SUMMABILITY ESTIMATES ON THE TRANSPORT DENSITY IN THE IMPORT-EXPORT TRANSPORT PROBLEM WITH RIEMANNIAN COST

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ABSTRACT. In this paper, we consider a mass transportation problem with transport cost given by a Riemannian metric in a bounded domain Ω, where a mass f^+ is sent to a location f^- in Ω with the possibility of importing or exporting masses from or to the boundary $\partial\Omega$. First, we study the L^p summability of the transport density σ in the Monge-Kantorovich problem with Riemannian cost between two diffuse measures f^+ and f^-. Using some technical geometrical estimates on the transport rays, we will show that σ belongs to $L^p(\Omega)$ as soon as the source measure f^+ and the target one f^- are both in $L^p(\Omega)$, for all $p \in [1, \infty]$. Moreover, we will prove that the transport density between a diffuse measure f^+ and its Riemannian projection onto the boundary (so, the target measure is singular) is in $L^p(\Omega)$ provided that $f^+ \in L^p(\Omega)$ and Ω satisfies a uniform exterior ball condition. Finally, we will extend the L^p estimates on the transport density σ to the case of a transport problem with import-export taxes.

1. Introduction

In the Monge-Kantorovich problem with Euclidean cost $c(x, y) = |x - y|$ (see [18, 20]), it is well known that an important role is played by the *transport density* σ, which represents the work for transporting the mass f^+ through each subset of Ω to the destination f^-. This measure σ also appears in the following minimal flow problem (which is called the Beckmann problem [1]):

$$\min\left\{ \int_\Omega |v(x)|\,\mathrm{d}x \,:\, v \in L^1(\Omega, \mathbb{R}^d),\ \nabla\cdot v = f := f^+ - f^- \text{ in } \Omega,\ v\cdot n = 0 \text{ on } \partial\Omega \right\}. \tag{1.1}$$

If v is an optimal flow in Problem (1.1) then one can show that $v = -\sigma\nabla u$, where u is a $1-$Lipschitz function with $|\nabla u| = 1$ $\sigma-$a.e. (this function u is called a Kantorovich potential, as it maximizes the dual of the Kantorovich problem; see [24] or Section 2). Hence, σ and u solve together the so-called Monge-Kantorovich system:

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f & \text{in } \Omega, \\ \sigma\nabla u\cdot n = 0 & \text{on } \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases} \tag{1.2}$$

In [8, 9, 10, 23], the authors have already studied the summability of this transport density σ. They proved that σ belongs to $L^p(\Omega)$ as soon as f^+ and f^- are in $L^p(\Omega)$, for all $p \in [1, \infty]$. While in [12], the author shows by a family of counter-examples that in general σ does not belong to $W^{1,p}(\Omega)$ (resp. $C^{0,\alpha}(\Omega)$ or $BV(\Omega)$) even if both $f^\pm$ are in $W^{1,p}(\Omega)$ (resp. $C^{0,\alpha}(\Omega)$ or $BV(\Omega)$).

On the other hand, the authors of [5, 14, 15, 22] considered the transport problem to the boundary. In this case, the Beckmann problem (1.1) becomes

$$\min\left\{ \int_\Omega |v(x)|\,\mathrm{d}x \,:\, v \in L^1(\Omega, \mathbb{R}^d),\ \nabla\cdot v = f^+ \text{ in } \mathring{\Omega} \right\}.$$

In addition, since the target measure f^- in the transport problem to the boundary is an arbitrary measure on $\partial\Omega$, then the Monge-Kantorovich system (1.2) will be complemented now with a Dirichlet boundary condition:

$$
\begin{cases} -\nabla\cdot(\sigma\nabla u) = f^+ & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \\ |\nabla u| \leq 1 & \text{in } \ \Omega, \\ |\nabla u| = 1 & \sigma-\text{a.e.} \end{cases} \tag{1.3}
$$

The pair (σ, u) in (1.3) models (in a statical or dynamical framework) the configuration of stable or growing sandpiles, where u gives the pile shape and σ stands for sliding layer. On the other hand, one can see that the optimal choice for the target measure f^- is to be equal the projection $P_\# f^+$ of f^+ onto $\partial\Omega$, where P denotes the projection map onto the boundary. But so, this yields that the measure σ in (1.3) is nothing else than the transport density between f^+ and $P_\# f^+$. As the target measure $P_\# f^+$ is now singular, then it is not clear whether the transport density σ belongs to $L^p(\Omega)$ or not even if $f^+ \in L^p(\Omega)$. However, the authors of [14] have already studied the L^p summability of this transport density; they proved that as soon as Ω satisfies a uniform exterior ball condition, the transport density σ in (1.3) is in $L^p(\Omega)$ provided that $f^+ \in L^p(\Omega)$, for all $p \in [1, \infty]$.

In this paper, we are mainly concerned with the summability of the transport density σ between two measures f^+ and f^- but in the case where the transport cost in the Monge-Kantorovich problem is given by a Riemannian metric $c(x, y) = d_k(x, y)$, where $k > 0$ is a continuous function over Ω. This transport problem is used to model a non-uniform cost for the movement (due to the presence of some geographical obstacles in Ω) and it has been already studied in [21, 2, 16, 11]. It is also known (see [21, 11]) that the equivalent minimal flow formulation becomes now to the following "weighted" Beckmann problem:

$$
\min\left\{ \int_\Omega k(x)|v(x)|\,\mathrm{d}x \,:\, v \in L^1(\Omega, \mathbb{R}^d),\ \nabla\cdot v = f^+ - f^- \ \text{ in } \ \Omega,\ v\cdot n = 0 \ \text{ on } \ \partial\Omega \right\}. \tag{1.4}
$$

Set $a = k^{-1}$. If v is a minimizer for Problem (1.4), then we have $v = -\sigma\, a\, \frac{\nabla u}{|\nabla u|}$, where u is the Kantorovich potential but now in the Monge-Kantorovich problem with Riemannian cost (we note that u is $1-$Lip with respect to the metric d_k and so, one has $|\nabla u| \leq k$). In particular, the pair (σ, u) solves

$$
\begin{cases} -\nabla\cdot[\sigma\, a\, \frac{\nabla u}{|\nabla u|}] = f & \text{in } \ \Omega, \\ \sigma\, a[\frac{\nabla u}{|\nabla u|}\cdot n] = 0 & \text{on } \ \partial\Omega, \\ |\nabla u| \leq k & \text{in } \ \Omega, \\ |\nabla u| = k & \sigma-\text{a.e.} \end{cases} \tag{1.5}
$$

In addition, we will study the summability of the transport density σ in the transport problem with Riemannian cost and with some additional import/export boundary costs $g^\pm$ (see [13, 19]). Under some assumptions on $g^\pm$, the problem (1.4) becomes

$$
\min\left\{ \int_\Omega k|v| + \int_{\partial\Omega} g^-\,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+\,\mathrm{d}\chi^+ : v \in L^1(\Omega, \mathbb{R}^d),\ \chi \in \mathcal{M}(\partial\Omega),\ \nabla\cdot v = f + \chi \ \text{ in } \ \overline{\Omega} \right\}. \tag{1.6}
$$

Here, the measures χ^+ and χ^- encode the import and export masses, respectively. Moreover, we note that the Monge-Kantorovich system (1.5) will be complemented with nonhomogeneous boundary conditions:

$$\begin{cases} -\nabla\cdot[\sigma\, a\,\frac{\nabla u}{|\nabla u|}] = f & \text{in } \Omega, \\ g^+ \le u \le g^- & \text{on } \partial\Omega, \\ |\nabla u| \le k & \text{in } \Omega, \\ |\nabla u| = k & \sigma-\text{a.e.} \end{cases} \tag{1.7}$$

The summability question of the transport density σ in (1.5) or (1.7) is an interesting one and it is not trivial since it requires to show some geometric estimates on the transport rays, which are now geodesics (not just straight lines as in the Euclidean case). To the best of our knowledge, there are no results in the literature concerning the summability of the transport density in the Riemannian case. We recall that the only known results in [8, 9, 10, 23, 14, 13] about the L^p summability of the transport density σ concern the Euclidean case (so, $k = 1$). Hence, the novelty of the present paper is the extension of these L^p estimates on σ to the general Riemannian case.

This paper is organized as follows. In Section 2, we will recall some well known facts, terminology and notations concerning the Monge-Kantorovich problem as well as the import-export transport problem with Riemannian cost, their duals and equivalent minimal flow formulations. In Section 3, we will show that under some assumptions on the metric k, the transport density σ between f^+ and f^- belongs to $L^p(\Omega)$ as soon as $f^\pm \in L^p(\Omega)$. While in Section 4, we will prove that the transport density σ between f^+ and its Riemannian projection onto the boundary $P_\# f^+$ is in $L^p(\Omega)$ provided that $f^+ \in L^p(\Omega)$ and Ω satisfies a uniform exterior ball condition. In Section 5, we will extend the L^p estimates proved in Section 4 on the transport density σ to the export transport problem (i.e., when there is an additional export tax g). Finally, we will collect in Section 6 the results of the previous sections, which will give L^p estimates on the transport density σ in the import-export transport problem with Riemannian cost.

2. Optimal transport problem with Riemannian cost

2.1. **The Monge-Kantorovich problem.** Let k be a positive continuous function on $\mathbb{R}^d$. Then, we denote by d_k the Riemannian metric associated with k:

$$d_k(x,y) = \min\left\{\int_0^1 k(\gamma(t))|\gamma'(t)|\,\mathrm{d}t : \gamma \in \mathrm{Lip}([0,1],\mathbb{R}^d),\ \gamma(0)=x \text{ and } \gamma(1)=y\right\},\ \forall\ x,\,y\in\mathbb{R}^d.$$

Given a compact domain $\Omega \subset \mathbb{R}^d$, let f^+ and f^- be two nonnegative Borel measures on Ω such that $f^+(\Omega) = f^-(\Omega)$. Then, we consider the Monge problem with Riemannian cost:

$$\inf\left\{\int_\Omega d_k(x,T(x))\,\mathrm{d}f^+(x)\ :\ T_\# f^+ = f^-\right\}, \tag{2.1}$$

where $T_\#$ denotes the pushforward operator acting on every Borel measure f^+ according to the formula $T_\# f^+(A) := f^+(T^{-1}(A))$, for all Borel set $A \subset \Omega$. Although this problem may have no solutions, its relaxed setting always has ones. The latter setting is the following Kantorovich problem with Riemannian cost:

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda(x,y)\ :\ \Lambda\in\mathcal{M}^+(\Omega\times\Omega),\ (\Pi_x)_\#\Lambda = f^+ \text{ and } (\Pi_y)_\#\Lambda = f^-\right\}, \tag{2.2}$$

where $\mathcal{M}^+(\Omega \times \Omega)$ denotes the set of all nonnegative Borel measures Λ on $\Omega \times \Omega$ and, Π_x and Π_y denote the projections with respect to x and y, respectively. In [17], the authors proved existence of an optimal transport map T for Problem (2.1) (or equivalently, an optimal transport plan Λ for Problem (2.2) which is concentrated on the graph of a map T) under the assumption that $f^+ \in L^1(\Omega)$. On the other hand, it is well known that Problem (2.2) admits a dual formulation which is the following (see [24]):

$$\sup\left\{ \int_\Omega u \,\mathrm{d}(f^+ - f^-) \,:\, |u(x) - u(y)| \le d_k(x,y),\ \forall\, x,\, y \in \Omega \right\}. \tag{2.3}$$

We recall that if a function u is $1-$Lipschitz with respect to the geodesic distance d_k, then one has $|\nabla u(x)| \le k(x)$ for almost every $x \in \Omega$, while the converse is true as soon as Ω is geodesically convex. Thanks to the duality min (2.2) = sup (2.3), we see that if Λ is an optimal transport plan and u is a Kantorovich potential (i.e. a maximizer of Problem (2.3)), then the following equality must hold

$$u(x) - u(y) = d_k(x,y), \text{ for all } (x,y) \in \mathrm{spt}(\Lambda). \tag{2.4}$$

Any maximal geodesic $\gamma_{x,y}$ between x and y that satisfies the equality (2.4) will be called a *transport ray*. In other words, an optimal transport plan Λ has to move the mass f^+ to f^- along these transport rays. From [17, Lemma 9], we have that two different transport rays cannot intersect at an interior point of one of them.

For an optimal transport plan Λ, we define a nonnegative measure σ_Λ on Ω (so-called *transport density*) which represents the amount of transport taking place in each region of Ω. Assume that Ω is geodesically convex. Then, this transport density σ_Λ is defined as follows:

$$< \sigma_\Lambda, \phi >:= \int_{\Omega\times\Omega} \int_0^1 \phi(\gamma_{x,y}(t))\, k(\gamma_{x,y}(t))\, |\gamma'_{x,y}(t)| \,\mathrm{d}t \,\mathrm{d}\Lambda(x,y), \text{ for all } \phi \in C(\Omega), \tag{2.5}$$

where $\gamma_{x,y}$ is the unique geodesic between x and y. From (2.5), we see that $\sigma_\Lambda(\Omega) = \min$ (2.2). On the other hand, we define a vector measure v_Λ (which is the vector version of $a\,\sigma_\Lambda$) as follows

$$< v_\Lambda, \xi >:= \int_{\Omega\times\Omega} \int_0^1 \xi(\gamma_{x,y}(t)) \cdot \gamma'_{x,y}(t) \,\mathrm{d}t \,\mathrm{d}\Lambda(x,y), \quad \text{for all } \ \xi \in C(\Omega, \mathbb{R}^d).$$

By [17, Lemma 10], one can show that if u is a Kantorovich potential, then u is differentiable at any interior point of a transport ray $\gamma_{x,y}$ and, one has $\nabla u(\gamma_{x,y}(t)) = -k(\gamma_{x,y}(t)) \frac{\gamma'_{x,y}(t)}{|\gamma'_{x,y}(t)|}$, for all $t \in (0,1)$. Hence, we have $v_\Lambda = -\sigma_\Lambda\, a\, \frac{\nabla u}{|\nabla u|}$. Moreover, by [21], one can show that v_Λ solves

$$\min\left\{ \int_\Omega k|v| \,:\, v \in \mathcal{M}(\Omega, \mathbb{R}^d),\ \nabla \cdot v = f \ \text{ in } \overline{\Omega} \right\}, \tag{2.6}$$

where $\mathcal{M}(\Omega, \mathbb{R}^d)$ is the set of vector measures over Ω, while the constraint $\nabla \cdot v = f$ is equivalent to say that $-\int_{\overline{\Omega}} \nabla\phi \cdot \mathrm{d}v = \int_{\overline{\Omega}} \phi \,\mathrm{d}f$, for all $\phi \in C^1(\overline{\Omega})$. In addition, the authors of [21] have proved that every minimizer of Problem (2.6) is in the form $v = v_\Lambda$, for some optimal transport plan Λ of Problem (2.2). In general, the optimal transport plan Λ in (2.2) is not unique. However, thanks to [17, Section 7], one can show that $\sigma_\Lambda := \sigma$ is unique (i.e., it does not depend on the choice of the optimal transport plan Λ) as soon as f^+ or f^- is in $L^1(\Omega)$. Moreover, $\sigma \in L^1(\Omega)$ and so, we see that the pair (σ, u) solves the Monge-Kantorovich system (1.5).

In Section 3, we prove L^p estimates on this transport density σ, for all $p \in [1,\infty]$. More precisely, we show the following statement: "σ belongs to $L^p(\Omega)$ as soon as $f^\pm \in L^p(\Omega)$ and, under the assumption that the metric k is of class $C^{1,1}$ and bounded away from zero".

2.2. **The transport problem to the boundary.** In [3, 4], the authors have studied the Monge-Kantorovich problem (2.1) in the presence of a Dirichlet region Σ (i.e., a region where transportation is free). Let us assume that $\Sigma = \partial\Omega$. So, we have a mass f^+ inside Ω that we want to transport into the boundary $\partial\Omega$, paying only the transport cost which will be given again by the Riemannian metric $d_k(x,y)$. In other words, we consider the following problem (2.7)

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda(x,y) \,:\, \Lambda \in \mathcal{M}^+(\Omega\times\Omega),\ (\Pi_x)_\#\Lambda = f^+ \text{ and } \operatorname{spt}[(\Pi_y)_\#\Lambda] \subset \partial\Omega\right\}.$$

This means that the target measure $(\Pi_y)_\#\Lambda$ is completely arbitrary on $\partial\Omega$. But, it is easy to see that the optimal choice for $(\Pi_y)_\#\Lambda$ is to be equal the projection $P_\# f^+$ of f^+ onto the boundary, where

$$P(x) := \operatorname{argmin}\{d_k(x,y),\, y \in \partial\Omega\}, \text{ for all } x \in \Omega.$$

Notice that P is a multivalued map, but it is a singleton at all the points x where the distance function to the boundary $d_k(x,\partial\Omega)$ is differentiable (so, at a.e. x). Then, one can see that $\Lambda := (Id, P)_\# f^+$ is the unique optimal transport plan in Problem (2.7) provided $f^+ \in L^1(\Omega)$. On the other hand, Problem (2.7) has a dual formulation, which is the following:

$$\sup\left\{\int_\Omega u\,\mathrm{d}f^+ \,:\, |\nabla u| \le k,\ u = 0 \text{ on } \partial\Omega\right\}. \tag{2.8}$$

Moreover, (2.6) becomes

$$\min\left\{\int_\Omega k|v| \,:\, v \in \mathcal{M}(\Omega,\mathbb{R}^d),\ \nabla\cdot v = f^+ \text{ in } \mathring{\Omega}\right\}, \tag{2.9}$$

where $\nabla\cdot v = f^+$ in $\mathring{\Omega}$ is in the sense that $-\int_\Omega \nabla\phi\cdot\mathrm{d}v = \int_\Omega \phi\,\mathrm{d}f^+$, for all $\phi \in C^1(\Omega)$ such that $\phi = 0$ on $\partial\Omega$. In addition, it is easy to see that $d_k(x,\partial\Omega)$ is the Kantorovich potential, $v_\Lambda := -\sigma_\Lambda\, a\,\frac{\nabla d_k(\cdot,\partial\Omega)}{|\nabla d_k(\cdot,\partial\Omega)|}$ is the unique optimal flow in Problem (2.9) and, the pair $(\sigma_\Lambda, d_k(\cdot,\partial\Omega))$ is the unique solution for the system:

$$\begin{cases} -\nabla\cdot[\sigma\, a\,\frac{\nabla u}{|\nabla u|}] = f^+ & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ |\nabla u| \le k & \text{in } \Omega,\\ |\nabla u| = k & \sigma-\text{a.e.} \end{cases} \tag{2.10}$$

One of the questions that we consider in this paper is whether the transport density σ in (2.10) is in $L^p(\Omega)$ or not when $f^+ \in L^p(\Omega)$. This result will generalize the one proved in [14] where the transport cost was assumed to be given by the Euclidean distance. The L^p estimates on this transport density σ will be introduced in Section 4 and they will require a different proof from the one given in [14], where the authors used some symmetrization techniques which do not work as soon as the transport cost is not uniform. Moreover, we note that the summability of this transport density σ will not follow from Section 3, since the target measure is now singular.

2.3. **The import-export transport problem.** Now, assume that we transport f^+ into f^- (f^+ and f^- do not have a priori the same total mass) but we have the possibility of transporting some mass to/from the boundary, paying the transport cost $d_k(x,y)$ plus an export tax $g^-(y)$ for each unit of mass that comes out from a point $y \in \partial\Omega$ and an import tax $-g^+(x)$ for each unit of mass that enters at the point $x \in \partial\Omega$. Moreover, we assume that g^+ and g^- satisfy the following inequality:

$$g^+(x) - g^-(y) \leq d_k(x,y), \text{ for all } x,\, y \in \partial\Omega. \tag{2.11}$$

Then, we minimize

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda + \int_{\partial\Omega} g^-(y)\,\mathrm{d}[(\Pi_y)_\#\Lambda] - \int_{\partial\Omega} g^+(x)\,\mathrm{d}[(\Pi_x)_\#\Lambda] \,:\, \Lambda \in \Pi(f^+,f^-)\right\} \tag{2.12}$$

where

$$\Pi(f^+,f^-) := \{\Lambda \in \mathcal{M}^+(\Omega\times\Omega),\, [(\Pi_x)_\#\Lambda]_{|\mathring{\Omega}} = f^+,\, [(\Pi_y)_\#\Lambda]_{|\mathring{\Omega}} = f^-\}.$$

From [11] and thanks to the assumption (2.11), one can show that Problem (2.12) reaches a minimum (if (2.11) is not satisfied then it is easy to see that inf (2.12) $= -\infty$). Moreover, this problem has a dual formulation with nonhomogeneous boundary conditions:

$$\sup\left\{\int_\Omega u\,\mathrm{d}(f^+ - f^-) \,:\, |\nabla u| \leq k,\ g^+ \leq u \leq g^- \ \text{ on } \partial\Omega\right\}. \tag{2.13}$$

In addition, one can show that the equivalent minimal flow formulation becomes (see [11]):

$$\min\left\{\int_\Omega k|v| + \int_{\partial\Omega} g^-\,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+\,\mathrm{d}\chi^+ \,:\, v \in \mathcal{M}(\Omega,\mathbb{R}^d),\ \chi^\pm \in \mathcal{M}^+(\partial\Omega),\ \nabla\cdot v = f + \chi \ \text{ in } \Omega\right\}. \tag{2.14}$$

If σ is the transport density in Problem (2.12) and u is a Kantorovich potential in Problem (2.13), then $v = -\sigma\, a\, \frac{\nabla u}{|\nabla u|}$ solves (2.14) and the pair (σ, u) solves (1.7).

From assumption (2.11), it is not difficult to see that one can always assume that an optimal transport plan Λ of (2.12) gives zero mass to $\partial\Omega\times\partial\Omega$. Then, one can divide Λ into three parts Λ_{ii}, Λ_{ib} and Λ_{bi}, where

$$\Lambda_{ii} := \Lambda \llcorner [\mathring{\Omega}\times\mathring{\Omega}], \quad \Lambda_{ib} := \Lambda \llcorner [\mathring{\Omega}\times\partial\Omega] \quad \text{and} \quad \Lambda_{bi} := \Lambda \llcorner [\partial\Omega\times\mathring{\Omega}].$$

Define

$$\nu^+ := (\Pi_x)_\#(\Lambda_{ib}) \quad \text{and} \quad \nu^- := (\Pi_y)_\#(\Lambda_{bi}).$$

So, we consider the following transport problems:

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda \,:\, \Lambda \in \Pi(f^+ - \nu^+, f^- - \nu^-)\right\}, \tag{2.15}$$

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda + \int_{\partial\Omega} g^-(y)\,\mathrm{d}[(\Pi_y)_\#\Lambda] \,:\, (\Pi_x)_\#\Lambda = \nu^+,\ \operatorname{spt}[(\Pi_y)_\#\Lambda] \subset \partial\Omega\right\}, \tag{2.16}$$

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda - \int_{\partial\Omega} g^+(x)\,\mathrm{d}[(\Pi_x)_\#\Lambda] \,:\, \operatorname{spt}[(\Pi_x)_\#\Lambda] \subset \partial\Omega,\ (\Pi_y)_\#\Lambda = \nu^-\right\}. \tag{2.17}$$

Hence, we have the following:

Proposition 2.1. *The plans Λ_{ii}, Λ_{ib} and Λ_{bi} solve (2.15), (2.16) and (2.17), respectively. In addition, $\Lambda_{ib} = (Id, T^-)_{\#}\nu^+$ and $\Lambda_{bi} = (T^+, Id)_{\#}\nu^-$, where*

$$T^-(x) := \mathit{argmin}\{d_k(x,y) + g^-(y) \,:\, y \in \partial\Omega\}$$

and

$$T^+(y) := \mathit{argmin}\{d_k(x,y) - g^+(x) \,:\, x \in \partial\Omega\}.$$

In particular, if χ^+ and χ^- denote the import/export masses, then we have $\chi^+ = T^+_{\#}\nu^-$ and $\chi^- = T^-_{\#}\nu^+$.

Proof. This follows immediately from the fact that Λ_{ii}, Λ_{ib} and Λ_{bi} are admissible in (2.15), (2.16) and (2.17), respectively. Moreover, if Λ_1, Λ_2 and Λ_3 minimize (2.15), (2.16) and (2.17), respectively, then $\Lambda_1 + \Lambda_2 + \Lambda_3$ minimizes Problem (2.12). $\square$

Finally, let σ_{ii}, σ_{ib} and σ_{bi} be the transport densities associated with the optimal transport plans Λ_{ii}, Λ_{ib} and Λ_{bi}, respectively. Then, it is clear that σ can be decomposed as the sum of these three transport densities $\sigma_{ii} + \sigma_{ib} + \sigma_{bi}$. Consequently, we will combine in Section 6 the results of Sections 3 & 5 to obtain L^p estimates on the transport densities σ_{ii}, σ_{ib} and σ_{bi}, then L^p estimates on σ.

3. L^p estimates on the transport density in the Monge-Kantorovich problem

The aim of this section is to study the L^p summability of the transport density σ between two L^p densities f^+ and f^- in the Monge-Kantorovich problem with Riemannian cost (2.2). The strategy of the proof (which is already used in [23]) is based on a displacement interpolation and an approximation by atomic measures. However, we note that it is not immediate to extend the L^p estimates in [23] to the Riemannian case since the transport rays are now geodesics and then, some technical geometrical estimates on the transport rays will be needed in order to prove L^p summability on σ.

Let k be a positive $C^{1,1}$ function on $\mathbb{R}^d$ such that $0 < k_{\min} \le k \le k_{\max} < \infty$. Assume that $\mathbb{R}^d$ is equipped with the Riemannian metric d_k. Let $\Omega \subset \mathbb{R}^d$ be a compact and geodesically convex domain. In the sequel, we will denote by exp the Riemannian exponential map, $[\cdot,\cdot]$ the Lie bracket and ∇ the Levi-Civita connection. We start by introducing the following geometric lemma which gives a lower bound on the Jacobian of the interpolation map $x \mapsto \gamma_{x,x_0}(t)$, where $\gamma_{x,x_0}(t)$ is the geodesic between x and x_0.

Lemma 3.1. *Fix a point $x_0 \in \Omega$. Then, there exists a constant $C < \infty$ depending only on d, $diam(\Omega)$, $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$ and $||\nabla^2 k||_\infty$ such that, for a.e. $x \in \Omega$, we have*

$$\det(D_x\gamma_{x,x_0}(t)) \ge (1-t)^C, \text{ for all } t \in [0,1]. \tag{3.1}$$

Proof. For $x \in \Omega$, we will denote by γ_x the unique geodesic between x and x_0, $\nu(x)$ the unit tangent vector to γ_x at x (the initial velocity) and $\tau(x) := d_k(x,x_0)$. Fix a point $s_\star \in \mathring{\Omega}$ and set $\chi := \{s \in B(s_\star,\varepsilon) \,:\, \tau(s) = \tau(s_\star) := \tau_\star\}$, where $\varepsilon > 0$ is small enough so that $\chi \subset \Omega$. For every $s \in \chi$, we denote by $\{e_1, ..., e_d\}$ an orthonormal basis such that $e_d = \nu(s)$. Let us parallel-transport along the geodesic γ_s to define a new family of orthonormal basis $\{e_1(\tau), ..., e_d(\tau)\}$ (so, we have $e_d(\tau) = \gamma_s'(\tau)$). For all $s \in \chi$ and $\tau \in [0,\tau_\star]$, we define

$$\Psi(s,\tau) := \gamma_s(\tau) = \exp_s[\tau\,\nu(s)].$$

Set $\Delta := \{x \in \gamma_s \,:\, s \in \chi\}$. Then, we will prove estimate (3.1) on Δ. Note that $x \in \Delta$ if and only if there exists a unique pair $(s,\tau) \in \chi \times [0,\tau_\star]$ such that $x = \Psi(s,\tau)$ and, we have $\gamma_{x,x_0}(t) = \Psi(s,(1-t)\tau + t\tau_\star)$, for all $t \in (0,1)$. Then, we get

$$\det(D_x\gamma_{x,x_0}(t))\det(D\Psi(s,\tau)) = (1-t)\det(D\Psi(s,(1-t)\tau + t\tau_\star)). \tag{3.2}$$

Now, we consider small variations of s on χ in the directions $e_1, ..., e_{d-1}$ that we will denote by $s+\delta e_1, ..., s+\delta e_{d-1}$. Then, we define the vector fields J_i ($i = 1, ..., d-1$) and J_d as follows:

$$J_i(s,\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(s+\delta e_i,\tau), \quad \text{for all } \ i \in \{1,...,d-1\},$$

and

$$J_d(s,\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(s,\tau+\delta).$$

Notice that, for every $i \in \{1,...,d-1\}$, the vector field J_i has been obtained by differentiating a family of geodesics depending on the parameter δ. Set

$$J(s,\tau) = (J_1(s,\tau),...,J_d(s,\tau)) \quad \text{and} \quad \mathcal{J}(s,\tau) = \det[J(s,\tau)].$$

Thanks to [25, Theorem 11.3], this Jacobian $\mathcal{J}$ cannot vanish, except possibly at the endpoints of the geodesic γ_s. Then, the formula for the differential with respect to τ of the determinant $\mathcal{J}(s,\tau)$ yields that

$$\mathcal{J}'(s,\tau) = \mathrm{tr}[J'(s,\tau)\,J(s,\tau)^{-1}]\,\mathcal{J}(s,\tau). \tag{3.3}$$

Let us denote by $\mathrm{d}\Psi$ the differential map of Ψ. The fact that $[\partial_{e_i},\partial_{e_d}] = 0$ implies that J_i and J_d commute since

$$[J_i,J_d] = [\mathrm{d}\Psi(\partial_{e_i}),\mathrm{d}\Psi(\partial_{e_d})] = \mathrm{d}\Psi[\partial_{e_i},\partial_{e_d}] = 0.$$

Then, we have

$$\nabla_{J_d}J_i - \nabla_{J_i}J_d = 0. \tag{3.4}$$

Yet,

$$J_i(s,\tau) = \sum_{j=1}^{d} J_{ji}(s,\tau)\,e_j(\tau).$$

Hence,

$$\nabla_{J_d}J_i = \nabla_{\gamma_s'}J_i = \sum_{j=1}^{d} J_{ji}'(s,\tau)\,e_j(\tau) + \sum_{j=1}^{d} J_{ji}(s,\tau)\,\nabla_{\gamma_s'}e_j(\tau).$$

Since γ_s is a geodesic and the Christoffel symbols vanish, then we have $\nabla_{\gamma_s'}e_j = \nabla_{\gamma_s'}\gamma_s' = 0$, for all $j \in \{1,...,d-1\}$. Consequently, we get that

$$\nabla_{J_d}J_i - \nabla_{J_i}J_d = \sum_{j=1}^{d} J_{ji}'\,e_j - \sum_{j=1}^{d} J_{ji}\,\nabla_{e_j}J_d.$$

Now, let V be the matrix in the basis $\{e_1(\tau),...,e_d(\tau)\}$ associated with the endomorphism $X \mapsto \nabla_X J_d$. So, V is the second fundamental form of the submanifold $\{\Psi(s,\tau) \,:\, s \in \chi\}$.

Then, one has

$$\nabla_{J_d} J_i - \nabla_{J_i} J_d = \sum_{j=1}^{d} J'_{ji}\, e_j - \sum_{j=1}^{d}\sum_{k=1}^{d} J_{ji}\, V_{kj}\, e_k = \sum_{j=1}^{d} [J'_{ji} - (VJ)_{ji}]\, e_j. \tag{3.5}$$

Combining (3.3), (3.4) & (3.5), we infer that

$$J' = VJ \qquad \text{and} \qquad \mathcal{J}' = \mathrm{tr}[V]\, \mathcal{J}.$$

Yet, it is well known (see [6]) that the distance function $d_k(\cdot, x_0)$ to the point x_0 is locally semi-concave in $\mathbb{R}^2 \backslash \{x_0\}$ with

$$D^2 d_k(x, x_0) \le \frac{C}{d_k(x, x_0)} I,$$

where the constant $C < \infty$ depends only on d, $\mathrm{diam}(\Omega)$, $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$ and $||\nabla^2 k||_\infty$. Yet, recalling the definition of J_d, we have $J_d = -a\, \frac{\nabla d_k(\cdot\,, x_0)}{|\nabla d_k(\cdot\,, x_0)|}$. So, this yields that

$$\mathrm{tr}[V] \ge \frac{-C}{\tau_\star - \tau}.$$

Hence,

$$\mathcal{J}'(s,\tau) \ge \frac{-C}{\tau_\star - \tau}\, \mathcal{J}(s,\tau).$$

Integrating in τ, we infer that

$$\frac{\mathcal{J}(s, (1-t)\tau + t\tau_\star)}{\mathcal{J}(s,\tau)} \ge (1-t)^C.$$

Recalling (3.2), we get

$$\det(D_x \gamma_{x,x_0}(t)) \ge (1-t)^C, \ \text{for all } \ t \in [0,1].$$

This concludes the proof. □

Now, we are ready to prove L^p estimates on the transport density σ.

Proposition 3.2. *Assume that $\Omega \subset \mathbb{R}^d$ is a compact, geodesically convex domain and that both f^+, $f^- \in L^p(\Omega)$, with $p \in [1, \infty]$. Then, the transport density σ between f^+ and f^- belongs to $L^p(\Omega)$.*

Proof. First, let us assume that the target measure f^- is finitely atomic and let us denote by $\{x_i : i = 1, ..., n\}$ its atoms. Let Λ be an optimal transport plan between f^+ and f^- and σ be the unique transport density between them (recall that the transport density is unique as soon as $f^+ \in L^1(\Omega)$). For every $t \in [0,1]$, set $f_t := (\Pi_t)_\#[d_k \cdot \Lambda]$ where $\Pi_t(x,y) = \gamma_{x,y}(t)$. As $|\gamma'_{x,y}(t)| = d_k(x,y)\, a(\gamma_{x,y}(t))$, for all $t \in (0,1)$, then by (2.5) we get that

$$\sigma = \int_0^1 f_t \,\mathrm{d}t. \tag{3.6}$$

Now, the aim is to show L^p estimates on f_t, for all $t \in (0,1)$. As f^- is atomic, then one can decompose Ω into essentially disjoint subsets Ω_i, $i \in \{1, ..., n\}$, such that for $\Lambda-$a.e. $(x,y) \in \Omega_i \times \Omega$, we have $y = x_i$. In other words, Ω_i represents the set of points x that will be transported to x_i. We note that if $x \in \Omega_i \cap \Omega_j$ (with $i \neq j$) then x is a double point, which means that x belongs to two different transport rays, but it is well known that the set of double points is Lebesgue-negligible (see [24, 17]). For all $\phi \in C(\Omega)$, we have

$$< f_t, \phi >= \sum_{i=1}^{n} \int_{\Omega_i \times \Omega} \phi(\gamma_{x,x_i}(t))\, d_k(x,x_i)\, \mathrm{d}\Lambda(x,y) = \sum_{i=1}^{n} \int_{\Omega_i} \phi(\gamma_{x,x_i}(t))\, d_k(x,x_i)\, \mathrm{d}f^+(x).$$

Fix $i \in \{1,...,n\}$. We consider the restriction f_t^i of f_t to Ω_i. Let us take a change of variable $z := z(x) = \gamma_{x,x_i}(t)$. We note that this map z is one-to-one thanks to the fact that two different transport rays cannot meet at intermediate points. Then, for all $\phi \in C(\Omega_i)$, we get that

$$< f_t^i, \phi >= \int_{\Omega_i} \phi(\gamma_{x,x_i}(t))\, d_k(x,x_i)\, \mathrm{d}f^+(x) = \int_{\Omega_i(t)} \phi(z)\, d_k(x,x_i)\, f^+(x)\, \mathcal{J}_t(x)^{-1} \mathrm{d}z,$$

where

$$\Omega_i(t) := \{\gamma_{x,x_i}(t) \,:\, x \in \Omega_i\}$$

and

$$\mathcal{J}_t(x) := \det[D_x \gamma_{x,x_i}(t)].$$

Hence, we have

$$f_t^i(z) = d_k(x,x_i)\, f^+(x)\, \mathcal{J}_t(x)^{-1}, \text{ for a.e. } z \in \Omega_i(t).$$

Consequently, we get

$$||f_t^i||^p_{L^p(\Omega_i(t))} = \int_{\Omega_i(t)} d_k(x,x_i)^p f^+(x)^p \mathcal{J}_t(x)^{-p}\, \mathrm{d}z = \int_{\Omega_i} d_k(x,x_i)^p f^+(x)^p \mathcal{J}_t(x)^{1-p}\, \mathrm{d}x.$$

So, we need to bound the Jacobian $\mathcal{J}_t$ from below (we note that in the Euclidean case, this is trivial since we have $\gamma_{x,x_i}(t) = (1-t)x + tx_i$ and so $\mathcal{J}_t(x) = (1-t)^d$). However, thanks to Lemma 3.1, we know that there is a uniform constant $C < \infty$ (which does not depend on i and n) such that

$$\mathcal{J}_t(x) \geq (1-t)^C.$$

Hence,

$$||f_t^i||^p_{L^p(\Omega_i(t))} \leq C^p (1-t)^{C(1-p)} ||f^+||^p_{L^p(\Omega_i)}, \text{ for all } i \in \{1,...,n\}.$$

Then, we have

$$||f_t||_{L^p(\Omega)} \leq C(1-t)^{\frac{C(1-p)}{p}} ||f^+||_{L^p(\Omega)}, \text{ for all } t \in (0,1). \tag{3.7}$$

We see that these L^p estimates in (3.7) on f_t do not depend on the target measure f^- (more precisely, on the number of atoms). Hence, by approximating f^- with a sequence of atomic measures, we get the same L^p estimates (3.7) on f_t in the case where f^- is an arbitrary measure. On the other hand, by symmetry it is obvious that one can also show the following L^p estimates on f_t but from the other side, i.e. by approximating f^+ with a sequence of atomic measures:

$$||f_t||_{L^p(\Omega)} \leq C\, t^{\frac{C(1-p)}{p}} ||f^-||_{L^p(\Omega)}, \text{ for all } t \in (0,1). \tag{3.8}$$

Combining (3.7) & (3.8), we infer that

$$||f_t||_{L^p(\Omega)} \leq C\, 2^{\frac{C(p-1)}{p}} \max\{||f^+||_{L^p(\Omega)}, ||f^-||_{L^p(\Omega)}\}, \text{ for all } t \in (0,1).$$

Recalling (3.6), we get

$$||\sigma||_{L^p(\Omega)} \leq C\,||f||_{L^p(\Omega)}.$$

However, there is an issue here: the previous L^p estimates on f_t for $t \leq \frac{1}{2}$ and $t \geq \frac{1}{2}$ have been obtained by discrete approximations of f^- and f^+, respectively. If the two approximations converge to two different transport plans between f^+ and f^-, then we cannot glue together the two estimates on f_t and deduce anything on σ. The idea is to use instead the transport cost $d_k(x,y)^{1+\varepsilon}$ (where $\varepsilon > 0$) since now the optimal transport plan Λ_ε in Problem (2.2) becomes unique (see [7]) and so, if $f_t^\varepsilon := (\Pi_t)[d_k^{1+\varepsilon} \cdot \Lambda_\varepsilon]$ then one can prove exactly as above the following L^p estimates on f_t^ε:

$$||f_t^\varepsilon||_{L^p(\Omega)} \leq C \max\{||f^+||_{L^p(\Omega)}, ||f^-||_{L^p(\Omega)}\}, \text{ for all } t \in (0,1). \tag{3.9}$$

Yet, it is not difficult to see that $\Lambda_\varepsilon \rightharpoonup \Lambda$, where Λ is an optimal transport plan in Problem (2.2) with transport cost $d_k(x,y)$. Moreover, $f_t^\varepsilon \rightharpoonup f_t$ and so passing to the limit when $\varepsilon \to 0$ in (3.9), we get

$$||f_t||_{L^p(\Omega)} \leq C \max\{||f^+||_{L^p(\Omega)}, ||f^-||_{L^p(\Omega)}\}, \text{ for all } t \in (0,1).$$

□

4. L^p estimates on the transport density in the transport problem to the boundary

Throughout this section, Ω is not necessarily geodesically convex but we always assume that between any two points x and y in Ω, there is a unique geodesic $\gamma_{x,y}$ (which is not a priori contained in Ω). So, the aim will be to study the L^p summability of the transport density σ in (2.7), i.e. between a density $f^+ \in L^p(\Omega)$ and its Riemannian projection onto the boundary $P_\# f^+$, where

$$P(x) := \operatorname{argmin}\{d_k(x,y) \,:\, y \in \partial\Omega\}.$$

Let $\Lambda := (I,P)_\# f^+$ be the unique optimal transport plan in Problem (2.7) and σ be the corresponding transport density. Let us denote by γ_x a (the) geodesic from x to $\partial\Omega$ (i.e. $\gamma_x = \gamma_{x,y}$, for some $y \in P(x)$). We note that this geodesic γ_x is unique at every point x where the distance function to the boundary $d_k(\cdot,\partial\Omega)$ is differentiable, then at almost every point x. From (2.5), we have

$$< \sigma, \phi > := \int_\Omega \int_0^1 \phi(\gamma_x(t))\, k(\gamma_x(t))\, |\gamma_x'(t)| f^+(x)\, \mathrm{d}t\, \mathrm{d}x, \quad \text{for all} \quad \phi \in C(\Omega).$$

Yet, one has

$$\gamma_x'(t) := -d_k(x,\partial\Omega)\, a(\gamma_x(t))\, \frac{\nabla d_k(\gamma_x(t),\partial\Omega)}{|\nabla d_k(\gamma_x(t),\partial\Omega)|}, \quad \text{for all} \quad t \in (0,1).$$

Hence,

$$< \sigma, \phi > := \int_\Omega \int_0^1 \phi(\gamma_x(t))\, d_k(x,\partial\Omega)\, f^+(x)\, \mathrm{d}t\, \mathrm{d}x, \quad \text{for all} \quad \phi \in C(\Omega).$$

This implies that

$$\sigma = \int_0^1 f_t\, \mathrm{d}t,$$

where

$$f_t := P_{t\#}[d_k(\cdot,\partial\Omega)\, f^+] \qquad \text{and} \qquad P_t(x) := \gamma_x(t),\ \text{for a.e. } x \in \Omega.$$

Now, we will find an explicit formula for the transport density σ. From the definition of f_t, we have

$$< f_t, \phi >= \int_\Omega \phi(P_t(x))\, d_k(x,\partial\Omega)\, f^+(x)\,\mathrm{d}x, \quad \text{for all} \quad \phi \in C(\Omega).$$

For a.e. $x \in \Omega$, set $y := P_t(x)$ (we note that this map P_t is one-to-one). Moreover, it is easy to see that $d_k(y,\partial\Omega) = (1-t)\, d_k(x,\partial\Omega)$. So, we get

$$< f_t, \phi >= \int_{\Omega_t} \phi(y)\,(1-t)^{-1} d_k(y,\partial\Omega)\, f^+(P_t^{-1}(y))\, \mathcal{J}_t(y)^{-1}\,\mathrm{d}y, \quad \text{for all} \quad \phi \in C(\Omega),$$

where

$$\Omega_t := P_t(\Omega) \quad \text{and} \quad \mathcal{J}_t(y) := [\det(DP_t(x))],\ \text{for a.e. } y \in \Omega_t.$$

This yields that

$$f_t(y) = (1-t)^{-1} d_k(y,\partial\Omega)\, f^+(P_t^{-1}(y))\, \mathcal{J}_t(y)^{-1},\ \text{ for a.e. } y \in \Omega_t.$$

Let us denote by $L_k(y)$ the length of the maximal transport ray passing through y. So, $y \in \Omega_t$ if and only if $d_k(y,\partial\Omega) \le (1-t)\, L_k(y)$. Set $r(y) := d_k(y,\partial\Omega)/L_k(y) \in [0,1]$. Hence, we infer that

$$\sigma(y) = \int_0^{1-r(y)} (1-t)^{-1} d_k(y,\partial\Omega)\, f^+(P_t^{-1}(y))\, \mathcal{J}_t(y)^{-1}\,\mathrm{d}t,\ \text{ for a.e. } y \in \Omega. \tag{4.1}$$

In order to prove L^p estimates on this transport density σ, we need to show (as in Section 3) a lower bound on the Jacobian $\mathcal{J}_t$. Before that, we recall the definition of a domain satisfying a uniform exterior ball condition.

Definition 4.1. *We say that a compact domain $\Omega \subset \mathbb{R}^d$ satisfies a uniform exterior ball condition of radius $R > 0$ if for every point $x \in \partial\Omega$, there is a point $x_0 \in \mathbb{R}^d\backslash\Omega$ such that $\overline{B(x_0,R)} \cap \Omega = \{x\}$.*

Under the assumptions that k is a $C^{1,1}$ function on $\mathbb{R}^d$ with $0 < k_{\min} \le k \le k_{\max} < \infty$ and that for every two points x and y in Ω there is a unique geodesic from x to y, we have the following:

Lemma 4.1. *Assume that Ω satisfies a uniform exterior ball condition of radius $R > 0$. Then, there exists a constant $C > 0$ depending only on d, R, diam(Ω), $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$ and $||\nabla^2 k||_\infty$ such that, for a.e. $x \in \Omega$, we have the following estimate:*

$$det(DP_t(x)) \ge C(1-t).$$

Proof. For every $s \in \partial\Omega$, let $\tau(s)$ be the length of the maximal unit geodesic γ_s starting at s with $P(\gamma_s(\tau)) = s$, for all $\tau \in [0,\tau(s)[$. Let $\nu(s)$ be the unit inner normal vector to $\partial\Omega$ at s. For almost every $x \in \Omega$, it is clear that there exists a unique pair $(s,\tau) \in \partial\Omega \times [0,\tau(s)[$ such that $x = \Psi(s,\tau) := \exp_s \tau\nu(s)$. For every $t \in [0,1]$, we have that $P_t(\Psi(s,\tau)) = \Psi(s,(1-t)\tau)$, for all $s \in \partial\Omega$ and $\tau \in [0,\tau(s)[$. Then, we get

$$\det(DP_t(\Psi(s,\tau)))\det(D\Psi(s,\tau)) = (1-t)\det(D\Psi(s,(1-t)\tau)). \tag{4.2}$$

For $s \in \partial\Omega$, let $\{e_1, ..., e_d\}$ be an orthonormal basis such that $e_d = \nu(s)$ and let us parallel-transport along the geodesic γ_s to define a new family of orthonormal basis $\{e_1(\tau), ..., e_d(\tau)\}$. Now, consider small variations of s on $\partial\Omega$ in the directions $e_1, ..., e_{d-1}$. Then, we define as in Lemma 3.1 the vector fields J_i $(i = 1, ..., d-1)$ and J_d:

$$J_i(s,\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(s + \delta e_i, \tau), \text{ for all } i \in \{1, ..., d-1\},$$

and

$$J_d(s,\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(s, \tau + \delta).$$

Set

$$J(s,\tau) = (J_1(s,\tau), J_2(s,\tau), ..., J_d(s,\tau)) \quad \text{and} \quad \mathcal{J}(s,\tau) = \det[J(s,\tau)].$$

We have

$$\mathcal{J}'(s,\tau) = \mathrm{tr}[J'(s,\tau)\, J(s,\tau)^{-1}]\, \mathcal{J}(s,\tau). \tag{4.3}$$

Similarly to Lemma 3.1, one can show that $J'(s,\tau)J(s,\tau)^{-1}$ is the second fundamental form of the submanifold $\{\Psi(s,\tau) : s \in \partial\Omega\}$. Yet, we have $J_d = a\, \frac{\nabla d_k(\cdot\,,\partial\Omega)}{|\nabla d_k(\cdot\,,\partial\Omega)|}$. Thanks to [6, Theorem 8.2.7], the distance function $d_k(\cdot, \partial\Omega)$ is semi-concave as soon as Ω satisfies a uniform exterior ball condition of radius $R > 0$ and $k \in C^{1,1}(\Omega)$ with $0 < k_{\min} \leq k \leq k_{\max} < \infty$. More precisely, there is a constant $C < \infty$ which only depends on d, R, $\mathrm{diam}(\Omega)$, $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$ and $||\nabla^2 k||_\infty$ such that

$$D^2[d_k(\cdot, \partial\Omega)] \leq CI.$$

Hence, we obtain

$$\mathrm{tr}[J'(s,\tau)J(s,\tau)^{-1}] \leq C.$$

By (4.3), we infer that

$$\mathcal{J}'(s,\tau) \leq C\, \mathcal{J}(s,\tau).$$

Integrating in τ, we get

$$\mathcal{J}(s,(1-t)\tau) \geq \mathcal{J}(s,\tau)\, e^{-Ct\tau}.$$

Recalling (4.2), this implies that

$$\det(DP_t(x)) \geq e^{-Ct\tau}(1-t).$$

This concludes the proof. $\square$

Under the assumptions of Lemma 4.1, we have the following L^p summability result on σ.

Proposition 4.2. *The transport density σ between f^+ and $P_{\#}f^+$ belongs to $L^p(\Omega)$ as soon as $f^+ \in L^p(\Omega)$, for all $p \in [1, \infty]$. Moreover, we have the following estimate:*

$$||\sigma||_{L^p(\Omega)} \leq C||f^+||_{L^p(\Omega)}.$$

Proof. From (4.1), we have

$$||\sigma||^p_{L^p(\Omega)} = \int_\Omega \left(\int_0^{1-r(y)} (1-t)^{-1} d_k(y,\partial\Omega)\, f^+(P_t^{-1}(y))\, \mathcal{J}_t(y)^{-1}\, \mathrm{d}t \right)^p \mathrm{d}y.$$

Using Hölder's inequality, we get

$$||\sigma||^p_{L^p(\Omega)}$$
$$\leq \int_\Omega \bigg(\int_0^{1-r(y)} (1-t)^{-q}\, d_k(y,\partial\Omega)^q \mathcal{J}_t(y)^{-1}\mathrm{d}t\bigg)^{p/q}\bigg(\int_0^{1-r(y)} f^+(P_t^{-1}(y))^p \mathcal{J}_t(y)^{-1}\mathrm{d}t\bigg)\mathrm{d}y.$$

Thanks to Lemma 4.1, we know that there is a constant $C>0$ such that, for a.e. $y\in\Omega_t$, we have the following estimate:

$$\mathcal{J}_t(y)\geq C(1-t).$$

Then, we infer that

$$||\sigma||^p_{L^p(\Omega)} \leq C^{-p/q}\int_\Omega \bigg(\int_0^{1-r(y)} \frac{d_k(y,\partial\Omega)^q}{(1-t)^{q+1}}\,\mathrm{d}t\bigg)^{p/q}\bigg(\int_0^{1-r(y)} f^+(P_t^{-1}(y))^p\, \mathcal{J}_t(y)^{-1}\,\mathrm{d}t\bigg)\mathrm{d}y.$$

Yet,

$$\bigg(\int_0^{1-r(y)} \frac{d_k(y,\partial\Omega)^q}{(1-t)^{q+1}}\,\mathrm{d}t\bigg)^{p/q} \leq \bigg(\frac{d_k(y,\partial\Omega)^q}{q\, r(y)^q}\bigg)^{p/q}\leq \frac{k^p_{\max}\,\mathrm{diam}(\Omega)^p}{q^{\,p/q}}.$$

Hence, we get

$$||\sigma||^p_{L^p(\Omega)} \leq \frac{(k_{\max}\,\mathrm{diam}(\Omega))^p}{(Cq)^{\,p/q}}\int_0^1\int_{\Omega_t} f^+(P_t^{-1}(y))^p\, \mathcal{J}_t(y)^{-1}\,\mathrm{d}y\,\mathrm{d}t.$$

This yields that

$$||\sigma||_{L^p(\Omega)}\leq C\,||f^+||_{L^p(\Omega)}. \quad \square$$

5. L^p estimates on the transport density in the export transport problem

In this section, we consider the transport problem to the boundary but in the case where we have an additional boundary cost. More precisely, we assume that we have a mass f^+ in the interior of Ω that we want to transport into the boundary $\partial\Omega$, minimizing the transport cost that is given by the Riemannian metric $d_k(x,y)$ plus an extra export tax $g(y)$ at the exit point $y\in\partial\Omega$, where $g:\partial\Omega\mapsto\mathbb{R}$ is $\lambda-$Lipschitz with respect to d_k with $\lambda<1$. In other words, we minimize

(5.1)
$$\min\bigg\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda(x,y)+\int_{\partial\Omega} g(y)\,\mathrm{d}[(\Pi_y)_\#\Lambda]\,:\,(\Pi_x)_\#\Lambda=f^+\ \text{ and }\ \mathrm{spt}[(\Pi_y)_\#\Lambda]\subset\partial\Omega\bigg\}.$$

Assume $f^+\in L^1(\Omega)$. We recall that $\Lambda:=(Id,T)_\# f^+$ is the unique optimal transport plan for Problem (5.1) where

$$T(x):=\mathrm{argmin}\{d_k(x,y)+g(y)\,:\,y\in\partial\Omega\},\ \text{for all}\ x\in\Omega.$$

We note that the transport density σ in (5.1) is well defined even if the domain Ω is not geodesically convex. Indeed, for every $x\in\Omega$, the geodesic $\gamma_{x,T(x)}$ lies in Ω; this follows from the fact that if $\gamma_{x,T(x)}$ intersects $\partial\Omega$ at a point $y\neq T(x)$ then we must have $d_k(x,y)+g(y)<d_k(x,T(x))+g(T(x))$, which contradicts the optimality of $T(x)$. On the other hand, we recall that the dual of Problem (5.1) is the following (see [13, 11]):

$$\sup\bigg\{\int_\Omega u\,\mathrm{d}f^+\,:\,|\nabla u|\leq k,\ u=g\ \text{ on }\ \partial\Omega\bigg\}. \tag{5.2}$$

Notice that $u(x) := \min\{d_k(x,y) + g(y) \,:\, y \in \partial\Omega\}$ is the Kantorovich potential in (5.2). In addition, the equivalent minimal flow formulation is now

$$\min\left\{ \int_\Omega k|v| + \int_{\partial\Omega} g\,\mathrm{d}\chi \,:\, v \in \mathcal{M}(\Omega,\mathbb{R}^d),\, \chi \in \mathcal{M}^+(\partial\Omega),\, \nabla\cdot v = f^+ - \chi \;\text{ in }\; \overline{\Omega} \right\}. \tag{5.3}$$

The vector measure $v = -\sigma\, a\, \frac{\nabla u}{|\nabla u|}$ is the unique optimal flow in Problem (5.3). And, the pair (σ, u) solves

$$\begin{cases} -\nabla\cdot[\sigma\, a\, \frac{\nabla u}{|\nabla u|}] = f^+ & \text{in } \;\Omega, \\ u = g & \text{on } \;\partial\Omega, \\ |\nabla u| \le k & \text{in } \;\Omega, \\ |\nabla u| = k & \sigma - \text{a.e.} \end{cases} \tag{5.4}$$

Now, the aim is to extend the L^p estimates of Section 4 to the transport density σ between f^+ and $T_\# f^+$. In the sequel, we will denote by γ_x the geodesic $\gamma_{x,T(x)}$, for a.e. $x \in \Omega$. For every $t \in [0,1]$, set $T_t(x) := \gamma_x(t)$, for a.e. $x \in \Omega$. Recalling (2.5) and the fact that $\gamma_x'(t) = -d_k(x,T(x))\, a(\gamma_x(t))\, \frac{\nabla u(\gamma_x(t))}{|\nabla u(\gamma_x(t))|}$ for all $t \in (0,1)$, one can show as in Section 4 that

$$\sigma(y) = \int_0^1 (1-t)^{-1} d_k(y,T(y))\, f^+(T_t^{-1}(y))\, [\det(DT_t(x))]^{-1}\,\mathrm{d}t, \quad \text{for a.e. } \; y \in T_t(\Omega).$$

Similarly to Lemma 4.1, we will show a lower bound on the Jacobian $\det(DT_t(x))$. Let us assume again that k is $C^{1,1}$ with $0 < k_{\min} \le k \le k_{\max} < \infty$, $\Omega \subset \mathbb{R}^d$ is a compact domain satisfying a uniform exterior ball condition of radius $R > 0$ and, we also assume that there is a unique geodesic between any two given points x and y in Ω. Then, under these assumptions, we have the following:

Lemma 5.1. *Assume that g is a C^1 function with $||\nabla g||_\infty < k_{\min}$ and, g is semi-concave. Then, there exists a constant $C > 0$ depending only on d, R, diam(Ω), $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$, $||\nabla^2 k||_\infty$, $||\nabla g||_\infty$ and $\sup[\nabla^2 g]$ (the smallest constant M such that $\nabla^2 g \le MI$) such that, for a.e. $x \in \Omega$, we have the following estimate:*

$$det(DT_t(x)) \ge C(1-t).$$

Proof. The proof is essentially the same introduced in Lemma 4.1 and so, we will omit some details. For every $s \in \partial\Omega$, we will denote again by $\tau(s)$ the length of the maximal unit geodesic γ_s starting at s with $T(\gamma_s(\tau)) = s$, for all $\tau \in [0,\tau(s)[$. Set $\nu(s) := a(s)\frac{\nabla u(s)}{|\nabla u(s)|}$, where u is the Kantorovich potential in (5.2). For a.e. $x \in \Omega$, there exists a unique pair $(s,\tau) \in \partial\Omega \times [0,\tau(s)[$ such that $x = \Psi(s,\tau) := \exp_s \tau\nu(s)$. Moreover, we have $T_t(\Psi(s,\tau)) = \Psi(s,(1-t)\tau)$, for all $(s,\tau) \in \partial\Omega \times [0,\tau(s)[$. Then,

$$\det(DT_t(\Psi(s,\tau)))\det(D\Psi(s,\tau)) = (1-t)\det(D\Psi(s,(1-t)\tau)).$$

For $s \in \partial\Omega$, let $\{e_1, ..., e_d\}$ be a basis (not necessarily orthonormal) such that $\{e_1, ..., e_{d-1}\}$ is an orthonormal basis of the tangent space to $\partial\Omega$ at s and $e_d = \nu(s)$. Let us parallel-transport along the geodesic γ_s to define a new family of basis $\{e_1(\tau), ..., e_d(\tau)\}$. We define again

$$J_i(s,\tau) = \frac{d}{d\delta}_{|\delta=0} \Psi(s + \delta e_i, \tau), \;\text{ for all } i \in \{1, ..., d-1\},$$

$$J_d(s,\tau) = \frac{d}{d\delta}_{|\delta=0}\Psi(s,\tau+\delta),$$

and

$$J(s,\tau) = (J_1(s,\tau), J_2(s,\tau), ..., J_d(s,\tau)), \qquad \mathcal{J}(s,\tau) = \det[J(s,\tau)].$$

But, we have

$$\nabla_{J_d} J_i = \sum_{j=1}^{d} J'_{ji}\, e_j + \sum_{j=1}^{d}\sum_{k=1}^{d} J_{ji}\,\Gamma^k_{dj}\, e_k = \sum_{j=1}^{d}\left[J'_{ji} + \sum_{k=1}^{d}\Gamma^j_{dk}\, J_{ki}\right] e_j,$$

where Γ^k_{dj} denote the Christoffel symbols. Let us denote again by V the matrix, in the basis $\{e_1(\tau), ..., e_d(\tau)\}$, associated with the endomorphism $X \mapsto \nabla_X J_d$. Then, we have

$$\nabla_{J_i} J_d = \sum_{j=1}^{d} (VJ)_{ji}\, e_j.$$

Hence,

$$J' = [V - \Gamma]J, \text{ where } \Gamma := (\Gamma^i_{dj})_{ij}.$$

In particular, we have

$$\mathcal{J}'(s,\tau) = \operatorname{tr}[V-\Gamma]\,\mathcal{J}(s,\tau).$$

Yet, we have $J_d = a\,\frac{\nabla u}{|\nabla u|}$. From [6, Theorem 8.2.7], the function u is semi-concave thanks to the assumptions that g is C^1 with $||\nabla g||_\infty < k_{\min}$ and that g is semi-concave. On the other hand, we have $\Gamma^d_{dd} = 0$ (since γ_s is a geodesic) and, as we parallel-transport along the geodesic γ_s, then one also has

$$\Gamma^i_{di} = \partial_{e_d} \log\left[\sqrt{\det[(k(\gamma_s(\tau))^2\, e_i(\tau)\cdot e_j(\tau))_{ij}]}\right] = 0.$$

Consequently, there is a constant C depending only on d, R, $\operatorname{diam}(\Omega)$, $k_{\min}$, $k_{\max}$, $||\nabla k||_\infty$, $||\nabla^2 k||_\infty$, $||\nabla g||_\infty$ and $\sup[\nabla^2 g]$ such that

$$\operatorname{tr}[V-\Gamma] \le C.$$

Then,

$$\mathcal{J}(s,(1-t)\tau) \ge \mathcal{J}(s,\tau)\, e^{-Ct\tau}.$$

Therefore, we get that

$$\det(DT_t(x)) \ge e^{-Ct\tau}(1-t).$$

□

Finally, we conclude the following:

Proposition 5.2. *Under the assumptions of Lemma 5.1, the transport density σ between f^+ and $T_\# f^+$ belongs to $L^p(\Omega)$ as soon as $f^+ \in L^p(\Omega)$, for all $p \in [1,\infty]$. In addition, there is a constant $C := C(d, R, diam(\Omega), k_{\min}, k_{\max}, ||\nabla k||_\infty, ||\nabla^2 k||_\infty, ||\nabla g||_\infty, \sup[\nabla^2 g]) < \infty$ such that*

$$||\sigma||_{L^p(\Omega)} \le C||f^+||_{L^p(\Omega)}.$$

6. L^p ESTIMATES ON THE TRANSPORT DENSITY IN THE IMPORT-EXPORT TRANSPORT PROBLEM

In this section, we collect the results of the previous sections to prove L^p estimates on the transport density σ in the import-export transport problem (2.12). Let g^+ and g^- be two continuous functions on $\partial\Omega$ and assume that there is a $\lambda < 1$ such that we have the following condition:

$$g^+(x) - g^-(y) \leq \lambda\, d_k(x,y), \text{ for all } x,\, y \in \partial\Omega. \tag{6.1}$$

Then, we consider

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\Lambda + \int_{\partial\Omega} g^-(y)\,\mathrm{d}[(\Pi_y)_\#\Lambda] - \int_{\partial\Omega} g^+(x)\,\mathrm{d}[(\Pi_x)_\#\Lambda] \,:\, \Lambda \in \Pi(f^+,f^-)\right\}. \tag{6.2}$$

Let Λ be an optimal transport plan in (6.2). Thanks to (6.1), it is not difficult to see that $\Lambda(\partial\Omega\times\partial\Omega) = 0$. Yet, from Section 2.3, we have

$$\Lambda = \Lambda_{ii} + \Lambda_{ib} + \Lambda_{bi}$$

where Λ_{ii} is an optimal transport plan in the Kantorovich problem between $f^+ - \nu^+$ and $f^- - \nu^-$ ($\nu^\pm$ are two parts of $f^\pm$), Λ_{ib} is the optimal transport plan in the export transport problem (5.1) of ν^+ to $\partial\Omega$ (see Proposition 2.1) and, Λ_{bi} is the optimal transport plan in the import transport problem from $\partial\Omega$ to ν^-.

Let σ, σ_{ii}, σ_{ib} and σ_{bi} be the transport densities associated with the optimal transport plans Λ, Λ_{ii}, Λ_{ib} and Λ_{bi}, respectively. Notice that these transport densities are well defined as soon as Ω is geodesically convex. However, we recall that the transport densities σ_{ib} and σ_{bi} are well defined even if Ω is not geodesically convex, but under the assumptions that g^+ and g^- are $\lambda-$Lip w.r.t. the distance d_k with $\lambda < 1$. Moreover, σ_{ii} is well defined provided that $g^+ = g^- = g$. Indeed, if the geodesic $\gamma_{x,y}$ intersects $\partial\Omega$ at (at least) two different points y' and x' with $d_k(x,y') < d_k(x,x')$, then it will be better (thanks to the fact that g is $\lambda-$Lip w.r.t. d_k with $\lambda < 1$) to export the mass which is located at x to $y' \in \partial\Omega$ and then import a mass from $x' \in \partial\Omega$ to y; but this yields to a contradiction with the optimality of Λ.

Assume that k is a $C^{1,1}$ function on $\mathbb{R}^d$ such that $0 < k_{\min} \leq k \leq k_{\max} < \infty$. Then, we have the following:

Proposition 6.1. *Assume that Ω is geodesically convex and f^+, $f^- \in L^p(\Omega)$. Then, the transport density σ_{ii} between $f^+ - \nu^+$ and $f^- - \nu^-$ belongs to $L^p(\Omega)$, for every $p \in [1,\infty]$.*

Proof. This follows immediately from Proposition 3.2 and the fact that $0 \leq \nu^\pm \leq f^\pm$, so $f^\pm - \nu^\pm \in L^p(\Omega)$. □

Proposition 6.2. *Assume that Ω satisfies a uniform exterior ball condition of radius $R > 0$ and that for every two points x and y in Ω, there is a unique geodesic from x to y. Let $g^\pm$ be two C^1 functions with $||\nabla g^\pm||_\infty < k_{\min}$, $\nabla^2 g^- \leq CI$ and $\nabla^2 g^+ \geq -CI$. Then, the transport density σ_{ib} (resp. σ_{bi}) belongs to $L^p(\Omega)$ as soon as $f^+ \in L^p(\Omega)$ (resp. $f^- \in L^p(\Omega)$), for all $p \in [1,\infty]$.*

Proof. This follows directly from Proposition 5.2. □

Consequently, we get the following L^p summability on the transport density σ:

Proposition 6.3. *Assume Ω is geodesically convex, $g^{\pm}$ are C^1 with $|\nabla g^{\pm}| < k_{\min}$, $\nabla^2 g^- \leq CI$ and $\nabla^2 g^+ \geq -CI$. Hence, the transport density σ in (6.2) belongs to $L^p(\Omega)$ as soon as $f^{\pm} \in L^p(\Omega)$, for all $p \in [1, \infty]$.*

Proof. This follows from Propositions 6.1 & 6.2 and the fact that $\sigma = \sigma_{ii} + \sigma_{ib} + \sigma_{bi}$. □

Finally, we also have the following:

Proposition 6.4. *Suppose that Ω has a uniform exterior ball condition of radius $R > 0$ and that there is a unique geodesic between any two points of Ω. Assume also that $g^+ = g^- = g$ is a $C^{1,1}$ function on $\partial\Omega$ such that $|\nabla g| < k_{\min}$. Then, the transport density σ is in $L^p(\Omega)$ provided that $f^{\pm} \in L^p(\Omega)$, for all $p \in [1, \infty]$.*

Proof. Since $g^+ = g^- = g$, the transport density σ is well defined even if the domain Ω is not geodesically convex. The rest follows again from Propositions 6.1 & 6.2. □

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