A CLASSICAL $\mathbb{S}^2$ SPIN SYSTEM WITH DISCRETE OUT-OF-PLANE ANISOTROPY: VARIATIONAL ANALYSIS AT SURFACE AND VORTEX SCALINGS

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Abstract. We consider a classical Heisenberg system of $\mathbb{S}^2$ spins on a square lattice of spacing ε. We introduce a magnetic anisotropy by constraining the out-of-plane component of each spin to take only finitely many values. Computing the Γ-limit of the energy functional as $\varepsilon \to 0$ we prove that, in the continuum description, the system concentrates energy at the boundary of sets in which the out-of-plane component of the spin is constant and that, in each of such phases the energy can further concentrate on finitely many points corresponding to vortex-like singularities of the in-plane components of the spins.

Keywords: Γ-convergence, interface energy, topological singularities

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Contents

1. Introduction

The classical Heisenberg model is a lattice spin model that associates to a configuration $u\colon \mathbb{Z}^2 \to \mathbb{S}^2$ of $\mathbb{S}^2$-spins the energy

$$H(u) := -\frac{1}{2} \sum_{|i-j|=1} u(i) \cdot u(j)\,. \tag{1.1}$$

While, as for all ferromagnetic models, the ground states of this model are constant spin configurations, the analysis of its low energy states, *i.e.*, spin configurations whose energy deviates from the energy of the ground states by a small (in terms of energy per spin) energy, is quite delicate. In fact, due to the $SO(3)$ symmetry of the model, low energy Heisenberg spins can form complicated topological excitations, known as skyrmions. They are the topological charges of this model and can be roughly considered as an higher dimensional analog of the vortex structures formed in planar rotator models (also known as classical XY models) in which the spins take values in $\mathbb{S}^1$ and the model has only $SO(2)$ symmetry. The variational analysis of the planar rotator model (and of some of its variants) has been the object of many recent studies [4, 6, 5, 7, 10, 11, 15, 28] and the behavior of its topological excited states, the vortices, has been well understood at several energy scalings thanks to the equivalence between the discrete XY spin model and the continuum Ginzburg-Landau model for $\mathbb{S}^1$ valued Sobolev maps, whose variational analysis has been developed in the last 30 years [1, 25, 26] (see also the monographs [12, 30]). In contrast, the variational analysis of Ginzburg-Landau type theories for $\mathbb{S}^2$-valued Sobolev maps has a more recent history [23, 27] and the variational equivalence of Heisenberg lattice models with a continuum theory has not yet been investigated.

This paper can be thought of as a first attempt to understand some of the analogies between the classical Heisenberg model and the classical XY model when, due to the presence of a discrete spin anisotropy, the symmetry group of the Heisenberg model is reduced to $SO(2)\times\mathbb{Z}_N$ and skyrmions cannot appear. We point out that the spin anisotropy in our classical model has microscopic (on scales much smaller than ε) quantum mechanical origins, it is induced by the presence of an external magnetic field, and can be controlled by an electric field. All these effects are neglected in our model in which we only focus on the geometric constraint induced by the anisotropy.

In what follows we describe in more details the main results of this paper.

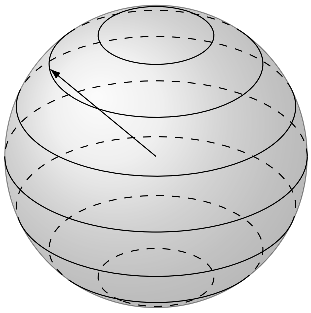

Figure 1. The codomain $\mathscr{S}^2_N$ of an admissible spin field in the case $N=7$.

Our analysis starts by localizing and scaling the energy (1.1) as follows. Given a regular open bounded set $\Omega\subset\mathbb{R}^2$ and a parameter $\varepsilon>0$, to every spin configuration $u\colon\varepsilon\mathbb{Z}^2\to\mathbb{S}^2$ we associate the energy per spin in Ω given by

$$H_\varepsilon(u,\Omega)=-\frac{1}{2}\sum_{\substack{\varepsilon i,\varepsilon j\in\varepsilon\mathbb{Z}^2\cap\Omega\\|i-j|=1}}u(\varepsilon i)\cdot u(\varepsilon j).$$

To enforce the anisotropic constraint in the Heisenberg model, we set $H_\varepsilon(u,\Omega)=+\infty$ unless the vertical component of the spin is constrained to a discrete set. To introduce such a class of admissible spins we assume u to be different from the north and the south poles and we collect its components (u^1,u^2,u^3) as $\big(\cos(\varphi(u))\widetilde{u},\sin(\varphi(u))\big)$, where $\widetilde{u}=u'/|u'|$, $u'=(u^1,u^2)$ and $\varphi(u)=\arcsin(u^3)\in(-\frac{\pi}{2},\frac{\pi}{2})$ is the latitude of u. The admissible spin configurations u are then defined to be those such that $u\in\mathscr{S}^2_N$, where $\mathscr{S}^2_N$ is the stratification of the unit sphere in N circles defined by

$$\mathscr{S}^2_N:=\Big\{y=\big(y',\sin(\varphi(y))\big)\in\mathbb{S}^2\ :\ \varphi(y)=-\tfrac{\pi}{2}+k\theta_N\,,\ k=1,\dots,N\Big\}\,,$$

where $\theta_N:=\frac{\pi}{N+1}$ (see Figure 1). We refer the energy $H_\varepsilon(u,\Omega)$ to its minimum by removing from each interaction energy between neighboring spins $-u(\varepsilon i)\cdot u(\varepsilon j)$ the energy -1 of two neighboring spins in a ground state configuration. We then divide by the number of lattice points in Ω which is of order $1/\varepsilon^2$ and we obtain a new energy per particle that we denote by $E_\varepsilon(u,\Omega)$ which is finite only on those spins valued in $\mathscr{S}^2_N$ on which it takes the form

$$E_\varepsilon(u,\Omega):=\frac{1}{2}\sum_{\substack{\varepsilon i,\varepsilon j\in\varepsilon\mathbb{Z}^2\cap\Omega\\|i-j|=1}}\varepsilon^2|u(\varepsilon i)-u(\varepsilon j)|^2\,. \tag{1.2}$$

The ground states u of this system have both discrete and continuous symmetries. They can be classified according to their latitude $\varphi(u)$, which can take the N admissible values $-\frac{\pi}{2}+k\theta_N$, $k=1,\dots,N$ and to the value of their normalized horizontal component $\widetilde{u}$ which belongs to $\mathbb{S}^1$. The energy needed to break such symmetries is of different orders.

The discrete symmetry of the vertical component of the admissible spins induces the existence of an energy regime, that we prove to be of order ε, at which phase separations can take place. In other words, as $\varepsilon \to 0$ spin configurations u_ε such that $E_\varepsilon(u_\varepsilon) \leq C\varepsilon$ are compact in $BV(\Omega, \mathscr{S}_N)$, hence they take finitely many values according to which Ω is partitioned in a finite union of sets of finite perimeter usually known as magnetic domains. On each of such sets, the limit spins have constant latitude $\bar\varphi \in \mathscr{L}_N := \big\{\ell_k := -\frac{\pi}{2} + k\theta_N\,,\ k = 1, \dots, N\big\}$ which jumps at the boundary of the partition. In this energetic regime, the horizontal components of u_ε are only weakly* compact in L^∞, hence in a set on which the latitude of a spin u takes the constant value $\bar\varphi$ the limit of its horizontal component can take any value in the disc of equation $|u'| \leq \cos(\bar\varphi)$. In Theorem 2.6 we prove that the Γ-limit of the scaled functional $E_\varepsilon(u)/\varepsilon$, carried out with respect to the L^1 convergence of the vertical components and the weak* convergence of the horizontal components of the spin, is proportional to the anisotropic perimeter of the boundaries (the interfaces of the magnetic domains, also known as magnetic domain walls) of the sets of the partition. The phenomenon of phase separation described above is similar to that happening in other systems with discrete symmetry as for instance in those Ising-like spin systems considered in [2, 10, 14, 16, 19].

The $SO(2)$ symmetry of the horizontal component of the spins in our model plays an important role at a lower energetic regime, *i.e.*, when $E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2|\log\varepsilon|$. As $\varepsilon^2|\log\varepsilon| \ll \varepsilon$, at the $\varepsilon^2|\log\varepsilon|$ regime the spin system cannot overcome the energetic barrier (of order ε) of the anisotropy transition explained above (the transition associated to the jump of the out-of-plane component of the spin field) (see Remark 2.5), hence the latitudes $\varphi(u_\varepsilon)$ converge strongly in L^1 to a constant latitude $\bar\varphi \in \mathscr{L}_N$ (here we are assuming that Ω is connected, otherwise the argument applies to each connected component of Ω). The normalized in-plane components $\widetilde{u}_\varepsilon$ of the spin field u_ε are associated to the relevant order parameter of the system, the discrete vorticity measures $\mu_{\widetilde{u}_\varepsilon}$ (see (2.6) for the definition) which keep track of a concentration phenomena already observed in the logarithmic scaling of the classical XY spin model. More precisely, in Theorem 2.10, we prove that, as $\varepsilon \to 0$, the vorticity measures $\mu_{\widetilde{u}_\varepsilon}$ converge to $\mu = \sum_{k=1}^M d_k\delta_{x_k}$, $(M \in \mathbb{N})$ with $d_k \in \mathbb{Z}$, $x_k \in \Omega$ and that the limit energy is proportional to $\cos^2(\bar\varphi)|\mu|(\Omega)$. The concentration of the vorticity measure $\mu_{\widetilde{u}_\varepsilon}$ on finitely many points can be read as the continuum description of the formation of finitely many singularities of a discrete spin whose out-of-plane component becomes constant and equal to $\bar\varphi \in \mathscr{L}_N$ while its in-plane component, constrained to be of length $\cos(\bar\varphi)$, winds d_k times around the points x_k in clockwise (if $d_k < 0$) or counter-clockwise (if $d_k > 0$) direction.

2. The model and main results

2.1. **Basic notation.** We let $\mathbb{R}^d$ be the d-dimensional Euclidean space with norm $|\cdot|$. The unit sphere in $\mathbb{R}^d$ is $\mathbb{S}^{d-1} := \{y \in \mathbb{R}^d\ :\ |y| = 1\}$. We let $e_1, \dots, e_d$ denote the vectors of the standard basis of $\mathbb{R}^d$. We write $B_r(x)$ for the open ball centered at x with radius r and we set $A_{r,R}(x) := B_R(x) \setminus \overline{B}_r(x)$.

For every $x = (x_1, x_2) \in \mathbb{R}^2$ we define $|x|_1 := |x_1| + |x_2|$. Given two unit vectors $y, z \in \mathbb{S}^2$, we write $\mathrm{d}_{\mathbb{S}^2}(y, z)$ for the geodesic distance on $\mathbb{S}^2$.

We let ι denote the imaginary unit. It will be used to identify unit vectors in $\mathbb{R}^2$ with complex numbers of the form $\exp(\iota\theta)$, $\theta \in \mathbb{R}$.

2.2. **BV-functions.** In this section we recall basic facts about functions of bounded variation. For more details we refer to the monograph [9].

Let $A \subset \mathbb{R}^d$ be an open set. A function $v \in L^1(A; \mathbb{R}^n)$ is a function of bounded variation if its distributional derivative $\mathrm{D}v$ is given by a finite matrix-valued Radon measure on A. We write $v \in BV(A; \mathbb{R}^n)$.

The space $BV_{\mathrm{loc}}(A; \mathbb{R}^n)$ is defined as usual. The space $BV(A; \mathbb{R}^n)$ is a Banach space when endowed with the norm $\|v\|_{BV(A)} = \|v\|_{L^1(A)} + |\mathrm{D}v|(A)$, where $|\mathrm{D}v|$ is the total variation measure of $\mathrm{D}v$. The total variation of $v\colon A \to \mathbb{R}$ with respect to the anisotropic norm $|\cdot|_1$ is denoted by $|\mathrm{D}v|_1$. For any Borel set $B \subset A$ the latter total variation can be written as $|Dv|_1(B) = \int_B \left|\frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}\right|_1 \mathrm{d}|Dv|$. If A is a bounded, open set with Lipschitz boundary, then $BV(A;\mathbb{R}^n)$ is compactly embedded in $L^1(A;\mathbb{R}^n)$. We say that a sequence v_n converges weakly* in $BV(A;\mathbb{R}^n)$ to v if $v_n \to v$ in $L^1(A;\mathbb{R}^n)$ and $\mathrm{D}v_n \overset{*}{\rightharpoonup} \mathrm{D}v$ in the sense of measures.

2.3. **Discrete spin fields.** We consider the square lattice $\varepsilon\mathbb{Z}^2$ with lattice spacing $\varepsilon > 0$. Given a non-empty set S, we will tacitly interpret maps $u\colon \varepsilon\mathbb{Z}^2 \to S$ as piecewise constant functions. More precisely, we let

$$\mathcal{PC}_\varepsilon(S) := \{u\colon \varepsilon\mathbb{Z}^2 \to S \ : \ u(x) = u(\varepsilon i) \text{ if } x \in \varepsilon i + [-\varepsilon/2, \varepsilon/2)^2 \text{ for some } i \in \varepsilon\mathbb{Z}^2\}\,, \quad (2.1)$$

and maps $u\colon \varepsilon\mathbb{Z}^2 \to S$ are in bijection with elements of $\mathcal{PC}_\varepsilon(S)$.

Given a vector $y = (y^1, y^2, y^3) \in \mathbb{S}^2$ we let $\varphi(y) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denote its latitude, *i.e.*,

$$\varphi(y) = \arcsin(y^3)\,.$$

We collect the components of y as follows:

$$y = \big(y', y^3\big) = \big(y', \sin(\varphi(y))\big)\,, \quad \text{where} \quad y' := (y^1, y^2)\,.$$

If $y \in \mathbb{S}^2$ is such that $\varphi(y) \neq \pm\frac{\pi}{2}$, then we can define the unit vector

$$\widetilde{y} := \frac{y'}{|y'|}.$$

Note that $|y'|^2 + \sin^2(\varphi(y)) = 1$, hence $|y'| = \cos(\varphi(y))$. We will often collect the components of y as follows:

$$y = \big(|y'|\widetilde{y}, y^3\big) = \big(\cos(\varphi(y))\widetilde{y}, \sin(\varphi(y))\big)\,.$$

We consider the stratification of the unit sphere in N circles given by

$$\mathscr{S}^2_N := \Big\{y = \big(y', \sin(\varphi(y))\big) \in \mathbb{S}^2 \ : \ \varphi(y) = -\tfrac{\pi}{2} + k\theta_N\,, \ k = 1, \ldots, N\Big\}\,,$$

where $\theta_N := \frac{\pi}{N+1}$. We stress that the north and south poles of $\mathbb{S}^2$ are excluded from the set $\mathscr{S}^2_N$. For future purposes, it is convenient to define the set of possible latitudes:

$$\mathscr{L}_N := \big\{\ell_k := -\tfrac{\pi}{2} + k\theta_N\,, \ k = 1, \ldots, N\big\}\,.$$

The following elementary lemma will be useful for our analysis.

Lemma 2.1. *Let $y, z \in \mathscr{S}^2_N$. Then*

$$|z - y|^2 \geq 4\frac{\sin^2\big(\frac{\theta_N}{2}\big)}{\theta_N}|\varphi(z) - \varphi(y)|\,.$$

Proof. Given $y \in \mathbb{S}^2$, we let $y = \big(\cos(\varphi(y))\widetilde{y}, \sin(\varphi(y))\big)$. Observe that for every $z = \big(\cos(\varphi(z))\widetilde{z}, \sin(\varphi(z))\big) \in \mathscr{S}^2_N$

$$|z - y|^2 = \big|\cos(\varphi(z))\widetilde{z} - \cos(\varphi(y))\widetilde{y}\big|^2 + \big|\sin(\varphi(z)) - \sin(\varphi(y))\big|^2$$

and $\mathbb{S}^1 \ni \widetilde{z} \mapsto \big|\cos(\varphi(z))\widetilde{z} - \cos(\varphi(y))\widetilde{y}\big|^2$ is minimized for $\widetilde{z} = \widetilde{y}$. Hence,

$$|z - y| \geq \Big|\big(\cos(\varphi(z))\widetilde{y}, \sin(\varphi(z))\big) - \big(\cos(\varphi(y))\widetilde{y}, \sin(\varphi(y))\big)\Big| \quad (2.2)$$

Observe that (2.2) and the identity $|a - b| = 2\sin\big(\frac{1}{2}\mathrm{d}_{\mathbb{S}^2}(a, b)\big)$ imply

$$\begin{aligned}|z - y|^2 &\geq \Big|\big(\cos(\varphi(z))\widetilde{y}, \sin(\varphi(z))\big) - \big(\cos(\varphi(y))\widetilde{y}, \sin(\varphi(y))\big)\Big|^2\\ &= 4\sin^2\Big(\frac{1}{2}\mathrm{d}_{\mathbb{S}^2}\Big(\big(\cos(\varphi(z))\widetilde{y}, \sin(\varphi(z))\big), \big(\cos(\varphi(y))\widetilde{y}, \sin(\varphi(y))\big)\Big)\Big)\\ &= 4\sin^2\Big(\frac{1}{2}|\varphi(z) - \varphi(y)|\Big)\,.\end{aligned}$$

Since $\varphi(y), \varphi(z) \in \mathscr{L}_N$, we have that $|\varphi(z) - \varphi(y)| = k\theta_N$ for some $k \in \mathbb{N}$, hence

$$4\sin^2\Big(\frac{1}{2}|\varphi(z) - \varphi(y)|\Big) = 4\sin^2\Big(\frac{k\theta_N}{2}\Big) \geq 4k\sin^2\Big(\frac{\theta_N}{2}\Big)\,,$$

where we used [18, Lemma 3.1]. We conclude that

$$|y-z|^2 \geq 4\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N}|\varphi(z)-\varphi(y)|\,.$$

□

Given an $\mathbb{S}^2$-valued spin field $u=(u^1,u^2,u^3)\colon \varepsilon\mathbb{Z}^2\to\mathbb{S}^2$, we collect its components as $u=\big(u',\sin(\varphi(u))\big)$, where $\varphi(u)$ is its latitude. To a spin field $u=\big(u',\sin(\varphi(u))\big)\colon\varepsilon\mathbb{Z}^2\to \mathscr{S}^2_N$ we associate the auxiliary $\mathbb{S}^1$-valued spin field $\widetilde{u}\colon\varepsilon\mathbb{Z}^2\to\mathbb{S}^1$ given by

$$\widetilde{u}:=\frac{u'}{|u'|}\,,$$

(Note that $\widetilde{u}$ is well-defined, since u differs from the north and south poles, hence $|u'|>0$.) Thus we write

$$u=\big(\cos(\varphi(u))\widetilde{u},\sin(\varphi(u))\big)\,.$$

We shall use the spin field $\widetilde{u}$ to define the vorticity measure $\mu_{\widetilde{u}}$ associated to u that is relevant for the problem.

2.4. **Description of the model.** Let $\Omega\subset\mathbb{R}^2$ be a bounded, open set with Lipschitz boundary. For every $u\colon\varepsilon\mathbb{Z}^2\to\mathbb{S}^2$ we set

$$E_\varepsilon(u,\Omega):=\frac{1}{2}\sum_{\substack{\varepsilon i,\varepsilon j\in\varepsilon\mathbb{Z}^2\cap\Omega\\|i-j|=1}}\varepsilon^2|u(\varepsilon i)-u(\varepsilon j)|^2 \tag{2.3}$$

if $u\colon\varepsilon\mathbb{Z}^2\to\mathscr{S}^2_N$ and $E_\varepsilon(u,\Omega):=+\infty$ otherwise. We will consider the energy as a functional $E_\varepsilon(\cdot,\Omega)\colon L^1(\Omega;\mathbb{R}^3)\to[0,+\infty]$ by interpreting spin fields $u\colon\varepsilon\mathbb{Z}^2\to\mathscr{S}^2_N$ as piecewise constant functions in $\mathcal{PC}_\varepsilon(\mathscr{S}^2_N)$.

2.5. **Vortices and the XY model.** We recall here some basic facts about discrete vorticity. Following [6], in order to define it, we introduce the projection $Q\colon\mathbb{R}\to2\pi\mathbb{Z}$ defined by

$$Q(t):=\operatorname{argmin}\{|t-s|\ :\ s\in2\pi\mathbb{Z}\}\,, \tag{2.4}$$

with the convention that, if the argmin is not unique, then we choose the one with minimal modulus. Then for every $t\in\mathbb{R}$ we define $\Psi(t):=t-Q(t)\in[-\pi,\pi]$.

Let $v\colon\varepsilon\mathbb{Z}^2\to\mathbb{S}^1$ and let $\varphi\colon\varepsilon\mathbb{Z}^2\to[0,2\pi)$ be the phase of v defined by the relation $v=\exp(\iota\varphi)$. For every $\varepsilon i\in\varepsilon\mathbb{Z}^2$, the discrete vorticity of v in the square $\varepsilon i+[0,\varepsilon]^2$ is defined by

$$\begin{aligned} d_v(\varepsilon i):=&\frac{1}{2\pi}\Big[\Psi\big(\varphi(\varepsilon i+\varepsilon e_1)-\varphi(\varepsilon i)\big)+\Psi\big(\varphi(\varepsilon i+\varepsilon e_1+\varepsilon e_2)-\varphi(\varepsilon i+\varepsilon e_1)\big)\\ &+\Psi\big(\varphi(\varepsilon i+\varepsilon e_2)-\varphi(\varepsilon i+\varepsilon e_1+\varepsilon e_2)\big)+\Psi\big(\varphi(\varepsilon i)-\varphi(\varepsilon i+\varepsilon e_2)\big)\Big]\,.\end{aligned} \tag{2.5}$$

As already noted in [6], it holds that $d_v\in\{-1,0,1\}$, *i.e.*, only vortices of degree ±1 can be present in the discrete setting. The discrete vorticity measure associated to v is given by

$$\mu_v:=\sum_{\varepsilon i\in\varepsilon\mathbb{Z}^2}d_v(\varepsilon i)\delta_{\varepsilon i+(\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\,. \tag{2.6}$$

Definition 2.2 (Flat convergence)**.** Let $A\subset\mathbb{R}^2$ be an open set. A sequence of finite Radon measures $\mu_j\in\mathcal{M}_b(A)$ converges *flat* to $\mu\in\mathcal{M}_b(A)$ if

$$\sup_{\substack{\psi\in C_c^\infty(A)\\\|\psi\|_{L^\infty}\leq1\,,\ \|\nabla\psi\|_{L^\infty}\leq1}}\Big|\int_A\psi\,\mathrm{d}(\mu_j-\mu)\Big|\to0\,.$$

In that case, we denote the convergence by $\mu_j\xrightarrow{\mathrm{f}}\mu$.

Given a bounded, open set with Lipschitz boundary $\Omega \subset \mathbb{R}^2$ and an $\mathbb{S}^1$-valued spin field $v\colon \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$, its XY energy is defined by

$$XY_\varepsilon(v,\Omega) := \frac{1}{2} \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \cap \Omega \\ |i-j|=1}} \varepsilon^2 |v(\varepsilon i) - v(\varepsilon j)|^2 .$$

We work with spin fields $v_\varepsilon\colon \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$ defined on the whole lattice $\varepsilon\mathbb{Z}^2$. We can always assume that $XY_\varepsilon(v_\varepsilon;\overline{\Omega}^\varepsilon) \leq C XY_\varepsilon(v_\varepsilon;\Omega)$, where $\overline{\Omega}^\varepsilon$ is the union of the squares $\varepsilon i + [0,\varepsilon]^2$ that intersect Ω. (If not, thanks to the Lipschitz regularity of Ω, we modify v_ε outside Ω in such a way that the energy estimate is satisfied, see [4, Remark 2].)

Remark 2.3. We recall that there is a strong relation between the number of discrete vortices and the XY-energy of a spin field. More precisely, under the assumption made above $XY_\varepsilon(v_\varepsilon;\overline{\Omega}^\varepsilon) \leq C XY_\varepsilon(v_\varepsilon;\Omega)$, there exists an universal constant $C' > 0$ such that for every $v_\varepsilon\colon \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$ we have

$$|\mu_{v_\varepsilon}|(\Omega) \leq \frac{C'}{\varepsilon^2} XY_\varepsilon(v_\varepsilon,\Omega) .$$

The reason is that a cell of $\varepsilon\mathbb{Z}^2$ with non-zero vorticity has non-aligned spins on the corners with a minimal distance for at least one couple of neighboring points, and thus carries an XY-energy larger than a strictly positive constant. For more details of the same estimate on the triangular lattice, see, *e.g.*, [11, Remark 3.1].

We recall the following compactness and lower bound for the XY model, see, *e.g.*, [4, Theorem 3] or [7, Theorem 3.1].

Proposition 2.4. *Let $v_\varepsilon\colon \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$ and assume that $XY_\varepsilon(v_\varepsilon,\Omega) \leq C\varepsilon^2|\log\varepsilon|$. Then there exists $M \in \mathbb{N}$ and a measure $\mu = \sum_{k=1}^M d_k \delta_{x_k}$ with $d_k \in \mathbb{Z}$ and $x_k \in \Omega$ such that, up to a non relabeled subsequence, $\mu_{v_\varepsilon} \xrightarrow{\mathrm{f}} \mu$ in Ω. Moreover*

$$2\pi|\mu|(\Omega) \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} XY_\varepsilon(v_\varepsilon,\Omega) .$$

2.6. **Surface scaling.** To present the results, we start by deducing some bounds provided by the energy.

Remark 2.5 (BV bound for $\varphi(u)$). We observe that the energy induces a bound on the total variation of the latitude $\varphi(u)$. Let $A \subset\subset \Omega$. By Lemma 2.1, given $u = \big(u', \sin(\varphi(u))\big)$, for ε small enough we have that

$$\begin{aligned} E_\varepsilon(u,\Omega) = \frac{1}{2} \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \cap \Omega \\ |i-j|=1}} \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2 &\geq 2\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N} \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \cap \Omega \\ |i-j|=1}} \varepsilon^2 |\varphi(u(\varepsilon i)) - \varphi(u(\varepsilon j))| \\ &\geq 4\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N} \varepsilon |\mathrm{D}\varphi(u)|_1(A) , \end{aligned}$$

where we used that the discrete energy counts each interaction twice. The previous inequality implies that sequences of spin fields $u_\varepsilon\colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$ satisfy

$$|\mathrm{D}\varphi(u_\varepsilon)|(A) \lesssim \frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon,\Omega) .$$

In the limit of the surface scaling, the domain Ω is partitioned in a finite union of sets of finite perimeter $(\Omega_{\bar\varphi})_{\bar\varphi \in \mathscr{L}_N}$ such that $\varphi = \bar\varphi$ a.e. in $\Omega_{\bar\varphi}$. This can be interpreted as follows: as $\varepsilon \to 0$, the spin field u_ε lies on the circle on $\mathbb{S}^2$ with latitude $\bar\varphi$ in most of the region $\Omega_{\bar\varphi}$. The limit energy is concentrated on the interfaces between the sets of the partition. The behavior of the first two components of the unit-vector field u is less rigid and we can control only the norm from above and a relaxation effect takes place. The precise statement is contained in the following theorem, for which we use the following notation: given $\varphi \in BV(\Omega;\mathbb{R})$, define

$$L^\infty(\Omega;\varphi) = \{v\colon \Omega \to \mathbb{R}^2 : |v| \leq \cos(\varphi) \text{ a.e. in } \Omega\} .$$

Theorem 2.6. *We have the following Γ-convergence result:*

(i) (Compactness) *Let $u_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$ and assume that $E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon$. Then there exists a function $\varphi \in BV(\Omega; \mathscr{L}_N)$ and $u' \in L^\infty(\Omega; \varphi)$ such that, up to a non-relabeled subsequence, $\varphi(u_\varepsilon) \to \varphi$ strongly in $L^1(\Omega;\mathbb{R})$ and $u'_\varepsilon \overset{*}{\rightharpoonup} u'$ in $L^\infty(\Omega;\mathbb{R}^2)$.*

(ii) (liminf inequality) *Let $u_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$, let $\varphi \in BV(\Omega;\mathscr{L}_N)$, and let $u' \in L^\infty(\Omega;\varphi)$. Assume that $\varphi(u_\varepsilon) \to \varphi$ strongly in $L^1(\Omega;\mathbb{R})$ and $u'_\varepsilon \overset{*}{\rightharpoonup} u'$ in $L^\infty(\Omega;\mathbb{R}^2)$. Then*

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon, \Omega) \geq \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N} |\mathrm{D}\varphi|_1(\Omega)\,.$$

(iii) (limsup inequality) *Let $\varphi \in BV(\Omega;\mathscr{L}_N)$ and $u' \in L^\infty(\Omega;\varphi)$. Then there exists a sequence of spin fields $u_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$ such that $\varphi(u_\varepsilon) \to \varphi$ strongly in $L^1(\Omega;\mathbb{R})$, $u'_\varepsilon \overset{*}{\rightharpoonup} u'$ in $L^\infty(\Omega;\mathbb{R}^2)$, and*

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon, \Omega) \leq \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N} |\mathrm{D}\varphi|_1(\Omega)\,.$$

Remark 2.7 (Limit as $N \to +\infty$). Let $E_{0,N}$ denote the Γ-limit of $\frac{1}{\varepsilon}E_\varepsilon(\cdot,\Omega)$ defined on $L^\infty(\Omega; B_1(0)) \times BV(\Omega;\mathscr{L}_N)$. Then one can prove that, as $N \to +\infty$, the sequence of functionals $\frac{1}{\theta_N}E_{0,N}$ Γ-converges with respect to weak*-convergence of u' and strong L^1-convergence of φ to the functional

$$E_{0,\infty}(u',\varphi) = \begin{cases} |\mathrm{D}\varphi|_1(\Omega) & \text{if } \varphi \in BV(\Omega;[-\pi/2,\pi/2]) \text{ and } u' \in L^\infty(\Omega;\varphi)\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Let us briefly sketch the argument: compactness of energy-bounded sequences follows from the compact embedding of BV in L^1. In order to prove the lower bound, it suffices to note that the anisotropic total variation is $L^1(\Omega)$-lower semicontinuous and that the condition $u'_n \in L^\infty(\Omega;\varphi_n)$ is stable when u'_n converges weakly* and φ_n converges strongly in $L^1(\Omega;\mathbb{R})$ (*cf.* the compactness proof of Theorem 2.6). The proof of the upper bound is slightly more technical. First one shows that one can approximate every piecewise constant map $\varphi \in BV(\Omega;[-\pi/2,\pi/2])$ with respect to strict convergence with piecewise constant maps $\varphi_N \in BV(\Omega;\mathscr{L}_N)$ using that the set $\mathscr{L}_N$ becomes dense in $[-\pi/2,\pi/2]$ as $N \to +\infty$. The approximating sequence $u'_N \in L^\infty(\Omega;\varphi_N)$ is defined by

$$u'_N(x) := \begin{cases} u'(x) & \text{if } |u'(x)| \leq \cos(\varphi_N(x))\,, \\ \frac{u'(x)}{|u'(x)|}\cos(\varphi_N(x)) & \text{otherwise,} \end{cases}$$

The L^1-convergence $\varphi_N \to \varphi$ implies that $u'_N \to u'$ even strongly in $L^1(\Omega;\mathbb{R}^2)$. This shows the upper bound for piecewise constant functions φ and $u' \in L^\infty(\Omega;\varphi)$. Then one uses the general fact that piecewise constant functions are dense in $BV(\Omega;[-\pi/2,\pi/2])$ with respect to strict convergence[1] and the same approximation of the component u' as above.

2.7. **Vortex scaling.** We carry out a finer analysis assuming that φ attains only one value $\bar\varphi \in \mathscr{L}_N$, *i.e.*, $\Omega = \Omega_{\bar\varphi}$. As no interface occurs, the scaled energy $\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon,\Omega)$ vanishes as $\varepsilon \to 0$, and thus it is reasonable to assume a stricter assumption on the energy scaling that seeks for a finer description of low-energy sequences. We study the scaled energy $\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon,\Omega)$ under the assumption $E_\varepsilon(u_\varepsilon,\Omega) \leq C\varepsilon^2|\log\varepsilon|$. On the one hand, by Remark 2.5 we get that $|\mathrm{D}\varphi(u_\varepsilon)|(A) \lesssim \varepsilon|\log\varepsilon| \to 0$ for every $A \subset\subset \Omega$. This is indeed compatible with the assumption that $\varphi(u_\varepsilon)$ converges to the constant function $\bar\varphi$. In the next remark we show that we have compactness for the discrete vorticity measures $\mu_{\tilde u_\varepsilon}$.

[1] To be more precise, this result is well-known for $BV(\Omega;\mathbb{R})$, but the case considered here follows by truncation.

Remark 2.8 (XY energy bound for $\widetilde{u}$). Let $u = (\cos(\varphi(u)\widetilde{u}, \sin(\varphi(u))) \colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$. We deduce a bound on the XY energy of $\widetilde{u} \colon \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$. By Lemma 2.9 below, we have that

$$\begin{aligned} E_\varepsilon(u,\Omega) &\geq \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2\cap\Omega \\ |i-j|=1}} \varepsilon^2 |u'(\varepsilon i) - u'(\varepsilon j)|^2 = \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2\cap\Omega \\ |i-j|=1}} \varepsilon^2 \big| |u'(\varepsilon i)|\widetilde{u}(\varepsilon i) - |u'(\varepsilon j)|\widetilde{u}(\varepsilon j)\big|^2 \\ &\geq \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2\cap\Omega \\ |i-j|=1}} \varepsilon^2 \min\{|u'(\varepsilon i)|^2, |u'(\varepsilon i)|^2\} |\widetilde{u}(\varepsilon i) - \widetilde{u}(\varepsilon j)|^2 \\ &\geq \varrho_N^2 \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2\cap\Omega \\ |i-j|=1}} \varepsilon^2 |\widetilde{u}(\varepsilon i) - \widetilde{u}(\varepsilon j)|^2 = \varrho_N^2 XY_\varepsilon(\widetilde{u}, \Omega)\,, \end{aligned}$$

where we set

$$\varrho_N^2 := \inf_{(y', \sin(\varphi(y))) \in \mathscr{S}^2_N} |y'|^2 = \sin^2(\theta_N)\,.$$

Thanks to Proposition 2.4, we deduce that a bound $E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2 |\log \varepsilon|$ gives compactness for the discrete vorticity measures $\mu_{\widetilde{u}_\varepsilon}$ with respect to flat convergence.

Lemma 2.9. *Let $a, b, c, d \in \mathbb{R}^d$. Assume that $|c| = |d|$. Then*

$$\big| |a|c - |b|d \big|^2 \geq \min\{|a|^2, |b|^2\} |c - d|^2.$$

Proof. Without loss of generality we assume that $|a| = \min\{|a|, |b|\}$, otherwise we exchange the role of $|a|c$ and $|b|d$. Then

$$\big||a|c - |b|d\big|^2 = \big||a|(c-d) + (|a|-|b|)d\big|^2 = |a|^2|c-d|^2 + \big||a|-|b|\big|^2|d|^2 + 2|a|\big(|a|-|b|\big)(c-d)\cdot d\,.$$

Observing that $|a| - |b| \leq 0$ and $(c-d)\cdot d = c\cdot d - |d|^2 \leq 0$, we have that

$$\big||a|c - |b|d\big|^2 \geq |a|^2|c-d|^2\,.$$

This concludes the proof. □

The precise behavior of the energy in the present scaling regime is contained in the following theorem.

Theorem 2.10. *Assume that Ω is connected*[2]*. Then we have the following:*

(i) (Compactness) *Let $u_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$ and assume that $E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2|\log\varepsilon|$. Then there exist a constant $\bar{\varphi} \in \mathscr{L}_N$ and a measure $\mu = \sum_{k=1}^M d_k \delta_{x_k}$ ($M \in \mathbb{N}$) with $d_k \in \mathbb{Z}$, $x_k \in \Omega$ such that, up to a non-relabeled subsequence, $\varphi(u_\varepsilon) \to \bar{\varphi}$ strongly in $L^1(\Omega;\mathbb{R})$ and $\mu_{\widetilde{u}_\varepsilon} \overset{\mathrm{f}}{\to} \mu$ in Ω.*

(ii) (liminf inequality) *Let $u_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$, let $\bar{\varphi} \in \mathscr{L}_N$, and let $\mu = \sum_{k=1}^M d_k \delta_{x_k}$ with $M \in \mathbb{N}$, $d_k \in \mathbb{Z}$, $x_k \in \Omega$. Assume that $\varphi(u_\varepsilon) \to \bar{\varphi}$ strongly in $L^1(\Omega;\mathbb{R})$ and that $\mu_{\widetilde{u}_\varepsilon} \overset{\mathrm{f}}{\to} \mu$ in Ω. Then*

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega) \geq 2\pi\cos^2(\bar{\varphi})|\mu|(\Omega)\,.$$

(iii) (limsup inequality) *Let $\bar{\varphi} \in \mathscr{L}_N$ and let $\mu = \sum_{k=1}^M d_k \delta_{x_k}$ with $M \in \mathbb{N}$, $d_k \in \mathbb{Z}$, $x_k \in \Omega$. Then there exists a sequence of spin fields $u_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$ such that $\varphi(u_\varepsilon) \to \bar{\varphi} \in \mathscr{L}_N$ strongly in $L^1(\Omega;\mathbb{R})$, $\mu_{\widetilde{u}_\varepsilon} \overset{\mathrm{f}}{\to} \mu$ in Ω, and*

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega) \leq 2\pi\cos^2(\bar{\varphi})|\mu|(\Omega)\,.$$

[2] If Ω is not connected, a similar result can be proved on each connected component. Due to the Lipschitz regularity and the boundedness of Ω there exist finitely many connected components and the boundary of each connected component is itself Lipschitz. These properties follow from the fact that the subgraph of a Lipschitz function is connected and a covering argument using the compactness of $\overline{\Omega}$.

2.8. **Gradient scaling.** To prove Theorem 2.10, we first need to prove the following result about the gradient scaling. Given $w\colon \varepsilon\mathbb{Z}^2 \to \mathbb{R}$, we let $A[w]\colon \mathbb{R}^2 \to \mathbb{R}$ denote its piecewise affine interpolation defined as the unique continuous function such that $A[w](\varepsilon i) = w(\varepsilon i)$ for all $i \in \mathbb{Z}^2$ and that is affine on all triangles of the form $\varepsilon i + \mathrm{co}(0, \varepsilon e_1, \varepsilon e_2)$ and $\varepsilon i + \mathrm{co}(0, -\varepsilon e_1, -\varepsilon e_2)$ with $i \in \mathbb{Z}$.

Theorem 2.11. *Assume that Ω is connected.*[3] *Then we have the following:*

i) (Compactness) *Let $u_\varepsilon\colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$ be such that $E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2$. Then there exist a constant $\bar\varphi \in \mathscr{L}_N$ and a map $u = (\cos(\bar\varphi)\widetilde{u}, \sin(\bar\varphi)) \in H^1(\Omega; \mathscr{S}^2_N)$ such that, up to a non-relabeled subsequence, $A[u_\varepsilon] \to u$ strongly in $L^2(\Omega; \mathbb{R}^3)$, and $A[u_\varepsilon] \rightharpoonup u$ weakly in $H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$. Moreover, $\varphi(u_\varepsilon) \to \bar\varphi$ strongly in $L^1(\Omega; \mathbb{R})$, $A[\widetilde{u}_\varepsilon] \to \widetilde{u}$ strongly in $L^2(\Omega; \mathbb{R}^2)$, and $A[\widetilde{u}_\varepsilon] \rightharpoonup \widetilde{u}$ in $H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$.*

ii) (liminf inequality) *Let $u_\varepsilon\colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$, let $\bar\varphi \in \mathscr{L}_N$, and let $u = (\cos(\bar\varphi)\widetilde{u}, \sin(\bar\varphi)) \in H^1(\Omega; \mathscr{S}^2_N)$ be such that $\varphi(u_\varepsilon) \to \bar\varphi$ strongly in $L^1(\Omega; \mathbb{R})$ and $A[u_\varepsilon] \to u$ strongly in $L^2(\Omega; \mathbb{R}^3)$. Then*

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega) \geq \int_\Omega |\nabla u|^2 \,\mathrm{d}x = \cos^2(\bar\varphi) \int_\Omega |\nabla \widetilde{u}|^2 \,\mathrm{d}x\,.$$

iii) (limsup inequality) *Let $\bar\varphi \in \mathscr{L}_N$ and let $u = (\cos(\bar\varphi)\widetilde{u}, \sin(\bar\varphi)) \in H^1(\Omega; \mathscr{S}^2_N)$. Then there exist $u_\varepsilon\colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$ such that $\varphi(u_\varepsilon) \to \bar\varphi$ strongly in $L^1(\Omega; \mathbb{R})$, $A[u_\varepsilon] \to u$ strongly in $L^2(\Omega; \mathbb{R}^3)$, and*

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega) \leq \int_\Omega |\nabla u|^2 \,\mathrm{d}x = \cos^2(\bar\varphi) \int_\Omega |\nabla \widetilde{u}|^2 \,\mathrm{d}x\,.$$

Remark 2.12. It will be clear from the proof that the sole information $A[u_\varepsilon] \to u \in H^1(\Omega; \mathscr{S}^2_N)$ in $L^2(\Omega; \mathbb{R}^3)$ implies the lower bound

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega) \geq \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,.$$

We will use this inequality in the proof of Theorem 2.10.

3. Proofs of the main results

3.1. **The surface scaling regime.**

Proof of Theorem 2.6. We prove the statements *(i)-(iii)* in separate steps.

Proof of *(i)* (compactness).
Due to Remark 2.5, the sequence $\varphi(u_\varepsilon)$ is bounded in $BV(A; \mathbb{R})$ for every $A \subset\subset \Omega$. Since it is also bounded in $L^\infty(\Omega; \mathbb{R})$, the equi-integrability and a diagonal argument show that (up to a subsequence) $\varphi(u_\varepsilon) \to \varphi$ in $L^1(\Omega; \mathbb{R})$ for some $\varphi \in BV(\Omega; \mathscr{L}_N)$. Moreover, u'_ε is bounded in $L^\infty(\Omega; \mathbb{R}^2)$, so that (up to a further subsequence) it converges weakly* to some function $u' \in L^\infty(\Omega; \mathbb{R}^2)$. It remains to show that $u' \in L^\infty(\Omega; \varphi)$. Fix $x \in \Omega$. Since $\varphi(u_\varepsilon) \to \varphi$ in $L^1(\Omega)$ as $\varepsilon \to 0$, from the weak*-lower semicontinuity of the L^1-norm we infer that for any $r > 0$

$$\fint_{B_r(x)} |u'| \,\mathrm{d}y \leq \liminf_{\varepsilon\to 0} \fint_{B_r(x)} |u'_\varepsilon| \,\mathrm{d}y = \liminf_{\varepsilon\to 0} \fint_{B_r(x)} \cos(\varphi(u_\varepsilon)) \,\mathrm{d}y = \fint_{B_r(x)} \cos(\varphi) \,\mathrm{d}y\,,$$

where we used the Dominated Convergence Theorem in the last step. Sending $r \to 0$, Lebesgue's differentiation theorem implies that $|u'(x)| \leq \cos(\varphi(x))$ for a.e. $x \in \Omega$, *i.e.*, $u' \in L^\infty(\Omega; \varphi)$.

Proof of *(ii)* (liminf inequality).
Fix $A \subset\subset \Omega$. Then due to Remark 2.5, for ε small enough we have that

$$\frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon, \Omega) \geq \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N} |\mathrm{D}\varphi(u_\varepsilon)|_1(A)\,.$$

[3]see Footnote 2

The anisotropic total variation on A is lower semicontinuous with respect to strong convergence in $L^1(A;\mathbb{R})$. Hence we conclude that

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon,\Omega)\geq \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N}|\mathrm{D}\varphi|_1(A)\,.$$

The arbitrariness of $A\subset\subset\Omega$ and the fact that $B\mapsto|\mathrm{D}\varphi|_1(B)$ is a Borel-measure yield the claim.

Proof of *(iii)* (limsup inequality).
We argue by gradual approximation. In a first step we assume that $\varphi\in BV(\Omega;\mathscr{L}_N)$ is given by the restriction to Ω of a function $\varphi\in\mathcal{PC}_\delta(\mathscr{L}_N)$ (*cf.* (2.1)) for some $\delta>0$. In particular, there exists a partition of $\mathbb{R}^2$ into half-open cubes of the form $x_0+[-\delta/2,\delta/2)^2$ such that φ is constant on each cube. Moreover, we assume that $u'\in L^\infty(\Omega;\varphi)$ is such that for each cube $x_0+[-\delta/2,\delta/2)^2$ of the partition associated to φ we have $|u'|=\cos(\varphi(x_0))$ and that there exists a further partition into smaller half-open cubes of the form $y_0+[-\eta/2,\eta/2)^2$ such that u' is constant on each of these cubes. In what follows we shall always assume that $\eta\ll\delta$ and to reduce notation we let $\mathcal{Q}_\eta(u')$ be the partition of $\mathbb{R}^d$ consisting of half-open cubes on which u' (and hence also φ) is constant. (We refer to the first picture in Figure 2.)

We construct the approximating sequence $u_\varepsilon\colon\varepsilon\mathbb{Z}^2\to\mathscr{S}^2_N$ locally on each cube $Q\in\mathcal{Q}_\eta(u')$ (see the second picture in Figure 2). First, we fix a projection $\Pi\colon\mathbb{R}^2\to\mathbb{R}^2$ such that for $x\in Q$ it holds that $\Pi(x)\in\partial Q$ with $|\Pi(x)-x|=\mathrm{dist}(x,\partial Q)$ (such a function is unique except on the diagonal lines connecting the corners of a cube Q). Then we define the latitude of u_ε on $\varepsilon\mathbb{Z}^2\cap Q$ by setting (see below an explanation of the formula)

$$\varphi(u_\varepsilon)(\varepsilon i):=\varphi(\Pi(\varepsilon i))+\min\Big\{\frac{\theta_N|\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon i,\partial Q)\rceil-1|}{|\varphi|_Q-\varphi(\Pi(\varepsilon i))|},1\Big\}\big(\varphi|_Q-\varphi(\Pi(\varepsilon i))\big)\,, \tag{3.1}$$

where $\lceil\cdot\rceil$ denotes the ceiling function and with a slight abuse of notation we set $\min\{x/0,1\}\cdot 0=0$ for all $x\geq 0$. Note that the function φ is defined pointwise on ∂Q since the cubes of the partition are half-open. In particular, on the top and right sides of ∂Q, the function φ takes the value of the cubes adjacent to Q (except in the top-right corner, but in Q there is no point that is projected by Π onto this corner). Moreover, note that $\varphi(u_\varepsilon)\in\mathscr{L}_N$. The third component is then defined by

$$u^3_\varepsilon(\varepsilon i)=\sin\big(\varphi(u_\varepsilon)(\varepsilon i)\big)\,.$$

Let us briefly explain the formula in (3.1) more in detail (see also the third picture in Figure 2): if εi is closest to a side of ∂Q that belongs to the half-open cube, then $\varphi(\Pi(\varepsilon i))=\varphi|_Q$. If εi is close to a side of ∂Q that does not belong to the half-open cube, then $\varphi(\Pi(\varepsilon i))$ corresponds to the value of φ on a neighboring cube and $\varphi(u_\varepsilon)$ is an interpolation between the values $\varphi|_Q$ and $\varphi(\Pi(\varepsilon i))$ towards the boundary with a step-wise increment of $\pm\theta_N$. Close to the corners of the cube Q, the projection Π is not continuous, but as we will see the interactions close to the corners are negligible due to the surface scaling of the energy. In fact, close to the corners the value of $\varphi(u_\varepsilon)$ could be chosen arbitrarily, see the third picture in Figure 2.

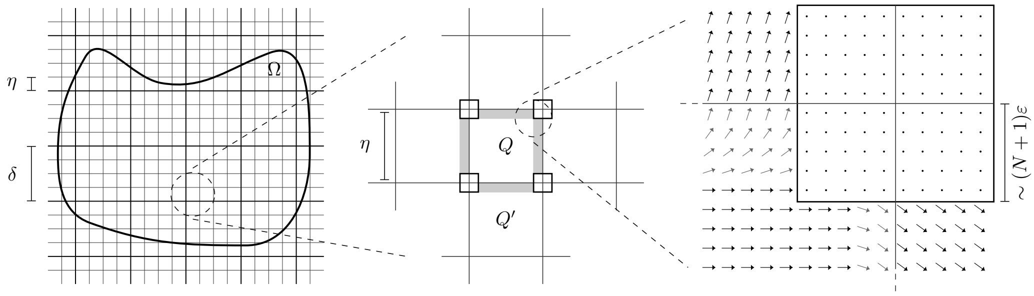

Figure 2. In the first picture: partition into cubes of size δ and of size η with $\eta \ll \delta$. In the second picture: the interpolation formula (3.1) occurs in the grey regions. In the third picture: the vectors represent the direction determined by the latitude $\varphi(u_\varepsilon)$ on a great circle of $\mathbb{S}^2$ passing through the poles; the change in each angle is $\pm\theta_N$; the value of the latitude close to the corner is not relevant as there the energy is negligible at the surface scaling.

Next, we define the component u'_ε on $\varepsilon\mathbb{Z}^2 \cap Q$. Since the condition $u_\varepsilon(\varepsilon i) \in \mathscr{S}^2_N$ must be satisfied for all points $\varepsilon i \in \varepsilon\mathbb{Z}^2$, we first construct a $\mathbb{S}^1$-valued function $\widetilde{u}_\varepsilon$ and then set $u'_\varepsilon := \sqrt{(1-|u^3_\varepsilon|^2)}\widetilde{u}_\varepsilon$. To define $\widetilde{u}_\varepsilon$, let $\theta \in [0, 2\pi)$ be such that $u'|_Q = \cos(\varphi|_Q)\exp(\iota\theta)$ and consider the piecewise affine function

$$\theta_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 \le t < 2N\varepsilon\,, \\ \varepsilon^{-\frac{1}{2}}(t-2N\varepsilon)\theta & \text{if } 2N\varepsilon \le t \le 2N\varepsilon + \varepsilon^{\frac{1}{2}}\,, \\ \theta & \text{if } t > 2N\varepsilon + \varepsilon^{\frac{1}{2}}\,. \end{cases}$$

For $\varepsilon i \in \varepsilon\mathbb{Z}^2 \cap Q$ define then

$$\widetilde{u}_\varepsilon(\varepsilon i) := \exp\big(\iota\theta_\varepsilon(\mathrm{dist}(\varepsilon i, \partial Q))\big)\,.$$

Let us identify the L^1-limit of u_ε on Q. First, let us consider the latitude $\varphi(u_\varepsilon)$. Note that $|\varphi|_Q - \varphi(\Pi(\varepsilon i))| \le N\theta_N$, so that

$$\varphi(u_\varepsilon(\varepsilon i)) = \varphi|_Q \quad \text{if } \lceil \varepsilon^{-1}\mathrm{dist}(\varepsilon i, \partial Q)\rceil \ge N+1\,. \tag{3.2}$$

In particular, $\varphi(u_\varepsilon(\varepsilon i)) = \varphi|_Q$ if $\mathrm{dist}(\varepsilon i, \partial Q) \ge (N+1)\varepsilon$, which together with uniform boundedness implies that $\varphi(u_\varepsilon) \to \varphi|_Q$ in $L^1(Q;\mathbb{R})$. Next, consider the function $\widetilde{u}_\varepsilon$. If $\mathrm{dist}(\varepsilon i, \partial Q) \ge 2N\varepsilon + \varepsilon^{\frac{1}{2}}$, then $\widetilde{u}_\varepsilon(\varepsilon i) = \exp(\iota\theta)$, so that due to uniform boundedness we have that $\widetilde{u}_\varepsilon \to \exp(\iota\theta)$ in $L^1(Q;\mathbb{R}^2)$. We conclude that

$$u'_\varepsilon = \cos(\varphi(u_\varepsilon))\widetilde{u}_\varepsilon \to \cos(\varphi|_Q)\exp(\iota\theta) = u'|_Q \quad \text{in } L^1(Q;\mathbb{R}^2)\,.$$

Globally, we deduce that $\varphi(u_\varepsilon) \to \varphi$ in $L^1(\Omega;\mathbb{R})$ and $u'_\varepsilon \to u'$ in $L^1(\Omega;\mathbb{R}^2)$ and due to uniform boundedness also weakly* in $L^\infty(\Omega;\mathbb{R}^2)$. Hence u_ε is an admissible candidate for a recovery sequence.

In the final step, we estimate the energy of u_ε.

Step 1: Interactions between different cubes.
Consider $\varepsilon i \in \varepsilon\mathbb{Z}^2 \cap Q$ and $\varepsilon j \in \varepsilon\mathbb{Z}^2 \cap Q'$ with $Q \neq Q'$ and assume without loss of generality that $i - j = e_2$ (the case $-e_2$ follows upon exchanging the roles of i and j, while the case $\pm e_1$ can be treated by similar arguments). Note that Q' is positioned below Q.

Let us first consider the case where $\Pi(\varepsilon j)$ does not belong to the top side of Q'. Since $|\varepsilon i - \varepsilon j| = \varepsilon$, the projection $\Pi(\varepsilon j)$ must belong either to the right or the left side of Q'. Then the projection of $\Pi(\varepsilon j)$ onto the top side of Q' is a corner point z of Q' at distance at most 2ε from εj. This implies that εj and εi are in the ball $B_{3\varepsilon}(z)$, where z is a corner point of Q'. Note that the number of lattice points of $\varepsilon\mathbb{Z}^2$ in the ball $B_{3\varepsilon}(z)$ is uniformly bounded with respect to ε and Q'. Hence, for these interactions we pay an energy

$$\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon, B_{3\varepsilon}(z)) \le C\varepsilon\,.$$

With an argument analogous to the previous one we treat the case where $\Pi(\varepsilon i)$ does not belong to the the bottom side of Q (which coincides with top side of Q'). Also in this case

εj and εi are in the ball $B_{3\varepsilon}(z)$, where z is a corner point of Q'. The energy paid for these interactions is $C\varepsilon$.

If instead $\Pi(\varepsilon i)$ and $\Pi(\varepsilon j)$ both belong to the top side of Q', then necessarily $\Pi(\varepsilon i) = \Pi(\varepsilon j)$ and therefore $\varphi(\Pi(\varepsilon i)) = \varphi(\Pi(\varepsilon j)) = \varphi|_Q$. For $\varepsilon i \in Q$ this implies that $\varphi(u_\varepsilon)(\varepsilon i) = \varphi|_Q$. For $\varepsilon j \in Q'$ we use that $0 < \mathrm{dist}(\varepsilon j, \partial Q') \leq \varepsilon$, so that $\lceil \varepsilon^{-1}\mathrm{dist}(\varepsilon j, \partial Q')\rceil - 1 = 0$ and therefore also $\varphi(u_\varepsilon)(\varepsilon j) = \varphi|_Q$. Moreover, the definition of the function θ_ε implies that $u'_\varepsilon(\varepsilon i) = \cos(\varphi|_Q)e_1 = u'_\varepsilon(\varepsilon j)$, so that $u_\varepsilon(\varepsilon i) = u_\varepsilon(\varepsilon j)$ and these interactions do not contribute to the energy.

To sum up, we can estimate the total energy by

$$\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon,\Omega) \leq \sum_{\substack{Q\in\mathcal{Q}_\eta(u')\\ Q\cap\Omega\neq\emptyset}} \frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon, Q) + C\varepsilon\#\{Q\in\mathcal{Q}_\eta(u') :\, Q\cap\Omega\neq\emptyset\}. \tag{3.3}$$

The second term vanishes when $\varepsilon \to 0$ and therefore it suffices to estimate the energy on a single cube Q. This will be the next part of the analysis.

Step 2: Interactions in a single cube.
Fix $\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \cap Q$. We distinguish the following two cases:

i) $\min\{\mathrm{dist}(\varepsilon i, \partial Q), \mathrm{dist}(\varepsilon j, \partial Q)\} \geq (N+1)\varepsilon$;
ii) $\min\{\mathrm{dist}(\varepsilon i, \partial Q), \mathrm{dist}(\varepsilon j, \partial Q)\} < (N+1)\varepsilon$.

We split the energy

$$\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon, Q) \leq \frac{1}{2}\sum_{\substack{\varepsilon i,\varepsilon j\in\varepsilon\mathbb{Z}^2\cap Q\\ |i-j|=1\\ \text{i) holds}}} \varepsilon|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 + \frac{1}{2}\sum_{\substack{\varepsilon i,\varepsilon j\in\varepsilon\mathbb{Z}^2\cap Q\\ |i-j|=1\\ \text{ii) holds}}} \varepsilon|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2. \tag{3.4}$$

In case i), by (3.2) we have that $u^3_\varepsilon(\varepsilon i) = u^3_\varepsilon(\varepsilon j) = \sin(\varphi|_Q)$ and therefore

$$\begin{aligned}|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 &= |u'_\varepsilon(\varepsilon i) - u'_\varepsilon(\varepsilon j)|^2 = \cos^2(\varphi|_Q)|\widetilde{u}_\varepsilon(\varepsilon i) - \widetilde{u}_\varepsilon(\varepsilon j)|^2\\ &\leq 2\cos^2(\varphi|_Q)\varepsilon^{-1}\theta^2|\mathrm{dist}(\varepsilon i, \partial Q) - \mathrm{dist}(\varepsilon j, \partial Q)|^2 \leq 2\cos^2(\varphi|_Q)\varepsilon\theta^2,\end{aligned} \tag{3.5}$$

where we used the 1-Lipschitz continuity of the distance function. This estimate can be improved when $\min\{\mathrm{dist}(\varepsilon i, \partial Q), \mathrm{dist}(\varepsilon j, \partial Q)\} \geq 2N\varepsilon + \varepsilon^{\frac{1}{2}}$, because in this case $\widetilde{u}_\varepsilon(\varepsilon i) = \widetilde{u}_\varepsilon(\varepsilon j)$ and hence the difference is zero. For ε small enough we have that $2N\varepsilon \leq \varepsilon^{\frac{1}{2}}$. Let us add the third condition, stronger that i),

i') $\min\{\mathrm{dist}(\varepsilon i, \partial Q), \mathrm{dist}(\varepsilon j, \partial Q)\} \geq \varepsilon^{\frac{1}{2}}$.

Then, by (3.5), we have that

$$\begin{aligned}\sum_{\substack{\varepsilon i,\varepsilon j\in\varepsilon\mathbb{Z}^2\cap Q\\ |i-j|=1\\ \text{i) holds}}} \varepsilon|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 &\leq \sum_{\substack{\varepsilon i,\varepsilon j\in\varepsilon\mathbb{Z}^2\cap Q\\ |i-j|=1\\ \text{i') holds}}} \varepsilon|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 + \sum_{\substack{\varepsilon i,\varepsilon j\in\varepsilon\mathbb{Z}^2\cap Q\\ |i-j|=1\\ \text{i) holds, i') does not}}} \varepsilon|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2\\ &\leq C\#\{\varepsilon i\in Q :\, \mathrm{dist}(x,\partial Q) \leq 2\varepsilon^{\frac{1}{2}}\}\varepsilon^2.\end{aligned} \tag{3.6}$$

Hence the first term in the right-hand side of (3.4) vanishes as $\varepsilon \to 0$, so we focus on the second term. From now on we assume that $\varepsilon i, \varepsilon j$ satisfy ii). We distinguish two cases:

Case 1: First assume that $\Pi(\varepsilon i)$ and $\Pi(\varepsilon j)$ belong to different sides of Q and let Π_i and Π_j denote the projection onto the sides that contain the points $\Pi(\varepsilon i)$ and $\Pi(\varepsilon j)$, respectively. Due to ii) and the fact that $|\varepsilon i - \varepsilon j| = \varepsilon$, the points $\Pi(\varepsilon i) = \Pi_i(\varepsilon i)$ and $\Pi(\varepsilon j) = \Pi_j(\varepsilon j)$ cannot belong to opposite sides, so that the point $\Pi_i(\Pi_j(\varepsilon i))$ is a corner point of Q. From the 1-Lipschitz continuity of the projections Π_i and Π_j, and from ii) we infer that

$$\begin{aligned}|\varepsilon i - \Pi_i(\Pi_j(\varepsilon i))| &\leq |\varepsilon i - \Pi_i(\varepsilon i)| + |\Pi_i(\varepsilon i) - \Pi_i(\Pi_j(\varepsilon i))| \leq (N+2)\varepsilon + |\varepsilon i - \Pi_j(\varepsilon i)|\\ &\leq (N+2)\varepsilon + |\varepsilon i - \varepsilon j| + |\varepsilon j - \Pi_j(\varepsilon j)| + |\Pi_j(\varepsilon j) - \Pi_j(\varepsilon i)| \leq (2N+6)\varepsilon\,.\end{aligned}$$

The same argument applies to εj and the corner point $\Pi_j(\Pi_i(\varepsilon j))$ and therefore the points $\varepsilon i, \varepsilon j$ belong to a ball of radius $C_N\varepsilon$ centered at a corner point of Q. Clearly those interactions are negligible as $\varepsilon \to 0$.

Case 2: In the second part of the analysis on Q, we assume that $\Pi(\varepsilon i)$ and $\Pi(\varepsilon j)$ belong to the same side S_0 of Q. In particular, we can assume that they belong to the relative interior of S_0 since otherwise the behavior is already covered by the above analysis. This additional assumption guarantees that $\varphi(\Pi(\varepsilon i)) = \varphi(\Pi(\varepsilon j)) =: \varphi_0$. Inserting this equality in the definition of $\varphi(u_\varepsilon)$ we obtain that

$$\begin{aligned}&\varphi(u_\varepsilon(\varepsilon i)) - \varphi(u_\varepsilon(\varepsilon j)) = (\varphi|_Q - \varphi_0)\\ &\times\left(\min\left\{\frac{\theta_N|\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon i,\partial Q)\rceil - 1|}{|\varphi|_Q-\varphi_0|},1\right\} - \min\left\{\frac{\theta_N|\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon j,\partial Q)\rceil - 1|}{|\varphi|_Q-\varphi_0|},1\right\}\right). \end{aligned}\tag{3.7}$$

If $\varphi|_Q = \varphi_0$, then the above difference vanishes. The same holds if $\varepsilon i - \varepsilon j$ is parallel to S_0, since then $\mathrm{dist}(\varepsilon i, \partial Q) = \mathrm{dist}(\varepsilon j, \partial Q)$. In both cases we obtain $u^3_\varepsilon(\varepsilon i) = u^3_\varepsilon(\varepsilon j)$. Moreover, since $2N \geq N+2$ for $N \geq 2$, property ii) implies that $\widetilde{u}_\varepsilon(\varepsilon i) = \widetilde{u}_\varepsilon(\varepsilon j) = e_1$ and hence $u_\varepsilon(\varepsilon i) = u_\varepsilon(\varepsilon j)$. We conclude that such interactions do not contribute to the energy.

Thus in case 2 we may assume that $\varphi|_Q \neq \varphi_0$ and that $\varepsilon i - \varepsilon j$ is orthogonal to S_0. Then $\mathrm{dist}(\varepsilon i, \partial Q) - \mathrm{dist}(\varepsilon j, \partial Q) = \pm\varepsilon$. Since φ takes values in the set $\mathscr{L}_N$, we can write

$$|\varphi|_Q - \varphi_0| = k\theta_N$$

for some $k \in \mathbb{N}\setminus\{0\}$. In particular, by (3.7) the difference $\varphi(u_\varepsilon(\varepsilon i)) - \varphi(u_\varepsilon(\varepsilon j))$ is non zero only if there exists n with $0 \leq n \leq k-1$ such that

$$\begin{aligned}|\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon i,\partial Q)\rceil - 1| = n \quad &\text{and} \quad |\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon j,\partial Q)\rceil - 1| = n+1 \quad \text{or}\\ |\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon i,\partial Q)\rceil - 1| = n+1 \quad &\text{and} \quad |\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon j,\partial Q)\rceil - 1| = n\,.\end{aligned}$$

Since we assume that $\varphi|_Q \neq \varphi_0$, we know that S_0 cannot be contained in the half-open cube Q. In particular, $\mathrm{dist}(\varepsilon i, \partial Q) > 0$ and therefore $\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon i,\partial Q)\rceil - 1 \geq 0$ and we can omit the modulus in the above inclusion, that is, there exists n with $1 \leq n \leq k$

$$\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon i,\partial Q)\rceil = n \quad \text{and} \quad \lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon i,\partial Q)\rceil = n+1 \quad \text{or} \tag{3.8}$$

$$\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon i,\partial Q)\rceil = n+1 \quad \text{and} \quad \lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon i,\partial Q)\rceil = n\,. \tag{3.9}$$

In this case, the linearity of $x \mapsto \min\{x,1\}$ on $[0,1]$ implies that

$$|\varphi(u_\varepsilon(\varepsilon i)) - \varphi(u_\varepsilon(\varepsilon j))| = \theta_N|\underbrace{\lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon i,\partial Q)\rceil - \lceil\varepsilon^{-1}\mathrm{dist}(\varepsilon j,\partial Q)\rceil}_{=\pm1}| = \theta_N. \tag{3.10}$$

Now we are in the position to evaluate the full difference $|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2$. Inspecting the proof of the second estimate in Lemma 2.1, we see that due to (3.10) and since $\widetilde{u}_\varepsilon(\varepsilon i) = \widetilde{u}_\varepsilon(\varepsilon j) = e_1$, all inequalities in that proof are in fact equalities and therefore we have that

$$|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 = 4\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N}|\varphi(u_\varepsilon(\varepsilon i)) - \varphi(u_\varepsilon(\varepsilon j))| = 4\sin^2\left(\tfrac{\theta_N}{2}\right).$$

Taking once more into account that $\mathrm{dist}(\varepsilon i, \partial Q) - \mathrm{dist}(\varepsilon j, \partial Q) = \pm\varepsilon$, the number of pairs $(\varepsilon i, \varepsilon j)$ satisfying (3.8) is given by $2k$ times the number of possible orthogonal lattice layers that are projected onto S_0. Hence the energetic contribution of all pairs $(\varepsilon i, \varepsilon j)$ such that $\Pi(\varepsilon i) = \Pi(\varepsilon j) \in S_0$ can be bounded by

$$\frac{1}{2}\frac{1}{\varepsilon}(2k)\frac{\mathcal{H}^1(S_0)+2\varepsilon}{\varepsilon}\varepsilon^2 4\sin^2\left(\tfrac{\theta_N}{2}\right) = 4\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N}|\varphi|_Q - \varphi_0|(\mathcal{H}^1(S_0)+2\varepsilon)\,. \tag{3.11}$$

Note that $|\varphi|_Q - \varphi_0|$ equals the jump amplitude of φ along the side S_0 and since S_0 is parallel to one of the coordinate axes, we have that

$$|\varphi|_Q - \varphi_0|\mathcal{H}^1(S_0) = \int_{S_0}|\varphi^+ - \varphi^-||\nu_\varphi|_1\,\mathrm{d}\mathcal{H}^1.$$

Moreover, the closure of any cube $Q \in \mathcal{Q}_\eta(u')$ such that $Q\cap\Omega \neq \emptyset$ is contained in $\Omega + B_{2\eta}(0)$. Hence, starting from (3.3) and taking into account (3.4), (3.6),x and that (3.11) can be non-zero only if S_0 is either the top or the right side of Q (so that the corresponding side is not taken into account by the energy on another cube), we deduce that

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon,\Omega)\le 4\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N}\int_{\Omega+B_{2\eta}(0)}|\varphi^+-\varphi^-||\nu_\varphi|_1\,\mathrm{d}\mathcal{H}^1.$$

Finally, fixing the piecewise constant functions u' and φ, we can refine the cube size η and repeat the above construction to obtain a diagonal sequence $u_\varepsilon:\varepsilon\mathbb{Z}^2\to\mathscr{S}^2_N$ such that $\varphi(u_\varepsilon)\to\varphi$ in $L^1(\Omega;\mathbb{R})$, $u'_\varepsilon\overset{*}{\rightharpoonup}u'$ in $L^\infty(\Omega;\mathbb{R}^2)$ and

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon,\Omega)\le 4\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N}\int_{\overline{\Omega}}|\varphi^+-\varphi^-||\nu_\varphi|_1\,\mathrm{d}\mathcal{H}^1=:4\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N}|\mathrm{D}\varphi|_1(\overline{\Omega}).$$

Using the abstract lower semicontinuity of the Γ-limsup and standard approximation results in the weak*-topology, we deduce that for φ as above and an arbitrary function $u'\in L^\infty(\Omega;\varphi)$ it holds that

$$\Gamma\text{-}\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}E_\varepsilon(\cdot,\Omega)(u',\varphi)\le 4\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N}|\mathrm{D}\varphi|_1(\overline{\Omega}).$$

Finally, given an arbitrary function $\varphi\in BV(\Omega;\mathscr{L}_N)$ and $u'\in L^\infty(\Omega;\varphi)$, we consider a sequence $\varphi_n\in\mathcal{PC}_{\delta_n}(\mathscr{L}_N)$ such that $\varphi_n\to\varphi$ in $L^1(\Omega;\mathbb{R})$ and $|\mathrm{D}\varphi_n|_1(\overline{\Omega})\to|\mathrm{D}\varphi|_1(\Omega)$ (see Proposition A.1) and define $u'_n\in L^\infty(\Omega;\varphi_n)$ by

$$u'_n=\begin{cases}u' & \text{on } \{\varphi_n=\varphi'\},\\ 0 & \text{otherwise.}\end{cases}$$

Since $\mathscr{L}_N$ is a finite set, the convergence $\varphi_n\to\varphi$ in $L^1(\Omega;\mathbb{R})$ implies that $|\{\varphi_n\neq\varphi\}|\to 0$ as $n\to+\infty$. Therefore we deduce that $u'_n\to u'$ in $L^1(\Omega;\mathbb{R}^2)$ and by boundedness also weakly* in $L^\infty(\Omega;\mathbb{R}^2)$, so that once again by the lower semicontinuity of the Γ-limsup we find that

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}E_\varepsilon(\cdot,\Omega)(u',\varphi)&\le\liminf_{n\to+\infty}\Gamma\text{-}\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}E_\varepsilon(\cdot,\Omega)(u'_n,\varphi_n)\\&\le\liminf_{n\to+\infty}4\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N}|\mathrm{D}\varphi_n|_1(\overline{\Omega})=4\frac{\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N}|\mathrm{D}\varphi|_1(\Omega).\end{aligned}$$

By the definition of the Γ-limsup this yields the property (iii) in Theorem 2.6 and we conclude the proof. □

3.2. **The gradient scaling regime.**

Proof of Theorem 2.11. We prove each part separately.

Proof of *(i)* (Compactness).
By assumption we know that $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon,\Omega)=0$. Hence from Theorem 2.6 it follows that (up to a subsequence) there exists $\varphi\in BV(\Omega;\mathbb{R})$ such that $\varphi(u_\varepsilon)\to\varphi$ in $L^1(\Omega;\mathbb{R})$ and $|\mathrm{D}\varphi|_1(\Omega)=0$. Since Ω is connected, we deduce that $\varphi=\bar{\varphi}\in\mathscr{L}_N$ is constant. Moreover, note that the definition of the piecewise affine interpolation implies that for any $\Omega'\subset\subset\Omega$ and ε small enough it holds that

$$C\ge\frac{1}{\varepsilon^2}E_\varepsilon(u_\varepsilon,\Omega)\ge\int_{\Omega'}|\nabla A[u_\varepsilon](x)|^2\,\mathrm{d}x\,,$$

where we used that $A[u_\varepsilon]$ is affine on triangles with volume $\varepsilon^2/2$. In particular, since the norms of u_ε and $A[u_\varepsilon]$ are uniformly bounded by 1, it follows that up to a further subsequence there exists $u\in H^1(\Omega';\mathscr{S}^2_N)$ such that $A[u_\varepsilon]\rightharpoonup u$ in $H^1(\Omega';\mathbb{R}^3)$ and strongly in $L^2(\Omega';\mathbb{R}^3)$. Since u_ε is equi-integrable, we can use the arbitrariness of $\Omega'\subset\subset\Omega$ and a diagonal argument to show that $u\in H^1(\Omega;\mathscr{S}^2_N)$ and in addition $A[u_\varepsilon]\to u$ in $L^2(\Omega;\mathbb{R}^3)$ and weakly in $H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$. Clearly we can write $u=(\cos(\bar{\varphi})\widetilde{u},\sin(\bar{\varphi}))$ with $\widetilde{u}\in H^1(\Omega;\mathbb{S}^1)$.

To control the behavior of the piecewise affine interpolation of $\widetilde{u}_\varepsilon$, we recall that, due to Remark 2.8, also $A[\widetilde{u}_\varepsilon]$ is bounded in $H^1(\Omega';\mathbb{R}^2)$ for every $\Omega'\subset\subset\Omega$. Repeating the arguments used above, it thus suffices to identify the L^1-limit along the subsequence chosen for

$A[u_\varepsilon]$. Due to [8, Proposition A.1 & Remark A.2] we know that the $L^1(\Omega;\mathbb{R}^2)$-convergence of $A[\widetilde{u}_\varepsilon]$ implies the $L^1(\Omega;\mathbb{R}^2)$ convergence of $\widetilde{u}_\varepsilon$ to the same limit. Hence it suffices to prove that $\widetilde{u}_\varepsilon \to \widetilde{u}$ in $L^1(\Omega;\mathbb{R}^2)$. By definition, it holds that

$$\widetilde{u}_\varepsilon = \frac{1}{\cos(\varphi(u_\varepsilon))} u'_\varepsilon \,.$$

Since the set $\mathscr{L}_N$ does not contain the values $\pm\pi/2$, it follows that $\frac{1}{\cos(\varphi(u_\varepsilon))} \to \frac{1}{\cos(\bar\varphi)}$ in $L^1(\Omega;\mathbb{R})$ and the convergence $A[u_\varepsilon] \to u$ in $L^1(\Omega;\mathbb{R}^3)$ implies the convergence $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^3)$. In particular, from uniform boundedness it follows that $\widetilde{u}_\varepsilon \to \frac{1}{\cos(\bar\varphi)} u' = \widetilde{u}$ in $L^1(\Omega;\mathbb{R}^2)$.

Proof of *(ii)* (liminf inequality).
Without loss of generality we assume that the liminf is finite and, up to a subsequence, it is a limit. Fix $\Omega' \subset\subset \Omega$. Then as in Step 1, for ε small enough we have that

$$\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon,\Omega) \geq \int_{\Omega'} |\nabla A[u_\varepsilon](x)|^2 \,\mathrm{d}x \,.$$

The weak lower semicontinuity of the right-hand side functional with respect to $L^2(\Omega;\mathbb{R}^3)$-convergence yields that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon,\Omega) \geq \int_{\Omega'} |\nabla u(x)|^2 \,\mathrm{d}x = \cos^2(\bar\varphi) \int_{\Omega'} |\nabla \widetilde{u}(x)|^2 \,\mathrm{d}x.$$

Letting $\Omega' \uparrow \Omega$ yields the claim.

Proof of *(iii)* (limsup inequality).
Fix $u = (\cos(\bar\varphi)\widetilde{u}, \sin(\bar\varphi)) \in H^1(\Omega;\mathscr{S}^2_N)$. Then $\widetilde{u} \in H^1(\Omega;\mathbb{S}^1)$. Using a local reflection argument (*cf.* the proof of [18, Lemma 3.4] for the details) we can assume that $\widetilde{u} \in H^1(\Omega';\mathbb{S}^1)$ for some open, connected set $\Omega' \supset\supset \Omega$ with smooth boundary. The density of smooth functions $C^\infty(\Omega';\mathbb{S}^1)$ in $H^1(\Omega';\mathbb{S}^1)$ allows us to reduce the proof of the limsup inequality to the case $\widetilde{u} \in C^\infty(\Omega';\mathbb{S}^1)$. The recovery sequence is then given by

$$u_\varepsilon(\varepsilon i) = (\cos(\bar\varphi)\widetilde{u}(\varepsilon i), \sin(\bar\varphi)).$$

A standard computation (that we leave to the reader) yields that for any $\Omega \subset\subset \Omega'' \subset\subset \Omega'$

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon,\Omega) = \cos^2(\bar\varphi) \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon} XY_\varepsilon(\widetilde{u},\Omega) \leq \cos^2(\bar\varphi) \int_{\Omega''} |\nabla\widetilde{u}(x)|^2 \,\mathrm{d}x.$$

Letting $\Omega'' \downarrow \overline{\Omega}$ we conclude the proof of the limsup inequality, keeping in mind that $|\partial\Omega| = 0$. □

3.3. **The vortex scaling regime.** We prove a result that will be fundamental for the proof of the theorem in the vortex scaling. It is based on the fact that $\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon)$ behaves like an interface energy.

Lemma 3.1. *Let $A' \subset\subset A \subset \mathbb{R}^2$ be bounded open sets with Lipschitz boundary. There exists a constant $C(A',N) > 0$ depending on A' and N such that for every $u_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$ and $\bar\varphi \in \mathscr{L}_N$ with $\varphi(u_\varepsilon) \to \bar\varphi$ strongly in $L^1(A;\mathbb{R})$, for ε small enough we have that*

$$\int_{A'} |\varphi(u_\varepsilon(x)) - \bar\varphi| \,\mathrm{d}x \leq C(A',N)\Big(\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon,A)\Big)^2.$$

Moreover, if A' is a ball $A' = B_\eta$, then the constant $C(A',N) = C(N)$ only depends on N.

Proof. We start by observing that

$$\int_{A'} |\varphi(u_\varepsilon(x)) - \bar\varphi| \,\mathrm{d}x = \int_{\{\varphi(u_\varepsilon)\neq\bar\varphi\}\cap A'} |\varphi(u_\varepsilon(x)) - \bar\varphi| \,\mathrm{d}x \leq \pi\mathcal{L}^2(\{\varphi(u_\varepsilon) \neq \bar\varphi\} \cap A'). \quad (3.12)$$

By the relative isoperimetric inequality (see [9, Remark 3.50]) we have that

$$\min\{\mathcal{L}^2(\{\varphi(u_\varepsilon) \neq \bar\varphi\} \cap A'), \mathcal{L}^2(A' \setminus \{\varphi(u_\varepsilon) \neq \bar\varphi\})\} \leq C(A')\big(\mathcal{H}^1(\partial^*\{\varphi(u_\varepsilon) \neq \bar\varphi\} \cap A')\big)^2, \quad (3.13)$$

where ∂^* is the reduced boundary and $C(A')$ is constant depending on A'. The constant $C(A')$ is independent of A' if is a ball, $A' = B_\eta$. By the assumptions, we have that

$$\mathcal{L}^2(\{\varphi(u_\varepsilon) \neq \bar{\varphi}\} \cap A') \leq \frac{1}{\theta_N} \int_A |\varphi(u_\varepsilon) - \bar{\varphi}| \, \mathrm{d}x \to 0 \,,$$

thus the minimum in (3.13) is attained at $\mathcal{L}^2(\{\varphi(u_\varepsilon) \neq \bar{\varphi}\} \cap A')$. This yields

$$\mathcal{L}^2(\{\varphi(u_\varepsilon) \neq \bar{\varphi}\} \cap A') \leq C(A')\big(\mathcal{H}^1(\partial^*\{\varphi(u_\varepsilon) \neq \bar{\varphi}\} \cap A')\big)^2. \tag{3.14}$$

To estimate $\mathcal{H}^1(\partial^*\{\varphi(u_\varepsilon) \neq \bar{\varphi}\} \cap A')$ we observe that Remark 2.5 gives

$$\begin{aligned} \mathcal{H}^1(\partial^*\{\varphi(u_\varepsilon) \neq \bar{\varphi}\} \cap A') &\leq \int_{\partial^*\{\varphi(u_\varepsilon)\neq\bar{\varphi}\}\cap A'} |\nu_{\varphi(u_\varepsilon)}|_1 \, \mathrm{d}\mathcal{H}^1 \\ &\leq \frac{1}{\theta_N} \int_{J_{\varphi(u_\varepsilon)}\cap A'} |\varphi(u_\varepsilon^+) - \varphi(u_\varepsilon^-)||\nu_{u_\varepsilon}|_1 \, \mathrm{d}\mathcal{H}^1 \\ &\leq \frac{1}{\theta_N} |\mathrm{D}\varphi(u_\varepsilon)|_1(A') \leq \frac{1}{4\sin^2\big(\frac{\theta_N}{2}\big)} \frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon, A) \,. \end{aligned}$$

The thesis follows by combining the previous inequality with (3.13) and (3.12). □

The next proof of Theorem 2.10 is based on the well-known ball construction [29, 24]. For the reader's convenience, in Appendix B we present a variant of this tool as presented in [11, Lemma 5.1]. We would like to stress that in what follows when we generically refer to "the ball construction" we mean the full geometric process contained in the proof of the Lemma B.1 and not only those properties that we have single out in its statement.

Proof of Theorem 2.10. Proof of *(i)* (Compactness).
Since $\varepsilon^2|\log\varepsilon| \ll \varepsilon$ as $\varepsilon \to 0$, we deduce from Theorem 2.6-*i)* that, up to a subsequence, $\varphi(u_\varepsilon) \to \varphi$ in $L^1(\Omega;\mathbb{R})$ for some $\varphi \in BV(\Omega;\mathscr{L}_N)$. Theorem 2.6-*ii)* then implies that $|\mathrm{D}\varphi|_1(\Omega) = 0$. Hence, due to the connectedness of Ω, there exists $\bar{\varphi} \in \mathscr{L}_N$ such that $\varphi \equiv \bar{\varphi}$ on Ω. The compactness of the discrete vorticity measures $\mu_{\widetilde{u}_\varepsilon}$ is a consequence of Remark 2.8 and the corresponding result for the XY model (see Proposition 2.4).

Proof of *(ii)* (liminf inequality).
To obtain the liminf inequality, we need to improve the lower bound given by Remark 2.8

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega) \geq 2\pi\varrho_N^2|\mu|(\Omega) \,,$$

which is sharp only in the case where $\varrho(\bar{\varphi}) = \varrho_N$, *i.e.*, $\bar{\varphi} = -\frac{\pi}{2} + \theta_N$ or $\bar{\varphi} = -\frac{\pi}{2} + N\theta_N$. To improve the lower bound, we will combine the ball construction with the result at the gradient scaling (Theorem 2.11) adapting a technique already proven to be useful in different contexts, see [21, 3, 11, 5].

Hereafter, we shall assume that

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega) < +\infty \,, \tag{3.15}$$

otherwise any lower bound is trivial. Moreover, up to extracting a subsequence we can assume that the liminf is a limit, and thus

$$E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2|\log\varepsilon| \,. \tag{3.16}$$

Step 1: (localization) By arguing locally in Ω close to each point of $\mathrm{supp}(\mu)$ and by the superadditivity of the liminf, we can assume without loss of generality that $0 \in \Omega$ and

$$\mu_{\widetilde{u}_\varepsilon} \overset{\mathrm{f}}{\to} d\delta_0 \,, \tag{3.17}$$

with $d \in \mathbb{Z}$. We fix open sets with Lipschitz boundary $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ with $0 \in \Omega'$.

Step 2: (setting up the ball construction) The plan is to apply the ball construction to the discrete vorticity measures $\mu_{\widetilde{u}_\varepsilon}$. We define here the initial family of balls $\mathcal{B}_\varepsilon$ and the increasing set function $\mathcal{E}$.

We consider the family of balls

$$\mathcal{B}_\varepsilon := \{B_{\varepsilon/2}(x) \ : \ x \in \operatorname{supp}(\mu_{\widetilde{u}_\varepsilon}) \cap \Omega\} \, .$$

Notice that each of these balls is contained in a square of the lattice $\varepsilon\mathbb{Z}^2$.

We define now the increasing set function required for the ball construction. For every $0 < r < R$ and for every $x \in \mathbb{R}^2$ such that the annulus $A_{r,R}(x) = B_R(x) \setminus \overline{B}_r(x)$ satisfies $A_{r,R}(x) \cap \bigcup_{B \in \mathcal{B}_\varepsilon} B = \emptyset$, we set

$$\mathcal{E}(\mathcal{B}_\varepsilon, \mu_{\widetilde{u}_\varepsilon}, A_{r,R}(x)) := |\mu_{\widetilde{u}_\varepsilon}(B_r(x))| \log \frac{R}{r} \, ,$$

and we extend $\mathcal{E}$ to every $A \in \mathcal{A}(\mathbb{R}^2)$ by

$$\mathcal{E}(\mathcal{B}_\varepsilon, \mu_{\widetilde{u}_\varepsilon}, A) := \sup \Big\{ \sum_{j=1}^N \mathcal{E}(\mathcal{B}_\varepsilon, \mu_{\widetilde{u}_\varepsilon}, A^j) : N \in \mathbb{N} \, , \ A^j = A_{r_j,R_j}(x_j) \, , \ A^j \cap \bigcup_{B \in \mathcal{B}_\varepsilon} B = \emptyset \, , \\ A^j \cap A^k = \emptyset \text{ for } j \neq k \, , \ A^j \subset A \text{ for all } j \Big\} \, . \tag{3.18}$$

The set function $\mathcal{E}$ has the following property. Given $A' \subset\subset A$ it holds that

$$\mathcal{E}(\mathcal{B}_\varepsilon, \mu_{\widetilde{u}_\varepsilon}, A') \leq \frac{C}{\varepsilon^2} XY_\varepsilon(\widetilde{u}_\varepsilon, A) \, , \tag{3.19}$$

for $\sqrt{2}\varepsilon < \operatorname{dist}(A', \partial A)$. The proof of the previous estimate can be found in [11, proof of Lemma 6.2]. It is based on the observation that the XY energy behaves like the squared L^2 norm of the gradient of the spin field and thus carries at least an energy proportional to $|\mu_{\widetilde{u}_\varepsilon}(B_r(x))| \log \frac{R}{r} = \mathcal{E}(\mathcal{B}_\varepsilon, \mu_{\widetilde{u}_\varepsilon}, A_{r,R}(x))$ in annuli $A_{r,R}(x)$ that do not contain vortices.

Step 3: (applying the ball construction) We apply the ball construction (Lemma B.1) to $\mathcal{B} = \mathcal{B}_\varepsilon$, $\mu = \mu_{\widetilde{u}_\varepsilon}$, and $\mathcal{E}$ defined in (3.18), which satisfy the assumptions (B1) and (B2). We let $\{\mathcal{B}_\varepsilon(t)\}_{t \geq 0}$ denote the family of balls satisfying (1)–(6) of Lemma B.1 with the choice

$$\sigma = \sigma_\varepsilon = 2\sqrt{2}\varepsilon \, . \tag{3.20}$$

For future use, it is convenient to count the number of balls in $\mathcal{B}_\varepsilon$. By Remark 2.3, Remark (2.8), and (3.16), we have that

$$\#\mathcal{B}_\varepsilon = \#\operatorname{supp}(\mu_{\widetilde{u}_\varepsilon}) \cap \Omega = |\mu_{\widetilde{u}_\varepsilon}|(\Omega) \leq \frac{C}{\varepsilon^2} XY_\varepsilon(\widetilde{u}_\varepsilon, \Omega) \leq \frac{C}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega) \leq C|\log \varepsilon| \, . \tag{3.21}$$

Thanks to (3.21), we estimate the sum of the radii of the balls in $\mathcal{B}_\varepsilon$ by

$$\mathcal{R}(\mathcal{B}_\varepsilon) \leq C\varepsilon |\log \varepsilon| \, . \tag{3.22}$$

By property (5) in Lemma B.1, by (3.20), and by (3.22), we have that

$$\mathcal{R}(\mathcal{B}_\varepsilon(t)) \leq (1+t)(\mathcal{R}(\mathcal{B}_\varepsilon) + \#\mathcal{B}_\varepsilon \sigma_\varepsilon) \leq C(1+t)\varepsilon|\log \varepsilon| \, . \tag{3.23}$$

Moreover, by the ball construction, and thanks to property (6) in Lemma B.1,

$$r(B) \geq (1+t)\frac{\varepsilon}{2} \quad \text{for every } B \in \mathcal{B}_\varepsilon(t) \, . \tag{3.24}$$

Finally, it is useful to define the set of merging times

$$\mathrm{T}_\varepsilon^{\mathrm{merg}} := \{t \in [0, +\infty) : \#\mathcal{B}_\varepsilon(t^+) < \#\mathcal{B}_\varepsilon(t^-) \text{ for every } t^-, t^+ \text{ such that } t^- < t < t^+\}$$

and to observe that there exists $K > 0$ such that, by the ball construction,

$$\#\mathrm{T}_\varepsilon^{\mathrm{merg}} \leq K |\log \varepsilon| \, . \tag{3.25}$$

In the following we let

$$U_\varepsilon(t) := \bigcup_{B \in \mathcal{B}_\varepsilon(t)} B \, . \tag{3.26}$$

We observe that by (3.19), by Remark 2.8, and by (3.16), the increasing set function $\mathcal{E}$ satisfies

$$\mathcal{E}(\mathcal{B}_\varepsilon, \mu_{\widetilde{u}_\varepsilon}, \Omega'' \setminus \overline{U}_\varepsilon(0)) \leq \frac{C}{\varepsilon^2} XY_\varepsilon(\widetilde{u}_\varepsilon, \Omega) \leq \frac{C}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega) \leq C|\log \varepsilon| \, . \tag{3.27}$$

Step 4: (choice of time for the ball construction) Let us fix $p \in (0,1)$. (At the very end of the proof we will let $p \to 1$.)

For $k = 0, \dots, \lfloor 2K|\log\varepsilon|\rfloor$ we set

$$\beta_p := \exp\Big(\frac{\sqrt{p}(1-\sqrt{p})}{2K}\Big), \quad t^k_{\varepsilon,p} := (\beta_p)^k \varepsilon^{\sqrt{p}-1} - 1\,, \tag{3.28}$$

and

$$\mathcal{K}_\varepsilon := \Big\{ k \in \{1, \dots, \lfloor 2K|\log\varepsilon|\rfloor\} : (t^{k-1}_{\varepsilon,p}, t^k_{\varepsilon,p}] \cap \mathrm{T}^{\mathrm{merg}}_\varepsilon = \emptyset \Big\}\,. \tag{3.29}$$

We observe that (3.25) implies that

$$\#\mathcal{K}_\varepsilon \geq \lfloor 2K|\log\varepsilon|\rfloor - \#\mathrm{T}^{\mathrm{merg}}_\varepsilon \geq K|\log\varepsilon| - 1 \geq \frac{K}{2}|\log\varepsilon|\,, \tag{3.30}$$

for $\varepsilon \leq e^{-2/K}$. We choose $k_\varepsilon \in \mathcal{K}_\varepsilon$ such that (recall that $U_\varepsilon(t)$ is defined in (3.26))

$$E_\varepsilon(u_\varepsilon, \Omega \cap U_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}) \setminus \overline{U}_\varepsilon(t^{k_\varepsilon - 1}_{\varepsilon,p})) \leq E_\varepsilon(u_\varepsilon, \Omega \cap U_\varepsilon(t^{k}_{\varepsilon,p}) \setminus \overline{U}_\varepsilon(t^{k-1}_{\varepsilon,p})) \quad \text{for every } k \in \mathcal{K}_\varepsilon\,.$$

We set

$$t_{\varepsilon,p} := t^{k_\varepsilon}_{\varepsilon,p}\,. \tag{3.31}$$

We will also use the notation $t^{k_\varepsilon}_{\varepsilon,p}$ when we want to stress the choice of the index k_ε. This choice yields, thanks to property (1) in Lemma B.1, (3.16), and (3.30)

$$\begin{aligned} E_\varepsilon(u_\varepsilon, \Omega \cap U_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}) \setminus \overline{U}_\varepsilon(t^{k_\varepsilon - 1}_{\varepsilon,p})) &\leq \frac{1}{\#\mathcal{K}_\varepsilon} \sum_{k \in \mathcal{K}_\varepsilon} E_\varepsilon(u_\varepsilon, \Omega \cap U_\varepsilon(t^{k}_{\varepsilon,p}) \setminus \overline{U}_\varepsilon(t^{k-1}_{\varepsilon,p})) \\ &\leq \frac{1}{\#\mathcal{K}_\varepsilon} E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2. \end{aligned} \tag{3.32}$$

Step 5: (auxiliary measures) We set

$$\mu_{\varepsilon,p} := \sum_{B \in \mathcal{B}_\varepsilon(t_{\varepsilon,p})} \mu_{\widetilde{u}_\varepsilon}(B)\delta_{x_B}\,, \tag{3.33}$$

where we let x_B denote the center of the ball B. We claim that the following bound and convergence statement hold true:

$$|\mu_{\varepsilon,p}|(\Omega') \leq C_p \quad \text{and} \quad \mu_{\widetilde{u}_\varepsilon} - \mu_{\varepsilon,p} \overset{\mathrm{f}}{\to} 0 \text{ in } \Omega'\,, \tag{3.34}$$

for some constant $C_p > 0$ depending on p.[4] By (3.17) we then have that

$$\mu_{\varepsilon,p} \overset{\mathrm{f}}{\to} d\delta_0\,. \tag{3.35}$$

We now prove the claim (3.34). By (3.27) and by property (3) in Lemma B.1, we have that for ε small enough

$$\begin{aligned} C|\log\varepsilon| &\geq \mathcal{E}(\mathcal{B}_\varepsilon, \mu_{\widetilde{u}_\varepsilon}, \Omega'' \setminus \overline{U}_\varepsilon(0)) \geq \mathcal{E}(\mathcal{B}_\varepsilon, \mu_{\widetilde{u}_\varepsilon}, \Omega' \cap U_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}) \setminus \overline{U}_\varepsilon(0)) \\ &\geq \sum_{\substack{B \in \mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}) \\ x_B \in \Omega'}} |\mu_{\widetilde{u}_\varepsilon}(B)| \log(1 + t^{k_\varepsilon}_{\varepsilon,p}) \geq |\mu_{\varepsilon,p}|(\Omega')\,|\log(1 + t^0_{\varepsilon,p})| \geq |\mu_{\varepsilon,p}|(\Omega')(1-\sqrt{p})|\log\varepsilon|\,. \end{aligned}$$

To prove the convergence in (3.34), we estimate the flat distance between $\mu_{\varepsilon,p}$ and $\mu_{\widetilde{u}_\varepsilon}$ with a standard argument (see, *e.g.*, [22, Lemma 2.2]). We let $\psi \in C^{0,1}_c(\Omega')$ be such that

[4] As a side note, (3.34) are the key estimates that yield compactness in the flat norm for the discrete vorticity measures.

$\|\psi\|_{L^\infty(\Omega')} \leq 1$, $\|\nabla\psi\|_{L^\infty(\Omega')} \leq 1$. Since the balls in $\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})$ are pairwise disjoint and contain the support of $\mu_{\widetilde{u}_\varepsilon}$ and $\mu_{\varepsilon,p}$,

$$\begin{aligned}\langle \mu_{\widetilde{u}_\varepsilon} - \mu_{\varepsilon,p}, \psi\rangle &= \sum_{\substack{B\in\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\\ x_B\in\Omega'}} \int_B \psi \,\mathrm{d}(\mu_{\widetilde{u}_\varepsilon} - \mu_{\varepsilon,p}) + \sum_{\substack{B\in\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\\ x_B\notin\Omega'}} \int_{B\cap\mathrm{supp}(\psi)} \psi \,\mathrm{d}\mu_{\widetilde{u}_\varepsilon}\\
&\leq \sum_{\substack{B\in\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\\ x_B\in\Omega'}} \mathrm{osc}_B(\psi)\big(|\mu_{\widetilde{u}_\varepsilon}| + |\mu_{\varepsilon,p}|\big)(\Omega') + \sum_{\substack{B\in\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\\ x_B\notin\Omega'}} \mathrm{osc}_B(\psi)|\mu_{\widetilde{u}_\varepsilon}|(\Omega')\\
&\leq 4\mathcal{R}(\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}))\big(|\mu_{\widetilde{u}_\varepsilon}| + |\mu_{\varepsilon,p}|\big)(\Omega')\,,\end{aligned}$$

where we used that for $x_B \notin \Omega'$ we can insert $\psi(x_B) = 0$ in the integral to estimate ψ via its oscillation on the ball B. Observe that by (3.23)

$$\mathcal{R}(\mathcal{B}_\varepsilon(t_{\varepsilon,p})) \leq (1 + t^{k_\varepsilon}_{\varepsilon,p})C\varepsilon|\log\varepsilon| \leq (\beta_p)^{2K|\log\varepsilon|}\varepsilon^{\sqrt{p}-1}C\varepsilon|\log\varepsilon| = C\varepsilon^p|\log\varepsilon|\,. \tag{3.36}$$

(In particular, this implies that all radii in our construction will be small.) Taking the supremum over ψ in the above inequality, by (3.36), (3.21), and the bound in (3.34), we get that

$$\|\mu_{\widetilde{u}_\varepsilon} - \mu_{\varepsilon,p}\|_{\mathrm{flat},\Omega'} \leq C\mathcal{R}(\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}))\big(|\mu_{\widetilde{u}_\varepsilon}| + |\mu_{\varepsilon,p}|\big)(\Omega') \leq C\varepsilon^p|\log\varepsilon|^2 \to 0\,,$$

whence also the convergence statement in (3.34).

Step 6: (modification in balls with zero net vorticity) We classify the balls of the family $\mathcal{B}_\varepsilon(t_{\varepsilon,p})$ into two subclasses

$$\begin{aligned}\mathcal{B}^{=0}_\varepsilon &:= \{B \in \mathcal{B}_\varepsilon(t_{\varepsilon,p}) : \mu_{\widetilde{u}_\varepsilon}(B) = 0\,,\ x_B \in \Omega'\}\,,\\
\mathcal{B}^{\neq 0}_\varepsilon &:= \{B \in \mathcal{B}_\varepsilon(t_{\varepsilon,p}) : \mu_{\widetilde{u}_\varepsilon}(B) \neq 0\,,\ x_B \in \Omega'\}\,.\end{aligned} \tag{3.37}$$

In what follows we modify the $\mathscr{S}^2_N$-valued spin field u_ε in the balls of $\mathcal{B}^{=0}_\varepsilon$ with an $\mathscr{S}^2_N$-valued spin field w_ε such that $\widetilde{w}_\varepsilon$ has no discrete vorticity in the balls of $\mathcal{B}^{=0}_\varepsilon$ and whose energy does not increase (asymptotically). More precisely, we prove that there exist a constant $c_p \in (0,1)$ and a sequence $w_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathscr{S}^2_N$ such that $w_\varepsilon = u_\varepsilon$ on $\Omega \setminus \bigcup_{B_R(x)\in\mathcal{B}^{=0}_\varepsilon} B_{c_pR}(x)$, $|\mu_{\widetilde{w}_\varepsilon}|(B) = 0$ for all $B \in \mathcal{B}^{=0}_\varepsilon$, and

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(w_\varepsilon, \Omega') \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega)\,.$$

We remark that the proof of the previous result is divided in two parts. In the first part we modify only the horizontal components $\widetilde{u}_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$ and we define $\widetilde{w}_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$ with zero discrete vorticity in the balls of $\mathcal{B}^{=0}_\varepsilon$ and whose XY energy does not increase with respect to $\widetilde{u}_\varepsilon$. In the second part of the proof we need to define the third component (or, equivalently, the latitude) of w_ε. This is done by modifying the third component of u_ε, setting it equal to the constant value $\bar{\varphi}$ inside balls contained in those belonging to $\mathcal{B}^{=0}_\varepsilon$ and chosen via a De Giorgi averaging argument.

We now prove what is stated above with the constant $c_p := \frac{\beta_p+1}{2\beta_p}$. Let us construct the spin field w_ε. Let $B_{R_\varepsilon}(x_\varepsilon) \in \mathcal{B}^{=0}_\varepsilon$. Since $k_\varepsilon \in \mathcal{K}_\varepsilon$, by (3.29) no merging occurs in the interval $(t^{k_\varepsilon-1}_{\varepsilon,p}, t^{k_\varepsilon}_{\varepsilon,p}]$ and therefore, according to the ball construction, there exists $B_{r_\varepsilon}(x_\varepsilon) \in \mathcal{B}_\varepsilon(t^{k_\varepsilon-1}_{\varepsilon,p})$ (*i.e.*, a ball with the same center in the family $\mathcal{B}_\varepsilon(t^{k_\varepsilon-1}_{\varepsilon,p})$). Note that, by (3.24) and by (3.28),

$$\frac{\varepsilon}{r_\varepsilon} \leq \frac{C}{1 + t^{k_\varepsilon-1}_{\varepsilon,p}} = \frac{C\varepsilon^{1-\sqrt{p}}}{(\beta_p)^{k_\varepsilon-1}} \leq C\varepsilon^{1-\sqrt{p}} \to 0\,. \tag{3.38}$$

Let r'_ε be the radius of the ball centred in x_ε at the last merging time $T \leq t^{k_\varepsilon-1}_{\varepsilon,p}$ (in the case no merging occurred before $t^{k_\varepsilon-1}_{\varepsilon,p}$, let $T = 0$). By construction, recalling (B.4) and (3.28), we deduce that

$$\frac{r_\varepsilon}{r'_\varepsilon} = \frac{1 + t^{k_\varepsilon-1}_{\varepsilon,p}}{1 + T}\,, \quad \frac{R_\varepsilon}{r'_\varepsilon} = \frac{1 + t^{k_\varepsilon}_{\varepsilon,p}}{1 + T} \implies \frac{R_\varepsilon}{r_\varepsilon} = \beta_p\,. \tag{3.39}$$

Note that, according to the ball construction, $\mu_{\widetilde{u}_\varepsilon}(B_{r_\varepsilon}(x_\varepsilon)) = 0$ and, by property (4) in Lemma B.1 and due to the choice $\sigma = 2\sqrt{2}\varepsilon$, $|\mu_{\widetilde{u}_\varepsilon}|(A_{r_\varepsilon - 2\sqrt{2}\varepsilon, R_\varepsilon + 2\sqrt{2}\varepsilon}(x_\varepsilon)) = 0$. The latter condition ensures that $|\mu_{\widetilde{u}_\varepsilon}|(\varepsilon i + [0,\varepsilon]^2) = 0$ for $i \in \mathbb{Z}^2$ such that $(\varepsilon i + [0,\varepsilon]^2) \cap A_{r_\varepsilon, R_\varepsilon}(x_\varepsilon) \neq \emptyset$. Furthermore, thanks to Remark 2.8 and (3.32), again appealing to the ball construction, we have that

$$XY_\varepsilon(\widetilde{u}_\varepsilon, A_{r_\varepsilon, R_\varepsilon}(x_\varepsilon)) \leq CE_\varepsilon(u_\varepsilon, A_{r_\varepsilon, R_\varepsilon}(x_\varepsilon)) \leq C\varepsilon^2 . \tag{3.40}$$

Therefore, we are in a position to apply the following extension result for discrete spin fields proven in [11, Lemma 3.5], see also [11, Remark 3.6].

Lemma 3.2. [11, Lemma 3.5] *There exists a universal constant $C_0 > 0$ such that the following holds true. Let $\varepsilon > 0$, $x_0 \in \mathbb{R}^2$, and $R > r > \varepsilon$, let $C_1 > 1$ and $v_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$ with $XY_\varepsilon(v_\varepsilon, A_{r,R}(x_0)) \leq C_1\varepsilon^2$, $\mu_{v_\varepsilon}(B_r(x_0)) = 0$, and $|\mu_{v_\varepsilon}|(\varepsilon i + [0,\varepsilon]^2) = 0$ for every $i \in \mathbb{Z}^2$ such that $(\varepsilon i + [0,\varepsilon]^2) \cap A_{r,R}(x_0) \neq 0$. Then there exists $\overline{v}_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$ such that for $\varepsilon < \frac{R-r}{C_0 C_1}$:*

- *$\overline{v}_\varepsilon = v_\varepsilon$ on $\varepsilon\mathbb{Z}^2 \setminus \overline{B}_{\frac{r+R}{2}}(x_0)$;*
- *$|\mu_{\overline{v}_\varepsilon}|(B_R(x_0)) = 0$;*
- *$XY_\varepsilon(\overline{v}_\varepsilon, B_R(x_0)) \leq C(r,R) XY_\varepsilon(v_\varepsilon, A_{r,R}(x_0))$, where $C(r,R) = C_0 \frac{R}{R-r}$.*

Applying the previous extension result to $\widetilde{u}_\varepsilon$, we obtain $\widetilde{w}_\varepsilon \colon \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$ such that $\widetilde{w}_\varepsilon = \widetilde{u}_\varepsilon$ on $\varepsilon\mathbb{Z}^2 \cap A_{c_p R_\varepsilon, R_\varepsilon}(x_\varepsilon)$ (observe that $\frac{r_\varepsilon + R_\varepsilon}{2} = c_p R_\varepsilon$), $|\mu_{\widetilde{w}_\varepsilon}|(B_{R_\varepsilon}(x_\varepsilon)) = 0$, and

$$XY_\varepsilon(\widetilde{w}_\varepsilon, B_{R_\varepsilon}(x_\varepsilon)) \leq C(r_\varepsilon, R_\varepsilon) XY_\varepsilon(\widetilde{u}_\varepsilon, A_{r_\varepsilon, R_\varepsilon}(x_\varepsilon)) \tag{3.41}$$

for ε small enough (*i.e.*, such that $\frac{\varepsilon}{r_\varepsilon} < \frac{\beta_p - 1}{C_0 C_1}$, *cf.* (3.38) and (3.39)). Note that (3.39) implies that

$$C(r_\varepsilon, R_\varepsilon) = C_0 \frac{R_\varepsilon}{R_\varepsilon - r_\varepsilon} = C_0 \frac{\beta_p r_\varepsilon}{\beta_p r_\varepsilon - r_\varepsilon} = C_0 \frac{\beta_p}{\beta_p - 1} = C(\beta_p) ,$$

i.e., it is independent of ε.

We need to define the latitude $\varphi(w_\varepsilon) \in \mathscr{L}_N$. Once $\varphi(w_\varepsilon)$ is defined, we set

$$w_\varepsilon := \big(\cos(\varphi(w_\varepsilon))\widetilde{w}_\varepsilon, \sin(\varphi(w_\varepsilon))\big) . \tag{3.42}$$

We define the latitude $\varphi(w_\varepsilon)$ by setting $\varphi(w_\varepsilon) := \bar{\varphi}$ inside a suitable ball contained in $B_{c_p R_\varepsilon}$ and $\varphi(w_\varepsilon) := \varphi(u_\varepsilon)$ outside the same ball. The selection of the ball is done via a De Giorgi averaging argument that we explain in detail here. It is convenient to introduce the auxiliary energy pertaining the third component of the spin fields

$$F_\varepsilon(\varphi, A) := \frac{1}{2} \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \cap A \\ |i-j|=1}} \varepsilon^2 |\sin(\varphi(\varepsilon i)) - \sin(\varphi(\varepsilon j))|^2. \tag{3.43}$$

We let $\sqrt{p} < \bar{p} < 1$ and

$$H_\varepsilon := \Big\lfloor \frac{1}{\varepsilon^{1-\bar{p}}} \Big\rfloor . \tag{3.44}$$

Note that, by (3.38) for ε small enough we have that

$$2\varepsilon H_\varepsilon \leq \frac{2\varepsilon}{\varepsilon^{1-\bar{p}}} = 2\varepsilon^{\bar{p}} \ll r_\varepsilon .$$

We consider H_ε equispaced circles in the annulus $A_{r_\varepsilon, c_p R_\varepsilon}(x_\varepsilon)$, *i.e.*, $\partial B_{r_\varepsilon^h}(x_\varepsilon)$ for $h = 1, \ldots, H_\varepsilon$ with

$$r_\varepsilon^h := r_\varepsilon + h \frac{c_p R_\varepsilon - r_\varepsilon}{H_\varepsilon + 1} .$$

Moreover, we consider the strips

$$S_\varepsilon^h := \partial B_{r_\varepsilon^h}(x_\varepsilon) + B_{3\varepsilon} \quad \text{for } h = 1, \ldots, H_\varepsilon .$$

Note that, for ε small enough,

$$S_\varepsilon^h \cap S_\varepsilon^{h'} = \emptyset \quad \text{for } h \neq h' \tag{3.45}$$

since, by (3.39) and (3.38),

$$\mathrm{dist}(S_\varepsilon^h, S_\varepsilon^{h'}) = \frac{c_p R_\varepsilon - r_\varepsilon}{H_\varepsilon + 1} - 6\varepsilon = \frac{\beta_p - 1}{2(H_\varepsilon+1)} r_\varepsilon - 6\varepsilon \geq \frac{C}{\varepsilon^{\bar p - 1}}\varepsilon^{\sqrt{p}} - 6\varepsilon = \frac{C}{\varepsilon^{\bar p - \sqrt{p}}}\varepsilon - 6\varepsilon > 0\,.$$

For $h = 1, \dots, H_\varepsilon$ and for $\varepsilon i \in \mathbb{Z}^2 \cap B_{R_\varepsilon}(x_\varepsilon)$ we let

$$\varphi_\varepsilon^h(\varepsilon i) := \begin{cases} \bar\varphi & \text{if } \varepsilon i \in \overline{B}_{r_\varepsilon^h}(x_\varepsilon)\,, \\ \varphi(u_\varepsilon(\varepsilon i)) & \text{if } \varepsilon i \in B_{R_\varepsilon}(x_\varepsilon) \setminus \overline{B}_{r_\varepsilon^h}(x_\varepsilon)\,. \end{cases}$$

We let $h_\varepsilon \in \{1, \dots, H_\varepsilon\}$ be such that

$$\int_{S_\varepsilon^{h_\varepsilon}} |\varphi(u_\varepsilon)) - \bar\varphi| \,\mathrm{d}x \leq \int_{S_\varepsilon^h} |\varphi(u_\varepsilon)) - \bar\varphi| \,\mathrm{d}x \quad \text{for } h = 1, \dots, H_\varepsilon\,.$$

We claim that the latitude $\varphi(w_\varepsilon) := \varphi_\varepsilon^{h_\varepsilon}$ satisfies the following inequality

$$F_\varepsilon(\varphi(w_\varepsilon), B_{R_\varepsilon}(x_\varepsilon)) \leq F_\varepsilon(\varphi(u_\varepsilon), A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)) + \frac{C}{H_\varepsilon}\int_{B_{R_\varepsilon}(x_\varepsilon)} |\varphi(u_\varepsilon) - \bar\varphi| \,\mathrm{d}x\,. \tag{3.46}$$

Indeed, since every point in $\mathbb{Z}^2$ has at most 4 neighbors, by definition of $\varphi_\varepsilon^{h_\varepsilon}$ we get that

$$\begin{aligned} F_\varepsilon(\varphi(w_\varepsilon), B_{R_\varepsilon}(x_\varepsilon)) &\leq F_\varepsilon(\varphi(u_\varepsilon), B_{R_\varepsilon}(x_\varepsilon) \setminus \overline{B}_{r_\varepsilon^{h_\varepsilon}}(x_\varepsilon)) + F_\varepsilon(\bar\varphi, \overline{B}_{r_\varepsilon^{h_\varepsilon}}(x_\varepsilon)) \\ &\quad + 2 \sum_{\substack{\varepsilon i \in \varepsilon\mathbb{Z}^2 \\ r_\varepsilon^{h_\varepsilon} \leq |\varepsilon i - x_\varepsilon| \leq r_\varepsilon^{h_\varepsilon} + \varepsilon}} \varepsilon^2 |\sin(\varphi(u_\varepsilon(\varepsilon i))) - \sin(\bar\varphi)|^2 \\ &\leq F_\varepsilon(\varphi(u_\varepsilon), A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)) + 2\int_{S_\varepsilon^{h_\varepsilon}} |\sin(\varphi(u_\varepsilon)) - \sin(\bar\varphi)|^2 \,\mathrm{d}x\,. \end{aligned} \tag{3.47}$$

Note that $|\sin(\varphi(u_\varepsilon)) - \sin(\bar\varphi)|^2 \leq \pi|\varphi(u_\varepsilon) - \bar\varphi|$. Due to the choice of h_ε we find that

$$\begin{aligned} F_\varepsilon(\varphi(w_\varepsilon), B_{R_\varepsilon}(x_\varepsilon)) &\leq F_\varepsilon(\varphi(u_\varepsilon), A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)) + \frac{2\pi}{H_\varepsilon}\sum_{h=1}^{H_\varepsilon}\int_{S_\varepsilon^h} |\varphi(u_\varepsilon) - \bar\varphi| \,\mathrm{d}x \\ &\leq F_\varepsilon(\varphi(u_\varepsilon), A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)) + \frac{2\pi}{H_\varepsilon}\int_{B_{R_\varepsilon}(x_\varepsilon)} |\varphi(u_\varepsilon) - \bar\varphi| \,\mathrm{d}x, \end{aligned}$$

where we used that the strips $(S_\varepsilon^h)_{h=1}^{H_\varepsilon}$ are pairwise disjoint. This concludes the proof of (3.46).

We are in a position to estimate the energy of w_ε in the ball B_{R_ε}. We exploit the inequality

$$\big|\alpha c - \beta d\big|^2 \leq 2\Big(\beta^2|c-d|^2 + |\alpha - \beta|^2|c|^2\Big) \quad \text{for } \alpha, \beta > 0 \text{ and } c, d \in \mathbb{R}^2$$

to obtain that (recall the definition of F_ε in (3.43))

$$\begin{aligned} &E_\varepsilon(w_\varepsilon, B_{R_\varepsilon}(x_\varepsilon)) = \frac{1}{2}\sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \cap B_{R_\varepsilon}(x_\varepsilon) \\ |i-j|=1}} \varepsilon^2 |w_\varepsilon(\varepsilon i) - w_\varepsilon(\varepsilon j)|^2 \\ &= \frac{1}{2}\sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \cap B_{R_\varepsilon}(x_\varepsilon) \\ |i-j|=1}} \varepsilon^2 |\cos(\varphi(w_\varepsilon(\varepsilon i)))\widetilde{w}_\varepsilon(\varepsilon i) - \cos(\varphi(w_\varepsilon(\varepsilon j)))\widetilde{w}_\varepsilon(\varepsilon j)|^2 + F_\varepsilon(\varphi(w_\varepsilon), B_{R_\varepsilon}(x_\varepsilon)) \\ &\leq 2XY_\varepsilon(\widetilde{w}_\varepsilon, B_{R_\varepsilon}(x_\varepsilon)) + \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \cap B_{R_\varepsilon}(x_\varepsilon) \\ |i-j|=1}} \varepsilon^2 \big|\cos(\varphi(w_\varepsilon(\varepsilon i))) - \cos(\varphi(w_\varepsilon(\varepsilon j)))\big|^2 \\ &\quad + F_\varepsilon(\varphi(w_\varepsilon), B_{R_\varepsilon}(x_\varepsilon))\,. \end{aligned} \tag{3.48}$$

We observe that $\cos(\varphi(w_\varepsilon)) = \sqrt{1 - \sin^2(\varphi(w_\varepsilon))}$ with $\varphi(w_\varepsilon) \in \mathscr{L}_N$. Note that $\mathscr{L}_N \subset [-\frac{\pi}{2} + \theta_N, \frac{\pi}{2} - \theta_N]$ and thus $\sin(\mathscr{L}_N) \subset [-\arcsin\big(\frac{\pi}{2} - \theta_N\big), \arcsin\big(\frac{\pi}{2} - \theta_N\big)] \subset\subset (-1, 1)$.

The function $s \mapsto \sqrt{1-s^2}$ is Lipschitz (with a Lipschitz constant depending on N) on $[-\arcsin\left(\frac{\pi}{2}-\theta_N\right), \arcsin\left(\frac{\pi}{2}-\theta_N\right)]$, hence

$$\big|\cos(\varphi(w_\varepsilon(\varepsilon i))) - \cos(\varphi(w_\varepsilon(\varepsilon j)))\big|^2 \leq C\big|\sin(\varphi(w_\varepsilon(\varepsilon i))) - \sin(\varphi(w_\varepsilon(\varepsilon j)))\big|^2 .$$

Therefore (3.48) reads

$$E_\varepsilon(w_\varepsilon, B_{R_\varepsilon}(x_\varepsilon)) \leq CXY(\widetilde{w}_\varepsilon, B_{R_\varepsilon}(x_\varepsilon)) + CF_\varepsilon(\varphi(w_\varepsilon), B_{R_\varepsilon}(x_\varepsilon)) .$$

By (3.41) and (3.46) we have that

$$\begin{aligned} &E_\varepsilon(w_\varepsilon, B_{R_\varepsilon}(x_\varepsilon)) \\ &\quad \leq CXY_\varepsilon(\widetilde{u}_\varepsilon, A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)) + CF_\varepsilon(\varphi(u_\varepsilon), A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)) + \frac{C}{H_\varepsilon}\int_{B_{R_\varepsilon}(x_\varepsilon)} |\varphi(u_\varepsilon) - \bar{\varphi}| \,\mathrm{d}x\,, \end{aligned}$$

which in turn implies, by Remark 2.8 and the definition of F_ε in (3.43),

$$E_\varepsilon(w_\varepsilon, B_{R_\varepsilon}(x_\varepsilon)) \leq CE_\varepsilon(u_\varepsilon, A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)) + \frac{C}{H_\varepsilon}\int_{B_{R_\varepsilon}(x_\varepsilon)} |\varphi(u_\varepsilon) - \bar{\varphi}| \,\mathrm{d}x\,.$$

We apply the previous construction in every ball $B_{R_\varepsilon}(x_\varepsilon) \in \mathcal{B}^{=0}_\varepsilon$ in order to obtain w_ε such that $w_\varepsilon = u_\varepsilon$ on $\Omega \setminus \bigcup_{\{x_\varepsilon\colon B_{R_\varepsilon}(x_\varepsilon)\in\mathcal{B}^{=0}_\varepsilon\}} \overline{B}_{c_pR_\varepsilon}(x_\varepsilon)$, $|\mu_{\widetilde{w}_\varepsilon}|(B_{R_\varepsilon}(x_\varepsilon)) = 0$ for all $B_{R_\varepsilon}(x_\varepsilon) \in \mathcal{B}^{=0}_\varepsilon$ and (we recall that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$). By the ball construction it holds that

$$\begin{aligned} &E_\varepsilon\Big(w_\varepsilon, \bigcup_{\{x_\varepsilon\colon B_{R_\varepsilon}(x_\varepsilon)\in\mathcal{B}^{=0}_\varepsilon\}} B_{R_\varepsilon}(x_\varepsilon)\Big) \\ &\quad \leq CE_\varepsilon(u_\varepsilon, \Omega\cap U_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}) \setminus U_\varepsilon(t^{k_\varepsilon-1}_{\varepsilon,p})) + \frac{C}{H_\varepsilon}\int_{\Omega''} |\varphi(u_\varepsilon) - \bar{\varphi}| \,\mathrm{d}x\,. \end{aligned} \tag{3.49}$$

We estimate the last integral by applying Lemma 3.1, by the definition of H_ε in (3.44), and by the energy bound (3.16):

$$\frac{1}{H_\varepsilon}\int_{\Omega''} |\varphi(u_\varepsilon) - \bar{\varphi}| \,\mathrm{d}x \leq \frac{1}{\frac{1}{\varepsilon^{1-\bar{p}}}-1} C(\Omega'',N)\Big(\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon,\Omega)\Big)^2 \leq C(\Omega'',N)\frac{\varepsilon^{1-\bar{p}}}{1-\varepsilon^{1-\bar{p}}}\varepsilon^2|\log\varepsilon|^2 .$$

Thanks to the previous inequality and to (3.32), for ε small enough (3.49) reads

$$E_\varepsilon\Big(w_\varepsilon, \bigcup_{\{x_\varepsilon\colon B_{R_\varepsilon}(x_\varepsilon)\in\mathcal{B}^{=0}_\varepsilon\}} B_{R_\varepsilon}(x_\varepsilon)\Big) \leq C\varepsilon^2 + C(\Omega'',N)\varepsilon^{1-\bar{p}}\varepsilon^2|\log\varepsilon|^2 .$$

We conclude that

$$\begin{aligned} \liminf_{\varepsilon\to0} \frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(w_\varepsilon,\Omega') &\leq \liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2|\log\varepsilon|}\Big(E_\varepsilon(u_\varepsilon,\Omega) + E_\varepsilon\Big(w_\varepsilon, \bigcup_{\{x_\varepsilon\colon B_{R_\varepsilon}(x_\varepsilon)\in\mathcal{B}^{=0}_\varepsilon\}} B_{R_\varepsilon}(x_\varepsilon)\Big)\Big) \\ &\leq \liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon,\Omega) + \limsup_{\varepsilon\to0}\Big(\frac{C}{|\log\varepsilon|} + C\varepsilon^{1-\bar{p}}|\log\varepsilon|\Big) \\ &= \liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon,\Omega)\,. \end{aligned}$$

This concludes the construction of w_ε.

Step 7: (limit of balls in $\mathcal{B}^{\neq0}_\varepsilon$) In view of (3.34), we have that $\#\mathcal{B}^{\neq0}_\varepsilon \leq C_p$ and therefore we can assume that (up to a subsequence) $\#\mathcal{B}^{\neq0}_\varepsilon = L$ for all $\varepsilon > 0$ for some $L \in \mathbb{N}$. Let $\mathcal{B}^{\neq0}_\varepsilon = \{B_{r^\ell_\varepsilon}(x^\ell_\varepsilon)\}_{\ell=1}^L$. By definition (3.33), we have that $\{x^1_\varepsilon,\dots,x^L_\varepsilon\}$ is the support of the measure $\mu_{\varepsilon,p}$. The points x^ℓ_ε converge (up to a subsequence) to points belonging to a finite set $\{0=\xi^1,\dots,\xi^{L'}\}$ contained in $\overline{\Omega}'$ with $L' \leq L$. Fix $\rho > 0$ such that $B_\rho \subset\subset \Omega'$ and $B_\rho(\xi^h)\cap B_\rho = \emptyset$ for all $h = 2,\dots,L'$. For $\varepsilon > 0$ small enough we have that either $B_{r^\ell_\varepsilon}(x^\ell_\varepsilon)\cap B_\rho = \emptyset$ or $B_{r^\ell_\varepsilon}(x^\ell_\varepsilon) \subset\subset B_\rho$. Furthermore, by (3.33), (3.35), and the fact that $|\mu|(\partial B_\rho) = 0$, for ε small enough we have that

$$\sum_{x^\ell_\varepsilon\in B_\rho} \mu_{\widetilde{u}_\varepsilon}(B_{r^\ell_\varepsilon}(x^\ell_\varepsilon)) = d\,. \tag{3.50}$$

We will prove that

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, B_\rho) \geq 2\pi\cos^2(\bar\varphi)|d|\,.$$

Since our estimate is local, we can assume, without loss of generality, that $|\mu_{\widetilde{u}_\varepsilon}|(\mathbb{R}^2\setminus B_\rho) = 0$, which implies that $x^\ell_\varepsilon \in B_\rho$, *i.e.*, $B_{r^\ell_\varepsilon}(x^\ell_\varepsilon) \subset\subset B_\rho$, for $\ell = 1,\dots,L$ and ε small enough.

Step 8: (lower bound with energy on annuli) We follow now an argument used, *e.g.*, in [21, 3, 11, 5] aimed at separating the scales of the radii of the balls charged by $\mu_{\widetilde{u}_\varepsilon}$.

Fix $0 < p' < p'' < p$ such that $\mathcal{R}(\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})) \leq \varepsilon^{p''}$ (this is possible due to (3.36)). We consider the function $g_\varepsilon \colon [p', p''] \to \{1,\dots,L\}$ such that $g_\varepsilon(q)$ gives the number of connected components of $\bigcup_{\ell=1}^L B_{\varepsilon^q}(x^\ell_\varepsilon)$. For each $\varepsilon > 0$, the function g_ε is monotonically non-decreasing so that it can have at most $\hat{L} \leq L-1$ discontinuity points. We let $\{q^\varepsilon_1,\dots,q^\varepsilon_{\hat{L}}\}$ denote these discontinuity points with

$$p' \leq q^\varepsilon_1 < \ldots < q^\varepsilon_{\hat{L}} \leq p''.$$

Up to a subsequence we may assume that $\hat{L}$ is independent of ε. Moreover, there exists a finite set $\mathfrak{D} = \{q_0,\dots,q_{\widetilde{L}+1}\}$ with $p' = q_0 < q_1 < \ldots < q_{\widetilde{L}+1} = p''$ such that $(q^\varepsilon_j)_\varepsilon$ converges to some point in $\mathfrak{D}$ as $\varepsilon\to 0$, for $j = 1,\dots,\hat{L}$. Note that we can always choose the set $\mathfrak{D}$ such that $\widetilde{L} \leq \hat{L}$. Let us fix $\lambda > 0$ with $4\lambda < \min_h(q_{h+1} - q_h)$. For $\varepsilon > 0$ small enough (that is, such that for $h' = 1,\dots,\hat{L}$ one has $|q^\varepsilon_{h'} - q_h| < \lambda$ for some $q_h \in \mathfrak{D}$) the function g_ε is constant in the interval $[q_h+\lambda, q_{h+1}-\lambda]$ with constant value M^ε_h, where $M^\varepsilon_h \leq L$. Up to extracting a subsequence, we assume that $M^\varepsilon_h = M_h$. For $h = 0,\dots,\widetilde{L}$ we construct a family of annuli as in [11, Lemma 6.7]. More precisely, we set

$$\alpha_h := q_h + \lambda\,, \quad \beta_h := q_{h+1} - \lambda\,.$$

The family of disjoint annuli $\{A^{h,m}_\varepsilon\}_{m=1}^{M_h}$ with $A^{h,m}_\varepsilon := B_{\varepsilon^{\alpha_h}}(z^{h,m}_\varepsilon) \setminus \overline{B}_{\varepsilon^{\beta_h}}(z^{h,m}_\varepsilon)$ satisfies that the sets in the family $\{\bigcup_{m=1}^{M_h} A^{h,m}_\varepsilon\}_{h=0}^{\widetilde{L}}$ are pairwise disjoint and

$$\bigcup_{\ell=1}^L B_{r^\ell_\varepsilon}(x^\ell_\varepsilon) \subset \bigcup_{m=1}^{M_h} B_{\varepsilon^{\beta_h}}(z^{h,m}_\varepsilon) \tag{3.51}$$

for $h = 0,\dots,\widetilde{L}$. Moreover, the points $z^{h,m}_\varepsilon$ are suitably chosen in $\varepsilon\mathbb{Z}^2 \cap \bigcup_{\ell=1}^L B_\varepsilon(x^\ell_\varepsilon)$.

The rest of the proof shows that the relevant energy of u_ε is concentrated on the annuli $A^{h,m}_\varepsilon$. Hence we estimate the energy from below by:

$$\frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega') \geq \sum_{h=0}^{\widetilde{L}} \sum_{m=1}^{M_h} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, A^{m,h}_\varepsilon)$$

and exploit the annuli $A^{h,m}_\varepsilon$ to prove the lower bound.

Note that, for ε small enough $A^{h,m}_\varepsilon \subset\subset B_\rho \subset \Omega'$ for $h = 0,\dots,\widetilde{L}$ and $m = 1,\dots,M_h$. Moreover, let us show that $|\mu_{\widetilde{u}_\varepsilon}(B_{\varepsilon^{\beta_h}}(z^{h,m}_\varepsilon))| \leq C$. Indeed, let $B_\varepsilon(x_\varepsilon) \in \mathcal{B}_\varepsilon$ be such that $x_\varepsilon \in B_{\varepsilon^{\beta_h}}(z^{h,m}_\varepsilon)$ (the measure $\mu_{\widetilde{u}_\varepsilon}$ only charges points as x_ε inside the set $B_{\varepsilon^{\beta_h}}(z^{h,m}_\varepsilon)$). By the ball construction, we have that $B_\varepsilon(x_\varepsilon)$ is contained in a ball $B_{r^\ell_\varepsilon}(x^\ell_\varepsilon)$ for some $\ell = 1,\dots,L$ (recall that $\mathcal{B}^{=0}_\varepsilon = \emptyset$). By (3.51), there exists $m' = 1,\dots,M_h$ such that $B_{r^\ell_\varepsilon}(x^\ell_\varepsilon) \subset B_{\varepsilon^{\beta_h}}(z^{h,m'}_\varepsilon)$. *A fortiori*, we have that $m = m'$, *i.e.*, $B_\varepsilon(x_\varepsilon) \subset B_{r^\ell_\varepsilon}(x^\ell_\varepsilon) \subset B_{\varepsilon^{\beta_h}}(z^{h,m}_\varepsilon)$ (otherwise we would have $B_\varepsilon(x_\varepsilon) \cap B_{\varepsilon^{\beta_h}}(z^{h,m}_\varepsilon) = \emptyset$, in contradiction with the fact that x_ε belongs to this intersection). From the previous argument and by (3.33), we deduce that

$$\begin{aligned}\mu_{\widetilde{u}_\varepsilon}(B_{\varepsilon^{\beta_h}}(z_\varepsilon^{h,m})) &= \mu_{\widetilde{u}_\varepsilon}\Big(\bigcup_{\substack{B\in\mathcal{B}_\varepsilon\\ B\subset B_{\varepsilon^{\beta_h}}(z_\varepsilon^{h,m})}} B\Big) = \sum_{\substack{B\in\mathcal{B}_\varepsilon\\ B\subset B_{\varepsilon^{\beta_h}}(z_\varepsilon^{h,m})}} \mu_{\widetilde{u}_\varepsilon}(B)\\ &= \sum_{\substack{B_{r_\varepsilon^\ell}(x_\varepsilon^\ell)\in\mathcal{B}_\varepsilon^{\neq 0}\\ B_{r_\varepsilon^\ell}(x_\varepsilon^\ell)\subset B_{\varepsilon^{\beta_h}}(z_\varepsilon^{h,m})}} \sum_{\substack{B\in\mathcal{B}_\varepsilon\\ B\subset B_{r_\varepsilon^\ell}(x_\varepsilon^\ell)}} \mu_{\widetilde{u}_\varepsilon}(B) = \sum_{\substack{B_{r_\varepsilon^\ell}(x_\varepsilon^\ell)\in\mathcal{B}_\varepsilon^{\neq 0}\\ B_{r_\varepsilon^\ell}(x_\varepsilon^\ell)\subset B_{\varepsilon^{\beta_h}}(z_\varepsilon^{h,m})}} \mu_{\widetilde{u}_\varepsilon}(B_{r_\varepsilon^\ell}(x_\varepsilon^\ell))\\ &= \mu_{\varepsilon,p}(B_{\varepsilon^{\beta_h}}(z_\varepsilon^{h,m}))\,.\end{aligned}$$

In view of (3.34), we have that $|\mu_{\widetilde{u}_\varepsilon}(B_{\varepsilon^{\beta_h}}(z_\varepsilon^{h,m}))| \le C$ for $h = 0,\dots,\widetilde{L}$ and $m = 1,\dots,M_h$. Therefore, up to extracting a further subsequence and thanks to (3.50), we have that

$$\mu_{\widetilde{u}_\varepsilon}(B_{\varepsilon^{\beta_h}}(z_\varepsilon^{h,m})) = d_{h,m} \in \mathbb{Z}\setminus\{0\} \quad \text{and} \quad \sum_{m=1}^{M_h} d_{h,m} = d\,, \tag{3.52}$$

with M_h and $d_{h,m}$ independent of ε.

Step 9: (blow-up of the annuli) We fix $h \in \{0,\dots,\widetilde{L}\}$ and $m \in \{1,\dots,M_h\}$. In this and in the next two steps we will show that

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, A_\varepsilon^{h,m}) \ge (\beta_h-\alpha_h)2\pi\cos^2(\bar\varphi)|d_{h,m}|\,. \tag{3.53}$$

We fix $R>1$ and, to simplify the notation, we write $\alpha := \alpha_h$, $\beta := \beta_h$ and $z = z_\varepsilon^{h,m}$.

We set $N_{\varepsilon,R} := \lfloor(\beta-\alpha)\frac{|\log\varepsilon|}{\log R}\rfloor$ and for $n = 1,\dots,N_{\varepsilon,R}$ define $A^{n,\varepsilon} := B_{R^n\varepsilon^\beta}(z)\setminus \overline{B}_{R^{n-1}\varepsilon^\beta}(z)$. We remark that $\bigcup_{n=1}^{N_{\varepsilon,R}} A^{n,\varepsilon} \subset B_{\varepsilon^\alpha}(z)\setminus\overline{B}_{\varepsilon^\beta}(z) = A_\varepsilon^{h,m}$. Let $\overline{n} = \overline{n}_{\varepsilon,R}$ be such that

$$E_\varepsilon(u_\varepsilon, A^{\overline{n},\varepsilon}) \le E_\varepsilon(u_\varepsilon, A^{n,\varepsilon}) \quad \text{for } n = 1,\dots,N_{\varepsilon,R}\,.$$

We let $\delta_\varepsilon := \frac{\varepsilon}{R^{\overline{n}-1}\varepsilon^\beta}$ and observe that

$$\varepsilon^\beta \le \frac{\varepsilon}{\delta_\varepsilon} \le \frac{\varepsilon}{\delta_\varepsilon}R \le \varepsilon^\alpha\,. \tag{3.54}$$

Then, defining the rescaled spin fields as $v_{\delta_\varepsilon}(\delta_\varepsilon i) := u_\varepsilon(\varepsilon i - z_\varepsilon^{h,m})$, $i\in\mathbb{Z}^2$, we have

$$\begin{aligned}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon, A_\varepsilon^{h,m}) &\ge \frac{1}{|\log\varepsilon|}\sum_{n=1}^{N_{\varepsilon,R}} \frac{1}{\varepsilon^2}E_\varepsilon(u_\varepsilon, A^{n,\varepsilon}) \ge \frac{1}{|\log\varepsilon|}\frac{N_{\varepsilon,R}}{\varepsilon^2}E_\varepsilon(u_\varepsilon, A^{\overline{n},\varepsilon})\\ &= \frac{1}{|\log\varepsilon|}\frac{N_{\varepsilon,R}}{\delta_\varepsilon^2}E_{\delta_\varepsilon}(v_{\delta_\varepsilon}, B_R\setminus\overline{B}_1)\,.\end{aligned}$$

Since $N_{\varepsilon,R} \ge (\beta-\alpha)\frac{|\log\varepsilon|}{\log R} - 1$, from the previous inequality we get that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon, A_\varepsilon^{h,m}) &\ge \liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\frac{(\beta-\alpha)\frac{|\log\varepsilon|}{\log R}-1}{\delta_\varepsilon^2}E_{\delta_\varepsilon}(v_{\delta_\varepsilon}, B_R\setminus\overline{B}_1)\\ &= \liminf_{\varepsilon\to 0}\Big(\frac{(\beta-\alpha)}{\log R} - \frac{1}{|\log\varepsilon|}\Big)\frac{1}{\delta_\varepsilon^2}E_{\delta_\varepsilon}(v_{\delta_\varepsilon}, B_R\setminus\overline{B}_1)\\ &\ge \frac{(\beta-\alpha)}{\log R}\liminf_{\varepsilon\to 0}\frac{1}{\delta_\varepsilon^2}E_{\delta_\varepsilon}(v_{\delta_\varepsilon}, B_R\setminus\overline{B}_1)\,.\end{aligned} \tag{3.55}$$

Step 10: (limit of rescaled variable and lower bound at gradient scaling) We identify the limit in $L^2(B_R\setminus\overline{B}_1;\mathbb{R}^3)$ of the rescaled spin fields v_{δ_ε}. First of all, we observe that (3.15) and (3.55) imply that

$$\liminf_{\varepsilon\to 0}\frac{1}{\delta_\varepsilon^2}E_{\delta_\varepsilon}(v_{\delta_\varepsilon}, B_R\setminus\overline{B}_1) < +\infty\,.$$

Up to the extraction of a subsequence (that we do not relabel), the above liminf is a limit, and thus $E_{\delta_\varepsilon}(v_{\delta_\varepsilon}, B_R\setminus\overline{B}_1) \le C\delta_\varepsilon^2$. Let us write $v_{\delta_\varepsilon} = \big(\cos(\varphi(v_{\delta_\varepsilon}))\widetilde{v}_{\delta_\varepsilon}, \sin(\varphi(v_{\delta_\varepsilon}))\big)$. We apply Theorem 2.11-*i)* to get a (non-relabeled) subsequence and a map $v\in H^1(B_R\setminus\overline{B}_1;\mathscr{S}^2_N)$

such that $A[v_{\delta_\varepsilon}] \to v$ strongly in $L^2(B_R \setminus \overline{B}_1;\mathbb{R}^3)$ and weakly in $H^1_{\mathrm{loc}}(B_R \setminus \overline{B}_1;\mathbb{R}^3)$. By Theorem 2.11-*ii)* (see also Remark 2.12) we have that

$$\liminf_{\varepsilon\to 0} \frac{1}{\delta_\varepsilon^2} E_{\delta_\varepsilon}(v_{\delta_\varepsilon}, B_R \setminus \overline{B}_1) \geq \int_{B_R\setminus\overline{B}_1} |\nabla v|^2 \,\mathrm{d}x\,. \tag{3.56}$$

Let us prove that $\varphi(v_{\delta_\varepsilon}) \to \bar\varphi$ strongly in $L^1(B_R\setminus\overline{B}_1;\mathbb{R})$ and thus $\varphi(v) = \bar\varphi$. We provide the details here as the specific model studied in the paper plays a role. By interpreting v_{δ_ε} and u_ε as piecewise constant functions, we have that for every $y \in B_R \setminus \overline{B}_1$

$$\varphi(v_{\delta_\varepsilon})(y) = \varphi\big(v_{\delta_\varepsilon}(y)\big) = \varphi\big(v_{\delta_\varepsilon}(\delta_\varepsilon \tfrac{y}{\delta_\varepsilon})\big) = \varphi\big(u_\varepsilon(\varepsilon \tfrac{y}{\delta_\varepsilon} y - z_\varepsilon^{h,m})\big)\,.$$

Changing variables $x = \frac{\varepsilon}{\delta_\varepsilon} y - z_\varepsilon^{h,m}$ and recalling (3.54), for ε small enough we obtain that

$$\begin{aligned}\int_{B_R\setminus\overline{B}_1} |\varphi(v_{\delta_\varepsilon})(y) - \bar\varphi| \,\mathrm{d}y &= \int_{B_R\setminus\overline{B}_1} |\varphi\big(u_\varepsilon(\tfrac{\varepsilon}{\delta_\varepsilon} y - z_\varepsilon^{h,m})\big) - \bar\varphi| \,\mathrm{d}y \\ &\leq \Big(\frac{\delta_\varepsilon}{\varepsilon}\Big)^2 \int_{B_{\varepsilon^\alpha}(z_\varepsilon^{h,m})\setminus\overline{B}_{\varepsilon^\beta}(z_\varepsilon^{h,m})} |\varphi(u_\varepsilon(x)) - \bar\varphi| \,\mathrm{d}x \\ &\leq \varepsilon^{-2\beta} \int_{B_\eta} |\varphi(u_\varepsilon(x)) - \bar\varphi| \,\mathrm{d}x\,,\end{aligned} \tag{3.57}$$

where $B_\eta \subset\subset \Omega$ is such that $B_{\varepsilon^\alpha}(z_\varepsilon^{h,m}) \subset\subset B_\eta$ for ε small enough. By Lemma 3.1 and (3.16) we have that

$$\int_{B_\eta} |\varphi(u_\varepsilon(x)) - \bar\varphi| \,\mathrm{d}x \leq C\Big(\frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon,\Omega)\Big)^2 \leq C\varepsilon^2 |\log\varepsilon|^2.$$

We can conclude the estimate in (3.57):

$$\int_{B_R\setminus\overline{B}_1} |\varphi(v_{\delta_\varepsilon})(y) - \bar\varphi| \,\mathrm{d}y \leq C\varepsilon^{-2\beta}\varepsilon^2 |\log\varepsilon|^2 = C\varepsilon^{2(1-\beta)}|\log\varepsilon|^2 \to 0\,.$$

where we recall that $\beta < 1$.

Since $\varphi(u) = \bar\varphi$, the lower bound (3.56) reads (*cf.* also Theorem 2.11-*ii)*)

$$\liminf_{\varepsilon\to 0} \frac{1}{\delta_\varepsilon^2} E_{\delta_\varepsilon}(v_{\delta_\varepsilon}, B_R \setminus \overline{B}_1) \geq \cos^2(\bar\varphi) \int_{B_R\setminus\overline{B}_1} |\nabla \widetilde v|^2 \,\mathrm{d}x\,,$$

which together with (3.55) implies that

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, A_\varepsilon^{h,m}) \geq \frac{(\beta-\alpha)}{\log R} \cos^2(\bar\varphi) \int_{B_R\setminus\overline{B}_1} |\nabla \widetilde v|^2 \,\mathrm{d}x\,. \tag{3.58}$$

Step 11: (degree of limit in the gradient scaling) We are now in a position to conclude the proof of (3.53). We observe that thanks to the ball construction we have that $|\mu_{\widetilde u_\varepsilon}|(B_{\varepsilon^\alpha}(z_\varepsilon^{h,m}) \setminus \overline{B}_{\varepsilon^\beta}(z_\varepsilon^{h,m})) = 0$. The latter equality, together with (3.54), and (3.52), yield

$$\begin{aligned}|\mu_{\widetilde v_{\delta_\varepsilon}}|(B_R \setminus \overline{B}_1) &= |\mu_{\widetilde u_\varepsilon}|(B_{R\frac{\varepsilon}{\delta_\varepsilon}}(z_\varepsilon^{h,m}) \setminus \overline{B}_{\frac{\varepsilon}{\delta_\varepsilon}}(z_\varepsilon^{h,m})) = |\mu_{\widetilde u_\varepsilon}|(B_{\varepsilon^\alpha}(z_\varepsilon^{h,m}) \setminus \overline{B}_{\varepsilon^\beta}(z_\varepsilon^{h,m})) = 0\,, \\ \mu_{\widetilde v_{\delta_\varepsilon}}(B_1) &= \mu_{\widetilde u_\varepsilon}(B_{\frac{\varepsilon}{\delta_\varepsilon}}(z_\varepsilon^{h,m})) = \mu_{\widetilde u_\varepsilon}(B_{\varepsilon^\beta}(z_\varepsilon^{h,m})) = d_{h,m}\,.\end{aligned}$$

These conditions and the convergence $A[\widetilde v_{\delta_\varepsilon}] \rightharpoonup \widetilde v$ weakly in $H^1_{\mathrm{loc}}(B_R\setminus\overline{B}_1;\mathbb{R}^2)$ imply by standard arguments (see, *e.g.*, [11, proof of Proposition 4.3] for more details) that $\deg(\widetilde v, \partial B_r) = d_{h,m}$ for a.e. $r \in [1,R]$. Minimizing among all $\widetilde v \in H^1(\Omega;\mathbb{S}^1)$ with $\deg(\widetilde v, \partial B_r) = d_{h,m}$, we conclude that (see also [11, proof of Proposition 4.3])

$$\cos^2(\bar\varphi) \int_{B_R\setminus\overline{B}_1} |\nabla \widetilde v|^2 \,\mathrm{d}x \geq 2\pi\cos^2(\bar\varphi)|d_{h,m}|\log R\,.$$

Putting together the last inequality and (3.58), we get (3.53), since

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, A_\varepsilon^{h,m}) \geq \frac{(\beta-\alpha)}{\log R} 2\pi\cos^2(\bar\varphi)|d_{h,m}|\log R = (\beta-\alpha)2\pi\cos^2(\bar\varphi)|d_{h,m}|\,.$$

Step 12: (conclusion) Summing (3.53) over h and m and using (3.52), yields

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon,\Omega') &\geq \sum_{h=0}^{\widetilde{L}}\sum_{m=1}^{M_h}(\beta_h-\alpha_h)2\pi\cos^2(\bar\varphi)|d_{h,m}|\\ &\geq \sum_{h=0}^{\widetilde{L}}(\beta_h-\alpha_h)2\pi\cos^2(\bar\varphi)|d|\\ &= (p''-p'-2(\widetilde{L}+1)\lambda)2\pi\cos^2(\bar\varphi)|d|\\ &= (p''-p'-2(\widetilde{L}+1)\lambda)2\pi\cos^2(\bar\varphi)|\mu|(\Omega)\,.\end{aligned}$$

Letting $\lambda\to 0$, $p'\to 0$, $p''\to p$, and $p\to 1$ in the previous inequality, we conclude the proof.

Proof of *(iii)* (limsup inequality).
Combining [4, Theorem 2 & Remark 3] and [6, Proposition 5.2], the Lipschitz-regularity of Ω implies that there exists a sequence $\widetilde{u}_\varepsilon\colon\varepsilon\mathbb{Z}^2\to\mathbb{S}^1$ such that $\mu_{\widetilde{u}_\varepsilon}\overset{\mathrm{f}}{\to}\mu$ in Ω and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2|\log\varepsilon|}XY_\varepsilon(\widetilde{u}_\varepsilon,\Omega)=2\pi|\mu|(\Omega).$$

We define $u_\varepsilon\colon\varepsilon\mathbb{Z}^2\to\mathscr{S}^2_N$ by $u_\varepsilon(\varepsilon i)=(\cos(\bar\varphi)\widetilde{u}_\varepsilon(\varepsilon i),\sin(\bar\varphi))$. Then $\varphi(u_\varepsilon(\varepsilon i))=\bar\varphi$ for all $\varepsilon i\in\varepsilon\mathbb{Z}^2$ and therefore $\varphi(u_\varepsilon)\to\bar\varphi$ in $L^1(\Omega;\mathbb{R})$. In order to estimate the energy of u_ε, note that

$$|u_\varepsilon(\varepsilon i)-u_\varepsilon(\varepsilon j)|^2=\cos(\bar\varphi)^2|\widetilde{u}_\varepsilon(\varepsilon i)-\widetilde{u}_\varepsilon(\varepsilon j)|^2,$$

which implies that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon,\Omega)=\cos^2(\bar\varphi)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2|\log\varepsilon|}XY_\varepsilon(\widetilde{u}_\varepsilon,\Omega)=2\pi\cos^2(\bar\varphi)|\mu|(\Omega).$$

This proves the limsup inequality in Theorem 2.10 and therefore we conclude the proof. □

Appendix A. An anisotropic density result for partitions

Similar to this paper, in many discrete-to-continuum approximations of interfacial problems the anisotropy of the lattice $\mathbb{Z}^2$ (or $\mathbb{Z}^d$) plays a fundamental role. Here we provide an approximation result for partitions that fits well the structure of the lattice $\mathbb{Z}^d$.

Proposition A.1. *Let $T\subset\mathbb{R}^m$ be a finite set and $u\in BV(\Omega;T)$. Then there exists a sequence $u_n\in BV_{\mathrm{loc}}(\mathbb{R}^d;T)$ such that*

$$u_n\in\mathcal{PC}_{\delta_n}:=\Big\{u_n(x)=u_n(i)\text{ for all }x\in i+[-\delta_n 2/2,\delta_n/2)^d\,,\ i\in\delta_n\mathbb{Z}^d\Big\}\,.$$

such that $u_n\to u$ in $L^1(\Omega;\mathbb{R}^m)$ and $|\mathrm{D}u_n|_1(\overline{\Omega})\to|\mathrm{D}u|_1(\Omega)$.

Proof. Fix a cube of the form $Q=(-k,k)^d$ with $k\in\mathbb{N}$ such that $Q\supset\Omega$ and set $\delta_n=2k/(2n+1)$ for $n\in\mathbb{N}$. Define then the functional

$$F_n(u,Q)=\begin{cases}\int_{Q\cap S_u}|u^+-u^-||\nu_u|_1\,\mathrm{d}\mathcal{H}^{d-1} & \text{if } u\in BV(Q;T)\cap\mathcal{PC}_{\delta_n}\,,\\ +\infty & \text{otherwise on } L^1(Q;T)\,.\end{cases}$$

Note that for $u\in\mathcal{PC}_{\delta_n}$ we can rewrite the functional $F_n(u,Q)$ as

$$F_n(u,Q)=\frac{1}{2}\sum_{\substack{i,j\in\delta_n\mathbb{Z}^d\cap Q\\|i-j|=1}}\delta_n^{d-1}|u(i)-u(j)|\,,$$

where we used that $|e_i|_1=1$ for all canonical basis vectors $e_i\in\mathbb{R}^d$ and that (up to boundary facets) Q can be written as the union of cubes of the form $i+[-\delta_n/2,\delta_n/2)^d$. In

[16, Theorem 5.8] it is proven that the Γ-limit in $L^1(Q)$ of such discrete functionals admits an integral representation of the form

$$F_0(u,Q) = \begin{cases} \displaystyle\int_{Q\cap S_u} \varphi(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} & \text{if } u \in BV(Q;T)\,, \\ +\infty & \text{otherwise on } L^1(\Omega;T)\,. \end{cases}$$

In the present periodic setting with finite range of interactions, the integrand φ can be obtained through the following multi-cell formula:

$$\varphi(a,b,\nu) = \lim_{t\to+\infty} \frac{1}{t^{d-1}} \inf\{F_1(u,tQ_\nu) : u \in \mathcal{PC}_1,\ u(i) = u_\nu^{a,b}(i) \text{ if } \operatorname{dist}(i,\partial(tQ_\nu)) \le 2\}\,;$$

cf. [16, Remarks 4.2 i) and 5.9]. Here, for $a,b \in T$ and $\nu \in \mathbb{S}^{d-1}$, the function $u_\nu^{a,b}$ is defined by

$$u_\nu^{a,b} = \begin{cases} a & \text{if } \langle x,\nu\rangle \ge 0\,, \\ b & \text{if } \langle x,\nu\rangle < 0\,. \end{cases}$$

Since $F_n(u,Q) = |\mathrm{D}u|_1(Q)$ for all $u \in BV(Q;T)\cap\mathcal{PC}_{\delta_n}$, it follows by comparison and the $L^1(Q)$-lower semicontinuity of $u \mapsto |\mathrm{D}u|_1(Q)$ that

$$\varphi(a,b,\nu) \ge |a-b||\nu|_1 \quad \text{for all } a,b\in T,\ \nu\in\mathbb{S}^{d-1}.$$

In order to show equality, we just have to provide a candidate for the minimization problem defining $\varphi(a,b,\nu)$ that has asymptotically an energy less or equal than $|a-b||\nu|_1$ as $t\to+\infty$. To this end, we define $u_t \in \mathcal{PC}_1$ by its values on $\mathbb{Z}^d$ setting $u_t(i) = u_\nu^{a,b}(i)$. Then clearly $|u_t(i)-u_t(j)| \in \{0,|a-b|\}$ and therefore

$$F_1(u_t,tQ_\nu) = |a-b|\#\underbrace{\{(i,j)\in(\mathbb{Z}^d\cap tQ_\nu)^2 : |i-j|=1,\ \langle i,\nu\rangle\ge 0,\ \langle j,\nu\rangle<0\}}_{=:I_\nu^t}\,.$$

Arguing as in the proof of [19, Proposition 3.4] one can show that $\limsup_{t\to+\infty}\#I_\nu^t \le |\nu|_1$, which then implies that $\varphi(a,b,\nu) = |a-b||\nu|_1$.

Summarizing, we proved that the Γ-limit of $u\mapsto F_n(u,Q)$ is given on $BV(Q;T)$ by

$$F_0(u,Q) = \int_{Q\cap S_u} |u^+-u^-||\nu_u|_1\,\mathrm{d}\mathcal{H}^{d-1} = |\mathrm{D}u|_1(Q).$$

Now given $u \in BV(\Omega;T)$, we can use a local reflection argument (*cf.* the proof of [18, Lemma 3.4] for details) to extend u to a function $u \in BV(Q;T)$ such that $|\mathrm{D}u|(\partial\Omega) = 0$. Applying the Γ-convergence result to the extended function, we find a recovery sequence $u_n \in \mathcal{PC}_{\delta_n}$ such that $u_n \to u$ in $L^1(Q;\mathbb{R}^3)$ (and thus in $L^1(\Omega;\mathbb{R}^3)$) and moreover $|\mathrm{D}u_n|_1(Q) \to |\mathrm{D}u|(Q)$. Since $|\mathrm{D}u|(\partial\Omega)=0$ (which implies $|\mathrm{D}u|_1(\partial\Omega)=0$), this also implies that $|\mathrm{D}u_n|_1(\overline{\Omega}) \to |\mathrm{D}u|_1(\overline{\Omega}) = |\mathrm{D}u|_1(\Omega)$ as claimed. □

Appendix B. The ball construction

For the reader's convenience we include in this appendix the ball construction as presented in [11, Lemma 5.1] which is a variant of the construction introduced in [29, 24].

Let $\mathcal{A}(\mathbb{R}^2)$ be the collection of open subsets of $\mathbb{R}^2$. Let $\mathcal{B} = \{B_{r_k}(x_k)\}_{k=1}^M$ be a finite family of pairwise disjoint open balls. Let $\mu = \sum_{k=1}^M d_k\delta_{x_k}$, $d_k \in \mathbb{Z}\setminus\{0\}$, $x_k\in\mathbb{R}^2$, and let $\mathcal{E}(\mathcal{B},\mu,\cdot)\colon \mathcal{A}(\mathbb{R}^2)\to[0,+\infty]$ be an increasing set function satisfying the following properties:

(B1) $\mathcal{E}(\mathcal{B},\mu,U\cup V) \ge \mathcal{E}(\mathcal{B},\mu,U)+\mathcal{E}(\mathcal{B},\mu,V)$ for every $U,V\in\mathcal{A}(\mathbb{R}^2)$ such that $U\cap V = \emptyset$.

(B2) for every annulus $A_{r,R}(x) = B_R(x)\setminus\overline{B}_r(x)$, with $0<r<R$ and $A_{r,R}(x)\cap\bigcup_{k=1}^M B_{r_k}(x_k) = \emptyset$, it holds

$$\mathcal{E}(\mathcal{B},\mu,A_{r,R}(x)) \ge |\mu(B_r(x))|\log\frac{R}{r}\,. \tag{B.1}$$

Given a ball B, we let $r(B)$ denote its radius. For a family of balls $\mathcal{B}$, we let $\mathcal{R}(\mathcal{B}) := \sum_{B\in\mathcal{B}} r(B)$.

Lemma B.1 (Ball construction)**.** *Let $\mathcal{B}$, μ, and $\mathcal{E}$ be as above. Let $\sigma > 0$. Then there exists a one-parameter family $\{\mathcal{B}(t)\}_{t\geq 0}$ of balls such that*

(1) *the following inclusions hold:*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in\mathcal{B}(t_1)} B \subset \bigcup_{B\in\mathcal{B}(t_2)} B\,, \quad \text{for every } 0\leq t_1\leq t_2\,;$$

(2) $\overline{B}\cap\overline{B}' = \emptyset$ *for every* $B, B'\in\mathcal{B}(t)$, $B\neq B'$, *and* $t\geq 0$*;*

(3) *for every* $0\leq t_1\leq t_2$ *and every* $U\in\mathcal{A}(\mathbb{R}^2)$ *we have that*

$$\mathcal{E}\Big(\mathcal{B},\mu,U\cap\Big(\bigcup_{B\in\mathcal{B}(t_2)} B\setminus\bigcup_{B\in\mathcal{B}(t_1)}\overline{B}\Big)\Big)\geq\sum_{\substack{B\in\mathcal{B}(t_2)\\ B\subset U}}|\mu(B)|\log\frac{1+t_2}{1+t_1}\,;$$

(4) $|\mu|(B_{r+\sigma}(x)\setminus B_{r-\sigma}(x)) = 0$ *for every* $B = B_r(x)\in\mathcal{B}(t)$ *and for every* $t\geq 0$*;*

(5) *for every* $t\geq 0$ *we have that* $\mathcal{R}(\mathcal{B}(t))\leq(1+t)\,(\mathcal{R}(\mathcal{B})+N\sigma)$*;*

(6) *for every* $t\geq 0$, $B\in\mathcal{B}$, *and* $B'\in\mathcal{B}(t)$ *with* $B\subset B'$ *we have that* $r(B')\geq(1+t)r(B)$.

Proof. In order to construct the family $\mathcal{B}(t)$, we closely follow the strategy of the ball construction due to Sandier [29] and Jerrard [24]. We adapt the argument in order to be sure that condition (4) holds true, *i.e.*, that the measure μ is supported far from the boundaries of the balls of the constructed family.

The ball construction consists in letting the balls alternatively expand and merge into each other. We let $T_0 := 0$ and we define the family $\mathcal{B}(T_0)$ by distinguishing the following two cases: If $\overline{B}_{r_i+\sigma}(x_i)\cap\overline{B}_{r_j+\sigma}(x_j)\neq\emptyset$ for some of the starting balls with $i,j\in\{1,\dots,N\}$, $i\neq j$, then the construction starts with a merging phase and $T_0 = 0$ is the first merging time. This phase consists in identifying a suitable partition $\{S^0_j\}_{j=1,\dots,N_0}$ of the family $\{B_{r_i+\sigma}(x_i)\}_{i=1}^N$ which satisfies the following: for each $j\in\{1,\dots,N_0\}$ there exists a ball $B_{r^0_j}(x^0_j)$ which contains all the balls in S^0_j and such that

i) $\overline{B}_{r^0_j}(x^0_j)\cap\overline{B}_{r^0_\ell}(x^0_\ell) = \emptyset$ for every $j,\ell\in\{1,\dots,N_0\}$, $j\neq\ell$,

ii) $r^0_j\leq\sum_{B\in S^0_j} r(B)$.

We then define

$$\mathcal{B}(T_0) := \{B_{r^0_j}(x^0_j)\colon j = 1,\dots,N_0\}\,. \tag{B.2}$$

If, instead, $\overline{B}_{r_i+\sigma}(x_i)\cap\overline{B}_{r_j+\sigma}(x_j) = \emptyset$ for every $i,j\in\{1,\dots,N\}$, $i\neq j$, then we let $N_0 := N$, $B_{r^0_j}(x^0_j) := B_{r_j+\sigma}(x_j)$ for $j = 1,\dots,N$ in (B.2), and we start with an expansion phase. During this first expansion phase, we let the balls expand without changing their centres, in such a way that the new radius $r^0_j(t)$ of the ball centred in x^0_j satisfies

$$\frac{r^0_j(t)}{r^0_j} = \frac{1+t}{1+T_0} = 1+t\,,$$

for every $t\geq T_0 = 0$ and every $j\in\{1,\dots,N_0\}$. We continue the first expansion phase as a long as

$$\overline{B}_{r^0_j(t)}(x_j)\cap\overline{B}_{r^0_\ell(t)}(x_\ell) = \emptyset \ \text{ for every } j,\ell\in\{1,\dots,N_0\}\,,\ j\neq\ell\,, \tag{B.3}$$

and we let T_1 denote the smallest $t\geq T_0 = 0$ such that (B.3) is violated. (Note that $T_1 > 0$.) At time T_1, following the same procedure described above, a merging phase starting from the balls $\{B_{r^0_j(T_1)}(x^0_j)\}_{j=1}^{N_0}$ begins, and it defines a new family of balls $\{B_{r^1_j}(x^1_j)\}_{j=1}^{N_1}$.

We iterate this procedure by alternating merging and expansion phases to obtain the following: a discrete set of times $\{T_0,\dots,T_K\}$, $K\leq N$; for each $k\in\{1,\dots,K\}$, a partition $\{S^k_j\}_{j=1}^{N_k}$ of $\{B_{r^{k-1}_j(T_k)}(x^{k-1}_j)\}_{j=1}^{N_{k-1}}$; for each subclass S^k_j, a ball $B_{r^k_j}(x^k_j)$, which contains the balls in S^k_j and such that the following properties are satisfied:

i) $\overline{B}_{r^k_j}(x^k_j)\cap\overline{B}_{r^k_\ell}(x^k_\ell) = \emptyset$ for every $j,\ell\in\{1,\dots,N_k\}$, $j\neq\ell$,

ii) $r^k_j\leq\sum_{B\in S^k_j} r(B)$.

For $t \geq 0$, the family $\mathcal{B}(t)$ is given by $\{B_{r_j^k(t)}(x_j^k)\}_{j=1}^{N_k}$ for $t \in [T_k, T_{k+1})$ and $k = 0, \dots, K$, where we set $T_{K+1} := +\infty$ (in other words, it consists of a single expanding ball for $t \geq T_K$). For every $t \in [T_k, T_{k+1})$ and for $j = 1, \dots, N_k$, the radii satisfy

$$\frac{r_j^k(t)}{r_j^k} = \frac{1+t}{1+T_k}\,. \tag{B.4}$$

Note that

$$\mathcal{R}(\mathcal{B}(T_0)) = \sum_{j=1}^{N_0} r_j^0 \leq \mathcal{R}(\mathcal{B}) + N\sigma\,. \tag{B.5}$$

It remains to check that conditions (1)–(5) hold true. By construction, it is clear that (1) and (2) are satisfied.

Let us prove (3). We note that, by (1),

$$\sum_{\substack{B\in\mathcal{B}(\tau_1)\\ B\subset U}} |\mu(B)| \geq \sum_{\substack{B\in\mathcal{B}(\tau_2)\\ B\subset U}} |\mu(B)| \qquad \text{for every } 0 < \tau_1 < \tau_2\,. \tag{B.6}$$

Let $t_1 < \bar{t} < t_2$. In view of (B.6), since $\mathcal{E}$ is an increasing sub-additive set-function, if we show that (3) holds true for the pairs $(t_1, \bar{t})$ and $(\bar{t}, t_2)$, then (3) also follows for t_1 and t_2. Therefore we can assume, without loss of generality, that $T_k \notin (t_1, t_2)$ for every $k = 1, \dots, K$. Let $t_1 < \tau < t_2$ and let $B \in \mathcal{B}(\tau)$. Then, there exists a unique ball $B' \in \mathcal{B}(t_1)$ such that $B' \subset B$. By construction $\mu(B) = \mu(B')$ and, by (B.1), we have that

$$\mathcal{E}(\mathcal{B}, \mu, B\setminus B') \geq |\mu(B')| \log\frac{1+\tau}{1+t_1} = |\mu(B)| \log\frac{1+\tau}{1+t_1}\,.$$

Summing up over all $B \in \mathcal{B}(\tau)$ with $B \subset U$ and using (B.6) yields

$$\mathcal{E}\Big(\mathcal{B}, \mu, U \cap \Big(\bigcup_{B\in\mathcal{B}(t_2)} B \setminus \bigcup_{B\in\mathcal{B}(t_1)} B\Big)\Big) \geq \sum_{\substack{B\in\mathcal{B}(\tau)\\ B\subset U}} |\mu(B)| \log\frac{1+\tau}{1+t_1} \geq \sum_{\substack{B\in\mathcal{B}(t_2)\\ B\subset U}} |\mu(B)| \log\frac{1+\tau}{1+t_1}\,.$$

Property (3) follows by letting $\tau \to t_2$.

Let us prove (4). Let $t \geq 0$ and let $B = B_r(x) \in \mathcal{B}(t)$. Let us fix an initial ball $B_{r_i}(x_i)$. By construction, $B_{r_i+\sigma}(x_i)$ is contained in some ball $B_{r'}(y) \in \mathcal{B}(t)$, *i.e.* $B_{r_i}(x_i) \subset B_{r'-\sigma}(y)$. Then $B_{r_i}(x_i) \cap B_{r+\sigma}(x) \subset B_{r-\sigma}(x)$, since condition (2) implies that $\overline{B}_{r'-\sigma}(y) \cap \overline{B}_{r+\sigma}(x) = \emptyset$ whenever $y \neq x$. This yields

$$B_{r+\sigma}(x) \cap \bigcup_{i=1}^N B_{r_i}(x_i) \subset B_{r-\sigma}(x) \qquad \Longrightarrow \qquad B_{r+\sigma}(x) \setminus B_{r-\sigma}(x) \subset B_{r+\sigma}(x) \setminus \bigcup_{i=1}^N B_{r_i}(x_i)\,.$$

Therefore

$$|\mu|(B_{r+\sigma}(x) \setminus B_{r-\sigma}(x)) \leq |\mu|\Big(\mathbb{R}^2 \setminus \bigcup_{i=1}^N B_{r_i}(x_i)\Big) = 0\,,$$

where we used the fact that μ is supported on $\{x_1, \dots, x_N\}$. This proves (4).

To prove (5), we start by observing that, by (B.4),

$$\mathcal{R}(\mathcal{B}(t)) = \sum_{j=1}^{N_k} r_j^k(t) = \sum_{j=1}^{N_k} \frac{1+t}{1+T_k} r_j^k = \frac{1+t}{1+T_k}\mathcal{R}(\mathcal{B}(T_k))$$

for every $t \in [T_k, T_{k+1})$ and every $k \in \{0, \dots, K\}$. It thus suffices to show that $\mathcal{R}(\mathcal{B}(T_k)) \leq (1+T_k)(\mathcal{R}(\mathcal{B}) + N\sigma)$ for every $k \in \{0, \dots, K\}$. For $k = 0$ this is a consequence of (B.5). For $k \geq 1$, it follows inductively by applying (5) for $t \in [T_{k-1}, T_k)$ and observing that

$$\begin{aligned}\mathcal{R}(\mathcal{B}(T_k)) &= \sum_{j=1}^{N_k} r_j^k \leq \sum_{j=1}^{N_k} \sum_{B\in S_j^k} r(B) = \sum_{j=1}^{N_{k-1}} r_j^{k-1}(T_k)\\ &= \limsup_{t\nearrow T_k} \mathcal{R}(\mathcal{B}(t)) \leq (1+T_k)\left(\mathcal{R}(\mathcal{B}) + N\sigma\right)\,,\end{aligned}$$

which follows from *ii*).

Finally, property (6) holds true by construction. □

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