Separation functions and mild topologies

Andrea C. G. Mennucci*

November 9, 2021

Abstract

Given M and N Hausdorff topological spaces, we study topologies on the space $C^0(M;N)$ of continuous maps $f : M \to N$. We review two classical such topologies, the *"strong"* and *"weak"* topology.

We propose a definition of *"mild topology"* that is coarser than the *"strong"* and finer than the *"weak"* topology. We compare properties of these three topologies, in particular with respect to proper continuous maps $f : M \to N$, and affine actions when $N = \mathbb{R}^n$.

To define the *"mild topology"* we use *"separation functions"*.

"Separation functions" are somewhat similar to the usual *"distance function* $d(x, y)$*"* in metric spaces (M, d), but have weaker requirements.

Separation functions are used to define *pseudo balls* that are a global base for a T2 topology; with some additional hypotheses we can define *"set separation functions"* that prove that the topology is T6; with some more hypotheses it is possible to prove that the topology is metrizable.

We provide some examples of usages of *separation functions*: one is the aforementioned case of the *mild topology* on $C^0(M;N)$. Another one can be used on topological manifolds.

1 Introduction

Let M and N be Hausdorff topological spaces, with topologies τ_M and τ_N. (Sometimes we may also assume that M or N be metric spaces with distances d_M or respectively d_N: then τ_M, or respectively τ_N, will be the associated topology).

In the third part §3 of the paper we will discuss a topology for the space $C^0(M;N)$ of continuous maps $f : M \to N$, that we will call *"mild topology"*.

To define and discuss the properties of this topology, we have developed a novel method, by using a family of *"separation functions"*; it is presented in the second part §2 of the paper.

Separation functions are somewhat similar to the usual *"distance function* $d(x, y)$*"* in metric spaces (M, d), but have weaker hypotheses (so they can be more manageable in some contexts). *Separation functions* are used to define *pseudo balls* that are a global base for a T2 topology (Thm. 2.3); with some additional hypotheses we will define in §2.5 *"set separation functions"* (similar to *set distance functions*) that prove that the topology is T6; with some more hypotheses it is possible to prove that the topology is metrizable (Theorems 2.21 and 2.30). Such will be the case for the *mild topology*.

We will also discuss the example of the *Sorgenfrey line* in §2.3; and in §2.12 how *"separation functions"* can be easily defined for topological manifolds, starting from the atlas of the manifold.

But first and foremost, we wish to explain why we may find useful a new topology on $C^0(M;N)$.

To this end, we start by recalling some fundamental definitions.

*Scuola Normale Superiore, Pisa, Italy (andrea.mennucci@sns.it)

[0]Keywords: continuous functions, proper maps, strong topology, weak topology, metrization, quasi metrics, asymmetric metrics, topological manifolds

[0]MSC 2020: 54C10 54C35 54E35 58D15 58D19 46T05 46T10

1.1 Topologies for continuous maps

We define the "graph" of the function $f : M \to N$ as

$$\text{graph}(f) \stackrel{\text{def}}{=} \{(x, y) \in M \times N : y = f(x)\} \quad .$$

We distinguish two fundamental examples of topologies for $C^0 = C^0(M; N)$.

Definition 1.1. *The* ***compact-open*** *topology, generated by sets of the form*

$$\{f \in C^0 : \ f(K) \subseteq U\}$$

where $K \subseteq M$ is compact and $U \subseteq N$ is open. (This collection of sets is a subbase *for the topology, but it does not always form a* base *for a topology.)*

It is also called the "topology of uniform convergence on compact sets" *or the* ***"weak topology"*** *in [2].*

We will write $C^0_W(M; N)$ to denote this topological space.

Definition 1.2. *The* ***graph topology*** *generated by sets of the form*

$$\{f \in C^0(M; N) : \textit{graph}(f) \subseteq U\}$$

where U runs through all open sets in $M \times N$. It is also called "wholly open topology" *in [4];* "fine" *or* "Whitney" *or* ***"strong topology"*** *in [2]. We will write $C^0_S(M; N)$ to denote this topological space.*

An equivalent definition of the strong topology can be formulated under additional hypotheses.

Proposition 1.3 (41.6 in [4], or Sec. 2.4 in [2])**.** *If M is paracompact and (N, d_N) is a metric space, then for $f \in C^0(M, N)$ the sets*

$$\{g \in C^0(M, N) : d_N(g(x), f(x)) < \varepsilon(x) \forall x \in M\}$$

form a basis of neighborhoods for the graph topology*, where ε runs through all positive continuous functions on M.*

Another way to state this result [1] *is to define the distances*

$$d_\varepsilon(f, g) \stackrel{\textit{def}}{=} \sup_{x \in M} \varepsilon(x) d_N(g(x), f(x))$$

then the topology generated by all these distances is the graph topology.

It is possible to define similar concepts for $C^r(M; N)$, the space of r times differentiable maps between two differentiable manifolds M, N; we do not detail the discussion.

The above topologies are invariant in this sense.

Proposition 1.4. *If $\Phi_M : \tilde{M} \to M$ and $\Phi_N : N \to \tilde{N}$ are homeomorphisms, then the map*

$$f \mapsto \Phi_N \circ f \circ \Phi_M$$

is a homeomorphism between $C^0(M; N)$ and $C^0(\tilde{M}; \tilde{N})$, where both spaces are endowed with the "weak", or respectively, the "strong" topology.

1.2 When $N = \mathbb{R}^n$

Let us suppose in this section that N is the standard Euclidean space $\mathbb{R}^n$; and that M is Hausdorff, locally compact and second countable [2].

We recall that any second countable Hausdorff space that is locally compact is paracompact, so the result 1.3 applies in the current context; moreover there exists a countable, locally finite, covering of open sets each with compact closure.

[1] Exercise 3 in Sec. 2.4 in [2]; to be compared with the "counterexample 1.1.8" in [1].

[2] (*I.e.* it admits a countable basis of open sets)

Definition 1.5. *$C^0_{loc}(M,\mathbb{R}^n)$ is the Frechét space where the topology is generated by the seminorms*

$$[f]_K = \sup_{x\in K} |f(x)|$$

for $K \subseteq M$ compact. If M is compact then it becomes the usual Banach space $C^0(M,\mathbb{R}^n)$ with the norm

$$\|f\| = \sup_{x\in M} |f(x)| \quad .$$

Proposition 1.6. *In the above hypotheses, $C^0_W(M;\mathbb{R}^n)$ coincides with $C^0_{loc}(M,\mathbb{R}^n)$.*

We can define also another topology.
For $K \subseteq M$ compact we define the subset

$$V_{0,K} = \{f \in C^0(M;\mathbb{R}^n) : \text{supp}(f) \subseteq K\}$$

of C^0 functions with support in K.
Each such $V_{0,K}$ is a closed subspace of C^0_{loc} so it is a Frechét space with the induced topology.
We can then define this topology.

Definition 1.7 (C^0_c topology)**.** *The $C^0_c(M;\mathbb{R}^n)$ topology is the strict inductive limit* [3] *with respect to the inclusions*

$$V_{0,K} \to C^0$$

for $K \subseteq M$ compact. A set W is open in the $C^0_c(M;\mathbb{R}^n)$ topology when for all $K \subseteq M$ compact, $W \cap V_{0,K}$ is open in $V_{0,K}$.

1.3 Properties

The weak topology has nice metrization properties.

Proposition 1.8. *Let N be metrizable with a complete metric, and let M be locally compact and second countable. Then $C^0_W(M,N)$ has a complete metric.*

This is proven in Theorem 4.1 in Sec. 2.4 in [2].

We ponder on these remarks, taken from [2].

Remark 1.9. *The topological space $C^0_S(M;N)$ resulting from the strong topology is the same as $C^0_W(M;N)$ if M is compact. If M is not compact, however, $C^0_S(M;N)$ can be an extremely large topology; for example when M,N are differentiable finite dimensional manifolds (of positive dimension), then it is not metrizable and in fact does not have a countable local base, at any point; and it has uncountably many connected components.*

In particular, if M is not compact, $C^0_S(M,\mathbb{R}^n)$ cannot be a topological vector space; but the connected component containing $f \equiv 0$ coincides with $C^0_c(M,\mathbb{R}^n)$.

1.4 Proper maps

Definition 1.11. *A proper map $f : M \to N$ is a continuous map such that for any $K \subseteq N$ compact we have that $f^{-1}(K)$ is compact in M.*

Obviously if M is compact then any continuous function is proper. More in general:

Lemma 1.12. *The family of proper maps $f : M \to N$ is both open and closed in the strong $C^0_S(M;N)$ topology.*

For the proof see *e.g.* Sec. 5.1 in [7] of Theorem 1.5 in Chap. 2 in [2].
It is easily seen that, in general, the proper functions are neither closed nor open in any C^r_{loc} topology.

Example 1.13. *Consider functions $f : \mathbb{R} \to \mathbb{R}$, and the weak topology; it is easy to show examples of sequences $f_n \to f$ such that*

[3] For the definition of *strict inductive limit* and its properties, we refer to 17G at page 148 in [3].

- *none of the f_n are proper, but f is,* e.g.

$$f_n(x) = n \,\text{arctan}(x/n)$$

and $f(x) = x$;

- *f_n are all proper but f is not,* e.g.

$$f_n(x) = \frac{1}{n} x^2$$

and $f(x) = 0$.

The above examples hold also in C^∞_{loc}, where convergence is: “local uniform convergence of all derivatives”.

1.5 Drawback of strong topologies

Given the above discussion, it would seem that, when dealing with proper maps, it would be best to use a strong topology. Strong topologies have drawbacks as well, alas.

In particular consider the case of maps $f : M \to \mathbb{R}^n$, let $C^0 = C^0(M; \mathbb{R}^n)$: we have some natural actions:

- the translation, given by the action

$$(v, f) \in \mathbb{R}^n \times C^0 \mapsto f + v \in C^0$$

- rotations

$$(A, f) \in \text{SO}(n) \times C^0 \mapsto Af \in C^0$$

- rescalings

$$(\lambda, f) \in I \times C^0 \mapsto \lambda f \in C^0$$

for $I = (0, \infty) \subset \mathbb{R}$;

- and in general, affine transformations

$$F, f \mapsto Ff$$

where $Fy = Ay + v$ is given by $A \in \text{GL}(\mathbb{R}^n)$, $v \in \mathbb{R}^n$.

These actions are continuous if C^0 is endowed with the weak topology; but may fail to be continuous if C^0 is endowed with the strong topology.

Another drawback that we already remarked that, when $N = \mathbb{R}^n$, then C^0 with the strong topology may fail to be a topological vector space.

1.6 Goals

One goal of this paper then is to define a new topology, called “mild topology”, that will share some good properties valid for the “weak” and the “strong” topology.

In a first draft of the paper, while developing the theory, it become clear that the underlying mathematical structure was becoming an interesting tool set in itself. So it was abstracted into a theory, developed in the next Sec. 2. This theory will then used to define the “mild topology” in Sec. 3.

2 Topology by separation functions

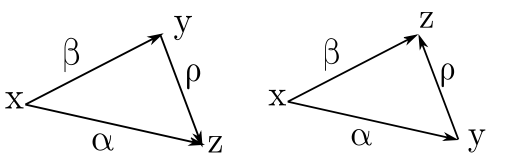
In this section, given a set X, we will use *“separation functions”* to define the topology.

Definition 2.1. *A family $d = (d_x)_{x\in X}$ of real positive functions*

$$d_x(y) : X \to [0, \infty]$$

*with $x \in X$, is a **family of separation functions** when*

- *$d_x(y) = 0$ iff $x = y$;*



pseudo triangle inequality (2.2) | pseudo reverse triangle inequality (2.3)

Figure 1: Representation of triangle inequalities

- *given $y \in X$ and $\alpha, \beta \in \mathbb{R}$ with $0 < \beta < \alpha$ there exists a function*

$$\rho_d = \rho_d(y, \alpha, \beta) > 0 \tag{2.1}$$

(called "modulus"*) such that, for all $x, z \in X$,*

$$d_x(y) \le \beta \wedge d_y(z) < \rho_d \Rightarrow d_x(z) < \alpha \tag{2.2}$$

$$d_x(y) \ge \alpha \wedge d_y(z) < \rho_d \Rightarrow d_x(z) > \beta \tag{2.3}$$

This condition (2.2) *will be called* "pseudo triangle inequality"*, while* (2.3) *will be called* "pseudo reverse triangle inequality"*, (See Figure 1)*

These are written as $d_x(y)$ and not $d(x, y)$ to remark that they do not satisfy the axioms of "distances": they are not required to be symmetric, and do not satisfy the standard triangle inequality.

Remark 2.2. *An* "asymmetric distance" *(*a.k.a. "quasi metric"*), is a function $b(x, y) : X^2 \to [0, +\infty]$ that satisfies the separation property $b(x, y) = 0$ iff $x = y$, and the standard triangle inequality, but it may fail to be symmetric. See [5, 6] and references therein. An* "asymmetric distance" *immediately provides a* family of separation functions *$d_x(y) = b(x, y)$ with $\rho_d(y, \alpha, \beta) = \alpha - \beta$.*

Theorem 2.3. *Let be given a set X with a* family of separation functions*, then we can define* "pseudo balls"

$$B(x, \varepsilon) = \{y \in X : d_x(y) < \varepsilon\}$$

these are then a global base for a T2 topology τ , and each $B(x, \varepsilon)$ is an open neighborhood of x in (X, τ).

Proof. Indeed the *pseudo triangle inequality* (2.2) tells that if $y \in B(x, \alpha)$ and $\beta = d_x(y)$ and $\rho_d = \rho_d(y, \alpha, \beta)$ then

$$B(y, \rho_d) \subseteq B(x, \alpha) \quad .$$

The *reverse pseudo triangle inequality* (2.3) tells that if $d_x(y) = \alpha > 0$ then

$$B(x, \beta) \cap B(y, \rho_d) = \emptyset \quad .$$

□

Proposition 2.4. *Each $d_x(y)$, for fixed x, is continuous on (X, τ)*

Proof. The Theorem 2.3 above readily implies that $d_x(y)$ is upper semi continuous: indeed we already know that

$$B(x, \varepsilon) = \{y \in X : d_x(y) < \varepsilon\}$$

is open. Let

$$V = \{z \in X : d_x(z) > \varepsilon\}$$

we want to prove that it is open; let $y \in V$ and let $\alpha = d_x(y) > \varepsilon$ let then $\varepsilon < \beta < \alpha$; the *reverse pseudo triangle inequality* (2.3) tells us that

$$B(y, \rho_d) \subseteq V \quad .$$

□

It is interesting to note that separation functions have a form of stability that makes them more manageable than distance functions.

Proposition 2.5. *Suppose that $d_x(y)$ is a separation function and $\varphi : [0,\infty] \to [0,\infty]$ is in the Θ class (defined in 2.11); let*

$$b_x(y) = \varphi(d_x(y))$$

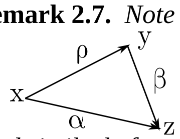
then $b_x(y)$ is a separation function.

Remark 2.6. *A similar proposition holds for distances when φ is also sub additive. But*

$$d_x(y) = |x-y|^2$$

is a separation function *on $\mathbb{R}$, and it is not a distance.*

Remark 2.7. *Note that we did not assume validity of this statement.*



"Given $x \in X$ and $\alpha > 0$ for any $0 < \beta < \alpha$ there exists $\rho > 0$ such that

$$d_x(y) < \rho \wedge d_y(z) \le \beta \Rightarrow d_x(z) < \alpha \quad \text{''}. \tag{2.4}$$

(and similarly for a "reverse" version).

This raises an (yet) unanswered question. Let us define

$$\tilde{d}_y(x) = d_x(y) \quad :$$

under which conditions $\tilde{d}_y(x)$ is a separation function?

2.1 On the modulus

Proposition 2.8. *There exists a* maximum modulus*: given $y \in X$ and $0 < \beta < \alpha$ define*

$$\begin{aligned}\hat{\rho}_d(y,\alpha,\beta) \overset{\text{def}}{=} \max \Big\{ & r \in [0,\infty] : \forall x, z \in X, \\ & (d_x(y) \le \beta \wedge d_y(z) < r \Rightarrow d_x(z) < \alpha) \wedge \\ & (d_x(y) \ge \alpha \wedge d_y(z) < r \Rightarrow d_x(z) > \beta) \Big\} \quad .\end{aligned}$$

Proof. This is maximum since we can write

$$\begin{aligned}\hat{\rho}_d(y,\alpha,\beta) &\overset{\text{def}}{=} \max A_{(y,\alpha,\beta)} \\ A_{(y,\alpha,\beta)} &= \bigcap_{x,z\in X} P_{(y,\alpha,\beta,x,z)} \cap R_{(y,\alpha,\beta,x,z)} \\ P_{(y,\alpha,\beta,x,z)} &\overset{\text{def}}{=} \Big\{ r \in [0,\infty] : (d_x(y) > \beta \vee d_y(z) \ge r \vee d_x(z) < \alpha) \Big\} \\ R_{(y,\alpha,\beta,x,z)} &\overset{\text{def}}{=} \Big\{ r \in [0,\infty] : (d_x(y) < \alpha \vee d_y(z) \ge r \vee d_x(z) > \beta) \Big\}\end{aligned}$$

and further yet

$$\begin{aligned}A_{(y,\alpha,\beta)} = [0,\infty] \cap & \left(\bigcap_{x,z\in X, d_x(y)\le\beta \wedge d_x(z)\ge\alpha} [0, d_y(z)] \right) \\ \cap & \left(\bigcap_{x,z\in X, d_x(y)\ge\alpha \wedge d_x(z)\le\beta} [0, d_y(z)] \right)\end{aligned}$$

that is an intersection of closed intervals starting from zero (included) and each containing $\rho_d(y,\alpha,\beta)$. □

Remark 2.9. *It is clear by the above formulas that $\hat{\rho}_d$ is weakly increasing in α and weakly decreasing in β; so we may assume this in the definition 2.1, with no loss of generality.*

If we add strict monotonicity and continuity, we obtain an interesting proposition.

Proposition 2.10. *Consider the* "pseudo triangle inequality" (2.2) *and these three additional conditions:* $\forall x, y, z \in X, \forall \alpha, \beta$ *with* $0 < \beta < \alpha$,

$$d_x(y) \le \beta \wedge d_y(z) \le \rho_d \Rightarrow d_x(z) \le \alpha \tag{2.5}$$

$$d_x(y) < \beta \wedge d_y(z) \le \rho_d \Rightarrow d_x(z) < \alpha \tag{2.6}$$

$$d_x(y) < \beta \wedge d_y(z) < \rho_d \Rightarrow d_x(z) < \alpha \tag{2.7}$$

where again $\rho_d = \rho_d(y, \alpha, \beta)$*, the same function* (2.1) *as used in* (2.2)*.*

Suppose that the function $\rho_d(y, \alpha, \beta)$*, for fixed* y*, is continuous in* α, β*, strictly increasing in* α *and strictly decreasing in* β*; then the three conditions* (2.2), (2.5), (2.6) *and* (2.7) *are equivalent: the validity of each one implies the other two. (The proof is in page 17)*

This happens for distances, where $\rho_d = \alpha - \beta$; and it happens in the next section 3.

A similar statement holds for the *"pseudo reverse triangle inequality"* (2.3), we skip it.

2.2 Fundamental family

In the above theory, we can assume that there are many *families of separation functions* $d_i = (d_{x,i})_{x \in X}$, for $i \in I$ a family of indexes (not depending on x); this is analogous to the framework in locally convex topological vector spaces, where we have multiple seminorms that are used to define multiple balls centered at zero (and then translated to each other point).

But in the following we will assume, for simplicity, that for each $x \in X$ there is exactly one *separation function*. So we have a **fundamental family of separation functions** $d = (d_x)_{x \in X}$. Then the topology satisfies the first countability axiom.

2.3 The Sorgenfrey line

The *Sorgenfrey line* is the set $X = \mathbb{R}$ with the topology generated by all the half-open intervals $[a, b)$, where $a, b \in \mathbb{R}, a < b$; this family is a global base for the topology. See example 51 in [10].

We can define

$$b(x, y) = \begin{cases} y - x & y \ge x \\ +\infty & y < x \end{cases} \tag{2.8}$$

this is an *"asymmetric distance"* that generates the above topology on $\mathbb{R}$; by 2.2 $d_x(y) = b(x, y)$ is a fundamental family of separation functions, with $\rho_d(y, \alpha, \beta) = \alpha - \beta$.

The *Sorgenfrey line* is a T6 space (a perfectly normal Hausdorff space); it is first-countable and separable, but not second-countable; so it is not metrizable.

This suggests that we need some extra hypotheses to obtain better properties.

2.4 Pseudo symmetry

We define a convenient class.

Definition 2.11 (Class Θ)**.** *We define the class* Θ *of functions*

$$\theta : [0, \infty] \to [0, \infty]$$

that are continuous, have $\theta(0) = 0$*, and are strictly increasing where they are finite. For such a* θ *we agree that* $\theta^{-1} \in \Theta$ *is so defined (with a slight abuse of notation)*

$$\theta^{-1}(s) \stackrel{def}{=} \sup\{t \in [0, \infty) : \theta(t) < s\} \quad ;$$

equivalently, if

$$D = \sup_{0 \le s < \infty} \theta(s)$$

then $\theta^{-1}(s)$ *is the usual inverse for* $s < D$*, otherwise* $\theta^{-1}(s) = +\infty$ [4]*; this implies that*

$$t = \theta^{-1}(s) \iff s = \theta(t)$$

[4] Indeed any $s \ge D$ cannot be equal to $\theta(t)$ for $t < \infty$, since θ is strictly increasing

whenever $s, t < \infty$.

Remark 2.12. *This is just a convenient choice, it allows us to simplify notation and analysis; for example the "tangent" function can be represented in* Θ *by defining it as*

$$\theta(t) = \begin{cases} \tan(t) & s < \pi/2 \\ +\infty & s \geq \pi/2 \end{cases} \quad ,$$

then its "inverse" is just $\theta^{-1}(s) = \arctan(s)$*, with* $\theta^{-1}(+\infty) = \pi/2$.

Definition 2.13. *Assume that associated to* X *we have a fundamental family* $(d_x)_{x \in X}$ *of* separation functions. *We will say that this family is* ***pseudo symmetric*** *if there exists a function* $\theta_d \in \Theta$ *such that*

$$d_y(x) \leq \theta_d \left(d_x(y)\right) \quad . \tag{2.9}$$

Remark 2.14. "Pseudo symmetry" *is useful because it tells us that the topology is equivalently generated by the inverse balls*

$$\tilde{B}(x, \varepsilon) = \{y \in X : d_y(x) < \varepsilon\} \quad ;$$

and it tells us that

$$\lim_{n \to \infty} x_n = x \iff d_x(x_n) \to_n 0 \iff d_{x_n}(x) \to_n 0 \quad .$$

Compare Definition 3.4 in [5].

2.5 Set separation function

Definition 2.15. *Mimicking the definition in metric space, we define, for* $A \subseteq X$*, the* ***set separation function***

$$d_A(y) \stackrel{def}{=} \inf_{x \in A} d_x(y) \quad . \tag{2.10}$$

Lemma 2.16. $d_A(y)$ *is continuous.*

Proof. We know that $d_A(y)$ is upper semi continuous, so we need to prove that it is lower semicontinuous, that is, that

$$V = \{y \in X : d_A(y) > \varepsilon\}$$

is open, for $\varepsilon \geq 0$; let $y \in V$ and let $\alpha = d_A(y) > \varepsilon$; let then $\varepsilon < \beta < \alpha$, by *reverse pseudo triangle inequality* (2.3) let $r = \rho_d(y, \alpha, \beta)$, we prove that

$$B(y, r) \subseteq V \quad ;$$

indeed for any $x \in A$ we have $d_x(y) \geq \alpha$ so if $d_y(z) < r$ then $d_x(z) > \beta$ hence $d_A(z) \geq \beta$. □

Remark 2.17. *Since*

$$\{y \in X : d_A(y) = 0\} \supseteq A$$

and the LHS is closed, in general we have

$$\{y \in X : d_A(y) = 0\} \supseteq \overline{A} \quad .$$

To obtain equality we add pseudo-symmetry.

Hypotheses 2.18. *From here on we assume that we have a fundamental family* $(d_x)_{x \in X}$ *of* separation functions *that is* pseudo symmetric.

Lemma 2.19. *Suppose that 2.18 holds then:*

$$\{y \in X : d_A(y) = 0\} = \overline{A}$$

Proof. We need to prove that

$$\{y \in X : d_A(y) = 0\} \subseteq \overline{A}$$

Let then $d_A(y) = 0$ this means that there is a sequence $(x_n)_n \subseteq A$ such that $d_{x_n}(y) \to_n 0$ by pseudo symmetry $y \in A$. □

Corollary 2.20. *In particular, when 2.18 holds the topological space is T6,* a.k.a. *a* perfectly normal Hausdorff space.

2.6 Metrization

We have then a first metrization Theorem, following the Urysohn Metrization Theorem.

Theorem 2.21. *Suppose 2.18 holds and the topological space (X, τ) is second countable, then it is metrizable.*

Proof. Indeed by 2.20 the space is T6. □

2.7 Diameter

Definition 2.22. *Let*

$$\text{diam}(A) \overset{def}{=} \sup_{x,y \in A} d_x(A)$$

be the ***diameter*** *of a set $A \subseteq X$.*

Lemma 2.23. *Suppose that 2.18 holds then:*

$$\lim_{\varepsilon \to 0} \text{diam}(B(y, \varepsilon)) = 0 \quad .$$

If ρ_d does not depend on y then the limit above is uniform in y.

Proof. The map $\varepsilon \mapsto \text{diam}(B(y, \varepsilon))$ is monotonic. Let us fix $\alpha > 0$ and $\beta = \alpha/2$; we will use (2.2) namely

$$d_x(y) \le \beta \wedge d_y(z) < \rho_d \Rightarrow d_x(z) < \alpha \quad .$$

There is $\varepsilon > 0$ small enough so that $\theta_d(\varepsilon) < \beta$ and $\varepsilon < \rho_d(y, \alpha, \beta)$. Choose any $x, z \in B(y, \varepsilon)$. By pseudo symmetry

$$x \in B(y, \varepsilon) \iff d_y(x) < \varepsilon \Rightarrow d_x(y) \le \theta_d(\varepsilon) < \beta$$

so by (2.2) $d_x(z) < \alpha$ hence $\text{diam}(B(y, \varepsilon)) \le \alpha$. □

2.8 Uniform modulus

Uniform modulus is the case when the function ρ_d does not depend on y. We saw in 2.23 that in this case we obtain stronger results. Here following are further results that use uniformity and pseudo symmetry.

Hypotheses 2.24. *From here on we assume that we have a fundamental family $d = (d_x)_{x \in X}$ of* separation functions *that is* pseudo symmetric, *and where the function ρ_d does not depend on y.*

Lemma 2.25. *Suppose 2.24 holds, then topological space (X, τ) is separable if and only if it is second countable.*

Proof. One implication is well known. Let $(x_n)_{n \in \mathbb{N}}$ be a dense subset of X, we prove that

$$B(x_n, 1/m)$$

is a global base, for $n, m \ge 1$ integers; to this end for any

$$B(x, \alpha)$$

that is an element of the global base (as by 2.3), and also a fundamental family of neighborhoods of x; we will prove that there is a ball as above such that

$$x \in B(x_n, 1/m) \subseteq B(x, \alpha) \quad .$$

Indeed we choose m such that $\text{diam}\, B(y, 1/m) < \alpha$ for any y; then by pseudo symmetry, there is an n such that $d_{x_n}(x) < 1/m$. □

2.9 Forward and reverse

Proposition 2.26. *When 2.24 holds, the* "pseudo triangle inequality" (2.2) *and the* "pseudo reverse triangle inequality" (2.3) *are equivalent.*

Proof. We prove that if (2.2) holds then (2.3) holds. (The other implication is similar). Consider

$$d_x(z) \geq \alpha \wedge d_z(y) < \tilde{\rho}_d \Rightarrow d_x(y) > \beta \tag{2.11}$$

that is just (2.3) switching x, z and replacing the modulus. We prove that for $0 < \beta < \alpha$ there exists $\tilde{\rho}_d = \tilde{\rho}_d(\alpha, \beta) > 0$ such that $\forall x, y, z \in X$, (2.11) holds.

The condition (2.2) equivalently tells us that

$$d_x(y) > \beta \vee d_y(z) \geq \rho_d \vee d_x(z) < \alpha$$

so if

$$d_x(z) \geq \alpha \wedge d_z(y) < \theta^{-1}(\rho_d)$$

then by pseudo symmetry (2.9)

$$d_y(z) \leq \theta_d\,(d_z(y)) < \rho_d$$

hence $d_x(y) > \beta$. We then define $\tilde{\rho}_d = \theta_d^{-1} \circ \rho_d$ with the convention in definition 2.11 □

2.10 Delta complement

We extract a lemma from the proof of the following Theorem 2.29.

Definition 2.27. *Assume 2.18. Given $C \subset X$ closed and $\delta > 0$ we define the*

$$F_\delta(C) = \{x \in X : d_C(x) > \delta\}$$

we call this "delta-complemented set" *since*

$$F_\delta(C) \cap C = \emptyset$$

and $F_\delta(C)$ is open, and

$$\bigcup_{\delta > 0} F_\delta(C) = X \setminus C \quad .$$

(All this follows from Lemma 2.16 and 2.19.) See figure 2 on the next page. Note also that for $0 < s < t$ we have

$$F_s(C) \supseteq F_t(C)$$

while for $C_1 \subseteq C_2$ closed sets

$$F_s(C_1) \supseteq F_s(C_2) \quad . \tag{2.12}$$

Lemma 2.28. *Assume 2.24. Let now $0 < s < t$ there exists an $\varepsilon > 0$ such that, for any $w \in X$, for any $C \subseteq X$ closed, if*

$$B(w, \varepsilon) \cap F_t(C) \neq \emptyset$$

then

$$B(w, \varepsilon) \subseteq F_s(C) \quad .$$

Proof. Let $0 < s < \tilde{s} < t$. By the reverse triangle inequality (2.3) let ρ be small enough so that

$$d_x(z) \geq t \wedge d_z(y) < \rho \Rightarrow d_x(y) > \tilde{s} \quad ;$$

then choose $\varepsilon > 0$ small enough so that by Lemma 2.23 we have

$$\forall w \in X, \forall z, y \in B(w, \varepsilon) \Rightarrow d_z(y) < \rho \quad .$$

Let us now suppose that

$$z \in B(w, \varepsilon) \cap F_t(C)$$

then for all $x \in C$ we have $d_x(z) > t$; consider any $y \in B(w, \varepsilon)$ then $d_z(y) < \rho$ we conclude that $d_x(y) > \tilde{s}$ hence $\inf_{x \in C} d_x(y) \geq \tilde{s}$ so that $y \in F_s(A)$. □

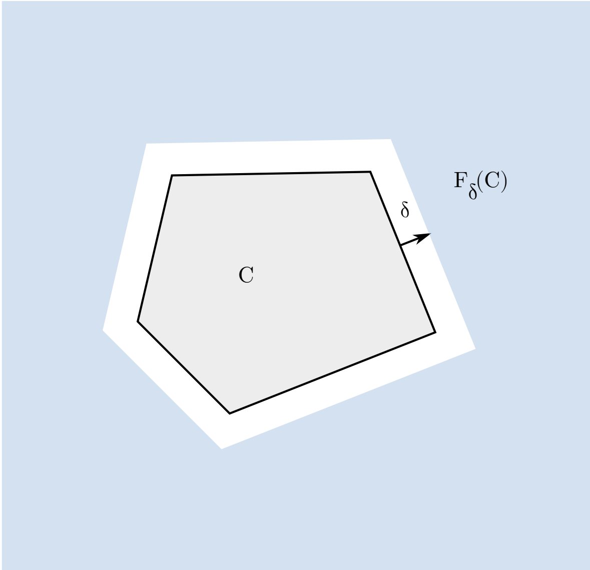


Figure 2: δ complement

2.11 Metrizability

Theorem 2.29. *Assume 2.24. If $\mathcal{A}$ is an open covering of X then there exists $\mathcal{E}$ an open covering of X that is countably locally finite, and it is a refinement of $\mathcal{A}$.*

(The proof is on page 18).

Theorem 2.30. *Assume 2.24. The topological space (X, τ) is metrizable.*

Proof. This follows from the above Theorem, and 2.20, and the Nagata–Smirnov metrization theorem (Sections 6-2 and 6-3 in [8]). ☐

2.12 Example: topological manifolds

As an example, we propose this construction.

Consider a topological manifold (X, τ) that is Hausdorff, second countable and locally euclidean with dimension m (See §36 in [8] for further details); then it is paracompact and σ-compact; so there exists an atlas of homeomorphisms

$$\varphi_i : V_i \to \mathbb{R}^m$$

for $i \in I$, where $I = \mathbb{N}$ or I finite, $V_i \subseteq X$ open with compact closure $\overline{V_i}$, and $(V_i)_{i\in I}$ is a locally finite open cover of X.

Theorem 2.31. *For $x, y \in X$ we can define*

$$d_x(y) = \min\{|\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} : i \in I \wedge x \in V_i \wedge y \in V_i\} \quad . \tag{2.13}$$

Note that the set on the right hand side is finite; if it is empty then $d_x(y) = +\infty$. These $d_x(y)$ are continuous (jointly in x, y) and are a fundamental family of separation functions that generate the topology τ of X and is symmetric.

(Although the above result seems intuitive, the proof is surprisingly long and intricate, so it was moved to Sect. A.3 on page 19. The modulus ρ_d can be defined to satisfy the requirements of Prop. 2.10.)

The family satisfies 2.18, consequently the *set separation function* $d_A(y)$ satisfies 2.16 and 2.19: this explicitly proves that the space is T6.

So Theorem 2.21 is an alternative proof that such a manifold X is metrizable.

Example 2.32. *In general the above $d_x(y)$ does not satisfy the triangle inequality, as in this example of a manifold covered with two charts, where*

$$x \in V_1 \setminus V_2 \quad , \quad y \in V_1 \cap V_2 \quad , \quad z \in V_2 \setminus V_1$$

and

$$d_x(y) < \infty \quad , \quad d_y(z) < \infty \quad , \quad d_x(z) = \infty \quad .$$

Example 2.33. *Consider $X = \mathbb{R}$ and cover it with charts having $V_n = (n-1, n+1)$ and*

$$\varphi_n(x) = \frac{1}{n}\psi(x-n) \quad , \quad \psi(x) = \frac{x}{(1-x^2)}$$

for $n \in \mathbb{Z}$ let then

$$x = n \ , \ y = n + \frac{1}{2} \ , \ z = n+1$$

so

$$d_x(y) = d_y(z) = \frac{2}{3n} \quad , \quad d_x(z) = \infty \quad .$$

This explains the importance of the dependence of ρ_d on y.

In some cases it may happen that we do not know an easy formula for the distance that metrizes the manifold X.

The metrization theorems, such as Nagata–Smirnov metrization theorem or Urysohn's metrization theorem, are usually proven by showing that there is an embedding of X into $\mathbb{R}^{\mathbb{N}}$; this embedding requires to use Urysohn's lemma to define countably many functions $f_n : X \to [0,1]$ (even when X is compact): while perfectly valid as a proof, it is not an easily manageable definition and unsuitable for for numerical algorithms.

If X is compact, then such manifold can be embedded in $\mathbb{R}^N$; so this can be used to define a distance on X, by carefully tracking how the embedding is defined (as *e.g.* in §36 in [8]); this plan could be carried on to provide an explicit formula for the distance, knowing the charts; in particular, we can assume that the atlas is finite, let $\#I$ be its cardinality, then such proof provides $N = (m+1)\#I$.

We remark that, at the same time, (2.13) gives us a very convenient definition of *separation functions* (also when X is not compact): those encode the idea of "nearness" and can be used in further proofs and/or for numerical algorithms.

3 Mild topology

In this section we propose a novel topology on the space of continuous functions $C^0(M;N)$.

To define the *mild topology* we need that (N, d_N) be a metric space.

We fix a distinguished point $\overline{p} \in N$.

Let $f, g \in C^0 = C^0(M;N)$, we define the *"mild separation"*

$$d_f^{\text{mild},\overline{p}}(g) \stackrel{\text{def}}{=} \sup_{x \in M} \frac{d_N(f(x), g(x))}{1 + d_N(f(x), \overline{p})} \quad . \tag{3.1}$$

For $f \in C^0$ and $\alpha > 0$ we define the *"mild pseudo ball"*

$$B^{\text{mild},\overline{p}}(f, \alpha) \stackrel{\text{def}}{=} \{g \in C^0 : d_f(g) < \alpha\} \tag{3.2}$$

We omit the superscripts "mild, $\overline{p}$" for ease of notation.

Definition 3.1. *The **mild topology** on C^0 is the topology generated by the above sets $B(f,\alpha)$. We will write $C^0_M(M;N)$ to denote this topological space.*

To justify these definitions, we prove these results.

Lemma 3.2 (Pseudo symmetry)**.**

$$d_g(f) \le \theta(d_f(g))$$

with

$$\theta(\alpha) = \begin{cases} \frac{\alpha}{1-\alpha} & \alpha < 1 \\ \infty & \alpha \ge 1 \end{cases} \quad .$$

Proof. Suppose $0 < \alpha < 1$ and $d_f(g) \le \alpha$ then

$$d_N(f(x), g(x)) \le \alpha\,(1 + d_N(f(x), \overline{p})) \le \alpha\,(1 + d_N(g(x), \overline{p}) + d_N(g(x), f(x)))$$

hence

$$(1-\alpha) d_N(f(x), g(x)) \le \alpha\,(1 + d_N(g(x), \overline{p})) \quad .$$

□

Lemma 3.3 (Pseudo triangle inequality). *Let $f, g, h \in C^0$ and $\alpha > 0$; if $d_f(g) \le \beta < \alpha$ and*

$$d_g(h) \le \rho_d(\alpha, \beta) \text{ with } \rho_d(\alpha, \beta) = \frac{\alpha - \beta}{1 + \beta}$$

then $d_f(h) \le \alpha$. This proves the pseudo triangle inequality *in the form* (2.5)*; then* (2.2) *follows from Prop. 2.10; and* (2.3) *follows from Prop. 2.26.*

Proof. $d_f(g) \le \beta$ means

$$d_N(f(x), g(x)) \le \beta(1 + d_N(f(x), \overline{p}))$$

moreover $d_g(h) \le \rho$ means

$$d_N(g(x), h(x)) \le \rho(1 + d_N(g(x), \overline{p}))$$

summing

$$\begin{aligned} d_N(f(x), h(x)) &\le d_N(f(x), g(x)) + d_N(g(x), h(x)) \le \\ &\le (\beta + \rho) + \beta d_N(f(x), \overline{p}) + \rho d_N(g(x), \overline{p}) \end{aligned}$$

but we have also

$$d_N(g(x), \overline{p}) \le d_N(g(x), f(x)) + d_N(f(x), \overline{p}) \le \beta + (1 + \beta) d_N(f(x), \overline{p})$$

substituting

$$d_N(f(x), h(x)) \le \Big(\beta + \rho(1 + \beta)\Big) + \Big(\beta + \rho(1 + \beta)\Big) d_N(f(x), \overline{p})$$

so we just need to find a ρ such that

$$\Big(\beta + \rho(1 + \beta)\Big) \le \alpha$$

and the one on the hypothesis will do.

□

Lemma 3.4. *The mild topology does not depend on the choice of $\overline{p} \in N$.*

Proof. Given $\overline{p}, \tilde{p} \in N$ for any β choosing

$$\alpha = \beta \frac{1}{1 + d_N(\overline{p}, \tilde{p})}$$

and we have

$$\alpha(1 + d_N(y, \overline{p})) \le \alpha(1 + d_N(y, \tilde{p}) + d_N(\tilde{p}, \overline{p})) \le \beta(1 + d_N(y, \tilde{p})) \quad ;$$

then reason as in Rem. 3.7. □

By the results in the previous section we obtain that

Corollary 3.5. *Those sets $B(f, \alpha)$ are a global base for the mild topology; this topology is metrizable.*

Proposition 3.6. *The mild topology is stronger than the weak topology; and it is weaker than the strong topology.*

Proof. • We show that the mild topology is stronger than the weak topology. Fix $\varepsilon > 0$ and a compact set $K \subseteq M$, let

$$\beta = \max_{x \in K} d_N(f(x), \overline{p})$$

and

$$\rho < \frac{\varepsilon}{1+\beta}$$

we know that if

$$g \in B(f, \rho)$$

then

$$\forall x \in K, \;\; d_N(f(x), g(x)) < \varepsilon \quad .$$

- The fact that the mild topology is weaker than the strong topology follows from Remark 3.7.

□

Remark 3.7. *We may also define the* "mild neighborhood"

$$\tilde{B}(f, \alpha) \stackrel{def}{=} \{g \in C^0 : \forall x \in M \;,\; d_N(f(x), g(x)) < \alpha(1 + d_N(f(x), \overline{p}))\} \tag{3.3}$$

Note that, for $0 < \beta < \alpha$

$$\tilde{B}(f, \beta) \subseteq B(f, \alpha) \subseteq \tilde{B}(f, \alpha) \quad , \tag{3.4}$$

so "mild neighborhoods" *can be used to define the* mild topology*; unfortunately, they may fail to be open.*

A "mild neighborhood" *can be built using the same method seen in the* graph topology *(see 1.2): indeed consider open sets of the form*

$$U = \{(x, y) \in M \times N : \forall x \in M \;,\; d_N(f(x), y) < \alpha(1 + d_N(f(x), \overline{p}))\}$$

for $f \in C^0, \alpha > 0$*, and then*

$$\tilde{B}(f, \alpha) = \{g \in C^0 : \mathit{graph}(g) \in U\}$$

so (3.4) *proves that the* mild topology *is coarser than the* strong topology.

Remark 3.8. *In general this topology is not separable, for example when* $N = M = \mathbb{R}$ *then setting* $f_s(x) = e^{sx}$ *we have*

$$d_{f_s}(f_t) = \begin{cases} 1 & s > t \\ \infty & s < t \end{cases} \quad ;$$

in these cases the topology does not satisfy the second countability axiom. (This is why we proved Theorem 2.30, that is based on Nagata–Smirnov metrization theorem; we cannot use Urysohn's metrization theorem to prove that the mild topology *is metrizable).*

3.1 Metrizability

We know by Theorem 2.30 that the topology is metrizable.

At first sight, a reasonable candidate for a distance that generates the mild topology may be

$$d_{\text{mild?}}(f, g) \stackrel{\text{def}}{=} \sup_{x \in M} \frac{d_N(f(x), g(x))}{1 + d_N(f(x), \overline{p}) + d_N(g(x), \overline{p})} \quad .$$

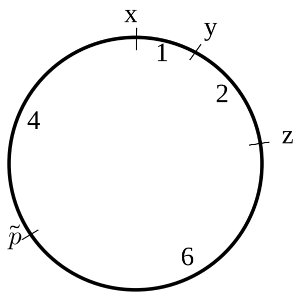
Note that $0 \leq d_{\text{mild?}}(f, g) \leq 1$

Lemma 3.9. *Obviously*

$$d_{\mathit{mild?}}(f, g) \leq d_f^{\mathit{mild}, \overline{p}}(g)$$

moreover for $0 < \alpha < 1$

$$\alpha = d_{\mathit{mild?}}(f, g) \Rightarrow d_f^{\mathit{mild}, \overline{p}}(g) \leq \frac{2\alpha}{1 - \alpha}$$



$$
\begin{aligned}
d_N(x,y) &= 1\\
d_N(y,z) &= 2\\
d_N(x,z) &= 3\\
d_N(\overline{p},x) &= 4\\
d_N(\overline{p},y) &= 5\\
d_N(\overline{p},z) &= 6
\end{aligned}
$$

Figure 3: Points along a circle N of length 13, and distances

Proof. If $0 < \alpha < 1$ and $d_{\text{mild?}}(f,g) \le \alpha$ then

$$d_N(fx), g(x)) \le \alpha\left(1 + d_N(f(x),\overline{p}) + d_N(g(x),\overline{p})\right) \le \alpha\left(1 + 2d_N(f(x),\overline{p}) + d_N(g(x),f(x))\right)$$

hence

$$(1-\alpha) d_N(f(x),g(x)) \le 2\alpha\left(1 + d_N(f(x),\overline{p})\right) \quad .$$

□

So if $d_{\text{mild?}}(f,g)$ is a distance, it will generate the mild topology.

But is it a distance? The formula is obviously symmetric, and we have

$$d_{\text{mild?}}(f,g) = 0 \iff f \equiv g \quad ;$$

the question is, does it satisfy the triangle inequality?

Consider then this formula

$$d_{\text{N?}}(x,y) \overset{\text{def}}{=} \frac{d_N(x,y)}{1 + d_N(x,\overline{p}) + d_N(y,\overline{p})}$$

for $x, y \in N$.

We note that $d_{\text{mild?}}(f,g)$ satisfies the triangle inequality if and only if $d_{\text{N?}}(x,y)$ does. (For one implication, consider constant functions; for the other, use standard properties of the supremum).

Unfortunately the quantity $d_{\text{N?}}(x,y)$ does not satisfy the triangle inequality for some choices of N as is seen in this example: let N be a circle of length 13 where the points are posed as in this drawing 3.

At the same time, consider the case when N is a Hilbert space and $\overline{p} = 0$: then

$$d_{\text{N?}}(x,y) \overset{\text{def}}{=} \frac{\|x-y\|_N}{1 + \|y\|_N + \|y\|_N} \quad .$$

Note that the formula is invariant for rotations, so it is enough to check the triangle inequality for $N = \mathbb{R}^3$; numerical experiments suggest that is it indeed a distance; to this end, we tested the triangle inequality with randomly sampled points, and tried to numerically minimize the difference

$$d_{\text{N?}}(x,y) + d_{\text{N?}}(y,z) - d_{\text{N?}}(x,z)$$

for $x, y, z \in \mathbb{R}^3$. We though could not prove it analytically. See addendum material for more information.

3.2 Properties of proper maps

Lemma 3.10. *Suppose that* (N, d_N) *is a* "proper metric space", i.e. *closed balls are compact.*

If $0 < \alpha < 1$ *and* $f \in B(g,\alpha)$, *then* g *is proper iff* f *is proper.*

Similarly for the "mild neighborhood" $\tilde{B}(f,\alpha)$ *defined in* 3.7.

Proof. Suppose that $d_g(f) = D < \infty$ and f is proper, we prove that g is proper; let $K \subseteq N$ be compact; let

$$R = \max_{y \in K} d_N(y, \overline{p})$$

let

$$H = \{y \in N : d_N(y, \overline{p}) \le D + R(D+1)\}$$

then H is compact and we prove that

$$g^{-1}(K) \subseteq f^{-1}(H)$$

so $g^{-1}(K)$ is compact. Indeed if $x \in g^{-1}(K)$ then $g(x) \in K$ so $d_N(g(x), \overline{p}) \le R$ hence

$$d_N(f(x), \overline{p}) \le d_N(g(x), \overline{p}) + d_N(g(x), f(x)) \le D + (D+1) d_N(g(x), \overline{p}) \le D + R(D+1) \quad .$$

Suppose now that g is proper and $d_g(f) < 1$ then by pseudo symmetry 3.2 $d_f(g) < \infty$ so f is proper.

□

Corollary 3.11. *The set of proper maps is both open and closed in the mild topology.*

3.3 Properties of affine actions

Theorem 3.12. *Suppose that $N = \mathbb{R}^n$, $d_N(x,y) = |y - x|$ is the usual Euclidean distance; endow $C^0(M; \mathbb{R}^n)$ with the* mild topology*: then the actions listed in Sec. 1.5 are (jointly) continuous.*

(The proof is in page 22).

Remark 3.13. *For some specific actions some extra information may be useful*

- *Rotation. If we choose $\overline{p} = 0$ for convenience in the definition eqn. (3.2), as is made possible by Lemma 3.4, then, given a rotation $R \in O(n)$, the map*

  $$f \in C^0 \mapsto Rf \in C^0$$

  is an "isometry"*: indeed*

  $$B(Rf, \alpha) = R\, B(f, \alpha)$$

  because

  $$d_{Rf} Rg = d_f g \quad .$$

  We also note that for $S, R \in O(n)$,

  $$|Rg(x) - Sg(x)| \le \|R - S\| |g(x)|$$

  so $d_{Rg}(Sg) \le \|R - S\|$ where $\|R - S\|$ is a matrix (operator) norm.

- *Rescaling. Let $s > 0$, let $m = \min\{1, s\}$, $M = \max\{1, s\}$ then*

  $$m\, d_f(g) \le d_{sf}(sg) \le M\, d_f(g)$$

  so

  $$f \in C^0 \mapsto sf \in C^0$$

  is again a homeomorphism. For the action

  $$s \in \mathbb{R} \mapsto sf \in C^0$$

  similarly

  $$|t - s| m \le d_{sf}(tf) \le |t - s| M \quad .$$

3.4 Caveats

We have then seen many good properties of the *mild topology*; there are some drawbacks though.

- The *mild topology* depends on the choice of distance d_N.
- It is not invariant w.r.t. homeomorphisms as in Prop. 1.4; it is invariant only for right action *i.e.* if $\Phi_M : \tilde{M} \to M$ is a homeomorphism then the map

$$f \mapsto f \circ \Phi_M$$

  is a homeomorphism between $C^0(M;N)$ and $C^0(\tilde{M};N)$, where both spaces are endowed with the *mild topology*.
- The space $C^0(M;N)$ with the mild topology may fail to be connected, since proper maps are open and closed.
- When $N = \mathbb{R}^n$ with Euclidean structure, the space $C^0(M;\mathbb{R}^n)$ with the mild topology is not in general a *topological vector space*; there are many reasons:
  - for $g \in C^0$ fixed, the map
$$f \in C^0 \mapsto g + f \in C^0$$
    may fail to be continuous. For example, consider $g(x) = -e^x$ and $C^0(\mathbb{R};\mathbb{R})$, $f(x) = e^x$, then the counter image of $B(0,1)$ is
$$\{h(x) + e^x : \sup_x |h(x)| < 1\}$$
    and it does not contain any ball $B(e^x, \varepsilon)$;
  - for $f \in C^0$ fixed, the map
$$\lambda \in \mathbb{R} \mapsto \lambda f \in C^0$$
    may fail to be continuous at $\lambda = 0$ (adapting the previous example);
  - the space may not be connected.

4 Conclusions

We have discussed a novel method to define topologies, by *separation functions*; we have shown that, even when the topology happens to be metrizable, it may happen that the actual metric is not known and/or that the *separation functions* are more manageable than the metric that metrizes the topology.

We have studied the *mild topology* $C^0(M;N)$; it has some good properties: proper maps are a closed and open subset of $C^0(M;N)$, as in the case of the *strong topology*; affine actions on $N = \mathbb{R}^n$ are continuous on $C^0(M;\mathbb{R}^n)$, as in the case of the *weak topology*.

It is possible to define similar concepts for $C^r(M;N)$, the space of r times differentiable maps between two differentiable manifolds M, N; similar properties hold, and can be extended to other interesting classes of maps such as: immersions, free immersions, submersions, embeddings, diffeomorphisms; this may be argument of a forthcoming paper.

A Proofs

A.1 Proof of Prop. 2.10

Proof. Fix $y \in X$. In the hypotheses, this form of implicit function theorem holds. Given $0 < \beta < \alpha$, let $r = \rho_d(y, \alpha, \beta)$ there are small open intervals I_α , $I_\beta \subseteq (0, \infty)$ with $\alpha \in I_\alpha$, $\beta \in I_\beta$ and a homeomorphic strictly increasing function $R : I_\alpha \to I_\beta$ such that,

$$\forall a \in I_\alpha\,,\ \forall b \in I_\beta\,, \quad 0 < a < b \quad \text{and} \quad r = \rho_d(y, a, b) \iff R(a) = b \quad .$$

We now prove the statement.

- Suppose that we want to prove

$$d_x(y) \le \beta \wedge d_y(z) \le r \Rightarrow d_x(z) \le \alpha \qquad (2.5)$$

knowing that (2.2) holds. If $d_y(z) < r$ then by (2.2) we readily get $d_x(z) < \alpha$; if $d_y(z) = r$ then, for $a > \alpha$ by hypotheses $r < \rho_d(x, \beta, a)$ so by (2.2) $d_x(z) < a$ and by arbitrariness of a this implies $d_x(z) \le \alpha$.

- Suppose that we want to prove (2.2) namely

$$d_x(y) \le \beta \wedge d_y(z) < r \Rightarrow d_x(z) < \alpha$$

knowing that (2.5) holds. Let $a < \alpha$ be near enough so that $d_y(z) < \rho_d(x, \beta, a) < r$ so by (2.5) $d_x(z) \le a < \alpha$.

- Suppose that we want to prove (2.5) namely

$$d_x(y) \le \beta \wedge d_y(z) \le r \Rightarrow d_x(z) \le \alpha$$

knowing that (2.6) holds. If $d_x(y) < \beta$ then by (2.6) we readily get $d_x(z) < \alpha$; if $d_x(y) = \beta$ then we choose $b \in I_\beta, b > \beta$ and $a = R^{-1}(b)$ so (2.6) $d_x(z) < a$ then we let $b \to \beta$ and we know that $a \to \alpha$.

- Suppose that we want to prove (2.6) namely

$$d_x(y) < \beta \wedge d_y(z) \le r \Rightarrow d_x(z) < \alpha$$

knowing that (2.5) holds. We choose $a \in I_\alpha, b \in I_\beta$ such that $a = R^{-1}(b)$ and $d_x(y) < b < \beta$ then by (2.5) $d_x(z) \le a < \alpha$.

- it is easily seen that (2.6) and (2.2) imply (2.7)

- Suppose that we want to prove (2.5) namely

$$d_x(y) \le \beta \wedge d_y(z) \le r \Rightarrow d_x(z) \le \alpha$$

knowing that (2.7) holds. We choose $a > \alpha, a \in I_\alpha$ so $R(a) > \beta$, then

$$b = \frac{\beta + R(a)}{2}$$

so $R(a) > b > \beta$ so $r < \rho_d(x, a, b)$: we can apply (2.7) in the form

$$d_x(y) < b \wedge d_y(z) < \rho_d(x, a, b) \Rightarrow d_x(z) < a$$

to obtain that $d_x(z) < a$: then we let $a \to \alpha$ and we know that $b \to \beta$ and $\rho_d(x, a, b) \to r$.

□

A.2 Proof of Theorem 2.29

Here is the proof of Theorem 2.29.

Proof. The proof is an adaptation of the analogous proof in Lemma 39.2 in [8]. We will use the *set separation function* to simplify some arguments.

- Given A open we define the *open erosion*

$$S_n(A) = \{x \in X : d_{X \setminus A}(x) > 1/n\} \quad .$$

In relation to Lemma 2.27 we note that

$$S_n(A) = F_{1/n}(X \setminus A) \quad ,$$

$S_n(A)$ is open, $S_n(A) \subseteq S_{n+1}(A)$, and

$$A = \bigcup_{n=1}^{\infty} S_n(A) \quad . \qquad (A.1)$$

- We also define the *closed erosion*

$$C_n(A) = \{x \in X : d_{X\setminus A}(x) \geq 1/(2n)\}$$

obviously $C_n(A)$ is closed and

$$C_n(A) \subseteq A \quad . \tag{A.2}$$

- We associate to $\mathcal{A}$ a well ordering $\preceq$.

- For each $U \in \mathcal{A}$ we define

$$T_n(U) = S_n(U) \setminus C_n\left(\bigcup_{Z\in\mathcal{A}, Z\prec U} Z\right) \quad ;$$

it is open.

- Let

$$\mathcal{E}_n = \{T_n(U) : U \in \mathcal{A}\} \quad .$$

- We we set $\mathcal{E} = \bigcup_n \mathcal{E}_n$, it is clearly a refinement of $\mathcal{A}$ since $T_n(U) \subseteq S_n(U) \subseteq U$.

- We prove that it is an open cover. Fix $x \in X$, let then U be the minimum $U \in \mathcal{A}$ such that $x \in U$, minimum according to the well ordering $\preceq$ of $\mathcal{A}$. If $V \prec U$, $x \notin V$ hence

$$x \notin \bigcup_{V\in\mathcal{A}, V\prec U} V$$

then by (A.2)

$$x \notin C_n\left(\bigcup_{Z\in\mathcal{A}, Z\prec U} Z\right) \quad ,$$

while for n large $x \in S_n(U)$ by (A.1) so $x \in T_n(U)$.

- We prove that $\mathcal{E}_n$ is locally finite; let indeed $x \in X$; choose $\varepsilon > 0$ small enough so that Lemma 2.28 holds for $s = 1/(2n), t = 1/n$.

  Let $w \in X$, consider any ball $B(w, \varepsilon)$: we show that it intersects at most one set $T_n(U)$, for $U \in \mathcal{A}$.

  Suppose that $B(w, \varepsilon)$ intersects any set in $\mathcal{E}_n$; Let U be the minimum (in the well ordering $\preceq$ of $\mathcal{A}$) of all $Z \in \mathcal{A}$ such that $T_n(Z)$ intersects $B(w, \varepsilon)$; we now show that $B(w, \varepsilon)$ does not intersect any $T_n(V)$ for $U \prec V$. Indeed by lemma 2.28 we know that

$$B(w, \varepsilon) \subseteq F_{1/2n}(X \setminus U) \subseteq C_n(U) \subseteq C_n\left(\bigcup_{Z\in\mathcal{A}, Z\prec V} Z\right) \quad .$$

□

A.3 Proof of Theorem 2.31

We now prove Theorem 2.31.

Proof.

- We define this useful notation: For any $w \in X$ let $J_w \subseteq I$ be the finite set

$$J_w = \{i \in I : w \in V_i\} \quad .$$

- By the definition we have symmetry $d_x(y) = d_y(x)$.

- It is also clear that

$$d_x(y) = 0 \iff x = y \quad .$$

- For $i \in I$ let
$$f_i(x,y) = \begin{cases} |\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} & x \in V_i \wedge y \in V_i \\ +\infty & \text{otherwise} \end{cases}$$
then each f_i is continuous and
$$d_x(y) = \min_{i \in I} f_i(x,y)$$
using the fact that the cover is locally finite, we know that in a small neighborhood of (x,y) the above is the minimum for i in a finite subset of I: then $d_x(y)$ are jointly continuous in x, y.

- This implies that the *"pseudo balls"*
$$B_d(x,\varepsilon) = \{y \in X : d_x(y) < \varepsilon\}$$
are open.

- We express them explicitly
$$B_d(x,\varepsilon) = \bigcup_{i \in J_x} \varphi_i^{-1}\left(B_{\mathbb{R}^n}(\varphi_i(x),\varepsilon)\right)$$
where $B_{\mathbb{R}^n}$ are standard balls in $\mathbb{R}^n$.

- We now show that they generate the topology, showing that for each $x \in X$, $W \subseteq X$ open with $x \in W$, there is $\varepsilon > 0$ such that
$$B_d(x,\varepsilon) \subseteq W$$
to this end, for each $i \in J_x$ consider
$$\varphi_i(V_i \cap W)$$
this is open and contains $\varphi_i(x)$ so there is ε_i such that
$$B_{\mathbb{R}^n}(\varphi_i(x),\varepsilon) \subseteq \varphi_i(V_i \cap W)$$
hence we let
$$\varepsilon = \min_{i \in I_x} \varepsilon_i \quad ,$$
so
$$\begin{gathered} d_x(y) < \varepsilon \Rightarrow \exists i (x \in V_i \wedge y \in V_i \wedge |\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} < \varepsilon) \\ \Rightarrow \exists i (x \in V_i \wedge y \in V_i \wedge \varphi_i(y) \in \varphi_i(V_i \cap W)) \Rightarrow y \in W \quad . \end{gathered}$$

- We want to prove (2.2) and (2.3), *i.e.* we fix $y \in X$, and $0 < \beta < \alpha$, we will show that there exists
$$\rho_d = \rho_d(y,\alpha,\beta)$$
such that, for all $x, z \in X$,
$$d_x(y) \le \beta \wedge d_y(z) < \rho_d \Rightarrow d_x(z) < \alpha \quad , \tag{(2.2)}$$
$$d_x(y) \ge \alpha \wedge d_y(z) < \rho_d \Rightarrow d_x(z) > \beta \quad . \tag{(2.3)}$$

- Let
$$A_y = \bigcap_{i \in J_y} V_j = \bigcap_{i \in I, y \in V_i} V_i$$
$$U_y = \bigcup_{i \in J_y} V_j = \bigcup_{i \in I, y \in V_i} V_i$$
that are open.

- Let us fix W open such that $\overline{W}$ is compact, $\overline{W}$ intersects only finitely many V_i and
$$y \in W \subseteq \overline{W} \subseteq A_y \quad .$$
Let $\tilde{J} = \{i \in I : V_i \cap \overline{W} \neq \emptyset\}$ be this finite set of indexes.

- As proven above, there is an $\tilde{\varepsilon} > 0$ such that

$$B_d(y, \tilde{\varepsilon}) \subseteq W$$

that is

$$\forall z, d_y(z) < \tilde{\varepsilon} \Rightarrow z \in W.$$

- Summarizing, for any $z \in X$, we have

$$d_y(z) < \tilde{\varepsilon} \Rightarrow \tilde{J} \supseteq J_z \supseteq J_y \quad . \tag{A.3}$$

- Note that $d_x(y) = \infty$ iff $x \notin U_y$; for $\varepsilon > 0$ and $\varepsilon < \tilde{\varepsilon}$ let

$$l(\varepsilon) \stackrel{\text{def}}{=} \inf_{x \notin U_y \wedge d_y(z) < \varepsilon} d_x(z) \quad ;$$

it is weakly decreasing in ε; we prove that

$$\lim_{\varepsilon \to 0} l(\varepsilon) = +\infty \quad .$$

In the above infimum we can assume that $x \in V_i$ for an $i \in \tilde{J}$; this follows from (A.3), indeed if for all $i \in \tilde{J}$ we have $x \notin V_i$ then, *a fortiori*, for all $i \in J_z$ we have $x \notin V_i$ hence $d_x(z) = +\infty$.
Let

$$Z = \bigcup_{i \in \tilde{J}} V_i$$

and we know that $\overline{Z}$ is compact

Suppose by contradiction that the above limit does not hold; there is then a sequence z_n with $d_y(z_n) < 1/n$ and $x_n \notin U_y$ such that $d_{x_n}(z_n)$ is bounded; since

$$x_n \in \overline{Z} \setminus U_y \quad , \quad z_n \in \overline{W}$$

we can extract converging sub sequences, up to renumbering the sequence we can assume that $x_n \to x, z_n \to z$ with

$$x \in \overline{Z} \setminus U_y \quad , \quad z \in \overline{W}$$

but then also $d_y(z_n) \to d_y(z) = 0$ so $y = z$; at the same time $d_{x_n}(z_n) \to_n d_x(y) = +\infty$ since $x \notin U_y$.

- There exists then a continuous decreasing function $\lambda : [0, \infty] \to [0, \tilde{\varepsilon})$ such that if $\varepsilon \le \lambda(\beta)$ then $l(\varepsilon) > \beta$.

- We prove this other result. Let

$$\psi_y(z) : W \to [0, \infty] \quad \psi_y(z) = \max_{i \in J_y} |\varphi_i(z) - \varphi_i(y)|_{\mathbb{R}^m}$$

and this is continuous as well, moreover $\psi_y(y) = 0$. For $t > 0$ there is then a $\delta(t) > 0$ such

$$z \in W \wedge d_y(z) < \delta(t) \Rightarrow \psi_y(z) < t \quad .$$

We can choose $\delta(t)$ to be continuous and increasing, and such that $\delta(t) < \tilde{\varepsilon}$.

- Let eventually

$$\rho_d(y, \alpha, \beta) = \min\{\delta(\alpha - \beta), \lambda(\beta)\, \alpha/(1 - \alpha)\} \quad .$$

Note that this satisfies the requirements of Prop. 2.10.

We will show that this solves the problem, in three steps.

Note that

$$d_y(z) < \rho_d \Rightarrow \psi_y(z) < \alpha - \beta$$

- We prove (2.2). Pick now a $x \in X$; if $d_x(y) \le \beta$ then there is an $i \in I$ such that $x \in V_i \wedge y \in V_i$ and

$$|\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} \le \beta$$

  if moreover $d_y(z) < \rho_d$ then $z \in W \subseteq V_i$ so the usual triangular inequality tells that

$$|\varphi_i(z) - \varphi_i(x)|_{\mathbb{R}^m} \le |\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} + |\varphi_i(y) - \varphi_i(z)|_{\mathbb{R}^m} \le \beta + \psi_y(z) < \alpha$$

  and this proves (2.2).

- Regarding (2.3), assuming $d_y(z) < \tilde{\varepsilon}$, we divide two case. If $d_x(y) < \infty$ then consider all $i \in I$ such that $x \in V_i \wedge y \in V_i$ but then $z \in V_i$, so

$$|\varphi_i(z) - \varphi_i(x)|_{\mathbb{R}^m} \ge |\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} - |\varphi_i(y) - \varphi_i(z)|_{\mathbb{R}^m} \ge \alpha - \psi_y(z)$$

  but knowing that $d_y(z) < \rho_d \le \delta(\alpha - \beta)$ we obtain $\psi_y(z) < \alpha - \beta$ so

$$|\varphi_i(z) - \varphi_i(x)|_{\mathbb{R}^m} > \beta$$

  and passing to the minimum in all such i we obtain $d_x(z) > \beta$.

- Assuming $d_y(z) < \tilde{\varepsilon}$ and $d_x(y) = \infty$ we know that $d_y(z) < \rho_d \le \lambda(\beta)$ implies $l(\varepsilon) > \beta$ so $d_x(z) > \beta$. □

In the above proof we understand why ρ_d must depend on y.

A.4 Proof of Prop. 3.12

Proof. Let $Fy = Ay + v$ the affine transformation given by $A \in \mathrm{GL}(\mathbb{R}^n)$, $v \in \mathbb{R}^n$; we know that

$$m|x| \le |Ax| \le M|x|$$

for example setting

$$M = \|A\| \quad , \quad m = \frac{1}{\|A^{-1}\|}$$

where $\|A\|$ is an operator norm for the linear transformation A.

We now estimate $d_{Ff}(Fg)$ noting that

$$\frac{|Fy - Fz|}{1 + |Fy|} \ge \frac{|A(y-z)|}{1 + |v| + |Fy|} \ge \frac{m|y-z|}{1 + |v| + M|y|} \ge \frac{|y-z|}{1+|y|} c_F \;\text{ with }\; c_F = \frac{m}{1 + |v| + M}$$

using the relation

$$\frac{1}{c + ab} \ge \frac{1}{(b+1)(a+c)}$$

valid for all $a, b, c > 0$; hence

$$d_{Ff}(Fg) \ge c_F \, d_f(g)$$

but also

$$d_{Ff}(Fg) \le \frac{1}{c_{F^{-1}}} \, d_f(g) \quad .$$

Let now $F_2 y = A_2 y + v_2$ then

$$d_{Fg}(F_2 g) \le \sup_{y \in \mathbb{R}^n} \frac{|F_2 y - Fy|}{1 + |Fy|} = \sup_{y \in \mathbb{R}^n} \frac{|F_2 F^{-1} y - y|}{1 + |y|} \le c_F(F_2)$$

where

$$c_F(F_2) = \|A_2 - A\| \, \|A^{-1}\| \, |v| + |v - v_2| \quad ;$$

this $c_F(F_2)$ is a "separation" that generates the topology of the space of affine maps: indeed having

$$F_2 F^{-1} y - y = A_2 A^{-1}(y - v) + v_2 - y = (A_2 A^{-1} - \mathbb{I})(y - v) + (v_2 - v)$$

(where $\mathbb{I}$ is the identity operator) hence

$$|F_2 F^{-1} y - y| \leq \|A_2 A^{-1} - \mathbb{I}\|(|y| + |v|) + |v_2 - v| \leq \|A_2 - A\| \|A^{-1}\|(|y| + |v|) + |v - v_2| \quad .$$

We eventually fix $\alpha > 0$, f and F; then choose $\beta < \alpha$ and $\rho = \rho(\alpha, \beta)$ as in Lemma 3.3. If

$$d_f(g) < c_{F^{-1}} \beta \ , \ c_F(F_2) < \rho$$

then

$$d_{Ff}(Fg) < \beta \ , \ d_{Fg}(F_2 g) < \rho$$

and then by Lemma 3.3 $d_{Ff}(F_2 g) < \alpha$. □

Contents

References

[1] Richard S. Hamilton. The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc. (N.S.)*, 7(1):65–222, 1982. ISSN 0273-0979. 1

[2] M.W. Hirsch. *Differential Topology*. Graduate Texts in Mathematics. Springer New York, 1997. ISBN 9780387901480. URL https://books.google.it/books?id=iSvnvOodWl8C. 1.1, 1.2, 1.3, 1, 1.3, 1.4

[3] John L. Kelley and Isaac Namioka. *Linear topological spaces*. Springer-Verlag, New York-Heidelberg, 1976. With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, and Kennan T. Smith, Second corrected printing, Graduate Texts in Mathematics, No. 36. 3

[4] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. ISBN 0-8218-0780-3. 1.2, 1.3

[5] A. C. G. Mennucci. On asymmetric distances. *Analysis and Geometry in Metric Spaces*, 1:200–231, 2013. *DOI: 10.2478/agms-2013-0004*. URL http://cvgmt.sns.it/person/109/. 2.2, 2.14

[6] ______. Geodesics in asymmetric metric spaces. *Analysis and Geometry in Metric Spaces*, 2:115–153, 2014. *DOI: 10.2478/agms-2014-0004*. URL http://cvgmt.sns.it/paper/340/. 2.2

[7] Peter W. Michor. *Manifolds of differentiable mappings*, volume 3 of *Shiva Mathematics Series*. Shiva Publishing Ltd., Nantwich, 1980. ISBN 0-906812-03-8. 1.4

[8] J.R. Munkres. *Topology*. Featured Titles for Topology. Prentice Hall, Incorporated, 2000. ISBN 9780131816299. URL https://books.google.it/books?id=XjoZAQAAIAAJ. 2.30, 2.12, 2.12, A.2

[9] Walter Rudin. *Functional analysis*. McGraw–Hill, 1973.

[10] L.A. Steen and J.A. Seebach. *Counterexamples in Topology*. Dover Books on Mathematics. Dover Publications, 2013. ISBN 9780486319292. URL https://books.google.it/books?id=Gc3DAgAAQBAJ. 2.3