Quantization of the Willmore Energy in Riemannian Manifolds

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Abstract

We show that the quantization of energy for Willmore spheres into closed Riemannian manifolds holds provided that the Willmore energy and the area be uniformly bounded. The analogous energy quantization result holds for Willmore surfaces of arbitrary genus, under the additional assumptions that the immersion maps weakly converge to a limiting (possibly branched, weak immersion) map from the same surface, and that the conformal structures stay within a compact domain of the moduli space.

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1 Introduction

Let $m \geq 3$ and (M^m, h) be a closed (compact without boundary) Riemannian manifold. For every smooth immersion $\vec{\Phi} : \Sigma \to M^m$, the (conformal) Willmore energy is defined by

$$W(\vec{\Phi}) = \int_\Sigma \left(|\vec{H}|^2 + K_h(\vec{\Phi}_* T\Sigma) \right) d\mathrm{vol}_g,$$

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where $g = \vec{\Phi}^* h$ is the induced metric by $\vec{\Phi}$ on Σ, $K_h(\vec{\Phi}_* T\Sigma)$ is the sectional curvature of the 2-plan induced by $\vec{\Phi}$, and $\vec{H}$ is the mean curvature vector, defined by

$$\vec{H} = \frac{1}{2} \sum_{i,j=1}^{2} g^{i,j} \, \vec{\mathbb{I}}_{i,j} \,,$$

where $\vec{\mathbb{I}}_{i,j}$ is the second fundamental form. This functional, first introduced in the Euclidean space by Poisson in 1814 ([50], see also the work of Sophie Germain [14]) in the context of non-linear elasticity, was rediscovered later by Blaschke and Thomsen [8] in the 1920's in the framework of conformal geometry and by Willmore [65] in 1965.

A key property of such a Lagrangian is the conformal invariance. Furthermore, it has been recently proved in [41] that the Willmore functional is the unique (up to linear combinations with topological terms) *conformally invariant* integral curvature energy for surfaces (in $\mathbb{R}^3$, it was already known that $(H^2 - K_g)d\text{vol}_g$ is up to scaling the only *pointwise* conformally invariant 2-form).

In addition to the aforementioned strong connection with conformal geometry, the Willmore functional in curved ambient spaces has remarkable links with other topics in mathematics and physics. For instance, the Willmore energy is the main term of the Hawking mass [16] in the framework of general relativity (see for instance [26, 13, 44]). Furthermore, it corresponds to the main term of the Nambu-Gotō action in string theory [51] and the renormalised area functional in the AdS/CFT correspondence [2, 3].

From the point of view of calculus of variations, as well as motivated by the aforementioned connections to physics, it is natural to investigate the existence and the properties of the *Willmore immersions* that are by definition the critical points of the Willmore energy. The standard first variation formulae show that critical points of W satisfy the Euler-Lagrange equation

$$\Delta_g^{\perp} \vec{H} - 2|\vec{H}|^2 \vec{H} + \mathscr{A}(\vec{H}) + \mathscr{R}_1^{\perp}(\vec{H}) - 2\widetilde{K}_h \vec{H} + 2\mathscr{R}_2(d\vec{\Phi}) + (DR)(d\vec{\Phi}) = 0 \,, \tag{1.1}$$

where $\mathscr{A}$ is the Simons operator, and the other terms are curvature functionals defined by

$$\mathscr{A}(\vec{w}) = -\frac{1}{2} \sum_{i,j=1}^{2} \left(\langle \vec{e}_i, \nabla_{\vec{e}_j} \vec{w} \rangle + \langle \vec{e}_j, \nabla_{\vec{e}_i} \vec{w} \rangle \right) \vec{\mathbb{I}}(\vec{e}_i, \vec{e}_j)$$

$$\mathscr{R}_1^{\perp}(\vec{w}) = \left(\sum_{i=1}^{2} R(\vec{w}, \vec{e}_i)\vec{e}_i \right)^{\perp}$$

$$\widetilde{K}_h = K_h(\vec{\Phi}_* T\Sigma)$$

$$\mathscr{R}_2(d\vec{\Phi}) = \sum_{i=1}^{2} \left(\langle R(\vec{\mathbb{I}}(\vec{e}_i, \vec{e}_1), \vec{e}_2)\vec{e}_2, \vec{e}_1 \rangle + \langle R(\vec{e}_1, \vec{\mathbb{I}}(\vec{e}_i, \vec{e}_2))\vec{e}_2, \vec{e}_1 \rangle \right)$$

$$(DR)(d\vec{\Phi}) = \sum_{i=1}^{m} \langle (\nabla_{\vec{v}_j} R)(\vec{e}_1, \vec{e}_2)\vec{e}_2, \vec{e}_1 \rangle \vec{v}_j \,,$$

where ∇ is the Levi-Civita connection, $(\vec{e}_1, \vec{e}_2)$ is an orthonormal moving frame of $\vec{\Phi}_*(T\Sigma)$, $(\vec{v}_1, \cdots, \vec{v}_m)$ is a local orthonormal frame of TM^m, and R is the Riemann curvature tensor of the ambient space (M^m, h). This equation was first obtained by Weiner ([64]) without the additional scalar curvature component whose derivation can be found in [42, Lemma 3.1] (see also [42, (3.13)]).

The literature about Willmore immersions in Riemannian manifolds (other than $\mathbb{R}^n$ or, equivalently by conformal invariance, S^n) is relatively recent and in expansion. The first existence results for Willmore spheres have been obtained in perturbative settings by the second author [38, 39]. Under the area-constraint condition, the existence and the geometric properties of Willmore-type spheres have been investigated by Lamm-Metzger-Schulze [26], Lamm-Metzger [24, 25], Laurain and the second author [27] and Eichmair-Körber [13]. Area-constrained Willmore tori of small area have been recently constructed by Ikoma, Malchiodi and the second author [19, 20]. All the aforementioned results are perturbative in nature, *i.e.* either the surfaces have sufficiently small area, or the ambient Riemannian metric is sufficiently close to either the Euclidean or the spherical metric.

The global problem of studying the existence of smooth immersed spheres minimising quadratic curvature functionals in Riemannian manifolds was initiated by the second author in collaboration with Kuwert and Schygulla [23] adapting Simon's ambient approach [59], and by the second author with Rivière [42, 43] via a parametric approach, proving the existence of area-constrained Willmore spheres in homotopy classes as well as the existence of Willmore spheres under various assumptions and constraints. Also, Chen-Li [10] proved the existence of stratified weak branched immersions of arbitrary genus minimising quadratic curvature functionals under various constraints (for weak immersions, refer to [45, 63] and to [55, 30] for works more in relationship with Willmore surfaces).

The main goal of this article is to generalise the quantization result of Bernard-Rivière [7] to the case of Willmore immersions in Riemannian manifolds. This result should be seen as the first step to generalise Rivière's min-max theory for Willmore spheres [56] to immersions with values into closed Riemannian manifolds. This extension is natural since, by conformal invariance of the Willmore energy, the quantization result in Euclidean spaces is equivalent to the energy quantization in the sphere S^n (for $n \geq 3$) equipped with its standard round metric.

Theorem A. *Let (M^m, h) be a smooth compact Riemannian manifold of dimension $m \geq 3$, and let $\{\vec{\Phi}_k\}_{k \in \mathbb{N}} \subset \mathrm{Imm}(S^2, M^m)$ be a sequence of Willmore immersions. Assume that*

$$\begin{cases} \limsup_{k \to \infty} W(\vec{\Phi}_k) < \infty \\ \limsup_{k \to \infty} \mathrm{Area}(\vec{\Phi}_k) < \infty. \end{cases} \tag{1.2}$$

Then, up to a subsequence, the following energy identity holds

$$\lim_{k \to \infty} W_{(M^m, h)}(\vec{\Phi}_k) = W_{(M^m, h)}(\vec{\Phi}_\infty) + \sum_{i=1}^{p} W_{(M^m, h)}(\vec{\Psi}_i) + \sum_{j=1}^{q} W_{\mathbb{R}^m}(\vec{\eta}_j) + \sum_{l=1}^{r} \left(W_{\mathbb{R}^m}(\vec{\zeta}_l) - 4\pi\, \theta_{0,l} \right), \tag{1.3}$$

where:

(1) *The map $\vec{\Phi}_\infty$ is a smooth Willmore immersion of S^2 into (M^m, h), possibly branched at finitely many points $a_1, \cdots, a_N \in S^2$.*

(2) *For any $j = 1, \ldots, p \in \mathbb{N}$, the map $\vec{\Psi}_i$ is a smooth, possibly branched Willmore immersion of S^2 into (M^m, h).*

(3) *For any $j = 1, \ldots, q \in \mathbb{N}$, $l = 1, \ldots, r \in \mathbb{N}$, the maps $\vec{\eta}_j : S^2 \to \mathbb{R}^m$ and $\vec{\zeta}_l : S^2 \to \mathbb{R}^m$ are smooth, possibly branched, Willmore immersions in $\mathbb{R}^m$ and $\theta_{0,l} = \theta_0(\vec{\zeta}_l, p_l) \in \mathbb{N}$ is the multiplicity of $\vec{\zeta}_l$ at some point $p_l \in \vec{\zeta}_l(S^2) \subset \mathbb{R}^n$.*

(4) *The map $\vec{\Phi}_\infty : S^2 \to M^m$ is obtained as follows: there exist a sequence of diffeomorphisms $\{f_k\}_{k \in \mathbb{N}}$ of S^2 such that $\vec{\Phi}_k \circ f_k$ is conformal and*

$$\vec{\Phi}_k \circ f_k \underset{k \to \infty}{\longrightarrow} \vec{\Phi}_\infty \quad \text{in } C^l_{\mathrm{loc}}(S^2 \setminus \{a_1, \cdots, a_N\}), \quad \forall l \in \mathbb{N}.$$

Furthermore, it holds

$$\lim_{k \to \infty} W_{(M^m, h)}(\vec{\Phi}_k) = W_{(M^m, h)}(\vec{\Phi}_\infty) \iff \vec{\Phi}_k \circ f_k \underset{k \to \infty}{\longrightarrow} \vec{\Phi}_\infty \text{ in } C^l(\Sigma), \quad \forall l \in \mathbb{N}.$$

Moreover, if $\lim_{k \to \infty} \mathrm{Area}(\vec{\Phi}_k) = 0$, then the first two terms in the right hand side of (1.3) are not present, i.e. using the same notation as above for $\vec{\eta}_j$, $\vec{\zeta}_l$ and $\theta_{0,l}$, it holds

$$\lim_{k \to \infty} W_{(M^m, h)}(\vec{\Phi}_k) = \sum_{j=1}^{q} W_{\mathbb{R}^m}(\vec{\eta}_j) + \sum_{l=1}^{r} \left(W_{\mathbb{R}^m}(\vec{\zeta}_l) - 4\pi\, \theta_{0,l} \right).$$

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Remarks 1.2. Let us denote $\vec{\xi}_k = \vec{\Phi}_k \circ f_k$, where f_k is given by (4) in Theorem A.

- The *Riemannian Willmore bubbles* $\vec{\Psi}_i : S^2 \to M^m$ are obtained as follows: for any $i \in \{1,\ldots,q\}$, there exist a sequence of positive Möbius transformations ψ_k^i of S^2 concentrating at one of $\{a_1,\cdots,a_N\}$ such that:

$$\vec{\xi}_k \circ \psi_k^i \xrightarrow[k\to\infty]{} \vec{\Psi}_i \quad \text{in } C_{\text{loc}}^l(S^2 \setminus \{a_1^i,\cdots,a_{N_i}^i\}), \quad \forall l \in \mathbb{N},$$

where $\{a_1^i,\cdots,a_{N_i}^i\}$ is a finite set of points in S^2.

- The *Euclidean Willmore bubbles* $\vec{\eta}_j, \vec{\zeta}_t : S^2 \to \mathbb{R}^m$ are obtained by the following blow up procedure: for any $s \in \{1,\ldots,p\}$ (resp. for any $t \in \{1,\ldots,q\}$), there exists a point $\overline{x}^j \in M$ (resp. $\overline{x}^l \in M$), there exist a sequence of positive Möbius transformations ψ_k^j (resp. ψ_k^l) of S^2 concentrating at one of $\{a_1,\cdots,a_N\}$, a sequence of rescalings $\lambda_k^j \to +\infty$ (resp. $\lambda_k^l \to +\infty$) and inversions Ξ_k^l of $\mathbb{R}^m$ such that:

$$\lambda_k^j \cdot \text{Exp}_{\overline{x}^j}^{-1} \circ \vec{\xi}_k \circ \psi_k^j \xrightarrow[k\to\infty]{} \vec{\eta}_j \quad \text{in } C_{\text{loc}}^l(S^2 \setminus \{a_1^j,\cdots,a_{N_j}^j\}), \quad \forall l \in \mathbb{N},$$

and, respectively,

$$\Xi_k^l \circ \lambda_k^l \cdot \text{Exp}_{\overline{x}^l}^{-1} \circ \vec{\xi}_k \circ \psi_k^l \xrightarrow[k\to\infty]{} \vec{\zeta}_t \quad \text{in } C_{\text{loc}}^l(S^2 \setminus \{a_1^l,\cdots,a_{N_l}^l\}), \quad \forall l \in \mathbb{N},$$

where $\{a_1^j,\cdots,a_{N_j}^j\}, \{a_1^l,\cdots,a_{N_l}^l\} \subset S^2$ are finite sets of points.

Arguing along the lines of the proof of Theorem A, one can prove the energy quantization for surfaces of arbitrary genus, under the assumption of $W^{2,2}$ weak convergence to a limit map and a bound on the conformal structures; the reader is referred to Theorem 6.1 for the precise statement.

Remarks 1.3. (1) Since the Gauss curvature is quantized as well (see (6.4), and refer to [7, Lemma V.1]), the quantization of energy stated in Theorem A and Theorem 6.1 also holds for a general quadratic curvature functional of the form

$$F_{\lambda_1,\lambda_2}(\vec{\Phi}) = \lambda_1 \int_\Sigma |\vec{H}|^2 \, d\text{vol}_g + \lambda_2 \int_\Sigma |\vec{h}_0|_{\text{WP}}^2 \, d\text{vol}_g, \tag{1.4}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$, where $\vec{h}_0$ is the Weingarten tensor (see for example the form introduced by Calabi in [9]) and $|\cdot|_{\text{WP}}$ the Weil-Petersson metric. Explicitly, we have in a conformal local chart $\vec{h}_0 = 2\pi_{\vec{n}}(\partial_z^2 \vec{\Phi})dz^2 = 2e^{2\lambda} \partial_z \left(e^{-2\lambda}\partial_z \vec{\Phi}\right) dz^2$, and for two 2-forms $\alpha = \varphi(z)dz^2$ and $\beta = \psi(z)dz^2$, we have

$$\langle \alpha, \beta \rangle_{WP} = e^{-4\lambda}\varphi(z)\overline{\psi(z)} = g^{-2} \otimes \alpha \otimes \overline{\beta}.$$

Indeed, the quantity

$$\int_\Sigma K_{g_k} d\text{vol}_{g_k}$$

is equal to $2\pi\chi(\Sigma)$ in the limit by Gauss-Bonnet Theorem and the smooth convergence of the conformal structures. The quantization of F_{λ_1,λ_2} also uses the quantization for the sectional curvature, but this result follows easily for this quantity is sub-critical.

(2) In [7] the boundedness of the area is not assumed because the authors work in $\mathbb{R}^n$. However, by a stereographic projection and by using the conformal invariance of the Willmore energy, their result is equivalent to the quantization of energy for S^n-valued maps, where S^n is equipped with its standard round metric. Indeed, for all immersion $\vec{\Phi} : \Sigma \to S^n$, the Willmore energy in the sphere is defined by

$$W_{S^n}(\vec{\Phi}) = \int_\Sigma \left(1 + |\vec{H}|^2\right) d\text{vol}_g,$$

which shows in particular that a uniform bound on the Willmore energy implies a uniform bound on the area. In particular, our assumption is a natural generalisation of the Euclidean result of Bernard-Rivière (notice that it holds in particular for any small enough perturbation of the round metric on the sphere).

(3) Consider the following assumption.

Assumption 1.4. The sectional curvature K_h of the ambient manifold (M, h) is bounded below by $\kappa_0 > 0$, *i.e.* there exists $\kappa_0 > 0$ such that $K_h(P) \geq \kappa_0$ for any 2-dimensional non-isotropic tangent plane $P \in \mathscr{G}_2(TM)$, where

$$K_h(P) = \frac{\langle R(\vec{v}, \vec{w})\vec{w}, \vec{v}\rangle}{|\vec{v}|^2|\vec{w}|^2 - \langle\vec{v}, \vec{w}\rangle^2},$$

and $P = \vec{v} \wedge \vec{w}$.

The assumption 1.4 implies that for any immersion $\vec{\Phi} : \Sigma \to (M^m, h)$, it holds

$$\text{Area}(\vec{\Phi}) \leq \frac{1}{\kappa_0} \int_\Sigma K_h(\vec{\Phi}_* T\Sigma) \, d\text{vol}_g \leq \frac{1}{\kappa_0} \int_\Sigma \left(|\vec{H}|^2 + K_h(\vec{\Phi}_* T\Sigma)\right) d\text{vol}_g \leq \frac{1}{\kappa_0} W(\vec{\Phi}),$$

which implies in particular that the bound on the area follows once a Willmore energy bound is in place. In particular, the theorem will hold for any small enough (in the C^2 topology) perturbation of the round metric on S^n *without the assumption on the uniform boundedness of the area*.

(4) Given a closed manifold M^m, for a generic Riemannian metric h (see [46]), one can control the area of an immersion $\vec{\Phi} : \Sigma \to (M^m, h)$ by its L^2-curvature energy ([4]; see also [40])

$$F_{1,1}(\vec{\Phi}) = \frac{1}{2} \int_\Sigma |\mathbb{I}|^2 \, d\text{vol}_g.$$

Therefore, the area bound in the assumption (1.2) can be dropped for a generic metric on a closed ambient manifold, when considering the quantization of energy for a functional F_{λ_1, λ_2}, with $\lambda_1, \lambda_2 > 0$. See (1.4) for the definition of F_{λ_1, λ_2}.

Some Ideas of the Proofs

Related literature on energy quantization and general strategy

The main results Theorem A and Theorem 6.1 shall be read in the context of other bubble-neck decomposition and energy quantization results. One can mention [57, 62, 21, 11, 49, 33, 52] in the setting of harmonic maps and other conformally invariant variational problems. A fundamental difference between the aforementioned energies and the Willmore functional is that, in the former, the corresponding Euler Lagrange equations are of second order, whilst the latter is a fourth-order problem.

As already mentioned, the first quantization result for the Willmore energy was obtained by Bernard-Rivière [7] for Willmore surfaces with bounded conformal structures and immersed in Euclidean ambient spaces. A first generalisation of [7] was established by Laurain-Rivière [29] in the case of Willmore immersions with degenerating conformal classes, still with values into $\mathbb{R}^n$. Also, Marque [36] obtained an improved convergence result in case of a single minimal bubble blown at a concentration point, and in [35] gave an explicit example of such a phenomenon. The crux of the proof of the quantization of energy is to obtain the *no-neck energy* property, once a suitable decomposition of the domain is performed. The idea is not restricted to the Willmore energy and applies to any quadratic energy (the Dirichlet energy [33], the Ginzburg-Landau energy [32, 31], the Euclidean Willmore energy [7, 29], horizontal 1/2-harmonic maps [34], etc). To fix ideas, consider an immersion $\vec{u} : (\Sigma, g) \to (M^m, h) \subset \mathbb{R}^n$ between Riemannian manifolds (where we assume without loss of generality that (M^m, h) is isometrically embedded into $\mathbb{R}^n$) and its Dirichlet energy given by

$$E(\vec{u}) = \frac{1}{2} \int_\Sigma |d\vec{u}|_g^2 \, d\text{vol}_g \,.$$

A neck region is conformally equivalent to an annulus $\Omega = B_R \setminus \overline{B}_r(0)$. We say that the *no-neck energy* property holds provided that for any neck-region $\Omega_k(1) = B_{R_k} \setminus \overline{B}_{r_k}(0) \subset \mathbb{C}$, we have

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \int_{\Omega_k(\alpha)} |\nabla \vec{u}_k|^2 dx = 0, \tag{1.5}$$

where, for all $0 < \alpha \leq 1$, $\Omega_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\alpha^{-1} r_k}(0)$. The idea of the proof is to use Lorentz spaces (see the Appendix 7 for more details) and the duality between $\mathrm{L}^{2,1}$ and $\mathrm{L}^{2,\infty}$, where $\mathrm{L}^{2,\infty}$ is the weak L^2 space and $\mathrm{L}^{2,1}$ its pre-dual which can be explicitly characterised. The duality implies in particular that for any measured space (X, μ) and any measurable maps $\vec{u}, \vec{v} : X \to \mathbb{R}^n$, it holds

$$\left| \int_\Omega \langle \vec{u}, \vec{v} \rangle \, d\mu \right| \leq \|\vec{u}\|_{\mathrm{L}^{2,1}(\Omega)} \|\vec{v}\|_{\mathrm{L}^{2,\infty}(\Omega)}. \tag{1.6}$$

The method proceeds into two steps. First, one proves for some $\alpha_0 > 0$ independent of $k \in \mathbb{N}$ an *a priori* estimate

$$\|d\vec{u}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))} \leq C \tag{1.7}$$

where $C > 0$ is independent of $k \in \mathbb{N}$. Then, one proves a *weak* quantization of energy, *i.e.* that

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \|\nabla \vec{u}_k\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha))} = 0, \tag{1.8}$$

then the duality inequality (1.6), (1.7), and (1.8) show that the energy quantization in (1.5) holds. Indeed:

$$\int_{\Omega_k(\alpha)} |\nabla \vec{u}_k|^2 dx \leq \|\nabla \vec{u}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} \|\nabla \vec{u}_k\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha))} \leq C \|\nabla \vec{u}_k\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha))}.$$

This approach was first developed by Lin-Rivière in the context of the Ginzburg-Landau functional [32, 31] and for harmonic maps [33]. It was used more recently for Willmore immersions with values into $\mathbb{R}^n$ by Bernard-Rivière [7] and Laurain-Rivière [29].

Difficulties and novelties of this paper

Although we build on Bernard-Rivière's work ([7]), new technical difficulties arise in the Riemannian setting. One key point is the need to introduce new Orlicz spaces that were not previously used in this context to our knowledge.

A fundamental ingredient in the proof of the energy quantization for Willmore immersions in Euclidean spaces [7] is the introduction of conservation laws by Rivière [53]: these permit to rewrite the fourth-order Willmore equation into a system of second-order Jacobian-type equations which can be handled with tools from integrability by compensation.

The first technical difficulty compared to [7] is that we do not get the existence of such an exact system of conservation laws for any critical immersion. Indeed, it is necessary to assume that the area is small enough to get the existence of a perturbed system of conservation laws (see [42, Lemma A.1 and Lemma A.2]), and Lemmas 3.6 and 3.14 in this paper. The perturbation is caused by the ambient curvature, and yields a complex-valued system of non-pure Jacobians (rather than real-valued systems of pure Jacobians as in the Euclidean setting). This is why, in order to prove the existence of the perturbed system of conservation laws in the neck region, we assume that the area is bounded and we prove that the area in neck-regions (and bubble regions too) is quantized, and therefore arbitrarily small.

The heart of the proof of Theorem 3.2 will be to establish refined estimates on the approximate conservation laws [42] satisfied by Willmore immersions. This will be achieved in Subsection 3.3.

A key technical difficult is to remove a $\log |z|$ term in estimate (3.67). It forced us to introduce a new Lorentz-type (or Orlicz) function space.

Another technical point is to obtain a suitable ε-regularity result for Willmore immersions with values into curved ambient spaces. We prove in Theorem 4.1 that there exists $\varepsilon_0 = \varepsilon_0(M^m, h) > 0$ with the following property: provided that $\vec{\Phi} : B(0,1) \to M^m$ is a weak Willmore immersion, the estimate

$$\text{Area}(\vec{\Phi}(B(0,1))) + \int_{B(0,1)} |\nabla \vec{n}|^2 dx \leq \varepsilon_0$$

implies that $\vec{\Phi} \in C^\infty(B(0,1))$ and, for all $k \in \mathbb{N}$, there exists $C_k < \infty$ such that

$$\left\| \nabla^k \vec{n} \right\|_{\mathrm{L}^\infty(B(0,\frac{1}{2}))} \leq C_k \left\| \nabla \vec{n} \right\|_{\mathrm{L}^2(B(0,1))}.$$

Notice that by the previous Remark 1.3 (3), the bound on the area is superfluous for a generic metric on M^m, or in the case of an ambient metric with positive sectional curvature.

The main technical difficulty is to obtain a pointwise $\mathrm{L}^{2,\infty}$ bound (see Theorem 3.13). In order to prove it, we will introduce a generalised Lorentz (or Orlicz-Lorentz) space modelled on $\mathrm{L}^{2,\infty}$ and named $\mathrm{L}^{2,\infty}_{\log^\beta}$ (where $0 \leq \beta \leq 1$ and $\mathrm{L}^{2,\infty}_{\log^0} = \mathrm{L}^{2,\infty}$) in the analysis. The reader is referred to Appendix 7 for more details on these Banach function spaces.

Another important step in the proof is to show that for holomorphic maps (the same proof works more generally for harmonic maps), the standard ε-regularity and scaling considerations giving that a $\mathrm{L}^{2,\infty}$ bound implies locally a $\mathrm{W}^{1,1} \cap \mathrm{L}^{2,1}$ estimate, hold more generally when one has a $\mathrm{L}^{2,\infty}_{\log^\beta}$ bound (see Lemma 7.7). Since the more classical improvement from $\mathrm{L}^{2,\infty}$ to $\mathrm{W}^{1,1} \cap \mathrm{L}^{2,1}$ for harmonic maps had several applications, it is natural to expect that the aforementioned sharpened improvement obtained in Lemma 7.7 will be useful also in other settings.

2   Notation and Preliminaries

Throughout the paper, (M^m, h) is a compact Riemannian manifold without boundary and Σ is a closed Riemann surface. Given a smooth immersion $\vec{\Phi} : \Sigma \to (M^m, h)$, we endow Σ with the pull-back metric $g = \vec{\Phi}^* h$. An important role will be played by the conformal structure associated to the metric g (for conformal structures on compact Riemann surfaces see for instance [22]). In particular we will assume that, given a sequence of immersions $\vec{\Phi}_k : \Sigma \to (M^m, h)$, the conformal structures associated to the pull-back metrics $g_k = \vec{\Phi}_k^* h$ are contained in a compact region of the moduli space. This assumption prevents the degeneration of the Riemann surface in the domain (see [7, 29]).

Without loss of generality, we can assume that the smooth immersion $\vec{\Phi} : \Sigma \to (M^m, h)$ is a conformal parametrisation, *i.e.* we can choose local coordinates (x_1, x_2) on Σ such that $g_{i,j} = e^{2\lambda} \delta_{i,j}$. The real valued function λ will be called *conformal factor*. Sometimes, taking advantage of the complex structure of a Riemann surface, it will be useful to switch the complex notation $z = x_1 + i\, x_2$.

Thanks to the Nash isometric embedding theorem, we can assume without loss of generality that $M^m \subset \mathbb{R}^n$, and that $h = \iota^* g_{\mathbb{R}^n}$, where $\iota : M^m \hookrightarrow \mathbb{R}^n$ is the Nash embedding. Since M^m is a compact manifold, we have in particular

$$\left\| \vec{\mathbb{I}}_{M^m} \right\|_{\mathrm{L}^\infty(M^m)} \leq C_0 < \infty, \tag{2.1}$$

where $\vec{\mathbb{I}}_{M^m}$ is the second fundamental form of ι. Given a smooth immersion $\vec{\Phi} : \Sigma \to (M^m, h)$ of the 2-dimensional surface Σ, we define the generalised Gauss map (see Hoffman-Osserman [18]) $\vec{n}_{\vec{\Phi}} : \Sigma \to$

$\Lambda^{m-2} TM^m$ by

$$\vec{n}_{\vec{\Phi}} = \star_h \frac{\partial_{x_1} \vec{\Phi} \wedge \partial_{x_2} \vec{\Phi}}{|\partial_{x_1} \vec{\Phi} \wedge \partial_{x_2} \vec{\Phi}|}$$

where $z = x_1 + i\,x_2$ are arbitrary local coordinates on Σ, and $\star_h : \Lambda^2 TM^m \to \Lambda^{m-2} TN^m$ is the linear Hodge operator associated to the metric h. We claim that the following formula holds:

$$\vec{n}_{\iota \circ \vec{\Phi}} = \iota_*(\vec{n}_{\vec{\Phi}}) \wedge (\vec{n}_\iota) \circ \vec{\Phi}. \tag{2.2}$$

Indeed, we have locally $\vec{n}_\iota = \vec{v}_1 \wedge \cdots \wedge \vec{v}_{n-m}$, and $\vec{n}_{\vec{\Phi}} = \vec{n}_1 \wedge \cdots \wedge \vec{n}_{m-2}$. If $\vec{e}_i = e^{-\lambda} \partial_{x_i} \vec{\Phi}$ in a conformal chart, we deduce by definition that $(\vec{e}_1, \vec{e}_2, \vec{n}_1, \cdots \vec{n}_{m-2})$ is an orthonormal basis of TM^m, which implies that $(\iota_*(\vec{e}_1), \iota_*(\vec{e}_2), \iota_*(\vec{n}_1), \cdots, \iota_*(\vec{n}_{m-2}), \vec{v}_1, \cdots, \vec{v}_{n-m})$ is an orthonormal basis of $\mathbb{R}^n$ and we deduce the claim (2.2).

Even if the main Theorem A concerns smooth immersions, some of the intermediate results that we will establish will hold more generally for weak conformal immersions. A weak conformal immersion of the unit ball $B(0,1) \subset \mathbb{R}^2$ is a map $\vec{\Phi} \in W^{1,\infty} \cap W^{2,2}(B(0,1), M^m)$ such that the a.e. well defined pullback metric $g = \vec{\Phi}^* h$ is conformal to the Euclidean metric on $B(0,1)$, i.e. $g_{i,j} = e^{2\lambda} \delta_{i,j}$ for some a.e. well defined function λ. Observe that the space of weak (conformal) immersions corresponds to the energy space for the Willmore functional W, thus it provides a natural functional analytic framework for the analysis and the calculus of variations of such an energy functional. Indeed the space of weak immersions with bounded area and Willmore energy satisfies useful pre-compactness properties. Let us recall the following pre-compactness result from [42], after [55, 30].

Theorem 2.1. *Let $\vec{\Phi}_k : S^2 \hookrightarrow (M^m, h)$ be a sequence of weak immersions of S^2 into the closed m-dimensional Riemannian manifold (M^m, h) and assume that the uniform area and Willmore bounds (1.2) hold. Then, up to pre-composing with suitable bi-Lipschitzian diffeomorphisms of S^2, one can assume that $\vec{\Phi}_k$ are conformally parametrised. Moreover,*

(1) ***Either**, $\mathrm{diam}\,(\vec{\Phi}_k(S^2)) \to 0$ and thus there exists a point $\overline{x} \in M$ such that, up to a subsequence, $\vec{\Phi}_k(S^2) \to \overline{x}$ in Hausdorff distance sense;*

(2) ***Or**, for every $k \in \mathbb{N}$, there exists a positive Möbius transformation f_k of S^2 such that, if we call*

$$\vec{\xi}_k = \vec{\Phi}_k \circ f_k$$

the reparametrised immersion and

$$\widetilde{\lambda}_k = \log |\partial_{x_1} \vec{\xi}_k| = \log |\partial_{x_2} \vec{\xi}_k|$$

the new conformal factor, the following holds (up to a subsequence):

(i) *There exists a finite set of points $\{a_1, \ldots, a_N\}$ such that for any compact subset $K \subset S^2 \setminus \{a_1, \ldots, a_N\}$*

$$\sup_{k \in \mathbb{N}} \left\| \widetilde{\lambda}_k \right\|_{L^\infty(K)} < \infty\,.$$

(ii) *There exists a conformal weak immersion $\vec{\xi}_\infty : S^2 \hookrightarrow (M^m, h)$, possibly branched at $\{a_1, \ldots, a_N\}$, such that*

$$\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty \quad \text{weakly in } W^{2,2}_{\mathrm{loc}}(S^2 \setminus \{a_1, \ldots, a_N\})\,. \tag{2.3}$$

Moreover,

$$W(\vec{\xi}_\infty) \leq \liminf_{k \to \infty} W(\vec{\xi}_k)\,.$$

(iii) *Furthermore, $W(\vec{\xi}_\infty) = \lim_{k \to \infty} W(\vec{\xi}_k)$ if and only if one can choose $\{a_1, \ldots, a_N\} = \emptyset$ in the above claims.*

Remark 2.2. Thanks to Simon's monotonicity formula [59], the first case in Theorem 2.1 is equivalent to $\mathrm{Area}(\vec{\Phi}_k(S^2)) \to 0$.

As a consequence of the ε-regularity Theorem 4.1 that we will prove later in the paper, if $\vec{\Phi}_k$ are Willmore spheres, then (2.3) can be improved to a $C^l_{\mathrm{loc}}(S^2 \setminus \{a_1, \ldots, a_N\})$ convergence for every $l \in \mathbb{N}$. The goal of the paper is to perform a fine analysis of $\vec{\xi}_k$, including the conformal factor, area, and Willmore energy, around the points $a_1, \ldots, a_N$, under the assumption that $\vec{\xi}_k$ are Willmore immersions. In order to simplify the notation, throughout the paper we will assume that the maps $\vec{\Phi}_k$ are already conformally parametrised and in the good gauge satisfying $\sup_{k \in \mathbb{N}} \|\lambda_k\|_{\mathrm{L}^\infty(K)} < \infty$, for every compact subset $K \subset S^2 \setminus \{a_1, \cdots, a_N\}$.

3 $\mathrm{L}^{2,1}$ Estimates on the Mean Curvature in the Neck Region

The first part of the proof of the main theorem is to establish $\mathrm{L}^{2,1}$ estimates on the mean curvature in the neck region (see Theorem 3.2). To this aim, in Subsection 3.1 we prove: a no-neck area property (in (3.24); see [43]), a Harnack-type inequality for the conformal factors (in (3.25); see [7]), an L^p quantization result for the conformal parameters for some $p > 2$ (see (3.29)) and, finally, uniform L^p estimates for the conformal parameters for some $p > 2$ (see (3.30)).

As we mentioned above, one of the fundamental parts of the proof of Theorem 3.2 is to obtain pointwise estimates for the tensors that appear in the approximate conversation laws from [42] (see Section 3.3).

Let $\left\{\vec{\Phi}_k\right\}_{k \in \mathbb{N}}$ be a sequence of Willmore immersions from a fixed closed Riemann surface Σ satisfying the hypotheses of Theorem A.

The crux will be to remove a $\log |z|$ term in one of the key estimates (more precisely, in (3.67)). This will take almost all of Subsection 3.4 and will be the most innovative and technical part of the paper, requiring the introduction of apparently new Lorentz-type function spaces. Let us stress that such a $\log |z|$ term is due to the curved ambient space and therefore was not present in the proof of the energy quantization for Willmore surfaces in Euclidean spaces [7].

Thanks to the hypothesis (1.2) of the theorem, we have

$$\Lambda = \sup_{k \in \mathbb{N}} \left(\mathrm{Area}(\vec{\Phi}_k(\Sigma)) + W(\vec{\Phi}_k) \right) < \infty \,. \tag{3.1}$$

Combining (2.2) with the triangle inequality, the Gauss equations and the conformal invariance of the Dirichlet energy, we get:

$$\int_\Sigma |d\vec{n}_{\iota \circ \vec{\Phi}}|^2_g d\mathrm{vol}_g \leq 2 \int_\Sigma |d\vec{n}_{\vec{\Phi}}|^2_g d\mathrm{vol}_g + 2 \int_\Sigma |d(\vec{n}_\iota \circ \vec{\Phi})|^2_g d\mathrm{vol}_g = 2 \int_\Sigma |d\vec{n}_{\vec{\Phi}}|^2_g d\mathrm{vol}_g + 2 \int_{\vec{\Phi}(\Sigma)} |\vec{\mathbb{I}}_{M^m}|^2_h d\mathrm{vol}_h$$

$$\leq 2 \int_\Sigma |d\vec{n}_{\vec{\Phi}}|^2_g d\mathrm{vol}_g + 2 C_0^2 \, \mathrm{Area}(\vec{\Phi}) = 2 \int_\Sigma \left(4|\vec{H}|^2 - 2K_g + 2 K_h(\vec{\Phi}_* T\Sigma) \right) d\mathrm{vol}_g + 2 C_0^2 \, \mathrm{Area}(\vec{\Phi})$$

$$\leq 8 \, W(\vec{\Phi}) + 2 C_0^2 \mathrm{Area}(\vec{\Phi}) + 4 \, \|K_h\|_{\mathrm{L}^\infty(M^m)} - 8\pi \, \chi(\Sigma) \,, \tag{3.2}$$

where we used estimate (2.1). Since M^m is a closed manifold, we deduce that the sectional curvature K_h of the smooth metric h on M^m is bounded. Therefore, the combination of (3.1) and (3.2) yields:

$$\sup_{k \in \mathbb{N}} \int_\Sigma |d\vec{n}_{\iota \circ \vec{\Phi}_k}|^2_g d\mathrm{vol}_g \leq (8 + 2C_0^2)\Lambda + 4 \, \|K_h\|_{\mathrm{L}^\infty(M^m)} - 8\pi \, \chi(\Sigma) = \Lambda(h) < \infty \,. \tag{3.3}$$

This allows us to apply the bubble-neck decomposition of Bernard-Rivière [7, Proposition III.1], and the other theorems of [7, Sections III and IV], since they do not use the Euler-Lagrange equation of Willmore surfaces in $\mathbb{R}^n$ and work for any sequence of smooth immersions of bounded Willmore energy.

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Lemma 3.1 (Bernard-Rivière, Lemma V.1 of [7]). *There exist constants $\varepsilon_0(n), C_0(n) > 0$ with the following property. If $0 < \varepsilon < \varepsilon_0(n)$, $0 < 4r < R < \infty$, $\Omega = B_R \setminus \overline{B}_r(0)$, and $\vec{\Phi} : \Omega \to \mathbb{R}^n$ is a conformal weak immersion satisfying the conditions*

$$
\begin{cases}
\|\nabla \vec{n}\|_{\mathrm{L}^{2,\infty}(\Omega)} \leq \varepsilon \\
\displaystyle\int_{\partial B_r(0)} |\nabla \vec{n}| \, d\mathscr{H}^1 \leq \varepsilon \\
\displaystyle\int_{B_R \setminus \overline{B}_{\frac{R}{2}}(0)} |\nabla \vec{n}|^2 dx + \int_{B_{2r} \setminus \overline{B}_r(0)} |\nabla \vec{n}|^2 dx \leq \varepsilon,
\end{cases}
\tag{3.4}
$$

then

$$
\left| \int_\Omega K_g d\mathrm{vol}_g \right| \leq C_0(n)\,\varepsilon,
$$

where $g = \vec{\Phi}^* g_{\mathbb{R}^n}$.

By the Theorema Egregium of Gauss, since $\iota : (M^m, h) \to (\iota(M^m), g_{\mathbb{R}^n})$ is an isometry, we deduce that we can apply this result to $\vec{\Phi}_k$. Hence, if $\vec{\Phi}_k$ is parametrising a neck-region $\Omega = B_R \setminus \overline{B}_r(0)$ and satisfies the hypothesis of Lemma 3.1, we deduce that

$$
\left| \int_\Omega K_{g_k} d\mathrm{vol}_{g_k} \right| \leq C_0(n)\varepsilon.
$$

Then, the other results of [7, Section V] where no Euler-Lagrange equation is used can be applied identically to $\{\iota \circ \vec{\Phi}_k\}_{k\in\mathbb{N}}$. The Liouville equation

$$
-\Delta \lambda_k = e^{2\lambda_k} K_{g_k}
$$

implies by the Adams-Morrey embedding (see [54]) that

$$
\sup_{k\in\mathbb{N}} \|\nabla \lambda_k\|_{\mathrm{L}^{2,\infty}(B(0,1))} \leq C.
\tag{3.5}
$$

Now, we will describe the differences from [7, Section VI] onwards.

In [7, Section VI], the conservative form of the Willmore equation for immersions in $\mathbb{R}^n$ discovered in [53] plays a fundamental role. In order to extend such analysis to the curved ambient setting, we will use the Euler-Lagrange equation in conservative form obtained in [42] for immersions into Riemannian manifolds. However, this requires to prove that the area in neck regions is small enough (in fact, we need a slightly stronger statement which will follow from this bound thanks to a Harnack inequality; see Lemma 3.6).

The next theorem is the main result of this section.

Theorem 3.2. *Let $\{r_k\}_{k\in\mathbb{N}}, \{R_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ be such that $r_k \xrightarrow[k\to\infty]{} 0$ and $R_k \xrightarrow[k\to\infty]{} R \in (0,\infty)$. Let $(M^m, h) \subset \mathbb{R}^n$ be a closed Riemannian manifold that we assume isometrically embedded in $\mathbb{R}^n$. For any $0 < \alpha \leq 1$, define the subset $\Omega_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\alpha^{-1} r_k}(0) \subset B(0, R_k)$. There exists constants $\varepsilon_0 = \varepsilon_0(n,h), \alpha_0 = \alpha_0(n,h) > 0$ with the following property. Let $\{\vec{\Phi}_k\}_{k\in\mathbb{N}} \subset C^\infty(B(0,R_k), M^m)$ be a sequence of Willmore disks satisfying:*

$$
\Lambda(h) = \sup_{k\in\mathbb{N}} \left(\|\nabla \lambda_k\|_{\mathrm{L}^{2,\infty}(B(0,R_k))} + \mathrm{Area}(\vec{\Phi}_k(B(0,R_k))) + \int_{B(0,R_k)} |\nabla \vec{n}_k|^2 dx \right) < \infty
\tag{3.6}
$$

$$
\sup_{s\in\left[r_k, \frac{R_k}{2}\right]} \int_{B_{2s} \setminus \overline{B}_s(0)} |\nabla \vec{n}_k|^2 dx \leq \varepsilon_0.
$$

Then we have

$$
\left\| e^{\lambda_k} \vec{H}_k \right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))} \leq C_0(n,h,\Lambda).
$$

The rest of the section will be devoted to the proof of Theorem 3.2. This will require to establish several results of independent interest.

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3.1 L^p Bounds and Quantization for the Conformal Parameters

The goal of this section is four-fold: we prove a no-neck area property (see (3.24)), establish a Harnack-type inequality for the conformal factors (see (3.25)), prove an L^p quantization result for the conformal parameters for some $p > 2$ (see (3.29)), and establish uniform L^p estimates for the conformal parameters for some $p > 2$ (see (3.30)).

Let us first recall the following lemma from [37], slightly generalising [7, Lemma IV.1].

Theorem 3.3. *There exists a positive real number $\varepsilon_1 = \varepsilon_1(n)$ with the following property. Let $0 < 2^6 r < R < \infty$ be fixed radii and $\vec{\Phi} : \Omega = B_R \setminus \overline{B}_r(0) \to \mathbb{R}^n$ be a weak immersion of finite total curvature such that*

$$\|\nabla\vec{n}\|_{L^{2,\infty}(\Omega)} \leq \varepsilon_1(n). \tag{3.7}$$

For all $\left(\dfrac{r}{R}\right)^{\frac{1}{2}} < \alpha < 1$, define $\Omega(\alpha) = B_{\alpha R} \setminus \overline{B}_{\alpha^{-1}r}(0)$. Then there exists a universal constant $C_1 = C_1(n)$ and $d \in \mathbb{R}$ (depending on $r, R, \vec{\Phi}$ but not on α) such that for all $\left(\dfrac{r}{R}\right)^{\frac{1}{3}} < \alpha < \dfrac{1}{4}$, we have

$$\|\nabla(\lambda - d\log|z|)\|_{L^{2,1}(\Omega(\alpha))} \leq C_1\left(\sqrt{\alpha}\,\|\nabla\lambda\|_{L^{2,\infty}(\Omega)} + \int_{\Omega}|\nabla\vec{n}|^2 dx\right) \tag{3.8}$$

and for all $r \leq \rho < R$, we have

$$\left|d - \frac{1}{2\pi}\int_{\partial B_\rho}\partial_\nu\lambda\,d\mathcal{H}^1\right| \leq C_1\left(\int_{B_{\max\{\rho,2r\}}\setminus\overline{B}_r(0)}|\nabla\vec{n}|^2 dx + \frac{1}{\log\left(\frac{R}{\rho}\right)}\int_{\Omega}|\nabla\vec{n}|^2 dx\right). \tag{3.9}$$

In particular, there exists a universal constant $C_1' = C_1'(n)$ with the following property: for all $\left(\dfrac{r}{R}\right)^{\frac{1}{3}} < \alpha < \dfrac{1}{4}$, there exists $A_\alpha \in \mathbb{R}$ such that

$$\|\lambda - d\log|z| - A_\alpha\|_{L^\infty(\Omega(\alpha))} \leq C_1'\left(\sqrt{\alpha}\,\|\nabla\lambda\|_{L^{2,\infty}(\Omega)} + \int_{\Omega}|\nabla\vec{n}|^2 dx\right). \tag{3.10}$$

Applying Theorem 3.3 to $\{\vec{\Phi}_k\}_{k\in\mathbb{N}}$ (that we see from now as a map $\vec{\Phi}_k : \Sigma \to \mathbb{R}^n$ such that $\vec{\Phi}_k(\Sigma) \subset M^m$ for all $k \in \mathbb{N}$), we deduce that in a neck region $\Omega_k(\alpha) = B(0, \alpha R_k) \setminus \overline{B}(0, \alpha^{-1}r_k)$ (where $\limsup\limits_{k\to\infty} R_k < \infty$), there exists for $k \in \mathbb{N}$ large enough $d_k \in \mathbb{R}$ and $A_k \in \mathbb{R}$ such that

$$\|\lambda_k - d_k\log|z| - A_k\|_{L^\infty(\Omega_k(\alpha))} \leq C_1'\left(\sqrt{\alpha}\,\|\nabla\lambda_k\|_{L^{2,\infty}(\Omega_k(\alpha))} + \int_{\Omega_k(1)}|\nabla\vec{n}_k|^2 dx\right) \leq C_1'' < \infty \tag{3.11}$$

thanks to (3.3) and (3.5). We deduce that:

$$e^{-2C_1''}e^{2A_k}|z|^{2d_k} \leq e^{2\lambda_k} \leq e^{2C_1''}e^{2A_k}|z|^{2d_k}, \quad \text{for all } z \in \Omega_k(\alpha). \tag{3.12}$$

Thanks to (3.1), we deduce that

$$\sup_{k\in\mathbb{N}}\int_{\Omega_k(\alpha)}e^{2A_k}|z|^{2d_k}|dz|^2 \leq \sup_{k\in\mathbb{N}}e^{2C_1''}\mathrm{Area}(\vec{\Phi}_k(\Omega_k(\alpha))) \leq e^{2C_1''}\Lambda < \infty. \tag{3.13}$$

Now, by the ε-regularity Theorem 4.1 to be proven below, we deduce that $\vec{\Phi}_k \xrightarrow[k\to\infty]{} \vec{\Phi}_\infty$ in $C^l_{\mathrm{loc}}(B(0,1) \setminus \{0\})$ for all $l \in \mathbb{N}$, where $\vec{\Phi}_\infty : B(0,1) \to \mathbb{R}^n$ is a branched immersion having at most a branch point at 0. Therefore, by [55, Lemma A.5] (see also [7]), there exist an integer $\theta_0 \geq 1$ and $\vec{A}_0 \in \mathbb{C}^n \setminus \{0\}$ such that

$$\partial_z\vec{\Phi}_\infty = \vec{A}_0 z^{\theta_0 - 1} + o(|z|^{\theta_0 - 1}). \tag{3.14}$$

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We also let $\beta_0 > 0$ such that

$$e^{2\lambda_\infty} = 2|\partial_z \vec{\Phi}_\infty|^2 = 2|\vec{A}_0|^2 |z|^{2\theta_0 - 2}(1 + o(1)) = \beta_0^2 |z|^{2\theta_0 - 2}(1 + o(1)).$$

Now, applying (3.9) to d_k and $\rho = \alpha \frac{R}{2}$, we deduce that

$$\limsup_{k \to \infty} \left| d_k - \frac{1}{2\pi} \int_{\partial B_{\alpha \frac{R}{2}}} \partial_\nu \lambda_k \, d\mathscr{H}^1 \right| \leq \Gamma_2 < \infty.$$

By the strong convergence, it follows that

$$\frac{1}{2\pi} \int_{\partial B_{\alpha \frac{R}{2}}} \partial_\nu \lambda_k \, d\mathscr{H}^1 \xrightarrow[k \to \infty]{} \frac{1}{2\pi} \int_{\partial B_{\alpha \frac{R}{2}}} \partial_\nu \lambda_\infty d\mathscr{H}^1 = \theta_0 - 1 + O(\alpha R)$$

and we deduce that $\{d_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ is a bounded sequence. Therefore, we can assume up to a subsequence that $d_k \xrightarrow[k \to \infty]{} d \in \mathbb{R}$.

Lemma 3.4. $d > -1$.

Proof. Without loss of generality, we can assume that $R_k \xrightarrow[k \to \infty]{} R$ such that $0 < R < \infty$. Furthermore, notice that by hypothesis, there is no energy concentration on $B_{2R_k} \setminus \overline{B}_{\frac{R_k}{2}}(0)$. Therefore, applying our ε-regularity result of Theorem 4.1, we deduce that $\vec{\Phi}_k \xrightarrow[k \to \infty]{} \vec{\Phi}_\infty$ in $C^l(B_{\frac{3}{2}R} \setminus \overline{B}_{\frac{2}{3}R}(0))$ for all $l \in \mathbb{N}$. Therefore, $\{\lambda_k\}_{k \in \mathbb{N}}$ is bounded on $\partial B_{R_k}(0)$, and since $d_k \xrightarrow[k \to \infty]{} d$, we deduce by (3.11) that $\{A_k\}_{k \in \mathbb{N}}$ is bounded. Therefore, setting $\underline{A} = \liminf_{k \to \infty} A_k$ and $\overline{A} = \limsup_{k \to \infty} A_k$, it holds that

$$-\infty < \underline{A} = \liminf_{k \to \infty} A_k \leq \limsup_{k \to \infty} A_k = \overline{A} < \infty. \tag{3.15}$$

Using once more the strong convergence in (3.12), (3.14) and (3.15) we obtain that for all $z \in B_R(0) \setminus \{0\}$,

$$e^{-2C_1''} e^{2\underline{A}} |z|^{2d} \leq |\vec{A}_0|^2 (1 + O(|z|)) |z|^{2\theta_0 - 2} \leq e^{2C_1''} e^{2\overline{A}} |z|^{2d}. \tag{3.16}$$

We argue by contradiction. First, assume that $d < -1$. Then, we get that for k large enough $d_k < -1$. Using that $r_k \to 0$ and recalling (3.12), we obtain that, for all $0 < \alpha < 1$, it holds

$$\text{Area}(\vec{\Phi}_k(\Omega_k(\alpha))) = \int_{\Omega_k(\alpha)} e^{2\lambda_k} |dz|^2 \geq e^{-2C_1''} \int_{\Omega_k(\alpha)} e^{2A_k} |z|^{2d_k} |dz|^2 \tag{3.17}$$

$$= \frac{\pi}{|d_k| - 1} e^{-2C_1''} e^{2A_k} \frac{1}{(\alpha^{-1} r_k)^{2(|d_k| - 1)}} \left(1 - \left(\frac{r_k}{\alpha^2 R_k} \right)^{2(|d_k| - 1)} \right) \xrightarrow[k \to \infty]{} \infty,$$

which contradicts (3.13). Thus $d \geq -1$ and we need only prove that $d \neq -1$. If $d_k = -1$, we have

$$\text{Area}(\vec{\Phi}_k(\Omega_k(\alpha))) = \int_{\Omega_k(\alpha)} e^{2\lambda_k} |dz|^2 \geq e^{-2C_1''} \int_{\Omega_k(\alpha)} e^{2A_k} \frac{|dz|^2}{|z|^2} = 2\pi e^{-2C_1''} e^{2A_k} \log\left(\frac{\alpha^2 R_k}{r_k} \right) \to \infty.$$

Therefore, we can assume that $d_k \neq -1$ for k large enough. By Fatou's lemma, (3.12) and (3.15), we deduce that, for all $0 < \varepsilon < R$, it holds

$$\liminf_{k \to \infty} \text{Area}(\vec{\Phi}_k(\Omega_k(\alpha))) \geq \liminf_{k \to \infty} \text{Area}(\vec{\Phi}_k(B_{\alpha R} \setminus \overline{B}_\varepsilon(0))) \geq e^{-2C_1''} \int_{B_{\alpha R} \setminus \overline{B}_\varepsilon(0)} e^{2\underline{A}} |z|^{2d} |dz|^2.$$

By hypothesis, $\text{Area}(\vec{\Phi}_k(\Omega_k(\alpha)))$ is bounded (see (3.13)), which implies that the function

$$(0, R) : \varepsilon \mapsto \int_{B_{\alpha R} \setminus \overline{B}_\varepsilon(0)} |z|^{2d} |dz|^2 \quad \text{is bounded.}$$

We deduce that $d > -1$, as desired. $\qquad \square$

Remark 3.5. Once the quantization of energy is established, it will imply *a posteriori* that (see [37])

$$d_k \underset{k\to\infty}{\longrightarrow} \theta_0 - 1 \geq 0. \tag{3.18}$$

In fact, [37, Theorem A] holds for an arbitrary immersion with values into $\mathbb{R}^n$ and yields that $d_k = \theta_0 - 1 \geq 0$. However, in the present setting of curved ambient space, we could not get (3.18) *a priori* (as it happens in the case of a flat ambient space). The origin of such a difficulty lies in the non-vanishing curvature that perturbs the system of conservation laws associated to the Willmore equation [42].

Let us also observe that the fact that a no-neck energy property implies the asymptotic integrality of d_k suggests that an hypothesis ensuring that $d_k > -1 + \varepsilon$ (for k large enough)—that follows from the uniform boundedness of the area—is necessary.

Now, since by [7, Proposition III.1] neck-regions are disjoint and finite unions of such annuli, if $\Omega_k(\alpha)$ is the whole neck-region associated to a concentration point a_i (where $1 \leq i \leq N$, and N is the number of concentration points), we have (see also [43]):

$$\lim_{\alpha\to 0} \limsup_{k\to\infty} \int_{\Omega_k(\alpha)} e^{2\lambda_k} |dz|^2 = 0. \tag{3.19}$$

Indeed, if $\Omega'_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\rho_k}(0)$, we deduce by Lemma 3.4 that there exists $\delta > 0$ such that $d_k > -1 + \delta$ for k large enough and

$$\int_{\Omega'_k(\alpha)} e^{2\lambda_k} |dz|^2 \leq e^{\overline{A}} \int_{B(0,\alpha R_k)} |z|^{2d_k} |dz|^2 = \frac{\pi e^{\overline{A}}}{d_k + 1} (\alpha R_k)^{2d_k + 2} \leq \frac{\pi e^{\overline{A}}}{\delta} (2\alpha R)^{\delta} \underset{\alpha\to 0}{\longrightarrow} 0.$$

Consider a typical bubble region

$$B(i, j, \alpha, k) = B_{\alpha^{-1} r_k^{i,j}}(x_k^{i,j}) \setminus \bigcup_{j' \in I^{i,j}} B_{\alpha r_k^{i,j}}(x_k^{i,j'}) \tag{3.20}$$

from the bubble-neck decomposition [7, Proposition III.1]. We refer to [7] for the precise statement and relevant definitions. For our purpose here it is sufficient to recall that

(1) $B_{r_k^{i,j}}(x_k^{i,j})$ corresponds to a bubble for $\vec{\Phi}_k$;

(2) the set of indices $I^{i,j}$ corresponds to the bubbles contained in $B_{r_k^{i,j}}(x_k^{i,j})$;

(3) the total number of bubbles is bounded: $\sup_{k\in\mathbb{N}} \text{card} \bigcup_{i,j} I^{i,j} < \infty$;

(4) $\lim_{k\to\infty} x_k^{i,j} \to a_i$, for every i, j.

(5) $\lim_{k\to\infty} r_k^{i,j} \to 0$, for every i, j.

From [7, (VIII.10)] (notice that this result does not use the Willmore equation, and therefore, holds in our Riemannian setting), a uniform Harnack inequality holds: for all $0 < \alpha < 1$, there exists $C_\alpha > 1$ such that for all $k \in \mathbb{N}$ large enough

$$\sup_{B(i,j,\alpha,k)} e^{2\lambda_k} \leq C_\alpha \inf_{B(i,j,\alpha,k)} e^{2\lambda_k}. \tag{3.21}$$

Therefore, the estimate (3.12) implies that there exists C'_α such that for all $z \in B(i, j, \alpha, k)$

$$e^{2\lambda_k(z)} \leq C'_\alpha (\alpha^{-1} r_k^{i,j})^{2d_k} \leq C''_\alpha \left(r_k^{i,j} \right)^{2d_k}. \tag{3.22}$$

Since $d_k \underset{k\to\infty}{\longrightarrow} d > -1$, we deduce that for all $0 < \alpha < 1$

$$\int_{B(i,j,\alpha,k)} e^{2\lambda_k} |dz|^2 \leq \pi C''_\alpha \left(r_k^{i,j} \right)^{2d_k + 2} \underset{k\to\infty}{\longrightarrow} 0. \tag{3.23}$$

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \int_{B(0,\alpha R_k)} e^{2\lambda_k} |dz|^2 = 0. \tag{3.24}$$

We also deduce that there exists $A \in \mathbb{R}$ such that, for all $k \in \mathbb{N}$ large enough, it holds for all $z \in \Omega_k(1/2)$:

$$e^{-A} |z|^{d_k} \le e^{\lambda_k(z)} \le e^A |z|^{d_k}. \tag{3.25}$$

For the next developments, we need to sharpen the above estimates to an $\mathrm{L}^{2,1}$ bound for the conformal parameter and an L^p quantization result for it.

Let $d \in \mathbb{R}$ and $f : B(0, R) \to \mathbb{R} \cup \{\infty\}$ be such that, for all $z \in B(0, R)$, it holds $f(z) = |z|^d$. If $d \le -1$, since $\| \cdot \|_{\mathrm{L}^{2,1}(X)} \ge 2\sqrt{2} \, \| \cdot \|_{\mathrm{L}^2(X)}$, we have:

$$\left\| |z|^d \right\|_{\mathrm{L}^{2,1}(B(0,R))} = \infty.$$

Recalling that for all measured space (X, μ), for all $1 < p < \infty$, it holds (see for example [37, Appendix 3.7.1])

$$\|f\|_{\mathrm{L}^{p,1}(X)} = \frac{p^2}{p-1} \int_0^\infty \mu \left(X \cap \{ x : |f(x)| > t \} \right)^{\frac{1}{p}} dt,$$

we have:

$$\left\| |z|^d \right\|_{\mathrm{L}^{2,1}(B(0,R))} = 4 \int_0^1 \mu(B(0,R))^{\frac{1}{2}} dt = 4\sqrt{\pi} R, \quad \text{for } d = 0,$$

$$\left\| |z|^d \right\|_{\mathrm{L}^{2,1}(B(0,R))} = 4 \int_0^{R^d} \mu(B(0,R))^{\frac{1}{2}} dt + 4 \int_{R^d}^\infty \mu(B(0, t^{\frac{1}{d}}))^{\frac{1}{2}} dt$$

$$= 4\sqrt{\pi} R^{1+d} + 4\sqrt{\pi} \frac{-d}{1+d} R^{1+d} = \frac{4\sqrt{\pi}}{1+d} R^{1+d}, \quad \text{for } -1 < d < 0,$$

$$\left\| |z|^d \right\|_{\mathrm{L}^{2,1}(B(0,R))} = 4 \int_0^{R^d} \mu(B(0,R) \setminus \overline{B}(0, t^{\frac{1}{d}}))^{\frac{1}{2}} dt = 4\sqrt{\pi} R^{1+d} \int_0^1 \sqrt{1 - s^{\frac{2}{d}}} \, ds$$

$$\le 4\sqrt{\pi} R^{1+d}, \quad \text{for } d > 0.$$

Combining the last estimates with (3.25), we deduce that, for all $k \in \mathbb{N}$ large enough, it holds

$$\left\| e^{\lambda_k} \right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} \le 4\sqrt{\pi} e^A \max \left\{ 1, \frac{1}{1+d_k} \right\} (\alpha R_k)^{1+d_k} \xrightarrow[k \to \infty]{} 4\sqrt{\pi} e^A \max \left\{ 1, \frac{1}{1+d} \right\} (\alpha R)^{1+d}. \tag{3.26}$$

By (3.22), we have

$$\left\| e^{\lambda_k} \right\|_{\mathrm{L}^{2,1}(B(i,j,\alpha,k))} \le 4\sqrt{\pi} \sqrt{C_\alpha''} \left(r_k^{i,j} \right)^{1+d_k} \xrightarrow[k \to \infty]{} 0. \tag{3.27}$$

The combination of (3.26) and (3.27) yields

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \left\| e^{\lambda_k} \right\|_{\mathrm{L}^{2,1}(B(0,\alpha R_k))} = 0. \tag{3.28}$$

We see that the estimate is a combination of (3.26) that controls the $L^{2,1}$ norm in the neck regions, and of (3.27) that controls the $L^{2,1}$ norm in the bubble regions.

Later on, we will need the following improvement of the quantization (3.28). Since $d_k \xrightarrow[k \to \infty]{} d > -1$, we deduce that there exists $0 < \varepsilon < 1$ and $N \in \mathbb{N}$ such that for all $k \ge N$, we have $d_k \ge -1 + \varepsilon$. In particular, this implies that

$$\int_{\Omega_k(\alpha)} e^{p\lambda_k(z)} |dz|^2 \le e^{pA} \int_{B(0,\alpha R_k)} |z|^{pd_k} |dz|^2 = 2\pi e^{pA} \frac{(\alpha R_k)^{2+pd_k}}{2 + pd_k}, \quad \text{for all } p < \frac{2}{1-\varepsilon}.$$

Using the Harnack inequality in bubble domains (3.21), we deduce that

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \left\| e^{\lambda_k} \right\|_{\mathrm{L}^p(B(0,\alpha R_k))} = 0, \quad \text{for all } p < \frac{2}{1-\varepsilon}. \tag{3.29}$$

Indeed, we have $2 + pd_k > 2 + p(-1+\varepsilon) > 0$ if and only if $p < \dfrac{2}{1-\varepsilon}$. In particular, we deduce that

$$e^{\lambda_k} \quad \text{is bounded in} \quad \mathrm{L}^p\left(B\left(0, R_k/2\right)\right) \text{ for all } p < \frac{2}{1-\varepsilon}. \tag{3.30}$$

3.2 Diameter Estimate in Bubbling Domains

In order to solve ∂_z-like equation, one needs to perform a trivialisation of the tangent bundle. Therefore, we need to make sure that $\vec{\Phi}_k(B(0,\alpha R_k))$ has a bounded diameter in $k \in \mathbb{N}$ that converges to 0 as $\alpha \to 0$.

Notice that in a neck region $\Omega_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\alpha^{-1}\rho_k}(0)$, we have

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \operatorname{diam}(\vec{\Phi}_k(B(0,\alpha R_k))) = 0.$$

Indeed, by the Harnack inequality in the neck region, we have on $\Omega_k(\alpha_0)$ for some $\alpha_0 > 0$ independent of $k \in \mathbb{N}$ and for all k large enough

$$e^{\lambda_k} = e^{A_k}|z|^{d_k}(1 + o(1)),$$

where $A_k \xrightarrow[k\to\infty]{} A \in \mathbb{R}$ and $d_k \xrightarrow[k\to\infty]{} d > -1$ by the above analysis. For simplicity, assume that there is a single bubble. Thanks to the Harnack inequality in each bubble domain and neck region ([7, (VIII.10) p. 131]), without loss of generality we can assume there is a single bubble. Now, by the Harnack inequality on the bubble domain, we deduce that for all $0 < \alpha < \alpha_0$, there exists $C_\alpha, C'_\alpha < \infty$ such that

$$C'_\alpha(\rho_k)^{d_k} \le C_\alpha^{-1} e^{\lambda_k(\alpha^{-1}\rho_k)} \le e^{\lambda_k} \le C_\alpha e^{\lambda_k(\alpha^{-1}\rho_k)} \le C'_\alpha(\rho_k)^{d_k} \qquad \text{in } B(0,\alpha^{-1}\rho_k).$$

In particular, for all $z \in B(0,\alpha R_k)$, we have

$$|\vec{\Phi}_k(z) - \vec{\Phi}_k(0)| \le \int_0^{|z|} e^{\lambda_k\left(r\frac{z}{|z|}\right)} dr \le C'_\alpha \int_0^{\alpha^{-1}\rho_k} (\rho_k)^{d_k} dr + 2\, e^{A_k} \int_{\alpha^{-1}\rho_k}^{|z|} r^{d_k} dr$$

$$\le \frac{C'_\alpha}{\alpha^{d_k}} \rho_k^{d_k+1} + \frac{2\, e^{A_k}}{d_k+1}|z|^{d_k+1},$$

and since $\rho_k \xrightarrow[k\to\infty]{} 0$, we have

$$\limsup_{k\to\infty} \sup_{z \in B(0,\alpha R_k)} |\vec{\Phi}_k(z) - \vec{\Phi}_k(0)| \le \frac{2\, e^{A}(\alpha R)^{d+1}}{d+1} \xrightarrow[\alpha\to 0]{} 0$$

since $d + 1 > 0$. Furthermore, since (M^m, h) is a compact manifold, its injectivity radius is strictly positive. Therefore, we deduce that there exists $\alpha_0 > 0$ such that for all $k \in \mathbb{N}$ large enough, $\vec{\Phi}_k(B(0,\alpha_0 R_k))$ is contained in a *fixed*—without loss of generality, we can assume that $\{\vec{\Phi}_k(0)\}_{k \in \mathbb{N}}$ converges to some point $p \in M^m$—domain V on which geodesic coordinates exist.

3.3 Refined Estimates on the Approximate Conservation Laws

We let as above $\{\vec{\Phi}_k\}_{k \in \mathbb{N}}$ be a sequence of smooth Willmore immersions satisfying the assumptions of the main Theorem A. In the next lemma, we generalise [42, Lemma A.1], by relaxing the L^∞ control to an $\mathrm{L}^{2,1}$ control.

Lemma 3.6. *There exists constants $\varepsilon_2(m), C_2(m) > 0$ with the following property. For all $j, k \in \{1, \cdots, m\}$, let $\gamma_j^k \in W^{1,2} \cap C^0(\mathbb{C})$ be such that $\operatorname{supp}(\gamma_j^k) \subset B(0,2)$ and $\left\|\gamma_j^k\right\|_{L^{2,1}(\mathbb{C})} \le \varepsilon_0$. For all $\vec{U} \in L^1_{\text{loc}}(\mathbb{C})$, define*

$$\left(\nabla_z \vec{U}\right)_j = \partial_z \vec{U}_j + \sum_{k=1}^m \gamma_j^k \vec{U}_k \qquad \text{in } \mathscr{D}'(\mathbb{C}), \text{ where } 1 \le j \le m.$$

Then for all $\vec{Y} \in \left(\dot{H}^{-1} + L^1\right)(\mathbb{C})$, there exists a unique $\vec{U} \in L^{2,\infty}(\mathbb{C})$ satisfying

$$\nabla_z \vec{U} = \vec{Y} \qquad \text{in } \mathscr{D}'(\mathbb{C})$$

Furthermore, we have the estimate

$$\left\|\vec{U}\right\|_{L^{2,\infty}(\mathbb{C})} \le C_2 \left\|\vec{Y}\right\|_{\dot{H}^{-1}+L^1(\mathbb{C})}.$$

Proof. As in [42], we use a fixed-point argument. For all $\vec{U} \in L^{2,\infty}(\mathbb{C})$, define

$$T(\vec{U}) = \left(-\frac{1}{\pi \bar{z}} * \left(\vec{Y}_j - \sum_{k=1}^m \gamma_j^k \vec{U}_k\right)\right)_{1 \le j \le m}.$$

By the Young inequality for weak L^p spaces (that follows from the classical Young inequality by interpolation) and the $L^{2,1}/L^{2,\infty}$ duality, we have for a universal $\Gamma_0 < \infty$

$$\left\|T(\vec{U}) + \frac{1}{\pi \bar{z}} * \vec{Y}\right\|_{L^{2,\infty}(\mathbb{C})} \le \sum_{j,k=1}^m \Gamma_0 \left\|\frac{1}{\pi \bar{z}}\right\|_{L^{2,\infty}(\mathbb{C})} \left\|\gamma_j^k \vec{U}_k\right\|_{L^1(\mathbb{C})} \le \frac{2\Gamma_0}{\sqrt{\pi}} \sum_{j,k=1}^m \left\|\gamma_j^k\right\|_{L^{2,1}(\mathbb{C})} \left\|\vec{U}_k\right\|_{L^{2,\infty}(\mathbb{C})}$$

$$\le \frac{2m\Gamma_0}{\sqrt{\pi}} \varepsilon_0 \sum_{k=1}^m \left\|\vec{U}_k\right\|_{L^{2,\infty}(\mathbb{C})} \le \frac{\Gamma_0 m^2}{\sqrt{\pi}} \varepsilon_0 \left\|\vec{U}\right\|_{L^{2,\infty}(\mathbb{C})}. \tag{3.31}$$

Choose

$$\varepsilon_0 = \frac{\sqrt{\pi}}{4m^2 \Gamma_0}.$$

Now, exactly as in [42], we get the estimate

$$\left\|\frac{1}{\pi \bar{z}} * \vec{Y}\right\|_{L^{2,\infty}(\mathbb{C})} \le \Gamma_1(m) \left\|\vec{Y}\right\|_{\dot{H}^{-1}+L^1(\mathbb{C})}, \tag{3.32}$$

where for all $u : \mathbb{R}^m \to \mathbb{R}^m$, we have

$$\|u\|_{\dot{H}^{-1}+L^1(\mathbb{R}^m)} = \inf\left\{\|u_1\|_{\dot{H}^{-1}(\mathbb{R}^m)} + \|u_2\|_{L^1(\mathbb{R}^m)} : u = u_1 + u_2\right\}.$$

Therefore, (3.31) and (3.32) implies that for all $\vec{U} \in L^{2,\infty}(\mathbb{C})$, we have

$$\left\|T(\vec{U})\right\|_{L^{2,\infty}(\mathbb{C})} \le \frac{1}{2} \left\|\vec{U}\right\|_{L^{2,\infty}(\mathbb{C})} + \Gamma_1(n) \left\|\vec{Y}\right\|_{\dot{H}^{-1}+L^1(\mathbb{C})}. \tag{3.33}$$

As in (3.31), for all $\vec{U}_1, \vec{U}_2 \in L^{2,\infty}(\mathbb{C})$, we have

$$\left\|T(\vec{U}_1) - T(\vec{U}_2)\right\|_{L^{2,\infty}(\mathbb{C})} \le \frac{1}{2} \left\|\vec{U}_1 - \vec{U}_2\right\|_{L^{2,\infty}(\mathbb{C})}. \tag{3.34}$$

Therefore, (3.33) and (3.34) prove that $T : L^{2,\infty}(\mathbb{C}) \to L^{2,\infty}(\mathbb{C})$ is a contraction, and therefore admits a fixed point by Banach contraction mapping Theorem. The estimate follows from (3.32).

$\square$

Now, let $f_k : \mathbb{C}^m \to \mathbb{C}^m$ be the linear map such that for all $X \in C^\infty(B(0, R_k), \mathbb{C}^m)$, we have

$$\nabla_z \vec{X} = \partial_z \vec{X} + f_k(\vec{X}) = \partial_z \vec{X} + \left(\sum_{l=1}^{m} \gamma_{j,k}^l \vec{X}_l \right)_{1 \leq j \leq m}, \tag{3.35}$$

where, denoting with $(\vec{\Phi}_k^1, \ldots, \vec{\Phi}_k^m)$ the components of $\vec{\Phi}_k$ in the local coordinates of M^m, we set:

$$\gamma_{j,k}^l = \sum_{q=1}^{m} \Gamma_{j,q}^l \partial_z \vec{\Phi}_k^q,$$

where $\Gamma_{j,q}^l$ are the Christoffel symbols of the ambient Riemannian manifold (M^m, h). We now fix some $\varepsilon_3(m) \leq \varepsilon_2(m)$ to be determined later. By the estimate (3.28), we deduce that there exists $\alpha_0 > 0$ and $N \in \mathbb{N}$ such that

$$\sup_{1 \leq j,l \leq m} \left\| \gamma_{j,k}^l \right\|_{\mathrm{L}^{2,1}(B(0,\alpha_0 R_k))} \leq \varepsilon_3(m) \leq \varepsilon_2(m), \quad \text{for all } k \geq N. \tag{3.36}$$

Recalling the $\mathrm{L}^{2,1}/\mathrm{L}^{2,\infty}$ duality, for all $X \in \mathrm{L}^{2,\infty}(B(0, R_k), \mathbb{C}^m)$ and for all $r \leq \alpha_0 R_k$ the following estimate holds:

$$\left\| f_k(\vec{X}) \right\|_{\mathrm{L}^1(B(0,r))} \leq \sum_{j,l=1}^{m} \left\| \gamma_j^l \right\|_{\mathrm{L}^{2,1}(B(0,r))} \left\| \vec{X}_l \right\|_{\mathrm{L}^{2,\infty}(B(0,r))} \leq \varepsilon_3(m) m^2 \left\| \vec{X} \right\|_{\mathrm{L}^{2,\infty}(B(0,r))}. \tag{3.37}$$

We also have the pointwise estimate

$$|f_k(\vec{X})(z)| \leq C_1(h) \, e^{\lambda_k(z)} |\vec{X}(z)|, \tag{3.38}$$

where $C_1(h) > 0$ only depends on h (the metric on M^m). Let $\vec{Y}_k : \Omega_k(\alpha_0) \to \mathbb{C}^m$ be defined by

$$\vec{Y}_k = i \left(\nabla_z \vec{H}_k - 3 \nabla_z^\perp \vec{H}_k - i \star_h \left(\nabla_z \vec{n}_k \wedge \vec{H}_k \right) \right). \tag{3.39}$$

Notice that $\vec{Y}_k$ is smooth, so in particular it is an element of $\mathrm{H}^{-1} + \mathrm{L}^1$. Thus, the extension $\vec{Y}_k$ to the whole $\mathbb{C}$ by setting $\vec{Y}_k \equiv 0$ on $\mathbb{C} \setminus \Omega_k(\alpha_0)$ is an element of $\dot{\mathrm{H}}^{-1} + \mathrm{L}^1(\mathbb{C})$. Making use of the $\mathrm{L}^{2,1}$ estimate (3.36), we are in position to apply Lemma 3.6 and deduce that there exists $\vec{L}_k \in \mathrm{L}^{2,\infty}(B(0, \alpha_0 R_k), \mathbb{C})$ such that

$$\begin{cases} \nabla_z \vec{L}_k = \vec{Y}_k & \text{in } B(0, \alpha_0 R_k) \\ \mathrm{Im}\,(\vec{L}_k) = 0 & \text{on } \partial B(0, \alpha_0 R_k). \end{cases} \tag{3.40}$$

Remark 3.7. The boundary condition for $\mathrm{Im}\,(\vec{L}_k)$ in (3.40) is obtained in the exact same way as in Lemma A.2 in [42] since $\vec{\Phi}_k$ is smooth (see Lemma 3.14 for more details). However, the $\mathrm{L}^{2,\infty}$ estimate obtained here by simply applying Lemma 3.6 will depend on k and for technical reasons we need to obtain a function $\vec{L}_k$ controlled in $\mathrm{L}^{2,\infty}$ independently of k. Indeed, without *a priori* estimates, since the boundary condition of $\mathrm{Re}\,(\vec{L}_k)$ cannot be prescribed, we would not be able to get a $\mathrm{L}^{2,\infty}$ control on $\mathrm{Re}\,(\vec{L}_k)$ (in [7], at a crucial step, the authors use the fact that the equation in $\nabla \vec{L}$ holds up to a constant, which allows to assume that some mean of $\vec{L}_k$ vanishes; however, if the Christoffel symbols do not vanish, the equation is not invariant by translation). This is due to the fact that in general, one cannot prescribe the full boundary condition in a $\overline{\partial}$ equation.

From the ε-regularity Theorem 4.1 we know that there exists a constant $C > 0$ independent of k and α_0 such that

$$e^{\lambda_k(z)} |\vec{Y}_k(z)| \leq C \frac{\delta_k(|z|)}{|z|} \leq \frac{C_0}{|z|^2}, \tag{3.41}$$

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where

$$\delta_k(r) = \left(\frac{1}{r^2} \int_{B_{2r} \setminus \overline{B}_{\frac{r}{2}}(0)} |\nabla \vec{n}_k|^2 dx \right)^{\frac{1}{2}}. \tag{3.42}$$

Indeed, since $\vec{H}_k$ satisfies the Euler-Lagrange equation

$$\mathscr{L}_{g_k} \vec{H}_k = \frac{1}{2} \nabla^*_{g_k} \left(\nabla_{g_k} \vec{H}_k - 3 \nabla^\perp_{g_k} \vec{H}_k + \star_h \left((*\nabla_{g_k} \vec{n}_k) \wedge \vec{H}_k \right) \right)$$
$$= -\mathscr{R}^\perp_1(\vec{H}_k) + 2 \widetilde{K}_h \vec{H}_k - 2 \mathscr{R}_2(d\vec{\Phi}_k) - (DR)(d\vec{\Phi}_k) - \mathscr{R}^\perp_3(\vec{H}_k),$$

which is uniformly elliptic in all dyadic annuli in $\Omega_k(1)$ thanks to the hypothesis of Theorem A and the Harnack inequality on the conformal parameter (Lemma 3.3, (3.10)), we deduce by standard elliptic regularity ([15], Theorem 3.9) that there exists a constant $\Gamma_0(n)$ such that

$$r \left\| \nabla \vec{H}_k \right\|_{\mathrm{L}^\infty(\partial B(0,r))} \leq \Gamma_0(n) \left(\left\| \vec{H}_k \right\|_{\mathrm{L}^\infty(B_{\frac{4}{3}r} \setminus \overline{B}_{\frac{3}{4}r}(0))} + r^2 \left\| \mathscr{L}_{g_k} \vec{H}_k \right\|_{\mathrm{L}^\infty(B_{\frac{4}{3}r} \setminus \overline{B}_{\frac{3}{4}r}(0))} \right). \tag{3.43}$$

With the ε-regularity Theorem 4.1, we deduce that

$$\left\| e^{\lambda_k} \vec{H}_k \right\|_{\mathrm{L}^\infty(B_{\frac{4}{3}r} \setminus \overline{B}_{\frac{3}{4}r}(0))} \leq \frac{\Gamma_1(n)}{r} \left\| e^{\lambda_k} \vec{H}_k \right\|_{\mathrm{L}^2(B_{2r} \setminus \overline{B}_{\frac{r}{2}}(0))}. \tag{3.44}$$

Now, since

$$e^{-A} |z|^{d_k} \leq e^{\lambda_k(z)} \leq e^A |z|^{d_k},$$

we deduce that for all $0 < \beta < 1$ and $r_k \leq \beta r < \beta^{-1} r < R_k$, we have for all $\beta r \leq |z| \leq \beta^{-1} r$

$$e^{-A} \alpha^{|d_k|} r^{d_k} \leq e^{\lambda_k(z)} \leq e^A \alpha^{-|d_k|} r^{d_k}.$$

And since $d_k \underset{k\to\infty}{\longrightarrow} d$, we deduce that there exists $B \in \mathbb{R}$ independent of k such that

$$\sup_{z \in B_{2r} \setminus \overline{B}_{\frac{r}{2}}(0)} e^{\lambda_k(z)} \leq e^B \inf_{z \in B_{2r} \setminus \overline{B}_{\frac{r}{2}}(0)} e^{\lambda_k(z)}, \quad \text{for all } 2r_k < r < \frac{R_k}{2}. \tag{3.45}$$

The combination of (3.44) and (3.45) gives

$$\left\| \vec{H}_k \right\|_{\mathrm{L}^\infty(B_{\frac{4}{3}r} \setminus \overline{B}_{\frac{3r}{4}}(0))} \leq e^B \frac{e^{-\lambda_k(r)}}{r} \Gamma_1(n) \left\| e^{\lambda_k} \vec{H}_k \right\|_{\mathrm{L}^2(B_{2r} \setminus \overline{B}_{\frac{r}{2}}(0))} \leq e^B e^{-\lambda_k(r)} \Gamma_1(n) \delta_k(r), \tag{3.46}$$

where $\lambda_k(r) = \lambda_k(r \cdot 1) = \lambda_k(r,0)$. Notice that since we use complex numbers for the argument of λ_k, the notation $\lambda_k(r)$ makes sense for all $r > 0$. Likewise, we have

$$\left\| \mathscr{L}_{g_k} \vec{H}_k \right\|_{\mathrm{L}^\infty(B_{\frac{4}{3}r} \setminus \overline{B}_{\frac{3}{4}r}(0))} \leq C(h) \left(1 + \left\| \vec{H}_k \right\|_{\mathrm{L}^\infty(B_{\frac{4}{3}r} \setminus \overline{B}_{\frac{3r}{4}}(0))} \right) \leq C(h) \left(1 + e^B e^{-\lambda_k(r)} \Gamma_1(n) \delta_k(r) \right). \tag{3.47}$$

Since $r\delta_k(r) \leq \sqrt{\Lambda(h)}$ (recall that $\Lambda(h)$ is defined in (3.6)), we finally deduce by (3.43), (3.46) and (3.47) that

$$\begin{cases} e^{\lambda_k(z)} |\vec{H}_k(z)| \leq |\nabla \vec{n}_k(z)| \leq C_4(n) \delta_k(|z|) & \text{for all } z \in \Omega_k\left(\frac{1}{2}\right) \\ e^{\lambda_k(z)} |\nabla \vec{H}_k(z)| \leq C_5(n,h) \dfrac{\delta_k(|z|)}{|z|} & \text{for all } z \in \Omega_k\left(\frac{1}{2}\right). \end{cases} \tag{3.48}$$

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Recalling that $\vec{Y}_k = i\left(\nabla_z \vec{H}_k - 3\nabla_{\bar{z}}^\perp \vec{H}_k - i\star_h \left(\nabla_z \vec{n}_k \wedge \vec{H}_k\right)\right)$, we deduce from (3.48), the Harnack inequality for λ_k, and the expansion (3.35) that

$$
\begin{aligned}
e^{\lambda_k(z)}|\vec{Y}_k(z)| &\le C\left(e^{\lambda_k(z)}|\nabla H_k(z)| + |\nabla \vec{n}_k|e^{\lambda_k(z)}|\vec{H}_k(z)| + e^{\lambda_k(z)}|\vec{H}_k(z)|\right.\\
&\quad \left.+ e^{\lambda_k(z)}|\nabla \vec{n}_k(z)|e^{\lambda_k(z)}|\vec{H}_k(z)|\right)\\
&\le C\left(C_5(n,h)\frac{\delta_k(|z|)}{|z|} + C_4(n)^2\delta_k(|z|)^2 + C_4(n)\delta_k(z) + C_4(n)e^A|z|^{d_k}\delta_k(|z|)\right)\\
&\le C'\frac{\delta_k(|z|)}{|z|}
\end{aligned}
$$

where we used $-1 < d_k$ for the last component, and the trivial estimate $\delta_k(|z|) \le \sqrt{\Lambda(h)}|z|^{-1}$.

For technical reasons, we will have to perform a disjunction of cases depending on the value of d_k. The analysis in the first case, $d_k \le 0$, will take several pages; the other cases will be discussed after (3.100). Notice that we consider the extension by 0 of $\vec{Y}_k|\Omega_k(\alpha_0)$ in the rest of this section. Let us state the main estimate that we will prove here.

Proposition 3.8. *Under the hypothesis of Theorem 3.2, there exists $C = C(m, h, \Lambda) > 0$, $\alpha_0 > 0$, and $\vec{V}_k : \mathbb{C} \to \mathbb{C}^m$ such that*

$$
\partial_z \vec{V}_k(z) = \vec{Y}_k \qquad \text{in } \Omega_k(\alpha_0)
$$

and satisfying the pointwise estimates

$$
\begin{cases}
e^{\lambda_k(z)}|\vec{V}_k(z)| \le \dfrac{C}{|z|}, & \text{for all } z \in \Omega_k(\alpha_0)\,,\\[2mm]
|z|^{d_k}|\vec{V}_k(z)| \le \dfrac{C}{|z|}, & \text{for all } z \in \Omega_k(\alpha_0)\,.
\end{cases}
\tag{3.49}
$$

Proof. **Analysis of Case 1: $d_k \le 0$**

If $d_k \le 0$, then we have *a fortiori*

$$
|\vec{Y}_k(z)| \le \frac{C_0}{|z|^2}.
\tag{3.50}
$$

First define $\vec{U}_k : \mathbb{C} \to \mathbb{C}^m$ by

$$
\vec{U}_k(z) = \frac{1}{\bar{z}^2}\left(-\frac{1}{\pi\bar{\zeta}} * \left(\bar{\zeta}^2 \vec{Y}_k(\zeta)\right)\right)(z) = -\frac{1}{\pi\bar{z}^2}\int_{\mathbb{C}}\frac{\bar{\zeta}^2\vec{Y}_k(\zeta)}{z-\zeta}|d\zeta|^2 = \frac{1}{2\pi i\bar{z}^2}\int_{\mathbb{C}}\frac{\bar{\zeta}^2\vec{Y}_k(\zeta)}{\zeta - z}d\bar{\zeta}\wedge d\zeta.
\tag{3.51}
$$

Lemma 3.9. *There exists constants $C_0, C_1 > 0$ and $a_{k,0} \in \mathbb{C}^m$, for $k \in \mathbb{N}$, with $\sup_{k\in\mathbb{N}}|a_{k,0}| < \infty$ such that*

$$
\left|\vec{U}_k(z) - \frac{a_{k,0}}{\bar{z}^2}\right| \le C_1\frac{\left|\log\left(\frac{R}{|z|}\right)\right|}{|z|} + \frac{8C_0}{|z|}, \quad \text{for all } z \in \mathbb{C}\,.
\tag{3.52}
$$

Proof. Since $\bar{z}^2$ is anti-holomorphic and $\vec{Y}_k$ is has compact support, we have that

$$
\partial_z \vec{U}_k(z) = \frac{1}{\bar{z}^2}\left(\delta_z * \left(\bar{\zeta}^2\vec{Y}_k(\zeta)\right)\right) = \frac{1}{\bar{z}^2}\cdot\bar{z}^2\vec{Y}_k(z) = \vec{Y}_k(z),
\tag{3.53}
$$

where δ_z is the Dirac mass in $z \in \mathbb{C}$. Write for simplicity $r = 2r_k$ and $R = \frac{R_k}{2}$.

Fix some $z \in B(0, R)$. First, if $0 < 2|z| \le r$, we have

$$
\vec{U}_k(z) = \frac{1}{2\pi i\bar{z}^2}\int_{B(0,R)\setminus\overline{B}(0,2|z|)}\frac{\bar{\zeta}^2\vec{Y}_k(\zeta)}{\zeta - z}d\bar{\zeta}\wedge d\zeta
\tag{3.54}
$$

and expanding, we get

$$\frac{1}{2\pi i}\int_{B(0,R)\setminus\overline{B}(0,2|z|)}\frac{\overline{\zeta}^2\vec{Y}_k(\zeta)}{\zeta - z}d\overline{\zeta}\wedge d\zeta = \sum_{l=0}^{\infty}\left(\frac{1}{\pi}\int_{B(0,R)\setminus\overline{B}(0,2|z|)}\frac{\overline{\zeta}^2\vec{Y}_k(\zeta)}{\overline{\zeta}^{l+1}}|d\zeta|^2\right)\overline{z}^l = \sum_{l=0}^{\infty}a_{k,l}\overline{z}^l\,. \quad (3.55)$$

For $l=0$, we have

$$\left|\int_{B(0,R)\setminus\overline{B}(0,2|z|)}\frac{\overline{\zeta}^2\vec{Y}_k(\zeta)}{\overline{\zeta}^{l+1}}|d\zeta|^2\right| \leq C_0\int_{B(0,R)}\frac{|d\zeta|^2}{|\zeta|} = 2\pi C_0 R, \quad (3.56)$$

and for $l=1$, we have

$$\left|\int_{B(0,R)\setminus\overline{B}(0,2|z|)}\frac{\overline{\zeta}^2\vec{Y}_k(\zeta)}{\overline{\zeta}^2}|d\zeta|^2\right| \leq 2\pi\int_{2|z|}^{R}\frac{dt}{t} = 2\pi\log\left(\frac{R}{2|z|}\right). \quad (3.57)$$

Therefore, by (3.54), (3.55), (3.56) and (3.57), we deduce that there exists a universal constant C_1 such that

$$\left|\overline{z}^2\vec{U}_k(z) - a_{k,0} - \sum_{l=2}^{\infty}a_{k,l}\overline{z}^l\right| \leq C_1|z|\log\left(\frac{R}{|z|}\right), \quad (3.58)$$

and $a_{k,0}$ is uniformly bounded for $k \geq N$.

Furthermore, we have for all $l \geq 2$

$$\left|\int_{B(0,R)\setminus\overline{B}(0,2|z|)}\frac{\overline{\zeta}^2\vec{Y}_k(\zeta)}{\zeta^{l+1}}|d\zeta|^2\right| \leq C_0\int_{B(0,R)\setminus\overline{B}(0,2|z|)}\frac{|d\zeta|^2}{|\zeta|^{l+1}} \leq 2\pi C_0\int_{2|z|}^{R}\frac{dt}{t^l} = \frac{2\pi C_0}{l-1}\left(\frac{1}{(2|z|)^{l-1}} - \frac{1}{R^{l-1}}\right), \quad (3.59)$$

which implies that

$$\left|\sum_{l=2}^{\infty}a_{k,l}\overline{z}^l\right| \leq 2C_0\sum_{l=2}^{\infty}\frac{1}{(2|z|)^{l-1}}\times|z|^l = 2C_0|z|. \quad (3.60)$$

Therefore, by (3.59) and (3.60) we get

$$\left|\vec{U}_k(z) - \frac{a_{k,0}}{\overline{z}^2}\right| \leq C_1\frac{\log\left(\frac{R}{|z|}\right)}{|z|} + \frac{2C_0}{|z|}. \quad (3.61)$$

Now, assume that $2|z| > r$. Then we have

$$\overline{z}^2\vec{U}_k(z) = \frac{1}{2\pi i}\int_{B(0,2|z|)\setminus\overline{B}(0,r)}\frac{\overline{\zeta}^2\vec{Y}_k(\zeta)}{\zeta - z}d\overline{\zeta}\wedge d\zeta + \frac{1}{2\pi i}\int_{B(0,R)\setminus\overline{B}(0,2|z|)}\frac{\overline{\zeta}^2\vec{Y}_k(\zeta)}{\zeta - z}d\overline{\zeta}\wedge d\zeta = \vec{u}_1(z) + \vec{u}_2(z). \quad (3.62)$$

Notice that this expansion holds, as by construction $\vec{Y}_k$ vanishes outside $B(0,R)\setminus\overline{B}(0,r)$. We first easily estimate

$$\left|\frac{1}{2\pi i}\int_{B(0,2|z|)\setminus\overline{B}(0,r)}\frac{\overline{\zeta}^2\vec{Y}_k(\zeta)}{\zeta - z}d\overline{\zeta}\wedge d\zeta\right| \leq \frac{C_0}{\pi}\int_{B(0,2|z|)}\frac{|d\zeta|^2}{|\zeta - z|} \leq \frac{C_0}{\pi}\int_{B(z,3|z|)}\frac{|d\zeta|^2}{|\zeta - z|} = 6C_0 R|z| \quad (3.63)$$

and the previous argument shows that (notice that this is the same constant $a_{k,0}$)

$$|\vec{u}_2(z) - a_{k,0}| \leq C_1|z|\left|\log\left(\frac{R}{|z|}\right)\right| + 2C_0|z|. \quad (3.64)$$

Finally, by (3.62), (3.63) and (3.64), we deduce that

$$\left|\vec{U}_k(z) - \frac{a_{k,0}}{\overline{z}^2}\right| \leq C_1\frac{\left|\log\left(\frac{R}{|z|}\right)\right|}{|z|} + \frac{8C_0}{|z|}, \quad (3.65)$$

which concludes the proof of the lemma. $\qquad\square$

Next, define $\vec{V}_k : \mathbb{C} \to \mathbb{C}^m$ by:

$$\vec{V}_k(z) = \vec{U}_k(z) - \frac{a_{k,0}}{\bar{z}^2}, \quad \text{for all } z \in \mathbb{C}. \tag{3.66}$$

Lemma 3.10. *Let $\vec{V}_k : \mathbb{C} \to \mathbb{C}^m$ be defined in (3.66). Then there exists $C_2 > 0$ such that*

$$\left| \vec{V}_k(z) \right| \leq \frac{C_2}{|z|} \left(\left| \log\left(\frac{R}{|z|} \right) \right| + 1 \right). \tag{3.67}$$

Moreover, $\nabla \mathrm{Im}\,(\vec{V}_k) \in \mathrm{L}^{2,\infty}(\mathbb{C})$ which shows in particular that $\mathrm{Im}\,(\vec{V}_k) \in \mathrm{W}^{1,(2,\infty)}(B(0, R_k))$. Furthermore, by the Sobolev embedding, it holds:

$$\mathrm{Im}\,(\vec{V}_k) \in \bigcap_{p < \infty} \mathrm{L}^p(B(0, R_k)).$$

Remark 3.11. The removal of the logarithmic term in the estimate (3.67) will be the main technical difficulty of the proof.

Proof. From Lemma 3.9, we get that $\partial_z \vec{V}_k = \vec{Y}_k$ on $\mathbb{C}$ and that there exists $C_2 > 0$ such that:

$$\left| \vec{V}_k(z) \right| \leq \frac{C_2}{|z|} \left(\left| \log\left(\frac{R}{|z|} \right) \right| + 1 \right), \quad \text{for all } z \in \mathbb{C}.$$

Furthermore, we have

$$\Delta \mathrm{Im}\,\left(\vec{V}_k \right) = 4\,\mathrm{Im}\,\left(\partial_{\bar{z}} \vec{Y}_k \right) = 4\,\mathrm{Im}\,\left(\nabla_{\bar{z}} \vec{Y}_k(z) \right) - 4\,\mathrm{Im}\,\left(\overline{f_k}(\vec{Y}_k) \right) \quad \text{in } \Omega_k(1/2),$$

whilst $\mathrm{Im}\,(\vec{V}_k)$ is harmonic in $B(0, \alpha_0^{-1} r_k)$. Recall that by [42, Lemma 3.2 and Theorem 3.1], the following identities hold:

$$\vec{Y}_k = i\left(\nabla_z \vec{H}_k - 3\,\nabla_z^\perp \vec{H}_k - i \star_h \left(\nabla_z \vec{n}_k \wedge \vec{H}_k \right) \right) = -2i\left(\nabla_z^\perp \vec{H}_k + \langle \vec{H}_k, \vec{H}_{k,0} \rangle \partial_{\bar{z}} \vec{\Phi}_k \right)$$

$$4e^{-2\lambda}\mathrm{Re}\,\left(\nabla_{\bar{z}}\left(\nabla_z^\perp \vec{H}_k + \langle \vec{H}_k, \vec{H}_{0,k} \rangle \partial_{\bar{z}} \vec{\Phi}_k \right) \right) = \Delta_g^\perp \vec{H}_k - 2|\vec{H}_k|^2 \vec{H} + \mathscr{A}(\vec{H}_k) + 8\,\mathrm{Re}\,\left(\langle R(\vec{e}_{\bar{z}}, \vec{e}_z) \vec{e}_z, \vec{H}_k \rangle \vec{e}_{\bar{z}} \right).$$

Therefore:

$$\mathrm{Im}\,(\nabla_{\bar{z}} \vec{Y}_k) = \mathrm{Im}\,\left(-2i \nabla_{\bar{z}}\left(\nabla_z^\perp \vec{H}_k + \langle \vec{H}_k, \vec{H}_{k,0} \rangle \partial_{\bar{z}} \vec{\Phi}_k \right) \right) = -2\,\mathrm{Re}\,\left(\nabla_{\bar{z}}\left(\nabla_z^\perp \vec{H}_k + \langle \vec{H}_k, \vec{H}_{k,0} \rangle \partial_{\bar{z}} \vec{\Phi}_k \right) \right)$$

$$= -\frac{1}{2} e^{2\lambda_k}\left(\Delta_{g_k}^\perp \vec{H}_k - 2|\vec{H}_k|^2 \vec{H} + \mathscr{A}(\vec{H}_k) \right) - 4e^{2\lambda_k}\,\mathrm{Re}\,\left(\langle R(\vec{e}_{\bar{z}}, \vec{e}_z) \vec{e}_z, \vec{H}_k \rangle \vec{e}_{\bar{z}} \right).$$

Using that $\vec{\Phi}_k$ is Willmore and using (1.1), we deduce that

$$\mathrm{Im}\,(\nabla_{\bar{z}} \vec{Y}_k) = \frac{1}{2} e^{2\lambda_k}\left(\mathscr{R}_1^\perp(\vec{H}_k) - 2\,\widetilde{K}_h \vec{H}_k + 2\,\mathscr{R}_2(d\vec{\Phi}_k) + (DR)(d\vec{\Phi}_k) - 8\,\mathrm{Re}\,\left(\langle R(\vec{e}_{\bar{z}}, \vec{e}_z) \vec{e}_z, \vec{H}_k \rangle \vec{e}_{\bar{z}} \right) \right). \tag{3.68}$$

Therefore, by (3.30) and Hölder's inequality, we get:

$$|\mathrm{Im}\,(\nabla_{\bar{z}} \vec{Y}_k)| \leq C e^{2\lambda_k(z)}\left(1 + |\vec{H}_k| \right) \in \bigcap_{p < \frac{1}{1-\varepsilon}} \mathrm{L}^p(B(0, R_k)), \tag{3.69}$$

where $\varepsilon > 0$ is such that $d > -1 + \varepsilon$. However, from (3.50) we have

$$\left| \overline{f_k}(\vec{Y}_k) \right| \leq C e^{\lambda_k} |\vec{Y}_k| \leq C' \frac{\delta_k(|z|)}{|z|} \leq \frac{C''}{|z|^2},$$

which does not suffice as one cannot obtain elliptic estimates from a $\mathrm{L}^{1,\infty}$ bound on the Laplacian. In order to circumvent this problem, we will argue differently. Recall that for any vector-field $\vec{X}$, we defined

$$\nabla_z \vec{X} = \partial_z \vec{X} + f_k(\vec{X}) = \partial_z \vec{X} + \left(\sum_{l=1}^m \gamma_{j,k}^l \vec{X}_l \right)_{1 \leq j \leq m},$$

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where

$$\gamma_{j,k}^l = \sum_{q=1}^{m} \Gamma_{j,q}^l \partial_z \vec{\Phi}_k^q.$$

Recalling that $\vec{Y}_k = i\left(\nabla_z \vec{H}_k - 3\nabla_z^\perp \vec{H}_k - i \star_h \left(\nabla_z \vec{n}_k \wedge \vec{H}_k\right)\right)$, we get

$$\vec{Y}_k = i\left(\partial_z \vec{H}_k - 3\pi_{\vec{n}_k}(\partial_z \vec{H}_k) - 2f_k(\vec{H}_k) - i \star_h (\partial_z \vec{n}_k \wedge \vec{H}_k) - i \star_h \left(G_k(\vec{n}_k) \wedge \vec{H}_k\right)\right)$$
$$= i\left(-2\,\partial_z \vec{H}_k + 3\,(\partial_z \pi_{\vec{n}_k})\,\vec{H}_k - 2f_k(\vec{H}_k) - i \star_h (\partial_z \vec{n}_k \wedge \vec{H}_k) - i \star_h \left(G_k(\vec{n}_k) \wedge \vec{H}_k\right)\right).$$

Since e^{λ_k}, and $e^{\lambda_k}\vec{H}_k$ and $|\nabla \vec{n}_k|$ are bounded in $\mathrm{L}^2(B(0,R_k))$ by hypothesis, we deduce that there exists a constant $C > 0$ such that:

$$\left|\overline{f_k}(\vec{Y}_k + 2i\,\partial_z \vec{H}_k)\right| \leq C\left(|\nabla \vec{n}_k|e^{\lambda_k}|\vec{H}_k| + e^{\lambda_k}|\vec{H}_k| + |\nabla \vec{n}_k|e^{\lambda_k}|\vec{H}_k| + e^{2\lambda_k}|\vec{H}_k|\right) \in \mathrm{L}^1(B(0,R_k)). \quad (3.70)$$

On the other hand, we have

$$\mathrm{Im}\left(\overline{f_k}(-2i\,\partial_z \vec{H}_k)\right)_j = \mathrm{Im}\left(-2i\sum_{q=1}^{m}\Gamma_{j,q}^l \partial_{\bar{z}}\vec{\Phi}_k^q \partial_z \vec{H}_{k,l}\right) = -2\sum_{q=1}^{m}\Gamma_{j,q}^l \mathrm{Re}\left(\partial_{\bar{z}}\vec{\Phi}_k^q \partial_z \vec{H}_{k,l}\right)$$

$$= -\frac{1}{2}\sum_{q=1}^{m}\Gamma_{j,q}^l \langle \nabla \vec{\Phi}_k^q, \nabla \vec{H}_{k,l}\rangle$$

$$= -\frac{1}{2}\,\mathrm{div}\left(\sum_{q=1}^{m}\Gamma_{j,q}^l \nabla\vec{\Phi}_k^q\,\vec{H}_{k,l}\right) + \frac{1}{2}\sum_{q=1}^{m}\langle\nabla\Gamma_{j,q}^l, \nabla\vec{\Phi}_k^q\rangle\vec{H}_{k,l} + \frac{1}{2}\sum_{q=1}^{m}\Gamma_{j,q}^l \Delta\vec{\Phi}_k^q\,\vec{H}_{k,l}$$

$$= -\frac{1}{2}\,\mathrm{div}\left(\sum_{q=1}^{m}\Gamma_{j,q}^l \nabla\vec{\Phi}_k^q\,\vec{H}_{k,l}\right) + \frac{1}{2}\sum_{q=1}^{m}\langle\nabla\Gamma_{j,q}^l, \nabla\vec{\Phi}_k^q\rangle\vec{H}_{k,l} + \sum_{q=1}^{m}\Gamma_{j,q}^l e^{2\lambda_k}\vec{H}_{k,q}\vec{H}_{k,l}.$$

Since $e^{\lambda_k}\vec{H}_k \in \mathrm{L}^2(B(0,R_k))$, we deduce that

$$\mathrm{Im}\left(\overline{f_k}(-2i\,\partial_z \vec{H}_k)\right) \in \mathrm{H}^{-1} + \mathrm{L}^1(B(0,R_k)). \quad (3.71)$$

Therefore, by (3.69), (3.70) and (3.71), we finally deduce that

$$\Delta\mathrm{Im}\left(\vec{V}_k\right) \in \mathrm{H}^{-1} + \mathrm{L}^1(\mathbb{C}).$$

Standard elliptic estimates imply that $\mathrm{Im}\left(\vec{V}_k\right) \in \mathrm{W}^{1,(2,\infty)}(\mathbb{C})$. Indeed, by the preceding estimates, we have a decomposition

$$\Delta\mathrm{Im}\left(\vec{V}_k\right) = \mathrm{div}(\vec{X}_k) + \vec{Z}_k,$$

where $\vec{X}_k \in \mathrm{L}^2(\mathbb{C})$ and $\vec{Z}_k \in \mathrm{L}^1(\mathbb{C})$. Therefore, making the decomposition $\mathrm{Im}\left(\vec{V}_k\right) = \vec{u}_k + \vec{v}_k$, where

$$\Delta\vec{u}_k = \mathrm{div}(\vec{X}_k) \quad \text{and} \quad \Delta\vec{v}_k = \vec{Z}_k,$$

we obtain:

$$\vec{v}_k(z) = \frac{1}{2\pi}\int_{\mathbb{C}} \Delta\vec{v}_k(\zeta)\log|z-\zeta|\,|d\zeta|^2 = \frac{1}{2\pi}\int_{\mathbb{C}} \vec{Z}_k(\zeta)\log|z-\zeta|\,|d\zeta|^2.$$

Clearly, it holds

$$\partial_z \vec{v}_k(z) = \frac{1}{4\pi}\int_{\mathbb{C}} \frac{\vec{Z}_k(\zeta)}{z-\zeta}|d\zeta|^2,$$

which, by Young inequality, yields:

$$\|\nabla \vec{v}_k\|_{L^{2,\infty}(\mathbb{C})} \leq C_Y \left\|\frac{1}{z}\right\|_{L^{2,\infty}(\mathbb{C})} \left\|\vec{Z}_k\right\|_{L^1(\mathbb{C})} \leq C.$$

Likewise, we have in the distributional sense

$$\partial_z \vec{u}_k(z) = \frac{1}{4\pi} \int_{\mathbb{C}} \frac{\operatorname{div}(\vec{X}_k)(\zeta)}{z - \zeta}|\zeta|^2.$$

Now, recall that

$$\mathscr{F}\left(z \mapsto \frac{1}{z}\right)(\xi) = -\frac{2\pi i}{\xi}.$$

Indeed, using Edwards' trick ([12, 5.15.8]; see also [58, VII, 7; 24]), namely that $\varepsilon^{-1}(1-|x|^{-\varepsilon}) \xrightarrow[\varepsilon \to 0]{} \log|x|$ in $\mathscr{S}'(\mathbb{R}^2)$ (the space of tempered distributions), it is easy to show that

$$\mathscr{F}\left(z \mapsto \log|z|\right)(\xi) = -2\pi \, \mathrm{f.p.}\frac{1}{|\xi|^2} - (2\pi)^2(\gamma - \log(2))\delta_0,$$

where f.p. stands for finite part of Hadamard. Therefore, we have

$$\mathscr{F}\left(z \mapsto \frac{1}{z}\right)(\xi) = 2\,\mathscr{F}\left(z \mapsto \partial_z\left(\log|z|\right)\right)(\xi) = i\,\overline{\xi}\,\mathscr{F}\left(z \mapsto \log|z|\right)(\xi) = -2\pi\, i\,\overline{\xi}\,\mathrm{f.p.}\frac{1}{|\xi|^2} = -\frac{2\pi\, i}{\xi}.$$

Therefore we have by the Plancherel identity

$$\int_{\mathbb{C}} |\nabla \vec{u}_k|^2 dx = 4 \int_{\mathbb{C}} |\partial_z \vec{u}_k|^2 \, |dz|^2 = \frac{1}{4\pi^2} \int_{\mathbb{C}} \left|\frac{1}{z} * \operatorname{div}(\vec{X}_k)\right|^2 |dz|^2 = \frac{1}{4\pi^2} \int_{\mathbb{C}} \frac{1}{|\xi|^2} \left|-i\,\xi_1 \vec{X}_{k,1} - i\,\xi_2 \vec{X}_{k,2}\right|^2 |d\xi|^2$$
$$\leq \frac{1}{2\pi^2} \int_{\mathbb{C}} |\vec{X}_k|^2 dx.$$

Finally, we deduce that $\nabla \mathrm{Im}\,(\vec{V}_k) \in L^{2,\infty}(\mathbb{C})$ which shows in particular that $\mathrm{Im}\,(\vec{V}_k) \in W^{1,(2,\infty)}(B(0,R_k))$. Furthermore, by the Sobolev embedding, we have

$$\mathrm{Im}\,(\vec{V}_k) \in \bigcap_{p<\infty} L^p(B(0,R_k)),$$

which concludes the proof of the lemma. $\qquad\square$

The next step will be to remove the $\log|z|$ term in the estimate (3.67).

3.4 Removal of the Logarithmic Singularity in the Estimate of $\vec{V}_k$

Removing the $\log|z|$ term in the inequality (3.67) is the most technical part of the proof of Theorem 3.2—notice that in the Euclidean setting [7, Lemma VII.1], no such difficulty arises for delicate averaging arguments and the ε-regularity allows one to directly obtain a $L^{2,\infty}$ control for $e^\lambda \vec{L}$. To this aim, we will introduce and use some (new) Lorentz-type functional space. Let us stress out that this difficulty is due to the curved ambient space and therefore was not present in the proof of the energy quantization for Willmore surfaces in Euclidean spaces [7].

Recall that here, we are still working under the assumption that $d_k \leq 0$. Coming back to the proof of Proposition 3.8, let us first make the decomposition

$$\mathrm{Im}\,(\vec{V}_k) = \vec{u}_k + \vec{v}_k + \vec{w}_k, \tag{3.72}$$

where

$$\begin{cases} \Delta \vec{u}_k = 4\,\mathrm{Im}\left(\nabla_{\bar{z}}\vec{Y}_k\right) & \text{in } B(0, R_k/2) \\ \vec{u}_k = 0 & \text{on } \partial B(0, R_k/2)\,, \end{cases}$$

while

$$\begin{cases} \Delta \vec{v}_k = -4\,\mathrm{Im}\left(\overline{f_k}(\vec{Y}_k)\right) & \text{in } B(0, R_k/2) \\ \vec{v}_k = 0 & \text{on } \partial B(0, R_k/2)\,, \end{cases}$$

and

$$\begin{cases} \Delta \vec{w}_k = 0 & \text{in } B(0, R_k/2) \\ \vec{w}_k = \mathrm{Im}\left(\vec{V}_k\right) & \text{on } \partial B(0, R_k/2). \end{cases} \tag{3.73}$$

By the bound (3.69), we have $\Delta \vec{u}_k \in \mathrm{L}^p(\mathbb{C})$ for all $p < \dfrac{1}{1-\varepsilon}$ which implies by Calderón-Zygmund estimates that $\vec{u}_k \in \mathrm{W}^{2,p}(B(0, R_k/2))$ for all $p < \dfrac{1}{1-\varepsilon}$. By standard elliptic regularity ([17], Chapter 3, 3.3), we also get $\vec{w}_k \in \mathrm{W}^{1,(2,\infty)}(B(0, R_k/2))$. Regarding $\vec{v}_k$, it holds

$$\partial_{\bar{z}}\left(z^2 \partial_z \vec{v}_k - \frac{1}{\pi}\int_{\mathbb{C}} \frac{\zeta^2 \mathrm{Im}\left(\overline{f_k}(\vec{Y}_k)(\zeta)\right)}{\zeta - z}|d\zeta|^2\right) = 0.$$

Furthermore, since $\mathrm{Im}\left(\overline{f_k}(\vec{Y}_k)\right) \in H^{-1} + L^1$, we deduce that $\vec{v}_k \in \mathrm{L}^{2,\infty}(\mathbb{C})$ (using the same proof as in the above estimate, or alternatively [17, Théorème (3.3.6)] for the L^1 estimate). Therefore, there exists a holomorphic function h_k such that

$$\partial_z \vec{v}_k = \frac{h_k(z)}{z^2} + \frac{1}{\pi z^2}\int_{\mathbb{C}} \frac{\zeta^2 \mathrm{Im}\left(\overline{f_k}(\vec{Y}_k)(\zeta)\right)}{\zeta - z}|d\zeta|^2.$$

By the estimates in the proof of Lemmas 3.9 and 3.10 (and the pointwise bounds on $\overline{f_k}(\vec{Y}_k)$ given in (3.70) and (3.71)), we deduce that there exists a constant $C > 0$ such that

$$\left|\frac{1}{\pi z^2}\int_{\mathbb{C}} \frac{\zeta^2 \mathrm{Im}\left(\overline{f_k}(\vec{Y}_k)(\zeta)\right)}{\zeta - z}|d\zeta|^2 - \frac{a_{0,k}}{z^2}\right| \leq C\frac{\left(\left|\log\left(\frac{R_k}{|z|}\right)\right| + 1\right)}{|z|}. \tag{3.74}$$

Taking the expansion $h_k(z) = b_{0,k} + O(|z|)$, from $\nabla \vec{v}_k \in \mathrm{L}^{2,\infty}(\mathbb{C})$ we deduce that

$$b_{0,k} = -a_{0,k}, \tag{3.75}$$

and that

$$\psi_k(z) = \frac{h_k(z) - b_{0,k}}{z^2} \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{C})\,, \quad \text{for all } p < 2. \tag{3.76}$$

More precisely, the estimate (3.74) shows that there exists a function $\psi_{1,k}$ and a constant $C_k > 0$ such that

$$|\psi_k(z) - \psi_{1,k}(z)| \in \mathrm{L}^{2,\infty}\left(B\left(0, \frac{R_k}{2}\right)\right) \tag{3.77}$$

and

$$|\psi_{1,k}(z)| \leq \frac{C_k}{|z|}\log\left(\frac{R_k}{|z|}\right). \tag{3.78}$$

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Therefore, we deduce that ψ_k admits the Laurent expansion

$$\psi_k(z) = \sum_{n \geq -1}^{\infty} a_{k,n} z^n. \tag{3.79}$$

Next, we will show that

$$\psi_k \in \mathrm{L}^{2,1}(B(0, \alpha R_k)). \tag{3.80}$$

The proof is quite involved and will make use of some (apparently new) Orlicz-type function spaces. For a more systematic discussion of generalised Lorentz spaces, the reader is referred to the Appendix 7.

First, write for simplicity $R = \dfrac{R_k}{2}$, and for all $\alpha > 0$, let $\varphi_\alpha : (0, R) \to \mathbb{R}$ be defined by

$$\varphi_\alpha(t) = \frac{t}{\log^\alpha\left(\frac{R}{t}\right)}.$$

Then we show in the appendix that such a function gives rise to an interesting family of Orlicz spaces whose properties are investigated in the appendix.

Estimating ψ_k

Recall that ψ_k was defined in (3.76). Thanks to Lemmas 7.7 and 7.8, recalling the estimates (3.77), (3.78) and (3.79), we deduce that $\psi_k \in \mathrm{L}^{2,\infty}_{\log}(B(0, \frac{R_k}{2}))$. Indeed, we first notice that since $\sqrt{t} \leq \Lambda_\alpha(t)$ for all $0 \leq \alpha \leq 1$ and $t \geq 0$, we have by Lemma 7.5 the continuous injection

$$\mathrm{L}^{2,\infty}(B(0, R)) = M(t \mapsto \sqrt{t}) \subset M(\Lambda_\alpha) = \mathrm{L}^{2,\infty}_{\log^\alpha}(B(0, R)). \tag{3.81}$$

Therefore, we have the decomposition $\psi_k(z) = \psi_{1,k}(z) + (\psi_k(z) - \psi_{1,k}(z)) = \psi_{1,k}(z) + \psi_{2,k}(z)$, where $\psi_{2,k} \in L^{2,\infty}(B(0, \frac{R_k}{2}))$, and $\psi_{1,k} \in \mathrm{L}^{2,\infty}_{\log}(B(0, \frac{R_k}{2}))$. By the continuous embedding $\mathrm{L}^{2,\infty}(B(0, \frac{R_k}{2})) \hookrightarrow \mathrm{L}^{2,\infty}_{\log}(B(0, \frac{R_k}{2}))$, we deduce that $\psi_k \in \mathrm{L}^{2,\infty}_{\log}(B(0, \frac{R_k}{2}))$. Since ψ_k is meromorphic with a pole of order at most 1, we can apply the previous Lemmas 7.7 and 7.8 to deduce the following estimate

$$\psi_k(z) - \frac{a_{-1,k}}{z} \in \mathrm{W}^{1,1} \cap \mathrm{L}^{2,1}\left(B\left(0, \frac{R_k}{4}\right)\right). \tag{3.82}$$

Notice that $a_{-1,k}$ is bounded for $k \geq N$. Indeed, $\Psi_k(z) = z\psi_k(z)$ is holomorphic, so the mean-value formula and (7.10) applied to $r = \frac{R_k}{2}$, yield that there exists $\rho \in [\frac{R}{2}, R]$ such that

$$|a_{-1,k}| = |\Psi_k(0)| = \frac{1}{2\pi\rho} \left| \int_{\partial B_\rho(0)} \Psi_k(z) dz \right| \leq \frac{1}{2\pi} \int_{\partial B_\rho(0)} |\psi_k| d\mathcal{H}^1$$

$$\leq \frac{1}{2\pi} \frac{2\sqrt{3\pi}}{\log(2)} \left(1 + \log\left(\frac{4}{\sqrt{3}}\right)\right) \|\psi_k\|_{\mathrm{L}^{2,\infty}_{\log}(B(0, \frac{R_k}{2}))}.$$

A Pointwise Estimate on $\vec{V}_k$

Using variants of Lemmas 7.7 and 7.8 for harmonic functions (see [29] and [37, Lemma 2.2, Lemma 2.3] for more details), we also deduce that $\vec{w}_k \in \mathrm{W}^{1,1} \cap \mathrm{L}^{2,1}(B(0, R_k/4))$ (recall that $\vec{w}_k$ is defined in (3.73)). Recalling the decomposition (3.72), we obtain the following decomposition on $B(0, R_k/4)$

$$\nabla \mathrm{Im}\left(\vec{V}_k\right) = h_{1,k}(z) + h_{2,k}(z), \tag{3.83}$$

where

$$|h_{1,k}(z)| \leq C \frac{\log\left(\frac{R_k}{|z|}\right)}{|z|}$$

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and $h_{2,k} \in \mathrm{W}^{1,1} \cap \mathrm{L}^{2,1}(B(0, R_k/4))$. More precisely, there exists a constant $C_0 > 0$ independent of k such that for all $k \geq N$,

$$\|h_{2,k}\|_{\mathrm{L}^{2,1}(B(0,R_k/4))} + \|\nabla h_{2,k}\|_{\mathrm{L}^1(B(0,R_k/4))} \leq C_0.$$

Now, identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can rewrite the equation for $\partial_z \vec{V}_k$ as

$$\begin{cases} \partial_x \mathrm{Re}\left(\vec{V}_k\right) + \partial_y \mathrm{Im}\left(\vec{V}_k\right) = 2\,\mathrm{Re}\left(\vec{Y}_k\right) \\ -\partial_y \mathrm{Re}\left(\vec{V}_k\right) + \partial_x \mathrm{Im}\left(\vec{V}_k\right) = 2\,\mathrm{Im}\left(\vec{Y}_k\right)\,, \end{cases} \tag{3.84}$$

or, equivalently,

$$\nabla \mathrm{Re}\left(\vec{V}_k\right) - i\,\nabla^\perp \mathrm{Im}\left(\vec{V}_k\right) = 2\,\overline{\vec{Y}_k}.$$

Therefore, we have

$$\begin{cases} \partial_x \mathrm{Re}\left(\vec{V}_k\right) = \partial_y \vec{H}_k - 3\sum_{j=1}^{n-2}\langle \partial_y \vec{H}_k, \vec{n}_{k,j}\rangle \vec{n}_{k,j} + \star_h\left(\partial_y \vec{n}_k \wedge \vec{H}_k\right) - \partial_y \mathrm{Im}\left(\vec{V}_k\right) \\ \partial_y \mathrm{Re}\left(\vec{V}_k\right) = -\partial_x \vec{H}_k + 3\sum_{j=1}^{n-2}\langle \partial_x \vec{H}_k, \vec{n}_{k,j}\rangle \vec{n}_{k,j} - \star_h\left(\partial_x \vec{n}_k \wedge \vec{H}_k\right) + \partial_x \mathrm{Im}\left(\vec{V}_k\right). \end{cases} \tag{3.85}$$

The identity (3.85) implies that

$$\partial_r \mathrm{Re}\left(\vec{V}_k\right) = \cos(\theta)\partial_x \mathrm{Re}\left(\vec{V}_k\right) + \sin(\theta)\partial_y \mathrm{Re}\left(\vec{V}_k\right) = \left(\cos(\theta)\partial_y \vec{H}_k - \sin(\theta)\partial_x \vec{H}_k\right)$$

$$- 3\sum_{j=1}^{n-2}\langle \cos(\theta)\partial_y \vec{H}_k - \sin(\theta)\partial_x \vec{H}_k, \vec{n}_{k,j}\rangle \vec{n}_{k,j} + \star_h\left((\cos(\theta)\partial_y \vec{n}_k - \sin(\theta)\partial_x \vec{n}_k) \wedge \vec{H}_k\right)$$

$$- \cos(\theta)\partial_y \mathrm{Im}\left(\vec{V}_k\right) + \sin(\theta)\mathrm{Re}\left(\vec{V}_k\right)$$

$$= \frac{1}{r}\partial_\theta \vec{H}_k - 3\pi_{\vec{n}_k}\left(\frac{1}{r}\partial_\theta \vec{H}_k\right) + \star_h\left(\frac{1}{r}\partial_\theta \vec{n}_k \wedge \vec{H}_k\right) - \frac{1}{r}\partial_\theta \mathrm{Im}\left(\vec{V}_k\right), \tag{3.86}$$

since for any smooth function $u : B(0,1) \to \mathbb{R}^n$, and $a, b \in \mathbb{C}$ we have

$$\frac{1}{r}\partial_\theta u = -\sin(\theta)\partial_x u + \cos(\theta)\partial_y u\,.$$

Now, recall the notation

$$\mathrm{Re}\left(\vec{V}_k\right)_\rho = \fint_{\partial B(0,\rho)} \mathrm{Re}\left(\vec{V}_k\right) d\mathscr{H}^1 = \frac{1}{2\pi\rho}\int_{\partial B(0,\rho)} \mathrm{Re}\left(\vec{V}_k\right) d\mathscr{H}^1, \quad \text{for all } \alpha_0^{-1} r_k \leq \rho \leq \alpha_0 R_k.$$

Since $\pi_{\vec{n}_k}(\vec{H}_k) = \vec{H}_k$, we have

$$\pi_{\vec{n}_k}\left(\partial_\theta \vec{H}_k\right) = \partial_\theta\left(\pi_{\vec{n}_k}(\vec{H}_k)\right) - (\partial_\theta \pi_{\vec{n}_k})(\vec{H}_k) = \partial_\theta \vec{H}_k - \left(\partial_\theta(\pi_{\vec{n}_k})(\vec{H}_k)\right). \tag{3.87}$$

Therefore, by (3.85), (3.86) and (3.87), we deduce that

$$\frac{d}{d\rho}\mathrm{Re}\left(\vec{V}_k\right)_\rho = \frac{1}{2\pi}\int_0^{2\pi} \partial_\rho \vec{V}_k(\rho,\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{3}{\rho}(\partial_\theta \pi_{\vec{n}_k})\left(\vec{H}_k\right) + \star_h\left(\frac{1}{\rho}\partial_\theta \vec{n}_k \wedge \vec{H}_k\right)\right)d\theta. \tag{3.88}$$

Arguing as in the proof of [7, Lemma VI.1] using the ε-regularity Theorem 4.1 (adapted from Rivière's original result [53]), we deduce that there exists $\varepsilon_4(n), C_4(n) > 0$ such that the following holds: if

$$\sup_{r_k < s < R_k}\int_{B_{2s}\setminus \overline{B}_s(0)} |\nabla \vec{n}_k|^2 dx \leq \varepsilon_4(n)$$

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then

$$|\nabla \vec{n}_k(z)|^2 \le C_4^2(n) \frac{1}{|z|^2} \int_{B_{2|z|} \setminus \overline{B}_{\frac{|z|}{2}}} |\nabla \vec{n}_k|^2 dx \le \frac{C_4^2(n) \varepsilon_3(n)}{|z|^2}, \quad \text{for all } 2r_k \le |z| \le \frac{R_k}{2}. \tag{3.89}$$

Recall from (3.42) that the function $\delta_k : (0, R_k) \to \mathbb{R}$ is defined as

$$\delta_k(\rho) = \left(\frac{1}{\rho^2} \int_{B_{2\rho} \setminus \overline{B}_{\frac{\rho}{2}}(0)} |\nabla \vec{n}_k|^2 dx \right)^{\frac{1}{2}} = \frac{\sqrt{15\pi}}{2} \left(\fint_{B_{2\rho} \setminus \overline{B}_{\frac{\rho}{2}}(0)} |\nabla \vec{n}_k|^2 dx \right)^{\frac{1}{2}}.$$

We have in particular

$$\rho \, \delta_k(\rho) \le \left(\int_{\Omega_k(1)} |\nabla \vec{n}_k|^2 dx \right)^{\frac{1}{2}} \le \sqrt{\Lambda(h)}. \tag{3.90}$$

Using Fubini's theorem, we deduce that, for all $0 < r < R < \frac{R_k}{2}$, it holds

$$\int_r^R \delta_k^2(\rho) \, \rho \, d\rho = \int_r^R \left(\int_{B_{2R} \setminus \overline{B}_{\frac{r}{2}}(0)} |\nabla \vec{n}_k|^2 \mathbf{1}_{\left\{ \frac{\rho}{2} < |x| < 2\rho \right\}} dx \right) \frac{d\rho}{\rho} = \int_{B_{2R} \setminus \overline{B}_{\frac{r}{2}}(0)} |\nabla \vec{n}_k|^2 \left(\int_{\frac{|x|}{2}}^{2|x|} \frac{d\rho}{\rho} \right) dx$$

$$= \log(4) \int_{B_{2R} \setminus \overline{B}_{\frac{r}{2}}(0)} |\nabla \vec{n}_k|^2 dx. \tag{3.91}$$

This implies by the Cauchy-Schwarz inequality that

$$\int_r^R \delta_k(\rho) \, \rho \, d\rho \le \left(\int_r^R \rho \, d\rho \right)^{\frac{1}{2}} \left(\int_r^R \delta_k^2(\rho) \, \rho \, d\rho \right)^{\frac{1}{2}} \le R_k \left(\int_{B_R \setminus \overline{B}_{\frac{r}{2}}(0)} |\nabla \vec{n}_k|^2 dx \right)^{\frac{1}{2}}. \tag{3.92}$$

Therefore, injecting the first inequality of (3.89) into (3.88) yields

$$\left| \frac{d}{d\rho} \text{Re} \, (\vec{V}_k)_\rho \right| \le C e^{-\lambda_k(\rho)} \delta_k^2(\rho).$$

This estimate implies that

$$\int_{2r_k}^{\frac{R_k}{2}} e^{\lambda_k(\rho)} \left| \frac{d}{d\rho} \text{Re} \, (\vec{V}_k)_\rho \right| \rho \, d\rho \le C \int_{\Omega_k(1)} |\nabla \vec{n}_k|^2 dx. \tag{3.93}$$

Define the function

$$a : (0, R_k/2) \to \mathbb{R}_+, \quad a(t) = \left| \text{Re} \left(\vec{V}_k \right)_t \right|.$$

As $|a'(t)| = \left| \frac{d}{dt} \text{Re} \left(\vec{V}_k \right)_t \right|$, the combination of (3.25) and (3.93) yields

$$\int_r^R t^{1+d_k} |a'(t)| dt \le C.$$

Since $d_k \xrightarrow[k \to \infty]{} d > -1$, from the identity

$$\int_r^R t^{1+d_k} a'(t) dt = R^{1+d_k} a(R) - r^{1+d_k} a(r) - (1 + d_k) \int_r^R t^{d_k} a(t) dt, \tag{3.94}$$

we deduce as in [7, p. 120] that

$$r^{1+d_k} a(r) \le \left(\frac{R_k}{2} \right)^{1+d_k} a \left(\frac{R_k}{2} \right) + C. \tag{3.95}$$

The estimate

$$|V_k(z)| \leq C_2 \frac{\log\left(\frac{R_k}{|z|}\right)}{|z|}$$

implies (independently of the estimate (3.95) which is tautological for the value $r = R_k/2$) that for all $|z| = \frac{R_k}{2}$, we have

$$|V_k(z)| \leq \frac{2C_2 \log(2)}{R_k}.$$

Taking the average on $\partial B(0, R_k/2)$, we deduce that

$$\left(\frac{R_k}{2}\right)^{1+d_k} a\left(\frac{R_k}{2}\right) \leq 2C_2 \log(2) R_k^{d_k}$$

which is bounded independently of $k \geq N$ since $d_k \xrightarrow[k\to\infty]{} d > -1$ and $R_k \xrightarrow[k\to\infty]{} R > 0$. Finally, we deduce that for all $2r_k < |z| < \frac{R_k}{2}$, it holds

$$|z|^{d_k} \left|\operatorname{Re}(\vec{V}_k)_{|z|}\right| \leq \frac{C}{|z|}. \tag{3.96}$$

Lemma 3.12. *For all $r > 0$ and $u \in \mathrm{W}^{1,1}(\partial B(0,r), \mathbb{R}^n)$, it holds*

$$\|u - u_r\|_{\mathrm{L}^\infty(\partial B(0,r))} \leq n \int_{\partial B(0,r)} |\nabla u| d\mathscr{H}^1. \tag{3.97}$$

Proof. By scaling invariance, we can assume that $r = 1$, which permits to see u as a 2π-periodic function $u = (u_1, \cdots, u_n) : [0, 2\pi] \to \mathbb{R}^n$. By the intermediate values theorem, for all $1 \leq i \leq n$, there exists $a_i \in [0, 2\pi]$ such that

$$u_i(a_i) = \fint_0^{2\pi} u_i \, d\mathscr{L}^1 = \frac{1}{2\pi} \int_0^{2\pi} u_i(\theta) d\theta.$$

Therefore, we have for all $1 \leq i \leq n$ and for all $\theta \in [0, 2\pi]$

$$\left|u_i(\theta) - \fint_0^{2\pi} u_i \, d\mathscr{L}^1\right| = |u_i(\theta) - u_i(a_i)| = \left|\int_{a_i}^\theta u_i'(t) dt\right| \leq \int_0^{2\pi} |u_i'(t)| dt,$$

Therefore, by the triangle inequality, we have

$$\left|u(\theta) - \fint_0^{2\pi} u \, d\mathscr{L}^1\right| \leq \sum_{i=1}^n \left|u_i(\theta) - \fint_0^{2\pi} u_i d\mathscr{L}^1\right| \leq \sum_{i=1}^n \int_0^{2\pi} |u_i'(t)| dt \leq n \int_0^{2\pi} |u'(t)| dt, \quad \text{for all } \theta \in [0, 2\pi].$$

Coming back to the initial inequality, we deduce that

$$\|u - u_r\|_{\mathrm{L}^\infty(\partial B(0,r))} \leq n \int_{\partial B(0,r)} \left|\frac{1}{r}\partial_\theta u\right| d\mathscr{H}^1 \leq n \int_{\partial B(0,r)} |\nabla u| d\mathscr{H}^1$$

which concludes the proof of the lemma. $\qquad\square$

Using the estimates $|\vec{Y}_k| \leq \frac{Ce^{-\lambda_k(z)}}{|z|^2}$, $|\nabla \operatorname{Im}(\vec{V}_k)| \leq \frac{C}{|z|}$ and (3.84), we deduce that for all $z \in \Omega_k(\frac{1}{4})$, we have

$$\left|\operatorname{Re}\left(\vec{V}_k(z)\right) - \operatorname{Re}\left(\vec{V}_k\right)_{|z|}\right| \leq n \int_{\partial B(0,|z|)} |\nabla \operatorname{Re}(\vec{V}_k)| d\mathscr{H}^1 = n \int_{\partial B(0,|z|)} \left|2\overline{\vec{Y}_k} + \nabla^\perp \operatorname{Im}(\vec{V}_k)\right| d\mathscr{H}^1$$

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$$\leq n \int_{\partial B(0,|z|)} \left(2|\vec{Y}_k| + |\nabla \mathrm{Im}\,(\vec{V}_k)| \right) d\mathscr{H}^1 \leq \frac{4\pi nC}{|z|^{1+d_k}} + 2\pi nC_5 \log\left(\frac{R_k}{|z|}\right)$$

$$+ n \int_{\partial B(0,|z|)} |h_2| d\mathscr{H}^1,$$

where h_2 is bounded in $\mathrm{W}^{1,1} \cap \mathrm{L}^{2,1}(B(0, R_k/4))$. In particular, by trace theory, we have $h_2 \in \mathrm{L}^1(\partial B(0,|z|))$ and for some universal constant independent of $|z|$, we have by the Cauchy-Schwarz inequality

$$\|h_2\|_{\mathrm{L}^1(\partial B(0,|z|))} \leq \Gamma_2 \left(\frac{1}{R_k} \|h_2\|_{\mathrm{L}^1(B(0,R_k/4))} + \|\nabla h_2\|_{\mathrm{L}^1(B(0,R_k/4))} \right)$$

$$\leq \Gamma_2 \left(\frac{\sqrt{\pi}}{4} \|h_2\|_{\mathrm{L}^2(B(0,R_k/4))} + \|\nabla h_2\|_{\mathrm{L}^1(B(0,R_k/4))} \right).$$

Therefore, we obtain the estimate

$$\left| \mathrm{Re}\,(\vec{V}_k(z)) - \mathrm{Re}\,\left(\vec{V}_k\right)_{|z|} \right| \leq \frac{4\pi nC}{|z|^{1+d_k}} + 2\pi nC_5 \log\left(\frac{R_k}{|z|}\right) + C,$$

where we recall that for all smooth $\varphi : \mathbb{R}^2 \to \mathbb{R}$, and $0 < r < \infty$, we have $\varphi_r = \fint_{\partial B(0,r)} \varphi \, d\mathscr{H}^1$ is the average of φ on the circle of radius r and centre 0. Finally, using that $d_k > -1 + \varepsilon$ and recalling the Harnack inequality (3.25), we deduce that

$$e^{\lambda_k(z)}|\mathrm{Re}\,(\vec{V}_k)(z)| \leq \frac{C}{|z|} + 2\pi nC_5 |z|^{-1+\varepsilon} \log\left(\frac{R_k}{|z|}\right) + C|z|^{-1+\varepsilon} \leq \frac{C}{|z|}. \tag{3.98}$$

The previous estimate on $\mathrm{Im}\,(\vec{V}_k)$ (see (3.83)) coupled with the same argument on averages implies that

$$e^{\lambda_k(z)}|\vec{V}_k(z)| \leq \frac{C}{|z|}, \quad \text{for all } z \in \Omega_k(1/4), \tag{3.99}$$

which, recalling the Harnack inequality (3.25) for the conformal parameters, in turn gives,

$$|z|^{d_k}|\vec{V}_k(z)| \leq \frac{C}{|z|}, \quad \text{for all } z \in \Omega_k(1/4). \tag{3.100}$$

Therefore, the result is finally proven for $d_k \leq 0$. Indeed, it represents (3.67) without the logarithm term, which is the intended result. We now move to the analysis of the remaining cases.

Analysis of the Other Cases

Let us recall for the sake of clarity (3.41)

$$e^{\lambda_k(z)}|\vec{Y}_k(z)| \leq C \frac{\delta_k(|z|)}{|z|} \leq \frac{C_0}{|z|^2}. \tag{3.101}$$

Now, assume that $d_k > 0$. We will distinguish the case $d_k \underset{k\to\infty}{\longrightarrow} 0$ and $d_k \underset{k\to\infty}{\longrightarrow} d > 0$.

Case 2: $d_k \underset{k\to\infty}{\longrightarrow} 0$. Define as above

$$\vec{U}_k(z) = \frac{1}{\bar{z}^2} \left(-\frac{1}{\pi\bar{\zeta}} \star \left(\bar{\zeta}^2 \vec{Y}_k(\zeta) \right) \right) = -\frac{1}{\pi\bar{z}^2} \int_{\mathbb{C}} \frac{\bar{\zeta}^2 \vec{Y}_k(\zeta)}{z - \zeta} |d\zeta|^2.$$

Notice that $\partial_z \vec{U}_k = \vec{Y}_k$. Fix some $z \in B(0, R)$, where we recall that $r = 2r_k$ and $R = \frac{R_k}{2}$. For $0 < 2|z| \leq r$, it holds

$$\vec{U}_k(z) = -\frac{1}{\pi\bar{z}^2} \int_{B(0,R)\setminus\overline{B}(0,2|z|)} \frac{\bar{\zeta}^2 \vec{Y}_k(\zeta)}{\zeta - z} |d\zeta|^2$$

$$= \frac{1}{\bar{z}^2} \sum_{l=0}^{\infty} \left(\frac{1}{\pi} \int_{B(0,R) \setminus \overline{B}(0,2|z|)} \frac{\bar{\zeta}^2 \vec{Y}_k(\zeta)}{\bar{\zeta}^{l+1}} |d\zeta|^2 \right) \bar{z}^l = \sum_{l=0}^{\infty} a_{k,l} \bar{z}^{l-2}.$$

Likewise, we have

$$|a_{k,0}| \leq 2\pi C_0 R.$$

Then, we estimate for $l = 1$

$$\left| \int_{B(0,R) \setminus \overline{B}(0,2|z|)} \frac{\bar{\zeta} \vec{Y}_k(\zeta)}{\bar{\zeta}^{l+1}} |d\zeta|^2 \right| \leq C_0 \int_{B(0,R) \setminus \overline{B}(0,2|z|)} \frac{|d\zeta|^2}{|\zeta|^{2+d_k}} = \frac{2\pi C_0}{d_k} \left(\frac{1}{(2|z|)^{d_k}} - \frac{1}{R^{d_k}} \right).$$

For $l \geq 2$, we have

$$\left| \int_{B(0,R) \setminus \overline{B}(0,2|z|)} \frac{\bar{\zeta}^2 \vec{Y}_k(\zeta)}{\bar{\zeta}^{l+1}} |d\zeta|^2 \right| \leq C_0 \int_{B(0,R) \setminus \overline{B}(0,2|z|)} \frac{|d\zeta|^2}{|\zeta|^{l+1+d_k}} = \frac{2\pi C_0}{l-1+d_k} \left(\frac{1}{(2|z|)^{l-1}} - \frac{1}{R^{l-1+d_k}} \right).$$

Therefore, it holds

$$\left| \sum_{l=2}^{\infty} a_{k,l} \bar{z}^l \right| \leq 2C_0 |z|.$$

We deduce that

$$\left| \vec{U}_k(z) - \frac{a_{k,0}}{\bar{z}^2} \right| \leq \frac{2C_0}{|z|} \frac{1}{d_k} \left(\frac{1}{(2|z|)^{d_k}} - \frac{1}{R^{d_k}} \right) + \frac{2C_0}{|z|}.$$

Recalling the definition (3.66) of $\vec{V}_k(z) = \vec{U}_k(z) - \frac{a_{k,0}}{\bar{z}^2}$, we deduce that for all $|z| \leq \frac{r}{2} \leq \frac{R}{4}$, it holds

$$|\vec{V}_k(z)| \leq \frac{2C_0 R^{-d_k}}{|z|} \frac{1}{d_k} \left(\left(\frac{R}{2|z|} \right)^{d_k} - 1 \right) + \frac{2C_0}{|z|} \leq \frac{4C_0}{|z|} \left(1 + \frac{1}{d_k} \left(\left(\frac{R}{2|z|} \right)^{d_k} - 1 \right) \right) \qquad (3.102)$$

for k large enough. Notice that for all $z \neq 0$ we have

$$\frac{1}{d_k} \left(\left(\frac{R}{2|z|} \right)^{d_k} - 1 \right) = \frac{1}{d_k} \left(e^{d_k \log\left(\frac{R}{2|z|} \right)} - 1 \right) = \log\left(\frac{R}{2|z|} \right) + O(d_k) \xrightarrow[k \to \infty]{} \log\left(\frac{R}{|z|} \right),$$

which suggests in the light of the previous discussion that the function above in the right-hand side of (3.102) belongs to an Orlicz space. Although one can effectively prove such an estimate by introducing a sequence of Orlicz spaces "converging" towards $\mathrm{L}_{\log}^{2,\infty}(B(0,R))$ and generalise Lemma 7.7, we will not need this fact.

Assume that $2|z| > r$. It holds

$$\bar{z}^2 \vec{U}_k(z) = \frac{1}{2\pi i} \int_{B(0,2|z|) \setminus \overline{B}(0,r)} \frac{\bar{\zeta}^2 \vec{Y}_k(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta + \frac{1}{2\pi i} \int_{B(0,R) \setminus \overline{B}(0,2|z|)} \frac{\bar{\zeta}^2 \vec{Y}_k(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta.$$

We first estimate directly

$$\left| \frac{1}{2\pi i} \int_{B(0,2|z|) \setminus \overline{B}(0,r)} \frac{\bar{\zeta}^2 \vec{Y}_k(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta \right| \leq \frac{C_0}{\pi} \int_{B(0,2|z|)} \frac{|d\zeta|^2}{|\zeta|^{d_k} |z - \zeta|}$$

$$= \frac{C_0}{\pi} \int_{B(0,\frac{|z|}{2})} \frac{|d\zeta|^2}{|\zeta|^{d_k} |z - \zeta|} + \frac{C_0}{\pi} \int_{B(0,2|z|) \setminus \overline{B}(0,\frac{|z|}{2})} \frac{|d\zeta|^2}{|\zeta|^{d_k} |z - \zeta|}.$$
(3.103)

For all $\zeta \in B(0, \frac{|z|}{2})$, we have by the triangle inequality $|z - \zeta| \geq |z| - |\zeta| \geq \frac{|z|}{2}$, which implies that

$$\int_{B(0,\frac{|z|}{2})} \frac{|d\zeta|^2}{|\zeta|^{d_k}|z-\zeta|} \leq \frac{2}{|z|} \int_{B(0,\frac{|z|}{2})} \frac{|d\zeta|^2}{|z|^{d_k}} = \frac{4\pi}{|z|} \int_0^{\frac{|z|}{2}} r^{1-d_k} dr = \frac{4\pi}{2-d_k}|z|^{1-d_k}, \tag{3.104}$$

whilst

$$\int_{B(0,2|z|)\setminus\overline{B}(0,\frac{|z|}{2})} \frac{|d\zeta|^2}{|\zeta|^{d_k}|z-\zeta|} \leq \frac{2^{d_k}}{|z|^{d_k}} \int_{B(z,3|z|)} \frac{|d\zeta|^2}{|z-\zeta|} = 2^{d_k+1} \cdot 3\pi|z|^{1-d_k}. \tag{3.105}$$

Since $d_k < 1$, the combination of (3.103), (3.104), (3.105) and (3.106) gives that

$$\left| \frac{1}{2\pi i} \int_{B(0,2|z|)\setminus\overline{B}(0,r)} \frac{\overline{\zeta}^2 \vec{Y}_k(\zeta)}{\overline{\zeta}-z} d\overline{\zeta} \wedge d\zeta \right| \leq 16C_0|z|^{1-d_k}. \tag{3.106}$$

The previous argument — that is, the estimates of the proof of Lemma 3.9 — shows that (for the same constant $a_{k,0}$)

$$\left| -\frac{1}{\pi\overline{z}^2} \int_{B(0,R)\setminus\overline{B}(0,2|z|)} \frac{\overline{\zeta}^2 \vec{Y}_k(\zeta)}{\overline{\zeta}-z} |d\zeta|^2 - \frac{a_{k,0}}{\overline{z}^2} \right| \leq \frac{4C_0}{|z|} \left(1 + \frac{1}{d_k} \left(\left(\frac{R}{2|z|} \right)^{d_k} - 1 \right) \right),$$

and

$$\left| \vec{U}_k(z) - \frac{a_{k,0}}{\overline{z}^2} \right| \leq \frac{4C_0}{|z|} \left(1 + \frac{1}{d_k} \left(\left(\frac{R}{2|z|} \right)^{d_k} - 1 \right) \right) + \frac{16C_0}{|z|^{1+d_k}}.$$

Finally, we get that for all $z \in B(0, \frac{R}{2})$ the following estimate holds:

$$|\vec{V}_k(z)| \leq \frac{4C_0}{|z|} \left(1 + \frac{1}{d_k} \left(\left(\frac{R}{2|z|} \right)^{d_k} - 1 \right) \right) + \frac{16C_0}{|z|^{1+d_k}}. \tag{3.107}$$

Case 3: $0 < d_k \leq 1$ and $d_k \xrightarrow[k\to\infty]{} d \in (0,1)$. The estimate (3.107) gives that

$$|z|^{d_k}|\vec{V}_k(z)| \leq \frac{4C_0}{|z|} \left(|z|^{d_k} + \frac{1}{d_k} \left((R^{d_k} - 1) \right) \right) + \frac{16C_0}{|z|} \leq \frac{C}{|z|}$$

Here, the hypothesis implies that $\frac{1}{d_k} \left((R^{d_k} - 1) \right)$ is bounded. Therefore, the claimed estimate holds uniformly in $k \in \mathbb{N}$.

Case 4: $d_k \xrightarrow[k\to\infty]{} d \geq 1$. By the previous cases, we can also assume that $d_k \geq 1$ for all $k \in \mathbb{N}$ too. Let $a \in \mathbb{N}$ such that $a \leq d_k < a + 1$, and define

$$\vec{U}_k(z) = \frac{1}{\overline{z}^{a+2}} \left(-\frac{1}{\pi\overline{z}} * \left(\overline{\zeta}^{a+2} \vec{Y}_k(\zeta) \right) \right) = -\frac{1}{\pi\overline{z}^{a+2}} \int_{\mathbb{C}} \frac{\overline{\zeta}^{a+2} \vec{Y}_k(\zeta)}{z-\zeta} |d\zeta|^2.$$

Observe that $\partial_z \vec{U}_k = \vec{Y}_k$. Fix some $z \in B(0, R)$, where we recall that $r = 2r_k$ and $R = \frac{R_k}{2}$. For $0 < 2|z| \leq r$, it holds

$$\vec{U}_k(z) = -\frac{1}{\pi\overline{z}^{a+2}} \int_{B(0,R)\setminus\overline{B}(0,2|z|)} \frac{\overline{\zeta}^{a+2} \vec{Y}_k(\zeta)}{\overline{\zeta}-z} |d\zeta|^2 = \frac{1}{\overline{z}^{a+2}} \sum_{l=0}^{\infty} \left(\frac{1}{\pi} \int_{B(0,R)\setminus\overline{B}(0,2|z|)} \overline{\zeta}^{a-l+1} \vec{Y}_k(\zeta)|d\zeta|^2 \right) \overline{z}^l$$

$$= \sum_{l=0}^{\infty} a_{k,l} \overline{z}^{l-a-2}.$$

First, we have

$$|a_{k,0}| \leq \frac{C_0}{\pi} \int_{B(0,R)\setminus\overline{B}(0,2|z|)} |\zeta|^{a-d_k-1}|d\zeta|^2 = \frac{C_0}{a+1-d_k}\left(R^{a+1-d_k} - (2|z|)^{a+1-d_k}\right).$$

Then, for $l = 1$:

$$\left| \frac{1}{\pi} \int_{B(0,R)\setminus\overline{B}(0,2|z|)} \overline{\zeta}^{a-l+1}\vec{Y}_k(\zeta)|d\zeta|^2 \right| \leq \frac{C_0}{\pi} \int_{B(0,R)\setminus\overline{B}(0,2|z|)} \frac{|d\zeta|^2}{|\zeta|^{2+d_k-a}}$$

$$= \begin{cases} 2C_0 \log\left(\dfrac{R}{2|z|}\right) & \text{if } d_k = a \\ \dfrac{2C_0}{d_k-a}\left(\dfrac{1}{(2|z|)^{d_k-a}} - \dfrac{1}{R^{d_k-a}}\right) & \text{if } d_k > a. \end{cases}$$

For $l \geq 2$, we have:

$$\left| \frac{1}{\pi} \int_{B(0,R)\setminus\overline{B}(0,2|z|)} \zeta^{a-l+1}\vec{Y}_k(\zeta)|d\zeta|^2 \right| \leq \frac{2C_0}{l-1}\frac{1}{(2|z|)^{d_k-a-1}}\frac{1}{(2|z|)^l}.$$

Therefore

$$\left| \sum_{l=2}^{\infty} a_{k,l}z^l \right| \leq 2C_0(2|z|)^{a+1-d_k}\sum_{l=2}^{\infty}\frac{1}{l-1}\frac{1}{2^l} = C_0(2|z|)^{a+1-d_k}\sum_{l=0}^{\infty}\frac{1}{l+1}\frac{1}{2^{l+1}} = \log(2)C_0(2|z|)^{a+1-d_k}.$$

We deduce that for $d_k = a$, it holds

$$\left| \vec{U}_k(z) - \frac{a_{k,0}}{\overline{z}^{a+2}} \right| \leq \frac{4C_0}{|z|^{d_k+1}} + 2C_0\frac{1}{|z|^{d_k+1}}\log\left(\frac{R}{2|z|}\right), \tag{3.108}$$

whilst for $d_k > a$,

$$\left| \vec{U}_k(z) - \frac{a_{k,0}}{\overline{z}^{a+2}} \right| \leq \frac{4C_0}{|z|^{1+d_k}} + \frac{2C_0}{d_k-a}\frac{1}{|z|^{a+1}}\left(\frac{1}{|z|^{d_k-a}} - \left(\frac{2}{R}\right)^{d_k-a}\right). \tag{3.109}$$

Now, if $2|z| > r$, we can make the same decomposition

$$\overline{z}^{a+2}\vec{U}_k(z) = \frac{1}{2\pi i}\int_{B(0,2|z|)\setminus\overline{B}(0,r)} \frac{\overline{\zeta}^{a+2}\vec{Y}_k(\zeta)}{\overline{\zeta-z}}d\overline{\zeta}\wedge d\zeta + \frac{1}{2\pi i}\int_{B(0,R)\setminus\overline{B}(0,2|z|)} \frac{\overline{\zeta}^{a+2}\vec{Y}_k(\zeta)}{\overline{\zeta-z}}d\overline{\zeta}\wedge d\zeta.$$

The second integral is estimated as above and we get

$$\left| \frac{1}{2\pi i}\int_{B(0,R)\setminus\overline{B}(0,2|z|)} \frac{\overline{\zeta}^2\vec{Y}_k(\zeta)}{\overline{\zeta-z}}d\overline{\zeta}\wedge d\zeta - \frac{a_{k,0}}{\overline{z}^{a+2}} \right| \leq \frac{4C_0}{|z|^{1+d_k}} + \frac{2C_0}{d_k-a}\frac{1}{|z|^{a+1}}\left(\frac{1}{|z|^{d_k-a}} - \left(\frac{2}{R}\right)^{d_k-a}\right). \tag{3.110}$$

Then, we have

$$\left| \frac{1}{2\pi i}\int_{B(0,2|z|)\setminus\overline{B}(0,r)} \frac{\overline{\zeta}^{a+2}\vec{Y}_k(\zeta)}{\overline{\zeta-z}}|d\zeta|^2 \right| \leq \frac{C_0}{\pi}\int_{B(0,2|z|)\setminus\overline{B}(0,r)} \frac{|d\zeta|^2}{|\zeta|^{d_k-a}|\zeta-z|} \leq 16C_0|z|^{a+1-d_k}$$

using the same proof given the estimate (3.106). Finally, we deduce that

$$|\vec{V}_k(z)| \leq \frac{4C_0}{|z|}\left(1 + \frac{1}{d_k}\left(\left(\frac{R}{2|z|}\right)^{d_k} - 1\right)\right) + \frac{16C_0}{|z|^{1+d_k}}. \tag{3.111}$$

Sub-case 1: $d_k \xrightarrow[k \to \infty]{} d$ with $a < d \leq a + 1$. Then the estimate (3.111) immediately implies the bound

$$|z|^{d_k} |\vec{V}_k(z)| \leq \frac{C}{|z|}, \quad \text{for all } z \in \Omega_k\left(\frac{1}{2}\right).$$

Sub-case 2: $d_k = a$. Then we can apply the same argument on averages as in the case $-1 < a \leq 0$ and the rest of the proof is unchanged (notice that the proof of the estimate on $\text{Im}\,(\vec{V}_k)$ is independent of the multiplicity d_k).

Sub-case 3: $d_k \xrightarrow[k \to \infty]{} a$ and $d_k > a$ for all $k \in \mathbb{N}$ (notice that it covers all remaining cases by the previous discussion). Then, we apply the same averaging argument as before the statement of Lemma 3.12 and find

$$r^{1+d_k} a(r) \leq \left(\frac{R_k}{2}\right)^{1+d_k} a\left(\frac{R_k}{2}\right) + C,$$

which shows by (3.109) that

$$r^{1+d_k} a(r) \leq 4C_0 + C.$$

Notice that if the estimate had been replaced by an estimate on $\Omega_k(\alpha_0)$ for all $0 < \alpha_0 < \frac{1}{2}$, we would have had instead

$$r^{1+d_k} a(r) \leq 4C_0 + \frac{2C_0}{d_k - a} \frac{1}{(\alpha_0 R_k)^{a+1}} \left(\frac{1}{(\alpha_0 R_k)^{d_k-a}} - \left(\frac{2}{R_k}\right)^{d_k-a}\right).$$

Since $d_k \xrightarrow[k \to \infty]{} a$ and $R_k \xrightarrow[k \to \infty]{} R \in (0, \infty)$, we deduce that

$$\frac{1}{(\alpha_0 R_k)^{d_k-a}} = e^{-(d_k-a)\log(\alpha_0 R_k)} = 1 - (d_k - a)\log(\alpha_0 R_k) + O\left((d_k - a)^2\right)$$

which gives that

$$\frac{1}{d_k - a} \left(\frac{1}{(\alpha_0 R_k)^{d_k-a}} - \left(\frac{2}{R_k}\right)^{d_k-a}\right) \xrightarrow[k \to \infty]{} \log\left(\frac{1}{2\alpha_0}\right),$$

showing that

$$\limsup_{k \in \to \infty} \left(4C_0 + \frac{2C_0}{d_k - a} \frac{1}{(\alpha_0 R_k)^{a+1}} \left(\frac{1}{(\alpha_0 R_k)^{d_k-a}} - \left(\frac{2}{R_k}\right)^{d_k-a}\right)\right) = 4C_0 + \frac{2C_0}{(\alpha_0 R)^{a+1}} \log\left(\frac{1}{2\alpha_0}\right) < \infty.$$

Finally, we deduce the estimate announced in Proposition 3.8. □

Estimates on $\vec{W}_k$ and $\vec{L}_k$

Theorem 3.13. *Under the hypothesis of Theorem 3.2, there exists $C_1(m, h, \Lambda)$, $\alpha_0 > 0$ and a measurable function $\vec{L}_k : B(0, \alpha_0 R_k) \to \mathbb{C}^m$ satisfying the equation*

$$\nabla_z \vec{L}_k = \vec{Y}_k \quad \text{on } \Omega_k(\alpha_0).$$

Moreover, the following decomposition holds: $\vec{L}_k = \vec{V}_k + \vec{W}_k$, where

$$e^{\lambda_k(z)} |\vec{V}_k(z)| \leq \frac{C_1}{|z|}, \quad \text{for all } z \in \Omega_k(\alpha_0),$$

and

$$\text{Im}\,(\vec{L}_k) \text{ and } \vec{W}_k \text{ are bounded in } \text{W}^{1,(2,\infty)}(B(0, \alpha_0 R_k)) \text{ uniformly in } k \in \mathbb{N}.$$

Proof. First, we will construct a function $\vec{U}_k$ on $B(0, \alpha_0 R_k)$, for $\alpha_0 > 0$, such that

$$\nabla_z \vec{U}_k = \partial_z \vec{U}_k + \vec{f}_k(\vec{U}_k) = \partial_z \vec{V}_k, \quad \text{on } \Omega_k(\alpha_0). \tag{3.112}$$

Note this is equivalent to

$$\vec{U}_k = \vec{V}_k - \frac{1}{\pi \bar{z}} * f_k(\vec{U}_k), \quad \text{on } \Omega_k(\alpha_0). \tag{3.113}$$

This will be achieved by a fixed point argument similar to the one of [42, Lemma A.1].

First, extend by 0 the restriction of $\vec{V}_k$ to $B(0, \alpha_0 R_k)$, and using a smooth non-negative cut-off η such that $\eta = 1$ on $B(0, \alpha_0 R_k)$ and $\operatorname{supp}(\eta) \subset B(0, 2\alpha_0 R_k)$, we can assume that $\operatorname{supp}(\gamma_j^l) \subset B(0, 2\alpha_0 R_k)$ for all $1 \leq j, l \leq m$. Then, define $\widetilde{\lambda}_k : \mathbb{C} \to \mathbb{R}$ such that

$$\widetilde{\lambda}_k(z) = \lambda_k(z)\eta(z) - (1 - \eta(z))\log(1 + |z|^2).$$

Consider the normed space

$$\mathrm{L}^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C}) = \mathrm{L}^1_{\mathrm{loc}}(\mathbb{C}) \cap \left\{ u : e^{\widetilde{\lambda}_k} u \in \mathrm{L}^{2,\infty}(\mathbb{C}) \right\}, \quad \text{for all } k \in \mathbb{N},$$

equipped with the norm

$$\|u\|_{\mathrm{L}^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})} = \left\| e^{\widetilde{\lambda}_k} u \right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})}.$$

One checks immediately that $\mathrm{L}^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})$ is a Banach space (for example, by the series criterion using that $\mathrm{L}^{2,\infty}(\mathbb{C})$ is a Banach space). Now, by the previous estimate (3.49), we deduce that $\vec{V}_k \in \mathrm{L}^{(2,\infty)}_{\widetilde{\lambda}_k}(\mathbb{C})$. We introduce the operator $T : \mathrm{L}^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C}) \to \mathrm{L}^{(2,\infty)}_{\widetilde{\lambda}_k}(\mathbb{C})$, defined by

$$T(\vec{U}) = \vec{V}_k - \frac{1}{\pi \bar{z}} * f_k(\vec{U}). \tag{3.114}$$

Using the Young inequality for convolution, for all $2 < p < \infty$, if $q < 2$ is such that

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{q} - 1,$$

using the support assumption on γ_j^l and that $|f_k(\vec{U})| \leq C_0 e^{\lambda_k}|\vec{U}|$, we deduce that

$$\left\| \frac{1}{\pi \bar{z}} * f_k(\vec{U}) \right\|_{\mathrm{L}^{p,\infty}(\mathbb{C})} \leq C(p) \left\| \frac{1}{\pi \bar{z}} \right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})} \left\| f_k(\vec{U}) \right\|_{\mathrm{L}^{\frac{2p}{p+2},\infty}(\mathbb{C})} = 2\sqrt{\pi}\, C(p) \left\| f_k(\vec{U}) \right\|_{\mathrm{L}^{\frac{2p}{p+2}}(B(0,2\alpha_0 R_k))}$$

$$\leq 2\sqrt{\pi}\, C(p) \|1\|_{\mathrm{L}^{\frac{p+2}{p}}(B(0,2\alpha_0 R_k))} \left\| f_k(\vec{U}) \right\|_{\mathrm{L}^{2,\infty}(B(0,2\alpha_0 R_k))}$$

$$\leq 2\sqrt{\pi}\, C_0 C(p) \pi^{\frac{p}{p+2}} (2\alpha_0 R_k)^{\frac{2p}{p+2}} \left\| \vec{U} \right\|_{\mathrm{L}^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})}. \tag{3.115}$$

Since $e^{\lambda_k} \in \mathrm{L}^p(B(0, R_k))$ for all $p < \dfrac{2}{1-\varepsilon}$, using Hölder's inequality we deduce that for all $2 < p < \dfrac{2}{1-\varepsilon}$ it holds

$$\left\| T(\vec{U}) - \vec{V}_k \right\|_{\mathrm{L}^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})} = \left\| e^{\widetilde{\lambda}_k}\left(T(\vec{U}) - \vec{V}_k \right) \right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})} \leq C(p) \left\| e^{\widetilde{\lambda}_k} \right\|_{\mathrm{L}^{p,\infty}(\mathbb{C})} \left\| T(\vec{U}) - \vec{V}_k \right\|_{\mathrm{L}^{\frac{p}{p-2},\infty}(\mathbb{C})}$$

$$\leq C(p) \left(C_{\alpha_0} \left\| r_k^{d_k} \right\|_{\mathrm{L}^p(B(0,\alpha_0^{-1} r_k))} + e^A \left\| |z|^{d_k} \right\|_{\mathrm{L}^p(B(0,2\alpha_0 R_k))} + \left\| \frac{1}{1+|z|^2} \right\|_{\mathrm{L}^2(\mathbb{C})} \right) \left\| \frac{1}{\pi \bar{z}} * f_k(\vec{U}) \right\|_{\mathrm{L}^{\frac{p}{p-2},\infty}(\mathbb{C})}$$

$$\leq C(p) \left(C_{\alpha_0} (2\pi)^{\frac{1}{p}} r_k^{\frac{2}{p}+d_k} + (2\pi)^{\frac{1}{p}} \frac{(2\alpha_0 R_k)^{\frac{2}{p}+d_k}}{(2+pd_k)^{\frac{1}{p}}} + \sqrt{\pi} \right)$$

$$\times\, 2\sqrt{\pi}\, C_0 C \left(\frac{p}{p-2}\right) \pi^{\frac{p-2}{3p-4}} (2\alpha_0 R_k)^{\frac{2p-4}{3p-4}} \left\|\vec{U}\right\|_{L^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})}.$$

In particular, choosing $p = 2 + \varepsilon < \frac{2}{1-\varepsilon}$, we obtain that there exists a constant $C(\varepsilon) > 0$ such that

$$\left\|T(\vec{U}) - \vec{V}_k\right\|_{L^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})} \leq C(\varepsilon) \alpha_0^{\frac{\varepsilon}{4}} \left\|\vec{U}\right\|_{L^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})}.$$

Therefore, taking $\alpha_0 = \left(\frac{1}{2C(\varepsilon)}\right)^{\frac{4}{\varepsilon}}$, we get that

$$\left\|T(\vec{U}) - \vec{V}_k\right\|_{L^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})} \leq \frac{1}{2} \left\|\vec{U}\right\|_{L^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})}.$$

Since $\vec{V}_k \in L^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})$, we conclude that $T : L^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C}) \to L^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})$ is a contraction which implies in particular that T admits a unique fixed point $\vec{U}_k$.

The function $\vec{U}_k \in L^{2,\infty}_{\widetilde{\lambda}_k}(\mathbb{C})$ satisfies

$$\vec{U}_k = \vec{V}_k - \frac{1}{\pi \bar{z}} * f_k(\vec{U}_k). \tag{3.116}$$

Since $f_k(\vec{U}_k) \in L^{2,\infty}(B(0, \alpha_0 R_k))$, the previous estimate (3.115) shows that

$$\frac{1}{\pi \bar{z}} * f_k(\vec{U}_k) \in \bigcap_{p < \infty} L^p(\mathbb{C}).$$

Since $e^{\lambda_k} \in L^p(B(0, \alpha_0 R_k))$ for all $p < \frac{2}{1-\varepsilon}$, we deduce by Hölder's inequality that

$$e^{\lambda_k} \left(\frac{1}{\pi \bar{z}} * f_k(\vec{U}_k)\right) \in \bigcap_{p < \frac{2}{1-\varepsilon}} L^p(B(0, \alpha_0 R_k)).$$

Now, recalling the pointwise estimate (3.49), we deduce that there exists $f : B(0, \alpha_0 R) \to \mathbb{R}$ (where $R = \sup_{k \in \mathbb{N}} R_k$) such that $f \in L^p(B(0, \alpha_0 R))$ for all $p < \frac{2}{1-\varepsilon}$ such that

$$e^{\lambda_k} |\vec{U}_k| \leq \frac{C}{|z|} + f(z) \qquad \text{for all } z \in \Omega_k(\alpha_0).$$

Then, using an estimate presented in the proof of [42, Lemma A.2], we deduce that

$$\left\|\nabla \left(\frac{1}{\pi \bar{z}} * f_k(\vec{U}_k)\right)\right\|_{L^{2,\infty}(\mathbb{C})} \leq C \left\|f_k(\vec{U}_k)\right\|_{L^{2,\infty}(\mathbb{C})} \leq C.$$

Since $\nabla \mathrm{Im}(\vec{V}_k) \in L^{2,\infty}(\mathbb{C})$ (this is where the removal of the logarithm term is crucial), by (3.116) we conclude that

$$\nabla \mathrm{Im}(\vec{U}_k) = \nabla \mathrm{Im}(\vec{V}_k) - \mathrm{Im}\left(\nabla \left(\frac{1}{\pi \bar{z}} * f_k(\vec{U}_k)\right)\right) \in L^{2,\infty}(\mathbb{C}).$$

Now, we can apply the exact same proof as Lemma A.1 of [42] (see also Lemma 3.14 for more details) to obtain by a similar contraction argument as above the existence of $\vec{W}_k \in W^{1,(2,\infty)}(\mathbb{C})$ such that

$$\begin{cases} \nabla_z \vec{W}_k = 0 & \text{in } B(0, \alpha_0 R_k) \\ \mathrm{Im}(\vec{W}_k) = \mathrm{Im}(\vec{U}_k) & \text{on } \partial B(0, \alpha_0 R_k). \end{cases}$$

Finally, defining $\vec{L}_k = \vec{U}_k - \vec{W}_k : B(0, \alpha_0 R_k) \to \mathbb{C}^n$, we deduce that

$$\begin{cases} e^{\lambda_k} \vec{L}_k \in \mathrm{L}^{2,\infty}(B(0, \alpha_0 R_k)) \\ \mathrm{Im}\left(\vec{L}_k\right) \in \mathrm{W}^{1,(2,\infty)}(B(0, \alpha_0 R_k)) \end{cases}$$

and

$$\begin{cases} \nabla_z \vec{L}_k = \vec{Y}_k & \text{in } B(0, \alpha_0 R_k) \\ \Delta\, \mathrm{Im}\left(\vec{L}_k\right) = 4\, \mathrm{Im}\left(\nabla_{\bar{z}} \vec{Y}_k\right) - 4\, \mathrm{Im}\left(f_k(\vec{Y}_k)\right) - 4\, \mathrm{Im}\left(\partial_{\bar{z}}\left(f_k(\vec{U}_k)\right)\right) & \text{in } B(0, \alpha_0 R_k) \\ \mathrm{Im}\left(\vec{L}_k\right) = 0 & \text{on } \partial B(0, \alpha_0 R_k). \end{cases}$$

Furthermore, there exists l_k bounded in $\mathrm{L}^p(B(0, \alpha_0 R_k))$ for all $p < \dfrac{2}{1-\varepsilon}$ such that

$$e^{\lambda_k(z)} |\vec{L}_k(z)| \leq \frac{C}{|z|} + l_k(z), \quad \text{on } \Omega_k(\alpha_0). \tag{3.117}$$

Now, we introduce a refinement of [42, Lemma A.2].

Lemma 3.14. *Let $2 < p < \infty$ and $1 < q < 2$ be fixed real numbers. There exists constants $\varepsilon_3(n, p, q) > 0$, $C_3(n, p, q) > 0$ with the following property. For all $j, k \in \{1, \cdots, n\}$, let $\gamma_j^k \in \mathrm{L}^p(\mathbb{C}) \cap \mathrm{W}^{1,(2,\infty)}(\mathbb{C})$ be such that $\mathrm{supp}\,(\gamma_j^k) \subset B(0, 2)$ and $\left\|\gamma_j^k\right\|_{\mathrm{L}^{2,1}(\mathbb{C})} \leq \varepsilon_0$. For all $\vec{U} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{C})$, define*

$$\left(\nabla_z \vec{U}\right)_j = \partial_z \vec{U}_j + \sum_{k=1}^{n} \gamma_j^k \vec{U}_k \qquad \text{in } \mathscr{D}'(\mathbb{C}), \text{ where } 1 \leq j \leq n.$$

Then for all $\vec{Y} \in \left(\mathrm{L}^1 \cap \mathrm{L}^{2,\infty}\right)(\mathbb{C})$ such that $\mathrm{Im}\,(\nabla_{\bar{z}} \vec{Y}) \in \mathrm{L}^q(\mathbb{C})$, provided that $\left\|\gamma_j^k\right\|_{\mathrm{L}^p(B(0,2))} \leq \varepsilon_3$, there exists a unique $\vec{U} \in \mathrm{W}^{1,(2,\infty)}(B(0,1))$ such that $\mathrm{Im}\,(\vec{U}) \in \mathrm{W}^{2,q}(B(0,1))$ satisfying

$$\begin{cases} \nabla_z \vec{U} = \vec{Y} & \text{in } \mathscr{D}'(B(0,1)) \\ \mathrm{Im}\,(\vec{U}) = 0 & \text{on } \partial B(0,1). \end{cases}$$

Furthermore, we have the estimate

$$\left\|\vec{U}\right\|_{\mathrm{W}^{1,(2,\infty)}(B(0,1))} \leq C_3 \left(\left\|\vec{Y}\right\|_{\mathrm{L}^1(\mathbb{C})} + \left\|\vec{Y}\right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})}\right).$$

Proof. As in the proof of Lemma 3.6, if T is defined as in (3.114), we have for all $\vec{U} \in \mathrm{L}^{2,\infty}(\mathbb{C})$ the estimate

$$\left\|T(\vec{U})\right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})} \leq C \left\|\vec{Y}\right\|_{\mathrm{L}^1(\mathbb{C})} + C \left\|\vec{U}\right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})}$$

and that for all $\vec{U}_1, \vec{U}_2 \in \mathrm{L}^{2,\infty}(\mathbb{C})$ if $\varepsilon_3 \leq \delta_0$ is small enough, we have

$$\left\|T(\vec{U}_1) - T(\vec{U}_2)\right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})} \leq \frac{1}{4} \left\|\vec{U}_1 - \vec{U}_2\right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})}.$$

By the same argument as [42, Lemma A.2], we deduce that

$$\left\|\nabla T(\vec{U})_j\right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})} \leq \Gamma_2 \left\|\vec{Y}_j - \sum_{k=1}^{n} \gamma_k^j \vec{U}_k\right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})} \leq \Gamma_2 \left\|\vec{Y}_j\right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})} + \sum_{j=1}^{n} \Gamma_2 \left\|\gamma_j^k \vec{U}_k\right\|_{\mathrm{L}^2(B(0,2))}$$

$$\leq \Gamma_2 \left\|\vec{Y}\right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})} + \sum_{j=1}^{n} \Gamma_2 \left\|\gamma_j^k\right\|_{\mathrm{L}^p(B(0,2))} \left\|\vec{U}_k\right\|_{\mathrm{L}^{\frac{2p}{p-2}}(B(0,2))}.$$

By the Sobolev embedding theorem, if $q < 2$, we have $W^{1,q}(B(0,2)) \hookrightarrow L^{q^*}(B(0,2))$, where

$$\frac{1}{q^*} = \frac{1}{q} - \frac{1}{2} = \frac{2-q}{2q}.$$

Since we want to impose $q^* = \dfrac{2p}{p-2}$ where $p > 2$, this implies that $q = p' = \frac{p}{p-1}$ Therefore, we deduce that there exists a universal constant Γ_4 such that

$$\left\| \vec{U}_k \right\|_{L^{\frac{2p}{p-2}}(B(0,2))} \leq \Gamma_4 \left(\left\| \vec{U}_k \right\|_{L^{\frac{p}{p-1}}(B(0,2))} + \left\| \nabla \vec{U}_k \right\|_{L^{\frac{p}{p-1}}(B(0,2))} \right).$$

By the $L^{\frac{2(p-1)}{p-2},1}/L^{\frac{2(p-1)}{p},\infty}$ duality applied to $|\vec{U}_k|^{\frac{p}{p-1}}$, we deduce that

$$\left\| \vec{U}_k \right\|_{L^{\frac{p}{p-1}}(B(0,2))} \leq \left\| 1 \right\|_{L^{\frac{2(p-1)}{p-2},1}(B(0,2))} \left\| \vec{U}_k \right\|_{L^{2,\infty}(B(0,2))} = 2^{\frac{3p-4}{p-1}} \frac{(p-1)^2}{p(p-2)} \pi^{\frac{p-2}{2(p-1)}} \left\| \vec{U}_k \right\|_{L^{2,\infty}(B(0,2))},$$

and likewise

$$\left\| \nabla \vec{U}_k \right\|_{L^{\frac{p}{p-1}}(B(0,2))} \leq 2^{\frac{3p-4}{p-1}} \frac{(p-1)^2}{p(p-2)} \pi^{\frac{p-2}{2(p-1)}} \pi^{\frac{p}{2(p-2)}} \left\| \nabla \vec{U}_k \right\|_{L^{2,\infty}(B(0,2))}.$$

Therefore, we get

$$\left\| \nabla T(\vec{U}) \right\|_{L^{2,\infty}(\mathbb{C})} \leq \Gamma_2 n \left\| \vec{Y} \right\|_{L^{2,\infty}(\mathbb{C})}$$
$$+ \Gamma_2 n^2 2^{\frac{3p-4}{p-1}} \frac{(p-1)^2}{p(p-2)} \pi^{\frac{p-2}{2(p-1)}} \varepsilon_4 \left(\left\| \vec{U}_k \right\|_{L^{2,\infty}(B(0,2))} + \left\| \nabla \vec{U}_k \right\|_{L^{2,\infty}(B(0,2))} \right),$$

and for all $\vec{U}_1, \vec{U}_2 \in W^{1,(2,\infty)}(\mathbb{C})$, the estimate

$$\left\| \nabla T(\vec{U}_1) - \nabla T(\vec{U}_2) \right\|_{L^{2,\infty}(\mathbb{C})} \leq \Gamma_2 n^2 2^{\frac{3p-4}{p-1}} \frac{(p-1)^2}{p(p-2)} \pi^{\frac{p-2}{2(p-1)}} \pi^{\frac{p}{2(p-2)}} \varepsilon_4 \left\| \vec{U}_1 - \vec{U}_2 \right\|_{W^{1,(2,\infty)}(\mathbb{C})}.$$

Taking

$$\varepsilon_4 = \min\left\{ \delta_0, \frac{1}{4} \left(\Gamma_2 n^2 2^{\frac{3p-4}{p-1}} \frac{(p-1)^2}{p(p-2)} \pi^{\frac{p-2}{2(p-1)}} \right)^{-1} \right\},$$

we deduce that

$$\left\| T(\vec{U}_1) - T(\vec{U}_2) \right\|_{W^{1,(2,\infty)}(\mathbb{C})} \leq \frac{1}{2} \left\| \vec{U}_1 - \vec{U}_2 \right\|_{W^{1,(2,\infty)}(\mathbb{C})}.$$

Therefore, T has a unique fixed point that we denote $\vec{U}_0$. Define by $f : \mathbb{C}^n \to \mathbb{C}^n$ the linear map such that for all $\vec{X} \in C^\infty(\mathbb{C}, \mathbb{C}^n)$, we have

$$\nabla_z \vec{X} = \partial_z \vec{X} + f(\vec{X}).$$

Now, to get the boundary condition, define the operator $S : W^{1,(2,\infty)}(B(0,1), \mathbb{C}^n) \to W^{1,(2,\infty)}(B(0,1), \mathbb{C}^n)$ such that for all $\vec{U} \in W^{1,(2,\infty)}(B(0,1), \mathbb{C}^n)$, $\vec{V} = S(\vec{U})$ is the unique solution of the equation

$$\begin{cases} \partial_z \vec{V} = -f(\vec{U}) & \text{in } B(0,1) \\ \text{Im}\,(\vec{V}) = \text{Im}\,(\vec{U}_0) & \text{on } \partial B(0,1). \end{cases} \tag{3.118}$$

Let us prove that this operator S is well-defined. First, by the Cauchy formula, if

$$\vec{V}_0(z) = \frac{1}{2\pi i} \int_{B(0,1)} \frac{f(\vec{U})}{\zeta - z} d\zeta \wedge d\bar{\zeta},$$

then we have $\partial_z \vec{V}_0 = -f(\vec{U})$ and $\vec{V}_0 = 0$ on $\partial B(0,1)$. Furthermore, by the previous convolution estimates, we have

$$\left\|\vec{V}_0\right\|_{W^{1,(2,\infty)}(B(0,1))} \leq \frac{1}{2}\left\|\vec{U}\right\|_{W^{1,(2,\infty)}(B(0,1))}. \tag{3.119}$$

Now, first solve in $W^{1,(2,\infty)}(B(0,1),\mathbb{R}^n)$ the equation

$$\begin{cases} \Delta\vec{V}_1 = 0 & \text{in } B(0,1) \\ \vec{V}_1 = \text{Im}\,(\vec{U}_0) & \text{on } \partial B(0,1). \end{cases}$$

Then we have by Calderón-Zygmund estimates $\vec{V}_1 \in W^{1,(2,\infty)}(B(0,1))$. If $\vec{V}_2$ is the harmonic conjugate of $\vec{V}_1$, then by interpolation theory, we also deduce that $\vec{V}_2 \in W^{1,(2,\infty)}(B(0,1))$ and that for some universal constant C, we have

$$\left\|\vec{V}_1\right\|_{W^{1,(2,\infty)}(B(0,1))} + \left\|\vec{V}_2\right\|_{W^{1,(2,\infty)}(B(0,1))} \leq C\left\|\text{Im}\,(\vec{U}_0)\right\|_{W^{1,(2,\infty)}(B(0,1))}.$$

By construction, the function $\vec{V}_3 = \vec{V}_1 + i\vec{V}_2 = i(\vec{V}_2 - i\vec{V}_1)$ is holomorphic, which implies that $\vec{V}_4 = i\overline{\vec{V}_3} = \vec{V}_2 + i\vec{V}_1$ is anti-holomorphic, *i.e.* $\partial_z\vec{V}_4 = 0$. Furthermore, we have by construction $\text{Im}\,(\vec{V}_4) = \vec{V}_1 = \text{Im}\,(\vec{U}_0)$ on $\partial B(0,1)$. Therefore, the function $\vec{V}_0 + \vec{V}_4$ is the unique solution to the system (3.118). Furthermore, by the estimate (3.119), S is a contraction, so we get a unique fixed point $\vec{U}_1$ of S, which satisfies $\nabla_z\vec{U}_1 = 0$ and $\text{Im}\,(\vec{U}_1) = \text{Im}\,\left(\vec{U}_0\right)$ on $\partial B(0,1)$. Therefore, the function $\vec{U} = \vec{U}_0 - \vec{U}_1$ is the unique solution to the system of the theorem. $\qquad\square$

Recall the expansion

$$\vec{L}_k = \vec{V}_k - \vec{W}_k - \frac{1}{\pi\overline{z}} * f_k(\vec{U}_k),$$

where $f_k(\vec{U}_k) \in L^{2,\infty}_{\lambda_k}(\mathbb{C})$, $\vec{W}_k \in W^{1,(2,\infty)}(B(0,\alpha_0 R_k))$. Moreover,

$$e^{\lambda_k(z)}|\vec{V}_k(z)| \leq \frac{C}{|z|}, \quad \text{for all } z \in \Omega_k(\alpha_0).$$

From [42, Lemma A.1], we deduce that

$$\frac{1}{\pi\overline{z}} * f_k(\vec{U}_k) \in W^{1,(2,\infty)}(\mathbb{C}),$$

which finally implies that

$$\vec{L}_k = \vec{V}_k + \widetilde{\vec{W}}_k, \quad \text{with } \widetilde{\vec{W}}_k \in W^{1,(2,\infty)}(B(0,\alpha_0 R_k)).$$

For simplicity, we shall rename $\widetilde{\vec{W}}_k$ as $\vec{W}_k$, and this concludes the proof of the proposition. $\qquad\square$

3.5   Conclusion of the Proof of Theorem 3.2

In the following, we introduce tensors

$$S_k \in W^{1,(2,\infty)}(B(0,\alpha_0 R_k),\mathbb{C})$$

and

$$\vec{R}_k \in W^{1,(2,\infty)}(B(0,\alpha_0 R_k),\Lambda^2\mathbb{C}^m)$$

that satisfy (up to lower order terms) Jacobian systems of equation, which allows one to use Wente-type inequalities to first upgrade the $L^{2,\infty}$ to an L^2 control, and later on, to an $L^{2,1}$ control. Since $\vec{H}_k$ can be

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written explicitly in terms of S_k and $\vec{R}_k$, the $\mathrm{L}^{2,1}$ control on ∇S_k and $\nabla \vec{R}_k$ will imply the $\mathrm{L}^{2,1}$ control on $e^{\lambda_k} \vec{H}_k$ (see (3.144)). The idea to introduce those new tensors goes back to Rivière's seminal article ([53]) where the existence of those tensors follows from the additional conservation laws satisfied by Willmore surfaces in Euclidean surfaces (that were later proved to come from Noether's theorem by Bernard [5] and the invariance of the Willmore energy under dilations and rotations). In Riemannian manifolds, we have almost conservation laws due to the curvature terms, but the introduction of those tensors permits to show similarly the smoothness of weak Willmore immersion ([42]). We also refer to Rivière's lectures notes for another presentation of the proof of the regularity of weak Willmore immersions ([54]).

By Lemma 3.14, we deduce that there exists $S_k \in \mathrm{W}^{1,(2,\infty)}(B(0, \alpha_0 R_k), \mathbb{C})$ such that

$$
\begin{cases}
\partial_z S_k = \left\langle \partial_z \vec{\Phi}_k, \overline{\vec{L}_k} \right\rangle & \text{in } B(0, \alpha_0 R_k) \\
\mathrm{Im}\,(S_k) = 0 & \text{on } \partial B(0, \alpha_0 R_k)
\end{cases}
\tag{3.120}
$$

Furthermore, by [42, (6.7)] we have

$$
\mathrm{Im}\,\left(\partial_{\overline{z}} \left(\langle \partial_z \vec{\Phi}_k, \overline{\vec{L}_k} \rangle \right) \right) = -\frac{1}{2} e^{2\lambda_k} \left\langle \vec{H}_k, \mathrm{Im}\,\left(\vec{L}_k \right) \right\rangle.
$$

Since $\mathrm{Im}\,(\vec{L}_k) \in \mathrm{L}^q(B(0, \alpha_0 R_k))$ for all $q < \infty$, $e^{\lambda_k} \vec{H}_k \in \mathrm{L}^2(B(0, \alpha_0 R_k))$ and $e^{\lambda_k} \in \mathrm{L}^p(\Omega_k(\alpha_0))$ for all $p < \frac{2}{1-\varepsilon}$, we deduce that

$$
e^{2\lambda_k} \left\langle \vec{H}_k, \mathrm{Im}\,\left(\vec{L}_k \right) \right\rangle \qquad \text{is bounded in } \mathrm{L}^q(B(0, \alpha_0 R_k)) \text{ for all } q < \frac{2}{2-\varepsilon}.
$$

Therefore, since

$$
\begin{cases}
\Delta \mathrm{Im}\,(S_k) = 4\,\mathrm{Im}\,\left(\partial_{\overline{z}} \left(\langle \partial_z \vec{\Phi}_k, \overline{\vec{L}_k} \rangle \right) \right) & \text{on } B(0, \alpha_0 R_k) \\
\mathrm{Im}\,(S_k) = 0 & \text{on } \partial B(0, \alpha_0 R_k),
\end{cases}
$$

the classical Calderón-Zygmund estimates give that

$$
\mathrm{Im}\,(S_k) \text{ is bounded in } \mathrm{W}^{2,q}(B(0, \alpha_0 R_k)) \text{ for all } q < \frac{2}{2-\varepsilon}.
$$

By the Sobolev embedding, we deduce that

$$
\nabla \mathrm{Im}\,(S_k) \text{ is bounded in } \mathrm{L}^p(B(0, \alpha_0 R_k)) \qquad \text{for all } p < \frac{2}{1-\varepsilon}.
$$

Therefore, for all $z \in \Omega_k(\alpha_0)$ it holds

$$
|\nabla \mathrm{Re}\,(S_k)| = 2|\partial_z \mathrm{Re}\,(S_k)| = 2 \left| -i\,\partial_z \mathrm{Im}\,(S_k) + \langle \partial_z \vec{\Phi}_k, \overline{\vec{L}_k} \rangle \right|
$$

$$
\leq |\nabla \mathrm{Im}\,(S_k)| + 2 e^{\lambda_k} |\vec{L}_k| \leq \frac{C}{|z|} + e^{\lambda_k} |g_k(z)|,
\tag{3.121}
$$

where $g_k \in \mathrm{W}^{1,(2,\infty)}(B(0, \alpha_0 R_k))$.

Now, using Lemma 3.14, we deduce that there exists $\vec{R}_k \in \mathrm{W}^{1,(2,\infty)}(B(0, \alpha_0 R_k), \Lambda^2 \mathbb{C}^m)$ such that

$$
\begin{cases}
\nabla_z \vec{R}_k = \partial_z \vec{\Phi}_k \wedge \vec{L}_k - 2i\,\partial_z \vec{\Phi}_k \wedge \vec{H}_k & \text{in } B(0, \alpha R_k) \\
\mathrm{Im}\,(\vec{R}_k) = 0 & \text{on } \partial B(0, \alpha R_k).
\end{cases}
\tag{3.122}
$$

Furthermore, by [42, (6.8)] we have

$$
\mathrm{Im}\,\left(\nabla_{\overline{z}} \left(\partial_z \vec{\Phi}_k \wedge \vec{L}_k - 2i\,\partial_z \vec{\Phi}_k \wedge \vec{H}_k \right) \right) = -\frac{1}{2} e^{2\lambda_k} \vec{H}_k \wedge \mathrm{Im}\,(\vec{L}_k) \in \mathrm{L}^q(\Omega_k(\alpha_0)) \qquad \text{for all } q < \frac{2}{2-\varepsilon}.
$$

This gives in turn

$$\Delta \operatorname{Im}(\vec{R}_k) = 4\, \partial_{\bar{z}}\partial_z \vec{R}_k = 4\operatorname{Im}\left(\partial_{\bar{z}}\left(\partial_z \vec{\Phi}_k \wedge \vec{L}_k - 2i\, \partial_z \vec{\Phi}_k \wedge \vec{H}_k\right)\right) - 4\operatorname{Im}\left(\partial_{\bar{z}}\left(F_k(\vec{R}_k)\right)\right)$$

$$= 4\operatorname{Im}\left(\nabla_{\bar{z}}\left(\partial_z \vec{\Phi}_k \wedge \vec{L}_k - 2i\, \partial_z \vec{\Phi}_k \wedge \vec{H}_k\right)\right) - 4\operatorname{Im}\left(\overline{F_k}\left(\partial_z \vec{\Phi}_k \wedge \operatorname{Im}(\vec{L}_k) - 2i\, \partial_z \vec{\Phi}_k \wedge \vec{H}_k\right)\right)$$

$$- 4\operatorname{Im}\left(\partial_{\bar{z}}\left(F_k(\vec{R}_k)\right)\right),$$

where $F_k(\vec{R}_k)$ is such that

$$\nabla_z \vec{R}_k = \partial_z \vec{R}_k + F_k(\vec{R}_k).$$

Notice that

$$\left|\overline{F_k}\left(\partial_z \vec{\Phi}_k \wedge \operatorname{Im}(\vec{L}_k) - 2i\, \partial_z \vec{\Phi}_k \wedge \vec{H}_k\right)\right| \le C e^{2\lambda_k}\left(|\vec{L}_k| + |\vec{H}_k|\right) \in \mathrm{L}^q(B(0, \alpha_0 R_k)) \quad \text{for all } q < \frac{2}{2-\varepsilon}.$$

Recalling the Sobolev embedding

$$\mathrm{W}^{1,(2,\infty)}(B(0,1)) \longhookrightarrow \bigcap_{q<\infty} \mathrm{L}^q(B(0,1)),$$

we have $\vec{R}_k \in \mathrm{L}^q$ for all $q < \infty$. Since $e^{\lambda_k} \in \mathrm{L}^p(\Omega_k(\alpha_0))$ for all $p < \frac{2}{1-\varepsilon}$, we deduce that it holds *a fortiori*

$$|F_k(\vec{R}_k)| \le C e^{\lambda_k}|\vec{R}_k| \in \bigcap_{p<\frac{2}{1-\varepsilon}} \mathrm{L}^p(\Omega_k(\alpha_0)).$$

Thus $\Delta\operatorname{Im}(\vec{R}_k) \in \mathrm{W}^{-1,p}(B(0, \alpha_0 R_k))$ for all $p < \dfrac{2}{1-\varepsilon}$ and, by Calderón-Zgymund estimates, we obtain that $\operatorname{Im}(\vec{R}_k) \in \mathrm{W}^{1,p}(B(0, \alpha_0 R_k))$ for all $p < \dfrac{2}{1-\varepsilon}$.

Next, we sharpen the last estimate. To this aim, we first prove a pointwise bound for $\operatorname{Re}(\vec{R}_k)$. Writing

$$\vec{R}_k = \vec{R}_k^1 \wedge \vec{R}_k^2,$$

we deduce that

$$\nabla_z \vec{R}_k = \partial_z \vec{R}_k + f_k(\vec{R}_k^1) \wedge \vec{R}_k^2 + \vec{R}_k^1 \wedge f_k(\vec{R}_k^2)$$

$$= \partial_z \vec{R}_k + \sum_{i,j,l=1}^m c_{i,j,l}\partial_z \vec{\Phi}_{k,l}\vec{R}_{k,i,j}\, e_i \wedge e_j,$$

where $(\vec{e}_1, \cdots, \vec{e}_m)$ is a basis of $\mathbb{R}^m$, and $c_{i,j,l} \in \mathbb{R}$ are finite linear combinations of Christoffel symbols with integer weights. We compute:

$$\partial_{\bar{z}} F_k(\vec{R}_k) = \sum_{i,j,l=1}^m \left(\partial_{\bar{z}}c_{i,j,l}\partial_z \vec{\Phi}_{k,l}\, \vec{R}_{k,i,j} + \frac{1}{2}c_{i,j,l}e^{\lambda_k}\vec{H}_k \cdot e^{\lambda_k}\vec{R}_{k,i,j} + c_{i,j,l}\partial_z \vec{\Phi}_{k,l}\partial_{\bar{z}}\vec{R}_{k,i,j}\right) e_i \wedge e_j$$

$$= \sum_{i,j,l=1}^n \left(\frac{1}{4}\left(\partial_x c_{i,j,l} + i\, \partial_y c_{i,j,l}\right)\left(\partial_x \vec{\Phi}_{k,l} - i\, \partial_y \vec{\Phi}_{k,l}\right)\left(\operatorname{Re}\left(\vec{R}_{k,i,j}\right) + i\operatorname{Im}\left(\vec{R}_{k,i,j}\right)\right)\right.$$

$$+ \frac{1}{2}c_{i,j,l}e^{\lambda_k}\vec{H}_k \cdot \left(\operatorname{Re}\left(\vec{R}_{k,i,j}\right) + i\operatorname{Im}\left(\vec{R}_{k,i,j}\right)\right)$$

$$\left.+ \frac{1}{4}c_{i,j,l}\left(\partial_x \vec{\Phi}_{k,l} - i\, \partial_y \vec{\Phi}_{k,l}\right)(\partial_x + i\, \partial_y)\left(\operatorname{Re}\left(\vec{R}_{k,i,j}\right) + i\operatorname{Im}\left(\vec{R}_{k,i,j}\right)\right)\right) e_i \wedge e_j$$

$$= \sum_{i,j,l=1}^m \left(\frac{1}{4}\left(\langle \nabla c_{i,j,l}, \nabla \vec{\Phi}_{k,l}\rangle + i\, \langle \nabla c_{i,j,l}, \nabla^\perp \vec{\Phi}_{k,l}\rangle\right)\left(\operatorname{Re}\left(\vec{R}_{k,i,j}\right) + i\operatorname{Im}\left(\vec{R}_{k,i,j}\right)\right)\right.$$

$$+ \frac{1}{2} c_{i,j,l} e^{\lambda_k} \vec{H}_k \cdot \left(\mathrm{Re}\left(\vec{R}_{k,i,j} \right) + i\,\mathrm{Im}\left(\vec{R}_{k,i,j} \right) \right)$$

$$+ \frac{1}{4} c_{i,j,l} \left(\partial_x \vec{\Phi}_{k,l} - i\, \partial_y \vec{\Phi}_{k,l} \right) \left(\partial_x \mathrm{Re}\left(\vec{R}_{k,i,j} \right) - \partial_y \mathrm{Im}\left(\vec{R}_{k,i,j} \right) \right.$$

$$+ i \left. \left(\partial_y \mathrm{Re}\left(\vec{R}_{k,i,j} \right) + \partial_x \mathrm{Im}\left(\vec{R}_{k,i,j} \right) \right) \right) \right) e_i \wedge e_j$$

$$= \sum_{i,j,l=1}^{m} \left(\frac{1}{4} \left(\langle \nabla c_{i,j,l}, \nabla \vec{\Phi}_{k,l} \rangle \mathrm{Re}\left(\vec{R}_{k,i,j} \right) - \langle \nabla c_{i,j,l}, \nabla^{\perp} \vec{\Phi}_{k,l} \rangle \mathrm{Im}\left(\vec{R}_{k,i,j} \right) \right) \right.$$

$$+ \frac{i}{4} \left(\langle \nabla c_{i,j,l}, \nabla^{\perp} \vec{\Phi}_{k,l} \rangle \mathrm{Re}\left(\vec{R}_{k,i,j} \right) + \langle \nabla c_{i,j,l}, \nabla \vec{\Phi}_{k,l} \rangle \mathrm{Im}\left(\vec{R}_{k,i,j} \right) \right)$$

$$+ \frac{1}{2} c_{i,j,l} e^{\lambda_k} \vec{H}_k \cdot e^{\lambda_k} \left(\mathrm{Re}\left(\vec{R}_{k,i,j} \right) + i\,\mathrm{Im}\left(\vec{R}_{k,i,j} \right) \right)$$

$$+ \frac{1}{4} c_{i,j,l} \left(\langle \nabla \vec{\Phi}_k, \nabla \mathrm{Re}\left(\vec{R}_{k,i,j} \right) \rangle + \langle \nabla \vec{\Phi}_k, \nabla^{\perp} \mathrm{Im}\left(\vec{R}_{k,i,j} \right) \rangle \right)$$

$$+ \frac{i}{4} c_{i,j,l} \left(-\langle \nabla \vec{\Phi}_{k,l}, \nabla^{\perp} \mathrm{Re}\left(\vec{R}_{k,i,j} \right) \rangle + \langle \nabla \vec{\Phi}_{k,l}, \nabla \mathrm{Im}\left(\vec{R}_{k,i,j} \right) \rangle \right) \right) e_i \wedge e_j .$$

Therefore, we have

$$\mathrm{Im}\left(\partial_{\bar{z}} F_k(\vec{R}_k) \right) = \frac{1}{4} \sum_{i,j,l=1}^{m} \left(\langle \nabla c_{i,j,l}, \nabla^{\perp} \vec{\Phi}_{k,l} \rangle \mathrm{Re}\left(\vec{R}_{k,i,j} \right) + \langle \nabla c_{i,j,l}, \nabla \vec{\Phi}_{k,l} \rangle \mathrm{Im}\left(\vec{R}_{k,i,j} \right) \right. \tag{3.123}$$

$$+ 2\, e^{\lambda_k} \vec{H}_k \cdot e^{\lambda_k} \mathrm{Im}\left(\vec{R}_{k,i,j} \right) + c_{i,j,l} \left(-\left\langle \nabla \vec{\Phi}_{k,l}, \nabla^{\perp} \mathrm{Re}\left(\vec{R}_{k,i,j} \right) \right\rangle + \left\langle \nabla \vec{\Phi}_{k,l}, \nabla \mathrm{Im}\left(\vec{R}_{k,i,j} \right) \right\rangle \right) \right) e_i \wedge e_j .$$

Since $\vec{R}_k \in \mathrm{L}^p(B(0, \alpha_0 R_k))$ for all $p < \infty$, we deduce that for all $i, j, l \in \{1, \cdots, m\}$

$$\langle \nabla c_{i,j,l}, \nabla^{\perp} \vec{\Phi}_{k,l} \rangle \mathrm{Re}\left(\vec{R}_{k,i,j} \right) + \langle \nabla c_{i,j,l}, \nabla \vec{\Phi}_{k,l} \rangle \mathrm{Im}\left(\vec{R}_{k,i,j} \right) \in \bigcap_{p < \frac{1}{1-\varepsilon}} \mathrm{L}^p(B(0, \alpha_0 R_k)),$$

Indeed, since for some $c_{i,j,l}^{\alpha,\beta,\gamma} \in \mathbb{N}$, we have

$$c_{i,j,l}(x) = \sum_{\alpha,\beta,\gamma} c_{i,j,l}^{\alpha,\beta,\gamma} \Gamma_{\alpha,\beta}^{\gamma}(\vec{\Phi}_k(x)), \quad ,$$

we deduce that

$$|\nabla c_{i,j,l}| \leq C_0 e^{\lambda_k},$$

so that $\langle \nabla c_{i,j,l}, \nabla^{\perp} \vec{\Phi}_{k,l} \rangle \in \bigcap_{p < \frac{1}{1-\varepsilon}} \mathrm{L}^p(B(0, \alpha_0 R_k))$. From $\mathrm{Im}\left(\vec{R}_k \right) \in C^0(B(0, \alpha_0 R_k))$, we deduce that

$$\left| e^{\lambda_k} \vec{H}_k \cdot e^{\lambda_k} \mathrm{Im}\left(\vec{R}_{k,i,j} \right) \right| \leq C|z|^{d_k} \delta_k(|z|) \leq C|z|^{d_k - 1} .$$

Since $\nabla \lambda_k \in \mathrm{L}^{2,\infty}$ and $\nabla \vec{R}_k \in \mathrm{L}^{2,\infty}$, using the previous L^p bound on e^{λ_k} (see (3.30)), we get that

$$\partial_{\bar{z}} F_k(\vec{R}_k) \in \mathrm{L}^q(B(0, \alpha_0 R_k)) \qquad \text{for all } q < \frac{2}{2-\varepsilon}.$$

Therefore, we have $\mathrm{Im}\left(\vec{R}_k \right) \in \mathrm{W}^{2,q}(B(0, \alpha_0 R_k))$ for all $q < \frac{2}{2-\varepsilon}$, and by Sobolev embedding, we get $\nabla \mathrm{Im}\left(\vec{R}_k \right) \in \mathrm{L}^p(B(0, \alpha_0 R_k))$ for all $p < \frac{2}{1-\varepsilon}$, which shows in particular that $\nabla \mathrm{Im}\left(\vec{R}_k \right) \in \mathrm{L}^{2,1}(B(0, \alpha_0 R_k))$ and $\mathrm{Im}\left(\vec{R}_k \right) \in C^0(B(0, \alpha_0 R_k))$, which suffices to our purpose. Indeed, by the pointwise bounds (3.48) and (3.117), we deduce that

$$|\nabla \mathrm{Re}\left(\vec{R}_k \right)| = 2|\partial_z \mathrm{Re}\left(\vec{R}_k \right)| \leq e^{\lambda_k} |\vec{L}_k| + 2e^{\lambda_k} |\vec{H}_k| + 2|F_k(\vec{R}_k)|$$

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$$\leq \frac{C}{|z|} + f_k(z) + C\delta_k(|z|) + Ce^{\lambda_k}|\vec{R}_k| \leq \frac{C}{|z|} + h_k(z), \tag{3.124}$$

where h_k is bounded in $\mathrm{L}^p(B(0, \alpha_0 R_k))$ for all $p < \dfrac{2}{1-\varepsilon}$.

Furthermore, $\vec{R}_k$ and S_k solve the system

$$\begin{cases} \nabla_z \vec{R}_k = i\left((-1)^{m+1} \star_h \left(\vec{n}_k \lrcorner \nabla_z \vec{R}_k\right) + (\partial_z S_k) \star_h \vec{n}_k\right) & \text{in } B(0, \alpha R_k) \\ \partial_z S_k = -i\langle \nabla_z \vec{R}_k, \star_h \vec{n}_k \rangle & \text{in } B(0, \alpha R_k). \end{cases} \tag{3.125}$$

Now, recall that there exists a linear map $F_k : \Lambda^2 \mathbb{R}^n \to \Lambda^2 \mathbb{R}^n$ such that

$$\nabla_z \vec{R}_k = \partial_z \vec{R}_k + F_k(\vec{R}_k),$$

where

$$|F_k(\vec{R}_k)| \leq Ce^{\lambda_k}|\vec{R}_k|. \tag{3.126}$$

Thus, we can rewrite the system (3.125) as

$$\begin{cases} \partial_z \vec{R}_k = i\left((-1)^{m+1} \star \left(\vec{n}_k \lrcorner \partial_z \vec{R}_k\right) + (\partial_z S_k) \star_h \vec{n}_k + i F_k(\vec{R}_k) + (-1)^{m+1} \star_h \left(\vec{n}_k \lrcorner F_k(\vec{R}_k)\right)\right) \\ \partial_z S_k = -i\langle \partial_z \vec{R}_k, \star_h \vec{n}_k \rangle - i\langle F_k(\vec{R}_k), \star_h \vec{n}_k \rangle. \end{cases} \tag{3.127}$$

Since $\vec{R}_k$ and S_k are bounded in $\mathrm{W}^{1,(2,\infty)}(B(0, \alpha R_k))$, we deduce by the Sobolev embedding that for all $1 < p < \infty$, there exists $C_p < \infty$ such that

$$\left\|\vec{R}_k\right\|_{\mathrm{L}^p(B(0,\alpha R_k))} + \|S_k\|_{\mathrm{L}^p(B(0,\alpha R_k))} \leq C_p < \infty, \quad \text{for all } 0 < \alpha < \alpha_0 \text{ and } k \geq N. \tag{3.128}$$

We prove the following lemma (see [7]). In order to make the notations easier to read, write

$$u_r = \fint_{\partial B(0,r)} u \, d\mathcal{H}^1 = \frac{1}{2\pi r} \int_{\partial B(0,r)} u \, d\mathcal{H}^1.$$

Lemma 3.15. *Let $k, m \in \mathbb{N}$, $u \in \mathrm{W}^{1,1}(B(0,1), \mathbb{C})$, $f \in \mathrm{L}^2(B(0,1), \mathbb{C})$, $\vec{v} \in \mathrm{W}^{1,(2,\infty)}(B(0,1), \Lambda^k \mathbb{C}^m)$, $\vec{w} \in \mathrm{W}^{1,2} \cap \mathrm{L}^\infty(B(0,1), \Lambda^k \mathbb{R}^m)$ such that*

$$\partial_z u = -i\left(\langle \partial_z \vec{v}, \vec{w} \rangle + f\right). \tag{3.129}$$

Let $0 < r < R < \infty$ and set $\Omega = B_R \setminus \overline{B}_r(0)$. Assume that $\mathrm{Im}\,(\vec{v}) \in \mathrm{W}^{1,2}(\Omega)$ and that

$$|\nabla \mathrm{Re}\,(\vec{v})(z)| \leq \frac{C_0}{|z|}, \quad \text{for all } r \leq |z| \leq R. \tag{3.130}$$

Then

$$\left(\int_r^R \left|\frac{d}{d\rho} \mathrm{Re}\,(u)_\rho\right|^2 \rho \, d\rho\right)^{\frac{1}{2}} \leq \sqrt{2\pi}\binom{n}{k} C_0 \|\nabla \vec{w}\|_{\mathrm{L}^2(\Omega)} + \frac{1}{\sqrt{2\pi}} \|\vec{w}\|_{\mathrm{L}^\infty(\Omega)} \|\nabla \mathrm{Im}\,(\vec{v})\|_{\mathrm{L}^2(\Omega)} + \frac{1}{\sqrt{2\pi}} \|f\|_{\mathrm{L}^2(\Omega)}. \tag{3.131}$$

Proof. Rewriting the equation (3.129) as

$$\partial_z \mathrm{Re}\,(u) = -i\left(\langle \partial_z \vec{v}, \vec{w} \rangle + \partial_z \mathrm{Im}\,(u) + f\right),$$

and recalling that $\partial_z = \frac{1}{2}(\partial_x - i\,\partial_y)$, we deduce that

$$\begin{cases} \partial_x \mathrm{Re}\,(u) = 2\,\mathrm{Re}\,\left(-i\left(\langle \partial_z \vec{v}, \vec{w} \rangle + \partial_z \mathrm{Im}\,(u) + f\right)\right) = 2\,\mathrm{Im}\,\left(\langle \partial_z \vec{v}, \vec{w} \rangle + \partial_z \mathrm{Im}\,(u) + f\right) \\ \partial_y \mathrm{Re}\,(u) = -2\,\mathrm{Im}\,\left(-i\left(\langle \partial_z \vec{v}, \vec{w} \rangle + \partial_z \mathrm{Im}\,(u) + f\right)\right) = 2\,\mathrm{Re}\,\left(\langle \partial_z \vec{v}, \vec{w} \rangle + \partial_z \mathrm{Im}\,(u) + f\right). \end{cases}$$

Recalling that

$$
\begin{cases}
\partial_r \mathrm{Re}\,(u) = \quad \cos(\theta)\,\partial_x \mathrm{Re}\,(u) + \sin(\theta)\,\partial_y \mathrm{Re}\,(u) \\
\dfrac{1}{r}\partial_\theta \mathrm{Re}\,(u) = -\sin(\theta)\,\partial_x \mathrm{Re}\,(u) + \cos(\theta)\,\partial_y \mathrm{Re}\,(u),
\end{cases}
$$

we get

$$
\partial_r \mathrm{Re}\,(u) = 2\cos(\theta)\mathrm{Im}\,\left(\langle \partial_z \vec{v}, \vec{w}\rangle + \partial_z \mathrm{Im}\,(u) + f\right) + 2\sin(\theta)\mathrm{Re}\,\left(\langle \partial_z \vec{v}, \vec{w}\rangle + \partial_z \mathrm{Im}\,(u) + f\right)
$$
$$
= \frac{2}{|z|}\mathrm{Im}\,\left(\langle z\,\partial_z \vec{v}, \vec{w}\rangle + z\,\partial_z \mathrm{Im}\,(u) + zf(z)\right). \tag{3.132}
$$

Now, notice that

$$
\frac{2}{|z|}\mathrm{Im}\,\left(\langle z\,\partial_z \vec{v}, \vec{w}\rangle\right) = \mathrm{Im}\,\left(\langle (\cos(\theta) + i\sin(\theta))\,(\partial_x - i\,\partial_y)\,\mathrm{Re}\,(\vec{v}), \vec{w}\rangle\right)
$$
$$
+ \mathrm{Im}\,\left(i\,\langle (\cos(\theta) + i\sin(\theta))\,(\partial_x - i\,\partial_y)\,\mathrm{Im}\,(\vec{w}), \vec{w}\rangle\right)
$$
$$
= \langle \sin(\theta)\partial_x\,\mathrm{Re}\,(\vec{v}) - \cos(\theta)\partial_y\mathrm{Re}\,(\vec{v}), \vec{w}\rangle + \langle \cos(\theta)\partial_x\mathrm{Im}\,(\vec{v}) + \sin(\theta)\partial_y\mathrm{Im}\,(\vec{v}), \vec{w}\rangle
$$
$$
= -\frac{1}{r}\langle \partial_\theta\mathrm{Re}\,(\vec{v}), \vec{w}\rangle + \langle \partial_r\mathrm{Im}\,(\vec{v}), \vec{w}\rangle.
$$

Using (3.132), the same computation for $z\,\partial_z\mathrm{Im}\,(u)$ yields

$$
\partial_r\,\mathrm{Re}\,(u) = -\frac{1}{r}\langle \partial_\theta\mathrm{Re}\,(\vec{v}), \vec{w}\rangle + \frac{1}{r}\partial_\theta\mathrm{Im}\,(u) + \langle \partial_r\mathrm{Im}\,(\vec{v}), \vec{w}\rangle + 2\,\mathrm{Im}\,\left(\frac{z}{|z|}f(z)\right). \tag{3.133}
$$

Therefore, we deduce that

$$
\frac{d}{dr}\mathrm{Re}\,(u)_r = \frac{1}{2\pi}\int_0^{2\pi}\partial_r\mathrm{Re}\,(u)(r,\theta)d\theta = -\frac{1}{2\pi}\int_0^{2\pi}\left\langle \frac{1}{r}\partial_\theta\mathrm{Re}\,(\vec{v}), \vec{w}\right\rangle d\theta + \frac{1}{2\pi r}\int_{\partial B(0,t)}\partial_\theta\mathrm{Im}\,(u)d\theta
$$
$$
+ \frac{1}{2\pi}\int_0^{2\pi}\langle \partial_r\mathrm{Im}\,(\vec{v}), \vec{w}\rangle + \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Im}\,\left(\frac{z}{|z|}f(z)\right)d\theta
$$
$$
= -\frac{1}{2\pi}\int_0^{2\pi}\left\langle \frac{1}{r}\partial_\theta\mathrm{Re}\,(\vec{v}), \vec{w} - \vec{w}_t\right\rangle d\theta + \frac{1}{2\pi}\int_0^{2\pi}\langle \partial_r\mathrm{Im}\,(\vec{v}), \vec{w}\rangle d\theta + \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Im}\,\left(\frac{z}{|z|}f(z)\right)d\theta.
$$

Notice that by the Cauchy-Schwarz inequality, we have for all $\varphi \in \mathrm{L}^2(\Omega,\mathbb{C})$

$$
\int_r^R\left|\int_0^{2\pi}\varphi\frac{d\theta}{2\pi}\right|^2\rho\,d\rho \le \frac{1}{2\pi}\int_r^R\int_0^{2\pi}|\varphi|^2\rho\,d\rho d\theta = \frac{1}{2\pi}\int_\Omega|\varphi|^2|dz|^2.
$$

Therefore, by the Minkowski inequality and (3.130) we have

$$
\left(\int_r^R\left|\frac{d}{d\rho}\mathrm{Re}\,(u)_\rho\right|^2\rho\,d\rho\right)^{\frac{1}{2}} \le \frac{C_0}{\sqrt{2\pi}}\left(\int_\Omega|\vec{w} - \vec{w}_{|z|}|^2\frac{|dz|^2}{|z|^2}\right)^{\frac{1}{2}} + \frac{1}{\sqrt{2\pi}}\left(\int_\Omega|\langle \partial_r\mathrm{Im}\,(\vec{v}), \vec{w}\rangle|^2|dz|^2\right)^{\frac{1}{2}}
$$
$$
+ \frac{1}{\sqrt{2\pi}}\left(\int_\Omega\left|\mathrm{Im}\,\left(\frac{z}{|z|}f(z)\right)\right|^2|dz|^2\right)^{\frac{1}{2}}
$$
$$
\le \frac{C_0}{\sqrt{2\pi}}\left(\int_\Omega|\vec{w} - \vec{w}_{|z|}|^2\frac{|dz|^2}{|z|^2}\right)^{\frac{1}{2}} + \frac{1}{\sqrt{2\pi}}\|\vec{w}\|_{\mathrm{L}^\infty(\Omega)}\|\nabla\mathrm{Im}\,(\vec{v})\|_{\mathrm{L}^2(\Omega)} + \frac{1}{\sqrt{2\pi}}\|f\|_{\mathrm{L}^2(\Omega)}.
$$

Now, by (3.97), we infer

$$
\|\vec{w} - \vec{w}_{|z|}\|_{\mathrm{L}^\infty(\partial B(0,|z|))} \le \binom{n}{k}\int_{\partial B(0,|z|)}|\nabla\vec{w}|d\mathcal{H}^1, \quad \text{for all } z \in B(0,1).
$$

Using the Cauchy-Schwarz inequality and the co-area formula twice, we deduce that

$$\int_{B_R\setminus\overline{B}_r(0)}|\vec{w}-\vec{w}_{|z|}|^2\frac{|dz|^2}{|z|^2}\leq\binom{n}{k}^2\int_{B_R\setminus\overline{B}_r(0)}\left(\int_{\partial B(0,|z|)}|\nabla\vec{w}|d\mathscr{H}^1\right)^2\frac{|dz|^2}{|z|^2}$$

$$\leq 2\pi\binom{n}{k}^2\int_{B_R\setminus\overline{B}_r(0)}\int_{\partial B(0,|z|)}|\nabla\vec{w}|^2\frac{|dz|^2}{|z|}=(2\pi)^2\binom{n}{k}^2\int_r^R\left(\int_{\partial B(0,t)}|\nabla\vec{w}|^2 d\mathscr{H}^1\right)dt$$

$$=(2\pi)^2\binom{n}{k}^2\int_{B_R\setminus\overline{B}_r(0)}|\nabla\vec{w}|^2|dz|^2.$$

We conclude that

$$\left(\int_r^R\left|\frac{d}{d\rho}\mathrm{Re}\,(u)_\rho\right|^2\rho\,d\rho\right)^{\frac{1}{2}}\leq\sqrt{2\pi}\binom{n}{k}C_0\|\nabla\vec{w}\|_{\mathrm{L}^2(\Omega)}+\frac{1}{\sqrt{2\pi}}\|\vec{w}\|_{\mathrm{L}^\infty(\Omega)}\|\nabla\mathrm{Im}\,(\vec{v})\|_{\mathrm{L}^2(\Omega)}+\frac{1}{\sqrt{2\pi}}\|f\|_{\mathrm{L}^2(\Omega)},$$

which concludes the proof of the lemma. $\qquad\qquad\square$

Applying Lemma 3.15 to S_k in the equation (3.127), we deduce that

$$\left(\int_{\alpha_0^{-1}r_k}^{\alpha R_k}\left|\frac{d}{d\rho}\mathrm{Re}\,(S_{k,\rho})\right|^2\rho\,d\rho\right)^{\frac{1}{2}}\leq n(n-1)\sqrt{\frac{\pi}{2}}C_3(n,\Lambda)\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}+\frac{1}{\sqrt{2\pi}}\left\|\nabla\mathrm{Im}\,(\vec{R}_k)\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}$$

$$+\frac{1}{\sqrt{2\pi}}\left\|\vec{R}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}.$$

Applying Lemma 3.15 to each of the $\frac{m(m-1)}{2}$ components of $\vec{R}_k$, we get by the Minkowski inequality that

$$\left(\int_{\alpha_0^{-1}r_k}^{\alpha_0 R_k}\left|\frac{d}{d\rho}\mathrm{Re}\,(\vec{R}_{k,\rho})\right|^2\rho\,d\rho\right)^{\frac{1}{2}}\leq n(n-1)\Gamma_1(n)C_3(n\Lambda)\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}$$

$$+\frac{n(n-1)}{2\sqrt{2\pi}}\left(\|\nabla\mathrm{Im}\,(S_k)\|_{\mathrm{L}^2(\Omega_k(\alpha))}+\left\|\nabla\mathrm{Im}\,(\vec{R}_k)\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}\right)+n(n-1)\sqrt{\frac{2}{\pi}}\left\|\vec{R}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}.$$

Recall now the following generalization (see [37]) of [7, Lemma VI.2] proved in [28] (see also [29]).

Lemma 3.16. *There exists a universal constant $R_0>0$ with the following property. Let $0<4r<R<R_0$, $\Omega=B(0,R)\setminus\overline{B}(0,r)\to\mathbb{R}$, $a,b:\Omega\to\mathbb{R}$ such that $\nabla a\in\mathrm{L}^{2,\infty}(\Omega)$ and $\nabla b\in\mathrm{L}^2(\Omega)$, and $\varphi:\Omega\to\mathbb{R}$ be a solution of*

$$\Delta\varphi=\nabla a\cdot\nabla^\perp b\qquad in\ \ \Omega.$$

For $r\leq\rho\leq R$, define

$$\varphi_\rho=\fint_{\partial B_\rho(0)}\varphi\,d\mathscr{H}^1=\frac{1}{2\pi\rho}\int_{\partial B_\rho(0)}\varphi\,d\mathscr{H}^1.$$

Then $\nabla\varphi\in\mathrm{L}^2(\Omega)$, and there exists a positive constant $C_0>0$ independent of $0<4r<R$ such that for all $\left(\frac{r}{R}\right)^{\frac{1}{2}}<\alpha<\frac{1}{2}$ it holds:

$$\|\nabla\varphi\|_{\mathrm{L}^2(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r})}\leq C_0\|\nabla a\|_{\mathrm{L}^{2,\infty}(\Omega)}\|\nabla b\|_{\mathrm{L}^2(\Omega)}+C_0\|\nabla\varphi_\rho\|_{\mathrm{L}^2(\Omega)}+C_0\|\nabla\varphi\|_{\mathrm{L}^{2,\infty}(\Omega)}.$$

From (3.128) we know that

$$\|S_k\|_{\mathrm{L}^2(\Omega_k(\alpha_0/2))}+\left\|\vec{R}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha_0/2))}\leq C.$$

Lemma 3.16, the system (3.127) and the Cauchy-Schwarz inequality imply that

$$\int_{2\alpha_0^{-1}r_k}^{\frac{\alpha_0 R_k}{2}} \left(\left| \frac{d}{d\rho} \mathrm{Re}\,(\vec{R}_{k,\rho}) \right| + \left| \frac{d}{d\rho} \mathrm{Re}\,(S_{k,\rho}) \right| \right) d\rho \leq C \int_{2\alpha_0^{-1}r_k}^{\frac{\alpha_0 R_k}{2}} \delta_k(\rho) \left(\int_{\partial B(0,\rho)} \left(|\nabla S_k| + |\nabla \vec{R}_k| \right) d\mathcal{H}^1 \right) d\rho$$

$$\leq C \left(\int_{2\alpha_0^{-1}r_k}^{\frac{\alpha_0 R_k}{2}} \delta_k^2(\rho)\rho\,d\rho \right)^{\frac{1}{2}} \left(\|S_k\|_{\mathrm{L}^2(\Omega_k(\alpha_0/2))} + \left\|\vec{R}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha_0/2))} \right) \leq C.$$

Furthermore, by the inequality (7.10), there exists $r_0 \in (\frac{\alpha_0 R_k}{4}, \frac{\alpha_0 R_k}{2})$ such that if $r = \frac{\alpha_0 R_k}{2}$, we have

$$\int_{\partial B_{r_0}(0)} \left(|S_k| + |\vec{R}_k| \right) d\mathcal{H}^1 \leq \frac{2\sqrt{3\pi}}{\log(2)} \left(\|S_k\|_{\mathrm{L}^{2,\infty}(B_{2r}\setminus\overline{B}_r(0))} + \left\|\vec{R}_k\right\|_{\mathrm{L}^{2,\infty}(B_{2r}\setminus\overline{B}_r(0))} \right)$$

$$\leq \frac{2\pi^{1-\frac{1}{2p}}(2r)^{1-\frac{1}{p}}}{\log(2)} \left(\|S_k\|_{\mathrm{L}^p(B_{2r}\setminus\overline{B}_r(0))} + \left\|\vec{R}_k\right\|_{\mathrm{L}^p(B_{2r}\setminus\overline{B}_r(0))} \right). \qquad (3.134)$$

Therefore, it holds

$$\left| \mathrm{Re}\,(\vec{R}_{k,r}) - \mathrm{Re}\,(\vec{R}_{k,r_0}) \right| + |\mathrm{Re}\,(S_{k,r}) - \mathrm{Re}\,(S_{k,r_0})| \leq C_n \int_{2\alpha_0^{-1}r_k}^{r_0} \left(\left| \frac{d}{d\rho} \mathrm{Re}\,(\vec{R}_{k,\rho}) \right| + \left| \frac{d}{d\rho} \mathrm{Re}\,(S_{k,\rho}) \right| \right) d\rho \leq C$$

and

$$|\mathrm{Re}\,(\vec{R}_{k,r})| + |\mathrm{Re}\,(S_{k,r})| \leq C + \frac{C}{(\alpha_0 R_k)^{\frac{1}{p}}} \left(\left\|\vec{R}_k\right\|_{\mathrm{L}^p(\Omega_k(\alpha_0/2))} + \|S_k\|_{\mathrm{L}^p(\Omega_k(\alpha_0/2))} \right).$$

In particular, we get that $\mathrm{Re}\,(\vec{R}_k)_\rho, \mathrm{Re}\,(S_k)_\rho \in \mathrm{L}^\infty([4\alpha_0^{-1}R_k, \frac{\alpha_0 R_k}{4}])$. We deduce that

$$\left| \mathrm{Re}\left(\vec{R}_k\right)(z) - \mathrm{Re}\left(\vec{R}_k\right)_{|z|} \right| \leq \frac{n(n-1)}{2} \int_{\partial B(0,|z|)} |\nabla \mathrm{Re}\left(\vec{R}_k\right)|\,d\mathcal{H}^1$$

$$\leq \frac{n(n-1)}{2} \int_{\partial B(0,|z|)} |\nabla \mathrm{Im}\,(\vec{R}_k)|\,d\mathcal{H}^1 + n(n-1) \int_{\partial B(0,|z|)} e^{\lambda_k} \left(|\vec{L}_k| + |\vec{H}_k| \right) d\mathcal{H}^1$$

$$+ C|z|^{d_k} \int_{\partial B(0,|z|)} \left| \mathrm{Re}\,(\vec{R}_k) \right| d\mathcal{H}^1 + C|z|^{d_k} \int_{\partial B(0,|z|)} |\mathrm{Im}\,(\vec{R}_k)|\,d\mathcal{H}^1$$

$$\leq C + |z|^{d_k} \int_{\partial B(0,|z|)} |\widetilde{W}_k|\,d\mathcal{H}^1 + C|z|^{d_k} \int_{\partial B(0,|z|)} |\mathrm{Re}\,(\vec{R}_k)|\,d\mathcal{H}^1. \qquad (3.135)$$

Now, easy scaling considerations show that for all $u \in \mathrm{W}^{1,1}(\mathbb{C})$, there exists a universal constant $C_1 < \infty$ such that for all $0 < r < \infty$,

$$\|u\|_{\mathrm{L}^1(\partial B(0,r))} \leq C_1 \left(\|\nabla u\|_{\mathrm{L}^1(B(0,r))} + \frac{1}{r} \|u\|_{\mathrm{L}^1(B(0,r))} \right). \qquad (3.136)$$

Assuming that we moreover have $u \in \mathrm{W}^{1,(2,\infty)}(\mathbb{C})$, we deduce by the Sobolev embedding that $u \in \mathrm{L}^p(\mathbb{C})$ for all $p < \infty$, which implies by Hölder's inequality that for all $\varepsilon > 0$, we have for some universal constant $\Gamma_\varepsilon < \infty$

$$\int_{B(0,r)} |u|dx \leq \|1\|_{\mathrm{L}^{\frac{1}{1-\varepsilon}}(B(0,r))} \|u\|_{\mathrm{L}^{\frac{1}{\varepsilon}}(B(0,r))} \leq \Gamma_\varepsilon r^{2(1-\varepsilon)} \|\nabla u\|_{\mathrm{L}^{2,\infty}(\mathbb{C})}. \qquad (3.137)$$

Now, using the $\mathrm{L}^{2,1}/\mathrm{L}^{2,\infty}$ duality, we get

$$\int_{B(0,r)} |\nabla u|dx \leq \|1\|_{\mathrm{L}^{2,1}(B(0,r))} \|\nabla u\|_{\mathrm{L}^{2,\infty}(B(0,r))} \leq 4\sqrt{\pi}\,r \|\nabla u\|_{\mathrm{L}^{2,\infty}(\mathbb{C})}. \qquad (3.138)$$

Finally, putting together (3.137) and (3.138), we deduce that for all $0 < r < \infty$ and for all $\varepsilon > 0$, there exists $C_\varepsilon < \infty$ such that

$$\|u\|_{\mathrm{L}^1(B(0,r))} \leq C_\varepsilon r^{1-\varepsilon} \|\nabla u\|_{\mathrm{L}^{2,\infty}(\mathbb{C})}. \qquad (3.139)$$

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Therefore, using that and $\widetilde{W}_k \in W^{1,(2,\infty)}(\mathbb{C})$ (making a controlled $W^{1,(2,\infty)}$ extension to $\mathbb{C}$ of our function defined on $B(0,\alpha_0 R_k)$), we deduce that

$$\int_{\partial B(0,|z|)} |\widetilde{W}_k| d\mathscr{H}^1 \leq C_\varepsilon' |z|^{1-\varepsilon} . \tag{3.140}$$

Since $d_k > -1 + \varepsilon$, we deduce by the L^∞ estimate for the means of $\mathrm{Re}\,(\vec{R}_k)$ and the combination of (3.135) and (3.140) that

$$|\mathrm{Re}\,(\vec{R}_k)(z)| \leq C + C|z|^{d_k} \int_{\partial B(0,|z|)} |\mathrm{Re}\,(\vec{R}_k)| d\mathscr{H}^1 .$$

Integrating this identity and using $d_k > -1 + \varepsilon$, we deduce that

$$\int_{\partial B(0,|z|)} |\mathrm{Re}\,(\vec{R}_k)| d\mathscr{H}^1 \leq 2\pi C |z| + 2\pi C |z|^{1+d_k} \int_{\partial B(0,|z|)} |\mathrm{Re}\,(\vec{R}_k)| d\mathscr{H}^1$$

$$\leq C|z| + C(\alpha_0 R_k)^\varepsilon \int_{\partial B(0,|z|)} |\mathrm{Re}\,(\vec{R}_k)| d\mathscr{H}^1, \tag{3.141}$$

which shows that for $\alpha_0 > 0$ small enough, we have

$$\int_{\partial B(0,|z|)} |\mathrm{Re}\,(\vec{R}_k)| d\mathscr{H}^1 \leq C|z|. \tag{3.142}$$

We conclude that $\mathrm{Re}\,(\vec{R}_k) \in L^\infty(B(0,\alpha_0 R_k))$. A similar argument (easier since we have a ∂_z equation and not a ∇_z one) finally yields that

$$\|S_k\|_{L^\infty(\Omega_k(\alpha_0/4))} + \left\|\vec{R}_k\right\|_{L^\infty(\Omega_k(\alpha_0/4))} \leq C. \tag{3.143}$$

To complete the proof, recall from [42] that in $B(0,\alpha_0 R_k)$ it holds:

$$\begin{cases} \Delta\left(\mathrm{Re}\,(\vec{R}_k)\right) = (-1)^n \star_h \left(\nabla\vec{n}_k \lrcorner \nabla^\perp \mathrm{Re}\,(\vec{R}_k)\right) - \star_h \left(\nabla\vec{n}_k \lrcorner \nabla^\perp (\mathrm{Re}\,(S_k))\right) + \vec{G}_{1,k} \\ \Delta\left(\mathrm{Re}\,(S_k)\right) = \langle \nabla(\star_h \vec{n}_k), \nabla^\perp \mathrm{Re}\,(\vec{R}_k)\rangle + G_{2,k} \end{cases}$$

for some $\vec{G}_{1,k}$ and $G_{2,k}$ which are bounded in $L^p(B(0,\alpha_0 R_k))$, for all $1 \leq p < 2$.

Recall also the following slight variant (see [37, Lemma 4.5]) from [28, Lemma 7].

Lemma 3.17. *Let $R_0 > 0$ be the constant of Lemma 3.16. Let $0 < 16r < R < R_0$, $\Omega = B(0,R) \setminus \overline{B}(0,r) \to \mathbb{R}$, $a,b : \Omega \to \mathbb{R}$ such that $\nabla a \in L^2(\Omega)$ and $\nabla b \in L^2(\Omega)$, and $\varphi : \Omega \to \mathbb{R}$ be a solution of*

$$\Delta\varphi = \nabla a \cdot \nabla^\perp b \quad \text{in } \Omega.$$

Assume that $\|\varphi\|_{L^\infty(\partial\Omega)} < \infty$. Then there exists a constant $C_1 > 0$ such that for all $\left(\frac{r}{R}\right)^{\frac{1}{2}} < \alpha < \frac{1}{4}$,

$$\|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^{2,1}(B_{\alpha R} \setminus \overline{B}_{\alpha^{-1} r}(0))} + \left\|\nabla^2\varphi\right\|_{L^1(B_{\alpha R} \setminus \overline{B}_{\alpha^{-1} r}(0))} \leq C_1 \left(\|\nabla a\|_{L^2(\Omega)} \|\nabla b\|_{L^2(\Omega)} + \|\varphi\|_{L^\infty(\partial\Omega)}\right).$$

From Lemma 3.17, we deduce that

$$\|\nabla S_k\|_{L^{2,1}(\Omega_k(\alpha_0/2))} + \left\|\nabla\vec{R}_k\right\|_{L^{2,1}(\Omega_k(\alpha_0/2))} + \left\|\nabla^2 S_k\right\|_{L^1(\Omega_k(\alpha_0/2))} + \left\|\nabla^2\vec{R}_k\right\|_{L^1(\Omega_k(\alpha_0/2))} \leq C.$$

Since $\mathrm{Im}\,(\vec{L}_k) \in W^{1,(2,\infty)}(B(0,\alpha_0 R_k))$ we deduce that $e^{\lambda_k} \mathrm{Im}\,(\vec{L}_k) \in L^{2+\varepsilon}(B(0,\alpha_0 R_k))$. Using the identity

$$e^{\lambda_k} \vec{H}_k = -\mathrm{Im}\left(\nabla_z \vec{R}_k \llcorner e^{-\lambda_k} \partial_{\bar{z}} \vec{\Phi}_k\right) - \frac{1}{2} e^{\lambda_k} \mathrm{Im}\,(\vec{L}_k)$$
$$- \mathrm{Re}\left(i\, e^{-\lambda_k} \partial_{\bar{z}} \vec{\Phi}_k \, \partial_z S_k\right) + \mathrm{Re}\left(\langle\partial_z\vec{\Phi}_k, \mathrm{Im}\,(\vec{L}_k)\rangle e^{-\lambda} \partial_{\bar{z}}\vec{\Phi}_k\right), \tag{3.144}$$

we finally get

$$\left\|e^{\lambda_k}\vec{H}_k\right\|_{L^{2,1}(\Omega_k(\alpha_0/4))} \leq C$$

which concludes the proof of Theorem 3.2. $\qquad\square$

4   Weak ε-regularity for Willmore Immersions with Values into Manifolds

.

In this section, by following [42] (that constitutes an alternative approach to Rivière's original strategy [53]), we generalise the ε-regularity for Willmore surfaces to the case of immersions with values into Riemannian manifolds (see Theorem 4.1). The ε-regularity is needed in the previous section to obtain pointwise estimates for $\vec{H}$ and $\nabla \vec{H}$ in annuli. For other ε-regularity results on the Willmore energy, we refer the reader to [48, Theorem 4.1], [36], [6].

This section is technically much easier than the previous one because we work on disks and therefore do not have to deal with singularities and improved Wente-type estimates.

By [42, Lemma 3.2 and Theorem 3.1], the following identities are satisfied for any smooth immersion $\vec{\Phi} : \Sigma \to (M^m, h)$

$$\vec{Y} = i \left(\nabla_z \vec{H} - 3 \nabla_z^\perp \vec{H} - i \star_h (\nabla_z \vec{n} \wedge \vec{H}) \right) = -2\, i \left(\nabla_z^\perp \vec{H} + \langle \vec{H}, \vec{H}_0 \rangle \partial_{\overline{z}} \vec{\Phi} \right)$$

$$4\, e^{-2\lambda} \mathrm{Re} \left(\nabla_{\overline{z}} \left(\nabla_z^\perp \vec{H} + \langle \vec{H}, \vec{H}_0 \rangle \partial_{\overline{z}} \vec{\Phi} \right) \right) = \Delta_g^\perp \vec{H} - 2|\vec{H}|^2 \vec{H} + \mathscr{A}(\vec{H}) + 8\, \mathrm{Re} \left(\langle R(\vec{e_{\overline{z}}}, \vec{e_z}) \vec{e_z}, \vec{H} \rangle \vec{e_{\overline{z}}} \right).$$

We deduce that

$$\mathrm{Im} \left(\nabla_{\overline{z}} \vec{Y} \right) = \mathrm{Im} \left(-2\, i\, \nabla_{\overline{z}} \left(\nabla_z^\perp \vec{H} + \langle \vec{H}, \vec{H}_0 \rangle \partial_{\overline{z}} \vec{\Phi} \right) \right) = -2\, \mathrm{Re} \left(\nabla_{\overline{z}} \left(\nabla_z^\perp \vec{H} + \langle \vec{H}, \vec{H}_0 \rangle \partial_{\overline{z}} \vec{\Phi} \right) \right)$$

$$= -\frac{1}{2} e^{2\lambda} \left(\Delta_g^\perp \vec{H} - 2|\vec{H}|^2 \vec{H} + \mathscr{A}(\vec{H}) \right) - 4 e^{2\lambda} \mathrm{Re} \left(\langle R(\vec{e_{\overline{z}}}, \vec{e_z}) \vec{e_z}, \vec{H} \rangle \vec{e_{\overline{z}}} \right).$$

Assuming that $\vec{\Phi}$ is a Willmore immersion, from (1.1) we deduce that

$$\mathrm{Im} \left(\nabla_{\overline{z}} \vec{Y} \right) = \frac{1}{2} e^{2\lambda} \left(\mathscr{R}_1^\perp(\vec{H}) - 2\, \widetilde{K}_h\, \vec{H} + 2\, \mathscr{R}_2(d\vec{\Phi}) + (DR)(d\vec{\Phi}) - 8\, \mathrm{Re} \left(\langle R(\vec{e_{\overline{z}}}, \vec{e_z}) \vec{e_z}, \vec{H} \rangle \vec{e_{\overline{z}}} \right) \right). \quad (4.1)$$

As before, let $f : \mathbb{C}^m \to \mathbb{C}^m$ be the linear map such that for all $X \in C^\infty(B(0,1), \mathbb{C}^m)$, it holds

$$\nabla_z \vec{X} = \partial_z \vec{X} + f(\vec{X}) = \partial_z \vec{X} + \left(\sum_{l=1}^m \gamma_l^j \vec{X}^l \right)_{1 \leq j \leq m},$$

where $\gamma_l^j = \sum_{q=1}^m \Gamma_{l,q}^j \partial_z \vec{\Phi}^q$ and $\Gamma_{l,q}^j$ are the Christoffel symbols of the ambient space (M^m, h).

Theorem 4.1. *Let (M^m, h) be a smooth closed Riemannian manifold. Assume that $(M^m, h) \hookrightarrow \mathbb{R}^n$ is isometrically embedded into $\mathbb{R}^n$. There exists constants $\varepsilon_0 > 0$, and constants $\{C_k\}_{k \in \mathbb{N}} \subset (0, \infty)$ with the following property. Let $\vec{\Phi} : B(0,1) \to (M^m, h)$ be a conformal Lipschitz map satisfying the weak form of the Willmore equation (4.1). Denote by $\vec{n} : B(0,1) \to \Lambda^{m-2} TM^m$ the Gauss map associated to $\vec{\Phi}$ and assume that*

$$\mathrm{Area}(\vec{\Phi}(B(0,1))) + \int_{B(0,1)} |\nabla \vec{n}|^2 dx \leq \varepsilon_0. \quad (4.2)$$

Then, for all $k \in \mathbb{N}$, it holds

$$\left\| \nabla^k \vec{n} \right\|_{L^\infty(B(0, \frac{1}{2}))}^2 \leq C_k \left(\mathrm{Area}(\vec{\Phi}(B(0,1))) + \int_{B(0,1)} |\nabla \vec{n}|^2 dx \right). \quad (4.3)$$

Proof. We will follow the proof of the regularity of weak Willmore immersions in manifolds [42, Theorem 6.1], replacing the main technical hypothesis by the weaker (4.2)—notice that the latter is exactly the assumption originally used in Rivière's work [53], once we replace the Euclidean ambient space by a sphere.

Step 0: Weak L^2 estimate of the conformal factor of $\vec{\Phi}$ (refer to [55, Theorem 1.1] for a similar proof). Notice that no assumptions on the smallness of a norm are necessary here.

Let $\lambda = \log\left(\frac{|\nabla\vec{\Phi}|^2}{2}\right)$ be the conformal factor of $\vec{\Phi}$. First, the Liouville equation implies that

$$-\Delta\lambda = e^{2\lambda}K_g. \tag{4.4}$$

Let u be the solution of

$$\begin{cases} -\Delta u = e^{2\lambda}K_g & \text{in } B(0,1) \\ \quad u = 0 & \text{on } \partial B(0,1). \end{cases}$$

Considering $\vec{\Phi}$ as a $\mathbb{R}^n$-valued map (using Nash embedding theorem as described in Section 2), if $\widetilde{n} : B(0,1) \to \Lambda^2\mathbb{R}^n$ is its unit normal, by standard elliptic regularity (see [17, Théorème 3.3.6]), we have

$$\|\nabla u\|_{\mathrm{L}^{2,\infty}(B(0,1))} \le C \left\| e^{2\lambda}K_g \right\|_{\mathrm{L}^1(B(0,1))} \le \frac{C}{2} \left\| \nabla\widetilde{n} \right\|_{\mathrm{L}^2(B(0,1))}$$
$$\le C'\left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \|\nabla\vec{n}\|_{\mathrm{L}^2(B(0,1))} \right). \tag{4.5}$$

By the Liouville equation, we deduce that $v = \lambda - u$ is a harmonic map. Therefore, the Caccioppoli inequality implies that

$$\|\nabla v\|_{\mathrm{L}^2(B(0,\frac{3}{4}))} \le \|v\|_{\mathrm{L}^2(B(0,\frac{5}{6}))}. \tag{4.6}$$

Now, by the mean-value formula, we have for all $x \in B(0,\frac{5}{6})$

$$v(x) = \fint_{B(x,\frac{1}{10})} v(y)dy,$$

which implies that

$$(v(x))^2 \le 2\left(\fint_{B(x,\frac{1}{10})} \lambda(y)dy \right)^2 + 2\left(\fint_{B(x,\frac{1}{10})} u(y)dy \right)^2$$
$$\le 2\left(\fint_{B(x,\frac{1}{10})} \lambda(y)dy \right)^2 + \frac{200}{\pi} \|u\|_{\mathrm{L}^2(B(0,1))}^2, \tag{4.7}$$

where we used the Cauchy-Schwarz inequality in the second inequality. Since u has zero trace on $\partial B(0,1)$, the Sobolev inequality implies that

$$\|u\|_{\mathrm{L}^2(B(0,1))} \le C \|\nabla u\|_{\mathrm{L}^{2,\infty}(B(0,1))} \le C\left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \|\nabla\vec{n}\|_{\mathrm{L}^2(B(0,1))} \right). \tag{4.8}$$

Now, Jensen's inequality implies by the convexity of the exponential function that

$$\exp\left(2\fint_{B(x,\frac{1}{10})} \lambda(y)dy \right) \le \fint_{B(x,\frac{1}{10})} e^{2\lambda(y)}dy \le \frac{100}{\pi} \int_{B(0,1)} e^{2\lambda(y)}dy.$$

Therefore, for all $x \in B(0,\frac{5}{6})$, we have

$$2\left(\fint_{B(x,\frac{1}{10})} \lambda(y)dy \right)^2 \le \frac{1}{2}\log\left(\frac{100}{\pi}\mathrm{Area}(\vec{\Phi}(B(0,1))) \right)^2, \tag{4.9}$$

and finally, (4.7), (4.8) and (4.9) show that

$$\|v\|_{\mathrm{L}^2(B(0,\frac{5}{6}))} \le C\left(\log\left(\frac{100}{\pi}\right) + \log\mathrm{Area}(\vec{\Phi}(B(0,1))) + \sqrt{\frac{200}{\pi}} \|u\|_{\mathrm{L}^2(B(0,1))} \right)$$

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$$\leq C\left(\log\left(\frac{100}{\pi}\right) + \log\mathrm{Area}(\vec{\Phi}(B(0,1))) + \mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \|\nabla\vec{n}\|_{\mathrm{L}^2(B(0,1))}\right). \quad (4.10)$$

Finally, by (4.5), (4.6) and (4.10) we deduce that

$$\|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(B(0,\frac{3}{4}))} \leq \|\nabla u\|_{\mathrm{L}^{2,\infty}(B(0,\frac{3}{4}))} + \|\nabla v\|_{\mathrm{L}^2(B(0,\frac{3}{4}))}$$

$$\leq C\left(1 + \left(\mathrm{Area}(\vec{\Phi}(B(0,1)))\right)^{\frac{1}{2}} + \|\nabla\vec{n}\|_{\mathrm{L}^2(B(0,1))} + \log\mathrm{Area}(\vec{\Phi}(B(0,1)))\right). \quad (4.11)$$

Step 1: Harnack inequality for the conformal parameter.

First, consider $\vec{\Phi}$ as a $\mathbb{R}^n$-valued map, and let $\widetilde{n} : B(0,1) \to \Lambda^{n-2}\mathbb{R}^n$ be the unit normal of $\vec{\Phi} : B(0,1) \to \mathbb{R}^n$. Then, we have by (2.2) the estimate

$$\int_{B(0,1)} |\nabla\widetilde{n}|^2 dx \leq 2\int_{B(0,1)} |\nabla\vec{n}|^2 dx + 2\|\mathbb{I}_{M^m}\|_{\mathrm{L}^\infty(M^m)}^2 \mathrm{Area}(\vec{\Phi}(B(0,1))) \leq 2(1+C_0^2)\varepsilon_0^2,$$

where $C_0 < \infty$ only depends on M^m. Therefore, provided that $2(1+C_0^2)\varepsilon_0^2 < \frac{8\pi}{3}$, we deduce that by Hélein's classical result ([17, Lemme (5.1.4)]), there exists a moving frame $(\vec{e}_1, \vec{e}_2) : B(0,1) \to S^{n-1}$ such that

$$\begin{cases} \widetilde{n} = \star(\vec{e}_1 \wedge \vec{e}_2) & \text{in } B(0,1) \\ \displaystyle\int_{B(0,1)} |\nabla\vec{e}_1|^2 dx + \int_{B(0,1)} |\nabla\vec{e}_2|^2 dx \leq 2\int_{B(0,1)} |\nabla\widetilde{n}|^2 dx. \end{cases}$$

Let $u \in \mathrm{W}^{1,1}(B(0,1), \mathbb{R})$ be the unique solution of the following Dirichlet boundary value problem:

$$\begin{cases} \Delta u = \nabla\vec{e}_1 \cdot \nabla^\perp\vec{e}_2 & \text{in } B(0,1) \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

Then, the improved Wente inequality ([17, Théorème (3.4.1)]) implies that

$$\|\nabla u\|_{\mathrm{L}^{2,1}(B(0,1))} + \|u\|_{\mathrm{L}^\infty(B(0,1))} \leq C_W \|\nabla\vec{e}_1\|_{\mathrm{L}^2(B(0,1))} \|\nabla\vec{e}_2\|_{\mathrm{L}^2(B(0,1))} \leq C_W \int_{B(0,1)} |\nabla\widetilde{n}|^2 dx. \quad (4.12)$$

If $v = \lambda - u$, the Sobolev embedding $\mathrm{W}^{1,(2,1)}(B(0,1)) \hookrightarrow L^\infty(B(0,1))$ and Wirtinger's inequality show that

$$\|v - \overline{v}\|_{\mathrm{L}^\infty(B(0,\frac{1}{2}))} \leq C \|\nabla v\|_{\mathrm{L}^{2,1}(B(0,\frac{1}{2}))}. \quad (4.13)$$

Since $v = \lambda - u$ is a harmonic map, there exists a universal constant $C < \infty$ such that

$$\|\nabla v\|_{\mathrm{L}^{2,1}(B(0,\frac{1}{2}))} \leq C \|\nabla v\|_{\mathrm{L}^{2,\infty}(B(0,\frac{3}{4}))}, \quad (4.14)$$

Therefore, (4.13) and (4.14) show that

$$\|v - \overline{v}\|_{\mathrm{L}^\infty(B(0,\frac{1}{2}))} \leq C \|\nabla v\|_{\mathrm{L}^{2,\infty}(B(0,\frac{3}{4}))}. \quad (4.15)$$

Thus, (4.12) and (4.15) imply that there exists $\overline{\lambda} \in \mathbb{R}$ such that

$$\left\|\lambda - \overline{\lambda}\right\|_{\mathrm{L}^\infty(B(0,\frac{1}{2}))} \leq C\left(\int_{B(0,1)} |\nabla\widetilde{n}|^2 dx + \|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(B(0,\frac{3}{4}))}\right)$$

$$\leq C\left(1 + \left(\mathrm{Area}(\vec{\Phi}(B(0,1)))\right)^{\frac{1}{2}} + \|\nabla\vec{n}\|_{\mathrm{L}^2(B(0,1))} + \log\mathrm{Area}(\vec{\Phi}(B(0,1)))\right), \quad (4.16)$$

where we used (4.11) in the second inequality. Finally, we deduce by (4.16)

$$\left\|e^\lambda\right\|_{\mathrm{L}^{2,1}(B(0,\frac{1}{2}))} \leq C\mathrm{Area}(\vec{\Phi}(B(0,1))^C \left\|e^{\overline{\lambda}}\right\|_{\mathrm{L}^{2,1}(B(0,\frac{1}{2}))} \leq C'e^{\overline{\lambda}} \leq C''\left\|e^\lambda\right\|_{\mathrm{L}^2(B(0,1))}. \quad (4.17)$$

Step 2: Diameter estimate.

Thanks to the Harnack inequality, we deduce that for all $z \in B(0,1)$, we have

$$|\vec{\Phi}(z) - \vec{\Phi}(0)| \leq \int_0^{|z|} e^{\lambda(r\frac{z}{|z|})} dr \leq C e^{\overline{\lambda}} |z| \leq C \mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} |z|.$$

Therefore, for some universal constant $C < \infty$, we have

$$\mathrm{diam}(\vec{\Phi}(B(0,1))) \leq C \mathrm{Area}(\vec{\Phi}(B(0,1))),$$

which implies that for $\varepsilon_0 > 0$ small enough, $\vec{\Phi}(B(0,1))$ is included in a geodesic domain of M^m, and that we can trivialise tensors in an exponential chart to get $\mathbb{R}^m$-valued maps.

Step 3: Construction of tensors $\vec{L} : B(0, \frac{1}{2}) \to \mathbb{C}^m$, $\vec{R} : B(0, \frac{1}{4}) \to \Lambda^2 \mathbb{C}^m$ and $S : B(0, \frac{1}{4}) \to \mathbb{C}$ coming from the almost conservation laws.

Let $\eta : \mathbb{C} \to \mathbb{R}_+$ be a smooth cut-off function such that $\eta = 1$ on $B(0, \frac{1}{2})$ and $\mathrm{supp}(\eta) \subset B(0,1)$. Let $\widetilde{\vec{Y}} = \eta \vec{Y}$, where

$$\vec{Y} = i \left(\nabla_z \vec{H} - 3 \nabla_z^\perp \vec{H} - \star_h (\nabla_z \vec{n} \wedge \vec{H}) \right).$$

Recall that we have

$$\nabla_z \vec{H} = \partial_z \vec{H} + f(\vec{H}),$$
$$\nabla_z \vec{n} = \partial_z \vec{n} + G(\vec{n})$$

where f and G are linear functions such that $|f(\vec{H})| \leq C e^\lambda |\vec{H}|$ and $|G(\vec{n})| \leq C e^\lambda$ for a universal constant $C > 0$. Therefore, noticing that

$$\nabla_z^\perp \vec{H} = \pi_{\vec{n}} \left(\partial_z \vec{H} + f(\vec{H}) \right) = \pi_{\vec{n}}(\partial_z \vec{H}) + f(\vec{H}) = \partial_z (\pi_{\vec{n}}(\vec{H})) + (\partial_z \pi_{\vec{n}})(\vec{H}) + f(\vec{H})$$
$$= \partial_z \vec{H} + (\partial_z \pi_{\vec{n}})(\vec{H}) + f(\vec{H}),$$

we can make the decomposition

$$\nabla_z \vec{H} - 3 \nabla_z^\perp \vec{H} - \star_h \left(\nabla_z \vec{n} \wedge \vec{H} \right) = -2 \partial_z \vec{H} - 3 (\partial_z \pi_{\vec{n}})(\vec{H}) - 2 f(\vec{H}) + \star_h \left(\partial_z \vec{n} \wedge \vec{H} \right) + \star_h (G(\vec{n}) \wedge \vec{H}).$$

Therefore, we have

$$\left\| -2 \partial_z \vec{H} \right\|_{\mathrm{H}^{-1}(B(0,1))} \leq C \left\| \vec{H} \right\|_{\mathrm{L}^2(B(0,1))},$$

while

$$\left\| -3 (\partial_z \pi_{\vec{n}})(\vec{H}) - 2 f(\vec{H}) + \star_h \left(\partial_z \vec{n} \wedge \vec{H} \right) + \star_h (G(\vec{n}) \wedge \vec{H}) \right\|_{\mathrm{L}^1(B(0,1))}$$
$$\leq C \left\| \nabla \vec{n} \right\|_{\mathrm{L}^2(B(0,1))} \left\| \vec{H} \right\|_{\mathrm{L}^2(B(0,1))} + C \left\| e^\lambda \vec{H} \right\|_{\mathrm{L}^1(B(0,1))}.$$

Therefore, we deduce that

$$\left\| \vec{Y} \right\|_{\dot{\mathrm{H}}^{-1} + \mathrm{L}^1(B(0,1))} \leq C \left(\left(1 + \left\| \nabla \vec{n} \right\|_{\mathrm{L}^2(B(0,1))} \right) \left\| \vec{H} \right\|_{\mathrm{L}^2(B(0,1))} + C \left\| e^\lambda \vec{H} \right\|_{\mathrm{L}^1(B(0,1))} \right).$$

Now, we estimate $\widetilde{\vec{Y}}$. Since $|\eta| \leq 1$, we have

$$\left\| \widetilde{\vec{Y}} - \eta \partial_z \vec{H} \right\|_{\dot{\mathrm{H}}^{-1} + \mathrm{L}^1(\mathbb{C})} \leq C \left\| \nabla \vec{n} \right\|_{\mathrm{L}^2(B(0,1))} \left\| \vec{H} \right\|_{\mathrm{L}^2(B(0,1))} + C \left\| e^\lambda \vec{H} \right\|_{\mathrm{L}^1(B(0,1))},$$

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while for all $\vec{\psi} \in C_c^\infty(\mathbb{C}, \mathbb{C}^m)$, we have

$$\int_{\mathbb{C}} \eta \, \partial_z \vec{H} \cdot \vec{\psi} \, |dz|^2 = - \int_{\mathbb{C}} \left(\vec{H} \cdot \vec{\psi} \, \partial_z \eta + \eta \, \vec{H} \cdot \partial_z \vec{\psi} \right) |dz|^2,$$

which implies that

$$\left\| \eta \, \partial_z \vec{H} \right\|_{H^{-1}(\mathbb{C})} \leq C \left(\left\| \vec{H} \right\|_{L^2(B_1 \setminus \overline{B}_{\frac{1}{2}}(0))} + \left\| \vec{H} \right\|_{L^2(B(0,1))} \right).$$

Finally, we deduce that

$$\left\| \widetilde{\vec{Y}} \right\|_{\dot{H}^{-1} + L^1(\mathbb{C})} \leq C \left(\left(1 + \| \nabla \vec{n} \|_{L^2(B(0,1))} \right) \left\| \vec{H} \right\|_{L^2(B(0,1))} + C \left\| e^\lambda \vec{H} \right\|_{L^1(B(0,1))} \right).$$

Therefore, provided that $\varepsilon_0 > 0$ is small enough, by **Step 2**, we can trivialise the tensors in a geodesic chart and apply Lemma 3.6 to deduce the existence of a unique $\widetilde{\vec{L}} \in L^{2,\infty}(\mathbb{C}, \mathbb{C}^m)$ such that

$$\nabla_z \widetilde{\vec{L}} = i \eta \left(\nabla_z \vec{H} - 3 \nabla_z^\perp \vec{H} - \star_h (\nabla_z \vec{n} \wedge \vec{H}) \right) \qquad \text{in } \mathbb{C}.$$

Furthermore, we have the estimate

$$\left\| \widetilde{\vec{L}} \right\|_{L^{2,\infty}(\mathbb{C})} \leq C \left(\left(1 + \| \nabla \vec{n} \|_{L^2(B(0,1))} \right) \left\| \vec{H} \right\|_{L^2(B(0,1))} + C \left\| e^\lambda \vec{H} \right\|_{L^1(B(0,1))} \right),$$

which implies by using twice the Harnack inequality that

$$\left\| e^\lambda \widetilde{\vec{L}} \right\|_{L^{2,\infty}(\mathbb{C})} \leq C \left(1 + \| \nabla \vec{n} \|_{L^2(B(0,1))} + \text{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} \right) \left\| e^\lambda \vec{H} \right\|_{L^2(B(0,1))}.$$

Using the identity

$$\text{Im}\left(\nabla_{\bar{z}} \vec{Y} \right) = \frac{1}{2} e^{2\lambda} \left(\mathscr{R}_1^\perp(\vec{H}) - 2 \, \widetilde{K}_h \, \vec{H} + 2 \, \mathscr{R}_2(d\vec{\Phi}) + (DR)(d\vec{\Phi}) - 8 \, \text{Re} \left(\langle R(\vec{e}_{\bar{z}}, \vec{e}_z) \vec{e}_z, \vec{H} \rangle \vec{e}_{\bar{z}} \right) \right),$$

by the Harnack inequality on the conformal parameter, we deduce that $\text{Im}\left(\nabla_{\bar{z}} \vec{Y} \right) \in L^2(B(0,1))$. Indeed, we have

$$\left\| \text{Im}\left(\nabla_{\bar{z}} \vec{Y} \right) \right\|_{L^2(B(0,1))} \leq C e^{\overline{\lambda}} \left(\text{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\| e^\lambda \vec{H} \right\|_{L^2(B(0,1))} \right)$$

$$\leq C \left(\text{Area}(\vec{\Phi}(B(0,1))) + \text{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} \left(\int_{B(0,1)} |\vec{H}|^2 d\text{vol}_g \right)^{\frac{1}{2}} \right).$$

Furthermore, the previous argument shows that

$$\left\| \partial_{\bar{z}} \eta \, \widetilde{\vec{Y}} \right\|_{\dot{H}^{-1} + L^1(\mathbb{C})} \leq C \left(\left(1 + \| \nabla \vec{n} \|_{L^2(B(0,1))} \right) \left\| \vec{H} \right\|_{L^2(B(0,1))} + C \left\| e^\lambda \vec{H} \right\|_{L^1(B(0,1))} \right).$$

Therefore, we deduce that

$$\left\| \text{Im}\left(\nabla_{\bar{z}} \widetilde{\vec{Y}} \right) \right\|_{\dot{H}^{-1} + L^1 + L^2(\mathbb{C})} \leq C \left(\text{Area}(\vec{\Phi}(B(0,1))) + \text{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} \left\| e^\lambda \vec{H} \right\|_{L^2(B(0,1))} \right)$$

$$+ C \left(\left(1 + \| \nabla \vec{n} \|_{L^2(B(0,1))} \right) \left\| \vec{H} \right\|_{L^2(B(0,1))} + C \left\| e^\lambda \vec{H} \right\|_{L^1(B(0,1))} \right).$$

Therefore, following the second half of the proof of [42, Lemma A.1], we deduce that there exists a unique $\vec{L} \in L^{2,\infty}(B(0, \frac{1}{2}), \mathbb{C}^m)$ such that $\widetilde{\vec{L}} \in L^{2,\infty}(\mathbb{C}, \mathbb{C}^m)$ such that

$$\begin{cases} \nabla_z \vec{L} = i \eta \left(\nabla_z \vec{H} - 3 \nabla_z^\perp \vec{H} - \star_h (\nabla_z \vec{n} \wedge \vec{H}) \right) & \text{in } B(0, 1/2) \\ \text{Im}\left(\vec{L} \right) = 0 & \text{on } \partial B(0, 1/2), \end{cases}$$

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which satisfies the estimate

$$\left\|e^\lambda \vec{L}\right\|_{\mathrm{L}^{2,\infty}(B(0,\frac{1}{2}))} + \left\|e^\lambda \nabla \mathrm{Im}\,(\vec{L})\right\|_{\mathrm{L}^{2,\infty}(B(0,\frac{1}{2}))} \leq C\left(1 + \|\nabla\vec{n}\|_{\mathrm{L}^2(B(0,1))} + \mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}}\right)\left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))}$$

$$+ C\,\mathrm{Area}(\vec{\Phi}(B(0,1)))\left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))}\right) \leq C\left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))}\right), \tag{4.18}$$

using the estimate (4.2). Therefore, we can now use Lemma 3.14 to deduce the existence of $S_k \in \mathrm{W}^{1,(2,\infty)}(B(0,\frac{1}{4}),\mathbb{C})$ such that

$$\begin{cases} \partial_z S = \left\langle \partial_z \vec{\Phi}, \overline{\vec{L}}\right\rangle & \text{in } B(0,1/4) \\ \mathrm{Im}\,(S) = 0 & \text{on } \partial B(0,1/4), \end{cases}$$

and

$$\|S\|_{\mathrm{W}^{1,(2,\infty)}(B(0,\frac{1}{4}))} \leq C\left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))}\right).$$

Notice that this equation implies that $\mathrm{Im}\,(S)$ satisfies the following system of equations

$$\begin{cases} \Delta \mathrm{Im}\,(S) = 4\,\mathrm{Im}\,\left(\partial_{\overline{z}}\left(\langle \partial_z \vec{\Phi}, \overline{\vec{L}}\rangle\right)\right) = -\dfrac{1}{2}e^{2\lambda}\left\langle \vec{H}, \mathrm{Im}\,\left(\vec{L}\right)\right\rangle & \text{in } B(0,1/4) \\ \mathrm{Im}\,(S) = 0 & \text{on } \partial B(0,1/4), \end{cases}$$

which implies by Calderón-Zygmund estimates that $\mathrm{Im}\,(S) \in \mathrm{W}^{2,q}(B(0,\frac{1}{4}))$ for all $q < 2$ and that

$$\|\mathrm{Im}\,(S)\|_{\mathrm{W}^{2,q}(B(0,\frac{1}{4}))} \leq C\left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,\frac{1}{2}))}\left\|e^\lambda \mathrm{Im}\,(S)\right\|_{\mathrm{L}^{q'}(B(0,\frac{1}{2}))}$$

$$\leq C\left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,\frac{1}{2}))} e^{\overline{\lambda}}\left\|\mathrm{Im}\,(\vec{L})\right\|_{\mathrm{L}^{q'}(B(0,\frac{1}{2}))}. \tag{4.19}$$

Since $\mathrm{Im}\,(\vec{L}) \in \mathrm{W}^{1,(2,\infty)}_0(B(0,\frac{1}{2}))$, we deduce by Sobolev embedding that for all $1 < q < 2$, there exists $C_q < \infty$ such that

$$\left\|\mathrm{Im}\,(\vec{L})\right\|_{\mathrm{L}^{q'}(B(0,\frac{1}{2}))} \leq C\left\|\nabla \mathrm{Im}\,(\vec{L})\right\|_{\mathrm{L}^{2,\infty}(B(0,\frac{1}{2}))} \leq Ce^{-\overline{\lambda}}\left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))}\right), \tag{4.20}$$

where we used (4.18). Putting together (4.19) and (4.20), we deduce that for all $1 \leq q < 2$, there exists $\Gamma_q < \infty$ such that

$$\|\mathrm{Im}\,(S)\|_{\mathrm{W}^{2,q}(B(0,\frac{1}{4}))} \leq \Gamma_q\left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))}\right)\left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))}$$

$$\leq \Gamma'_q\left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))}, \tag{4.21}$$

using the smallness hypothesis (4.2). Likewise, using Lemma 3.14, we deduce that there exists $\vec{R} \in \mathrm{W}^{1,(2,\infty)}(B(0,\frac{1}{4}))$ such that

$$\begin{cases} \nabla_z \vec{R} = \partial_z \vec{\Phi} \wedge \vec{L} - 2i\,\partial_z \vec{\Phi} \wedge \vec{H} & \text{in } B(0,1/4) \\ \mathrm{Im}\,(\vec{R}) = 0 & \text{on } \partial B(0,1/4), \end{cases} \tag{4.22}$$

and

$$\left\|\vec{R}\right\|_{\mathrm{W}^{1,(2,\infty)}(B(0,\frac{1}{4}))} \leq C\left(\left\|\partial_z \vec{\Phi} \wedge \vec{L} - 2i\,\partial_z \vec{\Phi} \wedge \vec{H}\right\|_{\mathrm{L}^1(B(0,\frac{1}{2}))} + \left\|\partial_z \vec{\Phi} \wedge \vec{L} - 2i\,\partial_z \vec{\Phi} \wedge \vec{H}\right\|_{\mathrm{L}^{2,\infty}(B(0,\frac{1}{2}))}\right)$$

$$\leq C\left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))}\right).$$

By [42, (6.8)] we have

$$\operatorname{Im}\left(\nabla_{\bar{z}}\left(\partial_z\vec{\Phi}\wedge\vec{L}-2i\,\partial_z\vec{\Phi}\wedge\vec{H}\right)\right)=-\frac{1}{2}e^{2\lambda}\vec{H}\wedge\operatorname{Im}\left(\vec{L}\right)\in\mathrm{L}^q(B(0,1/4))\qquad\text{for all }q<2,$$

which implies that

$$\begin{aligned}
\Delta\operatorname{Im}\left(\vec{R}\right)=4\,\partial_{\bar{z}}\partial_z\vec{R}&=4\operatorname{Im}\left(\partial_{\bar{z}}\left(\partial_z\vec{\Phi}\wedge\vec{L}-2i\,\partial_z\vec{\Phi}\wedge\vec{H}\right)\right)-4\operatorname{Im}\left(\partial_{\bar{z}}\left(F(\vec{R})\right)\right)\\
&=4\operatorname{Im}\left(\nabla_{\bar{z}}\left(\partial_z\vec{\Phi}\wedge\vec{L}-2i\,\partial_z\vec{\Phi}\wedge\vec{H}\right)\right)-4\operatorname{Im}\left(\overline{F}\left(\partial_z\vec{\Phi}\wedge\operatorname{Im}\left(\vec{L}\right)-2i\,\partial_z\vec{\Phi}\wedge\vec{H}\right)\right)\\
&\quad-4\operatorname{Im}\left(\partial_{\bar{z}}\left(F(\vec{R})\right)\right)\\
&=-2\,e^{2\lambda}\vec{H}\wedge\operatorname{Im}\left(\vec{L}\right)-4\operatorname{Im}\left(\overline{F}\left(\partial_z\vec{\Phi}\wedge\operatorname{Im}\left(\vec{L}\right)-2i\,\partial_z\vec{\Phi}\wedge\vec{H}\right)\right)-4\operatorname{Im}\left(\partial_{\bar{z}}\left(F(\vec{R})\right)\right).
\end{aligned}$$

We first estimate as previously

$$\left|\overline{F}\left(\partial_z\vec{\Phi}\wedge\operatorname{Im}\left(\vec{L}\right)-2i\,\partial_z\vec{\Phi}\wedge\vec{H}\right)\right|\le Ce^{2\lambda}\left(|\vec{L}|+|\vec{H}|\right)\in\mathrm{L}^q(B(0,1/4))\quad\text{for all }q<2.$$

Now, using (3.123),

$$\begin{aligned}
\operatorname{Im}\left(\partial_{\bar{z}}F(\vec{R})\right)=\frac{1}{4}\sum_{i,j,l=1}^m&\left(\langle\nabla c_{i,j,l},\nabla^\perp\vec{\Phi}_l\rangle\operatorname{Re}\left(\vec{R}_{i,j}\right)+\langle\nabla c_{i,j,l},\nabla\vec{\Phi}_l\rangle\operatorname{Im}\left(\vec{R}_{i,j}\right)\right.\\
&\left.+2\,e^\lambda\vec{H}\cdot e^\lambda\operatorname{Im}\left(\vec{R}_{i,j}\right)+c_{i,j,l}\left(-\left\langle\nabla\vec{\Phi}_l,\nabla^\perp\operatorname{Re}\left(\vec{R}_{i,j}\right)\right\rangle+\left\langle\nabla\vec{\Phi}_l,\nabla\operatorname{Im}\left(\vec{R}_{i,j}\right)\right\rangle\right)\right)e_i\wedge e_j,
\end{aligned}$$

we finally deduce by Calderón-Zygmund estimates that for all $q<2$, there exists $\Gamma_q''<\infty$ such that

$$\left\|\operatorname{Im}\left(\vec{R}\right)\right\|_{\mathrm{W}^{2,q}(B(0,\frac{1}{4}))}\le\Gamma_q''\left(\operatorname{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}}+\left\|e^\lambda\vec{H}\right\|_{\mathrm{L}^2(B(0,1))}\right).$$

Furthermore, $\vec{R}$ and S solve the system

$$\begin{cases}
\nabla_z\vec{R}=i\left((-1)^{n+1}\star_h\left(\vec{n}\lrcorner\nabla_z\vec{R}\right)+(\partial_z S)\star_h\vec{n}\right) & \text{in }B(0,1/4)\\
\partial_z S=-i\langle\nabla_z\vec{R},\star_h\vec{n}\rangle & \text{in }B(0,1/4).
\end{cases}\tag{4.23}$$

Step 4: Derivation of the Jacobian systems.

Since ∇ is the Levi-Civita connection associated to g, we deduce that

$$\partial_{z\bar{z}}^2 S=-i\,\partial_{\bar{z}}\langle\nabla_z\vec{R},\star_h\vec{n}\rangle=-i\langle\nabla_{\bar{z}}\nabla_z\vec{R},\star_h\vec{n}\rangle-i\langle\nabla_z\vec{R},\nabla_{\bar{z}}(\star_h\vec{n})\rangle.$$

Notice that

$$\Delta\vec{X}=2\left(\nabla_z\nabla_{\bar{z}}+\nabla_{\bar{z}}\nabla_z\right)\vec{X},$$

and that

$$\nabla_{\bar{z}}\nabla_z\vec{X}=\nabla_z\nabla_{\bar{z}}\vec{X}+R(\vec{e}_{\bar{z}},\vec{e}_z)\vec{X}.$$

In particular, we have

$$4\,\nabla_{\bar{z}}\nabla_z\vec{X}=\Delta\vec{X}+2\,R(\vec{e}_{\bar{z}},\vec{e}_z)\vec{X},$$

and

$$\begin{aligned}
4\operatorname{Re}\left(-i\langle\nabla_{\bar{z}}\nabla_z\vec{R},\star_h\vec{n}\rangle\right)&=\operatorname{Re}\left(-i\langle\Delta\vec{R},\star_h\vec{n}\rangle-2\,i\langle R(\vec{e}_{\bar{z}},\vec{e}_z)\vec{R},\star_h\vec{n}\rangle\right)\\
&=\langle\Delta\operatorname{Im}\left(\vec{R}\right),\star_h\vec{n}\rangle+\operatorname{Re}\left(-\frac{1}{2}i\langle R(\vec{e}_1+i\,\vec{e}_2,\vec{e}_1-i\,\vec{e}_2)\vec{R},\star_h\vec{n}\rangle\right)
\end{aligned}$$

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$$= \langle \Delta \mathrm{Im}\,(\vec{R}), \star_h \vec{n} \rangle + \mathrm{Re}\,\left(-\langle R(\vec{e}_1, \vec{e}_2)\vec{R}, \star_h \vec{n} \rangle \right)$$
$$= \langle \Delta \mathrm{Im}\,(\vec{R}), \star_h \vec{n} \rangle - \langle R(\vec{e}_1, \vec{e}_2)\mathrm{Re}\,(\vec{R}), \star_h \vec{n} \rangle. \tag{4.24}$$

Now, notice that for all function $a, b : B(0,1) \to \mathbb{R}$, it holds

$$4\,\mathrm{Re}\,(\partial_z a \cdot \partial_{\overline{z}} b) = \mathrm{Re}\,((\partial_{x_1} a - i\,\partial_{x_2} a)(\partial_{x_1} b + i\,\partial_{x_2} b)) = \partial_{x_1} a \cdot \partial_{x_1} b + \partial_{x_2} b \cdot \partial_{x_2} b = \nabla a \cdot \nabla b$$
$$4\,\mathrm{Re}\,(-i\,\partial_z a \cdot \partial_{\overline{z}} b) = \mathrm{Im}\,((\partial_{x_1} a - i\,\partial_{x_2} a)(\partial_{x_1} b + i\,\partial_{x_2} b)) = -\partial_{x_2} a \cdot \partial_{x_1} b + \partial_{x_1} a \cdot \partial_{x_2} b = \nabla^\perp a \cdot \nabla b,$$

since $\nabla^\perp a = (-\partial_{x_2} a, \partial_{x_1} a)$. In particular, for all $a : B(0,1) \to \mathbb{C}$ and $b : B(0,1) \to \mathbb{R}$, we have

$$4\,\mathrm{Re}\,(-i\,\partial_z a \cdot \partial_{\overline{z}} b) = 4\,\mathrm{Re}\,(-i\,\partial_z \mathrm{Re}\,(a) \cdot \partial_{\overline{z}} b) + 4\,\mathrm{Re}\,(\partial_z \mathrm{Im}\,(a) \cdot \partial_{\overline{z}} b) = \nabla^\perp \mathrm{Re}\,(a) \cdot \nabla b + \nabla \mathrm{Im}\,(a) \cdot \nabla b.$$

Therefore, we have

$$\Delta \mathrm{Re}\,(S) = 4\,\mathrm{Re}\,\left(-i\langle \nabla_z \vec{R}, \nabla_{\overline{z}}(\star_h \vec{n}) \rangle \right) + 4\,\mathrm{Re}\,\left(-i\langle \nabla_{\overline{z}} \nabla_z \vec{R}, \star_h \vec{n} \rangle \right)$$
$$= \langle \nabla^\perp \mathrm{Re}\,(\vec{R}), \nabla(\star_h \vec{n}) \rangle + \langle \nabla \mathrm{Im}\,(\vec{R}), \nabla(\star_h \vec{n}) \rangle + \langle \Delta \mathrm{Im}\,(\vec{R}), \star_h \vec{n} \rangle - \langle R(\vec{e}_1, \vec{e}_2)\mathrm{Re}\,(\vec{R}), \star_h \vec{n} \rangle. \tag{4.25}$$

Likewise, it holds

$$\nabla_{\overline{z}} \nabla_z \vec{R} = i(-1)^{n+1}\left(\nabla_{\overline{z}} \vec{n} \lrcorner \nabla_z \vec{R} \right) + (-1)^{n+1}\left(\star_h \vec{n} \lrcorner i\nabla_{\overline{z}} \nabla_z \vec{R} \right) + i\,\partial^2_{z\overline{z}} S \star_h \vec{n} + i\,\partial_z S \cdot \nabla_{\overline{z}}(\star_h \vec{n}),$$

and by what precedes, we have

$$4\,\mathrm{Re}\,\left(i\left(\nabla_{\overline{z}} \vec{n} \lrcorner \nabla_z \vec{R} \right) \right) = -\left(\nabla \vec{n} \lrcorner \nabla^\perp \mathrm{Re}\,(\vec{R}) \right) - \left(\nabla \vec{n} \lrcorner \mathrm{Im}\,(\vec{R}) \right).$$

We infer that

$$4\,\mathrm{Re}\,\left(\nabla_{\overline{z}} \nabla_z \vec{R} \right) = (-1)^m \left(\nabla \vec{n} \lrcorner \nabla^\perp \mathrm{Re}\,(\vec{R}) \right) + (-1)^m \left(\nabla \vec{n} \lrcorner \mathrm{Im}\,(\vec{R}) \right) + 4(-1)^{n+1}\mathrm{Re}\,\left(\star_h \vec{n} \lrcorner i\nabla_{\overline{z}} \nabla_z \vec{R} \right)$$
$$- \Delta \mathrm{Im}\,(S)(\star_h \vec{n}) - \nabla^\perp \mathrm{Re}\,(S) \cdot \nabla(\star_h \vec{n}) - \nabla \mathrm{Im}\,(S) \cdot \nabla(\star_h \vec{n}).$$

We also have

$$4\,\mathrm{Re}\,\left(\nabla_{\overline{z}} \nabla_z \vec{R} \right) = 2\,\mathrm{Re}\,\left(\nabla_{\overline{z}} \nabla_z \vec{R} + \nabla_z \nabla_{\overline{z}} \vec{R} \right) + 2\,\mathrm{Re}\,\left(R(\vec{e}_{\overline{z}}, \vec{e}_z)\vec{R} \right)$$
$$= \Delta \mathrm{Re}\,(\vec{R}) + \frac{1}{2}\,\mathrm{Re}\,\left(R(\vec{e}_1 + i\,\vec{e}_2, \vec{e}_1 - i\,\vec{e}_2)\vec{R} \right)$$
$$= \Delta \mathrm{Re}\,(\vec{R}) + R(\vec{e}_1, \vec{e}_2)\mathrm{Im}\,(\vec{R}).$$

Therefore, we infer that

$$\Delta \vec{R} = (-1)^m \left(\nabla \vec{n} \lrcorner \nabla^\perp \mathrm{Re}\,(\vec{R}) \right) + (-1)^m \left(\nabla \vec{n} \lrcorner \nabla \mathrm{Im}\,(\vec{R}) \right) + 4(-1)^{n+1}\mathrm{Re}\,\left(\star_h \vec{n} \lrcorner i\nabla_{\overline{z}} \nabla_z \vec{R} \right)$$
$$- \Delta \mathrm{Im}\,(S)(\star_h \vec{n}) - \nabla^\perp \mathrm{Re}\,(S) \cdot \nabla(\star_h \vec{n}) - \nabla \mathrm{Im}\,(S) \cdot \nabla(\star_h \vec{n}) - R(\vec{e}_1, \vec{e}_2)\mathrm{Im}\,(\vec{R}).$$

Now, by (4.24) we get that

$$4(-1)^{n+1}\mathrm{Re}\,\left(\star_h \vec{n} \lrcorner i\nabla_{\overline{z}} \nabla_z \vec{R} \right) = (-1)^m \left(\star_h \vec{n} \lrcorner \Delta \mathrm{Im}\,(\vec{R}) \right) + (-1)^{n+1}\left(\star_h \vec{n} \lrcorner R(\vec{e}_1, \vec{e}_2)\mathrm{Re}\,(\vec{R}) \right),$$

and we finally deduce that

$$\Delta \vec{R} = (-1)^m \left(\nabla \vec{n} \lrcorner \nabla^\perp \mathrm{Re}\,(\vec{R}) \right) - \nabla^\perp \mathrm{Re}\,(S) \cdot \nabla(\star_h \vec{n}) + (-1)^m \left(\nabla \vec{n} \lrcorner \nabla \mathrm{Im}\,(\vec{R}) \right) - \nabla \mathrm{Im}\,(S) \cdot \nabla(\star_h \vec{n})$$
$$+ (-1)^m \left(\star_h \vec{n} \lrcorner \Delta \mathrm{Im}\,(\vec{R}) \right) - \Delta \mathrm{Im}\,(S)(\star_h \vec{n}) + (-1)^{n+1}\left(\star_h \vec{n} \lrcorner R(\vec{e}_1, \vec{e}_2)\mathrm{Re}\,(\vec{R}) \right) - R(\vec{e}_1, \vec{e}_2)\mathrm{Im}\,(\vec{R}). \tag{4.26}$$

Step 5: Strong $\mathrm{L}^{2,1}$ estimate.

Now, as in the proof of [42, Theorem 6.1], we can consider $\vec{A} = (\text{Re}(\vec{R}), \text{Re}(S)) : B(0, \frac{1}{4}) \to \mathbb{R}^{\frac{m(m-1)}{2}+1}$, where $\text{Re}(\vec{R}) : B(0,1) \to \mathbb{R}^{\frac{m(m-1)}{2}}$ can be trivialised in a normal neighbourhood thanks to the diameter estimate of **Step 2**. Letting $N = \frac{m(m-1)}{2} + 1$, for all $1 \leq i \leq N$, as in [42], A_i solves a Jacobian-type equation of the form

$$\Delta A_i = \sum_{j=1}^{N} \nabla^\perp A_j \cdot \nabla B_{i,j} + F_i, \tag{4.27}$$

and the coupled system (4.25) and (4.26) shows that where $B_{i,j}$ are linear combinations with smooth coefficients of the unit normal $\vec{n}$, and F_i are smooth linear combinations of $\Delta \text{Im}(S)$, $\Delta \text{Im}(\vec{R})$, $\partial_{x_k} \vec{\Phi}_l \partial_{x_j} A_i$, A_i, and $\partial_{x_j} \text{Im}(S) \partial_{x_k}(\star_h \vec{n})_l$, and $\partial_{x_j} \text{Im}(\vec{R})_{i_1,i_2} \partial_{x_k}(\star_h \vec{n})_l$.

By the previous estimates, since $\text{Im}(\vec{R}), \text{Im}(S) \in \text{W}^{2,q}(B(0, \frac{1}{4}))$ for all $q < 2$, and $\vec{R}, S \in \text{W}^{1,(2,\infty)}$, we have for some universal constant $C < \infty$

$$\|F_i\|_{\text{L}^q(B(0,\frac{1}{4}))} \leq C \left(\text{Area}(\vec{\Phi}(B(0,1))^{\frac{1}{2}} + \left\| e^\lambda \vec{H} \right\|_{\text{L}^2(B(0,1))} \right).$$

Notice that each non-linearity can be absorbed to get a linear estimate thanks to the estimate (4.2). Now, make a decomposition

$$A_i = \varphi_i + \psi_i + \chi_i,$$

where

$$\begin{cases} \Delta \varphi_i = \displaystyle\sum_{j=1}^{N} \nabla^\perp A_j \cdot \nabla B_{i,j} & \text{in } B(0, 1/4) \\ \varphi_i = 0 & \text{on } B(0, 1/4), \end{cases}$$

and

$$\begin{cases} \Delta \psi_i = F_i & \text{in } B(0, 1/4) \\ C_i = 0 & \text{on } \partial B(0, 1/4). \end{cases}$$

By the refined $\text{L}^2/\text{L}^{2,\infty}$ Wente inequality (see [17, Théorème (3.4.5)]), we first get $\varphi \in \text{W}_0^{1,2}(B(0, 1/4))$ and

$$\|\nabla \varphi_i\|_{\text{L}^2(B(0,\frac{1}{4}))} \leq C \sum_{j=1}^{N} \|\nabla A_j\|_{\text{L}^{2,\infty}(B(0,\frac{1}{4}))} \|\nabla B_{i,j}\|_{\text{L}^2(B(0,\frac{1}{4}))}$$

$$\leq C \left(\text{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\| e^\lambda \vec{H} \right\|_{\text{L}^2(B(0,1))} \right).$$

Calderón-Zygmund estimates imply that $\psi_i \in \text{W}^{2,q}(B(0, \frac{1}{4}))$ for all $q < 2$, and that for all $q < 2$, there exists a universal constant $\Gamma_q < \infty$ such that

$$\|\psi_i\|_{\text{W}^{2,q}(B(0,\frac{1}{4}))} \leq \Gamma_q \|F_i\|_{\text{L}^q(B(0,\frac{1}{4}))} \leq \Gamma_q' \left(\text{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\| e^\lambda \vec{H} \right\|_{\text{L}^2(B(0,1))} \right).$$

Since χ_i is a harmonic function, we deduce that

$$\left\| \nabla^2 \chi_i \right\|_{\text{L}^1(B(0,\frac{1}{8}))} + \|\nabla \chi_i\|_{\text{L}^{2,1}(B(0,\frac{1}{8}))} + \|\chi_i\|_{\text{L}^\infty(B(0,1))} \leq C \|\chi_i\|_{\text{L}^{2,\infty}(B(0,\frac{1}{4}))}$$

$$\leq C \left(\text{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\| e^\lambda \vec{H} \right\|_{\text{L}^2(B(0,1))} \right),$$

where we used the two previous estimates. Therefore, since $A_i = \varphi_i + \psi_i + \chi_i$, we deduce that $\nabla A_i \in \text{L}^2(B(0, \frac{1}{8}))$, and using the afore-mentioned improved Wente inequality, we finally obtain the estimate $A_i \in \text{W}^{1,(2,1)}(B(0, \frac{1}{16})) \cap \text{L}^\infty(B(0, \frac{1}{16}))$, and

$$\|\nabla A_i\|_{\text{L}^{2,1}(B(0,\frac{1}{16}))} + \|A_i\|_{\text{L}^\infty(B(0,\frac{1}{16}))} \leq C \left(\text{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\| e^\lambda \vec{H} \right\|_{\text{L}^2(B(0,1))} \right).$$

Step 6: Morrey estimate and strong L^p estimate for (some) $p > 2$.

Now, in order to complete the bootstrap, we need to prove a L^p estimate on ∇A_i for some $p > 2$. We will use Morrey-type estimates. To simplify notations, notice that working with balls of decreasing but larger radii, we get the estimate

$$\left\|\nabla \vec{R}\right\|_{\mathrm{L}^{2,1}(B(0,\frac{4}{5}))} + \|\nabla S\|_{\mathrm{L}^{2,1}(B(0,\frac{4}{5}))} + \left\|\vec{R}\right\|_{\mathrm{L}^\infty(B(0,\frac{4}{5}))} + \|S\|_{\mathrm{L}^\infty(B(0,\frac{4}{5}))}$$
$$\leq C \left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))} \right).$$

Furthermore, by the previous discussion, for all $\varepsilon > 0$ there exists $0 < \rho_0 < \frac{1}{4}$ such that

$$\sup_{x \in B(0,\frac{1}{2})} \int_{B(x,\rho_0)} |\nabla B|^2 dx < \varepsilon^2.$$

Let $x \in B(0,\frac{1}{2})$, and $0 < \rho < \rho_0$. Making the same decomposition $A_i = \varphi_i + \psi_i + \chi_i$ as above but on $B(x,\rho)$ (replacing $B(0,\frac{1}{4})$ by $B(x,\rho)$), Wente's inequality shows that

$$\|\nabla \varphi_i\|_{\mathrm{L}^2(B(x,\rho))} \leq C\varepsilon \|\nabla A\|_{\mathrm{L}^2(B(x,\rho))},$$

whilst Calderón-Zygmund estimates imply that for all $q < 2$, there exists $C_q < \infty$ such that

$$\|\nabla \psi_i\|_{\mathrm{L}^2(B(x,\rho))} \leq C_q \rho^{\frac{2}{q'}} \|F_i\|_{\mathrm{L}^q(B(x,\rho))} \leq C_q' \rho^{\frac{2}{q'}} \left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))} \right).$$

In particular, for all $0 < \alpha < 1$, there exists $\Gamma_\alpha < \infty$ such that

$$\|\nabla \psi_i\|_{\mathrm{L}^2(B(x,\rho))} \leq \Gamma_\alpha \rho^{1-\alpha} \left(\mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))} \right).$$

Now, for all $0 < t < 1$, we have the sharp estimate

$$\|\nabla \chi_i\|_{\mathrm{L}^2(B(x,t\rho))} \leq t \|\nabla \chi_i\|_{\mathrm{L}^2(B(x,\rho))} \leq t \|\nabla A_i\|_{\mathrm{L}^2(B(x,\rho))},$$

where we used that χ_i is harmonic with $\chi_i = A_i$ on $\partial B(x,\rho)$, and thus χ_i has the least Dirichlet energy amongst maps of trace A_i on $\partial B(x,\rho)$. Finally, we deduce that

$$\|\nabla A_i\|_{\mathrm{L}^2(B(x,t\rho))} \leq C\varepsilon \|\nabla A\|_{\mathrm{L}^2(B(x,\rho))} + t \|\nabla A_i\|_{\mathrm{L}^2(B(x,\rho))} + \Gamma_\alpha \Lambda \rho^{1-\alpha},$$

where we wrote for simplicity

$$\Lambda = \mathrm{Area}(\vec{\Phi}(B(0,1)))^{\frac{1}{2}} + \left\|e^\lambda \vec{H}\right\|_{\mathrm{L}^2(B(0,1))}.$$

Now, using the elementary inequality

$$(a + b + c)^2 \leq 4a^2 + 4b^2 + 2c^2,$$

we deduce that

$$\int_{B(x,t\rho)} |\nabla A_i|^2 dx \leq 4C^2 \varepsilon^2 \int_{B(x,t\rho)} |\nabla A|^2 dx + 2t^2 \int_{B(x,\rho)} |\nabla A_i|^2 dx + 4\Gamma_\alpha^2 \Lambda^2 \rho^{2(1-\alpha)}.$$

Summing those estimates, we find that

$$\int_{B(x,t\rho)} |\nabla A|^2 dx = \sum_{i=1}^{N} \int_{B(x,t\rho)} |\nabla A_i|^2 dx \leq (4NC^2\varepsilon^2 + 2t^2) \int_{B(x,\rho)} |\nabla A|^2 dx + 4N\Gamma_\alpha^2 \Lambda^2 \rho^{2(1-\alpha)}.$$

Taking $t = \frac{1}{2}$ and $\varepsilon = \dfrac{\delta}{4\sqrt{N}C}$ for some $0 < \delta < 1$, we deduce that

$$\int_{B(x,\frac{\rho}{2})} |\nabla A|^2 dx \leq \left(\frac{1+\delta}{2} \right) \int_{B(x,\rho)} |\nabla A|^2 dx + C_\alpha \Lambda^2 \rho^{2(1-\alpha)}. \tag{4.28}$$

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This implies that ∇A belongs to a Morrey space. Indeed, let $0 < r < \frac{\rho_0}{2}$, and $i \in \mathbb{N}$ be such that

$$2^i r \leq \rho_0 < 2^{i+1} r.$$

Then, (4.28) implies that

$$
\begin{aligned}
\fint_{B(x,r)} |\nabla A|^2 dx &\leq \left(\frac{1}{2} + \delta\right) \fint_{B(x,2r)} |\nabla A|^2 dx + C_\alpha \Lambda^2 (2r)^{2(1-\alpha)} \\
&\leq \left(\frac{1+\delta}{2}\right)^2 \fint_{B(x,2r)} |\nabla A|^2 dx + C_\alpha (2r)^{2(1-\alpha)} \Lambda^2 \left(1 + \left(\frac{1+\delta}{2}\right) 4^{1-\alpha}\right) \\
&\leq \left(\frac{1+\delta}{2}\right)^i \fint_{B(x,2^i r)} |\nabla A|^2 dx + C_\alpha (2r)^{2(1-\alpha)} \Lambda^2 \left(\sum_{j=0}^{i} \left(\frac{1+\delta}{2}\right)^j (4^{1-\alpha})^j\right) \\
&\leq \left(\frac{1+\delta}{2}\right)^i \fint_{B(x,\rho_0)} |\nabla A|^2 dx + C_\alpha \Lambda^2 (2r)^{2(1-\alpha)} \frac{(2^{1-2\alpha}(1+\delta))^{i+1} - 1}{2^{1-2\alpha}(1+\delta) - 1}.
\end{aligned}
$$

Notice by the previous inequality that

$$i \leq \frac{\log\left(\frac{\rho_0}{r}\right)}{\log(2)} < i + 1.$$

We deduce that

$$\left(\frac{1+\delta}{2}\right)^i = \exp\left(-i \log\left(\frac{2}{1+\delta}\right)\right) \leq \exp\left(\left(1 - \frac{\log\left(\frac{\rho_0}{r}\right)}{\log(2)}\right) \log\left(\frac{2}{1+\delta}\right)\right) = \frac{2}{1+\delta} \left(\frac{r}{\rho_0}\right)^{\frac{\log\left(\frac{2}{1+\delta}\right)}{\log(2)}}.$$

Now, we have

$$
\begin{aligned}
\left(2^{1-2\alpha}(1+\delta)\right)^i &= \exp\left(i((1-2\alpha)\log(2) + \log(1+\delta))\right) \\
&\leq \exp\left(\log\left(\frac{\rho_0}{r}\right)((1-2\alpha+2\delta))\right) = \left(\frac{\rho_0}{r}\right)^{1-2\alpha+2\delta},
\end{aligned}
$$

where we used the inequality $\log(1 + x) \leq x$ for all $x > 0$. Finally, we infer that

$$\fint_{B(x,r)} |\nabla A|^2 dx \leq \frac{2}{1+\delta} \left(\frac{r}{\rho}\right)^{\frac{\log\left(\frac{2}{1+\delta}\right)}{\log(2)}} \fint_{B(x,\rho_0)} |\nabla A|^2 + C_\alpha \Lambda^2 \frac{2^{4(1-\alpha)}}{2^{1-2\alpha}(1+\delta) - 1} r^{1-2\delta} \rho_0^{1-2\alpha+\delta}.$$

Furthermore, since $0 < \alpha < 1$ is arbitrary, we deduce that for all $0 < \beta < 1$, there exists $0 < \alpha < 1$ and $\varepsilon_0 > 0$ such that

$$\fint_{B(x,r)} |\nabla A|^2 dx \leq 4 \left(\frac{r}{\rho_0}\right)^{1-\beta} \fint_{B(x,\rho_0)} |\nabla A|^2 dx + C_\beta \Lambda^2 r^{1-\beta} \leq C'_\beta \Lambda^2 r^{1-\beta},$$

which implies that

$$\sup_{x \in B(0,\frac{1}{2})} \left(\frac{1}{r^{1-\beta}} \int_{B(x,r)} |\nabla A|^2 dx\right)^{\frac{1}{2}} \leq C \Lambda < \infty. \tag{4.29}$$

In particular, A is $\frac{1-\beta}{2}$-Hölder-continuous on $B(0, \frac{1}{2})$, and we find that

$$\left\|A - \overline{A}\right\|_{L^\infty(B(0,\frac{1}{2}))} \leq C \Lambda, \tag{4.30}$$

where $\overline{A} = \fint_{B(0,\frac{1}{2})} A\, dx$ is the average of A on $B(0, \frac{1}{2})$. Now, first rewrite the set of equations as

$$\Delta A = \nabla B \cdot \nabla^\perp A + F,$$

where $B \in \mathrm{W}^{1,2}(B(0, 4/5), M_n(\mathbb{R}))$, and $F \in \mathrm{L}^q(B(0, 4/5))$. In particular, for all $q < 2$, there exists $C_q < \infty$ such that by Cauchy-Schwarz and Hölder's inequality, we have

$$
\begin{aligned}
\int_{B(x,r)} |\Delta A| dx &\leq \left(\int_{B(x,r)} |B|^2 dx \right)^{\frac{1}{2}} \left(\int_{B(x,r)} |\nabla A|^2 dx \right)^{\frac{1}{2}} + \int_{B(x,r)} |F| dx \\
&\leq C \left\| \nabla B \right\|_{\mathrm{L}^2(B(0,1))} \Lambda \, r^{\frac{1-\beta}{2}} + \left(\mathscr{L}^2(B(x,r)) \right)^{\frac{1}{q'}} \left\| F \right\|_{\mathrm{L}^q(B(x,r))} \\
&\leq C\Lambda \, r^{\frac{1-\beta}{2}} + \pi^{1-\frac{1}{q}} r^{2\left(1-\frac{1}{q}\right)} \Lambda \leq C\Lambda \, r^{\frac{1-\beta}{2}},
\end{aligned}
\tag{4.31}
$$

where we can take $q = \dfrac{4}{3+\beta} \in (1,2)$. Now, let η be a positive cut-off function such that $\eta = 1$ on $B(0, \frac{1}{4})$ and $\mathrm{supp}(\eta) \subset B(0, \frac{1}{2})$. Set $u = \eta(A - \overline{A})$. We have the following representation formula for all $x \in \mathbb{R}^2$:

$$
u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| \Delta u(y) dy.
$$

Therefore, we can write

$$
\nabla u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \Delta u(y) dy,
$$

and

$$
|\nabla u(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\Delta u(y)|}{|x - y|} dy,
$$

where the right-hand side is the convolution of $|\Delta \cdot|$ with a Riesz kernel. Now, recall the following result from [1, Theorem 5.1] (see also [54, Theorem VIII.3]). Let $0 < \gamma \leq 2$. For all $f \in \mathrm{L}^1(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, define

$$
\mathrm{M}_\gamma(f)(x) = \sup_{r>0} \frac{1}{\pi^{\frac{\gamma}{2}} r^\gamma} \int_{B(x,r)} |f(x)| dx.
$$

For $\gamma = 2$, $\mathrm{M}_\gamma = \mathrm{M}$ is the standard maximal function. Then, the following result holds:

Theorem 4.2 (Adams-Morrey embedding—Theorem 5.1 [1], see also Theorem VIII.3 in [54])**.** *Let* $0 < \gamma < 1$. *Then, there exists a constant* $C_\gamma < \infty$ *such that for all* $f \in \mathrm{W}^{2,1}(\mathbb{R}^2, \mathbb{R})$ *such that* $\mathrm{M}_\gamma(\Delta f) \in \mathrm{L}^\infty(\mathbb{R}^2)$, *we have* $\nabla f \in L^{\frac{2-\gamma}{1-\gamma}}(\mathbb{R}^2)$ *and*

$$
\left\| \nabla f \right\|_{\mathrm{L}^{\frac{2-\gamma}{1-\gamma}, \infty}(\mathbb{R}^2)} \leq C_\gamma \left\| \Delta f \right\|_{\mathrm{L}^1(\mathbb{R}^2)}^{\frac{1-\gamma}{2-\gamma}} \left\| \mathrm{M}_\gamma(\Delta f) \right\|_{\mathrm{L}^\infty(\mathbb{R}^2)}^{\frac{1}{2-\gamma}}.
\tag{4.32}
$$

Thanks to Theorem 4.2, we deduce that

$$
\left\| \nabla u \right\|_{\mathrm{L}^{\frac{2-\gamma}{1-\gamma}, \infty}(B(0, \frac{1}{2}))} \leq C_\gamma \left\| \Delta u \right\|_{\mathrm{L}^1(\mathbb{R}^2)}^{\frac{1-\gamma}{2-\gamma}} \left\| M_\gamma(\Delta u) \right\|_{\mathrm{L}^\infty(\mathbb{R}^2)}^{\frac{1}{2-\gamma}}
$$

where we wrote $\gamma = \dfrac{1-\beta}{2}$. Since $\eta = 1$ on $B(0, \frac{1}{4})$, we have in particular

$$
\left\| \nabla A \right\|_{\mathrm{L}^{\frac{2-\gamma}{1-\gamma}, \infty}(B(0, \frac{1}{4}))} \leq C_\gamma \left\| \Delta u \right\|_{\mathrm{L}^1(\mathbb{R}^2)}^{\frac{1-\gamma}{2-\gamma}} \left\| M_\gamma(\Delta u) \right\|_{\mathrm{L}^\infty(\mathbb{R}^2)}^{\frac{1}{2-\gamma}}.
\tag{4.33}
$$

First, by using (4.30), we estimate

$$
\frac{1}{r^\gamma} \int_{B(x,r)} |\Delta \eta| |A - \overline{A}| dx \leq C \left\| A - \overline{A} \right\|_{\mathrm{L}^\infty(B(0, \frac{1}{2}))} \min\{r^\gamma, 1\} \leq C \left\| A - \overline{A} \right\|_{\mathrm{L}^\infty(B(0, \frac{1}{2}))} \leq C\Lambda.
\tag{4.34}
$$

Likewise, by (4.31) and Cauchy-Schwarz inequality (recall that $2\gamma = 1 - \beta$), we have

$$\frac{1}{r^\gamma} \int_{B(x,r)} |\nabla\eta||\nabla A|dx \leq C \left(\frac{1}{r^{1-\beta}} \int_{B(x,r)} |\nabla A|^2 dx \right)^{\frac{1}{2}} \leq C\,\Lambda. \tag{4.35}$$

Finally, (4.31) yields

$$\frac{1}{r^\gamma} \int_{B(x,r)} \eta|\Delta A|dx \leq C\,\Lambda. \tag{4.36}$$

Likewise, we have

$$\|\Delta u\|_{\mathrm{L}^1(\mathbb{R}^2)} \leq C\,\Lambda, \tag{4.37}$$

and we finally deduce by (4.33), (4.34), (4.35), (4.36) and (4.37), that

$$\|\nabla A\|_{\mathrm{L}^{\frac{3+\beta}{1+\beta}}(B(0,\frac{1}{4}))} \leq C\,\Lambda. \tag{4.38}$$

Step 7: Bootstrap and conclusion.

We deduce that $\vec{R}, S \in \mathrm{W}^{1,p}$ for all $p < \frac{3+\beta}{1+\beta}$. Using the following pointwise identity

$$e^\lambda \vec{H} = -\mathrm{Im}\left(\nabla_z \vec{R} \llcorner e^{-\lambda} \partial_{\bar{z}} \vec{\Phi} \right) - \frac{1}{2} e^\lambda \mathrm{Im}\left(\vec{L} \right) - \mathrm{Re}\left(i\,e^{-\lambda} \partial_{\bar{z}} \vec{\Phi}\,\partial_z S \right) + \mathrm{Re}\left(\langle \partial_z \vec{\Phi}, \mathrm{Im}\,(\vec{L}) \rangle e^{-\lambda} \partial_{\bar{z}} \vec{\Phi} \right),$$

and observing that

$$\frac{3+\beta}{1+\beta} = 2 + \frac{1-\beta}{1+\beta} > 2 + \frac{1-\beta}{2} = 2 + \gamma,$$

we deduce

$$\left\| e^\lambda \vec{H} \right\|_{\mathrm{L}^{2+\gamma}(B(0,\frac{1}{4}))} \leq C\,\Lambda.$$

Using the classical Willmore equation, Calderón-Zygmund estimates permit to show that

$$\vec{H} \in \mathrm{W}^{1,p}\left(B\left(0, 1\frac{1}{8} \right) \right)$$

for all $p < \infty$ (see the end of the proof of [42, Theorem 6.1]) and (up to choosing slowly decreasing radii instead of the diadic choice made above) to prove that $\vec{\Phi} \in C^\infty(B(0,\frac{1}{2}))$ and satisfies the estimates (4.3). $\qquad\square$

5 $\mathrm{L}^{2,\infty}$ Quantization of Energy

The goal of this section is to prove the following result, establishing the $\mathrm{L}^{2,\infty}$ quantization of energy. The proof mimics the one found in [7] but has additional technical difficulties. We argue by contradiction and show that an absence of weak energy quantization for a non-trivial sequence of Willmore annuli (whose energy is uniformly bounded from below) allows us after rescaling to find a weak limit that has to be a constant map.

Theorem 5.1. *Let (M^m, h) be a closed m-dimensional Riemannian manifold. There exists $\varepsilon_0 > 0$ with the following property. Let $\{r_k\}_{k\in\mathbb{N}}, \{R_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ be such that $\limsup_{k\to\infty} R_k \in (0,\infty)$, $r_k \xrightarrow{k\to\infty} 0$ and set $\Omega_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\alpha^{-1} r_k}(0)$ for all $0 < \alpha \leq 1$. Let $\{\vec{\Phi}_k\}_{k\in\mathbb{N}} : B(0, R_k) \to (M^m, h)$ be a sequence of Willmore immersions such that*

$$\Lambda = \sup_{k\in\mathbb{N}} \left(\|\nabla\lambda_k\|_{\mathrm{L}^{2,\infty}(B(0,R_k))} + \int_{B(0,R_k)} |\nabla\vec{n}_k|^2 dx + \mathrm{Area}(\vec{\Phi}_k(B(0,R_k))) \right) < \infty,$$

and

$$\sup_{r_k < s < \frac{R_k}{2}} \int_{B_{2s} \setminus \overline{B}_s(0)} |\nabla \vec{n}_k|^2 dx \le \varepsilon_0.$$

Then

$$\limsup_{\alpha \to 0} \limsup_{k \to \infty} \||x| \nabla \vec{n}_k(x)\|_{\mathrm{L}^\infty(\Omega_k(\alpha))} = 0.$$

In particular, it holds

$$\limsup_{\alpha \to 0} \limsup_{k \to \infty} \|\nabla \vec{n}_k\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha))} = 0.$$

Proof. By the ε-regularity Theorem 4.1, we deduce that

$$|\nabla \vec{n}_k(z)|^2 \le \frac{C_1}{|z|^2} \int_{B_{2|z|} \setminus \overline{B}_{\frac{|z|}{2}}} |\nabla \vec{n}_k|^2 dx \le \frac{2C_1 \varepsilon_0}{|z|^2}.$$

In particular, we have

$$\|\nabla \vec{n}_k\|_{\mathrm{L}^{2,\infty}(\Omega_k(\frac{1}{2}))} \le 4\sqrt{\pi C_1 \varepsilon_0}.$$

Now, assume by contradiction that there exists $\varepsilon_1 > 0$, a sequence $\{\vec{\Phi}_k\}_{k \in \mathbb{N}} \subset C^\infty(B(0, R_k), M^m)$ of smooth Willmore immersions and a sequence $\{z_k\}_{k \in \mathbb{N}} \in \Omega_k(\frac{1}{2})$ such that

$$\log \left| \frac{|z_k|}{r_k} \right| \underset{k \to \infty}{\longrightarrow} \infty \qquad \text{and} \qquad \log \left| \frac{R_k}{|z_k|} \right| \underset{k \to \infty}{\longrightarrow} \infty \,,$$

and

$$|z_k| |\nabla \vec{n}_k(z_k)| \ge \varepsilon_1 > 0 \,. \tag{5.1}$$

Notice that, in particular, $z_k \underset{k \to \infty}{\longrightarrow} 0 \in \mathbb{C}$. Therefore, applying again the ε-regularity Theorem 4.1, we deduce

$$\int_{B_{2|z_k|} \setminus \overline{B}_{\frac{|z_k|}{2}}} |\nabla \vec{n}_k|^2 dx \ge C_0^{-1} |z_k|^2 |\nabla \vec{n}_k(z_k)|^2 \ge C_0^{-1} \varepsilon_1^2. \tag{5.2}$$

Using the previous result (Theorem 3.13 and the end of Section 3.13) and that $\limsup_{k \to \infty} R_k < \infty$, we get that there exist $\alpha_0 > 0$, $\vec{L}_k \in \mathrm{L}_{\lambda_k}^{2,\infty}(B(0, \alpha_0 R_k), \mathbb{C}^n)$, $S_k \in \mathrm{W}^{1,(2,\infty)}(B(0, \alpha_0 R_k), \mathbb{C})$ and $\vec{R}_k \in \mathrm{W}^{1,(2,\infty)}(B(0, \alpha_0 R_k), \Lambda^2 \mathbb{C}^n)$ such that

$$\begin{cases} \nabla_z \vec{L}_k = i \left(\nabla_z \vec{H}_k - 3 \pi_{\vec{n}_k}(\nabla_z \vec{H}_k) - i \star_h \left(\nabla_z \vec{n}_k \wedge \vec{H}_k \right) \right) & \text{in } \Omega_k(\alpha_0) \\ \partial_z S_k = \langle \partial_z \vec{\Phi}_k, \overline{\vec{L}_k} \rangle & \text{in } \Omega_k(\alpha_0) \\ \nabla_z \vec{R}_k = \partial_z \vec{\Phi}_k \wedge \overline{\vec{L}_k} - 2i \, \partial_z \vec{\Phi}_k \wedge \vec{H}_k & \text{in } \Omega_k(\alpha_0) \end{cases} \tag{5.3}$$

and satisfying the bounds:

$$\left\| e^{\lambda_k} \vec{L}_k \right\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha_0))} + \|\nabla S_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))} + \left\| \nabla \vec{R}_k \right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))} \le C$$

$$\left\| \mathrm{Im}\left(\vec{L}_k\right) \right\|_{\mathrm{W}^{1,(2,\infty)}(\Omega_k(\alpha_0))} + \|\mathrm{Im}(S_k)\|_{\mathrm{W}^{2,q}(B(0,\alpha_0 R_k))} + \left\| \mathrm{Im}\left(\vec{R}_k\right) \right\|_{\mathrm{W}^{2,q}(B(0,\alpha_0 R_k))} \le C_q, \ \text{ for all } q < \frac{2}{2 - \varepsilon}.$$

Furthermore, $\mathrm{Im}(S_k)$ and $\mathrm{Im}(\vec{R}_k)$ solve the equations

$$\begin{cases} \Delta \mathrm{Im}(S_k) = -2e^{2\lambda_k} \langle \vec{H}_k, \mathrm{Im}(\vec{L}_k) \rangle & \text{in } \Omega_k(\alpha_0) \\ \Delta \mathrm{Im}(\vec{R}_k) = 4 \mathrm{Im}\left(\nabla_{\overline{z}} \left(\partial_z \vec{\Phi}_k \wedge \vec{L}_k - 2i \, \partial_z \vec{\Phi}_k \wedge \vec{H}_k \right) \right) - 4 \mathrm{Im}\left(\partial_{\overline{z}} \left(F_k(\vec{R}_k) \right) \right) \\ \qquad - 4 \mathrm{Im}\left(\overline{F_k} \left(\partial_z \vec{\Phi}_k \wedge \mathrm{Im}(\vec{L}_k) - 2i \, \partial_z \vec{\Phi}_k \wedge \vec{H}_k \right) \right) & \text{in } \Omega_k(\alpha_0). \end{cases} \tag{5.4}$$

Define the function $\vec{\Psi}_k : \Omega_k(\alpha_0 |z_k|^{-1}) \to \mathbb{R}^n$ by

$$\vec{\Psi}_k(w) = e^{-\lambda_k(z_k) - \log|z_k|} \left(\vec{\Phi}_k(|z_k|w) - \vec{\Phi}_k(z_k) \right).$$

A direct computation shows that

$$\partial_w \vec{\Psi}_k(w) = |z_k| e^{-\lambda_k(z_k) - \log|z_k|} \partial_z \vec{\Phi}_k(|z_k|w) = e^{-\lambda_k(z_k)} \partial_z \vec{\Phi}_k(|z_k|w). \tag{5.5}$$

Therefore, the conformal parameter μ_k of $\vec{\Psi}_k$ satisfies

$$\mu_k(w) = \lambda_k(|z_k|w) - \lambda_k(z_k).$$

By the uniform Harnack inequality on the conformal parameters (see Theorem 3.3), there exists $C > 0$ independent of $k \geq N$ such that

$$\|\nabla (\lambda_k - d_k \log|z|)\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))} + \|\lambda_k - d_k \log|z|\|_{\mathrm{L}^\infty(\Omega_k(\alpha_0))} \leq C,$$

where $d_k \to d \in (-1, \infty)$, as $k \to \infty$. In particular, it holds

$$|\mu_k(w)| \leq |\lambda_k(|z_k|w) - d_k \log|z_k w|| + |d_k \log|z_k| - \lambda_k(z_k)| + |d_k \log|w|| \leq (|d| + 1)|\log|w|| + 2C,$$

which is uniformly bounded on any compact subset $K \subset \mathbb{C} \setminus \{0\}$. Now, by (5.5), we deduce that

$$\widetilde{\vec{n}}_k(w) = \vec{n}_{\vec{\Psi}_k}(w) = \vec{n}_k(|z_k|w). \tag{5.6}$$

Then, we compute

$$\partial^2_{w\bar{w}} \vec{\Psi}(w) = |z_k| e^{-\lambda_k(z_k)} \partial^2_{z\bar{z}} \vec{\Phi}_k(|z_k|w) = |z_k| e^{-\lambda_k(z_k)} \times 2 e^{2\lambda_k(|z_k|w)} \vec{H}_k(|z_k|w)$$
$$= 2|z_k| e^{2\lambda_k(|z_k|w) - \lambda_k(z_k)} \vec{H}_k(|z_k|w),$$

which implies that

$$\widetilde{\vec{H}}_k(w) = \frac{1}{2} e^{-2\mu_k(w)} \partial^2_{w\bar{w}} \vec{\Psi}_k(w) = |z_k| e^{\lambda_k(z_k)} \vec{H}_k(|z_k|w). \tag{5.7}$$

We deduce that

$$e^{\mu_k} \widetilde{\vec{H}}_k(w) = |z_k| e^{\lambda_k(z_k)} \vec{H}_k(|z_k|w),$$

which shows that, after the linear change of variable $z = |z_k|w$, it holds

$$\int_{\Omega_k(\alpha_0|z_k|^{-1})} e^{2\mu_k(w)} |\widetilde{\vec{H}}_k(w)|^2 |dw|^2 = \int_{\Omega_k(\alpha_0|z_k|^{-1})} |z_k|^2 e^{2\lambda_k(|z_k|w)} |\vec{H}_k(|z_k|w)|^2 |dw|^2$$
$$= \int_{\Omega_k(\alpha_0)} e^{2\lambda_k(z)} |\vec{H}_k(z)|^2 |dz|^2 = \int_{\Omega_k(\alpha_0)} |\vec{H}_k|^2 d\mathrm{vol}_{g_k}.$$

We deduce that

$$\partial_w \widetilde{\vec{H}}_k(w) = |z_k|^2 e^{\lambda_k(z_k)} \partial_z \vec{H}_k(|z_k|w) \tag{5.8}$$

and that

$$|z_k|^2 e^{\lambda_k(z_k)} \Gamma^j_{l,q}(\vec{\Phi}_k(|z_k|w)) \partial_z \vec{\Phi}_k(|z_k|w)^q \vec{H}_k(|z_k|w)^l = \Gamma^j_{l,q}(\vec{\Phi}_k(|z_k|w)) |z_k| \partial_z \vec{\Phi}_k(|z_k|w)^q \widetilde{\vec{H}}_k(w)^l. \tag{5.9}$$

Since $\{\mu_k\}_{k \in \mathbb{N}}$ is uniformly bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{C} \setminus \{0\})$, we deduce by the ε-regularity that $\{\widetilde{\vec{H}}_k\}_{k \in \mathbb{N}}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{C} \setminus \{0\})$. Furthermore, by the Harnack inequality (3.25), we deduce that

$$e^{-A} |z_k|^{d_k+1} |w|^{d_k} \leq |z_k| |\partial_z \vec{\Phi}_{k,q}(|z_k|w)| \leq e^A |z_k|^{d_k+1} |w|^{d_k}.$$

Since $d_k \underset{k\to\infty}{\longrightarrow} d > -1$, and $z_k \underset{k\to\infty}{\longrightarrow} 0$, whilst $\Gamma^j_{l,q}$ are bounded, we conclude that for all compact subset $K \subset \mathbb{C} \setminus \{0\}$, it holds

$$\lim_{k\to\infty} \left\| |z_k|^2 e^{\lambda_k(z_k)} \Gamma^j_{l,q}(\vec{\Phi}_k(|z_k|w)) \partial_z \vec{\Phi}_k(|z_k|w)^q \vec{H}_k(|z_k|w)^l \right\|_{\mathrm{L}^\infty(K)} = 0. \tag{5.10}$$

Now, the previous scaling considerations prompt us to introduce $\widetilde{\vec{L}}_k : \Omega_k(\alpha_0 |z_k|^{-1}) \to \mathbb{C}^n$ defined by

$$\widetilde{\vec{L}}_k(w) = |z_k| e^{\lambda_k(z_k)} \vec{L}_k(|z_k|w).$$

It is immediate to check that

$$e^{\mu_k} \widetilde{\vec{L}}_k(w) = |z_k| e^{\lambda_k(|z_k|w)} \vec{L}_k(|z_k|w) = |z_k| e^{\lambda_k(|z_k|w)} \vec{V}_k(|z_k|w) + |z_k| e^{\lambda_k(|z_k|w)} \vec{W}_k(|z_k|w),$$

where $\vec{W}_k \in \mathrm{W}^{1,(2,\infty)}(B(0,\alpha_0 R_k))$ and $\vec{V}_k$ satisfy for all $z \in \Omega_k(\alpha_0)$ the estimate

$$e^{\lambda_k(z)} |\vec{V}_k(z)| \le \frac{C}{|z|}$$

for some constant $C > 0$ independent of k. We deduce that

$$|z_k| e^{\lambda_k(|z_k|w)} \left| \vec{V}_k(|z_k|w) \right| \le |z_k| \times \frac{C}{||z_k|w|} = \frac{C}{|w|}.$$

Also, by defining $\widetilde{\vec{W}}_k : B(0, \alpha_0 |z_k|^{-1} R_k) \to \mathbb{C}^n$ as

$$\widetilde{\vec{W}}_k(w) = \vec{W}_k(|z_k|w),$$

we have

$$\left\| \nabla \widetilde{\vec{W}}_k \right\|_{\mathrm{L}^{2,\infty}(B(0,\alpha_0 |z_k|^{-1} R_k))} = \left\| \nabla \vec{W}_k \right\|_{\mathrm{L}^{2,\infty}(B(0,\alpha_0 R_k))} \le C,$$

giving that $\vec{W}_k$ is bounded in $\mathrm{W}^{1,(2,\infty)}(B(0, \alpha_0 |z_k|^{-1} R_k))$. Furthermore, by the Harnack inequality (3.25), we deduce that

$$|z_k| e^{\lambda_k(|z_k|w)} \le e^A |z_k|^{d_k+1} |w|^{d_k} \underset{k\to\infty}{\longrightarrow} 0 \qquad \text{in } \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{C} \setminus \{0\}).$$

Therefore, we get that

$$\left\| |z_k| e^{\lambda_k(|z_k|\cdot)} \widetilde{\vec{W}}_k \right\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha_0|z_k|^{-1}))} \underset{k\to\infty}{\longrightarrow} 0$$
$$|z_k| e^{\lambda_k(|z_k|\cdot)} \widetilde{\vec{W}}_k \underset{k\to\infty}{\longrightarrow} 0 \qquad \text{in } \mathrm{L}^p_{\mathrm{loc}}(\mathbb{C} \setminus \{0\}) \text{ for all } p < \infty.$$

Then, we have

$$\nabla \mathrm{Im}\,(\widetilde{\vec{L}}_k)(w) = |z_k|^2 e^{\lambda_k(z_k)} \nabla \mathrm{Im}\,(\vec{L}_k)(|z_k|w)$$

which implies that

$$\left\| \nabla \mathrm{Im}\,(\widetilde{\vec{L}}_k) \right\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha_0))} = |z_k| e^{\lambda_k(z_k)} \left\| \nabla \mathrm{Im}\,(\vec{L}_k) \right\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha_0))}$$

$$\le e^A |z_k|^{d_k+1} \left\| \nabla \mathrm{Im}\,(\vec{L}_k) \right\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha_0))} \underset{k\to\infty}{\longrightarrow} 0,$$

where we used $|z_k| \underset{k\to\infty}{\longrightarrow} 0$ and $d_k + 1 \underset{k\to\infty}{\longrightarrow} d + 1 > 0$. Finally, using (5.3), (5.8) and (5.9) we obtain:

$$\partial_w \widetilde{\vec{L}}_k - i \left(\partial_w \widetilde{\vec{H}}_k - 3 \pi_{\widetilde{n}_k}(\partial_w \vec{H}_k) - i \star_h \left(\partial_z \widetilde{\vec{n}}_k \wedge \widetilde{\vec{H}}_k \right) \right) = \vec{Z}_k \tag{5.11}$$

where

$$\left(\vec{Z}_k\right)_j = \sum_{l,q=1}^n |z_k|^2 e^{\lambda_k(z_k)} \Gamma_{j,q}^l(\vec{\Phi}_k(|z_k|w)) \partial_z \vec{\Phi}_{k,q}(|z_k|w) \Big(\vec{L}_{k,l}(|z_k|w)$$

$$- i\, \vec{H}_{k,l}(|z_k|w) + 3i\, \pi_{\vec{n}_k(|z_k|w)}(\vec{H}_k(|z_k|w))_l - \star_h \left(\vec{n}_k(|z_k|w) \wedge \vec{H}_k(|z_k|w) \right)_l \Big)$$

$$= \sum_{l,q=1}^n \Gamma_{j,q}^l(\vec{\Phi}_k(|z_k|w)) |z_k| \partial_z \vec{\Phi}_{k,q}(|z_k|w) \Big(\widetilde{L}_{k,l}(w) - i\, \widetilde{H}_{k,l}(w) + 3i\, \pi_{\widetilde{n}_k(w)}(\widetilde{\vec{H}}_k(w))_l$$

$$- \star \left(\widetilde{\vec{n}}_k(w) \wedge \widetilde{\vec{H}}_k(w) \right)_l \Big).$$

By the ε-regularity Theorem 4.1, as in (5.10), we get that for any compact subset $K \subset \mathbb{C} \setminus \{0\}$, it holds

$$\left\| \Gamma_{l,q}^j(\vec{\Phi}_k(|z_k|w)) |z_k| \partial_z \vec{\Phi}_k(|z_k|w)^q \Big(-i\, \widetilde{H}_k(w)^l + 3i\, \pi_{\widetilde{n}_k(w)}(\widetilde{\vec{H}}_k(w))^l - \star \left(\widetilde{\vec{n}}_k(w) \wedge \widetilde{\vec{H}}_k(w) \right)^l \Big) \right\|_{\mathrm{L}^\infty(K)} = 0.$$

Furthermore,

$$|\vec{L}_k(w)| \leq \frac{C}{|w|^{d_k+1}} + f_k(w),$$

where f_k is bounded in $\mathrm{W}^{1,(2,\infty)}$ and in particular in L^p for all $p < \infty$. Therefore, we have

$$|z_k|\, |\partial_z \vec{\Phi}_k(|z_k|w)|\, |\vec{L}_k(w)| \leq \frac{|z_k|^{d_k+1}}{|w|} + |z_k|^{d_k+1} |w|^{d_k} f_k(w)$$

which implies that for any compact subset $K \subset \mathbb{C} \setminus \{0\}$ and any $p < \infty$, it holds

$$\left\| \Gamma_{l,q}^j(\vec{\Phi}_k(|z_k|w)) |z_k| \partial_z \vec{\Phi}_k(|z_k|w)^q \widetilde{L}_k(w)^l \right\|_{\mathrm{L}^p(K)} \xrightarrow[k\to\infty]{} 0.$$

Now define $\widetilde{S}_k : \Omega_k(\alpha_0|z_k|^{-1}) \to \mathbb{C}$ and $\widetilde{\vec{R}}_k : \Omega_k(\alpha_0|z_k|^{-1}) \to \Lambda^2 \mathbb{C}^n$ by

$$\widetilde{S}_k(w) = S_k(|z_k|w), \qquad \widetilde{\vec{R}}_k(w) = \vec{R}_k(|z_k|w).$$

By scaling invariance, we have

$$\int_{\Omega_k(\alpha_0|z_k|^{-1})} |\nabla \vec{n}_{\vec{\Psi}_k}|^2 dx = \int_{\Omega_k(\alpha_0)} |\nabla \vec{n}_k|^2 dx \leq C$$

$$\left\| \nabla \widetilde{S}_k \right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0|z_k|^{-1}))} = \left\| \nabla S_k \right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))} \leq C$$

$$\left\| \nabla \widetilde{\vec{R}}_k \right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0|z_k|^{-1}))} = \left\| \nabla \vec{R}_k \right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))} \leq C.$$

Therefore, by the ε-regularity of Theorem 4.1 to deduce that for any compact subset $K \subset \mathbb{C} \setminus \{0\}$, and for any $l \in \mathbb{N}$, there exists $C_l(K) < \infty$ such that

$$\left\| \nabla^l \vec{n}_{\vec{\Psi}_k} \right\|_{\mathrm{L}^\infty(K)} \leq C_l(K).$$

Therefore, up to a subsequence, $\vec{\Psi}_k \xrightarrow[k\to\infty]{} \vec{\Psi}_\infty$ in $C^l_{\mathrm{loc}}(\mathbb{C} \setminus \{0\})$ and by lower semi-continuity and conformal invariance of the Dirichlet energy (or $\mathrm{L}^{2,1}$ norm of the gradient), we deduce that

$$\left\| e^{\mu_\infty} \widetilde{L}_\infty \right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})} + \left\| \nabla \widetilde{S}_\infty \right\|_{\mathrm{L}^{2,1}(\mathbb{C})} + \left\| \nabla \widetilde{\vec{R}}_\infty \right\|_{\mathrm{L}^{2,1}(\mathbb{C})} \leq C.$$

Furthermore, recalling that $\operatorname{Im}(S_k), \operatorname{Im}(\vec{R}_k) \in \mathrm{W}^{2,q}(B(0, \alpha_0 R_k))$ for all $q < \frac{2}{2-\varepsilon}$, and using that $\operatorname{Im}(\widetilde{S}_k)(w) = \operatorname{Im}(S_k)(|z_k|w)$, we deduce that for all $q < \frac{2}{1-\varepsilon}$

$$\int_{B(0, \alpha_0 |z_k|^{-1} R_k)} \left(|\nabla^2 \operatorname{Im}(\widetilde{S}_k)|^q + |\nabla^2 \operatorname{Im}(\vec{\widetilde{R}}_k)|^q \right) dx$$

$$= |z_k|^{2(q-1)} \int_{B(0, \alpha_0 R_k)} \left(|\nabla^2 \operatorname{Im}(S_k)|^q + |\nabla^2 \operatorname{Im}(R_k)|^q \right) dx \le 2(C_q)^q |z_k|^{2(q-1)} \xrightarrow[k \to \infty]{} 0,$$

It follows that $\nabla^2 \operatorname{Im}(\widetilde{S}_\infty) = 0$ and $\nabla^2 \operatorname{Im}(\vec{\widetilde{R}}_\infty) = 0$ and, since $\nabla \operatorname{Im}(\widetilde{S}_\infty) \in \mathrm{L}^{2,1}(\mathbb{C})$ and $\nabla \operatorname{Im}(\vec{\widetilde{R}}_\infty) \in \mathrm{L}^{2,1}(\mathbb{C})$, this implies that $\operatorname{Im}(\widetilde{S}_\infty)$ and $\operatorname{Im}(\vec{\widetilde{R}}_\infty)$ are constant. Recalling that (5.3) can be rewritten (see [42, Lemma 6.2]) as

$$\begin{cases} \nabla_z \vec{R}_k = (-1)^{m+1} \star_h \left(\vec{n}_k \lrcorner\, i \, \nabla_z \vec{R} \right) + i \, \partial_z S_k \, \star_h \vec{n}_k \\ \partial_z \vec{S}_k = -i \langle \nabla_z \vec{R}_k, \star_h \vec{n}_k \rangle, \end{cases}$$

an expansion similar to the one made in (5.11) shows that the system passes to the limit and yields

$$\begin{cases} \partial_z \vec{\widetilde{R}}_\infty = (-1)^{m+1} \star_h \left(\vec{\widetilde{n}}_\infty \lrcorner\, i \, \partial_z \vec{\widetilde{R}}_\infty \right) + i \, \partial_z \widetilde{S}_\infty \, \star_h \vec{n}_k \\ \partial_z \widetilde{S}_\infty = -i \langle \partial_z \vec{\widetilde{R}}_\infty, \star_h \vec{\widetilde{n}}_\infty \rangle. \end{cases}$$

Since both $\vec{\widetilde{R}}_\infty$ and $\widetilde{S}_\infty$ are real, this system can be rewritten as

$$\begin{cases} \nabla \vec{\widetilde{R}}_\infty = (-1)^m \star_h \left(\vec{\widetilde{n}}_\infty \lrcorner \nabla^\perp \vec{\widetilde{R}}_\infty \right) - \nabla^\perp \widetilde{S}_\infty \, \star_h \vec{\widetilde{n}}_\infty \\ \nabla \widetilde{S}_\infty = \langle \nabla^\perp \vec{\widetilde{R}}_\infty, \star_h \vec{\widetilde{n}}_\infty \rangle. \end{cases}$$

We deduce that the following Jacobian system holds:

$$\begin{cases} \Delta \vec{\widetilde{R}}_\infty = (-1)^m \star_h \left(\nabla \vec{\widetilde{n}}_\infty \lrcorner \nabla^\perp \vec{\widetilde{R}}_\infty \right) - \nabla^\perp \widetilde{S}_\infty \cdot \nabla \left(\star \vec{\widetilde{n}}_\infty \right) \\ \Delta \widetilde{S}_\infty = \langle \nabla^\perp \vec{\widetilde{R}}_\infty, \nabla \left(\star_h \vec{\widetilde{n}}_\infty \right) \rangle. \end{cases}$$

Using an improved Wente estimate as in [7], we deduce that

$$\left\| \nabla \vec{\widetilde{R}}_\infty \right\|_{\mathrm{L}^{2,1}(\mathbb{C})} + \left\| \nabla \widetilde{S}_\infty \right\|_{\mathrm{L}^{2,1}(\mathbb{C})} \le C \left(\left\| \nabla \vec{\widetilde{R}}_\infty \right\|_{\mathrm{L}^{2,1}(\mathbb{C})} + \left\| \nabla \widetilde{S}_\infty \right\|_{\mathrm{L}^{2,1}(\mathbb{C})} \right) \| \nabla \vec{\widetilde{n}}_\infty \|_{\mathrm{L}^{2,\infty}(\mathbb{C})}$$

$$\le C \varepsilon_0 \left(\left\| \nabla \vec{\widetilde{R}}_\infty \right\|_{\mathrm{L}^{2,1}(\mathbb{C})} + \left\| \nabla \widetilde{S}_\infty \right\|_{\mathrm{L}^{2,1}(\mathbb{C})} \right).$$

Therefore, taking $\varepsilon_0 = \frac{1}{2C}$, we get that $\vec{\widetilde{R}}_\infty$ and $\widetilde{S}_\infty$ are constant. Since $\operatorname{Im}(\vec{L}_\infty) = 0$, we have the identity

$$e^{2\mu_\infty} \vec{\widetilde{H}}_\infty = -\operatorname{Im}\left(\partial_z \vec{\widetilde{R}}_\infty \lrcorner \partial_{\bar{z}} \vec{\Psi}_\infty \right) - \operatorname{Re}\left(\partial_z \widetilde{S}_\infty (i \, \partial_{\bar{z}} \vec{\Psi}_\infty) \right) + \operatorname{Re}\left(\langle \partial_z \vec{\Psi}_\infty, \operatorname{Im}(\vec{L}_\infty) \rangle \partial_{\bar{z}} \vec{\Psi}_\infty \right) = 0.$$

We deduce that $\vec{\widetilde{H}}_\infty = 0$ which, by the exact same proof as in [7], yields that $\nabla \vec{\widetilde{n}}_\infty = 0$, contradicting the estimate

$$\int_{B_2 \setminus \overline{B}_1(0)} |\nabla \vec{\widetilde{n}}_\infty|^2 dx \ge C_0^{-1} \varepsilon_1^2$$

that passed to the limit thanks to the strong convergence on $\mathbb{C} \setminus \{0\}$. $\qquad \square$

6 Proof of the Main Theorem A

We are finally in position to prove the main result of the paper (Theorem A), namely the quantization of the Willmore energy for Willmore spheres in Riemannian manifolds. The proof will combine all the main technical results proved in the paper: the $L^{2,1}$ uniform bounds on the Willmore integrand in neck regions (Theorem 3.2), the ε-regularity Theorem 4.1, and the $L^{2,\infty}$ quantization of energy (Theorem 5.1).

Proof of Theorem A. Let us first consider the case when $\inf_{k \in \mathbb{N}} \text{Area}(\vec{\Phi}_k(S^2)) > 0$.

Thanks to the pre-compactness Theorem 2.1, we know that there exists a sequence of Lipschitzian diffeomorphisms $\{f_k\}$ of S^2 and a weak immersion $\vec{\xi}_\infty$ of $(S^2 \setminus \{a_1, \cdots, a_N\})$, possibly branched at the finitely many points $a_1, \cdots, a_N$, into (M^m, h) such that

$$\vec{\xi}_k = \vec{\Phi}_k \circ f_k \rightharpoonup \vec{\xi}_\infty \quad \text{weakly in } W^{2,2}_{\text{loc}}(S^2 \setminus \{a_1, \cdots, a_N\}) \,, \tag{6.1}$$

Thanks to the ε-regularity Theorem 4.1, we can improve the weak $W^{2,2}_{\text{loc}}$ convergence to local smooth convergence:

$$\vec{\xi}_k \underset{k \to \infty}{\longrightarrow} \vec{\xi}_\infty \qquad \text{in } C^l_{\text{loc}}(S^2 \setminus \{a_1, \cdots, a_N\}) \text{ for all } l \in \mathbb{N}. \tag{6.2}$$

Following verbatim the arguments at [7, pp.129–130], one can extend the map $\vec{\xi}_\infty$ to the whole S^2, so that the extension $\vec{\xi}_\infty : S^2 \to (M^m, h)$ realises a Willmore immersion of S^2 into (M^m, h), possibly branched at the points $a_1, \cdots, a_N$.

Now, in a neck-region conformally equivalent to $\Omega_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\alpha^{-1} r_k}(0)$, we can apply Theorem 3.2 and Theorem 5.1 to deduce that there exists $\alpha_0 > 0$ and $C > 0$ independent of k such that

$$\left\| e^{\lambda_k} \vec{H}_k \right\|_{L^{2,1}(\Omega_k(\alpha_0))} \leq C \quad \text{and} \quad \lim_{\alpha \to 0} \limsup_{k \to \infty} \|\nabla \vec{n}_k\|_{L^{2,\infty}(\Omega_k(\alpha))} = 0 \,.$$

By the $L^{2,1}/L^{2,\infty}$ duality, we deduce that for all $0 < \alpha < \alpha_0$

$$\int_{\Omega_k(\alpha)} e^{2\lambda_k} |\vec{H}_k|^2 dx \leq \left\| e^{\lambda_k} \vec{H}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} \left\| e^{\lambda_k} \vec{H}_k \right\|_{L^{2,\infty}(\Omega_k(\alpha))} \leq C \|\nabla \vec{n}_k\|_{L^{2,\infty}(\Omega_k(\alpha))} \,,$$

which implies that

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \int_{\Omega_k(\alpha)} |\vec{H}_k|^2 d\text{vol}_{g_k} = 0 \,. \tag{6.3}$$

By [7, Lemma V.1] (which does not use the Willmore equation and is valid for any weak immersion), we also have

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \left| \int_{\Omega_k(\alpha)} K_{g_k} d\text{vol}_{g_k} \right| = 0. \tag{6.4}$$

Moreover, by the proof of Theorem 3.2 (see (3.24)), it holds

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \text{Area}(\vec{\xi}_k(\Omega_k(\alpha))) = 0 \,. \tag{6.5}$$

Using the point-wise identity

$$|\nabla \vec{n}_k|^2 = 4|\vec{H}_k|^2 - 2K_{g_k} + 2K(\vec{\xi}_{k,*}(TS^2)) \,,$$

together with (6.3), (6.4) and (6.5), we deduce that

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \int_{\Omega_k(\alpha)} |\nabla \vec{n}_k|^2 dx = 0 \,. \tag{6.6}$$

This is the *no-neck energy* which will give below the desired quantization result.

Using that (M^m, h) is isometrically embedded into $\mathbb{R}^n$ and the conformal invariance of the Willmore energy to obtain a suitable convergence result for the energy of bubbles. Indeed, seeing $\vec{\Phi}_k : \Sigma \to (M^m, h) \subset \mathbb{R}^n$ as an immersion into $\mathbb{R}^n$, since by assumption $h = \iota^* g_{\mathbb{R}^n}$, where $\iota : M^m \hookrightarrow \mathbb{R}^n$ is the inclusion map, we deduce that for any open subset $\Omega \subset \Sigma$ it holds

$$\int_\Omega \left(|\vec{H}_k|^2 + K(\vec{\Phi}_{k,*}(T\Sigma)) \right) d\mathrm{vol}_{g_k} = W_{(M^m,h)}(\vec{\Phi}_k|\Omega) = W_{\mathbb{R}^n}(\iota \circ \vec{\Phi}_k|\Omega) = \int_\Omega |\vec{H}_{\iota \circ \vec{\Phi}_k}|^2 d\mathrm{vol}_{g_k},$$

Let

$$B(i, j, \alpha, k) = B_{\alpha^{-1}\rho_k^{i,j}}(x_k^{i,j}) \setminus \bigcup_{j' \in I^{i,j}} B_{\alpha \rho_k^{i,j}}(x_k^{i,j'})$$

be a bubble domain (for more details, see [7] or the discussion after (3.20)). Recall that $\rho_k^{i,j} \to 0$ as $k \to \infty$ and that the indices i, j vary within a finite set. The Harnack inequality (3.21) (proved in [7, Display (VIII.10)]) gives that for all $0 < \alpha < 1$, there exists $A_\alpha = A(i, j, \alpha) \geq 0$ such that

$$\sup_{z \in B(i,j,\alpha,k)} e^{\lambda_k(z)} \leq e^{A_\alpha} \inf_{z \in B(i,j,\alpha,k)} e^{\lambda_k(z)}. \tag{6.7}$$

Choose an arbitrary point $z_k^{i,j} \in B(i, j, \alpha, k)$ and set $\lambda(i, j, \alpha, k) = \lambda_k(z_k^{i,j})$. The uniform area bound implies that

$$\limsup_{k \to \infty} e^{2\lambda(i,j,\alpha,k)}(\rho_k^{i,j})^2 < \infty, \quad \text{for all } \alpha \in (0, 1). \tag{6.8}$$

Thus we have two cases.

Case 1. $\limsup\limits_{k \to \infty} e^{2\lambda(i,j,\alpha,k)}(\rho_k^{i,j})^2 > 0$, for some $\alpha \in (0, 1)$.

This case corresponds to a macroscopic bubbles forming in the region $B(i, j, \alpha, k)$. By performing a good gauge extraction procedure along the lines of [42, Lemma 4.1], we can find positive Möbius transformations $\widetilde{f}_k$ of S^2 such that the reparametrised immersions (up to a subsequence)

$$\widetilde{\vec{\xi}}_k = \vec{\xi}_k \circ \widetilde{f}_k : S^2 \to (M^m, h)$$

converge weakly in $W^{2,2}$ (and then smoothly, by the ε-regularity Theorem) outside finitely many points $\{a_1^{i,j}, \ldots, a_{N_{i,j}}^{i,j}\}$ to a Willmore immersion $\vec{\Psi}_{i,j} : S^2 \setminus \{a_1^{i,j}, \cdots, a_{N_{i,j}}^{i,j}\} \to (M^m, h)$:

$$\widetilde{\vec{\xi}}_k \xrightarrow[k \to \infty]{} \vec{\Psi}_{i,j} \quad \text{in } C_{\mathrm{loc}}^l \left(S^2 \setminus \{a_1^{i,j}, \cdots, a_{N_{i,j}}^{i,j}\} \right) \text{ for all } l \in \mathbb{N}. \tag{6.9}$$

Following verbatim the arguments at [7, pp. 129–130], one can extend the map to the whole S^2, so that the extension $\vec{\Psi}_{i,j} : S^2 \to (M^m, h)$ realises a Willmore immersion of S^2 into (M^m, h), possibly branched at finitely many points $a_1^{i,j}, \cdots, a_{N_{i,j}}^{i,j}$. Moreover, the no neck energy identity (6.6) ensures that

$$\lim_{\alpha \to 0} \lim_{k \to \infty} W_{(M,h)} \left(\vec{\xi}_k|B(i, j, k, \alpha) \right) = W_{(M,h)}(\vec{\Psi}_{i,j}). \tag{6.10}$$

Such a branched Willmore immersion corresponds to a Riemannian bubble $\vec{\Psi}_i$, $i \in \{1, \cdots, p\}$ in the statement of Theorem A.

Case 2. $\lim\limits_{k \to \infty} e^{2\lambda(i,j,\alpha,k)}(\rho_k^{i,j})^2 = 0$, for all $\alpha \in (0, 1)$.

In this case, there exists a point $\overline{x}_{i,j} \in M$ such that (again, up to a subsequence in k)

$$\vec{\xi}_k(B(i, j, \alpha, k)) \to \overline{x}_{i,j} \quad \text{in Hausdorff distance sense, as } k \to \infty, \text{ for all } \alpha \in (0, 1).$$

Let $\mathrm{Exp}_{\overline{x}_{i,j}} : B_\varepsilon^{\mathbb{R}^m}(0) \to M$ denote the exponential map of (M, h) based at the point $\overline{x}_{i,j}$. Consider the rescaled immersions (with values in $T_p M \simeq \mathbb{R}^m$)

$$\vec{\xi}_k^{i,j}(w) = e^{-\lambda(i,j,\alpha,k) - \log \rho_k^{i,j}} \mathrm{Exp}_{\overline{x}_{i,j}}^{-1} \left(\vec{\xi}_k(\rho_k^{i,j} w + z_k^{i,j}) - \vec{\xi}_k(z_k^{i,j}) \right), \forall w \in (\rho_k^{i,j})^{-1} \left(B(i, j, \alpha, k) - z_{i,j}^k \right) \subset \mathbb{C}.$$

It is easily seen that (6.7) implies

$$e^{-A_\alpha - 1} \le |\partial_w \vec{\xi}_k^{i,j}| \le e^{A_\alpha + 1}, \quad \text{for } k \in \mathbb{N} \text{ sufficiently large,}$$

and that the assumption of case 2 yields $e^{-\lambda(i,j,\alpha,k) - \log \rho_k^{i,j}} \to +\infty$.

Notice that $\vec{\xi}_k^{i,j}$ are Willmore immersions in $(\mathbb{R}^m, g(k,i,j))$, where the Riemannian metrics $g(k,i,j)$ converge to the Euclidean metric as $k \to \infty$, in $C_{\text{loc}}^l(\mathbb{R}^m)$ topology, for every $l \in \mathbb{N}$. Moreover, the scaling invariance of the Willmore functional implies that

$$W_{(M,h)}\left(\vec{\xi}_k | B(i,j,k,\alpha)\right) = W_{(\mathbb{R}^m, g(k,i,j))}\left(\vec{\xi}_k^{i,j} | (\rho_k^{i,j})^{-1}\left(B(i,j,\alpha,k) - z_{i,j}^k\right)\right). \tag{6.11}$$

Using the aforementioned $C_{\text{loc}}^l(\mathbb{R}^m)$ convergence of the ambient Riemannian metrics, one can immediately adapt the proof of the ε-regularity Theorem 4.1 to deduce that there exists a finite set of points $\left\{a_1^{i,j}, \cdots, a_{N_{i,j}}^{i,j}\right\} \subset \mathbb{C}$ with $N_{i,j} \in \mathbb{N}$ such that

$$\vec{\xi}_k^{i,j} \underset{k \to \infty}{\longrightarrow} \vec{\xi}_\infty^{i,j} \quad \text{in } C_{\text{loc}}^l\left(\mathbb{C} \setminus \left\{a_1^{i,j}, \cdots, a_{N_{i,j}}^{i,j}\right\}\right), \text{ for all } l \in \mathbb{N},$$

where $\vec{\xi}_\infty^{i,j}$ is a smooth Willmore immersion of $\mathbb{C} \setminus \left\{a_1^{i,j}, \cdots, a_{N_{i,j}}^{i,j}\right\}$ in the Euclidean space $\mathbb{R}^m$.
We can now follow verbatim the arguments in [7, pp. 130–132] and deduce that:

(1) In case $\int_\mathbb{C} |\nabla \vec{\xi}_\infty^{i,j}|^2 dx < \infty$, using the stereographic projection of S^2 to $\mathbb{C}$, the limit map $\vec{\xi}_\infty^{i,j}$ extends to a smooth Willmore immersion of S^2 into $\mathbb{R}^m$, possibly branched at $\left\{a_1^{i,j}, \cdots, a_{N_{i,j}}^{i,j}\right\}$. Moreover (see also (6.11)):

$$\lim_{\alpha \to 0} \lim_{k \to \infty} W_{(M,h)}\left(\vec{\xi}_k | B(i,j,k,\alpha)\right) = \lim_{\alpha \to 0} \lim_{k \to \infty} \int_{(\rho_k^{i,j})^{-1}\left(B(i,j,\alpha,k) - z_{i,j}^k\right)} |\vec{H}_{\vec{\xi}_k^{i,j}}|^2 e^{2\lambda_{\vec{\xi}_k^{i,j}}} dx$$
$$= W_{\mathbb{R}^m}(\vec{\xi}_\infty^{i,j}). \tag{6.12}$$

Such a branched Willmore immersion corresponds to a Euclidean bubble $\vec{\eta}_j$, $j \in \{1, \ldots, q\}$ in the statement of Theorem A.

(2) In case $\int_\mathbb{C} |\nabla \vec{\xi}_\infty^{i,j}|^2 dx = \infty$, one finds suitable inversions $\mathcal{I}_{k,i,j}, \mathcal{I}_{\infty,i,j}$ in $\mathbb{R}^m$ such that

$$\mathcal{I}_{k,i,j} \circ \vec{\xi}_k^{i,j} \to \mathcal{I}_{\infty,i,j} \circ \vec{\xi}_\infty^{i,j}, \qquad \text{in } C_{\text{loc}}^l\left(\mathbb{C} \setminus \left\{a_1^{i,j}, \cdots, a_{N_{i,j}}^{i,j}\right\}\right), \text{ for all } l \in \mathbb{N}.$$

Furthermore, one obtains that (pre-composing with the stereographic projection) $\mathcal{I}_{\infty,i,j} \circ \vec{\xi}_\infty^{i,j}$ extends to a smooth Willmore immersion of S^2 into $\mathbb{R}^m$, possibly branched at $\left\{a_1^{i,j}, \cdots, a_{N_{i,j}}^{i,j}\right\}$. Moreover,

$$\lim_{\alpha \to 0} \lim_{k \to \infty} W_{(M,h)}\left(\vec{\xi}_k | B(i,j,k,\alpha)\right) = \lim_{\alpha \to 0} \lim_{k \to \infty} \int_{(\rho_k^{i,j})^{-1}\left(B(i,j,\alpha,k) - z_{i,j}^k\right)} |\vec{H}_{\vec{\xi}_k^{i,j}}|^2 e^{2\lambda_{\vec{\xi}_k^{i,j}}} dx$$
$$= W_{\mathbb{R}^m}(\mathcal{I}_{\infty,i,j} \circ \vec{\xi}_\infty^{i,j}) - 4\pi \theta_{i,j}, \tag{6.13}$$

where $\theta_{i,j}$ is the integer density of $\mathcal{I}_{\infty,i,j} \circ \vec{\xi}_\infty^{i,j}$ at the image point $0 \in \mathbb{R}^m$.

Such an inverted (compact) branched Willmore immersion of S^2 corresponds to a Euclidean bubble $\vec{\zeta}_l$, $l \in \{1, \ldots, r\}$ in the statement of Theorem A.

The combination of (6.9), (6.10), (6.12) and (6.13) gives the desired energy identity (up to a subsequence in k):

$$\lim_{k \to \infty} W_{(M^m,h)}(\vec{\Phi}_k) = W_{(M^m,h)}(\vec{\xi}_\infty) + \sum_{i=1}^p W_{(M^m,h)}(\vec{\Psi}_i) + \sum_{j=1}^q W_{\mathbb{R}^m}(\vec{\eta}_j) + \sum_{l=1}^r \left(W_{\mathbb{R}^m}(\vec{\zeta}_l) - 4\pi \theta_{0,l}\right).$$

$\square$

Arguing along the lines of the proof of Theorem A, one can prove the following quantization result for surfaces of arbitrary genus, under the assumption of weak convergence to a limit surface and a bound on the conformal structures.

Theorem 6.1. *Let Σ be a closed Riemann surface, (M^m, h) be a smooth compact Riemannian manifold of dimension $m \geq 3$, and let $\{\vec{\Phi}_k\}_{k \in \mathbb{N}} \subset \mathrm{Imm}(\Sigma, M^m)$ be a sequence of conformally parametrised Willmore immersions. Assume that the conformal classes of $\left\{\vec{\Phi}_k^* h\right\}_{k \in \mathbb{N}}$ remain within a compact region of the moduli space of Σ and that there exists a weak, possibly branched, immersion $\vec{\Phi}_\infty : \Sigma \to (M^m, h)$ such that*

$$\vec{\Phi}_k \underset{k \to \infty}{\longrightarrow} \vec{\Phi}_\infty \quad \text{weakly in } \mathrm{W}^{2,2}_{\mathrm{loc}}(\Sigma \setminus \{a_1, \cdots, a_N\}) \text{ and weakly}^* \text{ in } \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Sigma \setminus \{a_1, \cdots, a_N\}) \qquad (6.14)$$

where $\{a_1, \cdots, a_N\} \subset \Sigma$ is a finite set. Then the following identity holds:

$$\lim_{k \to \infty} W_{(M^m, h)}(\vec{\Phi}_k) = W_{(M^m, h)}(\vec{\Phi}_\infty) + \sum_{i=1}^p W_{(M^m, h)}(\vec{\Psi}_i) + \sum_{j=1}^q W_{\mathbb{R}^m}(\vec{\eta}_j) + \sum_{l=1}^r \left(W_{\mathbb{R}^m}(\vec{\zeta}_l) - 4\pi\, \theta_{0,l} \right) ,$$

where:

(1) *The map $\vec{\Phi}_\infty$ is a smooth, possibly branched, Willmore immersion of Σ into (M^m, h) and*

$$\vec{\Phi}_k \underset{k \to \infty}{\longrightarrow} \vec{\Phi}_\infty \quad \text{in } C^l_{\mathrm{loc}}(\Sigma \setminus \{a_1, \cdots, a_N\}), \quad \forall l \in \mathbb{N}.$$

Furthermore, it holds

$$\lim_{k \to \infty} W_{(M^m, h)}(\vec{\Phi}_k) = W_{(M^m, h)}(\vec{\Phi}_\infty) \iff \vec{\Phi}_k \underset{k \to \infty}{\longrightarrow} \vec{\Phi}_\infty \text{ in } C^l(\Sigma), \quad \forall l \in \mathbb{N}.$$

(2) *The maps $\vec{\Psi}_i : S^2 \to (M^m, h)$ are smooth, possibly branched, Willmore immersions.*

(3) *The maps $\vec{\eta}_j : S^2 \to \mathbb{R}^m$ and $\vec{\zeta}_l : S^2 \to \mathbb{R}^m$ are smooth, possibly branched, Willmore immersions and $\theta_{0,l} = \theta_0(\vec{\zeta}_l, x_l) \in \mathbb{N}$ is the multiplicity of $\vec{\zeta}_l$ at some some point $x_l \in \mathbb{R}^m$.*

(4) *The Riemannian Willmore bubbles $\vec{\Psi}_i : S^2 \to M^m$ are obtained as follows: there exist a sequence of unit area and constant curvature metrics h_k on Σ conformally equivalent to $\vec{\xi}_k^* h$ and strongly converging in $C^l(\Sigma)$ such that for any $i \in \{1, \ldots, p\}$, there exists a sequence of points $x_k^i \in \Sigma$ converging to one of $\{a_1, \cdots, a_N\}$, a sequence of radii ρ_k^i converging to zero such that (in converging h_k conformal coordinates φ_k around the given point in $\{a_1, \cdots, a_N\}$):*

$$\vec{\xi}_k \circ \varphi_k(\rho_k^i y + \varphi_k^{-1}(x_k^i)) \underset{k \to \infty}{\longrightarrow} \vec{\Psi}_i \circ \pi^{-1}(y) \quad \text{in } C^l_{\mathrm{loc}}(\mathbb{C} \setminus \{a_1^i, \cdots, a_{N_i}^i\}), \quad \forall l \in \mathbb{N},$$

where π denotes the stereographic projection from S^2 into $\mathbb{C}$, and $\{a_1^i, \cdots, a_{N_i}^i\}$ is a finite set of points in the complex plane.

(5) *The Euclidean Willmore bubbles $\vec{\eta}_j, \vec{\zeta}_l : S^2 \to \mathbb{R}^m$ are obtained by the following blow-up procedure: there exist a sequence of unit area and constant curvature metrics h_k on Σ conformally equivalent to $\vec{\xi}_k^* h$ and strongly converging in $C^l(\Sigma)$ such that for any $s \in \{1, \ldots, p\}$ (resp. for any $t \in \{1, \ldots, q\}$), there exists a point $\overline{x}^j \in M$ (resp. $\overline{x}^l \in M$), there exists a sequence of points $x_k^j \in \Sigma$ (resp. $x_k^l \in \Sigma$) converging to one of $\{a_1, \cdots, a_N\}$, a sequence of radii ρ_k^j (resp. ρ_k^l) converging to zero, a sequence of rescalings $\lambda_k^j \to \infty$ (resp. $\lambda_k^l \to \infty$) and inversions Ξ_k^l of $\mathbb{R}^m$ such that (in converging h_k conformal coordinates φ_k around the given point in $\{a_1, \cdots, a_N\}$):*

$$\lambda_k^j \cdot \mathrm{Exp}_{\overline{x}^j}^{-1} \circ \vec{\xi}_k \circ \varphi_k(\rho_k^j y + \varphi_k^{-1}(x_k^j)) \underset{k \to \infty}{\longrightarrow} \vec{\eta}_j \circ \pi^{-1}(y) \quad \text{in } C^l_{\mathrm{loc}}(\mathbb{C} \setminus \{a_1^j, \cdots, a_{N_j}^j\}), \quad \forall l \in \mathbb{N},$$

and, respectively,

$$\Xi_k^l \circ \lambda_k^l \cdot \mathrm{Exp}_{\overline{x}^l}^{-1} \circ \vec{\xi}_k \circ \varphi_k(\rho_k^l y + \varphi_k^{-1}(x_k^l)) \underset{k \to \infty}{\longrightarrow} \vec{\zeta}_l \circ \pi^{-1}(y) \quad \text{in } C^l_{\mathrm{loc}}(\mathbb{C} \setminus \{a_1^l, \cdots, a_{N_l}^l\}), \quad \forall l \in \mathbb{N},$$

where π denotes the stereographic projection from S^2 into $\mathbb{C}$, and $\{a_1^j, \cdots, a_{N_j}^j\}, \{a_1^l, \cdots, a_{N_l}^l\}$ are finite sets of points in the complex plane.

7 Appendix

7.1 Lorentz and Orlicz Spaces

Let (X, μ) be a measured space. For any μ-measurable function $f : X \to \mathbb{R}^n$, we have

$$\int_X |f|^p d\mu = p \int_0^\infty t^{p-1} \lambda_f(t) dt \,,$$

where, for all $t > 0$, we denote

$$\lambda_f(t) = \mu \left(X \cap \{x : |f(x)| > t\} \right) \,.$$

Define the decreasing rearrangement $f_* : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ of f by

$$f_*(t) = \inf \left(\mathbb{R}_+ \cap \{s : \lambda_f(s) \leq t\} \right) \,.$$

It is clear from the definitions that for all $t > 0$

$$\mathscr{L}^1 \left(\mathbb{R}_+ \cap \{s : f_*(s) > t\} \right) = \lambda_f(t) \,, \quad \text{for all } t > 0 \,.$$

Applying twice the slicing formula, we deduce that

$$\int_X |f|^p d\mu = p \int_0^\infty t^{p-1} \lambda_f(t) dt = p \int_0^\infty t^{p-1} \mathscr{L}^1 \left(\mathbb{R}_+ \cap \{s : f_*(s) > t\} \right) dt = \int_0^\infty f_*^p(s) ds.$$

More generally, for all real-valued differentiable functions $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$, we have the integration by parts formula

$$\int_0^\infty \varphi(\lambda_f(t)) \psi'(t) dt = \int_0^\infty \varphi'(t) \psi \left(f_*(t) \right) dt. \tag{7.1}$$

Since this formula is not completely standard, we give a proof of it. Let f be a non-negative step function, then there exists $0 < a_1 < a_2 < \cdots < a_n < \infty$ and pair-wise disjoint measurable sets $A_1, \cdots, A_n$ such that

$$f = \sum_{i=1}^n a_i \mathbf{1}_{A_i}.$$

Following [61], defining $B_i = \bigcup_{j=i}^n A_j$, we have

$$f = \sum_{i=1}^n (a_i - a_{i-1}) \mathbf{1}_{B_i},$$

where $a_0 = 0$. Then it holds

$$\lambda_f = \sum_{i=1}^n \mu(B_i) \mathbf{1}_{[a_{i-1}, a_i]} \quad \text{and} \quad f_* = \sum_{i=1}^n a_i \mathbf{1}_{[\mu(B_{i+1}), \mu(B_i)]}.$$

Thanks to a discrete integration by parts with $B_{n+1} = \emptyset$, we deduce that

$$\int_0^\infty \varphi(\lambda_f(t)) \psi'(t) dt = \sum_{i=1}^n \int_{a_{i-1}}^{a_i} \varphi(\mu(B_i)) \psi'(t) = \sum_{i=1}^n \varphi(\mu(B_i)) \left(\psi(a_i) - \psi(a_{i-1}) \right)$$

$$= \sum_{i=1}^n \psi(a_i) \left(\varphi(\mu(B_i)) - \varphi(\mu(B_i + 1)) \right) = \sum_{i=1}^n \psi(a_i) \int_{\mu(B_{i+1})}^{\mu(B_i)} \varphi'(t) dt = \sum_{i=1}^n \int_{\mu(B_{i+1})}^{\mu(B_i)} \varphi'(t) \psi(f_*(t)) dt$$

$$= \int_0^\infty \varphi'(t) \psi(f_*(t)) dt.$$

The rest of the proof is the same as [61] if φ is unbounded or if (X, μ) is σ-finite, and given in [60] in the general case. Notice that the proof would hold unchanged only assuming that φ and ψ are absolutely continuous.

We will now define a class of Lorentz spaces (which can also be seen as generalisation of Orlicz spaces [47]) of interest in this paper (see [61] and [60]). Let $\mathscr{C}$ be the set of non-negative concave functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that φ is continuous at 0,

$$\varphi(0) = \lim_{t \to 0} \varphi(t) = 0$$

and $\varphi(t) > 0$ for all $t > 0$. For all measurable $f : X \to \mathbb{R}^n$ (of $f : X \to \mathbb{R} \cup \{\pm\infty\}$), define the norm

$$\|f\|_{N(\varphi)} = \int_0^\infty \varphi(\lambda_f(t))dt.$$

Now fix some integer $n \geq 1$ and let $\mathscr{M}(X)$ be the class of measurable $\mathbb{R}^n$-valued function on X. Define

$$N(\varphi) = \mathscr{M}(X) \cap \left\{ f : \|f\|_{N(\varphi)} < \infty \right\}.$$

Then we have the following result:

Theorem 7.1 (Steigerwalt-White [61], Steigerwalt [60])**.** *The functional $\| \cdot \|_{N(\varphi)}$ is a norm on $N(\varphi)$ and $(N(\varphi), \| \cdot \|_{N(\varphi)})$ is a Banach space.*

By the integration by parts formula (7.1), we deduce that

$$\|f\|_{N(\varphi)} = \int_0^\infty \varphi'(t) f_*(t)\, dt.$$

Now let $1 < p < \infty$ and $1 \leq q < \infty$. Define for all $t > 0$

$$f_{**}(t) = \fint_0^t f_*(s)ds = \frac{1}{t} \int_0^t f_*(s)ds.$$

Then, the Lorentz space $\mathrm{L}^{p,q}(X)$ is defined by

$$\mathrm{L}^{p,q}(X) \cap \left\{ f : \|f\|_{\mathrm{L}^{p,q}(X)} < \infty \right\}, \quad \text{where} \quad \|f\|_{\mathrm{L}^{p,q}(X)} = \left(\int_0^\infty t^{\frac{q}{p}-1} f_{**}^q(t)dt \right)^{\frac{1}{q}}.$$

It is a Banach space and the following semi-norm $| \cdot |_{\mathrm{L}^{p,q}(X)}$

$$|f|_{\mathrm{L}^{p,q}(X)} = \left(\int_0^\infty t^{\frac{q}{p}-1} f_*^q(t)dt \right)^{\frac{1}{q}}$$

is equivalent to $\| \cdot \|_{\mathrm{L}^{p,q}(X)}$. In the case $q = 1$, we have by Fubini's theorem

$$\|f\|_{\mathrm{L}^{p,1}(X)} = \frac{p}{p-1} |f|_{\mathrm{L}^{p,1}(X)}.$$

Using the integration by parts formula (7.1), with $\varphi(t) = \frac{p}{q} t^{\frac{p}{q}}$ and $\psi(t) = t^q$ we deduce that

$$\int_0^\infty t^{\frac{p}{q}-1} f_*^q(t)dt = \int_0^\infty q t^{q-1} \frac{p}{q} \lambda_f(t)^{\frac{q}{p}} dt = p \int_0^\infty t^{q-1} \lambda_f(t)^{\frac{q}{p}} dt.$$

This gives the well known fact that $\mathrm{L}^{p,p}(X) = \mathrm{L}^p(X)$ with equivalent norms. Taking instead $\varphi(t) = t^{\frac{1}{p}}$, we get that

$$\|f\|_{\mathrm{L}^{p,1}(X)} = \frac{p}{p-1} |f|_{\mathrm{L}^{p,1}(X)} = \frac{p}{p-1} \int_0^\infty t^{\frac{1}{p}-1} f_*(t)dt = \frac{p^2}{p-1} \|f\|_{N(\varphi)}.$$

Therefore, the spaces $N(\varphi)$ are generalisations of $\mathrm{L}^{p,1}$ spaces, but $\mathrm{L}^{p,q}$ spaces with $1 < q < \infty$ are not $N(\varphi)$-spaces.

Now, we will define generalisations of the weak L^p spaces or Marcinkiewicz spaces. Fix a σ-algebra $\mathscr{A} \subset X(\mu)$ and assume the following property: for all $A \in \mathscr{A}$, if $\mu(A) = \infty$, there exists $B \subset A$ such that $0 < \mu(B) < \infty$. For $\varphi \in \mathscr{C}$, set

$$\|f\|_{M(\varphi)} = \sup_{t>0} \left\{ \frac{1}{\varphi(t)} \int_0^t f_*(s)ds \right\}, \quad \text{for all } f \in \mathscr{M}(X). \tag{7.2}$$

If $1 \le p < \infty$, define $\mathrm{L}^{p,\infty}(X) = M(t^{1-\frac{1}{p}})$. It is known that $\mathrm{L}^{p,\infty}(X)$ is a Banach space equipped with this norm for $1 < p < \infty$ (and such a norm is denoted by $\|\cdot\|_{\mathrm{L}^{p,\infty}(X)}$). Furthermore, the following result holds.

Theorem 7.2 (Steigerwalt-White [61], Steigerwalt [60]). *Assume that $\varphi(t) = o(t)$ as $t \to \infty$. Then $M(\varphi)$ is a norm and $(M(\varphi), \|\cdot\|)$ is a Banach space.*

Remark 7.3. In [61], the authors first define the norm

$$\|f\|_{M(\varphi)} = \sup \left\{ \frac{1}{\varphi(\mu(A))} \int_A |f| d\mu : A \in \mathscr{A}_1 \right\} \tag{7.3}$$

where

$$\mathscr{A}_1 = \mathscr{A} \cap \{A : 0 < \mu(A) < \infty\}.$$

Then Theorem 7.2 holds with this norm without any restrictions on φ, and the authors show (Theorem 3.3) that (7.2) and (7.3) coincide if either $(X, \mathscr{A}, \mu)$ is σ-finite or if $\varphi(t) = o(t)$ as $t \to \infty$. Then they quote Steigerwalt's PhD thesis [60] where the result is proven without any hypothesis on X or φ. Notice that this result does not contradict the fact that $\mathrm{L}^{1,\infty}$ is not a Banach space (even with $X = \mathbb{R}$ and $\mu = \mathscr{L}^1$). Indeed, since $\mathrm{L}^{p,\infty}(X) = M(t^{1-\frac{1}{p}})$, we would have $\mathrm{L}^{1,\infty}(X) = M(1)$ but the function $\varphi(t) = 1$ is not admissible (it does not belong to $\mathscr{C}$) since it does not satisfy $\varphi(0) = 0$. However, taking $p = \infty$, we formally get $\mathrm{L}^{\infty,\infty}(X) = M(t) = N(t)^* = \mathrm{L}^1(X)^* = \mathrm{L}^\infty(X)$, so there is no new Lorentz space corresponding to $p = q = \infty$.

Now, it is known that for all $1 < p < \infty$ and $1 \le q \le \infty$, the dual space of $\mathrm{L}^{p,q}(X)$ is $\mathrm{L}^{p',q'}(X)$ with

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{q} + \frac{1}{q'} = 1.$$

Moreover, for all measurable $f, g : X \to \mathbb{R} \cup \{\pm\infty\}$ such that $f \in \mathrm{L}^{p,q}(X)$ and $g \in \mathrm{L}^{p',q'}(X)$, it holds

$$\left| \int_X fg \, d\mu \right| \le \|f\|_{\mathrm{L}^{p,q}(X)} \|g\|_{\mathrm{L}^{p',q'}(X)}.$$

For the generalised Lorentz space $N(\varphi)$, we have the following duality result.

Theorem 7.4 (Steigerwalt-White [61], Steigerwalt [60]). *For all $(f, g) \in N(\varphi) \times M(\varphi)$, it holds $fg \in \mathrm{L}^1(X, \mu)$ and*

$$\left| \int_X fg \, d\mu \right| \le \|f\|_{N(\varphi)} \|g\|_{M(\varphi)}.$$

In particular, $N(\varphi)^* = M(\varphi)$.

Notice that those results are consistent with the $\mathrm{L}^{p,1} - \mathrm{L}^{p',\infty}$ duality: indeed $\mathrm{L}^{q,\infty}(X) = M(t^{\frac{1}{q}})$, so that $\mathrm{L}^{p,1}(X)^* = N(t^{\frac{1}{p}})^* = M(t^{\frac{1}{p}}) = \mathrm{L}^{p',\infty}(X)$.

Finally, generalising both the Lorentz spaces and the Orlicz spaces, one can add a positive weight in the definition of $N(\varphi)$ which gives a new norm $N(\varphi, \psi)$ defined by

$$\|f\|_{N(\varphi,\psi)} = \int_0^\infty \varphi(\lambda_f(t))\psi(t) \, dt = \int_0^\infty \varphi'(t)\Psi(f_*(t)) \, dt, \quad \text{where } \Psi(t) = \int_0^t \psi(s) \, ds.$$

where the second identity follows from (7.1). Here, to make sure that $N(\varphi, \psi)$ is a norm, we must assume that $\psi(t) > 0$ for all $t > 0$ and $\varphi \in \mathscr{C}$. However, even this generalisation does not permit to recover the Lorentz spaces $\mathrm{L}^{p,q}(X)$ for $1 < q < \infty$.

We end this section with a trivial remark that we call lemma for convenience.

Lemma 7.5. *Let $\varphi, \psi \in \mathscr{C}$ be such that $\varphi \leq \psi$. Then, we have a continuous injection $N(\psi) \hookrightarrow N(\varphi)$ and a continuous injection $M(\varphi) \hookrightarrow M(\psi)$.*

Proof. Let $f \in N(\psi)$. Then, using the inequality $\varphi \leq \psi$ we have

$$\|f\|_{N(\varphi)} = \int_0^\infty \varphi(\lambda_f(t)) dt \leq \int_0^\infty \psi(\lambda_f(t)) dt = \|f\|_{N(\psi)},$$

which shows that $f \in N(\varphi)$, and that we have a continuous injection $N(\psi) \hookrightarrow N(\varphi)$. On the other hand, if $f \in M(\varphi)$, using $\varphi \leq \psi$, we get

$$\frac{1}{\psi(t)} \leq \frac{1}{\varphi(t)} \quad \text{for all } t > 0,$$

and

$$\|f\|_{M(\psi)} = \sup_{t>0} \left\{ \frac{1}{\psi(t)} \int_0^t f_*(s) ds \right\} \leq \sup_{t>0} \left\{ \frac{1}{\varphi(t)} \int_0^t f_*(s) ds \right\} = \|f\|_{M(\varphi)},$$

which shows that $f \in M(\psi)$, and implies the assertion on the continuity of the canonical injection $M(\varphi) \hookrightarrow M(\psi)$ mentioned in the lemma. □

7.2 A One-Parameter Family of Orlicz Spaces

Let $\alpha > 0$ and define

$$\varphi_\alpha(t) = \frac{t}{\log^\alpha \left(\frac{R}{t} \right)}. \tag{7.4}$$

Let $W : \mathbb{R}_+ \to \mathbb{R}_+$ be the Lambert function, which is the positive branch of the function $x \mapsto x\, e^x$ and that satisfies for all $x \geq 0$

$$W(x) e^{W(x)} = x \tag{7.5}$$

One easily checks that

$$\psi_\alpha(t) = R e^{-\alpha W \left(\frac{1}{\alpha} \left(\frac{R}{t} \right)^{\frac{1}{\alpha}} \right)}$$

is the inverse of φ_α. Notice that for $\alpha = 1$, the identity (7.5) gives

$$\psi_1(t) = t\, W \left(\frac{R}{t} \right).$$

Explicit computations give

$$\varphi_\alpha'(t) = \frac{1}{\log^\alpha \left(\frac{R}{t} \right)} + \frac{\alpha}{\log^{\alpha+1} \left(\frac{R}{t} \right)} > 0, \quad \varphi_\alpha''(t) = \frac{\alpha}{t \log^{\alpha+1} \left(\frac{R}{t} \right)} + \frac{\alpha(\alpha+1)}{t \log^{\alpha+2}(t)} > 0,$$

which show that φ_α is convex and strictly increasing. We deduce that for all $t > 0$, it holds

$$\mathscr{L}^2 \left(B(0, R) \cap \left\{ x : \frac{\log^\alpha \left(\frac{R}{|x|} \right)}{|x|} > t \right\} \right) = \mathscr{L}^2 \left(B(0, R) \cap \left\{ x : \varphi_\alpha(|x|) < \frac{1}{t} \right\} \right)$$

$$= \mathscr{L}^2\left(B(0,R) \cap \left\{x : |x| < \psi_\alpha\left(\frac{1}{t}\right)\right\}\right) = \pi \min\left\{R^2, \psi_\alpha^2\left(\frac{1}{t}\right)\right\} = \pi \min\left\{R^2, R^2 e^{-2\alpha W\left(\alpha^{-1}(Rt)^{\frac{1}{\alpha}}\right)}\right\}$$

$$= \pi R^2 e^{-2\alpha W\left(\alpha^{-1}(Rt)^{\frac{1}{\alpha}}\right)}.$$

The following asymptotic expansions hold:

$$\begin{cases} W(t) = t + O(t^2) & \text{when } t \to 0 \\ W(t) = \log(t) - \log\log(t) + o(1) & \text{when } t \to \infty. \end{cases}$$

Therefore, when $t \to 0$, we deduce that

$$e^{-2\alpha W\left(\alpha^{-1}(Rt)^{\frac{1}{\alpha}}\right)} = e^{-2(Rt)^{\frac{1}{\alpha}} + O(t^{\frac{2}{\alpha}})} \xrightarrow[t\to 0]{} 1,$$

so that

$$\lim_{t\to 0+} t^2 \mathscr{L}^2\left(B(0,R) \cap \left\{x : \frac{\log^\alpha\left(\frac{R}{|x|}\right)}{|x|} > t\right\}\right) = 0.$$

However, when $t \to \infty$, we have

$$e^{-2\alpha W\left(\alpha^{-1}(Rt)^{\frac{1}{\alpha}}\right)} = e^{-2\alpha\left(\log\left(\alpha^{-1}(Rt)^{\frac{1}{\alpha}}\right) - \log\log\left(\alpha^{-1}(Rt)^{\frac{1}{\alpha}}\right) + o(1)\right)}$$

$$= \frac{e^{2(\alpha\log(\alpha) - \log(R)) + o(1)}}{t^2} \log^{2\alpha}\left(\alpha^{-1}(Rt)^{\frac{1}{\alpha}}\right) = \frac{e^{A + o(1)}}{t^2}\left(\log^{2\alpha}(t) + O(1)\right).$$

for some $A \in \mathbb{R}$. This implies that

$$t^2 \mathscr{L}^2\left(B(0,R) \cap \left\{x : \frac{\log^\alpha\left(\frac{R}{|x|}\right)}{|x|} > t\right\}\right) = e^{A + o(1)}\left(\log^{2\alpha}(t) + O(1)\right) \xrightarrow[t\to\infty]{} \infty.$$

Note that

$$\lim_{t\to\infty}\left(\frac{t}{\log^\alpha(t)}\right)^2 \mathscr{L}^2\left(\mathbb{R}^2 \cap \left\{x : \frac{\log^\alpha\left(\frac{R}{|x|}\right)}{|x|} > t\right\}\right) \leq C < \infty.$$

This suggests that $u = u_\alpha = 1/\varphi_\alpha(|\cdot|)$ belongs to a Lorentz space. To determine its weight, we first compute the function $u_* : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$u_*(t) = \inf\left\{s > 0 : \mathscr{L}^2\left(B(0,R) \cap \{x : |u(x)| > s\}\right) \leq t\right\}.$$

For $t \leq \pi R^2$, it holds

$$\mathscr{L}^2\left(B(0,R) \cap \{x : |u(x)| > s\}\right) = \pi\psi^2\left(\frac{1}{s}\right) \leq t \iff \frac{1}{s} \leq \varphi\left(\sqrt{\frac{t}{\pi}}\right) \iff s \geq u\left(\sqrt{\frac{t}{\pi}}\right)$$

whilst $u_*(t) = 0$, for $t \geq \pi R^2$. Therefore,

$$u_*(t) = u\left(\sqrt{\frac{t}{\pi}}\right) = \sqrt{\frac{\pi}{t}}\log^\alpha\left(R\sqrt{\frac{\pi}{t}}\right), \quad \text{for } t \leq \pi R^2$$

and

$$u_*(t) = \begin{cases} \sqrt{\frac{\pi}{t}}\log^\alpha\left(R\sqrt{\frac{\pi}{t}}\right) & \text{for } t < \pi R^2 \\ 0 & \text{for } t \geq \pi R^2 \end{cases}$$

so that

$$\int_0^t u_*(s)\,ds = \sqrt{\pi}\left[2\sqrt{s}\log^\alpha\left(R\sqrt{\frac{\pi}{s}}\right)\right]_0^t + 2\pi\alpha R\int_0^t \frac{\log^{\alpha-1}\left(R\sqrt{\frac{\pi}{s}}\right)}{\sqrt{s}}\,ds$$

$$= 2\sqrt{\pi t}\log^\alpha\left(R\sqrt{\frac{\pi}{t}}\right) + 2\pi\alpha R\int_0^t \log^{\alpha-1}\left(R\sqrt{\frac{\pi}{s}}\right)\frac{ds}{\sqrt{s}},$$

which, for $\alpha = 1$, shows that

$$\int_0^t u_*(s)\,ds = 2\sqrt{\pi t}\log\left(R\sqrt{\frac{\pi}{t}}\right) + 4\pi R\sqrt{t}.$$

First assume that $\alpha \in \mathbb{N}$ and notice that, for $s = \pi R^2 v$, it holds:

$$\int_0^t u_*(s)\,ds = \pi R^2 \int_0^{\frac{t}{\pi R^2}} \frac{1}{R\sqrt{v}}\log^\alpha\left(\frac{1}{v}\right)du = \pi R\int_0^{\frac{t}{\pi R^2}} \frac{1}{\sqrt{v}}\log^\alpha\left(\frac{1}{v}\right)dv.$$

Now, we take α arbitrary and We introduce the integrals $I(\alpha, r)$ for $r \le 1$ defined by

$$I(\alpha, r) = \int_0^r \log^\alpha\left(\frac{1}{t}\right)\frac{dt}{\sqrt{t}}.$$

It holds

$$I(\alpha, r) = 2\sqrt{r}\log^\alpha\left(\frac{1}{r}\right) + 2\alpha\int_0^r \log^{\alpha-1}\left(\frac{1}{t}\right)\frac{dt}{\sqrt{t}} = 2\sqrt{r}\log^\alpha\left(\frac{1}{r}\right) + 2\alpha\, I(\alpha-1, r)$$

$$= 2\sqrt{r}\log^\alpha\left(\frac{1}{r}\right) + 4\alpha\sqrt{r}\log^{\alpha-1}\left(\frac{1}{r}\right) + 4\alpha(\alpha-1)I(\alpha-2, r).$$

Therefore, we deduce that provided that $\alpha = n + \beta$ with $n \in \mathbb{N}$ and $0 \le \beta < 1$, we have

$$I(\alpha, r) = \sum_{k=0}^{n-1} 2^{k+1}\left(\prod_{l=0}^{k-1}(n-l+\beta)\right)\sqrt{r}\log^{n-k+\beta}\left(\frac{1}{r}\right) + 2^n\prod_{k=0}^{n}(n-k+\beta)\int_0^r \log^\beta\left(\frac{1}{t}\right)\frac{dt}{\sqrt{t}}. \tag{7.6}$$

Now, we will estimate the last integral by the method of stationary phase. A change of variable $x = \log^\beta\left(\frac{1}{t}\right)$ shows that, for $p = \frac{1}{\beta} > 1$, it holds

$$\int_0^r \log^\beta\left(\frac{1}{t}\right)\frac{dt}{\sqrt{t}} = \int_{\log^\beta\left(\frac{1}{r}\right)}^\infty px^p e^{-\frac{x^p}{2}}\,dx = \left[-2x\,e^{-\frac{x^p}{2}}\right]_{\log^\beta\left(\frac{1}{r}\right)}^\infty + 2\int_{\log^\beta\left(\frac{1}{r}\right)}^\infty e^{-\frac{x^p}{2}}\,dx$$

$$= 2\sqrt{r}\log^\beta\left(\frac{1}{r}\right) + 2\log^\beta\left(\frac{1}{r}\right)\int_1^\infty e^{-\log\left(\frac{1}{r}\right)\frac{x^p}{2}}\,dx,$$

where we performed a linear change of variable in the last integral. Now, if $F : (0, \infty) \to \mathbb{R}$ is defined by

$$F(t) = \int_1^\infty e^{-t\frac{x^p}{2}}\,dx, \tag{7.7}$$

then we can directly apply the method of stationary phase since $p > 1$ (it will be clear that it fails for $p = 1$ since in this case, we have $F(t) = 2/t$). Indeed, if $\varphi : [1, \infty) \to \mathbb{R}$ is defined by $\varphi(x) = -\frac{x^p}{2}$, then φ is strictly decreasing and $\varphi''(1) = -\frac{p(p-1)}{2} < 0$, so the method of stationary phase (or rather Laplace's method) implies that

$$F(t) \underset{t\to\infty}{\sim} \sqrt{\frac{2\pi}{-\varphi''(1)}}\frac{e^{-\frac{t}{2}}}{\sqrt{t}} = \frac{4\pi}{p(p-1)}\frac{e^{-\frac{t}{2}}}{\sqrt{t}}.$$

Applying it to $t = \log\left(\frac{1}{r}\right)$, we deduce that

$$2\log^\beta\left(\frac{1}{r}\right)\int_1^\infty e^{-\log\left(\frac{1}{r}\right)\frac{x^p}{2}}\,dx \underset{r\to 0}{\sim} 2\beta\log^\beta\left(\frac{1}{r}\right)\sqrt{\frac{4\pi}{1-\beta}}\sqrt{\frac{r}{\log\left(\frac{1}{r}\right)}} = 2\beta\sqrt{\frac{4\pi}{1-\beta}}\sqrt{r}\log^{\beta-\frac{1}{2}}\left(\frac{1}{r}\right).$$

Therefore, the finitely many additional terms in the expansion (7.6) are negligible compared to the first one and we deduce that for $\alpha > 0$, we have

$$I(\alpha, r) \underset{r \to 0}{\sim} 2\sqrt{r} \log^\alpha \left(\frac{1}{r} \right). \tag{7.8}$$

In particular, choosing the function

$$\Lambda_\alpha(t) = \sqrt{t} \left(1 + \log_+^\alpha \left(R\sqrt{\frac{\pi}{t}} \right) \right), \tag{7.9}$$

we deduce that $u_\alpha \in M(\Lambda_\alpha) = \mathrm{L}^{2,\infty}_{\log^\alpha}(B(0,R))$, where

$$M(\Lambda_\alpha) = \mathrm{L}^1_{\mathrm{loc}}(B(0,R)) \cap \left\{ f : \|f\|_{M(\Lambda_\alpha)} = \sup_{t>0} \left(\frac{1}{\Lambda_\alpha(t)} \int_0^t f_*(s) ds \right) < \infty \right\}.$$

Likewise, we define the space $N(\Lambda_\alpha) = \mathrm{L}^{2,1}_{\log^\alpha}(B(0,R))$ by

$$N(\Lambda_\alpha) = \mathrm{L}^1_{\mathrm{loc}}(B(0,R)) \cap \left\{ f : \|f\|_{N(\Lambda_\alpha)} = \int_0^\infty \Lambda_\alpha \left(\mathcal{L}^2 \left(B(0,R) \cap \{x : |f(x)| > t\} \right) \right) dt < \infty \right\}.$$

Furthermore, notice that $\Lambda_\beta : (0,\infty) \to (0,\infty)$ is non-zero, concave (for $\beta \leq 1$), and that Λ_β extends continuously to 0 and that $\Lambda_\beta(0) = 0$. Therefore, we can apply the classical results of Steigerwalt and White to deduce ([61, Theorem 4.1 and Theorem 4.4], see also Theorem 7.4 in the appendix) that $N(\Lambda_\beta)^* = M(\Lambda_\beta)$, and that for all $(f,g) \in N(\Lambda_\beta) \times M(\Lambda_\beta)$, the product $fg \in \mathrm{L}^1(B(0,R))$ with

$$\left| \int_{B(0,R)} fg \, d\mathcal{L}^2 \right| \leq \|f\|_{N(\Lambda_\beta)} \|g\|_{M(\Lambda_\beta)}.$$

To see that Λ_β is concave, we can assume that $t \leq \pi R^2$ without loss of generality, so that up to a scaling Λ_β is concave if and only if

$$\psi(t) = \sqrt{t} \log^\beta \left(\frac{1}{t} \right)$$

is concave on $(0,1)$. We compute

$$\psi'(t) = \frac{1}{2\sqrt{t}} \log^\beta \left(\frac{1}{t} \right) - \frac{\beta}{\sqrt{t}} \log^{\beta-1} \left(\frac{1}{t} \right)$$

$$\psi''(t) = -\frac{1}{4t^{\frac{3}{2}}} \log^\beta \left(\frac{1}{t} \right) - \frac{\beta}{2t^{\frac{3}{2}}} \log^{\beta-1} \left(\frac{1}{t} \right) + \frac{\beta}{2t^{\frac{3}{2}}} \log^{\beta-1} \left(\frac{1}{t} \right) + \frac{\beta(\beta-1)}{t^{\frac{3}{2}}} \log^{\beta-2} \left(\frac{1}{t} \right)$$

$$= -\frac{1}{4t^{\frac{3}{2}}} \log^\beta \left(\frac{1}{t} \right) - \frac{\beta(1-\beta)}{t^{\frac{3}{2}}} \log^{\beta-2} \left(\frac{1}{t} \right) < 0$$

for all $0 < t < 1$ and $0 \leq \beta \leq 1$.

7.3 Technical Results for Holomorphic Functions with Orlicz Space Regularity

We will show that if u is a holomorphic function on $B(0,R)$, then a $\mathrm{L}^{2,\infty}_{\log^\beta}$ control implies a $\mathrm{W}^{1,1}$ control on $B(0,\alpha R)$ for all $\alpha < 1$. Since it does not seem standard to us, we give a full proof of this claim (in fact, we are not aware of a previous study of such spaces in the existing literature). We start with a slicing argument reminiscent of the proof of the Riemann-Lebesgue lemma.

Lemma 7.6. *Let* $0 < \beta \leq 1$ *and* $u \in L^{2,\infty}_{\log^\beta}(B(0,R))$. *Fix* $0 < \beta \leq 1$, $x \in B(0, \frac{R}{2})$ *and* $0 < r < \frac{R}{4}$. *Then, there exists* $\rho \in [r, 2r]$ *such that*

$$\int_{\partial B_\rho(x)} |u| d\mathcal{H}^1 \leq \frac{2\sqrt{3\pi}}{\log(2)} \left(1 + \log_+^\beta \left(\frac{1}{\sqrt{3}} \frac{R}{r} \right) \right) \|u\|_{\mathrm{L}^{2,\infty}_{\log^\beta}(B_{2r}\setminus \overline{B}_r(x))} \tag{7.10}$$

Proof. By the co-area formula, we have

$$\int_{B_{2r}\setminus\overline{B}_r(x)}|u(x)|dx = \int_r^{2r}\left(\rho\int_{\partial B_\rho(x)}|u|d\mathscr{H}^1\right)\frac{d\rho}{\rho} \geq \log(2)\inf_{r<\rho<2r}\left(\rho\int_{\partial B_\rho(x)}|u|d\mathscr{H}^1\right).$$

Therefore, there exists $\rho \in (r, 2r)$ such that

$$\rho\int_{\partial B_\rho(x)}|u|d\mathscr{H}^1 \leq \frac{1}{\log(2)}\int_{B_{2r}\setminus\overline{B}_r(x)}|u(x)|dx.$$

Now, Using the $\mathrm{L}^{2,1}_{\log^\beta}/\mathrm{L}^{2,\infty}_{\log^\beta}$ duality (see Theorem 7.4 in the appendix, or [61, Theorem 4.4]), we get

$$\int_{B_{2r}\setminus\overline{B}_r(x)}|u(x)|dx \leq \|1\|_{\mathrm{L}^{2,1}_{\log^\beta}(B_{2r}\setminus\overline{B}_r(0))}\|u\|_{\mathrm{L}^{2,\infty}_{\log^\beta}(B_{2r}\setminus\overline{B}_r(0))}.$$

Notice that

$$\lambda_1(t) = \mathscr{L}^2\left(B_{2r}\setminus\overline{B}_r(x)\cap\{x : 1 > t\}\right) = \begin{cases}3\pi r^2 & \text{if } t < 1 \\ 0 & \text{if } t \geq 1.\end{cases}$$

We have by definition

$$\|1\|_{\mathrm{L}^{2,1}_{\log}(B_{2r}\setminus\overline{B}_r(x))} = \int_0^\infty (\lambda_1(t))^{\frac{1}{2}}\left(1 + \log_+^\beta\left(R\sqrt{\frac{\pi}{\lambda_1(t)}}\right)\right)dt = \sqrt{3\pi}r\left(1 + \log_+^\beta\left(\frac{1}{\sqrt{3}}\frac{R}{r}\right)\right).$$

Finally, we deduce that

$$\rho\int_{\partial B_\rho(x)}|u|d\mathscr{H}^1 \leq \frac{\sqrt{3\pi}}{\log(2)}r\left(1 + \log_+^\beta\left(\frac{1}{\sqrt{3}}\frac{R}{r}\right)\right)\|u\|_{\mathrm{L}^{2,\infty}_{\log^\beta}(B_{2r}\setminus\overline{B}_r(x))},$$

and that, *a fortiori*, for all $x \in B(0, \frac{R}{2})$ and $0 < r \leq \frac{R}{2}$ such that $B(x, 2r) \subset B(0, R)$, there exists $\rho \in [r, 2r]$ such that

$$\int_{\partial B_\rho(x)}|u|d\mathscr{H}^1 \leq \frac{2\sqrt{3\pi}}{\log(2)}\left(1 + \log_+^\beta\left(\frac{1}{\sqrt{3}}\frac{R}{r}\right)\right)\|u\|_{\mathrm{L}^{2,\infty}_{\log^\beta}(B_{2r}\setminus\overline{B}_r(x))}. \tag{7.11}$$

$\square$

Thanks to this result, we will now be able to show a variant of [28, Lemma 10] (see also [29, Lemma B.1] and [37, Lemma 2.2]).

Lemma 7.7. *Let $u : B(0, R) \to \mathbb{C}$ be a holomorphic function and fix some $0 \leq \alpha < 1$, $0 \leq \beta \leq 1$. Assume that $u \in \mathrm{L}^{2,\infty}_{\log^\beta}(B(0, R))$. Then $u \in \mathrm{L}^2(B(0, \alpha R))$ and there exists a universal constant Γ_0 (independent of α and β) such that*

$$\|u\|_{\mathrm{L}^2(B(0,\alpha R))} \leq \Gamma_0\frac{\alpha}{(1-\alpha)}\left(1 + \log^\beta\left(\frac{1}{1-\alpha}\right)\right)\|u\|_{\mathrm{L}^{2,\infty}_{\log^\beta}(B(0,R))}.$$

Proof. Write

$$u(z) = \sum_{n=0}^\infty a_n z^n.$$

Notice that $u = \partial_z v$, where

$$v(z) = \sum_{n=1}^\infty b_n z^n = \sum_{n=1}^\infty \frac{a_{n-1}}{n}z^n. \tag{7.12}$$

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First, using the estimate (7.10) applied to ∇v at a point $z \in \partial B(0, \alpha R)$ with $r = \frac{1}{2}(1 - \alpha)R$, we deduce by the mean-value formula that for some $\rho \in [\frac{1}{2}(1 - \alpha)R, (1 - \alpha)R]$, we have

$$
|\nabla v(z)| = |2\,\partial_z v(z)| = \left| \frac{1}{\pi \rho} \int_{\partial B(z,\rho)} \partial_\zeta v(\zeta) d\mathscr{H}^1 \right|
$$

$$
\leq \frac{4}{\log(2)} \sqrt{\frac{3}{\pi}} \frac{1}{(1-\alpha)R} \left(1 + \log^\beta \left(\frac{2}{\sqrt{3}} \frac{1}{1-\alpha} \right) \right) \|\nabla v\|_{\mathrm{L}^{2,\infty}_{\log^\beta}(B(0,R))}
$$

$$
= \frac{8}{\log(2)} \sqrt{\frac{3}{\pi}} \frac{1}{(1-\alpha)R} \left(1 + \log^\beta \left(\frac{2}{\sqrt{3}} \frac{1}{1-\alpha} \right) \right) \|u\|_{\mathrm{L}^{2,\infty}_{\log^\beta}(B(0,R))}, \tag{7.13}
$$

where we used $|\nabla v|^2 = 4|\partial_z v|^2$ by the holomorphy of v. Therefore, via integration by parts and using that both v and $\overline{v}$ are harmonic, we deduce that

$$
\int_{B(0,\alpha R)} |u(z)|^2 |dz|^2 \leq \frac{1}{2} \int_{B(0,\alpha R)} |\nabla v|^2 dx = \frac{1}{2} \int_{B(0,\alpha R)} \operatorname{div}\left(\overline{v} \nabla v \right) = \frac{1}{2} \int_{\partial B(0,\alpha R)} \overline{v}\, \partial_\nu v \, d\mathscr{H}^1
$$

$$
= \frac{1}{2} \int_{\partial B(0,\alpha R)} \overline{(v - v_{\alpha R})} \partial_\nu v \, d\mathscr{H}^1, \tag{7.14}
$$

where for all $0 < \rho < R$, we set

$$
v_\rho = \fint_{\partial B_\rho(0)} v \, d\mathscr{H}^1.
$$

Thanks to the L^∞ bound (7.13) and the Sobolev embedding $\mathrm{H}^{\frac{1}{2}}(S^1) \hookrightarrow \mathrm{L}^1(B(0,1))$, there exists a uniform constant $C_0 > 0$ such that

$$
\|\nabla v\|^2_{\mathrm{L}^2(B(0,\alpha R))} = \left| \int_{\alpha R} \overline{(v - v_{\partial B_{\alpha R}})} \partial_\nu v \, d\mathscr{H}^1 \right| \leq \|v - v_{\alpha R}\|_{\mathrm{L}^1(\partial B_{\alpha R}(0))} \|\nabla v\|_{\mathrm{L}^\infty(\partial B_{\alpha R(0)})}
$$

$$
\leq C_0 \alpha R \|v\|_{\mathrm{H}^{\frac{1}{2}}(\partial B_{\alpha R}(0))} \times \frac{8}{\log(2)} \sqrt{\frac{3}{\pi}} \frac{1}{(1-\alpha)R} \left(1 + \log^\beta \left(\frac{2}{\sqrt{3}} \frac{1}{1-\alpha} \right) \right) \|u\|_{\mathrm{L}^{2,\infty}_{\log^\beta}(B(0,R))}
$$

$$
\leq \frac{8 C_0}{\log(2)} \sqrt{\frac{3}{\pi}} \frac{\alpha}{(1-\alpha)} \left(1 + \log^\beta \left(\frac{2}{\sqrt{3}} \frac{1}{1-\alpha} \right) \right) \|\nabla v\|_{\mathrm{L}^2(B(0,\alpha R))} \|u\|_{\mathrm{L}^{2,\infty}_{\log^\beta}(B(0,R))}.
$$

Therefore, we get

$$
\|u\|_{\mathrm{L}^2(B(0,\alpha R))} \leq \frac{4 C_0}{\log(2)} \sqrt{\frac{3}{\pi}} \frac{\alpha}{(1-\alpha)} \left(1 + \log^\beta \left(\frac{2}{\sqrt{3}} \frac{1}{1-\alpha} \right) \right) \|u\|_{\mathrm{L}^{2,\infty}_{\log^\beta}(B(0,R))}
$$

which concludes the proof of the lemma. $\qquad \square$

We will also need the following lemma.

Lemma 7.8. *Let $R > 0$ and $u : B(0, R) \to \mathbb{C}$ by a holomorphic function such that $u \in \mathrm{L}^2(B(0,R))$. Then, for all $0 < \alpha < 1$, the following estimates hold:*

$$
\|u\|_{\mathrm{L}^{2,1}(B(0,\alpha R))} \leq \frac{4\alpha}{1 - \alpha^2} \|u\|_{\mathrm{L}^2(B(0,R))}
$$

$$
\|\nabla u\|_{\mathrm{L}^1(B(0,\alpha R))} \leq 4\sqrt{\pi} \frac{\alpha^2}{(1 - \alpha^2)^{\frac{3}{2}}} \|u\|_{\mathrm{L}^2(B(0,R))}.
$$

Proof. Let $\{a_n\}_{n \in \mathbb{N}}$ be such that

$$
u(z) = \sum_{n=0}^\infty a_n z^n.
$$

Taking complex coordinates, we deduce that

$$\int_{B(0,R)} |u(z)|^2 |dz|^2 = 2\pi \sum_{n=0}^{\infty} \int_0^R |a_n|^2 \rho^{2n+1} d\rho = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} R^{2(n+1)}.$$

A direct computation shows that for all $n \geq 0$, we have

$$\||z|^n\|_{\mathrm{L}^{2,1}(B(0,R))} = 4\sqrt{\pi} R^{n+1}.$$

Therefore, we deduce by the triangle inequality and the Cauchy-Schwarz inequality that

$$\|u\|_{\mathrm{L}^{2,1}(B(0,\alpha R))} \leq 4\sqrt{\pi} \sum_{n=0}^{\infty} |a_n| (\alpha R)^{n+1} \leq 4\sqrt{\pi} \left(\sum_{n=0}^{\infty} (n+1)\alpha^{2(n+1)} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} R^{2(n+1)} \right)^{\frac{1}{2}}$$

$$= \frac{4\alpha}{1-\alpha^2} \|u\|_{\mathrm{L}^2(B(0,R))}.$$

Then, we compute

$$\partial_z u(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n.$$

Using that

$$\||z|^n\|_{\mathrm{L}^1(B(0,R))} = \frac{2\pi}{n+2} R^{n+2}, \quad \text{for all } n \geq 0, \tag{7.15}$$

we deduce that

$$\frac{1}{2} \|\nabla u\|_{\mathrm{L}^1(B(0,\alpha R))} = \|\partial_z u\|_{\mathrm{L}^1(B(0,\alpha R))} \leq 2\pi \sum_{n=0}^{\infty} \frac{(n+1)}{(n+2)} |a_{n+1}| (\alpha R)^{n+2}$$

$$\leq 2\pi \left(\sum_{n=0}^{\infty} \frac{(n+1)^2}{n+2} \alpha^{2(n+2)} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{|a_{n+1}|^2}{n+2} R^{2(n+2)} \right) \leq 2\sqrt{\pi} \frac{\alpha^2}{(1-\alpha^2)^{\frac{3}{2}}} \|u\|_{\mathrm{L}^2(B(0,R))}$$

where we used the following identities valid for $|a| < 1$

$$\sum_{n=0}^{\infty} \frac{(n+1)^2}{n+2} a^n = \sum_{n=0}^{\infty} (n+1) a^n - \sum_{n=0}^{\infty} \frac{n+1}{n+2} a^n = \sum_{n=0}^{\infty} (n+1) a^n - \sum_{n=0}^{\infty} a^n + \sum_{n=0}^{\infty} \frac{a^n}{n+2}$$

$$= \frac{1}{(1-a^2)} - \frac{1}{1-a} + \sum_{n=0}^{\infty} \frac{a^n}{n+2},$$

and

$$\sum_{n=0}^{\infty} \frac{a^n}{n+2} = \frac{1}{a^2} \sum_{n=0}^{\infty} \frac{a^{n+2}}{n+2} = \frac{1}{a^2} \left(\sum_{n=0}^{\infty} \frac{a^{n+1}}{n+1} - a \right) = \frac{1}{a^2} \left(-\log(1-a) - a \right)$$

which shows that

$$\sum_{n=0}^{\infty} \frac{(n+1)^2}{n+2} a^{n+2} = \frac{a^3}{(1-a)^2} - \log(1-a) - a \leq \frac{a^3}{(1-a)^2} + \frac{a^2}{2(1-a)} = \frac{a^2 + a^3}{2(1-a)^3}.$$

<div style="text-align: right">□</div>

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