REGULARITY OF THE TRANSPORT DENSITY IN THE OPTIMAL TRANSPORT PROBLEM WITH RIEMANNIAN COST

SAMER DWEIK

Abstract. In this paper, we consider a mass transportation problem with transport cost given by a smooth positive Riemannian metric in a bounded domain Ω where a mass f^+ is sent to a location f^- in Ω (with the possibility of importing/exporting masses from/to the boundary $\partial\Omega$). First, we study the L^p summability of the transport density σ between two regular measures f^+ and f^-. By a geometrical proof, we show that σ belongs to $L^p(\Omega)$ as soon as the source measure f^+ and the target one f^- are both in $L^p(\Omega)$, for all $p \in [1, \infty]$. Moreover, we prove that the transport density in the transport problem to the boundary (i.e. between a mass f^+ and its Riemannian projection onto the boundary, so the target measure is singular) is in $L^p(\Omega)$ provided $f^+ \in L^p(\Omega)$ and Ω satisfies a uniform exterior ball condition. Finally, we collect these results to obtain L^p estimates on the transport density σ in the Riemannian import/export transport problem.

1. Introduction

In the Monge-Kantorovich problem with Euclidean cost $c(x, y) = |x - y|$ (see [18, 20]), it is well known that an important role is played by the *transport density* σ, which represents the work for transporting the mass f^+ through each subset of Ω to f^-. This measure σ also appears in the following minimal flow problem (which is called the Beckmann problem [1]):

$$\min\left\{\int_\Omega |v(x)|\,\mathrm{d}x \,:\, v \in L^1(\Omega, \mathbb{R}^d),\ \nabla\cdot v = f := f^+ - f^- \text{ in } \Omega,\ v\cdot n = 0 \text{ on } \partial\Omega\right\}. \tag{1.1}$$

More precisely, if v is an optimal flow in Problem (1.1) then one can show that $v = -\sigma\nabla u$, where u is a 1−Lipschitz function with $|\nabla u| = 1$ σ−a.e. (this function u is called a Kantorovich potential, since it maximizes the dual of the Kantorovich problem; see [24]). Hence, σ and u solve together the so-called Monge-Kantorovich system:

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f & \text{in } \Omega,\\ \sigma\nabla u\cdot n = 0 & \text{on } \partial\Omega,\\ |\nabla u| \le 1 & \text{in } \Omega,\\ |\nabla u| = 1 & \sigma-\text{a.e.} \end{cases} \tag{1.2}$$

In [8, 9, 10, 23], the authors have already studied the regularity of this transport density σ. They proved that σ belongs to $L^p(\Omega)$ as soon as f^+ and f^- are in $L^p(\Omega)$, for all $p \in [1, \infty]$. While in [12], the author proved by a family of counter examples that in general σ does not belong to $W^{1,p}(\Omega)$ (resp. $C^{0,\alpha}(\Omega)$) even if $f^\pm \in W^{1,p}(\Omega)$ (resp. $C^{0,\alpha}(\Omega)$).

On the other hand, the authors of [5, 14, 15, 22] considered the transport problem to the boundary with Euclidean cost. In this case, the Beckmann formulation (1.1) becomes

$$\min\left\{\int_\Omega |v(x)|\,\mathrm{d}x \,:\, v \in L^1(\Omega, \mathbb{R}^d),\ \nabla\cdot v = f^+ \text{ in } \mathring{\Omega}\right\}.$$

As the target measure in this transport problem is an arbitrary measure on $\partial\Omega$, then the system (1.2) will be complemented now with a Dirichlet boundary condition:

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f^+ & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases} \tag{1.3}$$

We note that the pair (σ, u) in the system (1.3) models (in a statical or dynamical framework) the configuration of stable or growing sandpiles, where u gives the pile shape and σ stands for sliding layer. On the other hand, one can show that the optimal choice in the transport problem to the boundary is to take the target measure f^- equals the projection $P_\# f^+$ of f^+ onto $\partial\Omega$, where P denotes the projection map onto the boundary. Then, the measure σ in (1.3) is nothing else than the transport density in the classical Monge-Kantorovich system (1.2) between f^+ and $P_\# f^+$. As the target measure $P_\# f^+$ is now singular, then it is not clear whether the transport density σ belongs to $L^p(\Omega)$ or not even if $f^+ \in L^p(\Omega)$. However, the authors of [14] have already studied the L^p summability of this transport density; they proved that as soon as Ω satisfies a uniform exterior ball condition, the transport density σ in (1.3) is in $L^p(\Omega)$ provided that $f^+ \in L^p(\Omega)$, for all $p \in [1, \infty]$.

In this paper, we are mainly concerned with the L^p summability of the transport density σ between two masses f^+ and f^- in the transport problem with Riemannian cost $c(x,y) = d_k(x,y)$. In fact, this transport problem is used to model a non-uniform cost for the movement (due, for instance, to geographical obstacles) and it has been already considered in several papers [21, 2, 16, 11]. In [21], it has been shown that the equivalent minimal flow formulation of the Monge-Kantorovich problem with Riemannian metric k is the following "weighted" Beckmann problem:

$$\min\left\{ \int_\Omega k(x)|v(x)|\,\mathrm{d}x \,:\, v \in L^1(\Omega, \mathbb{R}^d),\ \nabla\cdot v = f \text{ in } \Omega,\ v\cdot n = 0 \text{ on } \partial\Omega \right\}. \tag{1.4}$$

More precisely, if v is an optimal flow in Problem (1.4) then we have $v = -k^{-1}\sigma\frac{\nabla u}{|\nabla u|}$, where u is the Kantorovich potential in the transport problem with Riemannian metric k (so, u is $1-$Lip w.r.t. the distance d_k and $|\nabla u| = k$ $\sigma-$a.e.). In particular, the pair (σ, u) solves the following weighted Monge-Kantorovich system:

$$\begin{cases} -\nabla\cdot[k^{-1}\sigma\frac{\nabla u}{|\nabla u|}] = f & \text{in } \Omega, \\ k^{-1}\sigma\frac{\nabla u}{|\nabla u|}\cdot n = 0 & \text{on } \partial\Omega, \\ |\nabla u| \le k & \text{in } \Omega, \\ |\nabla u| = k & \sigma - \text{a.e.} \end{cases} \tag{1.5}$$

In addition, we study the L^p summability of the transport density σ in the import/export transport problem with Riemannian cost (see also [13]). In this case, Problem (1.4) becomes

$$\min\left\{ \int_\Omega k(x)|v(x)|\,\mathrm{d}x \,:\, v \in L^1(\Omega, \mathbb{R}^d),\ \nabla\cdot v = f^+ - f^- \text{ in } \mathring{\Omega} \right\}.$$

Moreover, the weighted Monge-Kantorovich system (1.5) becomes with a Dirichlet boundary condition:

$$
\begin{cases}
-\nabla\cdot[k^{-1}\,\sigma\frac{\nabla u}{|\nabla u|}]=f & \text{in }\ \Omega,\\
u=0 & \text{on }\ \partial\Omega,\\
|\nabla u|\le k & \text{in }\ \Omega,\\
|\nabla u|=k & \sigma-\text{a.e.}
\end{cases}
\tag{1.6}
$$

In fact, the summability question of the transport density σ in (1.5) or (1.6) is an interesting one and it is non-trivial since it requires to show some geometric estimates on the transport rays, which are now geodesics (and not just straight lines as in the Euclidean case). To the best of our knowledge, there are no results in the literature concerning the summability of this transport density (we recall that the only known results about the L^p summability of σ concern the Euclidean case, see [8, 9, 10, 23, 14, 13]). Hence, the novelty of the present paper is to extend the L^p estimates on σ to the Riemannian case.

The paper is organized as follows. In Section 2, we recall some well known facts, terminology and notations concerning some optimal transport problems with Riemannian cost, their dual formulations, as well as their equivalent minimal flow formulations. In Section 3, we will show that the transport density σ between f^+ and f^- belongs to $L^p(\Omega)$ as soon as $f^\pm\in L^p(\Omega)$. In Section 4, we prove that the transport density σ between f^+ and its Riemannian projection on the boundary $P_\# f^+$ is in $L^p(\Omega)$ provided $f^+\in L^p(\Omega)$ and Ω satisfies a uniform exterior ball condition. In Section 5, we extend the L^p estimates of Section 4 to the case where we have an additionally boundary cost in the transport problem. Finally, Section 6 gives the application of the results of the previous sections to the regularity of the transport density σ in the general Riemannian import/export transport problem.

2. Transport problem with Riemannian cost

2.1. **The Monge-Kantorovich problem with Riemannian cost.** Let k be a smooth (say $C^{1,1}$) positive function on $\mathbb{R}^d$. We denote by d_k the Riemannian metric associated with k:

$$d_k(x,y)=\min\left\{\int_0^1 k(\gamma(t))|\gamma'(t)|\,\mathrm{d}t:\gamma\in\mathrm{Lip}([0,1],\mathbb{R}^d),\,\gamma(0)=x\text{ and }\gamma(1)=y\right\},\,\forall\,x,\,y\in\mathbb{R}^d.$$

Given a compact domain $\Omega\subset\mathbb{R}^d$, let f^+ and f^- be two nonnegative Borel measures on Ω such that $f^+(\Omega)=f^-(\Omega)$. Then, we consider the Monge problem with Riemannian cost:

$$\inf\left\{\int_\Omega d_k(x,T(x))\,\mathrm{d}f^+(x)\ :\ T_\# f^+=f^-\right\},\tag{2.1}$$

where $T_\#$ denotes the pushforward operator acting on every Borel measure f^+ according to the formula $T_\# f^+(B):=f^+(T^{-1}(B))$, for all Borel set $B\subset\Omega$. Although this problem may have no solutions, its relaxed setting always has ones. The latter setting is the following Kantorovich problem with Riemannian cost:

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda(x,y)\ :\ \Lambda\in\mathcal{M}^+(\Omega\times\Omega),\ (\Pi_x)_\#\Lambda=f^+\ \text{ and }\ (\Pi_y)_\#\Lambda=f^-\right\},\tag{2.2}$$

where $\mathcal{M}^+(\Omega\times\Omega)$ denotes the set of nonnegative Borel measures Λ on $\Omega\times\Omega$, Π_x and Π_y are the projections w.r.t. x and y, respectively. In [17], the authors proved existence of an optimal transport map T (or equivalently, an optimal transport plan Λ which is concentrated on the graph of a map T) under the assumption that $f^+ \in L^1(\Omega)$. On the other hand, the problem (2.2) admits a dual formulation

$$\sup\left\{ \int_\Omega u \,\mathrm{d}(f^+ - f^-) \,:\, |u(x) - u(y)| \leq d_k(x,y),\ \forall\, x,\, y \in \Omega \right\}. \tag{2.3}$$

We note that if a function u is $1-$Lipschitz with respect to the geodesic distance d_k, then one has $|\nabla u(x)| \leq k(x)$ for almost every $x \in \Omega$, while the converse is true as soon as Ω is geodesically convex. Thanks to the duality min (2.2) = sup (2.3), we see that if Λ is an optimal transport plan and u is a Kantorovich potential (i.e. a maximizer for Problem (2.3)), then the following equality must hold

$$u(x) - u(y) = d_k(x,y),\ \text{for all } (x,y) \in \operatorname{spt}(\Lambda). \tag{2.4}$$

Any maximal geodesic $\gamma_{x,y}$ between x and y that satisfies the equality (2.4) will be called a transport ray. In other words, any optimal transport plan Λ has to move the mass along these transport rays. From [17, Lemma 9], we have that two different transport rays cannot intersect at an interior point of one of them.

For an optimal transport plan Λ, we define a nonnegative measure σ_Λ on Ω (so-called *transport density*) which represents the amount of transport taking place in each region of Ω. Assume that Ω is geodesically convex. Then, this transport density σ_Λ is defined as follows:

$$< \sigma_\Lambda, \phi > := \int_{\Omega\times\Omega} \int_0^1 \phi(\gamma_{x,y}(t))\, k(\gamma_{x,y}(t))\, |\gamma_{x,y}'(t)| \,\mathrm{d}t \,\mathrm{d}\Lambda(x,y),\ \text{ for all } \phi \in C(\Omega), \tag{2.5}$$

where $\gamma_{x,y}$ is the unique geodesic between x and y. From (2.5), we also see that the following holds

$$\sigma_\Lambda(B) := \int_{\Omega\times\Omega} \mathcal{H}_k^1(B \cap \gamma_{x,y}) \,\mathrm{d}\Lambda(x,y),\ \text{ for all Borel set } B \subset \Omega,$$

where $\mathcal{H}_k^1(\gamma) := \int_0^1 k(\gamma(t))|\gamma'(t)| \,\mathrm{d}t$ and $\sigma_\Lambda(\Omega) = \min$ (2.2). On the other hand, we define a vector measure v_Λ, which is the vector version of $k^{-1}\,\sigma_\Lambda$, as follows

$$< v_\Lambda, \xi > := \int_{\Omega\times\Omega} \int_0^1 \xi(\gamma_{x,y}(t)) \cdot \gamma_{x,y}'(t) \,\mathrm{d}t \,\mathrm{d}\Lambda(x,y), \quad \text{for all } \ \xi \in C(\Omega, \mathbb{R}^d).$$

By [17, Lemma 10], one can show that if u is a Kantorovich potential, then u is differentiable at any interior point of a transport ray $\gamma_{x,y}$ and one has $\nabla u(\gamma_{x,y}(t)) = -k(\gamma_{x,y}(t))\frac{\gamma_{x,y}'(t)}{|\gamma_{x,y}'(t)|}$, for all $t \in (0,1)$. Hence, we have $v_\Lambda = -k^{-1}\sigma_\Lambda \frac{\nabla u}{|\nabla u|}$. Moreover, by [21], one can show that v_Λ solves

$$\min\left\{ \int_\Omega k|v| \,:\, v \in \mathcal{M}(\Omega, \mathbb{R}^d),\ \nabla \cdot v = f \ \text{ in } \ \Omega \right\}, \tag{2.6}$$

where $\mathcal{M}(\Omega, \mathbb{R}^d)$ is the set of vector measures on Ω and the constraint $\nabla \cdot v = f$ is equivalent to say that $-\int_{\bar{\Omega}} \nabla\phi \cdot \mathrm{d}v = \int_{\bar{\Omega}} \phi \,\mathrm{d}f$, for all $\phi \in C^1(\bar{\Omega})$. In addition, the authors of [21] proved that every solution of Problem (2.6) is of the form $v = v_\Lambda$, for some optimal transport plan Λ (in general, the optimal transport plan Λ is not unique). However, thanks to [17, Section 7], one can show that $\sigma_\Lambda := \sigma$ is unique (i.e. it does not depend on the choice of the optimal transport plan Λ) as soon as f^+ or f^- is in $L^1(\Omega)$. Finally, we note that the primal-dual optimality conditions in the above problems can be written in the following PDE form:

$$\begin{cases} -\nabla\cdot[k^{-1}\,\sigma\frac{\nabla u}{|\nabla u|}] = f & \text{in } \Omega, \\ k^{-1}\sigma\frac{\nabla u}{|\nabla u|}\cdot n = 0 & \text{on } \partial\Omega, \\ |\nabla u| \le k & \text{in } \Omega, \\ |\nabla u| = k & \sigma-\text{a.e.} \end{cases} \tag{2.7}$$

In Section 3, we will prove L^p estimates on the transport density σ in the system (2.7). More precisely, we show that under some assumptions on the metric k the following statement holds: if f^+ and f^- are both in $L^p(\Omega)$, then σ is also in $L^p(\Omega)$, for all $p \in [1,\infty]$.

2.2. **The transport problem to the boundary with Riemannian cost.** In [3, 4], the authors studied the transport problem in the presence of a Dirichlet region Σ (i.e., a region where transportation is free). Here, we assume that $\Sigma = \partial\Omega$. So, we have a mass f^+ inside Ω that we transport to the boundary $\partial\Omega$ paying only the transport cost which is given by a Riemannian metric (in Section 5, we study the transport problem with boundary cost). In other words, we consider the following problem

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda(x,y)\ :\ \Lambda\in\mathcal{M}^+(\Omega\times\Omega),\ (\Pi_x)_\#\Lambda = f^+ \ \text{and}\ \operatorname{spt}[(\Pi_y)_\#\Lambda]\subset\partial\Omega\right\}. \tag{2.8}$$

This means that the target measure $(\Pi_y)_\#\Lambda$ is completely arbitrary on $\partial\Omega$. But, it is easy to see that the optimal choice for $(\Pi_y)_\#\Lambda$ is to be equal the projection $P_\# f^+$ of f^+ onto the boundary, where

$$P(x) := \operatorname{argmin}\{d_k(x,y),\, y\in\partial\Omega\},\ \text{for all } x\in\Omega.$$

Notice that P is a multivalued map, but it is a singleton at all the points x where the distance function to the boundary $d_k(\cdot,\partial\Omega)$ is differentiable (so, at a.e. x). Then, $\Lambda := (I,P)_\# f^+$ is the unique optimal transport plan in Problem (2.8) if $f^+\in L^1(\Omega)$. On the other hand, Problem (2.8) has a dual formulation, which is the following:

$$\sup\left\{\int_\Omega u\,\mathrm{d}f^+\ :\ |\nabla u|\le k,\ u=0\ \text{on}\ \partial\Omega\right\}. \tag{2.9}$$

Moreover, Problem (2.6) becomes

$$\min\left\{\int_\Omega k|v|\ :\ v\in\mathcal{M}(\Omega,\mathbb{R}^d),\ \nabla\cdot v = f^+\ \text{in}\ \mathring{\Omega}\right\}, \tag{2.10}$$

where $\nabla\cdot v = f^+$ in $\mathring{\Omega}$ is in the sense that $-\int_\Omega \nabla\phi\cdot\mathrm{d}v = \int_\Omega \phi\,\mathrm{d}f^+$, for all $\phi\in C^1(\Omega)$ such that $\phi = 0$ on $\partial\Omega$. In addition, it is easy to see that $d_k(\cdot,\partial\Omega)$ is the Kantorovich potential, $v_\Lambda := -k^{-1}\sigma_\Lambda\frac{\nabla d_k(\cdot,\partial\Omega)}{|\nabla d_k(\cdot,\partial\Omega)|}$ is the unique optimal flow for Problem (2.10), and the pair $(\sigma_\Lambda, d_k(\cdot,\partial\Omega))$ is the unique solution for the following system:

$$\begin{cases} -\nabla\cdot[k^{-1}\,\sigma\frac{\nabla u}{|\nabla u|}] = f^+ & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| \le k & \text{in } \Omega, \\ |\nabla u| = k & \sigma-\text{a.e.} \end{cases} \tag{2.11}$$

One of the questions that we consider in this paper is whether the transport density σ in (2.11) is in $L^p(\Omega)$ or not when $f^+\in L^p(\Omega)$. This result will generalize the one given in [14] where the transport cost was assumed to be given by the Euclidean distance. We note that the L^p estimates on the transport density σ in the Riemannian case require a different proof from the one introduced in [14], where the authors used some symmetrization techniques which do not work when the transport cost is non-uniform. Moreover, the L^p summability of this transport density σ will not follow from Section 3, since the target measure is now singular. These L^p estimates on σ will be introduced in Section 4 (see also Section 5 for the summability of σ in the transport problem with boundary cost).

2.3. **The import/export transport problem with Riemannian cost.** In [19, 13], the authors studied an import/export transportation problem, where we have two masses f^+ and f^- (which do not have a priori the same total mass) in the interior of Ω and we transport f^+ to f^- with the possibility to import/export masses from/to the boundary, paying the transport cost $c(x,y) = d_k(x,y)$ for each unit of mass x that moves to the destination y (plus possibly two additional costs on the boundary; see Section 6). In other words, we consider the following minimization problem

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda\,,\ \Lambda\in\mathcal{M}^+(\Omega\times\Omega)\ :\ [(\Pi_x)_\#\Lambda]_{|\mathring{\Omega}} = f^+,\ [(\Pi_y)_\#\Lambda]_{|\mathring{\Omega}} = f^-\right\}. \tag{2.12}$$

It is well known (see, for instance, [11]) that Problem (2.12) has a dual formulation which is the following

$$\sup\left\{\int_\Omega u\,\mathrm{d}(f^+ - f^-)\ :\ |\nabla u|\le k,\ u=0\ \text{ on }\ \partial\Omega\right\}. \tag{2.13}$$

Moreover, if Λ is an optimal transport plan for Problem (2.12), then the vector measure v_Λ solves

$$\min\left\{\int_\Omega k|v|\ :\ v\in\mathcal{M}(\Omega,\mathbb{R}^d),\ \nabla\cdot v = f^+ - f^-\ \text{ in }\ \mathring{\Omega}\right\}. \tag{2.14}$$

In addition, the transport density σ_Λ in Problem (2.12) with the Kantorovich potential u in Problem (2.13) solve together

$$\begin{cases} -\nabla\cdot[k^{-1}\,\sigma\frac{\nabla u}{|\nabla u|}] = f & \text{in } \ \Omega,\\ u=0 & \text{on } \ \partial\Omega,\\ |\nabla u|\le k & \text{in } \ \Omega,\\ |\nabla u| = k & \sigma-\text{a.e.}\end{cases} \tag{2.15}$$

In Section 6, we will study the L^p summability of the transport density σ in (2.15). More precisely, we will show that σ can be decomposed into three transport densities: σ_{ii} which represents the transport density from a part of f^+ to a part of f^-, σ_{ib} is the transport density in the export transport problem (i.e. from a part of f^+ to the boundary) and σ_{bi} is the transport density in the import transport problem (i.e. from the boundary to a part of f^-). Hence, we will combine the results of Sections 3, 4 & 5 to obtain L^p estimates on the transport densities σ_{ii}, σ_{ib} and σ_{bi}, then L^p estimates on σ. Finally, we also get L^p estimates on the transport density σ in the case where we have some extra boundary costs g^+ and g^-, i.e. when the system (2.15) becomes

$$\begin{cases} -\nabla\cdot[k^{-1}\,\sigma\frac{\nabla u}{|\nabla u|}] = f & \text{in } \ \Omega,\\ g^+\le u\le g^- & \text{on } \ \partial\Omega,\\ |\nabla u|\le k & \text{in } \ \Omega,\\ |\nabla u| = k & \sigma-\text{a.e.}\end{cases}$$

3. L^p estimates on the transport density in the Monge-Kantorovich problem with Riemannian cost

The aim of this section is to study the L^p summability of the transport density σ between two L^p densities f^+ and f^- in the Monge-Kantorovich problem with Riemannian cost (2.2). The strategy of the proof, which is already used in [23], is based on a displacement interpolation and an approximation by atomic measures. However, we recall that it is not immediate to extend the L^p estimates in [23] to the Riemannian case since the transport rays are geodesics and then, some technical geometric estimates will be needed in order to prove L^p summability on σ.

First, we need to introduce a geometric lemma which gives a lower bound on the Jacobian of the interpolation map $x \mapsto \gamma_{x,x_0}(t)$. Let k be a positive $C^{1,1}$ function on $\mathbb{R}^d$ such that $0 < k_{\min} \le k \le k_{\max} < \infty$. Let the manifold $\mathbb{R}^d$ be equipped with the Riemannian metric d_k and, let $\Omega \subset \mathbb{R}^d$ be a compact and geodesically convex domain. In the sequel, we will denote by exp the Riemannian exponential map, $[\cdot,\cdot]$ the Lie bracket and ∇ the Levi-Civita connection.

Lemma 3.1. *Fix a point $x_0 \in \Omega$. Then, there exists a constant $C < \infty$ depending only on d, diam(Ω), $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$ and $||D^2k||_\infty$ such that, for a.e. $x \in \Omega$, we have*

$$\det(D_x\gamma_{x,x_0}(t)) \ge (1-t)^C, \text{ for all } t \in [0,1]. \tag{3.1}$$

Proof. For $x \in \Omega$, we will denote by γ_x the unique geodesic between x and the point x_0, $\nu(x)$ the unit tangent vector to γ_x at x and $\tau(x) := d_k(x,x_0)$. Fix a point $x' \in \mathring{\Omega}$ and set $\chi := \{\tilde{x} \in B(x',\varepsilon) \,:\, \tau(\tilde{x}) = \tau(x')\}$, where $\varepsilon > 0$ is small enough so that $\chi \subset \Omega$. Let $(e_1,...,e_d)$ be an orthonormal basis such that $e_d = \nu(\tilde{x})$ and let us parallel-transport along the geodesic $\gamma_{\tilde{x}}$ to define a new family of orthonormal basis $(e_1(\tau),...,e_d(\tau))$ (so, we have $e_d(\tau) = \gamma'_{\tilde{x}}(\tau)$). For all $\tilde{x} \in \chi$ and $\tau \in [0,\tau(x')]$, we define

$$\Psi(\tilde{x},\tau) := \gamma_{\tilde{x}}(\tau) = \exp_{\tilde{x}}[\tau\,\nu(\tilde{x})].$$

Set $\Delta := \{x \in \gamma_{\tilde{x}} \,:\, \tilde{x} \in \chi\}$. We will prove the estimate (3.1) on Δ. Notice that $x \in \Delta$ if and only if there exists a unique pair $(\tilde{x},\tau) \in \chi \times [0,\tau(x')]$ such that $x = \Psi(\tilde{x},\tau)$ and, we have $\gamma_{x,x_0}(t) = \Psi(\tilde{x},(1-t)\tau + t\tau(x'))$, for all $t \in (0,1)$. Then, we get

$$\det(D_x\gamma_{x,\,x_0}(t))\det(D\Psi(\tilde{x},\tau)) = (1-t)\det(D\Psi(\tilde{x},(1-t)\tau + t\tau(x'))). \tag{3.2}$$

Now, we consider small variations of $\tilde{x}$, on χ, in the directions $e_1,...,e_{d-1}$ denoted by $\tilde{x}+\delta e_1,...,\ \tilde{x}+\delta e_{d-1}$. Then, we define the vector fields J_i $(i = 1,...,d-1)$ and J_d as follows:

$$J_i(\tilde{x},\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(\tilde{x}+\delta e_i,\tau), \quad \text{for all } \ i \in \{1,...,d-1\},$$

and

$$J_d(\tilde{x},\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(\tilde{x},\tau+\delta).$$

Notice that, for every $i \in \{1,...,d-1\}$, the vector field J_i has been obtained by differentiating a family of geodesics depending on the parameter δ. Set

$$J(\tilde{x},\tau) = (J_1(\tilde{x},\tau),...,J_d(\tilde{x},\tau)) \quad \text{and} \quad \mathcal{J}(\tilde{x},\tau) = \det[J(\tilde{x},\tau)].$$

In fact, this Jacobian $\mathcal{J}$ cannot vanish, except possibly at the endpoints of the geodesic $\gamma_{\tilde{x}}$ (see [25, Theorem 11.3]). Then, the formula for the differential with respect to τ of the determinant $\mathcal{J}(\tilde{x},\tau)$ yields

$$\mathcal{J}'(\tilde{x},\tau) = \mathrm{tr}[J'(\tilde{x},\tau)\,J(\tilde{x},\tau)^{-1}]\,\mathcal{J}(\tilde{x},\tau). \tag{3.3}$$

Now, let us denote by $\mathrm{d}\Psi$ the differential map of Ψ. The fact that $[\partial_{e_i}, \partial_{e_d}] = 0$ implies that J_i and J_d commute, since

$$[J_i, J_d] = [\mathrm{d}\Psi(\partial_{e_i}), \mathrm{d}\Psi(\partial_{e_d})] = \mathrm{d}\Psi[\partial_{e_i}, \partial_{e_d}] = 0.$$

Then, we have

$$\nabla_{J_i} J_d = \nabla_{J_d} J_i. \tag{3.4}$$

Yet,

$$J_i(\tilde{x}, \tau) = \sum_{j=1}^{d} J_{ji}(\tilde{x}, \tau) e_j(\tau).$$

Hence,

$$\nabla_{J_d} J_i = \nabla_{\gamma'_{\tilde{x}}} J_i = \sum_{j=1}^{d} J'_{ji}(\tilde{x}, \tau) e_j(\tau) + \sum_{j=1}^{d} J_{ji}(\tilde{x}, \tau) \nabla_{\gamma'_{\tilde{x}}} e_j(\tau).$$

Thanks to the fact that $\gamma_{\tilde{x}}$ is a geodesic and the Christoffel symbols vanish, we then have $\nabla_{\gamma'_{\tilde{x}}} e_j = \nabla_{\gamma'_{\tilde{x}}} \gamma'_{\tilde{x}} = 0$, for all $j \in \{1, ..., d-1\}$. Consequently, we get

$$\nabla_{J_d} J_i = \sum_{j=1}^{d} J'_{ji}(\tilde{x}, \tau) e_j(\tau). \tag{3.5}$$

On the other hand, we have

$$\nabla_{J_i} J_d = \sum_{j=1}^{d} J_{ji}(\tilde{x}, \tau) \nabla_{e_j(\tau)} J_d.$$

Now, let V be the matrix, in the basis $\{e_1(\tau), ..., e_d(\tau)\}$, associated with the endomorphism $X \mapsto \nabla_X J_d$ (so, V is the second fundamental form of the submanifold $\{\Psi(\tilde{x}, \tau) \,:\, \tilde{x} \in \chi\}$). Then, one has

$$\nabla_{J_i} J_d = \sum_{j=1}^{d} \sum_{k=1}^{d} J_{ji}(\tilde{x}, \tau) V_{kj}\, e_k(\tau) = \sum_{k=1}^{d} (VJ)_{ki}\, e_k(\tau). \tag{3.6}$$

Combining (3.4), (3.5) & (3.6), we get

$$J' = VJ.$$

Recalling (3.3), this implies that

$$\mathcal{J}'(\tilde{x}, \tau) = \mathrm{tr}[V]\, \mathcal{J}(\tilde{x}, \tau).$$

Yet, it is well known that the distance function $d_k(\cdot, x_0)$ is locally semi-concave in $\mathbb{R}^2 \backslash \{x_0\}$ with $D^2[d_k(\cdot, x_0)](\Psi(\tilde{x}, \tau)) \leq \frac{C}{\tau(x')-\tau} I$, for some constant $C < \infty$ depending only on d, $\mathrm{diam}(\Omega)$, $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$ and $||D^2 k||_\infty$ (see, for instance, [6]). Recalling the definition of J_d, we have $J_d(\tilde{x}, \tau) = -k(\Psi(\tilde{x}, \tau))^{-1} \frac{\nabla d_k(\Psi(\tilde{x}, \tau), x_0)}{|\nabla d_k(\Psi(\tilde{x}, \tau), x_0)|}$. This yields that

$$\mathrm{tr}[V] \geq \frac{-C}{\tau(x') - \tau}.$$

Hence,

$$\mathcal{J}'(\tilde{x}, \tau) \geq \frac{-C}{\tau(x') - \tau}\, \mathcal{J}(\tilde{x}, \tau).$$

We infer that

$$\log[\mathcal{J}(\tilde{x},(1-t)\tau+t\tau(x'))]-\log[\mathcal{J}(\tilde{x},\tau)]\geq C\log(1-t).$$

Then,

$$\frac{\mathcal{J}(\tilde{x},(1-t)\tau+t\tau(x'))}{\mathcal{J}(\tilde{x},\tau)}\geq(1-t)^{C}.$$

Recalling (3.2), we get

$$\det(D_x\gamma_{x,x_0}(t))\geq(1-t)^{C},\ \text{ for all }\ t\in[0,1].$$ □

Proposition 3.2. *Let $k\in C^{1,1}(\mathbb{R}^d)$ with $0<k_{\min}\leq k\leq k_{\max}<\infty$. Assume that $\Omega\subset\mathbb{R}^d$ is a compact, geodesically convex domain and f^+, $f^-\in L^p(\Omega)$. Then, the transport density σ between f^+ and f^- belongs to $L^p(\Omega)$, for every $p\in[1,\infty]$.*

Proof. First, let us assume that the target measure f^- is finitely atomic and let us denote by $(x_i)_{i=1,\dots,n}$ its atoms. Let Λ be an optimal transport plan from f^+ to f^- and let σ be the unique transport density between them (recall that the transport density is unique as soon as $f^+\in L^1(\Omega)$). Set $f_t:=(\Pi_t)_{\#}[d_k\cdot\Lambda]$ where $\Pi_t(x,y)=\gamma_{x,y}(t)$, for all $t\in[0,1]$. As $|\gamma'_{x,y}(t)|=k(\gamma_{x,y}(t))^{-1}d_k(x,y)$, for all $t\in(0,1)$, then by (2.5) we get that

$$\sigma=\int_0^1 f_t\,\mathrm{d}t. \tag{3.7}$$

The aim now is to show L^p estimates on f_t, for all $t\in(0,1)$. As f^- is atomic, then one can decompose Ω into essentially disjoint subsets Ω_i, $i\in\{1,...,n\}$, such that for Λ−a.e. $(x,y)\in\Omega_i\times\Omega$, we have $y=x_i$. In other words, Ω_i represents the set of points x that will be transported to x_i. For all $\phi\in C(\Omega)$, we have

$$<f_t,\phi>=\sum_{i=1}^{n}\int_{\Omega_i\times\Omega}\phi(\gamma_{x,x_i}(t))\,d_k(x,x_i)\,\mathrm{d}\Lambda(x,y)=\sum_{i=1}^{n}\int_{\Omega_i}\phi(\gamma_{x,x_i}(t))\,d_k(x,x_i)\,\mathrm{d}f^+(x).$$

Fix $i\in\{1,...,n\}$. We consider the restriction f_t^i of f_t to Ω_i. Let us take a change of variable $z:=z(x)=\gamma_{x,x_i}(t)$. We note that this map z is one-to-one thanks to the fact that two different transport rays cannot meet at intermediate points. Then, for all $\phi\in C(\Omega_i)$, we get

$$<f_t^i,\phi>=\int_{\Omega_i}\phi(\gamma_{x,x_i}(t))\,d_k(x,x_i)\,\mathrm{d}f^+(x)=\int_{\Omega_i(t)}\phi(z)\,d_k(x,x_i)\,f^+(x)\,\mathcal{J}_t(x)^{-1}\mathrm{d}z,$$

where

$$\Omega_i(t):=\{\gamma_{x,x_i}(t)\,:\,x\in\Omega_i\}$$

and

$$\mathcal{J}_t(x):=\det[D_x\gamma_{x,x_i}(t)].$$

Hence, we have

$$f_t^i(z)=d_k(x,x_i)\,f^+(x)\,\mathcal{J}_t(x)^{-1},\ \text{ for a.e. }z\in\Omega_i(t).$$

Consequently, we get

$$||f_t^i||^p_{L^p(\Omega_i(t))}=\int_{\Omega_i(t)}d_k(x,x_i)^p f^+(x)^p\mathcal{J}_t(x)^{-p}\,\mathrm{d}z=\int_{\Omega_i}d_k(x,x_i)^p f^+(x)^p\mathcal{J}_t(x)^{1-p}\,\mathrm{d}x.$$

So, we need a lower bound on the Jacobian $\mathcal{J}_t$ (we note that in the Euclidean case, this is trivial since we have $\gamma_{x,x_i}(t)=(1-t)x+tx_i$ and so $\mathcal{J}_t(x)=(1-t)^d$). However, thanks to Lemma 3.1, there is a uniform constant $C<\infty$ (which does not depend on i and n) such that

$$\mathcal{J}_t(x)\geq(1-t)^{C}.$$

Hence,

$$||f_t^i||^p_{L^p(\Omega_i(t))} \leq C^p(1-t)^{C(1-p)}||f^+||^p_{L^p(\Omega_i)}, \text{ for all } i \in \{1,...,n\}.$$

Then, we have

$$||f_t||_{L^p(\Omega)} \leq C(1-t)^{\frac{C(1-p)}{p}}||f^+||_{L^p(\Omega)}, \text{ for all } t \in (0,1). \tag{3.8}$$

We see that these L^p estimates in (3.8) on f_t do not depend on the target measure f^- (more precisely, on the number of atoms). Hence, by approximating f^- with a sequence of atomic measures, we get the same L^p estimates (3.8) on f_t in the case where f^- is an arbitrary measure. On the other hand, by symmetry, it is obvious that one can also show the following L^p estimates on f_t but from the other side, i.e. by approximating f^+ with a sequence of atomic measures:

$$||f_t||_{L^p(\Omega)} \leq C\, t^{\frac{C(1-p)}{p}}||f^-||_{L^p(\Omega)}, \text{ for all } t \in (0,1). \tag{3.9}$$

Combining (3.8) & (3.9), we infer that

$$||f_t||_{L^p(\Omega)} \leq C\, 2^{\frac{C(p-1)}{p}} \max\{||f^+||_{L^p(\Omega)}, ||f^-||_{L^p(\Omega)}\}, \text{ for all } t \in (0,1).$$

Recalling (3.7), we get

$$||\sigma||_{L^p(\Omega)} \leq C\, ||f||_{L^p(\Omega)}.$$

However, there is an issue here: the previous L^p estimates on f_t for $t \leq \frac{1}{2}$ and $t \geq \frac{1}{2}$ have been obtained by discrete approximations of f^- and f^+, respectively. If the two approximations converge to two different transport plans between f^+ and f^-, then we cannot glue together the two estimates on f_t and deduce anything on σ. The idea is to consider instead the transport cost $d_k(x,y)^{1+\varepsilon}$ (where $\varepsilon > 0$) since now the Kantorovich problem will have a unique optimal transport plan Λ_ε (see, for instance, [7]) and so, if $f_t^\varepsilon := (\Pi_t)[d_k^{1+\varepsilon} \cdot \Lambda_\varepsilon]$ then one can prove exactly as above the following L^p estimates on f_t^ε:

$$||f_t^\varepsilon||_{L^p(\Omega)} \leq C \max\{||f^+||_{L^p(\Omega)}, ||f^-||_{L^p(\Omega)}\}, \text{ for all } t \in (0,1). \tag{3.10}$$

Yet, it is not difficult to see that when $\varepsilon \to 0$, $\Lambda_\varepsilon \rightharpoonup \Lambda$ for some optimal transport plan Λ of Problem (2.2) (with transport cost $d_k(x,y)$). Moreover, $f_t^\varepsilon \rightharpoonup f_t$ and passing to the limit when $\varepsilon \to 0$ in (3.10), we get

$$||f_t||_{L^p(\Omega)} \leq C \max\{||f^+||_{L^p(\Omega)}, ||f^-||_{L^p(\Omega)}\}, \text{ for all } t \in (0,1). \qquad \square$$

4. L^p ESTIMATES ON THE TRANSPORT DENSITY IN THE TRANSPORT PROBLEM TO THE BOUNDARY

Throughout this section, we assume that between any two points x and y in Ω, there is a unique geodesic $\gamma_{x,y}$. The aim will be to study the L^p summability of the transport density σ in (2.11), i.e. between a density $f^+ \in L^p(\Omega)$ and its Riemannian projection onto the boundary $P_\# f^+$, where

$$P(x) := \operatorname{argmin}\{d_k(x,y) \,:\, y \in \partial\Omega\}.$$

Let $\Lambda := (I,P)_\# f^+$ be the unique optimal transport plan for Problem (2.8) and σ be the corresponding transport density. Let us denote by γ_x a (the) geodesic from x to $\partial\Omega$ (i.e. $\gamma_x = \gamma_{x,y}$, for some $y \in P(x)$). We note that this geodesic γ_x is unique at every point x where the distance function to the boundary $d_k(\cdot,\partial\Omega)$ is differentiable, then at almost every point x. From (2.5), one has

$$< \sigma,\phi >:= \int_\Omega \int_0^1 \phi(\gamma_x(t))\, k(\gamma_x(t))\, |\gamma_x'(t)| f^+(x)\,\mathrm{d}t\,\mathrm{d}x, \quad \text{for all} \quad \phi \in C(\Omega).$$

Yet, we have

$$\gamma_x'(t) := -k(\gamma_x(t))^{-1} d_k(x,\partial\Omega)\, \frac{\nabla d_k(\gamma_x(t),\partial\Omega)}{|\nabla d_k(\gamma_x(t),\partial\Omega)|}, \ \text{ for all } \ t \in (0,1).$$

Hence,

$$< \sigma,\phi >:= \int_\Omega \int_0^1 \phi(\gamma_x(t))\, d_k(x,\partial\Omega)\, f^+(x)\,\mathrm{d}t\,\mathrm{d}x, \quad \text{for all} \quad \phi \in C(\Omega).$$

This implies that

$$\sigma = \int_0^1 f_t\,\mathrm{d}t,$$

where

$$f_t := P_{t\#}[d_k(\cdot,\partial\Omega) f^+] \ \text{ and } \ P_t(x) := \gamma_x(t), \ \text{for a.e. } x \in \Omega.$$

The aim now is to find an explicit formula for the transport density σ. From the definition of f_t, we have

$$< f_t,\phi >= \int_\Omega \phi(P_t(x))\, d_k(x,\partial\Omega)\, f^+(x)\,\mathrm{d}x, \quad \text{for all} \quad \phi \in C(\Omega).$$

For a.e. $x \in \Omega$, set $y := P_t(x)$ (we note that this map P_t is one-to-one). Moreover, it is easy to see that $d_k(y,\partial\Omega) = (1-t)\, d_k(x,\partial\Omega)$. So, we get

$$< f_t,\phi >= \int_{\Omega_t} \phi(y)\, (1-t)^{-1} d_k(y,\partial\Omega)\, f^+(P_t^{-1}(y))\, \mathcal{J}_t(y)^{-1}\,\mathrm{d}y, \quad \text{for all} \ \ \phi \in C(\Omega),$$

where

$$\Omega_t := P_t(\Omega) \ \text{ and } \ \mathcal{J}_t(y) := [\det(DP_t(x))], \ \text{for a.e. } y \in \Omega_t.$$

This yields that

$$f_t(y) = (1-t)^{-1} d_k(y,\partial\Omega)\, f^+(P_t^{-1}(y))\, \mathcal{J}_t(y)^{-1}, \ \text{ for a.e. } y \in \Omega_t.$$

For a.e. $y \in \Omega$, let us denote by $L_k(y)$ the length of the maximal geodesic $\gamma : [0,1] \mapsto \Omega$ passing through y such that $P(\gamma(t)) = P(y)$, for all $t \in (0,1)$. So, $y \in \Omega_t$ if and only if $d_k(y,\partial\Omega) \le (1-t)\, L_k(y)$. Hence, one has

$$\sigma(y) = \int_0^{1-\frac{d_k(y,\partial\Omega)}{L_k(y)}} \frac{d_k(y,\partial\Omega)}{1-t}\, f^+(P_t^{-1}(y))\, \mathcal{J}_t(y)^{-1}\,\mathrm{d}t, \ \text{ for a.e. } y \in \Omega. \tag{4.1}$$

In order to prove L^p estimates on the transport density σ, we need to show (as in Section 3) a lower bound on the Jacobian $\mathcal{J}_t$. Before that, we introduce the following:

Definition 4.1. *We say that a compact domain $\Omega \subset \mathbb{R}^d$ satisfies a uniform exterior ball condition of radius $r > 0$ if for every point $x \in \partial\Omega$, there is a point $a \in \mathbb{R}^d\backslash\Omega$ such that $\overline{B(a,r)} \cap \Omega = \{x\}$.*

Lemma 4.1. *Let k be a $C^{1,1}$ function on $\mathbb{R}^d$ such that $0 < k_{\min} \leq k \leq k_{\max} < \infty$. Let $\Omega \subset \mathbb{R}^d$ be a compact domain satisfying a uniform exterior ball condition of radius r and, assume that for every two points x and y in Ω, there is a unique geodesic from x to y. Then, there exists a constant $C > 0$ depending only on d, r, $diam(\Omega)$, $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$ and $||D^2 k||_\infty$ such that, for a.e. $x \in \Omega$, we have the following estimate:*

$$det(DP_t(x)) \geq C(1-t).$$

Proof. For every $s \in \partial\Omega$, let $\tau(s)$ be the length of the maximal unit geodesic γ_s starting from s with $P(\gamma_s(\tau)) = s$, for all $\tau \in [0, \tau(s)]$. Let $\nu(s)$ be the unit inner normal vector to $\partial\Omega$ at s. For almost every $x \in \Omega$, it is clear that there exists unique $s \in \partial\Omega$ and $\tau \in [0, \tau(s)]$ such that $x = \Psi(s,\tau) := \exp_s \tau\nu(s)$. For every $t \in [0,1]$, we have that $P_t(\Psi(s,\tau)) = \Psi(s,(1-t)\tau)$, for all $s \in \partial\Omega$ and $\tau \in [0,\tau(s)]$. Then, we get

$$\det(DP_t(\Psi(s,\tau)))\det(D\Psi(s,\tau)) = (1-t)\det(D\Psi(s,(1-t)\tau)). \tag{4.2}$$

For $s \in \partial\Omega$, let $(e_1, ..., e_d)$ be an orthonormal basis such that $e_d = \nu(s)$ and let us parallel-transport along the geodesic γ_s to define a new family of orthonormal basis $(e_1(\tau), ..., e_d(\tau))$. Now, consider small variations of s, on $\partial\Omega$, in the directions $e_1, ..., e_{d-1}$. Then, we define as in Lemma 3.1 the vector fields J_i $(i = 1, ..., d-1)$ and J_d :

$$J_i(s,\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(s+\delta e_i,\tau), \text{ for all } i \in \{1, ..., d-1\},$$

and

$$J_d(s,\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(s,\tau+\delta).$$

Set

$$J(s,\tau) = (J_1(s,\tau), J_2(s,\tau), ..., J_d(s,\tau)) \quad \text{and} \quad \mathcal{J}(s,\tau) = \det[J(s,\tau)].$$

We have

$$\mathcal{J}'(s,\tau) = \mathrm{tr}[J'(s,\tau)\,J(s,\tau)^{-1}]\,\mathcal{J}(s,\tau). \tag{4.3}$$

In fact, one can show (see Lemma 3.1) that $J'(s,\tau)J(s,\tau)^{-1}$ is the second fundamental form of the submanifold $\{\Psi(s',\tau) : s' \in \partial\Omega \cap B(s,\varepsilon)\}$, where $\varepsilon > 0$ is small enough. Yet, we have $J_d(s,\tau) = k(\Psi(s,\tau))^{-1}\frac{\nabla d_k(\Psi(s,\tau),\partial\Omega)}{|\nabla d_k(\Psi(s,\tau),\partial\Omega)|}$. Moreover, the distance function $d_k(\cdot,\partial\Omega)$ is semi-concave as soon as Ω satisfies a uniform exterior ball condition of radius $r > 0$ and $k \in C^{1,1}(\Omega)$ with $0 < k_{\min} \leq k \leq k_{\max} < \infty$ (see [6, Theorem 8.2.7]). More precisely, there is a constant C which only depends on d, r, $\mathrm{diam}(\Omega)$, $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$ and $||D^2 k||_\infty$ such that $D^2[d_k(\cdot,\partial\Omega)] \leq CI$. Hence, we obtain that

$$\mathrm{tr}[J'(s,\tau)J(s,\tau)^{-1}] \leq C.$$

By (4.3), we get

$$\mathcal{J}'(s,\tau) \leq C\,\mathcal{J}(s,\tau).$$

This implies that

$$\mathcal{J}(s,(1-t)\tau) \geq \mathcal{J}(s,\tau)\,e^{-Ct\tau}.$$

Recalling (4.2), we infer that

$$\det(DP_t(x)) \geq e^{-Ct\tau}(1-t).$$

□

Proposition 4.2. *Under the assumptions of Lemma 4.1, the transport density σ between f^+ and $P_\# f^+$ belongs to $L^p(\Omega)$ as soon as $f^+ \in L^p(\Omega)$, for all $p \in [1,\infty]$. Moreover, there is a constant $C := C(d,\, r,\, diam(\Omega),\, k_{\min},\, k_{\max},\, ||\nabla k||_\infty,\, ||D^2 k||_\infty) < \infty$ such that the following estimate holds:*

$$||\sigma||_{L^p(\Omega)} \le C||f^+||_{L^p(\Omega)}.$$

Proof. From (4.1), we have

$$||\sigma||^p_{L^p(\Omega)} = \int_\Omega \bigg(\int_0^{1-\frac{d_k(y,\partial\Omega)}{L_k(y)}} \frac{d_k(y,\partial\Omega)}{1-t}\, f^+(P_t^{-1}(y))\, \mathcal{J}_t(y)^{-1}\, \mathrm{d}t \bigg)^p \mathrm{d}y.$$

Using Hölder's inequality, we get

$$||\sigma||^p_{L^p(\Omega)} \le \int_\Omega \bigg(\int_0^{1-\frac{d_k(y,\partial\Omega)}{L_k(y)}} \frac{d_k(y,\partial\Omega)^q}{(1-t)^q} \mathcal{J}_t(y)^{-1} \mathrm{d}t \bigg)^{p/q} \bigg(\int_0^{1-\frac{d_k(y,\partial\Omega)}{L_k(y)}} f^+(P_t^{-1}(y))^p \mathcal{J}_t(y)^{-1} \mathrm{d}t \bigg) \mathrm{d}y.$$

Thanks to Lemma 4.1, there is a constant $C > 0$ such that, for a.e. $y \in \Omega_t$, we have the following estimate:

$$\mathcal{J}_t(y) \ge C(1-t).$$

Then, we infer that

$$||\sigma||^p_{L^p(\Omega)} \le C^{-p/q} \int_\Omega \bigg(\int_0^{1-\frac{d_k(y,\partial\Omega)}{L_k(y)}} \frac{d_k(y,\partial\Omega)^q}{(1-t)^{q+1}}\, \mathrm{d}t \bigg)^{p/q} \bigg(\int_0^{1-\frac{d_k(y,\partial\Omega)}{L_k(y)}} f^+(P_t^{-1}(y))^p\, \mathcal{J}_t(y)^{-1}\, \mathrm{d}t \bigg) \mathrm{d}y.$$

Yet,

$$\bigg(\int_0^{1-\frac{d_k(y,\partial\Omega)}{L_k(y)}} \frac{d_k(y,\partial\Omega)^q}{(1-t)^{q+1}}\, \mathrm{d}t \bigg)^{p/q} \le \frac{k^p_{\max}\, \mathrm{diam}(\Omega)^p}{q^{p/q}}.$$

Hence, we get

$$||\sigma||^p_{L^p(\Omega)} \le \frac{(k_{\max}\, \mathrm{diam}(\Omega))^p}{(Cq)^{p/q}} \int_0^1 \int_\Omega f^+(P_t^{-1}(y))^p\, \mathcal{J}_t(y)^{-1}\, \mathrm{d}y\, \mathrm{d}t.$$

This yields that

$$||\sigma||_{L^p(\Omega)} \le C\, ||f^+||_{L^p(\Omega)}.$$ □

5. L^p estimates on the transport density in the transport problem with boundary cost

In this section, we consider the transport problem to the boundary but in the case where we have an extra boundary cost. More precisely, we assume that we have a mass f^+ in the interior of a domain Ω (we assume again that we have uniqueness of geodesics between points of Ω) that we transport to the boundary $\partial\Omega$, minimizing the transport cost that is given by the Riemannian metric $d_k(x,y)$ plus an export tax $g(y)$ at the exit point $y \in \partial\Omega$, where $g : \partial\Omega \mapsto \mathbb{R}^+$ is $\lambda-$Lipschitz with respect to d_k with $\lambda < 1$. In other words, we minimize

(5.1)
$$\min \bigg\{ \int_{\Omega\times\Omega} d_k(x,y)\, \mathrm{d}\Lambda(x,y) + \int_{\partial\Omega} g(y)\, \mathrm{d}[(\Pi_y)_\#\Lambda] \,:\, (\Pi_x)_\#\Lambda = f^+ \text{ and } \mathrm{spt}[(\Pi_y)_\#\Lambda] \subset \partial\Omega \bigg\}.$$

Assume $f^+ \in L^1(\Omega)$. It is not difficult to see that $\Lambda := (I,T)_\# f^+$ is the unique optimal transport plan for Problem (5.1), where

$$T(x) := \mathrm{argmin}\{d_k(x,y) + g(y) \,:\, y \in \partial\Omega\}, \text{ for all } x \in \Omega.$$

In fact, one can show that $T(x)$ is a singleton at a.e. x. Moreover, we note that the transport density σ_Λ is well defined even if the domain Ω is not geodesically convex. Indeed, for every $x \in \Omega$, the geodesic $\gamma_{x,T(x)}$ lies in Ω; this follows from the fact that if $\gamma_{x,T(x)}$ intersects $\partial\Omega$ at a point $y \neq T(x)$, then we must have $d_k(x,y) + g(y) < d_k(x,T(x)) + g(T(x))$; which contradicts the optimality of $T(x)$. On the other hand, one can show that the dual of Problem (5.1) is the following (see [13, 11]):

$$\sup\left\{ \int_\Omega u \,\mathrm{d}f^+ \,:\, |\nabla u| \leq k,\ u = g \text{ on } \partial\Omega \right\}. \tag{5.2}$$

Notice that $u(x) := \min\{d_k(x,y) + g(y) \,:\, y \in \partial\Omega\}$ is the Kantorovich potential. In addition, the equivalent minimal flow formulation is now

$$\min\left\{ \int_\Omega k|v| + \int_{\partial\Omega} g \,\mathrm{d}\chi \,:\, v \in \mathcal{M}(\Omega,\mathbb{R}^d),\, \chi \in \mathcal{M}^+(\partial\Omega),\, \nabla\cdot v = f^+ - \chi \text{ in } \bar{\Omega} \right\} \tag{5.3}$$

while, the vector measure $v_\Lambda = -k^{-1}\sigma_\Lambda \frac{\nabla u}{|\nabla u|}$ is the unique optimal flow for Problem (5.3). We also see that the pair (σ_Λ, u) solves

$$\begin{cases} -\nabla\cdot[k^{-1}\,\sigma\frac{\nabla u}{|\nabla u|}] = f^+ & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \\ |\nabla u| \leq k & \text{in } \Omega, \\ |\nabla u| = k & \sigma - \text{a.e.} \end{cases} \tag{5.4}$$

In this section, we extend the L^p estimates of Section 4 to the transport density σ between f^+ and $T_\# f^+$ (or equivalently, the measure σ in (5.4)). We will denote by γ_x the geodesic $\gamma_{x,T(x)}$, for a.e. $x \in \Omega$, and we set $T_t(x) := \gamma_x(t)$, for a.e. $x \in \Omega$ and all $t \in [0,1]$. Recalling (2.5), the estimates in Section 4 and the fact that $\gamma_x'(t) = -k(\gamma_x(t))^{-1} d_k(x,T(x))\,\frac{\nabla u(\gamma_x(t))}{|\nabla u(\gamma_x(t))|}$, for all $t \in (0,1)$ and a.e. $x \in \Omega$, one can show that for a.e. $y \in \Omega$,

$$\sigma(y) = \int_0^1 (1-t)^{-1} d_k(y,T(y))\, f^+(T_t^{-1}(y))\,[\det(DT_t(x))]^{-1}\, 1_{T_t(\Omega)}(y)\,\mathrm{d}t.$$

Recalling the estimates in Proposition 4.2, we will be able to prove L^p summability on σ as soon as we show a uniform lower bound on the Jacobian $\det(DT_t(x))$.

Lemma 5.1. *Assume that k is a $C^{1,1}$ function on $\mathbb{R}^d$ with $0 < k_{\min} \leq k \leq k_{\max} < \infty$, $\Omega \subset \mathbb{R}^d$ is a compact domain satisfying a uniform exterior ball condition of radius $r > 0$ (we also assume that there is a unique geodesic between any two points of Ω), and g is a C^1 semi-concave function on $\partial\Omega$ with $|\nabla g| \leq \lambda < k_{\min}$. Then, there exists a constant $C > 0$ depending only on d, r, λ, diam(Ω), $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$, $||D^2 k||_\infty$ and $\sup[D^2 g]$ such that, for a.e. $x \in \Omega$, we have the following estimate:*

$$det(DT_t(x)) \geq C(1-t).$$

Proof. The proof is essentially similar to the one introduced in Lemma 4.1 and so, we will omit some details. For every $s \in \partial\Omega$, we will denote by $\tau(s)$ the length of the maximal unit geodesic γ_s starting from s with $T(\gamma_s(\tau)) = s$, for all $\tau \in [0,\tau(s)]$. Set $\nu(s) := k^{-1}(s)\frac{\nabla u(s)}{|\nabla u(s)|}$, where u is the Kantorovich potential in Problem (5.2). For a.e. $x \in \Omega$, there exists unique $s \in \partial\Omega$ and $\tau \in [0,\tau(s)]$ such that $x = \Psi(s,\tau) := \exp_s \tau\nu(s)$. Moreover, we have $T_t(\Psi(s,\tau)) = \Psi(s,(1-t)\tau)$, for all $s \in \partial\Omega$ and $\tau \in [0,\tau(s)]$. Then,

$$\det(DT_t(\Psi(s,\tau)))\det(D\Psi(s,\tau)) = (1-t)\det(D\Psi(s,(1-t)\tau)).$$

For $s \in \partial\Omega$, let $(e_1, ..., e_d)$ be a basis (not necessarily orthonormal) such that $(e_1, ..., e_{d-1})$ is an orthonormal basis of the tangent space to $\partial\Omega$ at s and $e_d = \nu(s)$. Let us parallel-transport along the geodesic γ_s to define a new family of basis $(e_1(\tau), ..., e_d(\tau))$. We define again

$$J_i(s,\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(s+\delta e_i,\tau), \ \text{ for all } i \in \{1,...,d-1\},$$

$$J_d(s,\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(s,\tau+\delta),$$

and

$$J(s,\tau) = (J_1(s,\tau), J_2(s,\tau), ..., J_d(s,\tau)), \qquad \mathcal{J}(s,\tau) = \det[J(s,\tau)].$$

We have

$$\nabla_{J_d} J_i = \sum_{j=1}^d J'(s,\tau)_{ji} e_j(\tau) + \sum_{j=1}^d\sum_{k=1}^d J(s,\tau)_{ji}\,\Gamma^k_{dj}\,e_k(\tau) = \sum_{k=1}^d \left[J'(s,\tau)_{ki} + \sum_{j=1}^d \Gamma^k_{dj} J(s,\tau)_{ji}\right] e_k(\tau),$$

where Γ^k_{dj} denote the Christoffel symbols. Let us denote again by V the matrix, in the basis $\{e_1(\tau), ..., e_d(\tau)\}$, associated with the endomorphism $X \mapsto \nabla_X J_d$. Then, we have

$$\nabla_{J_i} J_d = \sum_{k=1}^d (VJ)_{ki}\, e_k(\tau).$$

Hence,

$$J' = [V - \Gamma]J, \ \text{ where } \ \Gamma := (\Gamma^i_{dj})_{ij}.$$

In particular, we have

$$\mathcal{J}'(s,\tau) = \operatorname{tr}[V-\Gamma]\,\mathcal{J}(s,\tau).$$

Yet, we have $J_d(s,\tau) = k(\Psi(s,\tau))^{-1}\frac{\nabla u(\Psi(s,\tau))}{|\nabla u(\Psi(s,\tau))|}$. Thanks again to [6, Theorem 8.2.7], the function u is semi-concave as soon as Ω satisfies a uniform exterior ball condition of radius r, $k \in C^{1,1}(\Omega)$ with $0 < k_{\min} \le k \le k_{\max} < \infty$ and g is a C^1 semi-concave function on $\partial\Omega$ with $|\nabla g| \le \lambda < k_{\min}$. On the other hand, we have $\Gamma^i_{dd} = 0$ (since γ_s is a geodesic) and, as we parallel-transport along the geodesic γ_s, then one has

$$\Gamma^i_{di} = \partial_{e_d} \log\left[\sqrt{\det[(k(\gamma_s(\tau))^2\, e_i(\tau)\cdot e_j(\tau))_{ij}]}\right] = 0.$$

Consequently, there is a constant C depending only on d, r, λ, $\operatorname{diam}(\Omega)$, $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$, $||D^2k||_\infty$ and $\sup[D^2g]$ such that

$$\operatorname{tr}[V-\Gamma] \le C.$$

Then,

$$\mathcal{J}(s,(1-t)\tau) \ge \mathcal{J}(s,\tau)\,e^{-Ct\tau}.$$

Finally, we get that

$$\det(DT_t(x)) \ge e^{-Ct\tau}(1-t). \qquad \square$$

Proposition 5.2. *Under the assumptions of Lemma 5.1, the transport density σ between f^+ and $T_\# f^+$ belongs to $L^p(\Omega)$ as soon as $f^+ \in L^p(\Omega)$, for all $p \in [1,\infty]$. In addition, there is a constant $C := C(d, r, \lambda, diam(\Omega), k_{\min}, k_{\max}, ||\nabla k||_\infty, ||D^2k||_\infty, \sup[D^2g]) < \infty$ such that*

$$||\sigma||_{L^p(\Omega)} \le C||f^+||_{L^p(\Omega)}.$$

6. Application to the import/export transport problem

In this section, we collect the results of the previous sections to prove L^p estimates on the transport density σ in the import/export transport problem (2.12). More generally, we consider here the transport problem with import/export taxes, i.e. we transport a mass f^+ to the location f^- in Ω with the possibility of transporting some mass to/from the boundary, paying the transport cost $d_k(x,y)$, plus an extra boundary cost $g^-(y)$ for each unit of mass that comes out from a point $y \in \partial\Omega$ (the export tax) or $-g^+(x)$ for each unit of mass that enters at the point $x \in \partial\Omega$ (the import tax). We assume that g^+ and g^- satisfy the following inequality:

$$g^+(x) - g^-(y) < d_k(x,y), \text{ for all } x,\, y \in \partial\Omega. \tag{6.1}$$

Then, we minimize

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda + \int_{\partial\Omega} g^-(y)\,\mathrm{d}[(\Pi_y)_\#\Lambda] - \int_{\partial\Omega} g^+(x)\,\mathrm{d}[(\Pi_x)_\#\Lambda] \,:\, \Lambda \in \Pi(f^+,f^-)\right\} \tag{6.2}$$

where

$$\Pi(f^+,f^-) := \{\Lambda \in \mathcal{M}^+(\Omega\times\Omega),\, [(\Pi_x)_\#\Lambda]_{|\mathring{\Omega}} = f^+,\, [(\Pi_y)_\#\Lambda]_{|\mathring{\Omega}} = f^-\}.$$

From [11], Problem (6.2) reaches a minimum (thanks to assumption (6.1)). Moreover, this problem has a dual formulation with nonhomogeneous boundary conditions:

$$\sup\left\{\int_\Omega u\,\mathrm{d}(f^+ - f^-) \,:\, |\nabla u| \le k,\; g^+ \le u \le g^- \text{ on } \partial\Omega\right\}. \tag{6.3}$$

In addition, one can show that the equivalent minimal flow formulation becomes the following:

$$\min\left\{\int_\Omega k|v| + \int_{\partial\Omega} g^-\,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+\,\mathrm{d}\chi^+ \,:\, v \in \mathcal{M}(\Omega,\mathbb{R}^d),\, \chi^\pm \in \mathcal{M}^+(\partial\Omega),\, \nabla\cdot v = f + \chi \text{ in } \Omega\right\}. \tag{6.4}$$

And, if σ is the transport density in Problem (6.2) and u is the Kantorovich potential in Problem (6.3), then the pair (σ,u) solves

$$\begin{cases} -\nabla\cdot[k^{-1}\,\sigma\frac{\nabla u}{|\nabla u|}] = f^+ - f^- & \text{in } \Omega,\\ g^+ \le u \le g^- & \text{on } \partial\Omega,\\ |\nabla u| \le k & \text{in } \Omega,\\ |\nabla u| = k & \sigma-\text{a.e.} \end{cases} \tag{6.5}$$

The aim now is to study the L^p summability of the transport density σ in (6.5). First of all, by assumption (6.1), it is not difficult to show that any optimal transport plan Λ of Problem (6.2) gives zero mass to $\partial\Omega\times\partial\Omega$. Then, one can devide Λ into three parts Λ_{ii}, Λ_{ib} and Λ_{bi}, where

$$\Lambda_{ii} := \Lambda\cdot 1_{\mathring{\Omega}\times\mathring{\Omega}},\; \Lambda_{ib} := \Lambda\cdot 1_{\mathring{\Omega}\times\partial\Omega} \text{ and } \Lambda_{bi} := \Lambda\cdot 1_{\partial\Omega\times\mathring{\Omega}}.$$

Set

$$\nu^+ := (\Pi_x)_\#(\Lambda_{ib}) \quad\text{and}\quad \nu^- := (\Pi_y)_\#(\Lambda_{bi}).$$

Then, we consider the following transport problems:

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda \,:\, \Lambda \in \Pi(f^+ - \nu^+, f^- - \nu^-)\right\}, \tag{6.6}$$

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda + \int_{\partial\Omega} g^-(y)\,\mathrm{d}[(\Pi_y)_\#\Lambda] \,:\, (\Pi_x)_\#\Lambda = \nu^+,\ \mathrm{spt}[(\Pi_y)_\#\Lambda] \subset \partial\Omega\right\}, \tag{6.7}$$

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda - \int_{\partial\Omega} g^+(x)\,\mathrm{d}[(\Pi_x)_\#\Lambda] \,:\, \mathrm{spt}[(\Pi_x)_\#\Lambda] \subset \partial\Omega,\ (\Pi_y)_\#\Lambda = \nu^-\right\}. \tag{6.8}$$

Hence, we have the following:

Proposition 6.1. *The plans Λ_{ii}, Λ_{ib} and Λ_{bi} solve (6.6), (6.7) and (6.8), respectively. In addition, $\Lambda_{ib} = (Id, T^-)_\#\nu^+$ and $\Lambda_{bi} = (T^+, Id)_\#\nu^-$, where $T^-(x) := argmin\{d_k(x,y) + g^-(y) : y \in \partial\Omega\}$ and $T^+(y) := argmin\{d_k(x,y) - g^+(x) : x \in \partial\Omega\}$. In particular, if χ^+ and χ^- denote the import/export masses, then we have $\chi^+ = T^+_\#\nu^-$ and $\chi^- = T^-_\#\nu^+$.*

Proof. This follows immediately from the fact that Λ_{ii}, Λ_{ib} and Λ_{bi} are admissible in (6.6), (6.7) and (6.8), respectively. Moreover, if Λ_1, Λ_2 and Λ_3 minimize (6.6), (6.7) and (6.8), respectively, then $\Lambda_1 + \Lambda_2 + \Lambda_3$ minimizes Problem (6.2). □

Let σ_{ii}, σ_{ib} and σ_{bi} be the transport densities associated with the transport plans Λ_{ii}, Λ_{ib} and Λ_{bi}, respectively. We note that these transport densities are well defined as soon as Ω is geodesically convex. However, we recall that the transport densities σ_{ib} and σ_{bi} are well defined even if Ω is not geodesically convex, but under the assumptions that $g^\pm$ are $\lambda-$Lip w.r.t. to the Riemannian metric d_k with $\lambda < 1$. Moreover, σ_{ii} is well defined if $g^+ = g^- := g$. Indeed, for $\Lambda-$a.e. (x,y), the geodesic $\gamma_{x,y}$ lie in Ω since if this is not the case, then $\gamma_{x,y}$ must intersect $\partial\Omega$ at two different points y' and x' with $d_k(x,y') < d_k(x,x')$ and so, it is better in this case (thanks to the fact that g is $\lambda-$Lip w.r.t. to d_k with $\lambda < 1$) to export the mass which is located at x to $y' \in \partial\Omega$ and then import a mass from $x' \in \partial\Omega$ to y; but this yields to a contradiction with the optimality of Λ.

Proposition 6.2. *Let k be a $C^{1,1}$ function on $\mathbb{R}^d$ such that $0 < k_{\min} \le k \le k_{\max} < \infty$. Assume that Ω is geodesically convex and f^+, $f^- \in L^p(\Omega)$. Then, the transport density σ_{ii} between $f^+ - \nu^+$ and $f^- - \nu^-$ belongs to $L^p(\Omega)$, for every $p \in [1,\infty]$.*

Proof. This follows immediately from Proposition 3.2 and the fact that $\nu^+ = (\Pi_x)_\#(\Lambda_{ib}) \le [(\Pi_x)_\#\Lambda]_{|\mathring{\Omega}} = f^+$ and $\nu^- = (\Pi_y)_\#(\Lambda_{bi}) \le [(\Pi_y)_\#\Lambda]_{|\mathring{\Omega}} = f^-$. □

Proposition 6.3. *Let k be a $C^{1,1}$ function on $\mathbb{R}^d$ such that $0 < k_{\min} \le k \le k_{\max} < \infty$. Assume that Ω satisfies a uniform exterior ball condition of radius $r > 0$ and for every two points x and y in Ω, there is a unique geodesic from x to y. Let $g^\pm$ be two C^1 functions with $|\nabla g^\pm| \le \lambda < k_{\min}$, $D^2 g^- \le CI$ and $D^2 g^+ \ge -CI$. Then, the transport density σ_{ib} (resp. σ_{bi}) belongs to $L^p(\Omega)$ as soon as $f^+ \in L^p(\Omega)$ (resp. $f^- \in L^p(\Omega)$), for all $p \in [1,\infty]$.*

Proof. This follows directly from Proposition 5.2. □

Consequently, we get the following L^p summability on the transport density σ in (6.5):

Proposition 6.4. *Let $k \in C^{1,1}(\mathbb{R}^d)$ with $0 < k_{\min} \le k \le k_{\max} < \infty$. Assume that Ω is a geodesically convex domain and, $g^\pm$ are two C^1 functions with $|\nabla g^\pm| \le \lambda < k_{\min}$, $D^2 g^- \le CI$ and $D^2 g^+ \ge -CI$. Then, the transport density σ between $f^+ + \chi^+$ and $f^- + \chi^-$ (where $\chi^\pm$ are the import/export masses on $\partial\Omega$) belongs to $L^p(\Omega)$ as soon as $f^\pm \in L^p(\Omega)$, for all $p \in [1,\infty]$.*

Proof. This follows from Propositions 6.2 & 6.3 and the fact that $\sigma = \sigma_{ii} + \sigma_{ib} + \sigma_{bi}$. □

Finally, we also have the following:

Proposition 6.5. *Let $k \in C^{1,1}(\mathbb{R}^d)$ with $0 < k_{\min} \leq k \leq k_{\max} < \infty$. Suppose that Ω is a compact domain satisfying a uniform exterior ball condition of radius r and that there is a unique geodesic between any two points of Ω. Let $g^\pm = g$ be a $C^{1,1}$ function with $|\nabla g| \leq \lambda < k_{\min}$. Then, the transport density σ between $f^+ + \chi^+$ and $f^- + \chi^-$ is in $L^p(\Omega)$ provided that $f^\pm \in L^p(\Omega)$, for all $p \in [1, \infty]$.*

Proof. We recall that if we have uniqueness of geodesics between points of Ω and if $g^+ = g^- = g$ with $|\nabla g| \leq \lambda < k_{\min}$, then the transport density σ is well defined even if the domain Ω is not geodesically convex. The rest follows immediately from Propositions 6.2 & 6.3. □

Acknowledgments: the author would like to thank Prof. Pierre Pansu for his fruitful suggestions and his help in proving the geometric estimates.

References

[1] M. Beckmann, A continuous model of transportation, *Econometrica* **20**, 643–660, 1952.
[2] S. Bianchini and F. Cavalletti, The Monge problem for distance cost in geodesic spaces, *Commun. Math. Phys.* 318, 615–673, 2013.
[3] G. Bouchitté and G. Buttazzo, Characterization of optimal shapes and masses through Monge-Kantorovich equation, *J. Eur. Math. Soc.* 3 (2), 139–168, 2001.
[4] G. Buttazzo, É. Oudet and E. Stepanov, Optimal transportation problems with free Dirichlet regions, in *Variational methods for discontinuous structures*, 41–65, vol 51 of PNLDE, Birkhäuser, Basel, 2002.
[5] P. Cannarsa and P. Cardaliaguet, Representation of equilibrium solutions to the table problem for growing sandpiles, *J. Eur. Math. Soc.* 6, 1–30.
[6] P. Cannarsa and C. Sinestrari, *Semi-Concave functions, Hamilton-Jacobi equations and optimal control*, in Progress in Nonlinear Differential Equations and Their Applications.
[7] F. Cavalletti and M. Huesmann, Existence and uniqueness of optimal transport maps, *Ann I. H. Poincaré*, AN (32), 1367–1377, 2015.
[8] L. De Pascale, L. C. Evans and A. Pratelli, Integral estimates for transport densities, *Bull. of the London Math. Soc.* 36, n. 3, pp. 383–395, 2004.
[9] L. De Pascale and A. Pratelli, Regularity properties for Monge Transport Density and for Solutions of some Shape Optimization Problem, *Calc. Var. Par. Diff. Eq.* 14, n. 3, pp. 249–274, 2002.
[10] L. De Pascale and A. Pratelli, Sharp summability for Monge Transport density via Interpolation, *ESAIM Control Optim. Calc. Var.* 10, n. 4, pp. 549–552, 2004.
[11] S. Dweik, Weighted Beckmann problem with boundary costs, *Quarterly of applied mathematics*, 76, 601–609, 2018.
[12] S. Dweik, Lack of regularity of the transport density in the Monge problem, *Journal de mathématiques pures et appliquées*, 118, 219–234, 2018.
[13] S. Dweik, Optimal transportation with boundary costs and summability estimates on the transport density, *Journal of Convex Analysis*, 25(1), 135–160, 2016.
[14] S. Dweik and F. Santambrogio, Summability estimates on transport densities with Dirichlet regions on the boundary via symmetrization techniques, *ESAIM Control Optimisation and Calculus of Variations 24 (3)*, 1167–1180, 2017.
[15] S. Dumont and N. Igbida, On a dual formulation for the growing sandpile problem, *Euro. J. Appl. Math.*, 20 (02), pp. 169–185, 2009.
[16] M. Feldman and R. McCann, Uniqueness and transport density in Monge's mass transportation problem, *Calc. Var. Par. Diff. Eq.* 15, n. 1, pp. 81–113, 2002.
[17] M. Feldman and R. McCann, Monge's transport problem on a Riemannian manifold, *Trans. Amer. Math. Soc.* 354, 1667–1997, 2002.
[18] L. Kantorovich, On the transfer of masses, *Dokl. Acad. Nauk. USSR*, (37), 7–8, 1942.
[19] J.M.Mazon, J.Rossi and J.Toledo, An optimal transportation problem with a cost given by the euclidean distance plus import/export taxes on the boundary, *Rev. Mat. Iberoam.* 30, no. 1, 1–33, 2014.
[20] G. Monge, Mémoire sur la théorie des déblais et des remblais, *Histoire de lâAcadémie Royale des Sciences de Paris*, 666–704, 1781.
[21] A. Pratelli, Equivalence between some definitions for the optimal mass transport problem and for the transport density on manifolds, *Ann. Mat. Pura Appl.* 184, no. 2, 215–238, 2005.
[22] L. Prigozhin, Variational model of sandpile growth, *Euro. J. Appl. Math.*, 7, 225–236, 1996.

[23] F. Santambrogio, Absolute continuity and summability of transport densities: simpler proofs and new estimates, *Calc. Var. Par. Diff. Eq.*, 36, 343–354, 2009.
[24] F. Santambrogio, *Optimal Transport for Applied Mathematicians*, in *Progress in Nonlinear Differential Equations and Their Applications* 87, Birkhäuser, Basel, 2015.
[25] C. Villani, *Topics in Optimal Transportation*, Graduate Studies in Mathematics. Vol. 58, 2003.

S. Dweik: Laboratoire de Mathématiques de Versailles, UVSQ, Université Paris-Saclay, 45 avenue des Etats-Unis, F-78035 Versailles Cedex.
Email address: samer.dweik@uvsq.fr