STABILITY RESULTS FOR THE ROBIN LAPLACIAN ON NONSMOOTH DOMAINS

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Abstract. We formulate a generalization of the Laplace equation under Robin boundary conditions on a large class of possibly nonsmooth domains by dealing with the trace term appearing in the variational formulation from the point of view of the theory of functions of bounded variation. Admissible domains may have inner boundaries, i.e., inner cracks. In dimension two, we formulate a stability result for the elliptic problems under domain variation: with this aim, we introduce a notion of perimeter (Robin perimeter) which is tailored to count the inner boundaries with the appropriate natural multiplicity.

Keywords: Robin Laplacian, rectifiable sets, functions of bounded variations, lower semicontinuity, Hausdorff convergence.

2000 Mathematics Subject Classification: 26A45, 28A75, 35J20.

Contents

1. Introduction

The class of open sets in the plane

$$\mathcal{O}_m(\mathbb{R}^2) := \{\Omega \subseteq \mathbb{R}^2 \,:\, \Omega \text{ is open and bounded and } \mathbb{R}^2 \setminus \Omega \text{ has at most } m \text{ connected components}\} \tag{1.1}$$

(where $m \geq 1$) proved to be an ideal framework to study the stability of the Laplace equations with Dirichlet or Neumann boundary conditions under variation of the domain. The case of Dirichlet conditions was considered in the pioneering paper [22], while the case of Neumann boundary conditions has been addressed in [10, 11, 13, 15]. The kind of geometric perturbation considered there is given by the Hausdorff complementary topology $\mathcal{H}^c$ on the class of open sets (see Section 2) which allows to deal with nonsmooth domains and to consider singular perturbations of the domains involved.

The aim of this paper is to show that the class (1.1) is also a natural framework to study the stability of the Laplace operator under Robin boundary conditions. However, in order to achieve this objective, a natural question is to understand what is the meaning of a Robin problem in an arbitrary open set which does not enjoy any smoothness.

In general, for $\Omega \subset \mathbb{R}^d$ the Robin boundary value problem reads

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where f is a locally square integrable function, $\beta > 0$, and ν denotes the exterior normal. If Ω is sufficiently smooth, the solution u belongs to the Sobolev space $H^1(\Omega)$ and satisfies the weak formulation

$$\forall \varphi \in H^1(\Omega) \, : \, \int_\Omega \nabla u \cdot \nabla \varphi \, dx + \beta \int_{\partial\Omega} u\varphi \, d\mathcal{H}^{d-1} = \int_\Omega f\varphi \, dx, \tag{1.3}$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure, and the integral on the boundary involves the traces of the functions which are well defined elements in $L^2(\partial\Omega)$. From a variational point of view, the solution is the unique minimizer of the functional on $H^1(\Omega)$ given by

$$F(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{\beta}{2} \int_{\partial\Omega} u^2 \, d\mathcal{H}^{d-1} - \int_\Omega fu \, dx. \tag{1.4}$$

In order to formulate the problem on the possibly non smooth domains we preliminary need to specify the exact meaning of the boundary terms in the energy (1.4).

The issue of defining the Robin-Laplace boundary value problem in a non smooth setting has been addressed in [16], where the functional framework to settle problem (1.2) is identified with the abstract completion $V(\Omega)$ of the space

$$V_0(\Omega) := H^1(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega)$$

under the norm

$$\|u\|_{V_0} := \|u\|_{H^1(\Omega)} + \|u_{|\partial\Omega}\|_{L^2(\partial\Omega;\mathcal{H}^{d-1})},$$

for which a continuous embedding

$$j_0 : V_0(\Omega) \to L^{\frac{2d}{d-1}}(\Omega)$$

is available.

The space $V(\Omega)$ provides a Hilbert-space functional setting for a generalization of the boundary value problem (1.2): the bilinear form appearing on the left hand side of (1.3) can be lifted by completion to a continuous bilinear form $a : V(\Omega) \times V(\Omega) \to \mathbb{R}$, and also the embedding j_0 can be lifted to an embedding $j : V(\Omega) \to L^{\frac{2d}{d-1}}(\Omega)$, so that the natural generalization of (1.3) proposed in [16] takes the form

$$\forall \varphi \in V(\Omega) \, : \, a(u, \varphi) = \int_\Omega f j(\varphi) \, dx.$$

The space $V(\Omega)$ can be clearly identified with a subspace of pairs $(u, w) \in H^1(\Omega) \times L^2(\partial\Omega; \mathcal{H}^{d-1})$, the function w being in some sense a sort of weak trace of u, recovered in the completion procedure through the boundary values of the continuous up to the boundary functions approximating u. As shown in [3], it may happen that for Ω too irregular, elements of the form $(0, v)$ belong to $V(\Omega)$, so that a function could have several weak traces on $\partial\Omega$. This is due to some measure theoretic issues concerning $\partial\Omega$, i.e., the presence of a large number of points with density zero with respect to Ω (see also the considerations at the end Section 3 of [5]): this suggests that a control on the full topological boundary is somehow excessive.

A further drawback is that the space $V(\Omega)$ does not "see" inner cracks: more precisely, if a regular domain R contains a sufficiently smooth crack Γ, it turns out that $V(R \setminus \Gamma) = H^1(R)$, and this fact produces unnatural instability results. For instance, consider the following sequence of open sets

$$\Omega_n := [(0,1) \times (0,1)] \cup \left[\left(1 + \frac{1}{n}, 2 + \frac{1}{n} \right) \times (0,1) \right] \longrightarrow [(0,1) \times (0,1)] \cup [(1,2) \times (0,1)] =: \Omega.$$

The limit set Ω coincides with the rectangle $R := (0,2) \times (0,1)$ from which we remove the inner crack $\Gamma := \{1\} \times (0,1)$, so that the Robin problem for Ω reduces to the classical one on R: we immediately see that the solutions on Ω_n with a right hand side equal to 1 do not converge. Simple counterexamples can be constructed involving connected approximating domains.

In Section 3 we provide a new formulation of the Robin-Laplace problem (1.2) on non-smooth domains in which we deal with the boundary terms relying on the properties of the space of *function of bounded variation* $BV(\mathbb{R}^d)$ (see Section 2): this is suggested by the simple observation that for Ω sufficiently regular and $u \in H^1(\Omega)$, the extension $u1_\Omega$ by zero outside the domain is such that

$$u1_\Omega \in BV(\mathbb{R}^d)$$

and the boundary terms in (1.3) and (1.4) are connected with the traces of the BV function $u1_\Omega$ on the set $\partial\Omega$, one of them being zero. Equivalently, the boundary terms are described by the *jump part* of the derivative of $u1_\Omega$. This point of view revealed to be very fruitful, providing a new way to approach issues of shape optimization under Robin boundary conditions based on the theory of *free discontinuity problems* (see e.g. [5, 7, 8]).

More precisely, we consider open bounded sets $\Omega \subseteq \mathbb{R}^d$ with[1]

$$\mathcal{H}^{d-1}(\cup_i \partial\Omega^i) < +\infty, \tag{1.5}$$

where $\{\Omega^i\}_{i\in\mathbb{N}}$ denotes the family of connected components of Ω, and choose a $\mathcal{H}^{d-1}$-countably rectifiable subset $\Gamma_\Omega \subseteq \cup_{i\in\mathbb{N}}\partial\Omega^i$ with maximal $\mathcal{H}^{d-1}$-measure (see Theorem 2.1).

We define the Robin space of Ω as

$$\mathcal{R}(\Omega) := \left\{u \in H^1(\Omega) \,:\, u1_\Omega \in BV(\mathbb{R}^d), \gamma_r(u1_\Omega), \gamma_l(u1_\Omega) \in L^2(\Gamma_\Omega)\right\},$$

where $\gamma_r(u1_\Omega)$ and $\gamma_l(u1_\Omega)$ are the $\mathcal{H}^{d-1}$-a.e. well defined traces of the BV function $u1_\Omega$ on the suitably oriented rectifiable set Γ_Ω.

The weak reformulation of the Robin-Laplace problem we consider is then

$$\forall\varphi \in \mathcal{R}(\Omega) \,:\, \int_\Omega \nabla u \cdot \nabla\varphi\, dx + \beta \int_{\Gamma_\Omega} \left[\gamma_r(u1_\Omega)\gamma_r(\varphi 1_\Omega) + \gamma_l(u1_\Omega)\gamma_l(\varphi 1_\Omega)\right] d\mathcal{H}^{d-1} = \int_\Omega f\varphi\, dx$$

with associated functional

$$F(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{\beta}{2}\int_{\Gamma_\Omega} \left[|\gamma_r(u1_\Omega)|^2 + |\gamma_l(u1_\Omega)|^2\right] d\mathcal{H}^{d-1} - \int_\Omega fu\, dx.$$

It turns out that $\mathcal{R}(\Omega)$ has a natural Hilbert space structure (see Proposition 3.8), and that the Robin problem is well posed with a resolvent operator which is compact from $L^2(\Omega)$ to $\mathcal{R}(\Omega)$ (see Theorem 3.10), sharing thus the same features of the extension of [16]: clearly it reduces to the classical problem when Ω is regular.

In our approach the set Γ_Ω replaces the full topological boundary, and the boundary values are recovered by considering the traces of the global BV function $u1_\Omega$, whose jump set is contained in Γ_Ω (see Lemma 3.5). In this generality, "inner" boundaries (like inner cracks) are taken into account with two possible trace values: this is particularly suited when dealing with sequences of converging domains, as suggested by the case of a domain with a cavity shrinking to a compactly contained hypersurface. Finally, the choice of Γ_Ω or of its orientation (to distinguish between γ_r and γ_l) are only instrumental, as different sets or orientations lead to the same problem (Remarks 3.2 and 3.11).

In order to address stability issues under domain perturbation, we shall restrict our attention to the two dimensional setting by considering the family of sets from (1.1) given by

$$\mathcal{A}_m(\mathbb{R}^2) := \{\Omega \in \mathcal{O}_m(\mathbb{R}^2) \,:\, \mathcal{H}^1(\cup_i \partial\Omega^i) < +\infty\}, \tag{1.6}$$

and show (Proposition 4.3) that this class fits the framework of our generalization of the Robin-Laplace problems. In this case, we have the precise description of Γ_Ω as

$$\Gamma_\Omega = \cup_i \partial\Omega^i.$$

[1]Of course, if $\mathcal{H}^{d-1}(\partial\Omega) < +\infty$ then (1.5) is satisfied, while the reciprocal may be false. However, in practice, hypothesis (1.5) might be relaxed into $\forall i, \mathcal{H}^{d-1}(\partial\Omega_i) < +\infty$.

In order to have stability of the problems along a Hausdorff converging sequence of domains, we in general need some conditions on the behaviour of the boundaries, which in a classical setting amounts to the convergence of the perimeters as shown in [9] (otherwise simple counterexamples can be constructed). As in our approach some parts of the boundaries "count twice", we need an adapted version of the perimeter (Definition 5.1) which we call *Robin perimeter* and which is given by

$$Per_{\mathcal{R}}(\Omega) := \int_{\Gamma_\Omega} [\gamma_r(1_\Omega) + \gamma_l(1_\Omega)]\, d\mathcal{H}^1.$$

Again this geometric quantity is recovered in the same spirit as the boundary terms appearing in the Robin problems, that is by using the global BV function 1_Ω: for regular domains, it reduces to the classical perimeter $\mathcal{H}^1(\partial\Omega)$. The definition can be extended to general open sets in $\mathbb{R}^d$ under a suitable control on the topological boundary, but the properties which hold in the class $\mathcal{A}_m(\mathbb{R}^2)$ fail in general (see Remark 5.6). Related notions of perimeter have been considered by Cerf in [12] in the study of the lower semicontinuous envelope of the Hausdorff measure for the approximation by smooth sets, and by Henrot and Zucco in [21] in relationship with the Minkowski content.

Our main stability result (Theorem 6.1) shows that for a sequence of domains $(\Omega_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_m(\mathbb{R}^2)$ contained in a fix open and bounded set D such that

$$\Omega_n \stackrel{\mathcal{H}^c}{\to} \Omega \qquad \text{and} \qquad Per_{\mathcal{R}}(\Omega_n) \to Per_{\mathcal{R}}(\Omega),$$

then

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(D)$$

and

$$\nabla u_n 1_{\Omega_n} \to \nabla u 1_\Omega \qquad \text{strongly in } L^2(D;\mathbb{R}^2),$$

where u_n and u are the solutions of the Robin problems associated to $f \in L^2(D)$ on Ω_n and Ω. The stability is a consequence of some compactness properties of the admissible domains (Theorem 4.8) and of the associated Robin spaces (Proposition 4.10): the two-dimensional setting is fundamental as the boundaries of the connected components of the domains are union of connected curves, whose lengths are lower semicontinuous under Hausdorff convergence thanks to Gołąb Theorem (see Theorem 2.4).

The paper is organized as follows. In Section 2 we fix the notation and recall some basic notions employed throughout the paper. In Section 3 we formulate our weak version of the Robin-Laplace problem. In Section 4 we collect some basic properties of the class of admissible two dimensional domains (1.6) which are fundamental to study the stability of the problems. The Robin perimeter is introduced in Section 5, and its relation with the classical perimeter (or better to $\mathcal{H}^1(\Gamma_\Omega)$) is studied in detail (Proposition 5.4 and Proposition 5.8): the stability of the Robin perimeter along a converging sequence entails continuity information concerning the associated Robin spaces (Proposition 5.7). The main stability results (Theorem 6.1 and Theorem 6.2) are finally contained in Section 6.

2. Notation and preliminary results

In this section we introduce the basic notation and recall some notions employed in the rest of the paper.

Basic notation. If $E \subseteq \mathbb{R}^d$, we will denote with $|E|$ its d-dimensional Lebesgue measure, and by $\mathcal{H}^{d-1}(E)$ its $(d-1)$-dimensional Hausdorff measure: we refer to [19, Chapter 2] for a precise definition, recalling that for sufficiently regular sets $\mathcal{H}^{d-1}$ coincides with the usual area measure. Moreover, we denote by E^c the complementary set of E, and by 1_E its characteristic function, i.e., $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ otherwise. If u is a function defined on E, we will denote with $u1_E$ the extension of u to $\mathbb{R}^d$ which is equal to zero outside E.

If $A \subseteq \mathbb{R}^d$ is open and $1 \le p \le +\infty$, we denote by $L^p(A)$ the usual space of p-summable functions on A with norm indicated by $\|\cdot\|_p$. $H^1(A)$ will stand for the Sobolev space of functions in $L^2(A)$ whose gradient in the sense of distributions belongs to $L^2(A,\mathbb{R}^d)$. Finally $\mathcal{M}_b(A;\mathbb{R}^d)$

will denote the space of $\mathbb{R}^d$-valued Radon measures on A, which can be identified with the dual of $\mathbb{R}^d$-valued continuous functions on A vanishing at the boundary.

Countably rectifiable sets. We say that $E \subseteq \mathbb{R}^d$ is $\mathcal{H}^{d-1}$-countably rectifiable if

$$E = N \cup \bigcup_{i \in \mathbb{N}} E_i$$

where $\mathcal{H}^{d-1}(N) = 0$ and $E_i \subseteq \mathcal{M}_i$, where $\mathcal{M}_i$ is a C^1-hypersurface of $\mathbb{R}^d$. It is not restrictive to assume that the sets E_i are mutually disjoint.

We will make use of the following result (see e.g. [17, Theorem 5.7]).

Theorem 2.1. *Let $E \subseteq \mathbb{R}^d$ be such that $\mathcal{H}^{d-1}(E) < +\infty$. Then there exists a $\mathcal{H}^{d-1}$-countably rectifiable set $\Gamma \subseteq E$ such that*

$$\mathcal{H}^{d-1}(\Gamma) = \max\{\mathcal{H}^{d-1}(R) \,:\, R \subseteq E \text{ is } \mathcal{H}^{d-1}\text{-countably rectifiable}\}.$$

Proof. Let $m := \sup\{\mathcal{H}^{d-1}(R) \,:\, R \subseteq E \text{ is } \mathcal{H}^{d-1}\text{-countably rectifiable}\}$. By assumption $m \leq \mathcal{H}^{d-1}(E)$. Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{H}^{d-1}$-countably rectifiable subsets of E with

$$\mathcal{H}^{d-1}(R_n) \to m.$$

Then setting $\Gamma := \cup_n R_n$, we have easily that $\Gamma \subseteq E$ is $\mathcal{H}^{d-1}$-countably rectifiable and $\mathcal{H}^{d-1}(\Gamma) = m$. □

The following rectifiablity property will be important for our analysis (see [20]).

Theorem 2.2. *Let $K \subseteq \mathbb{R}^d$ be compact, connected and with $\mathcal{H}^1(K) < +\infty$. Then K is $\mathcal{H}^1$-countably rectifiable.*

Hausdorff convergence. The family $\mathcal{K}(\mathbb{R}^d)$ of closed sets in $\mathbb{R}^d$ can be endowed with the Hausdorff metric d_H defined by

$$d_H(K_1, K_2) := \max\left\{ \sup_{x\in K_1} \operatorname{dist}(x, K_2), \sup_{y \in K_2} \operatorname{dist}(y, K_1) \right\}$$

with the conventions $\operatorname{dist}(x, \emptyset) = +\infty$ and $\sup \emptyset = 0$, so that $d_H(\emptyset, K) = 0$ if $K = \emptyset$ and $d_H(\emptyset, K) = +\infty$ if $K \neq \emptyset$.

The Hausdorff metric has good compactness properties (see [2, Theorem 4.4.15]).

Proposition 2.3 (Compactness). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets contained in a fixed compact set of $\mathbb{R}^d$. Then there exists a compact set $K \subseteq \mathbb{R}^d$ such that up to a subsequence*

$$K_n \to K \qquad \text{in the Hausdorff metric.}$$

Let us set for $m \geq 1$

$$\begin{aligned} \mathcal{K}_m(\mathbb{R}^d) := \{K \subset \mathbb{R}^d \,:\, K &\text{ is compact,} \\ &\text{with at most } m \text{ connected components and } \mathcal{H}^1(K) < +\infty\}. \end{aligned} \tag{2.1}$$

For our analysis we will need the following property due to Gołąb: for the proof we refer the reader to [20, Theorem 3.18] or [2, Theorem 4.4.17].

Theorem 2.4 (Gołąb). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}_m(\mathbb{R}^d)$ with*

$$K_n \to K \qquad \text{in the Hausdorff metric.}$$

Then $K \in \mathcal{K}_m(\mathbb{R}^d)$ and

$$\mathcal{H}^1(K) \leq \liminf_n \mathcal{H}^1(K_n).$$

In order to study the behaviour of Robin problems under general domain variations, we will use the *Hausdorff complementary topology* on the family of open sets of $\mathbb{R}^d$ which is defined as follows. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets in $\mathbb{R}^d$. We say that Ω_n converges to the open set $\Omega \subseteq \mathbb{R}^d$ in the Hausdorff complementary topology and write

$$\Omega_n \xrightarrow{\mathcal{H}^c} \Omega$$

if for every closed ball $B \subseteq \mathbb{R}^d$ we have

$$B \cap \Omega_n^c \to B \cap \Omega^c \quad \text{in the Hausdorff metric on } \mathcal{K}(\mathbb{R}^d).$$

Functions of bounded variation and sets of finite perimeter. We say that $u \in BV(\mathbb{R}^d)$ if $u \in L^1(\mathbb{R}^d)$ and its derivative in the sense of distributions is a finite Radon measure on $\mathbb{R}^d$, i.e., $Du \in \mathcal{M}_b(\mathbb{R}^d;\mathbb{R}^d)$. $BV(\mathbb{R}^d)$ is called the space of *functions of bounded variation* on $\mathbb{R}^d$. $BV(\mathbb{R}^d)$ is a Banach space under the norm $\|u\|_{BV(\mathbb{R}^d)} := \|u\|_{L^1(\mathbb{R}^d)} + \|Du\|_{\mathcal{M}_b(\mathbb{R}^d;\mathbb{R}^d)}$. We call $|Du|(\mathbb{R}^d) := \|Du\|_{\mathcal{M}_b(\mathbb{R}^d;\mathbb{R}^d)}$ the *total variation* of u. We refer the reader to [1] for an exhaustive treatment of the space BV.

If $u \in BV(\mathbb{R}^d)$, then the measure Du can be decomposed canonically (and uniquely) as

$$Du = D^a u + D^j u + D^c u.$$

The measure $D^a u$ is the absolutely continuous part (with respect to the Lebesgue measure) of the derivative: the associated density is denoted by $\nabla u \in L^1(\mathbb{R}^d;\mathbb{R}^d)$. The measure $D^j u$ is the *jump part* of the derivative and it turns out that

$$D^j u = (u^+ - u^-) \otimes \nu_u \, \mathcal{H}^{d-1} \lfloor J_u.$$

Here J_u is the *jump set* of u, ν_u is the normal to J_u, while $u^\pm$ are the upper and lower approximate limits at x. It turns out that J_u is a $\mathcal{H}^{d-1}$-countably rectifiable set: if we choose the orientation given by a normal vector field ν_u we have $\mathcal{H}^{d-1}$-a.e.

$$u^+ = \gamma_r(u) \qquad \text{and} \qquad u^- = \gamma_l(u)$$

where $\gamma_r(u)$ and $\gamma_l(u)$ are the right and left traces of u on the rectifiable set J_u. Finally $D^c u$ is called the *Cantor part* of the derivative, and it vanishes on sets which are σ-finite with respect to $\mathcal{H}^{d-1}$. Clearly $D^j u + D^c u$ is the singular part $D^s u$ of Du with respect to $\mathcal{L}^d$.

We will use the following result.

Theorem 2.5. *The following items hold true.*

(a) Compact embedding. *The space $BV(\mathbb{R}^d)$ is embedded in $L^p(\mathbb{R}^d)$ for every $1 \le p \le \frac{d}{d-1}$, the immersion being locally compact if $p < \frac{d}{d-1}$. More precisely for every $u \in BV(\mathbb{R}^d)$*

$$\|u\|_{L^{d/d-1}(\mathbb{R}^d)} \le C_d |Du|(\mathbb{R}^d) \tag{2.2}$$

for some $C_d > 0$.

(b) Lower semicontinuity of the total variation. *If $(u_n)_{n\in\mathbb{N}}$ is bounded in $BV(\mathbb{R}^d)$ and $u_n \to u$ strongly in $L^1_{loc}(\mathbb{R}^d)$, then*

$$|Du|(\mathbb{R}^d) \le \liminf_n |Du_n|(\mathbb{R}^d).$$

We will make use also of the sets of finite perimeter. If $|E| < +\infty$, then E has finite perimeter in $\mathbb{R}^d$ if and only if $1_E \in BV(\mathbb{R}^d)$. The perimeter of E is defined as

$$Per(E) = |D1_E|(\mathbb{R}^d).$$

Inequality (2.2) reduces to the isoperimetric inequality (for $|E| < +\infty$)

$$|E|^{\frac{d-1}{d}} \le C_d Per(E),$$

and the compact immersion says that sequences of bounded sets with bounded perimeters is relatively compact in $L^1(\mathbb{R}^d)$. It turns out that

$$D1_E = \nu_E \mathcal{H}^{d-1} \lfloor \partial^* E, \qquad Per(E) = \mathcal{H}^{d-1}(\partial^* E),$$

where $\partial^* E$ is called the *reduced boundary* of E, and ν_E is the associated inner approximate normal (see [1, Section 3.5]). We have that $\partial^* E \subseteq \partial E$, but the topological boundary can in in general be much larger than the reduced one. If $\Omega \subseteq \mathbb{R}^d$ is open and bounded with $\mathcal{H}^{d-1}(\partial\Omega) < +\infty$, then Ω has finite perimeter with $Per(\Omega) \leq \mathcal{H}^{d-1}(\partial\Omega)$.

Γ-convergence. Let us recall the definition of De Giorgi's Γ*-convergence* in metric spaces: we refer the reader to [14] for an exhaustive treatment of this subject. Let (X, d) be a metric space. We say that $F_n : X \to \overline{\mathbb{R}}$ Γ-converges to $F : X \to \overline{\mathbb{R}}$ (as $n \to +\infty$) if for all $u \in X$ the following items hold true.

(i) (Γ*-liminf inequality*) For every sequence $(u_n)_{n\in\mathbb{N}}$ converging to u in X,

$$\liminf_n F_n(u_n) \geq F(u).$$

(ii) (Γ*-limsup inequality*) There exists a sequence $(u_n)_{n\in\mathbb{N}}$ converging to u in X, such that

$$\limsup_n F_n(u_n) \leq F(u).$$

The function F is called the Γ-limit of the sequence $(F_n)_{n\in\mathbb{N}}$ (with respect to d), and we write $F = \Gamma\!-\lim_n F_n$. The following result holds true (see [14, Corollary 7.20 and Theorem 8.5])

Theorem 2.6. *Let X be a separable and metric space, and let $F_n : X \to \overline{\mathbb{R}}$.*

(a) *There exist $F : X \to \overline{\mathbb{R}}$ and a subsequence $(F_{n_k})_{k\in\mathbb{N}}$ such that*

$$F_{n_k} \xrightarrow{\Gamma} F.$$

(b) *If $x_n \in X$ is a minimizer of F_n such that $x_n \to x \in X$, then x is a minimizer of F.*

3. The Robin-Laplace operator in a non smooth setting

In this section we provide a general framework to deal with the Robin boundary value problem for the Laplace operator on bounded connected open sets whose topological boundary has finite area: the boundary can be thus irregular, and in particular may present inner cracks. More generally, we formulate the problem for open sets whose connected components have topological boundaries suitably controlled in $\mathcal{H}^{d-1}$-measure (see (3.1) below): this is because we are interested in stability issues, where connectedness can be lost very easily along a converging sequence of domains.

3.1. **Functional setting.** Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, and let $\{\Omega^j\}_{j\in\mathbb{N}}$ denote the family of its connected components. We assume

$$\mathcal{H}^{d-1}\left(\bigcup_{j\in\mathbb{N}} \partial\Omega^j\right) < +\infty. \tag{3.1}$$

According to Theorem 2.1, let us choose

$$\Gamma_\Omega \subseteq \bigcup_{j\in\mathbb{N}} \partial\Omega^j,\ \mathcal{H}^{d-1}\text{-countably rectifiable subset with maximal } \mathcal{H}^{d-1}\text{-measure.} \tag{3.2}$$

Note that the set Γ_Ω is non trivial: indeed, since every connected component Ω^j has finite perimeter, by maximality we have

$$\mathcal{H}^{d-1}\left((\cup_j \partial^*\Omega^j) \setminus \Gamma_\Omega\right) = 0,$$

so that Γ_Ω contains at least (up to $\mathcal{H}^{d-1}$-negligible sets) the union of the reduced boundaries of each connected component. On the other hand, it can be that

$$\mathcal{H}^{d-1}(\Gamma_\Omega \setminus \cup_j \partial^*\Omega^j) > 0,$$

and this is the case in which for example the connected components present inner cracks (which are not part of the reduced boundary, as their points have density one).

We may write

$$\Gamma_\Omega = N \cup \bigcup_{i=0}^{\infty} \Gamma_i, \tag{3.3}$$

where $\mathcal{H}^{d-1}(N) = 0$, while for every $i \in \mathbb{N}$ the Borel sets Γ_i are subsets of a $\mathcal{C}^1$ -manifold $\mathcal{M}_i$, and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$.

It is not restrictive, up to reducing $\mathcal{M}_i$, to assume that $\mathcal{M}_i$ is orientable with associated normal vector filed ν_i, and that two continuous trace operators from $BV(\mathbb{R}^d)$ to $L^1(\mathcal{M}_i)$, the "left" and "right" traces, are defined.

For every $v \in BV(\mathbb{R}^d)$, let us denote by $\gamma_r^i(v), \gamma_l^i(v)$ the "right" and "left" traces of v on Γ_i, using the orientation associated to ν_i. By general theory of BV functions it is known that

$$Dv\lfloor\Gamma_\Omega = \sum_i [\gamma_r^i(v) - \gamma_l^i(v)]\nu_i \mathcal{H}^{d-1}\lfloor\Gamma_i. \tag{3.4}$$

We define global "right" and "left" traces on the full Γ_Ω by setting

$$\gamma_r(v) := \sum_i \gamma_r^i(v) \qquad \text{and} \qquad \gamma_l(v) := \sum_i \gamma_l^i(v). \tag{3.5}$$

Definition 3.1 (**Robin function space**)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set satisfying* (3.1) *and* (3.2)*. We set*

$$\mathcal{R}(\Omega) := \left\{ u \in H^1(\Omega) \,:\, u1_\Omega \in BV(\mathbb{R}^d), \gamma_r(u1_\Omega), \gamma_l(u1_\Omega) \in L^2(\Gamma_\Omega) \right\}.$$

We call $\mathcal{R}(\Omega)$ the Robin space associated to Ω.

Remark 3.2. Notice that if we choose another maximal rectifiable subset $\tilde{\Gamma}_\Omega$ of $\bigcup_{j\in\mathbb{N}} \partial\Omega^j$, then we immediately have

$$\mathcal{H}^{d-1}(\Gamma_\Omega \Delta \tilde{\Gamma}_\Omega) = 0$$

where $A\Delta B$ denotes the symmetric difference. We conclude that the Robin space individuated by $\tilde{\Gamma}_\Omega$ is the same as that associated to Γ_Ω.

Remark 3.3. Notice that if Ω has a Lipschitz boundary, being a bounded set, then it has a finite number of connected components, the boundary is $\mathcal{H}^{d-1}$-countably rectifiable and we may choose $\Gamma_\Omega = \partial\Omega$. Thus $\mathcal{R}(\Omega)$ reduces to $H^1(\Omega)$, being Ω an extension domain: choosing the orientation of $\partial\Omega$ given by the exterior normal, we get $\gamma_r(u1_\Omega) = 0$ while $\gamma_l(u1_\Omega)$ reduces to the classical trace of Sobolev functions.

Remark 3.4. The distinction between right and left traces depends clearly on the orientation given to the subsets involved in the decomposition of Γ_Ω. They will play a symmetrical role in the Robin boundary value problem formulated in Section 3.2, showing that the decomposition is only instrumental for the formulation of the problem.

In the following lemma we collect some basic properties of the extension by zero outside the domain of functions in the Robin spaces.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set satisfying* (3.1) *and* (3.2)*, and let $u \in \mathcal{R}(\Omega)$. Then the following items hold true.*

(a) *Concerning the function $u1_\Omega \in BV(\mathbb{R}^d)$ we have*

$$D^a(u1_\Omega) = \nabla u 1_\Omega,$$

$D^s(u1_\Omega)$ is supported on Γ_Ω with

$$|D^s(u1_\Omega)| \leq [|\gamma_r(u1_\Omega)| + |\gamma_l(u1_\Omega)|]\ \mathcal{H}^{d-1}\lfloor\Gamma_\Omega,$$

and

$$\|u1_\Omega\|_{BV(\mathbb{R}^d)}| \leq |\Omega|^{1/2}\|u\|_{H^1(\Omega)} + \mathcal{H}^{d-1}(\Gamma_\Omega)^{1/2}[\|\gamma_r(u1_\Omega)\|_{L^2(\Gamma_\Omega)} + \|\gamma_l(u1_\Omega)\|_{L^2(\Gamma_\Omega)}].$$

(b) *We have* $u^2 1_\Omega \in BV(\mathbb{R}^d)$ *with*

$$\|u^2 1_\Omega\|_{BV(\mathbb{R}^d)} \le \|u\|^2_{H^1(\Omega)} + \int_{\Gamma_\Omega} \left[|\gamma_r(u1_\Omega)|^2 + |\gamma_l(u1_\Omega)|^2\right] d\mathcal{H}^{d-1}, \tag{3.6}$$

and there exists $C = C(d, |\Omega|)$ *such that*

$$\|u\|^2_{L^2(\Omega)} \le C\left[\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^d))} + \int_{\Gamma_\Omega} \left[|\gamma_r(u1_\Omega)|^2 + |\gamma_l(u1_\Omega)|^2\right] d\mathcal{H}^{d-1}\right]. \tag{3.7}$$

Proof. Let us start with point (a). It suffices to prove the result for $u \ge 0$. By [1, Proposition 4.4], for every $n, k \in \mathbb{N}$ we have

$$w_n^k := (u \wedge n)1_{\cup_{i\le k}\Omega^i} = (u1_{\cup_{i\le k}\Omega^i}) \wedge n \in BV(\mathbb{R}^d)$$

with

$$\nabla w_n^k = \nabla u 1_{\{u<n\}\cap\cup_{i\le k}\Omega^i} \qquad \text{and} \qquad J_{w_n^k} \subseteq \cup_{i\le k}\partial\Omega^i.$$

Since $J_{w_n^k} \subset \cup_i \partial\Omega^i$, by the maximality of Γ_Ω we deduce that

$$J_{w_n^k} \subseteq \Gamma_\Omega \qquad \text{up to } \mathcal{H}^{d-1}\text{-negligible sets.}$$

Moreover, concerning the traces on Γ_Ω we have

$$\gamma_r(w_n^k) \le \gamma_r(u1_\Omega) \qquad \text{and} \qquad \gamma_l(w_n^k) \le \gamma_l(u1_\Omega)$$

so that

$$|D^s w_n^k| \le [|\gamma_r(u1_\Omega)| + |\gamma_l(u1_\Omega)|]\, \mathcal{H}^{d-1} \lfloor \Gamma_\Omega$$

as measures on $\mathbb{R}^d$. The result follows noting that for $n, k \to \infty$

$$w_n^k \to u1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^d)$$

while

$$\begin{aligned}|Dw_n^k|(\mathbb{R}^d) &\le \int_\Omega |\nabla u|\,dx + \int_{\Gamma_\Omega} [|\gamma_r(u1_\Omega)| + |\gamma_l(u1_\Omega)|]\, d\mathcal{H}^{d-1}\\ &\le |\Omega|^{1/2}\|\nabla u\|_{L^2(\Omega;\mathbb{R}^d)} + \mathcal{H}^{d-1}(\Gamma_\Omega)^{1/2}[\|\gamma_r(u1_\Omega)\|_{L^2(\Gamma_\Omega)} + \|\gamma_l(u1_\Omega)\|_{L^2(\Gamma_\Omega)}].\end{aligned}$$

Let us come to point (b), again assuming without loss of generality that $u \ge 0$. Proceeding again by truncation we have by the chain rule in BV (see [1, Theorem 3.96])

$$v_n := (u1_\Omega \wedge n)^2 \in BV(\mathbb{R}^d)$$

with $\nabla v_n = 2u\nabla u 1_{\{u<n\}\cap\Omega}$ and $J_{v_n} \subseteq \Gamma_\Omega$ up to $\mathcal{H}^{d-1}$-negligible sets. In particular, concerning the traces on Γ_Ω we have

$$\gamma_r(v_n) \le [\gamma_r(u1_\Omega)]^2 \qquad \text{and} \qquad \gamma_l(v_n) \le [\gamma_l(u1_\Omega)]^2.$$

We conclude that

$$v_n \to u^2 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^d)$$

and since from the lower semicontinuity of the total variation

$$\begin{aligned}|D(u^2 1_\Omega)|(\mathbb{R}^d) \le \liminf_n |Dv_n|(\mathbb{R}^d) &= \liminf_n \left[\int_{\mathbb{R}^d} |\nabla v_n|\,dx + |D^s v_n|(\mathbb{R}^d)\right]\\ &\le 2\int_\Omega u|\nabla u|\,dx + \int_{\Gamma_\Omega} \left[|\gamma_r(u)|^2 + |\gamma_l(u)|^2\right] d\mathcal{H}^{d-1}\end{aligned}$$

we get that $u^2 1_\Omega \in BV(\mathbb{R}^d)$, and (3.6) follows estimating the first integral in the last term of the previous inequality with $\|u\|^2_{H^1(\Omega)}$.

Finally by Sobolev embedding of $BV(\mathbb{R}^d)$ into $L^{\frac{d}{d-1}}(\mathbb{R}^d)$ we can write for every $\varepsilon > 0$

$$\begin{aligned}\|u\|^2_{L^2(\Omega)} &\le |\Omega|^{\frac{1}{d}}\|u^2\|_{L^{d/d-1}(\Omega)} \le C_d|\Omega|^{\frac{1}{d}}|D(u^2 1_\Omega)|(\mathbb{R}^d)\\ &\le C_d|\Omega|^{\frac{1}{d}}\left[\varepsilon\|u\|^2_{L^2(\Omega)} + C_\varepsilon\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^d)} + \int_{\Gamma_\Omega} \left[|\gamma_r(u)|^2 + |\gamma_l(u)|^2\right] d\mathcal{H}^{d-1}\right],\end{aligned}$$

where C_ε is a suitable constant. Then (3.7) follows by choosing ε small enough. □

Remark 3.6. The proof of point (a) shows that if $u \in H^1(\Omega) \cap L^\infty(\Omega)$, then $u \in \mathcal{R}(\Omega)$. In particular we always have $1_\Omega \in \mathcal{R}(\Omega)$.

Remark 3.7. The best constant in (3.7) is related to the first Robin eigenvalue of the ball $B^{|\Omega|}$ with the same volume of Ω: indeed the previous argument shows that $u^2 1_\Omega$ belongs to the class $SBV(\mathbb{R}^d)$ of special functions of bounded variation, so that the Faber-Krahn inequality proved in [5] yields

$$\lambda_{1,1}(B^{|\Omega|})\|u\|^2_{L^2(\Omega)} \le \left[\|\nabla u\|^2_{L^2(\Omega)} + \|\gamma_r(u)\|^2_{L^2(\Gamma_\Omega)} + \|\gamma_l(u)\|^2_{L^2(\Gamma_\Omega)}\right],$$

where $\lambda_{1,1}$ is the first Robin eigenvalue associated to the parameter $\beta = 1$.

We endow $\mathcal{R}(\Omega)$ with the following scalar product

$$(u,v)_{\mathcal{R}} := (u,v)_{H^1(\Omega)} + (\gamma_r(u1_\Omega), \gamma_r(v1_\Omega))_{L^2(\Gamma_\Omega)} + (\gamma_l(u1_\Omega), \gamma_l(v1_\Omega))_{L^2(\Gamma_\Omega)}. \tag{3.8}$$

Proposition 3.8. *$\mathcal{R}(\Omega)$ is a Hilbert space with respect to the scalar product* (3.8). *Moreover the immersion*

$$\mathcal{R}(\Omega) \hookrightarrow L^2(\Omega)$$

is compact.

Proof. Let us firstly check the completeness of $\mathcal{R}(\Omega)$. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{R}(\Omega)$. Clearly $(u_n)_{n\in\mathbb{N}}$, $(\gamma_r(u_n 1_\Omega))_{n\in\mathbb{N}}$ and $(\gamma_l(u_n 1_\Omega))_{n\in\mathbb{N}}$ are Cauchy sequences in $H^1(\Omega)$ and $L^2(\Gamma_\Omega)$, respectively. Hence there exist $u \in H^1(\Omega)$, $w_r, w_l \in L^2(\Gamma_\Omega)$ such that

$$u_n \to u \quad \text{strongly in } H^1(\Omega),$$

$$\gamma_r(u_n 1_\Omega) \to w_r \quad \text{strongly in } L^2(\Gamma_\Omega), \tag{3.9}$$

and

$$\gamma_l(u_n 1_\Omega) \to w_l \quad \text{strongly in } L^2(\Gamma_\Omega). \tag{3.10}$$

According to Lemma 3.5, the measure $D^s(u_n 1_\Omega)$ is supported on Γ_Ω, so that thanks to (3.4)

$$|D^s(u_n 1_\Omega)|(\mathbb{R}^d) = \|\gamma_r(u_n 1_\Omega) - \gamma_l(u_n 1_\Omega)\|_{L^1(\Gamma_\Omega)}.$$

Since Ω is bounded and Γ_Ω has finite $\mathcal{H}^{d-1}$-measure, we may write

$$\begin{aligned}
\|u_n 1_\Omega - u_m 1_\Omega\|_{BV(\mathbb{R}^d)} &= \|\nabla u_n - \nabla u_m\|_{L^1(\Omega)} \\
&\quad + \|\gamma_r(u_n 1_\Omega - u_m 1_\Omega) - \gamma_l(u_n 1_\Omega - u_m 1_\Omega)\|_{L^1(\Gamma_\Omega)} + \|u_n - u_m\|_{L^1(\Omega)} \\
&\le C\left[\|u_n - u_m\|_{H^1(\Omega)} + \|\gamma_r(u_n 1_\Omega) - \gamma_r(u_m 1_\Omega)\|_{L^2(\Gamma_\Omega)} + \|\gamma_l(u_n 1_\Omega) - \gamma_l(u_m 1_\Omega)\|_{L^2(\Gamma_\Omega)}\right],
\end{aligned}$$

for some $C > 0$ depending on Ω, so that $(u_n 1_\Omega)_{n\in\mathbb{N}}$ is a Cauchy sequence in $BV(\mathbb{R}^d)$ with

$$u_n 1_\Omega \to u 1_\Omega \qquad \text{strongly in } BV(\mathbb{R}^d). \tag{3.11}$$

Taking into account (3.3), for every $i \in \mathbb{N}$ we deduce from (3.11) and the continuity of the trace operators that

$$\gamma_r^i(u_n 1_\Omega) \to \gamma_r^i(u 1_\Omega), \qquad \text{and} \qquad \gamma_l^i(u_n 1_\Omega) \to \gamma_l^i(u 1_\Omega) \qquad \text{strongly in } L^1(\Gamma_i).$$

We conclude from the very definition (3.5) of the global traces on Γ_Ω and from (3.9),(3.10) that

$$w_r = \gamma_r(u 1_\Omega) \qquad \text{and} \qquad w_l = \gamma_l(u 1_\Omega).$$

We conclude that $u \in \mathcal{R}(\Omega)$ and

$$u_n \to u \qquad \text{strongly in } \mathcal{R}(\Omega),$$

which shows that $\mathcal{R}(\Omega)$ is a Hilbert space.

Let us come to the compact embedding in $L^2(\Omega)$. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{R}(\Omega)$. By Lemma 3.5 we have that $u_n 1_\Omega$ is bounded in $BV(\mathbb{R}^d)$ so that, up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges strongly in $L^1(\Omega)$. Moreover also $u_n^2 1_\Omega$ is bounded in $BV(\mathbb{R}^d)$, so that using the embedding of $BV(\mathbb{R}^d)$ into $L^{d/d-1}(\mathbb{R}^d)$, we conclude that $(u_n)_{n\in\mathbb{N}}$ converges strongly also in $L^2(\Omega)$. ∎

3.2. **The Generalized Robin-Laplace operator.** We generalize the Robin-Laplacian boundary value problem to open sets satisfying the structural assumption (3.1) in the following way.

Definition 3.9 (Generalized Robin boundary value problem). *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set satisfying* (3.1)*. Given $\beta > 0$ and $f \in L^2(\Omega)$, we say that $u \in \mathcal{R}(\Omega)$ is a solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega \end{cases}$$

if for every test function $v \in \mathcal{R}(\Omega)$ we have

$$\mathcal{L}_\beta(u, v) = (f, v)_{L^2(\Omega)}, \tag{3.12}$$

where

$$\mathcal{L}_\beta(u, v) := (\nabla u, \nabla v)_{L^2(\Omega)} + \beta(\gamma_r(u), \gamma_r(v))_{L^2(\Gamma_\Omega)} + \beta(\gamma_l(u), \gamma_l(v))_{L^2(\Gamma_\Omega)},$$

where Γ_Ω is given according to (3.2)*.*

The following existence and uniqueness result holds true.

Theorem 3.10 (Existence and uniqueness of a solution). *Problem* (3.12) *admits a unique solution $u \in \mathcal{R}(\Omega)$ and the resolvent operator*

$$L^2(\Omega) \to \mathcal{R}(\Omega)$$

is compact, with a norm depending only on d, $|\Omega|$ and β. Finally u is the unique minimizer in $\mathcal{R}(\Omega)$ of the functional

$$F(v) := \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \frac{\beta}{2}\int_{\Gamma_\Omega} \left[|\gamma_r(u)|^2 + |\gamma_l(u)|^2\right]\,d\mathcal{H}^{d-1} - \int_\Omega fu\,dx.$$

Proof. Existence, uniqueness and the variational characterization come immediately from the fact that the bilinear symmetric form $\mathcal{L}_\beta$ is continuous and coercive on the Hilbert space $\mathcal{R}(\Omega)$, while

$$v \mapsto (f, v)_{L^2(\Omega)}$$

is a linear and continuous functional on $\mathcal{R}(\Omega)$. Coercivity is a consequence of (3.7).

If u_f is the solution associated to f we have

$$\|u_f\|^2_{\mathcal{R}(\Omega)} \le C\mathcal{L}_\beta(u_f, u_f) = C\int_\Omega u_f f\,dx \le C\|u_f\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \le C\|u_f\|_{\mathcal{R}(\Omega)}\|f\|_{L^2(\Omega)}$$

so that

$$\|u_f\|_{\mathcal{R}(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

Here $C = C(d, |\Omega|, \beta)$, so that the operator norm depends only on the dimension, the volume $|\Omega|$, and the Robin parameter β.

Compactness of the resolvent operator follows by classical arguments. If $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, we may assume that

$$f_n \rightharpoonup f \qquad \text{weakly in } L^2(\Omega)$$

for some $f \in L^2(\Omega)$. Let u_n be the solution associated to f_n. Then $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{R}(\Omega)$, so that up possibly to a further subsequence

$$u_n \rightharpoonup u \qquad \text{weakly in } \mathcal{R}(\Omega)$$

for some $u \in \mathcal{R}(\Omega)$. In view of the compact embedding into $L^2(\Omega)$ given by Proposition 3.8, we have

$$u_n \to u \qquad \text{strongly in } L^2(\Omega).$$

Clearly u is the solution associated to f, with

$$\mathcal{L}_\beta(u_n, u_n) = (f_n, u_n)_{L^2(\Omega)} \to (f, u)_{L^2(\Omega)} = \mathcal{L}_\beta(u, u).$$

Since $u \mapsto \left(\mathcal{L}_\beta(u, u) + \|u\|_{L^2(\Omega)^2}\right)^{1/2}$ is a norm equivalent to that of $\mathcal{R}(\Omega)$, we conclude that

$$u_n \to u \qquad \text{strongly in } \mathcal{R}(\Omega).$$

□

Remark 3.11 (Independence from the choice of Γ_Ω and its orientation). Thanks to Remark 3.2, and since the role of the left and right trace in the formulation (3.12) of the problem is symmetrical (especially in the formula of the functional F), we conclude that the general Robin problem is independent of the choice of the set Γ_Ω and of its orientation.

Remark 3.12 (Classical setting). In the case Ω is Lipschitz regular, the formulation (3.12) reduces clearly to the standard weak formulation of the Robin-Laplacian.

Remark 3.13 (Decomposition in connected components). Note as well that under assumption (3.1) if $\Omega = \cup_i \Omega^i$ is decomposed in its connected components, than the solution of the Robin problem on Ω equals on each connected component the solution of the corresponding Robin problem on the component. As well, the spectrum of the Robin-Laplacian on Ω is the union of the spectra of each Ω^i, as in the classical setting.

4. Admissible domains in dimension two and compactness results.

In this section we study in detail some properties of a class of open sets in dimension two on which the Robin generalized problem of Section 3 is well posed. In particular, since we are interested in stability results under the variation of the domains, we look for compactness results concerning the Robin spaces as the open sets vary.

The class of admissible domains is the following.

Definition 4.1 (The class $\mathcal{A}_m(\mathbb{R}^2)$ of admissible domains). *Given $m \in \mathbb{N}$, $m \geq 1$, we denote with $\mathcal{A}_m(\mathbb{R}^2)$ the class of open and bounded sets $\Omega \subset \mathbb{R}^2$ such that Ω^c has at most m connected components and*

$$\mathcal{H}^1(\cup_i \partial\Omega^i) < +\infty,$$

where $\{\Omega^i\}_{i\in\mathbb{N}}$ denotes the family of its connected components. We set

$$\Gamma_\Omega = \cup_i \partial\Omega^i. \tag{4.1}$$

Remark 4.2. In the rest of the paper, we will use the following intuitive fact: if $\Omega \in \mathcal{A}_m(R^2)$, and Ω^i is one of its connected components, then $\partial\Omega^i$ has at most m connected components, i.e., if Ω is bounded, $\partial\Omega^i \in \mathcal{K}_m(\mathbb{R}^2)$. We have not been able to find a simple proof of this fact, nor to individuate a precise reference.

A non elementary proof of this fact uses the following argument from algebraic topology. It is not restrictive to prove the result for $\Omega \in \mathcal{A}_m(\mathbb{R}^2)$ connected. For $\varepsilon > 0$, let us consider the open sets A_ε and B_ε made up of the points whose distance from Ω and Ω^c is less than ε. For ε small enough, we have that A_ε is connected and B_ε has at most m connected components. Then we can use the Mayer-Vietoris exact sequence for the homology to obtain the short exact sequence

$$0 \longrightarrow H_0(A_\varepsilon \cap B_\varepsilon) \xrightarrow{i_1} H_0(A_\varepsilon) \oplus H_0(B_\varepsilon) \xrightarrow{i_2} H_0(\mathbb{R}^2) \longrightarrow 0,$$

as $H_1(\mathbb{R}^2)$ is trivial. From the connectedness assumptions we get $dim H_0(\mathbb{R}^2) = 1$, $dim(H_0(A_\varepsilon)) = 1$ and $dim(H_0(B_\varepsilon)) \leq m$. From the exactness of the sequence we get that i_1 is injective, i_2 is surjective and that $Im\, i_1$ is isomorphic to $Ker\, i_2$. We deduce that $H_0(A_\varepsilon \cap B_\varepsilon)$ is isomorphic to $Ker\, i_2$, which has a dimension at most m, so that $A_\varepsilon \cap B_\varepsilon$ has at most m connected components. Letting $\varepsilon \to 0$, we infer that $\partial\Omega$ has at most m connected components.

The generalized Robin problems of Section 3 are well defined, the maximal rectifiable set involved in the definition being given precisely by (4.1) (so that the the notation is well chosen).

Proposition 4.3 (Admissible domains and Robin problems). *Let $\Omega \in \mathcal{A}_m(\mathbb{R}^2)$. Then the generalized Robin-Laplace problem of Definition 3.9 is well posed and the set Γ_Ω given in* (4.1) *satisfies* (3.2).

Proof. Let $\Omega \in \mathcal{A}_m(\mathbb{R}^2)$ and let $\{\Omega^i\}_{i\in\mathbb{N}}$ denote the family of its connected components. Since Ω^i is connected and $(\Omega^i)^c$ has at most m connected components (see Remark 4.2), we deduce that $\partial\Omega^i \in \mathcal{K}_m(\mathbb{R}^2)$ (see (2.1)). Since by Theorem 2.2 elements of $\mathcal{K}_m(\mathbb{R}^2)$ are $\mathcal{H}^1$-countably rectifiable, we conclude that

$$\Gamma_\Omega := \cup_i \partial\Omega^i$$

is an admissible choice in (3.2). $\square$

4.1. **A collection of some useful technical results.** The results below are not new. However, as we did not find any reference, for the clearness of the paper we give a proof here.

Proposition 4.4. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of equibounded open sets in $\mathbb{R}^d$ such that*

$$\Omega_n \overset{\mathcal{H}^c}{\to} \Omega \qquad \text{and} \qquad \partial\Omega_n \overset{\mathcal{H}}{\to} K.$$

The following items hold true.

(a) $\partial\Omega \subseteq K$.
(b) $\overline{\Omega}_n \xrightarrow{\mathcal{H}} \overline{\Omega} \cup K$.
(c) *If $\mathcal{H}^{d-1}(\partial\Omega_n) \leq C$ and $|K| = 0$, then*

$$1_{\Omega_n} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^d). \tag{4.2}$$

Proof. Let us start with (a). Let $x \in \partial\Omega$, and by contradiction suppose that $x \notin K$. Hence there exists $\varepsilon > 0$ such that

$$\overline{B}(x,\varepsilon) \cap K = \emptyset.$$

Hence, for n large enough

$$B(x,\varepsilon) \cap \partial\Omega_n = \emptyset.$$

We distinguish two cases.

(i) If $B(x,\varepsilon) \subseteq \Omega_n$, then passing to the limit in view of the convergence of the domains we have $B(x,\varepsilon) \subseteq \Omega$, so that $x \notin \partial\Omega$, which is a contradiction.
(ii) If $B(x,\varepsilon) \subseteq \Omega_n^c$, then passing to the limit we have $B(x,\varepsilon) \subseteq \Omega^c$, so that $x \in int(\Omega^c)$, which is again a contradiction.

Let us pass to the proof of (b). Up to a subsequence we may assume

$$\overline{\Omega}_n \xrightarrow{\mathcal{H}} P.$$

Notice that $\overline{\Omega} \cup K \subseteq P$. Indeed $K \subseteq P$, while if $x \in \overline{\Omega} \setminus K$, by point (i) we have that $x \in \Omega$, which yields $x \in \Omega_n$ for n large, and so $x \in P$.

On the other hand let us show that $P \subseteq \overline{\Omega} \cup K$. Since by point (i) we have $\overline{\Omega} \cup K = \Omega \cup K$, it suffices to see that

$$P \setminus \Omega \subseteq K.$$

Let $x \in P \setminus \Omega$, and let $y_n \in \overline{\Omega}_n$ be such that

$$y_n \to x.$$

Since $x \in \Omega^c$, by definition of Hausdorff complementary topology there exists $z_n \in \Omega_n^c$ such that

$$z_n \to x.$$

As the segment with extremes y_n and z_n intersects $\partial\Omega_n$, we find $\xi_n \in \partial\Omega_n$ such that

$$\xi_n \to x,$$

which yields $x \in K$. We conclude that $P = \overline{\Omega} \cup K$. Since P is well determined, we do not need to pass to subsequences, so that point (b) follows.

Finally, let us come to point (c). Assume $\Omega_n \subseteq D$ with D smooth, open and bounded. Let us first prove that

$$|\Omega_n| \to |\Omega|. \tag{4.3}$$

Since $\Omega_n \overset{\mathcal{H}^c}{\to} \Omega$, we easily get

$$|\Omega| \leq \liminf_n |\Omega_n|,$$

so that it is sufficient to prove that

$$|\Omega| \geq \limsup_n |\Omega_n|.$$

Since $|K| = 0$, by point (a) we get in particular $|\partial\Omega| = 0$. Clearly we have

$$|D \setminus P| \leq \liminf_n |D \setminus \overline{\Omega}_n|,$$

so that

$$\limsup_n |\overline{\Omega}_n| \le |P|.$$

Hence, since $|K| = |\partial\Omega| = |\partial\Omega_n| = 0$

$$|\Omega| = |\overline{\Omega} \cup K| = |P| \ge \limsup_n |\overline{\Omega}_n| = \limsup_n |\Omega_n \cup \partial\Omega_n| = \limsup_n |\Omega_n|,$$

which gives yields (4.3).

Let us come to the proof of (4.2). Since $\mathcal{H}^{d-1}(\partial\Omega_n) < +\infty$, we have that Ω_n has finite perimeter in $\mathbb{R}^d$ with

$$Per(\Omega_n) \le \mathcal{H}^{d-1}(\partial\Omega_n).$$

Since $\Omega_n \subseteq D$, by the compactness of sets with finite perimeter, up to a subsequence we may thus assume that

$$1_{\Omega_n} \to 1_E \qquad \text{strongly in } L^1(\mathbb{R}^d) \tag{4.4}$$

for some set of finite perimeter $E \subseteq D$, the convergence being also pointwise almost everywhere.

Notice that $\Omega \subseteq E$. Indeed, for every compact set $C \subseteq \Omega$, by Hausdorff convergence we get $C \subseteq \Omega_n$ for n large enough. This yields $1_{\Omega_n} = 1$ on C so that $1_E = 1$ a.e. on C. Then $C \subseteq E$ up to negligible sets, and by the arbitrariness of C we deduce $\Omega \subseteq E$ up to negligible sets. Taking into account (4.3) and (4.4) we deduce

$$|E \setminus \Omega| = |E| - |\Omega| = |\Omega| - |\Omega| = 0.$$

This yields $1_\Omega = 1_E$ a.e., so that (4.2) follows. $\square$

Remark 4.5. Simple examples show that the inclusion $\partial\Omega \subseteq K$ can be strict and that the strong convergence of the characteristic functions may fail: for example if $\{q_n\}_{n\in\mathbb{N}}$ is dense in D open and bounded, then $\Omega_n := D \setminus \cup_{k\le n}\{q_k\}$ $\mathcal{H}^c$-converges to $\Omega := \emptyset$: in this case $K = \bar{D}$, while $|\Omega_n| = |D|$ and $|\Omega| = 0$.

The following corollary in a two dimensional setting will be useful.

Corollary 4.6. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of equibounded open sets in $\mathbb{R}^2$ such that*

$$\Omega_n \xrightarrow{\mathcal{H}^c} \Omega, \qquad \partial\Omega_n \text{ has at most } m \text{ connected components}, \qquad \text{and} \qquad \limsup_n \mathcal{H}^1(\partial\Omega_n) < +\infty.$$

Then

$$1_{\Omega_n} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^2). \tag{4.5}$$

Proof. By assumption we have $\partial\Omega_n \in \mathcal{K}_m(\mathbb{R}^2)$ (see (2.1)). Up to a subsequence we may assume that

$$\partial\Omega_n \xrightarrow{\mathcal{H}} K \in \mathcal{K}_m(\mathbb{R}^2).$$

In view of Gołąb Theorem (see Theorem 2.4) we may write

$$\mathcal{H}^1(K) \le \liminf_n \mathcal{H}^1(\partial\Omega_n) < +\infty,$$

so that in particular we get $|K| = 0$. The conclusion follows by Proposition 4.4. $\square$

In order to establish our main compactness result for sequences of admissible open sets, we will need the following lemma which holds in general dimension d.

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set, and let us denote by $\{\Omega^i\}_{i\in\mathbb{N}}$ the family of its connected components. Assume that*

$$|\Omega| \ge C_1 \qquad \text{and} \qquad \mathcal{H}^{d-1}\left(\cup_i \partial\Omega^i\right) \le C_2$$

for some $C_1, C_2 > 0$. Then there exists a connected component $\Omega^{\bar{i}}$ such that

$$|\Omega^{\bar{i}}| \ge C_3,$$

where C_3 depends only on C_1, C_2 and d.

Proof. Notice that

$$\sum_i Per(\Omega^i) \leq 2\mathcal{H}^{d-1}(\cup_i \partial\Omega^i). \tag{4.6}$$

Since in view of the isoperimetric inequality we can write

$$|\Omega^i|^{\frac{d-1}{d}} \leq C_d Per(\Omega^i),$$

we infer

$$\frac{C_1}{m_\Omega^{\frac{1}{d}}} \leq \frac{|\Omega|}{m_\Omega^{\frac{1}{d}}} = \sum_i \frac{|\Omega^i|}{m_\Omega^{\frac{1}{d}}} \leq \sum_i |\Omega^i|^{\frac{d-1}{d}} \leq C_d \sum_i Per(\Omega^i) \leq 2C_d C_2,$$

where $m_\Omega := \max_{i\in\mathbb{N}} |\Omega^i|$. We thus conclude

$$m_\Omega \geq \left(\frac{C_1}{2C_d C_2}\right)^d$$

and the result follows.

Inequality (4.6) follows by noticing that the sets $\partial^*\Omega^i \cap \partial^*\Omega^j$ for $i \neq j$ and $\partial^*\Omega^h \setminus \cup_{k\neq h}\partial^*\Omega^k$ have $\mathcal{H}^{d-1}$-negligible overlapping (as the connected components are disjoint and with finite perimeter), so that we get

$$\begin{aligned}\sum_i Per(\Omega^i) &= \sum_i \mathcal{H}^{d-1}(\partial^*\Omega^i) = \sum_i \left[\mathcal{H}^{d-1}(\partial^*\Omega^i \setminus \cup_{j\neq i}\partial^*\Omega^j) + \sum_{j\neq i}\mathcal{H}^{d-1}(\partial^*\Omega^i \cap \partial^*\Omega^j)\right] \\ &= \sum_i \left[\mathcal{H}^{d-1}(\partial^*\Omega^i \setminus \cup_{j\neq i}\partial^*\Omega^j)\right] + 2\sum_{i\neq j}\mathcal{H}^{d-1}(\partial^*\Omega^i \cap \partial^*\Omega^j) \\ &\leq 2\mathcal{H}^{d-1}(\cup_i \partial^*\Omega^i) \leq 2\mathcal{H}^{d-1}(\cup_i \partial\Omega^i).\end{aligned}$$

□

4.2. **A compactness result in $\mathcal{A}_m(\mathbb{R}^2)$.** We are now in a position to state our main compactness result for sequences of open sets in $\mathcal{A}_m(\mathbb{R}^2)$.

Theorem 4.8 (Compactness for the admissible domains). *Let $(\Omega_n)_{n\in\mathbb{N}}$ be an equibounded sequence in $\mathcal{A}_m(\mathbb{R}^2)$ such that*

$$\limsup_n \mathcal{H}^1(\Gamma_{\Omega_n}) < +\infty. \tag{4.7}$$

Then there exists $\Omega \in \mathcal{A}_m(\mathbb{R}^2)$ such that up to a subsequence $\Omega_n \xrightarrow{\mathcal{H}^c} \Omega$ with

$$1_{\Omega_n} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^2). \tag{4.8}$$

and

$$\mathcal{H}^1(\Gamma_\Omega) \leq \liminf_n \mathcal{H}^1(\Gamma_{\Omega_n}). \tag{4.9}$$

Proof. Up to a subsequence, we have that

$$\Omega_n \xrightarrow{\mathcal{H}^c} \Omega$$

for some open bounded set $\Omega \subseteq \mathbb{R}^2$ such that Ω^c has at most m connected components. We need to show that (4.8) and (4.9) hold true, so that in particular $\Omega \in \mathcal{A}_m(\mathbb{R}^2)$.

Step 1. We claim that up to a subsequence we can write for every $k \geq 1$

$$\Omega_n = A_n^1 \cup \cdots \cup A_n^k \cup R_n^k \qquad \text{and} \qquad \Omega = A^1 \cup \cdots \cup A^k \cup R^k \tag{4.10}$$

such that the following items hold true:

(a) A^i and R^k are union of connected components of Ω;
(b) A_n^i are connected components of Ω_n, while R_n^k is union of connected components of Ω_n;

(c) we have for $n \to \infty$

$$A_n^i \overset{\mathcal{H}^c}{\to} A^i \text{ for } i = 1, \dots, k, \qquad \text{and} \qquad R_n^k \overset{\mathcal{H}^c}{\to} R^k,$$

while for $k \to \infty$

$$R^k \overset{\mathcal{H}^c}{\to} \emptyset \qquad \text{and} \qquad |R^k| \to 0.$$

Let us denote with $\{\Omega^i\}_{i \geq 1}$ the family of the connected components of Ω. Then we can write

$$\Omega_n = A_n^1 \cup R_n^1,$$

and up to a subsequence we can assume

$$A_n^1 \overset{\mathcal{H}^c}{\to} A^1 \qquad \text{and} \qquad R_n^1 \overset{\mathcal{H}^c}{\to} R^1,$$

where A^1 and R^1 are union of connected components of Ω, $\Omega^1 \subseteq A^1$,

$$\Omega = A^1 \cup R^1,$$

and A_n^1 is a connected component of Ω_n.

Let $i_1 > 1$ be the first index such that $\Omega^{i_1} \not\subseteq A^1$, so that $\Omega^j \subseteq A^1$ for every $j < i_1$. Clearly $\Omega^{i_1} \subseteq R^1$, so that, proceeding as above, we can write

$$R_n^1 = A_n^2 \cup R_n^2, \qquad R^1 = A^2 \cup R^2, \qquad \Omega^{i_1} \subseteq A^2$$

with, up to a further subsequence,

$$A_n^2 \overset{\mathcal{H}^c}{\to} A^2, \qquad \text{and} \qquad R_n^2 \overset{\mathcal{H}^c}{\to} R^2.$$

Again A_n^2 is a connected component of Ω_n, while A^2 and R^2 are union of connected components of Ω.

In general, iterating the construction and employing a diagonal argument, we can find a strictly increasing sequence of indices $(i_k)_{k \geq 1}$ and a unique subsequence such that there exist decompositions

$$\Omega_n = A_n^1 \cup \cdots \cup A_n^k \cup R_n^k, \qquad \Omega = A^1 \cup \cdots \cup A^k \cup R^k$$

with $\Omega^j \subseteq A^1 \cup \cdots \cup A^k$ for every $j < i_k$, $\Omega^{i_k} \subseteq R^k$,

$$A_n^h \overset{\mathcal{H}^c}{\to} A^h \text{ for } h = 1, \dots, k, \qquad \text{and} \qquad R_n^k \overset{\mathcal{H}^c}{\to} R^k.$$

Notice that

$$R^k \subseteq \Omega \setminus \bigcup_{j < i_k} \Omega^j$$

so that the last relation of point (c) follows.

Step 2. Let us employ the decomposition (4.10) of Step 1. Since the open sets A_n^h are connected components of Ω_n, we infer that $\partial A_n^h \in \mathcal{K}_m(\mathbb{R}^2)$ (see (2.1)), as $(A_n^h)^c$ has at most m connected components (see Remark 4.2). In particular, thanks to (4.7) we have for every n

$$\mathcal{H}^1(\partial A_n^h) \leq C.$$

In view of Corollary 4.6 we deduce that

$$1_{A_n^h} \to 1_{A^h} \qquad \text{strongly in } L^1(\mathbb{R}^2) \tag{4.11}$$

for every $h \geq 1$.

We claim that

$$\lim_n |\Omega_n| = |\Omega|. \tag{4.12}$$

Since it is always true that $\liminf_n |\Omega_n| \geq |\Omega|$ in view of the Hausdorff convergence, let us assume by contradiction that there exists $\varepsilon > 0$ such that for n large

$$|\Omega_n| \geq |\Omega| + \varepsilon.$$

In view of (4.11) we infer for every $k \geq 1$

$$\liminf_n |R_n^k| = \liminf_n \left[|\Omega_n| - \sum_{h=1}^k |A_n^h| \right] \geq |\Omega| + \varepsilon - \sum_{h=1}^k |A^h| = |R^k| + \varepsilon.$$

Let us select $n_k \to \infty$ such that

$$R_{n_k}^k \stackrel{\mathcal{H}^c}{\to} \emptyset. \tag{4.13}$$

In view of Lemma 4.7 there exists $\tilde{\Omega}_{n_k}$ connected component of Ω_{n_k} contained in $R_{n_k}^k$ such that

$$|\tilde{\Omega}_{n_k}| \geq \delta$$

for some $\delta > 0$ independent of k. Since $\partial\tilde{\Omega}_{n_k}$ has at most m connected components, thanks to Corollary 4.6 which entails the stability of the measures, we have that, up to a further subsequence, the sequence $\{\tilde{\Omega}_{n_k}\}$ has a nontrivial Hausdorff limit, which is in contradiction with (4.13). Convergence (4.12) is thus established.

Step 3: Conclusion. Let us prove (4.8) and (4.9). Taking into account the decompositions (4.10) and the relations (4.11) and (4.12) we have

$$\lim_n |R_n^k| = |R^k|.$$

We thus infer using again (4.11)

$$\limsup_n \|1_{\Omega_n} - 1_\Omega\|_{L^1(\mathbb{R}^2)} \leq 2|R^k| \stackrel{k\to\infty}{\to} 0,$$

which yields (4.8).

Up to a further subsequence, we have that

$$\partial A_n^i \stackrel{\mathcal{H}^c}{\to} K^i$$

for some $K^i \in \mathcal{K}_m(\mathbb{R}^2)$ with $\partial A^i \subseteq K^i$. Using Gołąb theorem (see Theorem 2.4) we have

$$\mathcal{H}^1(\cup_{i\leq k}\partial A^i)) \leq \mathcal{H}^1(\cup_{i\leq k} K^i) \leq \liminf_n \mathcal{H}^1(\cup_{i\leq k}\partial A_n^i) \leq \liminf_n \mathcal{H}^1(\Gamma_{\Omega_n})$$

so that (4.9) follows letting $k \to \infty$ since $\Gamma_\Omega = \cup_i \partial A^i$. □

4.3. **Compactness for the Robin spaces.** We concentrate now on compactness results for the Robin spaces $\mathcal{R}(\Omega_n)$ as Ω_n varies in the admissible class $\mathcal{A}_m(\mathbb{R}^2)$.

It will be useful to talk about traces on elements K of $\mathcal{K}_m(\mathbb{R}^2)$ (see (2.1)) using the same notation employed for the generalized Robin-Laplace operator: for $v \in BV(\mathbb{R}^2)$ we will consider right and left traces $\gamma_r(v), \gamma_l(v)$ as done at the beginning of Section 3.1, associated to a suitable decomposition and orientation of K. It turns out that

$$\{v^+, v^-\} = \{\gamma_r(v), \gamma_l(v)\} \qquad \mathcal{H}^1\text{-a.e. on } K, \tag{4.14}$$

where $v^\pm$ are the upper and lower approximate limits.

The following result, which is fundamental for our analysis, is a reformulation in our context of [6, Lemma 19].

Lemma 4.9. *Let $D \subseteq \mathbb{R}^2$ be an open and bounded set and $m \geq 1$, and let $(K_n)_{n\in\mathbb{N}}, K \subset D$ be compact sets in $\mathcal{K}_m(\mathbb{R}^2)$ such that*

$$K_n \stackrel{\mathcal{H}}{\to} K \qquad \text{and} \qquad \limsup_n \mathcal{H}^1(K_n) < +\infty.$$

Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $BV(D) \cap H^1(D \setminus K_n)$ with

$$\limsup_n \left[\|v_n\|_{H^1(D\setminus K_n)} + \int_{K_n} \left[|\gamma_r(v_n)|^2 + |\gamma_l(v_n)|^2 \right] d\mathcal{H}^1 \right] < +\infty.$$

Then there exists $v \in BV(D) \cap H^1(D \setminus K)$ such that

$$v_n \to v \qquad \text{strongly in } L^2_{loc}(D) \tag{4.15}$$

$$\nabla v_n \rightharpoonup \nabla v \qquad \textit{weakly in } L^2(D;\mathbb{R}^2), \tag{4.16}$$

and

$$\int_K [|\gamma_r(v)|^2 + |\gamma_l(v)|^2]\, d\mathcal{H}^1 \le \liminf_n \int_{\partial K_n} [|\gamma_r(v_n)|^2 + |\gamma_l(v_n)|^2]\, d\mathcal{H}^1. \tag{4.17}$$

Proof. Since

$$\begin{aligned}\|v_n\|_{BV(D)} &\le \|\nabla v_n\|_{L^1(D\setminus K_n;\mathbb{R}^2)} + \|v_n\|_{L^1(D\setminus K_n)} + \int_{K_n} [|\gamma_r(v_n)| + |\gamma_l(v_n)|]\, d\mathcal{H}^1 \\ &\le C\left[\|v_n\|_{H^1(D\setminus K_n)} + \int_{K_n} [|\gamma_r(v_n)|^2 + |\gamma_l(v_n)|^2]\, d\mathcal{H}^1\right],\end{aligned}$$

we have that $(v_n)_{n\in\mathbb{N}}$ is bounded in $BV(D)$, with

$$\{v_n^+, v_n^-\} = \{\gamma_r(v_n), \gamma_l(v_n)\} \qquad \mathcal{H}^1\text{-a.e. on } K_n,$$

where $v_n^\pm$ are the upper and lower approximate limits. From [6, Lemma 19], we know that there exists $v \in H^1(D \setminus K)$ such that up to a subsequence (4.15) and (4.16) hold true and

$$\int_K [(v^+)^2 + (v^-)^2]\, d\mathcal{H}^1 \le \liminf_n \int_{K_n} [(v_n^+)^2 + (v_n^-)^2]\, d\mathcal{H}^1. \tag{4.18}$$

From the bounds obtained above, we infer that $v \in BV(D)$, and (4.17) is a consequence of (4.18) in view of (4.14). □

The following result holds true.

Proposition 4.10 (Compactness). *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of equibounded sets in $\mathcal{A}_m(\mathbb{R}^2)$ with*

$$\limsup_n \mathcal{H}^1(\Gamma_{\Omega_n}) < +\infty,$$

and $\Omega_n \overset{\mathcal{H}^c}{\to} \Omega$ for some $\Omega \in \mathcal{A}_m(\Omega)$.

Let $u_n \in \mathcal{R}(\Omega_n)$ be such that

$$\limsup_n \left[||\nabla u_n||^2_{L^2(\Omega_n;\mathbb{R}^2)} + \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2\right] d\mathcal{H}^1 \right] < +\infty. \tag{4.19}$$

Then there exists $u \in \mathcal{R}(\Omega)$ such that up to a subsequence

$$u_n 1_{\Omega_n} \to u 1_\Omega \quad \textit{strongly in } L^2(\mathbb{R}^2), \tag{4.20}$$

$$\nabla u_n 1_{\Omega_n} \rightharpoonup \nabla u 1_\Omega \quad \textit{weakly in } L^2(\mathbb{R}^2;\mathbb{R}^2), \tag{4.21}$$

and

$$\int_{\Gamma_\Omega} \left[|\gamma_r(u1_\Omega)|^2 + |\gamma_l(u1_\Omega)|^2\right] d\mathcal{H}^1 \le \liminf_n \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2\right] d\mathcal{H}^1. \tag{4.22}$$

Proof. Let $R > 0$ be such that $\Omega_n \subset B_R(0)$ for every $n \in \mathbb{N}$. Consider $v_n = u_n 1_{\Omega_n}$ and $D := B_{2R}(0)$. By Lemma 3.5 and taking into account (4.19), we get that $(v_n)_{n\in\mathbb{N}}$ and $(v_n^2)_{n\in\mathbb{N}}$ are bounded in $BV(D)$, so that up to a subsequence

$$v_n \to v \qquad \text{strongly in } L^2(D)$$

for some $v \in BV(D)$ with $v^2 \in BV(D)$. In view of Theorem 4.8, we also have that

$$1_{\Omega_n} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^2), \tag{4.23}$$

from which we deduce

$$v = v 1_\Omega$$

Setting $u := v_{|\Omega}$, from the Hausdorff convergence of the sets and (4.23) we get easily that $u \in H^1(\Omega)$ and that (4.20) and (4.21) hold true.

Let $\{\Omega_n^i\}_{i\in\mathbb{N}}$ denote the family of connected components of Ω_n. Up to a subsequence, we may assume that for every $i \in \mathbb{N}$ we have

$$\Omega_n^i \xrightarrow{\mathcal{H}^c} A^i$$

where A^i are union of connected components of Ω with $\Omega = \cup_i A^i$. Moreover, we may assume

$$\partial\Omega_n^i \xrightarrow{\mathcal{H}} K^i$$

for some $K^i \in \mathcal{K}_m(\mathbb{R}^2)$ (see (2.1)). From Proposition 4.4, we have $\partial A^i \subseteq K^i$, while from Corollary 4.6 we have

$$1_{\Omega_n^i} \to 1_{A^i} \qquad \text{strongly in } L^1(\mathbb{R}^2).$$

For every $k \geq 1$ let us consider

$$w_n^k := u_n 1_{\cup_{i\leq k}\Omega_n^i}.$$

We can interpret w_n^k as an element of $BV(D) \cap H^1(D \setminus \cup_{i\leq k}\partial\Omega_n^i)$, for which we have

$$w_n^k \to u1_{\cup_{i\leq k}A^i} \qquad \text{strongly in } L^2(D)$$

and

$$\nabla w_n^k \rightharpoonup \nabla u1_{\cup_{i\leq k}A^i} \qquad \text{weakly in } L^2(D;\mathbb{R}^2).$$

By the lower semicontinuity of Lemma 4.9 we get that

$$\begin{aligned}
\int_{\cup_{i\leq k}\partial A^i} \left[|\gamma_r(u1_{\cup_{i\leq k}A^i})|^2 + |\gamma_l(u1_{\cup_{i\leq k}A^i})|^2\right] d\mathcal{H}^1 &\leq \int_{\cup_{i\leq k}K^i} \left[|\gamma_r(u1_{\cup_{i\leq k}A^i})|^2 + |\gamma_l(u1_{\cup_{i\leq k}A^i})|^2\right] d\mathcal{H}^1 \\
&\leq \liminf_n \int_{\cup_{i\leq k}\partial\Omega_n^i} \left[|\gamma_r(w_n^k)|^2 + |\gamma_l(w_n^k)|^2\right] d\mathcal{H}^1 \\
&\leq \liminf_n \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2\right] d\mathcal{H}^1,
\end{aligned}$$

so that for every $h \leq k$

$$\begin{aligned}
&\int_{\cup_{i\leq h}\partial A^i} \left[|\gamma_r(u1_{\cup_{i\leq k}A^i})|^2 + |\gamma_l(u1_{\cup_{i\leq k}A^i})|^2\right] d\mathcal{H}^1 \\
&\qquad\qquad \leq \liminf_n \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2\right] d\mathcal{H}^1.
\end{aligned}$$

Inequality (4.22) follows by letting $k \to +\infty$ and then $h \to +\infty$ recalling that $\Gamma_\Omega = \cup_i \partial A^i$. We thus conclude that $u \in \mathcal{R}(\Omega)$, and the proof is finished. ☐

5. Robin perimeter

In this section we introduce a notion of perimeter for the open sets in the admissible class $\mathcal{A}_m(\mathbb{R}^2)$ (see Definition 4.1) which is tailored to the study of Robin problems on varying domains.

Definition 5.1 (Robin perimeter). *For $\Omega \in \mathcal{A}_m(\mathbb{R}^2)$ let us set*

$$Per_{\mathcal{R}}(\Omega) := \int_{\Gamma_\Omega} \left[|\gamma_r(1_\Omega)| + |\gamma_l(1_\Omega)|\right] d\mathcal{H}^1,$$

where Γ_Ω is given by (4.1).

From an intuitive point of view, this notion of perimeter marks a difference between "external" and "internal" boundaries, assigning to inner cracks a multiplicity equal to two.

The intuitive inequality

$$\mathcal{H}^1(\Gamma_\Omega) \leq Per_{\mathcal{R}}(\Omega)$$

holds true (see Proposition 5.4), but requires some measure theoretic arguments to take care of the fact that boundaries involved are possibly irregular.

The Robin perimeter $Per_{\mathcal{R}}$ is lower semicontinuous under the Hausdorff complementary topology (see Proposition 5.5); moreover it turns out that its convergence is sufficient to guarantee the continuity of the Robin energies (see Proposition 5.7), which is a key fact to prove stability results for the associated generalized boundary value problems.

We start with the following result.

Proposition 5.2. *Let $\Omega \in \mathcal{A}_m(\mathbb{R}^2)$ be connected. Then $\mathcal{H}^1$-a.e. point of $\partial\Omega$ has density either* 1 *or* $1/2$ *with respect to Ω.*

Proof. It suffices to show the result for every connected component K of $\partial\Omega$. We know from Theorem 2.2 that K is $\mathcal{H}^1$-countably rectifiable. Then K admits an approximate tangent line at $\mathcal{H}^1$-a.e. $x \in K$, i.e., as $\varepsilon \to 0^+$ we have

$$\mathcal{H}^1 \lfloor K_{x,\varepsilon} \overset{*}{\rightharpoonup} \mathcal{H}^1 \lfloor l_x \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\mathbb{R}^2), \tag{5.1}$$

where

$$K_{x,\varepsilon} := \frac{1}{\varepsilon}[K - x] := \left\{ \frac{y-x}{\varepsilon} \,:\, y \in K \right\},$$

and l_x is a line through the origin, the so called *approximate tangent line*.

Since Ω has finite perimeter (as in this case $\Gamma_\Omega = \partial\Omega$ and so $\mathcal{H}^1(\partial\Omega) < +\infty$), we have that $\mathcal{H}^1$-a.e. point in $\mathbb{R}^2$ has either density 1, 1/2 or 0 with respect to Ω. In order to conclude, it suffices to show that at $\mathcal{H}^1$-a.e. point of K where the approximate tangent line exists, the density cannot be zero.

We divide the proof in several steps.

Step 1: Some geometric properties of K. Let $x \in K$ be a point which admits an approximate tangent line l_x. We claim that for every $m \in \mathbb{N}$, $m \geq 1$

$$K_{x,\varepsilon} \cap \overline{Q}_m(0) \to l_x \cap \overline{Q}_m(0) \qquad \text{in the Hausdorff metric}, \tag{5.2}$$

where $Q_m(0)$ is the square with center 0 and side m. This property can be found in Step 2 of the proof of [4, Proposition 2.6], and we report here the argument for completeness.

Up to a translation, we may assume $x = 0$ and write K_ε and l in place of $K_{x,\varepsilon}$ and l_x. Given any sequence $\varepsilon_n \to 0$, by the compactness of Hausdorff convergence and using a diagonal argument, we can find a subsequence $(\varepsilon_{n_h})_{h\in\mathbb{N}}$ such that for every $m \in \mathbb{N}$, $m \geq 1$

$$K_{\varepsilon_{n_h}} \cap \bar{Q}_m(0) \to K_0^m \qquad \text{in the Hausdorff metric.}$$

It is readily checked that for every $m \geq 1$

$$K_0^m \subseteq K_0^{m+1} \qquad \text{and} \qquad K_0^m \cap Q_m(0) = K_0^{m+1} \cap Q_m(0). \tag{5.3}$$

Let us set $K_0 := \bigcup_{m=1}^\infty K_0^m$. The conclusion follows by showing that

$$K_0 = l. \tag{5.4}$$

(a) We have $K_0 \subseteq l$. Indeed, assume by contradiction that $\xi \in K_0 \setminus l$ with $\overline{B}_\eta(\xi) \cap l = \emptyset$. Using the measure convergence (5.1), we obtain that

$$\mathcal{H}^1(K_{\varepsilon_{n_h}} \cap B_\eta(\xi)) \to 0. \tag{5.5}$$

But $K_{\varepsilon_{n_h}}$ is connected by arcs (see [20, Lemma 3.12]), so that the points $\xi_{n_h} \in K_{\varepsilon_{n_h}}$ such that $\xi_{n_h} \to \xi$ are connected to 0 through an arc contained in $K_{\varepsilon_{n_h}}$, against (5.5).

(b) We have on the contrary $l \subseteq K_0$. Indeed, assume by contradiction that $\xi \in l \setminus K_0$. Then there exists $\eta > 0$ such that $K_{\varepsilon_{n_h}} \cap B_\eta(\xi) = \emptyset$ for h large, against (5.1).

In view of (5.3) and (5.4) we deduce that for $\varepsilon \to 0$ and for every $m \geq 1$

$$K_\varepsilon \cap \bar{Q}_m(0) \to l \cap \bar{Q}_m(0) \qquad \text{in the Hausdorff metric},$$

i.e., convergence (5.2) holds true.

Step 2: a blow up argument. Let $x \in K$ satisfy (5.1) and (5.2). Let us show that the density of x with respect to Ω cannot be zero.

Up to a rototranslation, we may assume $x = 0$ and that the approximate tangent line is horizontal. We write K_ε and l in place of $K_{x,\varepsilon}$ and l_x. By contradiction, assume that the origin has density zero for Ω, i.e., for $\varepsilon \to 0^+$

$$1_{\Omega_\varepsilon} \to 0 \qquad \text{strongly in } L^1_{loc}(\mathbb{R}^2), \tag{5.6}$$

where $\Omega_\varepsilon := \frac{1}{\varepsilon}\Omega$.

From (5.1) we have that

$$\mathcal{H}^1(K_\varepsilon \cap Q_1(0)) \to \mathcal{H}^1(l \cap Q_1(0)) = 1. \tag{5.7}$$

For every $\eta > 0$, by (5.2) we deduce that for ε small enough

$$K_\varepsilon \cap \bar{Q}_1(0) \subseteq S_\eta := \{x \in \bar{Q}_1(0) \,:\, |x_2| < \eta\}.$$

In particular we get that $\partial\Omega_\varepsilon \cap \bar{Q}_1(0) \subset S_\eta$, so that in view of (5.6) we get that for ε small enough

$$\Omega_\varepsilon \cap Q_1(0) \subset S_\eta. \tag{5.8}$$

Assume that for every $|a| < 1$ we have for ε small enough

$$\Omega_\varepsilon \cap Q_1(0) \cap \{(a,s) \,:\, s \in \mathbb{R}\} \neq \emptyset, \tag{5.9}$$

i.e. the intersection of $\Omega_\varepsilon \cap Q_1(0)$ with the vertical line through $(a,0)$ in not empty. In particular thanks to (5.8) we deduce that

$$K_\varepsilon^a := K_\varepsilon \cap Q_1(0) \cap \{(a,s) \,:\, s \in \mathbb{R}\}$$

contains at least two points. Then we get in view of the area formula

$$\liminf_{\varepsilon\to 0} \mathcal{H}^1(K_\varepsilon \cap Q_1(0)) \geq \liminf_{\varepsilon\to 0} \int_{-1/2}^{1/2} \mathcal{H}^0(K_\varepsilon^a)\, da \geq 2 \tag{5.10}$$

against (5.7).

If (5.9) is violated for some $a_0 \in]-1,1[$ along a sequence $\varepsilon_n \to 0$, then we have that

$$\Omega_{\varepsilon_n} \cap Q_1(0) \cap \{(a,s) \,:\, s \in \mathbb{R}\} \neq \emptyset,$$

for every $a \neq a_0$: otherwise, if for example $a > a_0$, then $\Omega_{\varepsilon_n} \cap Q_1(0) \cap \{x \in \mathbb{R}^2 \,:\, a_0 < x_1 < a\}$ would be separated from the rest of Ω_{ε_n}, against its connectedness. We can therefore repeat the argument of (5.10) along the sequence ε_n and get again a contradiction. $\square$

Remark 5.3. Note that the connectedness assumption for Ω in Proposition 5.2 is essential for the conclusion. Indeed consider a sequence of points q_n in the unit square $Q_1(0) \subseteq \mathbb{R}^2$ such that for every $k \geq 1$

$$\bigcup_{n\geq k} \{q_n\} \subset \left\{ x = (x_1,x_2) \in \mathbb{R}^2 \,:\, 1 - \frac{1}{k} < x_1 < 1 \right\} \qquad \text{and} \qquad l := \{1\} \times [0,1] \subseteq \overline{\bigcup_{n\geq k} \{q_n\}}.$$

We can find $r_n > 0$ so small that the disks $B_{r_n}(q_n)$ are mutually disjoint, and such that setting

$$\Omega := \bigcup_n B_{r_n}(q_n),$$

then the points of $l \subset \partial\Omega$ are of density zero with respect to Ω.

We are now in a position to compare the Robin perimeter with the length of the boundary.

Proposition 5.4. *Let $\Omega \in \mathcal{A}_m(\mathbb{R}^2)$. Then*

$$\mathcal{H}^1(\Gamma_\Omega) \leq Per_{\mathcal{R}}(\Omega) \leq 2\mathcal{H}^1(\Gamma_\Omega). \tag{5.11}$$

Proof. Let $\{\Omega^i\}_{i\in\mathbb{N}}$ denote the family of connected components of Ω. By Proposition 5.2 we have that $\mathcal{H}^1$-a.e. $x \in \partial\Omega^i$ has not density zero with respect to Ω^i so that we deduce

$$|\gamma_r(1_\Omega)(x)| + |\gamma_l(1_\Omega)(x)| \geq |\gamma_r(1_{\Omega^i})(x)| + |\gamma_l(1_{\Omega^i})(x)| \geq 1.$$

Therefore

$$Per_{\mathcal{R}}(\Omega) = \int_{\Gamma_\Omega} [|\gamma_r(1_\Omega)| + |\gamma_l(1_\Omega)|]\, d\mathcal{H}^1 \geq \mathcal{H}^1(\Gamma_\Omega).$$

The second inequality comes from the fact that we have always $|\gamma_r(1_\Omega)(x)| + |\gamma_l(1_\Omega)(x)| \leq 2$. $\square$

The following lower semicontinuity result for the Robin perimeter holds true.

Proposition 5.5 (**Lower semicontinuity of** $Per_{\mathcal{R}}$). *Let* $(\Omega_n)_{n\in\mathbb{N}}$ *be a sequence of equibounded sets in* $\mathcal{A}_m(\mathbb{R}^2)$ *with* $\Omega_n \stackrel{\mathcal{H}^c}{\to} \Omega$ *for some* $\Omega \in \mathcal{A}_m(\Omega)$. *Then*

$$Per_{\mathcal{R}}(\Omega) \le \liminf_n Per_{\mathcal{R}}(\Omega_n).$$

Proof. We may assume $Per_{\mathcal{R}}(\Omega_n) \le C$ for some $C > 0$. Thanks to Proposition 5.4, we deduce that $\mathcal{H}^1(\Gamma_{\Omega_n}) \le C$. The result then follows from Proposition 4.10 applied to the functions 1_{Ω_n} which converge to 1_Ω thanks to Theorem 4.8. □

Remark 5.6. Notice that the definition of $Per_{\mathcal{R}}$ can be naturally extended to general open sets in $\mathbb{R}^d$ satisfying (3.1). However, even in dimension two (outside the class $\mathcal{A}_m(\mathbb{R}^2)$), lower semicontinuity results with respect to $\mathcal{H}^c$-convergence fail (the case of weaker notions of convergence is not clear): counterexamples in dimension two involve sequences of inner cracks which violate Gołąb lower semicontinuity theorem.

The following lemma deals with the convergence for the Robin energies under the convergence of the Robin perimeter.

Proposition 5.7 (**Continuity of the Robin energy**). *Let* $(\Omega_n)_{n\in\mathbb{N}}$ *be an equibounded sequence in* $\mathcal{A}_m(\mathbb{R}^2)$ *such that*

$$\Omega_n \stackrel{\mathcal{H}^c}{\to} \Omega \qquad \text{and} \qquad Per_{\mathcal{R}}(\Omega_n) \to Per_{\mathcal{R}}(\Omega)$$

for some $\Omega \in \mathcal{A}_m(\mathbb{R}^2)$. *Let* $u_n \in \mathcal{R}(\Omega_n)$ *and* $u \in \mathcal{R}(\Omega)$ *be given according to Proposition 4.10 with*

$$\limsup_n \|u_n\|_\infty < +\infty.$$

Then

$$\lim_n \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2\right] d\mathcal{H}^1 = \int_{\Gamma_\Omega} \left[|\gamma_r(u 1_\Omega)|^2 + |\gamma_l(u 1_\Omega)|^2\right] d\mathcal{H}^1.$$

Proof. In view of (5.11) we get

$$\limsup_n \mathcal{H}^1(\Gamma_{\Omega_n}) \le \limsup_n Per_{\mathcal{R}}(\Omega_n) = Per_{\mathcal{R}}(\Omega) \le 2\mathcal{H}^1(\Gamma_\Omega) < +\infty \tag{5.12}$$

so that we may apply Proposition 4.10 to get

$$\int_{\Gamma_\Omega} \left[|\gamma_r(u 1_\Omega)|^2 + |\gamma_l(u 1_\Omega)|^2\right] d\mathcal{H}^1 \le \liminf_n \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2\right] d\mathcal{H}^1. \tag{5.13}$$

Let us consider

$$v_n := \sqrt{4M^2 - u_n^2},$$

where

$$M > \limsup_n \|u_n\|_\infty.$$

Clearly $v_n \in H^1(\Omega_n)$ with

$$\nabla v_n = -\frac{u_n \nabla u_n}{\sqrt{4M^2 - u_n^2}}.$$

Moreover we can write

$$v_n 1_{\Omega_n} = \Phi((4M^2 - u_n^2)1_{\Omega_n}),$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz map with $\Phi(0) = 0$ and $\Phi(s) = \sqrt{s}$ if $s \ge 3M^2$, so that by the chain rule in BV [1, Theorem 3.96] we infer $v_n 1_{\Omega_n} \in BV(\mathbb{R}^2)$. Moreover we have

$$\gamma_r(v_n) = \sqrt{4M^2 \gamma_r(1_{\Omega_n}) - \gamma_r(u_n)^2},$$

and a similar formula holds for the left traces. We thus conclude $v_n \in \mathcal{R}(\Omega_n)$.

Similarly

$$v := \sqrt{4M^2 - u^2} \in \mathcal{R}(\Omega)$$

and

$$\gamma_r(v) = \sqrt{4M^2 \gamma_r(1_\Omega) - \gamma_r(u)^2},$$

with an analogous relation for the left traces.

In view of (5.12) and thanks to Theorem 4.8 we have that

$$1_{\Omega_n} \to 1_\Omega \quad \text{strongly in } L^2(\mathbb{R}^2),$$

so that we infer

$$v_n 1_{\Omega_n} \to v 1_\Omega \qquad \text{strongly in } L^2(\mathbb{R}^2)$$

and

$$\nabla v_n 1_{\Omega_n} \rightharpoonup \nabla v 1_\Omega \qquad \text{weakly in } L^2(\mathbb{R}^2;\mathbb{R}^2).$$

Moreover

$$\limsup_n \left[||\nabla v_n||^2_{L^2(\Omega_n;\mathbb{R}^2)} + \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(v_n)|^2 + |\gamma_l(v_n)|^2 \right] \, d\mathcal{H}^1 \right] < +\infty.$$

From Proposition 4.10 we deduce

$$\begin{aligned} &4M^2 Per_{\mathcal{R}}(\Omega) - \int_{\Gamma_\Omega} \left[|\gamma_r(u1_\Omega)|^2 + |\gamma_l(u1_\Omega)|^2 \right] \, d\mathcal{H}^1 \\ &\qquad \le \liminf_n \left[4M^2 Per_{\mathcal{R}}(\Omega_n) - \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2 \right] \, d\mathcal{H}^1 \right] \end{aligned}$$

which gives

$$\int_{\Gamma_\Omega} \left[|\gamma_r(u1_\Omega)|^2 + |\gamma_l(u1_\Omega)|^2 \right] \, d\mathcal{H}^1 \ge \limsup_n \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2 \right] d\mathcal{H}^1,$$

in view of the assumption $Per_{\mathcal{R}}(\Omega_n) \to Per_{\mathcal{R}}(\Omega)$. The conclusion follows taking into account (5.13). □

We conclude the section by showing that the convergence of the length of the boundaries entails that of the Robin perimeter, providing thus a simple geometric property which ensures the stability of $Per_{\mathcal{R}}$.

Proposition 5.8. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be an equibounded sequence in $\mathcal{A}_m(\mathbb{R}^2)$ such that*

$$\Omega_n \stackrel{\mathcal{H}^c}{\to} \Omega \qquad \text{and} \qquad \mathcal{H}^1(\Gamma_{\Omega_n}) \to \mathcal{H}^1(\Gamma_\Omega). \tag{5.14}$$

Then

$$Per_{\mathcal{R}}(\Omega_n) \to Per_{\mathcal{R}}(\Omega).$$

Proof. Let $R > 0$ be such that $\Omega_n \subset B_R(0)$ for every $n \in \mathbb{N}$, and set $D := B_{2R}(0)$. Thanks to Theorem 4.8 we have that

$$1_{\Omega_n} \to 1_\Omega \quad \text{strongly in } L^2(D).$$

In view of Proposition 5.5 we obtain that

$$Per_{\mathcal{R}}(\Omega) \le \liminf_n Per_{\mathcal{R}}(\Omega_n). \tag{5.15}$$

To prove the reverse inequality, we proceed as in the proof of Lemma 4.10. Let $\{\Omega_n^i\}_{i\in\mathbb{N}}$ denote the family of connected components of Ω_n. Up to a subsequence, we may assume that for every $i \in \mathbb{N}$ we have

$$\Omega_n^i \stackrel{\mathcal{H}^c}{\to} A^i,$$

where A^i are union of connected components of Ω with $\Omega = \cup_i A^i$. Moreover, we may assume

$$\partial\Omega_n^i \stackrel{\mathcal{H}}{\to} K^i$$

for some $K^i \in \mathcal{K}_m(\mathbb{R}^2)$ (see (2.1)). From Proposition 4.4, we have $\partial A^i \subseteq K^i$, while from Corollary 4.6 we have

$$1_{\Omega_n^i} \to 1_{A^i} \qquad \text{strongly in } L^1(\mathbb{R}^2).$$

Gołąb Theorem (see Theorem 2.4) entails that

$$\mathcal{H}^1(\cup_{i\le k}\partial A^i) \le \liminf_n \mathcal{H}^1(\cup_{i\le k}\partial\Omega_n^i) = \liminf_n \left[\mathcal{H}^1(\Gamma_{\Omega_n}) - \mathcal{H}^1(\Gamma_{\Omega_n} \setminus \cup_{i\le k}\partial\Omega_n^i) \right].$$

Recalling that $\Gamma_\Omega = \cup_i \partial A^i$, the convergence (5.14) implies

$$\limsup_n \mathcal{H}^1(\Gamma_{\Omega_n} \setminus \cup_{i\le k}\partial\Omega_n^i) \le e_k \tag{5.16}$$

where $e_k \to 0$. Let us consider

$$v_n^k := 1 - 1_{\cup_{i\le k}\Omega_n^i},$$

which we can interpret as a function in $BV(D) \cap H^1(D \setminus \cup_{i\le k}\partial\Omega_n^i)$ for which

$$v_n^k \to v^k := 1 - 1_{\cup_{i\le k}A^i} \qquad \text{strongly in } L^2(D),$$

and

$$\nabla v_n^k \rightharpoonup 0 \qquad \text{weakly in } L^2(D;\mathbb{R}^2).$$

By the lower semicontinuity of Proposition 4.9 we get

$$\begin{aligned} \int_{\cup_{i\le k}\partial A^i} \left[|\gamma_r(v^k)|^2 + |\gamma_l(v^k)|^2\right] d\mathcal{H}^1 &\le \int_{\cup_{i\le k}K^i} \left[|\gamma_r(v^k)|^2 + |\gamma_l(v^k)|^2\right] d\mathcal{H}^1 \\ &\le \liminf_n \int_{\cup_{i\le k}\partial\Omega_n^i} \left[|\gamma_r(v_n^k)|^2 + |\gamma_l(v_n^k)|^2\right] d\mathcal{H}^1. \end{aligned} \tag{5.17}$$

Notice that

$$\gamma_r(v_n^k) = 1 - \gamma_r(1_{\cup_{i\le k}\Omega_n^i}) \qquad \mathcal{H}^1\text{-a.e. on } \cup_{i\le k}\partial\Omega_n^i$$

with the same relation for the left traces, and that analogous relations hold for v^k on $\cup_{i\le k}\partial A^i$. Since the traces involved are either 0 or 1, we can remove the squares in (5.17) and write in view convergence (5.14) and inequality (5.16),

$$\begin{aligned} &\mathcal{H}^1(\cup_{i\le k}\partial A^i) - \int_{\cup_{i\le k}\partial A^i} \left[|\gamma_r(1_{\cup_{i\le k}A^i}) + |\gamma_l(1_{\cup_{i\le k}A^i})|\right] d\mathcal{H}^1 \\ &\qquad \le \mathcal{H}^1(\Gamma_\Omega) - \limsup_n \int_{\cup_{i\le k}\partial\Omega_n^i} \left[|\gamma_r(1_{\cup_{i\le k}\Omega_n^i})| + |\gamma_l(1_{\cup_{i\le k}\Omega_n^i})|\right] d\mathcal{H}^1 \\ &\qquad\qquad \le \mathcal{H}^1(\Gamma_\Omega) - \limsup_n \int_{\Gamma_{\Omega_n}} [|\gamma_r(1_{\Omega_n})| + |\gamma_l(1_{\Omega_n})|]\, d\mathcal{H}^1 + 2e_k \end{aligned}$$

from which, letting $k \to \infty$ we conclude

$$\begin{aligned} \limsup_n Per_{\mathcal{R}}(\Omega_n) &= \limsup_n \int_{\Gamma_{\Omega_n}} [|\gamma_r(1_{\Omega_n})| + |\gamma_l(1_{\Omega_n})|]\, d\mathcal{H}^1 \\ &\le \int_{\Gamma_\Omega} [|\gamma_r(1_\Omega)| + |\gamma_l(1_\Omega)|]\, d\mathcal{H}^1 = Per_{\mathcal{R}}(\Omega). \end{aligned} \tag{5.18}$$

The conclusion follows gathering (5.15) and (5.18). □

Remark 5.9. Notice that the convergence of the Robin perimeter does not entail in general that of the length of the boundaries. For example one can consider the case of $\Omega_n := B_R(0) \setminus C_n$, where C_n is a cavity shrinking smoothly to the diameter $L := [-R/2, R/2] \times \{0\}$. In this case we have for $\Omega := B_R(0) \setminus L$

$$Per_{\mathcal{R}}(\Omega_n) = \mathcal{H}^1(\partial\Omega_n) \to 2\pi R + 2R = Per_{\mathcal{R}}(\Omega)$$

while

$$\mathcal{H}^1(\partial\Omega) = 2\pi R + R.$$

6. Stability results for the generalized Robin-Laplacian

In this section we formulate our main stability result for the Robin-Laplace problems of Definition (3.9) on open sets varying in the class $\mathcal{A}_m(\mathbb{R}^2)$.

Theorem 6.1 (The stability result). *Let $D \subset \mathbb{R}^2$ an open bounded set, and let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_m(\mathbb{R}^2)$ and $\Omega \in \mathcal{A}_m(\mathbb{R}^2)$ such that $\Omega_n, \Omega \subset D$,*

$$\Omega_n \overset{\mathcal{H}^c}{\to} \Omega \qquad \text{and} \qquad Per_{\mathcal{R}}(\Omega_n) \to Per_{\mathcal{R}}(\Omega).$$

Then for all $f \in L^2(D)$ we have that

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(D),$$

and

$$\nabla u_n 1_{\Omega_n} \to \nabla u 1_\Omega \qquad \text{strongly in } L^2(D;\mathbb{R}^2),$$

where $u_n \in \mathcal{R}(\Omega_n)$ and $u \in \mathcal{R}(\Omega)$ are the solutions of the Robin problems associated to f on Ω_n and Ω respectively

Proof. For every $A \in \mathcal{A}_m(\mathbb{R}^2)$ with $A \subset D$, let us consider

$$F_A : L^2(D) \to \mathbb{R} \cup \{+\infty\}$$

such that

$$F_A(v) := \frac{1}{2}\int_A |\nabla v|^2\,dx + \frac{\beta}{2}\int_{\Gamma_A} \left[|\gamma_r(v1_A)|^2 + |\gamma_l(v1_A)|^2\right]\,d\mathcal{H}^1 - \int_A fv\,dx$$

if $v = 0$ a.e. on $D \setminus A$ and $v_{|A} \in \mathcal{R}(A)$, while

$$F_A(v) := +\infty$$

otherwise in $L^2(D)$. By Theorem 3.10 we have that v is a minimizer of F_A if and only if $v = 0$ a.e. on $D \setminus A$ and $v_{|A}$ is the solution of the Robin problem relative to f.

We claim that

$$F_{\Omega_n} \to F_\Omega \tag{6.1}$$

in the sense of Γ-convergence with respect to the strong topology of $L^2(D)$ (see Section 2). Assuming the claim, the stability result can be proved as follows. Let u_n be the solution of the Robin problem on Ω_n relative to f. First of all, thanks to Lemma 3.5 we may write

$$\frac{1}{2}\int_{\Omega_n} |\nabla u_n|^2\,dx + \frac{\beta}{2}\int_{\Gamma_{\Omega_n}} \left[\gamma_r(u_n 1_{\Omega_n})^2 + \gamma_l(u_n 1_{\Omega_n})^2\right]\,d\mathcal{H}^1 \le \|f\|_{L^2(D)}\|u_n\|_{L^2(\Omega_n)}$$
$$\le C\left[\frac{1}{2}\int_{\Omega_n} |\nabla u_n|^2\,dx + \frac{\beta}{2}\int_{\Gamma_{\Omega_n}} \left[\gamma_r(u_n 1_{\Omega_n})^2 + \gamma_l(u_n 1_{\Omega_n})^2\right]\,d\mathcal{H}^1\right]^{\frac{1}{2}},$$

where C is independent of n (since $|\Omega_n| \le |D|$), so that

$$\limsup_n \left[\|\nabla u_n\|_{L^2(\Omega_n;\mathbb{R}^2)} + \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2\right]\,d\mathcal{H}^1\right] < +\infty.$$

From Proposition 4.10 we get that there exists $u \in \mathcal{R}(\Omega)$ such that, up to subsequence

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(D),$$

$$\nabla u_n 1_{\Omega_n} \rightharpoonup \nabla u 1_\Omega \quad \text{weakly in } L^2(\mathbb{R}^2;\mathbb{R}^2), \tag{6.2}$$

and

$$\int_{\Gamma_\Omega} \left[|\gamma_r(u1_\Omega)|^2 + |\gamma_l(u1_\Omega)|^2\right]\,d\mathcal{H}^1 \le \liminf_n \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2\right] d\mathcal{H}^1.$$

By the Γ-convergence result (6.1), we infer that $u1_\Omega$ is the minimizer of F_Ω, so that u is the solution of the Robin problem on Ω relative to f. Since

$$F_{\Omega_n}(u_n) \to F_\Omega(u).$$

the weak convergence of (6.2) is indeed strong. Finally, since there is no need to pass to a subsequence, the result follows.

In order to conclude the proof, let us prove claim (6.1) by checking the inequalities of Γ-convergence in two separate steps.

Step 1: Γ-liminf inequality. Let us check that for $v_n \to v$ strongly in $L^2(D)$ we have

$$F_\Omega(v) \le \liminf_n F_{\Omega_n}(v_n). \tag{6.3}$$

It is not restrictive to assume, possibly passing to a subsequence, that

$$F_{\Omega_n}(v_n) \le C.$$

We infer that $v_n = u_n 1_{\Omega_n}$ for some $u_n \in \mathcal{R}(\Omega_n)$ with

$$\limsup_n \left[\|\nabla u_n\|^2_{L^2(\Omega_n;\mathbb{R}^2)} + \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2 \right] \, d\mathcal{H}^1 \right] < +\infty.$$

From Proposition 4.10 we deduce that there exists $u \in \mathcal{R}(\Omega)$ such that $v = u 1_\Omega$ with

$$\nabla u_n 1_{\Omega_n} \rightharpoonup \nabla u 1_\Omega \quad \text{weakly in } L^2(\mathbb{R}^2;\mathbb{R}^2),$$

and

$$\int_{\Gamma_\Omega} \left[|\gamma_r(u 1_\Omega)|^2 + |\gamma_l(u 1_\Omega)|^2 \right] \, d\mathcal{H}^1 \le \liminf_n \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2 \right] d\mathcal{H}^1.$$

We conclude that (6.3) holds true.

Step 2: Γ-limsup inequality. Given $v \in L^2(D)$, let us check that there exists $v_n \to v$ strongly in $L^2(D)$ such that

$$\limsup_n F_{\Omega_n}(v_n) \le F_\Omega(v). \tag{6.4}$$

It is not restrictive to assume $F_\Omega(v) < +\infty$, so that $v = u 1_\Omega$ for some $u \in \mathcal{R}(\Omega)$. Proceeding by truncation and employing a diagonal argument, we may also assume that $\|u\|_\infty \le M$.

From Theorem 4.8 we have

$$1_{\Omega_n} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^2).$$

Then the sequence of domains $(\Omega_n)_{n\in\mathbb{N}}$ fits into the framework of [11, Theorem 4.1, Remark 5.2], so that we can find $u_n \in H^1(\Omega_n)$ such that

$$u_n 1_{\Omega_n} \to u 1_\Omega \quad \text{strongly in } L^2(D) \tag{6.5}$$

and

$$\nabla u_n 1_{\Omega_n} \to \nabla u 1_\Omega \quad \text{strongly in } L^2(D,\mathbb{R}^2). \tag{6.6}$$

By truncation, we may assume that also u_n is such that $\|u_n\|_\infty \le M$. As a consequence we get $u_n \in \mathcal{R}(\Omega_n)$. Thanks to the convergence of the perimeters, in view of Proposition 5.7 we infer

$$\lim_n \int_{\Gamma_{\Omega_n}} \left[|\gamma_r(u_n 1_{\Omega_n})|^2 + |\gamma_l(u_n 1_{\Omega_n})|^2 \right] \, d\mathcal{H}^1 = \int_{\Gamma_\Omega} \left[|\gamma_r(u 1_\Omega)|^2 + |\gamma_l(u 1_\Omega)|^2 \right] \, d\mathcal{H}^1. \tag{6.7}$$

Gathering (6.5), (6.6) and (6.7), we get that the Γ-limsup inequality (6.4) follows. □

The previous stability result is fundamental to obtain the convergence of the resolvent of the generalized Robin-Laplace operators and of the related spectra. Let us consider the bounded linear operators on $L^2(D)$

$$R_{\Omega,\beta} : L^2(D) \to L^2(D)$$

such that

$$R_{\Omega,\beta}(f) := u_f 1_\Omega,$$

where u_f is the solution of the Robin problem on Ω relative to f.

Theorem 6.2 (**Convergence of the resolvent operator and stability of the spectrum).** *Under the assumptions of Theorem 6.1 we have*

$$R_{\Omega_n,\beta} \longrightarrow R_{\Omega,\beta} \quad \textit{strongly in } \mathcal{L}(L^2(D)).$$

In particular, for every $k \geq 1$ we have

$$\lambda_{k,\beta}(\Omega_n) \longrightarrow \lambda_{k,\beta}(\Omega),$$

where $\lambda_{k,\beta}(A)$ is the k-th eigenvalue on the domain A with parameter β.

Proof. The convergence in the operator norm is equivalent to the following relation

$$\sup_{f\in L^2(D),\|f\|_{L^2(D)}\leq 1} \|R_{\Omega_n,\beta}(f) - R_{\Omega,\beta}(f)\|_{L^2(D)} \to 0. \tag{6.8}$$

The convergence of the eigenvalues is then a standard consequence of the convergence of the resolvents (see e.g. [18, Lemma XI.9.5]).

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence such that $\|f_n\|_{L^2(D)} \leq 1$ and

$$\|R_{\Omega_n,\beta}(f_n) - R_{\Omega,\beta}(f_n)\|_{L^2(D)} = \sup_{f\in L^2(D),\|f\|_{L^2(D)}\leq 1} \|R_{\Omega_n,\beta}(f) - R_{\Omega,\beta}(f)\|_{L^2(D)}.$$

The existence of f_n follows easily from the compactness property of solutions given by Theorem 3.10. Since f_n is bounded in $L^2(D)$, we may assume that up to a subsequence there exists $f \in L^2(D)$, such that

$$f_n \rightharpoonup f \qquad \text{weakly in } L^2(D).$$

We may write

$$\begin{aligned}
&\|R_{\Omega_n,\beta}(f_n) - R_{\Omega,\beta}(f_n)\|_{L^2(D)} \\
&\leq \|R_{\Omega_n,\beta}(f_n) - R_{\Omega_n,\beta}(f)\|_{L^2(D)} + \|R_{\Omega_n,\beta}(f) - R_{\Omega,\beta}(f)\|_{L^2(D)} + \|R_{\Omega,\beta}(f) - R_{\Omega,\beta}(f_n)\|_{L^2(D)} \\
&\qquad\qquad =: I_n + II_n + III_n.
\end{aligned}$$

From Theorem 6.1 we get that

$$II_n = \|R_{\Omega_n,\beta}(f) - R_{\Omega,\beta}(f)\|_{L^2(D)} \to 0.$$

We observe that III_n is a particular case of I_n with $\Omega_n = \Omega$ fixed, hence to conclude it is sufficient to prove that $I_n \to 0$. We prove it in two steps.

Step 1. We have

$$R_{\Omega_n,\beta}(f_n) \rightharpoonup R_{\Omega_n,\beta}(f) \qquad \text{weakly in } L^2(D). \tag{6.9}$$

Notice that $R_{\Omega,\beta}$ is self-adjoint for every Ω: indeed from the weak formulation of the problem we have for every $f, g \in L^2(D)$

$$\int_D R_{\Omega,\beta}(f) g \, dx = \mathcal{L}_\beta(R_{\Omega,\beta}(g), R_{\Omega,\beta}(f)) = \mathcal{L}_\beta(R_{\Omega,\beta}(f), R_{\Omega,\beta}(g)) = \int_\Omega f R_{\Omega,\beta}(g)\, dx.$$

We can then write for every $\varphi \in L^2(D)$

$$\int_D [R_{\Omega_n,\beta}(f_n) - R_{\Omega_n,\beta}(f)] \cdot \varphi \, dx = \int_D R_{\Omega_n,\beta}(f_n - f) \cdot \varphi \, dx = \int_D (f_n - f) \cdot R_{\Omega_n,\beta}(\varphi)\, dx \to 0,$$

the last convergence coming from the fact that

$$R_{\Omega_n,\beta}(\varphi) \to R_{\Omega,\beta}(\varphi) \qquad \text{strongly in } L^2(D)$$

thanks to Theorem 6.1. Convergence (6.9) is thus proved.

Step 2. By Theorem 3.10 we have

$$\|R_{\Omega_n,\beta}(f_n) - R_{\Omega_n,\beta}(f)\|_{\mathcal{R}(\Omega_n)} = \|R_{\Omega_n,\beta}(f_n - f)\|_{\mathcal{R}(\Omega_n)} \leq C(\beta, |\Omega_n|)\|f_n - f\|_{L^2(D)} \leq C,$$

where C does not depend on n, as $|\Omega_n| \to |\Omega|$. Then from Proposition 4.10 we infer that up to a subsequence

$$R_{\Omega_n,\beta}(f_n) - R_{\Omega_n,\beta}(f) \to v1_\Omega \qquad \text{strongly in } L^2(D)$$

for some $v \in \mathcal{R}(\Omega)$. But from Step 1 we get that $v = 0$, so that the conclusion follows. $\square$

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