_M$. Notice that, if $p \neq q$, then $|z^{j_p}_n - z^{j_q}_n| \to \infty$ as $n \to \infty$ and thus we have that $\{z^{j_p}_n + Q_R\}_p$ are disjoint sets for large n and

$$\int_{z^{j_p}_n + Q_R} \int_{z^{j_q}_n + Q_R} \frac{1}{|x-y|^{N+s}}\, dx\, dy \xrightarrow[n\to\infty]{} 0 \tag{6.9}$$

where Q_R is the cube of side R. Then, by using the similar argument to the one shown in the proof of the Γ-liminf inequality in Proposition 6.1 with (6.9), we have the following computation: let $\varepsilon \in (0,\,1)$ and, from (g5), we can choose a constant $R_\varepsilon > 1$ such that $g(x) \le \frac{\varepsilon}{|x|^{N+s}}$ for any $|x| \ge R_\varepsilon$. Then it holds that

$$\begin{aligned}
&\liminf_{n\to\infty} \big(P_s(F_n) - P_{g_{\lambda_n}}(F_n)\big) \\
&\ge (1-\varepsilon) \liminf_{n\to\infty} \int_{F_n \cap A^{M,R}_n} \int_{F^c_n} \frac{\chi_{\{|x-y| \ge r^\varepsilon_n\}}(x,\,y)}{|x-y|^{N+s}}\, dx\, dy \\
&\quad + (1-\varepsilon) \liminf_{n\to\infty} \int_{F_n \setminus A^{M,R}_n} \int_{A^{M,R}_n \setminus F_n} \frac{\chi_{\{|x-y| \ge r^\varepsilon_n\}}(x,\,y)}{|x-y|^{N+s}}\, dx\, dy \\
&\quad\quad + (1-\varepsilon) \liminf_{n\to\infty} 2 \sum_{p\neq q} \int_{z^{j_p}_n + Q_R} \int_{z^{j_q}_n + Q_R} \frac{\chi_{\{|x-y| \ge r^\varepsilon_n\}}(x,\,y)}{|x-y|^{N+s}}\, dx\, dy
\end{aligned} \tag{6.10}$$

for any $\varepsilon \in (0,\,1)$ where we set $r^\varepsilon_n := \lambda_n^{-1} R_\varepsilon$ for each n and $A^{M,R}_n := \cup_{p=1}^M (z^{j_p}_n + Q_R)$.

From (4.8), (6.10), and the lower semi-continuity of P_s in the L^1_{loc}-topology, we obtain

$$
\begin{aligned}
&\liminf_{n\to\infty} \left(P_s(F_n) - P_{g_{\lambda_n}}(F_n)\right)\\
&\geq (1-\varepsilon) \liminf_{n\to\infty} \sum_{p=1}^{M} \left(\int_{F_n \cap (z_n^{i_p}+Q_R)} \int_{F_n^c} \frac{\chi_{\{|x-y|\geq r_n^{\varepsilon}\}}(x,\, y)}{|x-y|^{N+s}}\, dx\, dy \right.\\
&\quad \left. + \int_{F_n \setminus (z_n^{j_p}+Q_R)} \int_{(z_n^{j_p}+Q_R)\setminus F_n} \frac{\chi_{\{|x-y|\geq r_n^{\varepsilon}\}}(x,\, y)}{|x-y|^{N+s}}\, dx\, dy \right)\\
&\geq (1-\varepsilon) \sum_{p=1}^{M} P_s(G^{j_p}; Q_R)
\end{aligned}
$$

for any $\varepsilon \in (0,\, 1)$. Letting $R \to \infty$, $M \to \infty$, and $\varepsilon \to 0$, we finally conclude that the inequality (6.8) holds true. Therefore, we conclude that the existence of sets $\{G^j\}_j$ satisfying (6.7) is true.

Step 2. We show that, under (g1), (g2), and (g5), each G^j actually coincides, up to translations and negligible sets, with the ball with volume $|G^j|$ whenever $|G^j| > 0$, where $\{G^j\}$ is given in the previous step.

Indeed, we first set B_j as the ball of radius $r_j := |B_1|^{-1/N}\, |G^j|^{1/N}$ for each j. Then, from (6.7) and the minimality of F_n, we have that

$$
\begin{aligned}
\sum_j \left(P_s(G^j) - P_s(B_j)\right) &\leq \liminf_{n\to\infty} \widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n) - \sum_j P_s(B_i)\\
&\leq P_s(B_1) - \sum_j \left(\frac{|G^j|}{|B_1|}\right)^{\frac{N-s}{N}} P_s(B_1)\\
&\leq P_s(B_1) - P_s(B_1) \left(\sum_j \frac{|G^j|}{|B_1|}\right)^{\frac{N-s}{N}} = 0.
\end{aligned}
\tag{6.11}
$$

From the isoperimetric inequality of P_s, we know that $P_s(B_j) \leq P_s(G^j)$ for any j and the equality holds if and only if $G^j = B_j$ up to translation and negligible sets. Hence, from (6.11), we conclude the proof of Step 2.

Step 3. We finally show that, under (g1), (g2), and (g5), there exists j_0 such that $|G^j| = 0$ for any $j \neq j_0$.

Indeed, from the isoperimetric inequality of the fractional perimeter P_s and (6.7), we obtain the following:

$$
\begin{aligned}
\liminf_{n\to\infty} \widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n) &\geq \sum_j P_s(G^j)\\
&\geq \sum_j \frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}} |G^j|^{\frac{N-s}{N}}\\
&\geq \frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}} \left(\sum_j |G^j|\right)^{\frac{N-s}{N}} = P_s(B_1).
\end{aligned}
\tag{6.12}
$$

In addition, from the definition of $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ and the minimality of each F_n, we have that

$$
\widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n) \leq P_s(B_1) \tag{6.13}
$$

for any $n \in \mathbb{N}$. Thus, from (6.12) and (6.13), we obtain

$$
P_s(B_1) = \sum_j P_s(G^j) = \frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}} \left(\sum_j |G^j|\right)^{\frac{N-s}{N}}, \quad \sum_j |G^j| = |B_1|. \tag{6.14}
$$

Thanks to the isoperimetric inequality for P_s, we get that each set G^j is a ball of volume $|G^j|$. Moreover, from the first equality in (6.14) it also follows that there exists j_0 such that $|G^j| = 0$ for any $j \neq j_0$, so that G^{j_0} coincides with the ball B_1, up to translations.

Therefore, taking into account all of the steps in the above, we may conclude that there exists points $\{z'_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that, up to extracting a subsequence, we have

$$\chi_{F_n - z'_n} \xrightarrow[n\to\infty]{} \chi_{B_1} \quad \text{in } L^1_{loc}.$$

From Brezis-Lieb lemma in [6], we obtain that the convergence

$$\chi_{F_n - z'_n} \xrightarrow[n\to\infty]{} \chi_{B_1} \quad \text{in } L^1_{loc}$$

holds in L^1 sense. Finally, we may repeat the above argument for any subsequence of $\{F_n\}_{n\in\mathbb{N}}$ and therefore, we conclude that Theorem 2.6 is valid.

□

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