Balanced-Viscosity solutions to infinite-dimensional multi-rate systems

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Abstract. We consider generalized gradient systems with rate-independent and rate-dependent dissipation potentials. We provide a general framework for performing a vanishing-viscosity limit leading to the notion of parametrized and true Balanced-Viscosity solutions that include a precise description of the jump behavior developing in this limit. Distinguishing an elastic variable u having a viscous damping with relaxation time ε^α and an internal variable z with relaxation time ε we obtain different limits for the three cases $\alpha \in (0,1)$, $\alpha = 1$ and $\alpha > 1$. An application to a delamination problem shows that the theory is general enough to treat nontrivial models in continuum mechanics.

Keywords: balanced-viscosity solution, reparametrized solutions, energy-dissipation principle, generalized gradient systems.

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List of symbols.

$\mathbf{U}, \mathbf{Z}$	state spaces	(1.1)
$\mathbf{Q} = \mathbf{U}\times\mathbf{Z}$	overall state space	
$\mathbf{U}_{\mathrm{e}} \subset \mathbf{U},\ \mathbf{Z}_{\mathrm{e}} \Subset \mathbf{Z}$	energy spaces	Hyp. 4.1
$\mathbf{Z}_{\mathrm{ri}} \supset \mathbf{Z}$	space for 1-homogeneous dissipation potential	Hyp. 4.1
$\mathcal{R} : \mathbf{Z}_{\mathrm{ri}} \to [0,\infty)$	1-homogeneous dissipation potential	Hyp. 4.2
$\mathcal{V}_{\mathsf{u}} : \mathbf{U} \to [0,\infty),\ \ \mathcal{V}_{\mathsf{z}} : \mathbf{Z} \to [0,\infty)$	viscous dissipation potentials	Hyp. 4.2
$\mathcal{V}_{\mathsf{x}}^* : \mathbf{X}^* \to [0,\infty),\ \mathbf{X} \in \{\mathbf{U},\mathbf{Z}\}$	Legendre-Fenchel conjugate of $\mathcal{V}_{\mathsf{x}}$ for $\mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$	Def. 3.1
$\mathcal{W}_{\mathsf{z}}^* : \mathbf{Z}^* \to [0,\infty)$	conjugate of $\mathcal{R}+\mathcal{V}_{\mathsf{z}}$	(4.17)
$\mathcal{V}_{\mathsf{x}}^{\lambda},\ \mathsf{x} \in \{\mathsf{u},\mathsf{z}\},\ \lambda \in (0,\infty)$	rescaled viscous dissipation potentials	(1.5a)
$\Psi_{\varepsilon,\alpha} = \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha} + \mathcal{R} + \mathcal{V}_{\mathsf{z}}^{\varepsilon}$	overall viscous potential	(1.7)
$\mathcal{E} : [0,T] \times \mathbf{U}\times\mathbf{Z} \to (-\infty,+\infty]$	driving energy functional	Hyp. 4.3
$\mathcal{S}_E,\ E > 0$	energy sublevels	(4.8)
$\partial_q\mathcal{E}$	Fréchet subdifferential of $\mathcal{E}(t,\cdot)$	(4.10)
$\mathscr{S}_{\mathsf{x}}^* : [0,T]\times\mathrm{D} \to [0,\infty]$	generalized slope functional for $\mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$	(4.19)
$\mathfrak{A}_{\mathsf{x}}^*(t,q),\ (t,q) \in [0,T]\times\mathrm{D}$	set of minimizers for the slope $\mathscr{S}_{\mathsf{x}}^*(t,q),\ \mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$	(4.21)
$\mathscr{G}^\alpha[t,q],\ (t,q) \in [0,T]\times\mathrm{D}$	sets of positivity for the slopes at (t,q)	(5.19)
$\mathcal{B}_\psi$	B-function associated with a dissipation potential ψ	(3.1)
$\mathfrak{b}_\psi$	vanishing-viscosity contact potential assoc. with ψ	(3.4)
$\mathfrak{B}_\varepsilon^\alpha$, $\varepsilon \geq 0$,	rescaled joint B-function	(3.14)
$\mathfrak{M}_\varepsilon^\alpha$, $\varepsilon > 0$,	rescaled joint M-function	(5.1)
$\mathfrak{M}_0^\alpha$	(limiting) rescaled joint M-function	(5.7)
$\mathfrak{M}_0^{\alpha,\mathrm{red}}$	reduced rescaled joint M-function	(5.11)
$\mathscr{A}([a,b];[0,T]\times\mathbf{Q})$	admissible parametrized curves from $[a,b]$ to $[0,T]\times\mathbf{Q}$	Def. 5.6
$\mathcal{A}_t(q_0,q_1)$	admissible transition curves betw. q_0 and q_1 at time t	Def. 5.6
Σ_α	contact set	(5.40)
$\mathrm{A}_{\mathsf{u}}\mathrm{C}_{\mathsf{z}} = \mathrm{A}_{\mathsf{u}} \cap \mathrm{C}_{\mathsf{z}}$	evolution regimes with $\mathrm{A} \in \{\mathrm{E},\mathrm{V},\mathrm{B}\}$ and $\mathrm{C} \in \{\mathrm{R},\mathrm{V},\mathrm{B}\}$	(5.43)
$\mathrm{Var}_{\mathcal{R}}$	$\mathcal{R}$-variation	(6.2)
$\mathrm{J}[q]$	jump set of a true BV solution	Def. 6.4
$\mathrm{cost}_{\mathfrak{M}_0^\alpha}$	Finsler cost induced by $\mathfrak{M}_0^\alpha$	(6.6)
$\mathrm{Var}_{\mathfrak{M}_0^\alpha}$	total variation induced by $\mathfrak{M}_0^\alpha$	(6.8)

1. Introduction

In this paper we address rate-independent limits of viscous evolutionary systems that are motivated by applications in solid mechanics. These systems can be described in terms of two variables $u \in \mathbf{U}$ and $z \in \mathbf{Z}$; throughout, we shall assume that the state spaces

$$\mathbf{U} \text{ and } \mathbf{Z} \text{ are (separable) reflexive Banach spaces.} \tag{1.1}$$

Typically, u is the displacement, or the deformation of the body, whereas z is an internal variable specific of the phenomenon under investigation, in accordance with the theory of *generalized standard materials*, see [HaN75].

1.1. Rate-independent systems.

Under very slow loading rates, one often assumes that u satisfies a *static* balance law that arises as Euler-Lagrange equation from minimizing the energy functional $\mathcal{E}$ with respect to u. The evolution of z is governed by a (doubly nonlinear) subdifferential inclusion featuring the z-derivative of the energy and the viscous force in form of the subdifferential $\partial\mathcal{R}$ of a dissipation potential $\mathcal{R}$:

$$\mathrm{D}_u\mathcal{E}(t,u(t),z(t)) = 0 \qquad \text{in } \mathbf{U}^*, \qquad t\in(0,T), \tag{1.2a}$$

$$\partial\mathcal{R}(z'(t)) + \mathrm{D}_z\mathcal{E}(t,u(t),z(t)) \ni 0 \qquad \text{in } \mathbf{Z}^*, \qquad t\in(0,T)\,. \tag{1.2b}$$

If $\mathcal{R}:\mathbf{Z}\to[0,\infty]$ is positively homogeneous of degree 1, i.e. $\mathcal{R}(\lambda z') = \lambda\mathcal{R}(z')$ for all $\lambda>0$, then the system (1.2) is called *rate-independent*, and the triple $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{R})$ is called a rate-independent system, cf. [MiR15].

Here, $\partial\mathcal{R}:\mathbf{Z}\rightrightarrows\mathbf{Z}^*$ denotes the subdifferential of convex analysis for the nonsmooth functional $\mathcal{R}$, whereas, throughout this introduction, for simplicity we will assume that the (proper) energy functional $\mathcal{E}:[0,T]{\times}\mathbf{U}{\times}\mathbf{Z}\to(-\infty,\infty]$, which is smooth with respect to time, is additionally smooth with respect to both variables u and z. System (1.2) reflects the ansatz that energy is dissipated through changes of the internal variable z only: in particular, the doubly nonlinear evolution inclusion (1.2b) balances the dissipative frictional forces from $\partial\mathcal{R}(z')$ with the restoring force $\mathrm{D}_z\mathcal{E}(t,u,z)$. Despite the assumed smoothness of $(u,z)\mapsto\mathcal{E}(t,u,z)$, system (1.2) is only formally written: due to the 1-homogeneity of $\mathcal{R}$, one can in general expect only BV-time regularity for z. Thus z may have jumps as a function of time and the pointwise derivative z' in the subdifferential inclusion (1.2b) need not be defined. This has motivated the development of various weak solution concepts for system (1.2).

Energetic solutions were advanced in the late '90s in [MiT99, MTL02, MiT04] for *abstract* rate-independent systems, and in the context of phase transformations in solids. In the realm of crack propagation, an analogous notion of evolution was pioneered in [FrM98] and later further developed in [DaT02] with the concept of 'quasistatic evolution'. Due to its flexibility, the energetic concept has been successfully applied to a wide scope of problems, see e.g. [MiR15] for a survey.

1.2. The vanishing-viscosity approach.

However, it has been observed that the energetic notion may fail to provide a feasible description of the system behavior at jumps, in the case of a *nonconvex* driving energy. This fact has motivated the introduction of an alternative weak solvability concept, first suggested in [EfM06] and based on the vanishing-viscosity regularization of the rate-independent system as a selection criterion for mechanically feasible weak solutions. In the context of system (1.2), this 'viscous regularization' involves a second (lower semicontinuous, convex) dissipation potential $\mathcal{V}_\mathsf{z}:\mathbf{Z}\to[0,+\infty)$, with superlinear growth at infinity; to fix ideas, we may think of a *quadratic* potential. The vanishing-viscosity approach then consists in performing the asymptotic analysis of solutions to the *rate-dependent* system

$$\mathrm{D}_u\mathcal{E}(t,u(t),z(t)) = 0 \qquad \text{in } \mathbf{U}^*, \qquad t\in(0,T), \tag{1.3a}$$

$$\partial\mathcal{R}(z'(t)) + \partial\mathcal{V}_\mathsf{z}(\varepsilon z'(t)) + \mathrm{D}_z\mathcal{E}(t,u(t),z(t)) \ni 0 \qquad \text{in } \mathbf{Z}^*, \qquad t\in(0,T)\,, \tag{1.3b}$$

as the *viscosity* parameter $\varepsilon\to0^+$. System (1.3) now features *two rates*: in addition to that of the external loading, scaling as $\varepsilon^0=1$, the internal rate of the system, set on the faster scale ε, is revealed. In diverse (finite-dimensional, infinite-dimensional, and even metric) setups, cf. [EfM06, MRS09, MRS12a, MiZ14, MRS16a] (see also [KnZ21] and [RSV21]), solutions to the 'viscous system' have been shown to converge to a different type of solution of (1.2), which we shall refer to as *Balanced-Viscosity* solution, featuring a better description of the jumps of the system. In parallel, the vanishing-viscosity approach has proved to be a robust method in manifold applications, ranging from plasticity (cf. e.g. [DDS11, BFM12, FrS13]), to fracture [KMZ08, LaT11, Neg14], damage and fatigue [KRZ13, CrL16, ACO19], and to optimal control [SWW17] to name a few.

This paper revolves around a different, but still of *vanishing-viscosity type*, solution notion for system (1.2). Indeed, we are going to regularize it by considering a viscous approximation of (1.2a), besides the viscous approximation (1.3b) of (1.2b). Therefore, we will address the asymptotic analysis as $\varepsilon\to0^+$ of the system of doubly nonlinear differential inclusions

$$\partial\mathcal{V}_\mathsf{u}^{\varepsilon^\alpha}(u'(t)) + \mathrm{D}_u\mathcal{E}(t,u(t),z(t)) \ni 0 \qquad \text{in } \mathbf{U}^* \qquad \text{for a.a. } t\in(0,T), \tag{1.4a}$$

$$\partial\mathcal{R}(z'(t)) + \partial\mathcal{V}_\mathsf{z}^{\varepsilon}(z'(t)) + \mathrm{D}_z\mathcal{E}(t,u(t),z(t)) \ni 0 \qquad \text{in } \mathbf{Z}^* \qquad \text{for a.a. } t\in(0,T), \tag{1.4b}$$

where for $\mathsf{x}\in\{\mathsf{u},\mathsf{z}\}$ we have set

$$\mathcal{V}_\mathsf{x}^\lambda(w) := \frac{1}{\lambda}\mathcal{V}_\mathsf{x}(\lambda w) \text{ for } \lambda\in(0,\infty) \;\text{ and }\; \mathcal{V}_\mathsf{x}^\infty(w) := \begin{cases} 0 & \text{for } w=0, \\ \infty & \text{for } w\neq 0 \end{cases} \tag{1.5a}$$

(the functional $\mathcal{V}_\mathsf{x}^\infty$ will indeed come into play later on, cf. (1.13)). Throughout we assume that $\mathcal{V}_\mathsf{x}$ satisfies $\mathcal{V}_\mathsf{x}(0)=0$, $\partial\mathcal{V}_\mathsf{x}(0)=\{0\}$, and has superlinear growth, which implies that $\mathcal{V}_\mathsf{x}^0$ and $\mathcal{V}_\mathsf{x}^\infty$ are indeed the Mosco limits of $\mathcal{V}_{\mathsf{x}}^{\lambda}$ for $\lambda \to 0^+$ and $\lambda \to \infty$, respectively. We will use that the subdifferentials take the form

$$\partial\mathcal{V}_{\mathsf{x}}^{\lambda}(w) = \partial\mathcal{V}_{\mathsf{x}}(\lambda w) \text{ for } \lambda \in [0,\infty) \quad\text{and}\quad \partial\mathcal{V}_{\mathsf{x}}^{\infty}(w) = \begin{cases} \mathbf{X}^* & \text{for } w = 0, \\ \emptyset & \text{for } w \neq 0. \end{cases} \tag{1.5b}$$

The parameter α in (1.4a) determines which of the two variables u and z relaxes faster to *equilibrium* and *rate-independent* evolution, respectively. Hence, following the finite-dimensional work [MRS16b] we shall refer to (1.4) as a *multi-rate* system, with the time scale $\varepsilon^0 = 1$ of the external loading and the (possibly different) relaxation times ε and ε^α of the variables z and u.

From a broader perspective, with our analysis we aim to contribute to the investigation of *coupled* rate-dependent/rate-independent phenomena, a topic that has attracted some attention over the last decade. In this connection, we may mention the study of systems with a *mixed* rate-dependent/rate-independent character (typically, a rate-independent flow rule for the internal variable coupled with the momentum balance, with viscosity and inertia, for the displacements, and possibly with the heat equation), see the series of papers by T. Roubíček [Rou09, Rou10, Rou13a, Rou13b, RoT15b], among others. There, a weak solvability notion, still of *energetic type*, was advanced, cf. also [RoT17, MaM16].

However, unlike in those contributions, in our 'modeling' approach the balanced interplay of rate-dependent and rate-independent behavior does not stem from coupling equations with a rate-dependent and a rate-independent character. Instead, it emerges through the asymptotic analysis as $\varepsilon \to 0^+$ of the 'viscous' system (1.4), which leads to a solution of the rate-independent one (1.2) that is 'reminiscent of viscosity', *in both variables* u and z, in the description of the system behavior at jumps. This 'full' vanishing-viscosity approach, also involving the displacement variable u, has been already carried out for a model for fracture evolution with pre-assigned crack path in [Rac12], as well as in the context of perfect plasticity [DaS14, Ros18] and delamination [Sca17]. With different techniques, based on an alternating minimization scheme, the emergence of viscous behavior both for the displacement and for the internal variable is demonstrated in [KnN17] for a phase-field type fracture model.

In this mainstream, in [MRS16b] we have addressed the vanishing-viscosity analysis of (1.4) in a preliminary finite-dimensional setting, with $\mathbf{U} = \mathbb{R}^n$ and $\mathbf{Z} = \mathbb{R}^m$, and for a smooth energy $\mathcal{E} \in \mathrm{C}^1([0,T]\times\mathbb{R}^n\times\mathbb{R}^m)$, with the aim of emphasizing the role of viscosity in the description of the jump behavior of the limiting rate-independent system. Even in this significantly simplified setup, the analysis in [MRS16b] conveyed how the balanced interplay of the different relaxation rates in (1.4) enters in the description of the jump dynamics of the rate-independent system. In particular, it showed that viscosity in u and viscosity z determine the jump transition path in different ways depending on whether the parameter α is strictly bigger than, or equal to, or strictly smaller than 1.

The aim of this paper is to thoroughly extend the results from [MRS16b] to an *infinite-dimensional* and *non-smooth* setting, suited for the application of this vanishing-viscosity approach to models in solid mechanics. What is more, we will also broaden the analysis in [MRS16b], which is confined to the case of *quadratic* 'viscous' dissipation potentials, to a fairly general class of potentials $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$.

1.3. Our results.

Throughout most of this paper, we will confine the discussion to the *abstract rate-independent* system $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ arising in the vanishing-viscosity limit of (1.4). The notation looks a bit extensive, but has the advantage of emphasizing the dependence of the solution concept on the energy functional $\mathcal{E}$, the three different types of dissipation $\mathcal{V}_{\mathsf{u}}$, $\mathcal{R}$, and $\mathcal{V}_{\mathsf{z}}$, and the parameter $\alpha > 0$. This also explains the name "Balanced-Viscosity solution" that suggests the appearance of the viscous effects by balancing the influence of $\mathcal{R}$, $\mathcal{V}_{\mathsf{x}}$, and $\mathcal{V}_{\mathsf{z}}$ in such a way that the energy-dissipation balance remains true. Of course, using the abbreviation "BV solution" should remind us about the fact that these solutions may not be continuous but may have jumps as functions of time.

In our opinion, in that general framework the main ideas underlying the vanishing-viscosity approach are easier to convey. Indeed, we aim to provide some possible recipes for the application of this approach to concrete rate-independent limiting processes, where of course the 'abstract techniques' may have to be suitably adjusted to the specific situation. For this, we will strive to work in a fairly general setup,

(1) encompassing nonsmoothness of the energies $u \mapsto \mathcal{E}(t,u,z)$ and $z \mapsto \mathcal{E}(t,u,z)$ through the usage of suitable subdifferentials $\partial_{\mathsf{u}}\mathcal{E} : [0,T]\times\mathbf{U}\times\mathbf{Z} \rightrightarrows \mathbf{U}^*$ and $\partial_{\mathsf{z}}\mathcal{E} : [0,T]\times\mathbf{U}\times\mathbf{Z} \rightrightarrows \mathbf{Z}^*$ in place of the Gâteau derivatives $\mathrm{D}_u\mathcal{E}$ and $\mathrm{D}_z\mathcal{E}$, and
(2) allowing for a wide range of 'viscous dissipation potentials' $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$. In particular, we shall allow for a much broader class of dissipation potentials $\mathcal{V}_{\mathsf{z}}$ than those considered in [MRS16a].

The first cornerstone of our vanishing-viscosity analysis is the observation that the viscous system (1.4) has the structure of a *generalized gradient system* (cf. [Mie16]): indeed, it rewrites as

$$\partial\Psi_{\varepsilon,\alpha}(q'(t)) + \mathrm{D}_q\mathcal{E}(t,q(t)) \ni 0 \qquad \text{in } \mathbf{Q}^* \quad \text{for a.a. } t \in (0,T) \tag{1.6}$$

with $q = (u, z) \in \mathbf{Q} = \mathbf{U}\times\mathbf{Z}$ and

$$\Psi_{\varepsilon,\alpha}(q') = (\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha} \oplus (\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon}))(q') := \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}(u') + \mathcal{R}(z') + \mathcal{V}_{\mathsf{z}}^{\varepsilon}(z'). \tag{1.7}$$

In turn, (1.6) can be equivalently formulated using the single *energy-dissipation balance*

$$\mathcal{E}(t, q(t)) + \int_s^t \widetilde{\mathfrak{M}}_\varepsilon^\alpha(r, q(r), q'(r))\,\mathrm{d}r = \mathcal{E}(s, q(s)) + \int_s^t \partial_t\mathcal{E}(r, q(r))\,\mathrm{d}r \tag{1.8}$$

for all $0 \le s \le t \le T$, featuring the M-function

$$\widetilde{\mathfrak{M}}_\varepsilon^\alpha(t, q, q') := \Psi_{\varepsilon,\alpha}(q') + \Psi_{\varepsilon,\alpha}^*(-\mathrm{D}_q\mathcal{E}(t, q)) \tag{1.9}$$

with the Legendre-Fenchel conjugate $\Psi_{\varepsilon,\alpha}^*$ of $\Psi_{\varepsilon,\alpha}$. This reformulation is often referred to *energy-dissipation principle*; the germs of this idea trace back to E. De Giorgi's *variational theory for gradient flows* in [Amb95], see also [AGS08, Prop. 1.4.1] and [Mie16, Thm. 3.2]. In our setup, it is based on the validity of a suitable chain rule for $\mathcal{E}$, which will be thoroughly discussed in the sequel. From (1.8) we obtain the basic a priori estimates on a sequence $(u_\varepsilon, z_\varepsilon)_\varepsilon$ of solutions to (1.4). Together with the additional bound

$$\int_0^T \|u_\varepsilon'(t)\|_{\mathbf{U}}\,\mathrm{d}t \le C, \tag{1.10}$$

we are able to reparametrize the curves $(q_\varepsilon)_\varepsilon = (u_\varepsilon, z_\varepsilon)_\varepsilon$ by their "dissipation arclength" $\mathsf{s}_\varepsilon : [0, T] \to [0, \mathsf{S}_\varepsilon]$ given by

$$\mathsf{s}_\varepsilon(t) := \int_0^t \Big(1+\widetilde{\mathfrak{M}}_\varepsilon^\alpha(r, q_\varepsilon(t), q_\varepsilon'(r))+\|u_\varepsilon'(r)\|_{\mathbf{U}}\Big)\,\mathrm{d}r\,.$$

Reparametrization was first advanced in [EfM06] as a tool to capture the viscous transition paths, at jumps, in the rate-independent limit. With this aim, first of all we observe that, setting $\mathsf{t}_\varepsilon := \mathsf{s}_\varepsilon^{-1} : [0, , \mathsf{S}_\varepsilon] \to [0, T]$ and $\mathsf{u}_\varepsilon := u_\varepsilon \circ \mathsf{t}_\varepsilon$, $\mathsf{z}_\varepsilon := z_\varepsilon \circ \mathsf{t}_\varepsilon$, the rescaled curves $(\mathsf{t}_\varepsilon, \mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon)_\varepsilon$ satisfy a reparametrized version of (1.8). Using the first main results of this paper presented in Theorems 5.11 and 5.14, we are able to pass to the limit in this reparametrized energy balance as $\varepsilon \to 0^+$ and obtain a triple $(\mathsf{t}, \mathsf{q}) = (\mathsf{t}, \mathsf{u}, \mathsf{z}) : [0, \mathsf{S}] \to [0, T]\times\mathbf{U}\times\mathbf{Z}$ satisfying the energy-dissipation balance

$$\begin{aligned} &\mathcal{E}(\mathsf{t}(s_2), \mathsf{q}(s_2)) + \int_{s_1}^{s_2} \mathfrak{M}_0^\alpha(\mathsf{t}(s), \mathsf{q}(s), \mathsf{t}'(s), \mathsf{q}'(s))\,\mathrm{d}s \\ &= \mathcal{E}(\mathsf{t}(s_1), \mathsf{q}(s_1)) + \int_{s_1}^{s_2} \partial_t\mathcal{E}(s, \mathsf{q}(s), \mathsf{t}'(s))\,\mathrm{d}s \quad \text{for all } 0 \le s_1 \le s_2 \le \mathsf{S}\,, \end{aligned} \tag{1.11}$$

which encodes all the information on the behavior of the limiting rate-independent system in the expression of the 'time-space dissipation function' $\mathfrak{M}_0^\alpha$, thoroughly investigated in Section 5.1. We shall call a triple $(\mathsf{t}, \mathsf{u}, \mathsf{z})$ complying with (1.11) a *parametrized Balanced-Viscosity* (pBV, for short) solution to the rate-independent system $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$.

We highlight two main properties of this solution concept that follow from the special form of $\mathfrak{M}_0^\alpha$:

- When a solution does not jump, i.e. when the function t of the artificial time s, recording the (slow) external time scale, fulfills $\mathsf{t}'(s) > 0$, the term $\mathfrak{M}_0^\alpha(\mathsf{t}, \mathsf{q}, \mathsf{t}', \mathsf{q}')$ is finite if and only if u is *stationary* and z is *locally stable*, i.e.

$$-\mathrm{D}_u\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s), \mathsf{z}(s)) = 0 \text{ in } \mathbf{U}^* \quad \text{and} \quad -\mathrm{D}_z\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s), \mathsf{z}(s)) \in \partial\mathcal{R}(0) \text{ in } \mathbf{Z}^*.$$

  Because of the local character of the second condition, the unfeasible jumps that may occur in 'energetic solutions' via their 'global stability' are thus avoided.
- The function $\mathfrak{M}_0^\alpha$ in (1.11) comprises the contributions of the dissipation potentials $\mathcal{R}$, $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$ by condensing the viscous effects into a description of the limiting jump behavior that can occur only if $\mathsf{t}'(s) = 0$, i.e. the slow external time is frozen. For example, if the dissipation potentials $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$ are p-homogeneous (i.e. $\mathcal{V}_{\mathsf{x}}(\lambda\mathsf{x}') = \lambda^p\mathcal{V}_{\mathsf{x}}(\mathsf{x}')$ for $\lambda > 0$), then for $\alpha = 1$ and $\mathsf{t}' = 0$ we have

$$\begin{aligned} &\mathfrak{M}_0^1(\mathsf{t}, (\mathsf{u}, \mathsf{z}), 0, (\mathsf{u}', \mathsf{z}')) = \mathcal{R}(\mathsf{z}') \\ &\quad + \widehat{c}_p\,\big(\mathcal{V}_{\mathsf{z}}(\mathsf{z}')+\mathcal{V}_{\mathsf{u}}(\mathsf{u}')\big)^{1/p}\,\big(\mathcal{V}_{\mathsf{u}}^*(-\mathrm{D}_u\mathcal{E}(\mathsf{t}, \mathsf{u}, \mathsf{z}))+\mathcal{W}_{\mathsf{z}}^*(-\mathrm{D}_z\mathcal{E}(\mathsf{t}, \mathsf{u}, \mathsf{z}))\big)^{1-1/p} \end{aligned} \tag{1.12}$$

  (see Example 5.3). The symmetric role of $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$ in (1.12) arises because of $\alpha = 1$ and reflects the fact that, at a jump, the system may switch to a viscous regime where *both* dissipation mechanisms intervene in the evolution of u and z, respectively. In contrast, for $\alpha > 1$ and $\alpha < 1$, the M-function $\mathfrak{M}_0^\alpha$ shows the different roles of $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$, cf. (5.12).

These features are even more apparent in the characterization of a suitable class of pBV solutions in terms of a system of subdifferential inclusions that has the very same structure as the original viscous system (1.4) as provided by Theorem 5.20. This result shows that a triple $(\mathsf{t},\mathsf{u},\mathsf{z}) : [0,\mathsf{S}] \to [0,T]\times\mathbf{U}\times\mathbf{Z}$ is an *enhanced* pBV solution if and only if there exist measurable functions $\lambda_\mathsf{u}, \lambda_\mathsf{z} : [0,\mathsf{S}] \to [0,\infty]$ such that for almost all $s \in (0,\mathsf{S})$ we have

$$\begin{aligned} \partial\mathcal{V}_\mathsf{u}^{\lambda_\mathsf{u}(s)}(\mathsf{u}'(s)) + \mathrm{D}_u\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s)) \ni 0 \text{ in } \mathbf{U}^*, \\ \partial\mathcal{R}(\mathsf{z}'(s)) + \partial\mathcal{V}_\mathsf{z}^{\lambda_\mathsf{z}(s)}(\mathsf{z}'(s)) + \mathrm{D}_z\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s)) \ni 0 \text{ in } \mathbf{Z}^*, \end{aligned} \tag{1.13a}$$

$$\mathsf{t}'(s)\,\frac{\lambda_\mathsf{u}(s)}{1+\lambda_\mathsf{u}(s)} = 0 \quad\text{and}\quad \mathsf{t}'(s)\,\frac{\lambda_\mathsf{z}(s)}{1+\lambda_\mathsf{z}(s)} = 0, \tag{1.13b}$$

$$\begin{cases} \lambda_\mathsf{u}(s)\,\dfrac{1}{1+\lambda_\mathsf{z}(s)} = 0 & \text{for } \alpha > 1, \\ \lambda_\mathsf{u}(s) = \lambda_\mathsf{z}(s) & \text{for } \alpha = 1, \\ \dfrac{1}{1+\lambda_\mathsf{u}(s)}\,\lambda_\mathsf{z}(s) = 0 & \text{for } \alpha \in (0,1). \end{cases} \tag{1.13c}$$

In (1.13b) and (1.13c) we use the obvious conventions $\frac{\infty}{1+\infty} = 1$ and $\frac{1}{1+\infty} = 0$, respectively. Condition (1.13b) entails that the coefficients $\lambda_\mathsf{u}(s)$ and $\lambda_\mathsf{z}(s)$ of the 'viscous terms' in (1.13a) are allowed to be nonzero only when $\mathsf{t}'(s) = 0$, i.e. viscous behavior may manifest itself only at jumps happening now at a fixed time $t_* = \mathsf{t}(s)$ for $s \in [s_0,s_1]$. Conditions (1.13c) reveal that the onset of viscous effects in u and/or in z depends on whether u relaxes to equilibrium faster (case $\alpha > 1$), with the same speed (case $\alpha = 1$), or more slowly (case $\alpha < 1$), than z relaxes to local stability. In particular, the case $\lambda_\mathsf{x} = \infty$ leads to a blocking of the variable $\mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$, i.e. $\mathsf{x}'(s) = 0$ and $\partial\mathcal{V}_\mathsf{x}^\infty(0) = \mathbf{Z}_\mathrm{e}^*$. These aspects will be thoroughly explored in Sections 2 and 5.5.

Finally, in analogy with the case of the 'single-rate' vanishing-viscosity approach developed in [MRS12a, MRS16a], here as well we introduce "true *Balanced-Viscosity solutions*" (shortly referred to as *BV solutions*) as the *non-parametrized counterpart* to pBV solutions, see Definition 6.5. These solutions are functions of the *original time variable* $t \in [0,T]$ and fulfill an energy balance that again encompasses the contribution of the viscous dissipation potentials $\mathcal{V}_\mathsf{u}$ and $\mathcal{V}_\mathsf{z}$ to the description of energy dissipation at jump times of the solution. We are going to show that true BV solutions are related to pBV solutions in a canonical way, see Theorem 6.15. What is more, in Theorems 6.8 and 6.12 we provide general assumptions that guarantee that all pointwise-in-time limits of a family of (*non-parametrized*) viscous solutions $q_{\varepsilon_k} : [0,T] \to \mathbf{Q}$, for $\varepsilon_k \to 0^+$, is indeed a BV solution.

We emphasize that the definition of BV solutions is independent of the vanishing-viscosity approach. This independence guarantees that the solution concept is indeed stable under parameter variations in the way shown in [MRS13, Thm. 4.8] for generalized gradient systems (cf. also [MRS12b, Thm. 4.2]). Otherwise, doing the limit $\varepsilon \to 0^+$ first and then a parameter limit $\delta \to \delta_*$ it is not possible to show that the obtained limit curve is a vanishing-viscosity limit for fixed δ_*, see Remark 6.9. In principle, our general definition of (parametrized) BV solutions for limiting rate-independent systems can be used and analyzed independently of the vanishing-viscosity approach. However, to avoid overburdening the present work we do not following this line and restrict ourselves to situations where existence of solutions can be established exactly by these methods. After all, this is the mechanical motivation for considering such solution classes.

1.4. Application to a model for delamination.

In Section 8 we show that our existence results for pBV solutions, characterized by (1.13), and (true) BV solutions apply to a rate-independent process modeling delamination between two elastic bodies in adhesive contact along a prescribed interface. For a first approach to energetic solutions for this delamination problem, we refer to [KMR06]. A systematic approach to BV solutions for a multi-rate system involving elastoplasticity and damage is given in [CrR21].

The vanishing-viscosity analysis for the viscously regularized delamination model poses nontrivial challenges due to the presence of various maximal monotone nonlinearities, in the displacement equation and in the flow rule for the delamination variable z, which for instance render the constraints $z(t,x) \in [0,1]$ and the unidirectionality of the evolution. In particular, the main challenge is to obtain the a priori estimate (1.10) uniformly in ε when taking the vanishing-viscosity limit. For this, it is necessary to carefully regularize the viscous system. Because of the relatively weak coupling between the displacement equation and the flow rule for z, the smoothened system possesses a *semilinear structure* that allows us to apply the techniques developed in [Mie11, Sec. 4.4] and [MiZ14, Sec. 2], see Section 8.4.

1.5. Plan of the paper. In Section 2 we introduce a prototype of the coupled systems that we aim to mathematically model through the Balanced-Viscosity concept. In this simplified context, avoiding technicalities we illustrate the notion of (parametrized) BV solution and its mechanical interpretation.

Section 3 contains some auxiliary tools on that will be central for the rest of the paper. It revolves around the construction of *vanishing-viscosity contact potential* that will be relevant for describing the dissipative behavior of the viscously regularized system in the multi-rate case with 1, ε, and ε^α. In fact, it will enter into the definition of the function $\mathfrak{M}_0^\alpha$ in (1.11). Since in this paper we will extend the analysis of [MRS16b] to general viscous dissipation potentials, we will not be able to explicitly calculate the related vanishing-viscosity contact potential except for particular cases. Thus, a large part of Section 3 will focus on the derivation of general properties of contact potentials that will lay the ground for the study of the dissipation function $\mathfrak{M}_0^\alpha$.

In Section 4 we thoroughly establish the setup for our analysis, specifying the basic conditions on the spaces, on the energy functional, and on the dissipation potentials. Moreover, Theorem 4.8 recalls the existence result from [MRS13] for the viscous system (1.4). Section 4.6 is devoted to the derivation of a priori estimates for the solutions $(u_\varepsilon, z_\varepsilon)_\varepsilon$ to (1.4) that are uniform with respect to the parameter ε.

Section 5.1 entirely revolves around the functional $\mathfrak{M}_0^\alpha$ that has a central role in the definition of both pBV and true BV solutions. In particular, (i) we motivate its definition as the Mosco limit of the family of the time-integrated dissipation functional appearing in (1.8), and (ii) relying on the results from Section 3 we compute the limit $\mathfrak{M}_0^\alpha$ explicitly and investigate its properties. In the subsequent subsections we give the definition of parametrized Balanced-Viscosity solution to the rate-independent system $(\mathbf{U}{\times}\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$, state our existence results in Theorem 5.11 (and Theorem 5.14 for enhanced pBV solutions), and present the characterizations of pBV in terms of the subdifferential inclusions (1.13), cf. Theorem 5.20.

In Section 6 we introduce *true* BV solutions and state our existence result Theorem 6.8 (and Theorem 6.12 for enhanced BV solutions). In particular, we show that these solutions are obtained by taking the vanishing-viscosity limit in system (1.4) written in the *real time* variable $t \in [0, T]$. We also gain further insight into the description of the jump dynamics provided by true BV solutions.

The proofs of the main results of Sections 5 and 6 are carried out in Section 7.

Section 8 shows that our abstract setup is suitable to handle a concrete application to in solid mechanics. In particular, in Theorem 8.1 we prove the existence of enhanced parametrized and true BV solutions for a viscoelastic model with delamination along a prescribed interface.

1.6. General notations. Throughout the paper, for a given Banach space X, we will denote its norm by $\|\cdot\|_X$. For product spaces $X\times\cdots\times X$, we will often (up to exceptions) simply write $\|\cdot\|_X$ in place of $\|\cdot\|_{X\times\cdots\times X}$. By $\langle\cdot,\cdot\rangle_X$ we shall denote both the duality pairing between X and X^* and the scalar product in X, if X is a Hilbert space.

We shall use the symbols c, c', C, C', etc., whose meaning may vary even within the same line, to denote various positive constants depending only on known quantities. Furthermore, the symbols I_i, $i = 0, 1, ...$, will be used as place-holders for terms involved in the various estimates: we warn the reader that we will not be self-consistent with the numbering, so that, for instance, the symbol I_1 will occur several times with different meanings.

2. A prototypical class of coupled systems

In this section we illustrate the notion of parametrized BV solution for a prototypical and simple class of coupled systems to which the existence and characterization results obtained in the sequel will apply. In particular, it contains a model combining linearized viscoelasticity and viscoplasticity. We shall confine the discussion to the particular case in which the ambient spaces

$$\mathbf{U} \text{ and } \mathbf{Z} \text{ are Hilbert spaces,} \tag{2.1a}$$

the viscous dissipation potentials are *quadratic*, namely

$$\mathcal{V}_{\mathsf{u}} : \mathbf{U} \to [0,\infty);\ u' \mapsto \frac{1}{2}\langle \mathbb{V}_{\mathsf{u}} u', u'\rangle, \qquad \mathcal{V}_{\mathsf{z}} : \mathbf{Z} \to [0,\infty);\ z' \mapsto \frac{1}{2}\langle \mathbb{V}_{\mathsf{z}} z', z'\rangle, \tag{2.1b}$$

with bounded linear and symmetric operators $\mathbb{V}_{\mathsf{x}} : \mathbf{Z}_{\mathrm{e}} \to \mathbf{Z}_{\mathrm{e}}^*$, and the driving energy functional is of the form

$$\mathcal{E}(t,u,z) := \frac{1}{2}\langle \mathbb{A}u, u\rangle_{\mathbf{U}} + \langle \mathbb{B}u, z\rangle_{\mathbf{Z}} + \frac{1}{2}\langle \mathbb{G}u, u\rangle_{\mathbf{U}} - \langle f(t), u\rangle_{\mathbf{U}} - \langle g(t), z\rangle_{\mathbf{Z}}, \tag{2.1c}$$

where $\mathbb{A} : \mathbf{U} \to \mathbf{U}^*$ and $\mathbb{G} : \mathbf{Z} \to \mathbf{Z}^*$ are linear, bounded and self-adjoint, $\mathbb{B} : \mathbf{U} \to \mathbf{Z}^*$ is linear and bounded, and $(f, g) : [0, T] \to \mathbf{U}^* {\times} \mathbf{Z}^*$ are smooth time-dependent applied forces. Moreover, we assume that the block operator $\begin{pmatrix} \mathbb{A} & \mathbb{B}^* \\ \mathbb{B} & \mathbb{G} \end{pmatrix}$ is positive semidefinite. Together with the 1-homogeneous potential $\mathcal{R} : \mathbf{Z} \to [0, \infty)$ the viscous system (1.4) reads

$$\varepsilon^\alpha \mathbb{V}_{\mathsf{u}} u' + \mathbb{A}u + \mathbb{B}^* z = f(t) \quad \text{in } \mathbf{U} \quad \text{for a.a. } t \in (0, T), \tag{2.2a}$$

$$\partial\mathcal{R}(z') + \varepsilon \mathbb{V}_{\mathsf{z}} z' + \mathbb{B}u + \mathbb{G}z = g(t) \quad \text{in } \mathbf{Z} \quad \text{for a.a. } t \in (0, T), \tag{2.2b}$$

with $\mathbb{V}_{\mathsf{x}}$ from (2.1b). It will be important to allow for coercivity of $\mathcal{R}$ on a Banach space $\mathbf{Z}_{\mathrm{ri}}$ such that $\mathbf{Z} \subset \mathbf{Z}_{\mathrm{ri}}$ continuously and $\mathcal{R}(z') \geq c\|z'\|_{\mathbf{Z}_{\mathrm{ri}}}$ for all $z' \in \mathbf{Z}_{\mathrm{ri}}$.

Example 2.1 (Linearized elastoplasticity with hardening)**.** *Let the elastoplastic body occupy a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$: linearized elastoplasticity is described in terms of the displacement $u : \Omega \to \mathbb{R}^d$ with $u(t) \in \mathbf{U} = \mathrm{H}^1_0(\Omega)$ for simplicity and in terms of the symmetric, trace-free plastic strain tensor $z : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{dev}} := \{ z \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \,|\, \mathrm{tr}(z) = 0 \}$. The driving energy functional $\mathcal{E} : [0, T]{\times}\mathbf{U}{\times}\mathbf{Z} \to \mathbb{R}$ with $\mathbf{Z} = \mathrm{L}^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{dev}})$ is defined by*

$$\mathcal{E}(t, u, z) := \int_\Omega \big(\tfrac{1}{2}(e(u){-}z){:}\mathbb{C}(e(u){-}z) {+} \tfrac{1}{2} z{:}\mathbb{H}z \big) \,\mathrm{d}x - \langle f(t), u\rangle_{H^1_0(\Omega;\mathbb{R}^d)}$$

with $e(u)$ the linearized symmetric strain tensor, $\mathbb{C} \in \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\mathbb{H} \in \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{devm}})$ are the positive definite and symmetric elasticity and hardening tensors, respectively, and $f : [0, T] \to \mathrm{H}^{-1}(\Omega; \mathbb{R}^d)$ a time-dependent volume loading. The dissipation potentials are

$$\mathcal{R}(z') = \int_\Omega \sigma_{\mathrm{yield}} |z'| \,\mathrm{d}x, \quad \mathcal{V}_{\mathsf{u}}(u') := \int_\Omega \tfrac{1}{2} e(u') : \mathbb{D}e(u') \,\mathrm{d}x, \quad \mathcal{V}_{\mathsf{z}}(z') := \int_\Omega \tfrac{1}{2} z' : \mathbb{V}z' \,\mathrm{d}x$$

where $\sigma_{\mathrm{yield}} > 0$ is the yield stress and $\mathbb{D} \in \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\mathbb{V} \in \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{devm}})$ are the symmetric and positive definite viscoelasticity and viscoplasticity tensors, respectively.

Hence, system (2.2) translates into

$$\begin{aligned} -\mathrm{div}\big(\varepsilon^\alpha \mathbb{D}e(u') + \mathbb{C}(e(u){-}z)\big) &= f(t) \quad \text{in } \Omega{\times}(0, T), \\ \sigma_{\mathrm{yield}} \,\mathrm{Sign}(z') + \varepsilon\mathbb{V}z' + \mathrm{dev}\big(\mathbb{C}(z{-}e(u))\big) + \mathbb{H}z &\ni 0 \quad \text{in } \Omega{\times}(0, T). \end{aligned}$$

*where "*dev*" projects to the deviatoric part, namely* $\mathrm{dev}\, A = A - \frac{1}{d}(\mathrm{tr}\, A)\, I$.

For the system $(\mathbf{U}{\times}\mathbf{Z}, \mathcal{E}, \mathcal{V}^{\varepsilon^\alpha}_{\mathsf{u}} {+} \mathcal{R} {+} \mathcal{V}^{\varepsilon}_{\mathsf{z}})_{\varepsilon\downarrow 0}$ from (2.1), featuring 2-positively homogeneous dissipation potentials, the time-space dissipation function $\mathfrak{M}^\alpha_0$ that enters into the definition of (parametrized) Balanced Viscosity solution can be explicitly computed (cf. Example 5.3 ahead). Nonetheless, here we can give an even more transparent illustration of (parametrized) pBV solutions in terms of their differential characterization (1.13). The upcoming Theorem 5.20 states that a triple $(\mathsf{t}, \mathsf{u}, \mathsf{z})$ is an (*enhanced*) parametrized BV solution if and only if it solves, for almost all $s \in (0, \mathsf{S})$,

$$\begin{aligned} \lambda_{\mathsf{u}}(s)\mathbb{V}_{\mathsf{u}}\mathsf{u}'(s) + \mathbb{A}\mathsf{u}(s) + \mathbb{B}^*\mathsf{z}(s) &\ni f(\mathsf{t}(s)) \quad \text{in } \mathbf{U}^*, \\ \partial\mathcal{R}(\mathsf{z}'(s)) + \lambda_{\mathsf{z}}(s)\mathbb{V}_{\mathsf{z}}\mathsf{z}'(s) + \mathbb{B}\,\mathsf{u}(s) + \mathbb{G}\,\mathsf{z}(s) &\ni g(\mathsf{t}(s)) \quad \text{in } \mathbf{Z}^*, \end{aligned} \tag{2.3}$$

joint with the 'switching conditions' (1.13b)–(1.13c) on the measurable functions λ_{u}, $\lambda_{\mathsf{z}} : (0, \mathsf{S}) \to [0, \infty]$. Here "$\infty\mathbb{V}_{\mathsf{z}}\mathsf{z}'$" has to be interpreted in the sense of $\partial\mathcal{V}^\infty_{\mathsf{z}}(\mathsf{z}')$, see (1.5b).

We recall that (1.13b) simply ensures that, if the system is not jumping (i.e., $\mathsf{t}'(s) > 0$), then viscosity does not come into action, i.e. $\lambda_{\mathsf{u}}(s) = \lambda_{\mathsf{z}}(s) = 0$. This means that $\mathsf{u}(s)$ is in 'E'quilibrium with respect to $\mathsf{z}(s)$ and the loading $f(\mathsf{t}(s))$, whereas z evolves according to the truly 'R'ate-independent evolution $\partial\mathcal{R}(\mathsf{z}') + \mathbb{B}\mathsf{u} + \mathbb{G}\mathsf{z} \ni g$, hence we will denote this evolution regime by $\mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}}$ in Section 5.5.

Conditions (1.13c) differ in the three cases $\alpha = 1$, $\alpha > 1$ and $\alpha \in (0, 1)$ and indeed show how the (possibly different) relaxation rates of the variables u and z influence the system behavior at jumps, see Section 5.5 for a full discussion of the occurring evolution regimes.

For $\underline{\alpha = 1}$ the variables u and z relax with the same rate: at a jump, the system *may* switch to a viscous regime where the viscosity in u and in z are involved *equally*, since the coefficients λ_{u} and λ_{z} modulating the 'V'iscosity terms $\mathbb{V}_{\mathsf{u}}\mathsf{u}'$ and $\mathbb{V}_{\mathsf{z}}\mathsf{z}'$ coincide. This evolution regime will be denoted V_{uz}.

For $\underline{\alpha > 1}$ the switching condition (1.13c) imposes that either $\lambda_{\mathsf{z}} = \infty$ (i.e. $\mathsf{z}' = 0$) or that $\lambda_{\mathsf{u}} = 0$ (so that u is at equilibrium). Indeed, since u relaxes 'V'iscously faster to equilibrium than z to rate-independent evolution, z is 'B'locked until u has reached the equilibrium: and we call this evolution regime $\mathrm{V}_{\mathsf{u}}\mathrm{B}_{\mathsf{z}}$. After that u is in 'E'quilibrium and z may have a 'V'iscous transition with $\lambda_{\mathsf{z}} > 0$, a regime denoted by $\mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}$. Moreover, under suitable conditions on the operators $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{G}$ which in particular ensure that the functional $\mathcal{E}(t, \cdot, z)$ from (2.1c) is uniformly convex, the arguments from

[MRS16b, Prop. 5.5] may be repeated for the system $(\mathbf{U}{\times}\mathbf{Z}, \mathcal{E}, \mathcal{V}_\mathsf{u}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_\mathsf{z}^\varepsilon)_{\varepsilon\downarrow 0}$ defined via (2.1). Hence, it is possible to show that the regime $\mathrm{V_u B_z}$ can only occur once in the initial phase, while u never leaves equilibrium afterwards, i.e. only $\mathrm{E_u R_z}$ and $\mathrm{E_u V_z}$ are possible.

For $\alpha \in (0,1)$ the variable z relaxes faster than u, which leads to the two viscous regimes: (i) $\mathrm{B_u V_z}$ where u is blocked ($\lambda_\mathsf{u} = \infty$) while z evolves viscously, and (ii) $\mathrm{V_u R_z}$ where u relaxes to equilibrium while z stays in locally stable states ($\lambda_\mathsf{z} = 0$). For $\alpha \in (0,1)$ these two regimes and $\mathrm{E_u R_z}$ may occur more than once in the evolution of the system.

3. Some auxiliary tools for dissipation potentials

In this section we prepare a series of useful tools for handling the balanced effect of the different dissipation potentials. They will be essential for the upcoming analysis and may be interesting elsewhere.

Definition 3.1 (Primal and dual dissipation potentials)**.** *Let X be a reflexive Banach space. Then, a function $\psi : X \to [0,\infty]$ is called a* (primal) dissipation potential*, if*

$$\psi \text{ is convex, lower semicontinuous (lsc, for short) and } \psi(0) = 0.$$

The dual dissipation potential *$\psi^* : X^* \to [0,\infty]$ is defined via Legendre-Fenchel conjugation as*

$$\psi^*(\xi) := \sup\left\{ \langle \xi, v\rangle - \psi(v) \,\middle|\, v \in X \right\}.$$

Note that ψ^* is indeed again a dissipation potential, and we have $(\psi^*)^* = \psi$. In this section, we allow for functionals ψ taking the value ∞ as well as degenerate functionals such that $\psi(v) = 0$ for $v \neq 0$. With ψ we associate the *B-function*

$$\mathcal{B}_\psi : (0,\infty){\times}X{\times}[0,\infty) \to [0,\infty], \qquad \mathcal{B}_\psi(\tau, v, \sigma) := \tau\psi\left(\frac{v}{\tau}\right) + \tau\sigma\,. \tag{3.1}$$

We highlight the rescaling properties of $\mathcal{B}_\psi$ as follows

$$\mathcal{B}_\psi(\tau, v, \sigma) = \tau\,\mathcal{B}_\psi(1, \frac{1}{\tau}v, \sigma) = \frac{1}{\delta}\mathcal{B}_\psi(\delta\tau, \delta v, \sigma) \quad \text{for all } \delta > 0. \tag{3.2}$$

We will use that the functional $\mathcal{B}_\psi(\cdot,\cdot,\sigma)$ is convex for all $\sigma \geq 0$. To see this, we consider $\tau_0,\ \tau_1 \in (0,\infty)$, $v_0, v_1 \in X$, and $\theta \in [0,1]$ and set $\tau_\theta := (1{-}\theta)\tau_0 + \theta\tau_1 > 0$ and $v_\theta := (1{-}\theta)v_0 + \theta v_1$. With this we find

$$\begin{aligned}
\mathcal{B}_\psi(\tau_\theta, v_\theta, \sigma) &= \tau_\theta\,\psi\left(\frac{v_\theta}{\tau_\theta}\right) + \tau_\theta\sigma = \tau_\theta\,\psi\Big(\frac{(1{-}\theta)\tau_0}{\tau_\theta}\,\frac{1}{\tau_0}v_0 + \frac{\theta\tau_1}{\tau_\theta}\,\frac{1}{\tau_1}v_1\Big) + \tau_\theta\sigma \\
&\overset{(1)}{\leq} \tau_\theta\Big(\frac{(1{-}\theta)\tau_0}{\tau_\theta}\psi\left(\frac{v_0}{\tau_0}\right) + \frac{\theta\tau_1}{\tau_\theta}\psi\left(\frac{v_1}{\tau_1}\right)\Big) + (1{-}\theta)\tau_0\sigma + \theta\tau_1\sigma \\
&= (1{-}\theta)\mathcal{B}_\psi(\tau_0, v_0, \sigma) + \theta\mathcal{B}_\psi(\tau_1, v_1, \sigma),
\end{aligned} \tag{3.3}$$

where in $\overset{(1)}{\leq}$ we used the convexity of ψ. We next define the functional

$$\mathfrak{b}_\psi : X{\times}[0,\infty) \to [0,\infty]; \qquad \mathfrak{b}_\psi(v,\sigma) := \inf\left\{ \mathcal{B}_\psi(\tau, v, \sigma) \,\middle|\, \tau > 0 \right\}. \tag{3.4}$$

We shall refer to the functional $\mathfrak{b}_\psi$ as *vanishing-viscosity contact potential* associated with ψ, in accordance with the terminology used in [MRS12a]. As we will see, $\mathfrak{b}_\psi$ will be handy for describing the interplay of vanishing viscosity and time rescaling upon taking the limit of (1.4).

3.1. Properties of vanishing-viscosity contact potentials $\mathfrak{b}_\psi$.

For arbitrary dissipation potentials ψ, we define the *rate-independent part* $\psi_{\mathrm{ri}} : X \to [0,\infty]$ via

$$\psi_{\mathrm{ri}}(v) = \lim_{\gamma\to 0^+} \frac{1}{\gamma}\psi(\gamma v) = \sup\left\{ \langle \eta, v\rangle_X \,\middle|\, \eta \in \partial\psi(0) \right\}. \tag{3.5}$$

The following results are slight variants of the results in [MRS12a, Thm. 3.7].

Proposition 3.2 (Properties of vanishing-viscosity contact potentials)**.** *Assume that the dissipation potential $\psi : X \to [0,\infty]$ is superlinear, i.e.*

$$\lim_{\|v\|_X\to\infty} \frac{\psi(v)}{\|v\|_X} = \infty\,. \tag{3.6}$$

Then, $\mathfrak{b}_\psi$ has the following properties:

(b1) *$\mathfrak{b}_\psi(v,\sigma) = 0$ implies $\sigma = 0$ or $v = 0$. Moreover, $\mathfrak{b}_\psi(0,\sigma) = 0$ for all $\sigma \geq 0$.*

(b2) *For all $v \in X$ the function $\mathfrak{b}_\psi(v,\cdot) : [0,\infty) \to [0,\infty]$ is nondecreasing and concave. For $v \neq 0$ and $\sigma > 0$ the infimum in the definition of $\mathfrak{b}_\psi$ is attained at a value $\tau_{v,\sigma} \in (0,\infty)$. Moreover, for all $v \neq 0$ and $\sigma > 0$ we have $\mathfrak{b}_\psi(v,\sigma) > \mathfrak{b}_\psi(v,0) = \psi_{\mathrm{ri}}(v)$.*

(b3) *For all* $\sigma \geq 0$ *the function* $\mathfrak{b}_\psi(\cdot,\sigma) : X \to [0,\infty)$ *is positively 1-homogeneous, lsc, and convex.*
(b4) *If* $\psi = \phi + \varphi$ *where* ϕ *is 1-homogeneous, then* $\mathfrak{b}_{\phi+\varphi}(v,\sigma) = \phi(v) + \mathfrak{b}_\varphi(v,\sigma)$.
(b5) *For all* $(v,\eta) \in X{\times}X^*$ *we have* $\mathfrak{b}_\psi(v,\psi^*(\eta)) \geq \langle \eta, v\rangle_X$.

Proof. The main observation is that the function $g_{v,\sigma} : (0,\infty) \ni \tau \mapsto \tau\psi(\frac{1}{\tau}v) + \tau\sigma$ is convex (cf. (3.3)) and takes only nonnegative values. For $\sigma > 0$ we have $g_{v,\sigma}(\tau) \to \infty$ for $\tau \to \infty$, and for $v \neq 0$ we have $g_{v,\sigma}(\tau) \to \infty$ for $\tau \to 0^+$ due to superlinearity of ψ.

Part (b1): If $\mathfrak{b}_\psi(v,\sigma) = \inf g_{v,\sigma}(\cdot) = 0$, the infimum must either be realized for $\tau \to 0^+$ or for $\tau \to \infty$. In the first case, the value of σ does not matter, but the superlinearity of ψ gives $\tau\psi(\frac{1}{\tau}v) \to \infty$, unless $v = 0$. In the second case we have $\tau\sigma \to \infty$, unless $\sigma = 0$. The relation $\mathfrak{b}_\psi(0,\sigma) = 0$ is obvious.

Part (b2): The first two statements follow because $\mathfrak{b}_\psi(v,\cdot)$ is the infimum of a family of functions that are increasing and concave in σ. For $v \neq 0$ and $\sigma > 0$ the minimum of $g_{v,\sigma}(\tau)$ is achieved at a $\tau_{v,\sigma} \in (0,\infty)$ as $g_{v,\sigma}(\tau) \to \infty$ on both sides (i.e., as $\tau \to 0^+$ and $\tau \to \infty$). Thus, $\mathfrak{b}_\psi(v,\sigma) \geq \mathfrak{b}_\psi(v,0) + \sigma\tau_{v,\sigma} > \mathfrak{b}_\psi(v,0)$ as desired. The relation $\mathfrak{b}_\psi(v,0) = \psi_{\mathrm{ri}}(v)$ follows easily from the convexity of ψ.

Part (b3): The positive 1-homogeneity $\mathfrak{b}_\psi(\gamma v,\sigma) = \gamma\mathfrak{b}_\psi(v,\sigma)$ for all $\gamma > 0$ follows by replacing τ by $\tau\gamma$. Convexity is obtained as follows. For fixed $v_0, v_1 \in X$, $\theta \in (0,1)$, and $\sigma \geq 0$, we choose $\varepsilon > 0$ and find $\tau_0, \tau_1 > 0$ such that for $j \in \{0,1\}$ we have

$$\tau_j\, \psi\big(\frac{1}{\tau_j} v_j, \sigma\big) + \tau_j\sigma \leq \mathfrak{b}_\psi(v_j,\sigma) + \varepsilon. \tag{3.7}$$

Here we assumed without loss of generality $\mathfrak{b}_\psi(v_j,\sigma) < \infty$ since otherwise there is nothing to be shown. Now we set $v_\theta = (1{-}\theta)v_0 + \theta v_1$ and $\tau_\theta = (1{-}\theta)\tau_0 + \theta\tau_1 > 0$. Using the convexity (3.3) of the functional $\mathcal{B}_\psi(\cdot,\cdot,\sigma)$, we obtain

$$\begin{aligned}\mathfrak{b}_\psi(v_\theta,\sigma) \leq \mathcal{B}_\psi(\tau_\theta, v_\theta,\sigma) &\leq (1{-}\theta)\mathcal{B}_\psi(\tau_0,v_0,\sigma) + \theta\mathcal{B}_\psi(\tau_1,v_1,\sigma)\\ &\leq (1{-}\theta)\mathfrak{b}_\psi(v_0,\sigma) + \theta\mathfrak{b}_\psi(v_1) + \varepsilon,\end{aligned}$$

with the last inequality due to (3.7). Since $\varepsilon > 0$ was arbitrary, this is the desired result.

To prove lower semicontinuity, we use the special way $\mathfrak{b}_\psi$ is constructed and that ψ is lsc. For all sequences $v_j \to v_*$ and $\sigma_j \to \sigma_*$ we have to show

$$\mathfrak{b}_\psi(v_*,\sigma_*) \leq \alpha := \liminf_{j\to\infty} \mathfrak{b}_\psi(v_j,\sigma_j)$$

We may assume $\alpha < \infty$ and $\mathfrak{b}_\psi(v_*,\sigma_*) > 0$, since otherwise the desired estimate is trivial.

The case $\sigma_* = 0$ is simple, as we have

$$\alpha = \liminf_{j\to\infty} \mathfrak{b}_\psi(v_j,\sigma_j) \geq \liminf_{j\to\infty} \mathfrak{b}_\psi(v_j,0) \geq \liminf_{j\to\infty} \psi_{\mathrm{ri}}(v_j) \geq \psi_{\mathrm{ri}}(v_*) = \mathfrak{b}_\psi(v_*,0) = \mathfrak{b}_\psi(v_*,\sigma_*).$$

It remains to consider the case $v_* \neq 0$ and $\sigma_* > 0$. Since $\|v_j\| \geq \|v_*\|/2 > 0$ and $\sigma_j \geq \sigma_*/2 > 0$ for sufficiently large j, we see that the optimal $\tau_j = \tau_{v_j,\sigma_j}$ lie in a set $[1/M, M] \Subset (0,\infty)$. Thus, choosing a subsequence (not relabeled), we may assume $\tau_j \to \tau_\circ$ and obtain lower semicontinuity by using $\frac{1}{\tau_j}v_j \to \frac{1}{\tau_\circ}v_*$ as follows:

$$\alpha = \liminf_{j\to\infty} \mathfrak{b}_\psi(v_j,\sigma_j) = \liminf_{j\to\infty} \left(\tau_j\psi\big(\frac{1}{\tau_j}v_j\big) + \tau_j\sigma_j\right) \geq \tau_\circ\psi\big(\frac{1}{\tau_\circ}v_*\big) + \tau_\circ\sigma_* \geq \mathfrak{b}_\psi(v_*,\sigma_*).$$

Part (b4): The formula for $\mathfrak{b}_{\phi+\varphi}$ follows from a direct calculation.

Part (b5): We have $g_\tau(v,\psi^*(\eta)) = \tau\Big(\psi\big(\frac{1}{\tau}v\big) + \psi^*(\eta)\Big) \geq \tau\big(\langle\eta, \frac{1}{\tau}v\rangle\big) = \langle\eta, v\rangle$, and taking the infimum over $\tau > 0$ gives the result. Thus, Proposition 3.2 is proved. □

There is a canonical case in which $\mathfrak{b}_\psi$ can be given explicitly, namely the case that $\mathfrak{b}_\psi(v)$ only depends on the Banach-space norm $\|v\|$. In that case we have an explicit expression for $\mathfrak{b}_\psi$ and the functional $X \times X^* \ni (v,\eta) \mapsto \mathfrak{b}_\psi(v,\psi^*(\eta))$.

Lemma 3.3 (Dissipation potentials depending on the norm)**.** *Assume that* ψ *is given in the form* $\psi(v) = \zeta(\|v\|)$, *where* $\zeta : [0,\infty) \to [0,\infty]$ *satisfies* $\zeta(0) = 0$ *and is lsc, nondecreasing, convex, and superlinear. Setting* $\zeta'(0) = \lim_{h\to 0^+} \frac{1}{h}\zeta(h)$ *we have the identities*

$$\begin{aligned}\mathfrak{b}_\psi(v,\sigma) &= \|v\|\kappa_\zeta(\sigma) \text{ with } \kappa_\zeta(\sigma) := \inf\big\{\, \tau\zeta(1/\tau) + \tau\sigma \,\big|\, \tau > 0 \,\big\},\\ \mathfrak{b}_\psi(v,\psi^*(\xi)) &= \|v\| \max\big\{\zeta'(0)\,,\, \|\xi\|_*\big\}.\end{aligned} \tag{3.8}$$

Proof. The first statement is trivial for $v = 0$. For $v \neq 0$ we can replace τ by $\tau\|v\|$ and obtain the desired product form with $\|v\|$ as the first factor.

To obtain the second statement in (3.8) we first observe that $\psi^*(\xi) = \zeta^*(\|\xi\|_*)$ with $\zeta^*(r) = \sup\big\{\, r\rho - \zeta(\rho) \,\big|\, \rho \geq 0 \,\big\}$. As ζ is superlinear $\zeta^*(r)$ is finite for all $r \geq 0$, and $\zeta^*(r) = 0$ for $r \in [0, \zeta'(0)]$. Secondly, we characterize κ_ζ by using the following estimate

$$\kappa_\zeta(\zeta^*(r)) = \inf\big\{\, \tau\big(\zeta(\tfrac{1}{\tau}) + \zeta^*(r)\big) \,\big|\, \tau > 0 \,\big\} \geq \inf\big\{\, \tau\big(\tfrac{1}{\tau} r\big) \,\big|\, \tau > 0 \,\big\} = r.$$

The inequality is even an identity if the infimum is attained, which is the case of $\frac{1}{\tau} \in \partial\zeta^*(r)$ for some τ. Thus, we have attainment whenever $\zeta^*(r) > 0$, whereas for $r \in [0, \zeta'(0)]$, where $\zeta^*(r) = 0$, we have non-attainment but find $\kappa_\zeta(0) = \zeta'(0)$. Together we arrive at $\kappa_\zeta(\zeta^*(r)) = \max\{\zeta'(0), r\}$ (see also [LMS18, Sec. 2.3]), and $\mathfrak{b}(v, \psi^*(\xi)) = \|v\|\kappa_\zeta(\zeta^*(\|\xi\|_*))$ gives the desired result. □

The above result shows that the estimate $\mathfrak{b}_\psi(v, \psi^*(\xi)) \geq \langle \xi, v\rangle$ in (b4) improves to

$$\mathfrak{b}_\psi(v, \psi^*(\xi)) \geq \|\xi\|_*\|v\| \tag{3.9}$$

in certain cases, in particular in the metric approach used in [RMS08, MRS09]. As some of the following examples show, the latter estimate is not true in general, and that is why we will derive general lower bounds on the vanishing-viscosity contact potential $\mathfrak{b}_\psi$ in Section 3.3.

Example 3.4 (The function $\mathfrak{b}_\psi$ for some special cases)**.** *The following cases give some intuition about the vanishing-viscosity contact potential $\mathfrak{b}_\psi$.*

(A) Assume that ψ is positively p-homogeneous with $p \in (1, \infty)$, i.e. $\psi(\gamma v) = \gamma^p\psi(v)$ for all $\gamma > 0$ and $v \in X$. Then, we have

$$\mathfrak{b}_\psi(v, \sigma) = \big(\psi(v)\big)^{1/p}\, \hat{c}_p\, \sigma^{1/p'}, \quad \text{where } \hat{c}_p = p^{1/p}(p')^{1/p'} \text{ and } \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.10}$$

In particular, for $\psi(v) = \frac{1}{p}\|v\|^p$ we find

$$\mathfrak{b}_\psi(v, \sigma) = \|v\|\big(p'\sigma\big)^{1/p'} \quad \text{and} \quad \mathfrak{b}_\psi(v, \psi^*(\eta)) = \|v\|\|\eta\|_*\,.$$

(B) On $X = \mathbb{R}^2$ consider $\psi(v) = \frac{1}{2}(av_1^2 + bv_2^2)$ with $a, b > 0$. Then,

$$\mathfrak{b}_\psi(v, \sigma) = \big(av_1^2 + bv_2^2\big)^{1/2}\, (2\sigma)^{1/2} \quad \text{and} \quad \mathfrak{b}_\psi(v, \psi^*(\xi)) = \big(av_1^2 + bv_2^2\big)^{1/2}\big(\frac{1}{a}\xi_1^2 + \frac{1}{b}\xi_2^2\big)^{1/2}.$$

If $\mathbb{R}^2$ is equipped with the Euclidean norm $\|\cdot\|$, then $\mathfrak{b}_\psi(v, \psi^*(\xi)) \geq \big(\frac{\min\{a,b\}}{\max\{a,b\}}\big)^{1/2}\|\xi\|_*\|v\|$, but estimate (3.9) fails, while $\mathfrak{b}_\psi(v, \psi^*(\xi)) \geq \langle \xi, v\rangle$ obviously holds.

(C) Again for $X = \mathbb{R}^2$ consider $\psi(v) = \frac{1}{2}v_1^2 + \phi(v_2)$ with

$$\phi(s) = \begin{cases} \frac{1}{2}s^2 & \text{for } |s| \leq 1, \\ \frac{1}{4}(|s|+1)^2 - \frac{1}{2} & \text{for } |s| \geq 1, \end{cases} \quad \text{and} \quad \phi^*(r) = \begin{cases} \frac{1}{2}r^2 & \text{for } |r| \leq 1, \\ r^2 - |r| + \frac{1}{2} & \text{for } |r| \geq 1. \end{cases}$$

An explicit calculation leads to the expression

$$\mathfrak{b}_\psi(v, \sigma) = \begin{cases} \|v\|\sqrt{2\sigma} & \text{for } v_1^2 \geq (2\sigma - 1)v_2^2, \\ \frac{1}{2}\sqrt{2v_1^2 + v_2^2}\,\sqrt{4\sigma - 1} + \frac{|v_2|}{2} & \text{for } \sigma \geq 1/2 \text{ and } v_1^2 \leq (2\sigma - 1)v_2^2. \end{cases}$$

Using $\psi^*(\xi) = \frac{1}{2}\xi_1^2 + \phi^*(\xi_2)$ we find $\mathfrak{b}_\psi(v, \psi^*(\xi)) = \|\xi\|_*\|v\|$ whenever $\|\xi\|_* \leq 1$. However, the explicit form of $\mathfrak{b}_\psi$ shows that, in general, $\mathfrak{b}_\psi(v, \psi^*(\xi))$ cannot be expressed in terms of $\|v\|$ and $\|\xi\|_*$ alone. With $\psi^*(\xi) \geq \frac{1}{2}\|\xi\|_*^2$ and $\mathfrak{b}_\psi(v, \sigma) \geq \frac{1}{2}\big(\sqrt{4\sigma - 1} + 1\big)\|v\|$ for $\sigma \geq 1/2$ we obtain $\mathfrak{b}_\psi(v, \psi^*(\xi)) \geq \|v\|\frac{1}{2}\big(\sqrt{2\|\xi\|_*^2 - 1} + 1\big)$ for $\|\xi\|_* \geq 1$.

(D) We still look at the case $X = \mathbb{R}^2$ with the Euclidean norm $\|v\| = \big(v_1^2 + v_2^2\big)^{1/2}$ and

$$\psi(v) = \frac{1}{2}\,v_1^2 + \frac{1}{4}\,v_2^4 \quad \text{and} \quad \psi^*(\xi) = \frac{1}{2}\,\xi_1^2 + \frac{4}{3}\,|\xi_2|^{4/3}.$$

In principle, we can calculate $\mathfrak{b}_\psi(v_1, v_2, \sigma)$ explicitly, however, it suffices to use (A) giving

$$\mathfrak{b}_\psi(v_1, 0, \sigma) = |v_1|(2\sigma)^{1/2} \quad \text{and} \quad \mathfrak{b}_\psi(0, v_2, \sigma) = |v_2|\big(\tfrac{4}{3}\sigma\big)^{3/4}.$$

Inserting $\sigma = \psi^*(\xi_1, \xi_2)$ and inserting the "wrong directions" with $\langle \xi, v\rangle = 0$ we find

$$\mathfrak{b}_\psi(v_1, 0, \psi^*(\xi_1, \xi_2)) = \big(\tfrac{8}{3}\big)^{1/2}|v_1|\,|\xi_2|^{2/3} \quad \text{and} \quad \mathfrak{b}_\psi(0, v_2, \psi^*(\xi_1, \xi_2)) = \big(\tfrac{2}{3}\big)^{3/4}|v_2|\,|\xi_1|^{3/2}.$$

Clearly, there cannot be a constant $c_0 > 0$ *such that* $\mathfrak{b}_\psi(v, \psi^*(\xi)) \geq c_0\|v\|\,\|\xi\|$ *for all* $v, \xi \in \mathbb{R}^2$. *Of course, the relations are compatible with (b4) in Proposition 3.2, i.e.* $\mathfrak{b}_\psi(v, \psi^*(\xi)) \geq \langle \xi, v\rangle$.

As we will see, the vanishing-viscosity contact potentials $\mathfrak{b}_\psi$, which were developed for the case of two-rate problems (with time scales 1 and ε) in [MRS12a], are also relevant to describe the limiting behavior of B-functions in the multi-rate case with time scales 1, ε, and ε^α. For this, we will use the concept of (sequential) Mosco convergence, which we recall here for a sequence of functionals $\mathscr{F}_n : X \to (-\infty, +\infty]$ defined in a Banach space X.

Definition 3.5 (Mosco convergence)**.** *We say that* $\mathscr{F} : X \to (-\infty, +\infty]$ *is the* Mosco limit *of the functionals* $(\mathscr{F}_n)_n$ *as* $n \to \infty$, *and write* $\mathscr{F}_n \overset{\mathrm{M}}{\to} \mathscr{F}$ *in* X, *if the following two conditions hold:*

$$\Gamma\text{-}\liminf \text{ estimate}: \qquad x_n \rightharpoonup x \textit{ weakly in } X \implies \mathscr{F}(x) \leq \liminf_{n\to\infty} \mathscr{F}_n(x_n); \tag{3.11a}$$

$$\Gamma\text{-}\liminf \text{ estimate}: \quad \forall\, x \in X\ \exists\, (x_n)_n \subset X: \quad x_n \to x \textit{ strongly in } X \textit{ and } \mathscr{F}(x) \geq \limsup_{n\to\infty} \mathscr{F}_n(x_n). \tag{3.11b}$$

3.2. Mosco convergence for the joint B-functions $\mathfrak{B}_\varepsilon^\alpha$.

In view of the vanishing-viscosity analysis of (1.4), we now work with two dissipation potentials $\psi_{\mathsf{u}} : \mathbf{U} \to [0, \infty]$ and $\psi_{\mathsf{z}} : \mathbf{Z} \to [0, \infty]$, with $\mathbf{U}$ and $\mathbf{Z}$ the state spaces from (1.1). In Section 5.1, we will indeed confine the discussion to the choices $\psi_{\mathsf{u}} := \mathcal{V}_{\mathsf{u}}$ and $\psi_{\mathsf{z}} := \mathcal{R} + \mathcal{V}_{\mathsf{z}}$, but here we want to keep the discussion more general and in particular allow for ψ_{u} to have a nontrivial rate-independent part, too.

When constructing the associated B-function we have to take care of the different scalings namely $\psi_{\mathsf{u}}^{\varepsilon^\alpha}$ and $\psi_{\mathsf{z}}^{\varepsilon}$ in the sense of (1.5a), i.e. $\psi^\lambda(v) = \frac{1}{\lambda}\psi(\lambda v)$. Indeed, since the conjugate function $(\psi^\lambda)^*$ satisfies the simple scaling law $(\psi^\lambda)^*(\xi) = \frac{1}{\lambda}\psi^*(\xi)$, the B-function $\mathcal{B}_{\psi^\lambda}$ obeys the scaling relations

$$\mathcal{B}_{\psi^\lambda}(\tau, v, \frac{1}{\lambda}\sigma) = \mathcal{B}_\psi(\frac{\tau}{\lambda}, v, \sigma) = \frac{1}{\lambda}\,\mathcal{B}_\psi(\tau, \lambda v, \sigma), \tag{3.12}$$

where we used (3.2) for the last step. Our definition of the associated B-function for the sum

$$\Psi_{\varepsilon,\alpha} : \mathbf{U}\times\mathbf{Z} \to [0, \infty]; \quad \Psi_{\varepsilon,\alpha}(u', z') := \frac{1}{\varepsilon^\alpha}\psi_{\mathsf{u}}(\varepsilon^\alpha u') + \frac{1}{\varepsilon}\psi_{\mathsf{z}}(\varepsilon z')$$

will be denoted by the symbol $\mathfrak{B}_{\Psi_{\varepsilon,\alpha}}$, see (3.13) below. We emphasize that we deviate from the construction set forth in (3.1), since (3.13) applies (3.12) for each component individually. Hence, we introduce

$$\mathfrak{B}_{\Psi_{\varepsilon,\alpha}}(\tau, u', z', \sigma_{\mathsf{u}}, \sigma_{\mathsf{z}}) := \frac{1}{\varepsilon^\alpha}\,\mathcal{B}_{\psi_{\mathsf{u}}}(\tau, \varepsilon^\alpha u', \sigma_{\mathsf{u}}) + \frac{1}{\varepsilon}\,\mathcal{B}_{\psi_{\mathsf{z}}}(\tau, \varepsilon z', \sigma_{\mathsf{z}}) \tag{3.13a}$$

$$= \mathcal{B}_{\psi_{\mathsf{u}}}\big(\frac{\tau}{\varepsilon^\alpha}, u', \sigma_{\mathsf{u}}\big) + \mathcal{B}_{\psi_{\mathsf{z}}}\big(\frac{\tau}{\varepsilon}, z', \sigma_{\mathsf{z}}\big). \tag{3.13b}$$

Subsequently, we will use the short-hand notation $\mathfrak{B}_\varepsilon^\alpha$ in place of $\mathfrak{B}_{\Psi_{\varepsilon,\alpha}}$ and extend $\mathfrak{B}_\varepsilon^\alpha$ to allow for the value $\tau = 0$, defining *rescaled joint B-function* $\mathfrak{B}_\varepsilon^\alpha : [0, \infty)\times\mathbf{U}\times\mathbf{Z}\times[0, \infty)^2 \to [0, \infty]$ via

$$\mathfrak{B}_\varepsilon^\alpha(\tau, u', z', \sigma_{\mathsf{u}}, \sigma_{\mathsf{z}}) := \begin{cases} \dfrac{\tau}{\varepsilon^\alpha}\,\psi_{\mathsf{u}}\Big(\dfrac{\varepsilon^\alpha}{\tau}\,u'\Big) + \dfrac{\tau}{\varepsilon^\alpha}\,\sigma_{\mathsf{u}} + \dfrac{\tau}{\varepsilon}\,\psi_{\mathsf{z}}\Big(\dfrac{\varepsilon}{\tau}\,z'\Big) + \dfrac{\tau}{\varepsilon}\,\sigma_{\mathsf{z}} & \text{for } \tau > 0, \\ \infty & \text{for } \tau = 0. \end{cases} \tag{3.14}$$

We highlight that $\mathfrak{B}_\varepsilon^\alpha$ is relevant for the *coupled* system (1.4), hence the name rescaled *joint* B-function.

The next result shows that the Mosco limit $\mathfrak{B}_0^\alpha$ of the B-functions $(\mathfrak{B}_\varepsilon^\alpha)_\varepsilon$ always exists and can be expressed in terms of the potentials $\mathfrak{b}_{\psi_{\mathsf{u}}}$, $\mathfrak{b}_{\psi_{\mathsf{z}}}$, and $\mathfrak{b}_{\psi_{\mathsf{u}}\oplus\psi_{\mathsf{z}}}$. We emphasize that $(\tau, u', z') \mapsto \mathfrak{B}_0^\alpha(\tau, u', z', \sigma_{\mathsf{u}}, \sigma_{\mathsf{z}})$ is 1-homogeneous, which reflects the rate-independent character of the limiting procedure.

Proposition 3.6 (Mosco limit $\mathfrak{B}_0^\alpha$ of the family $\mathfrak{B}_\varepsilon^\alpha$)**.** *Let ψ_u and ψ_z satisfy* (3.6) *and assume $\alpha > 0$. Then, $\mathfrak{B}_\varepsilon^\alpha$ Mosco converge to the limit $\mathfrak{B}_0^\alpha : [0,\infty)\times\mathbf{U}\times\mathbf{Z}\times[0,\infty)^2 \to [0,\infty]$ that is given as follows:*

$$\tau > 0: \qquad \mathfrak{B}_0^\alpha(\tau, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z}) = \begin{cases} \big(\psi_\mathsf{u}\big)_{\mathrm{ri}}(u') + \big(\psi_\mathsf{z}\big)_{\mathrm{ri}}(z') & \text{for } \sigma_\mathsf{u} = \sigma_\mathsf{z} = 0, \\ \infty & \text{otherwise;} \end{cases}$$

$$\tau = 0,\ \alpha > 1: \ \mathfrak{B}_0^\alpha(0, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z}) = \begin{cases} \big(\psi_\mathsf{u}\big)_{\mathrm{ri}}(u') + \mathfrak{b}_{\psi_\mathsf{z}}(z', \sigma_\mathsf{z}) & \text{for } \sigma_\mathsf{u} = 0, \\ \mathfrak{b}_{\psi_\mathsf{u}}(u', \sigma_\mathsf{u}) & \text{for } \sigma_\mathsf{u} > 0 \text{ and } z' = 0, \\ \infty & \text{otherwise;} \end{cases}$$

$$\tau = 0,\ \alpha = 1: \ \mathfrak{B}_0^1(0, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z}) \ = \ \mathfrak{b}_{\psi_\mathsf{u}\oplus\psi_\mathsf{z}}((u', z'), \sigma_\mathsf{u}{+}\sigma_\mathsf{z});$$

$$\tau = 0,\ \alpha < 1: \ \mathfrak{B}_0^\alpha(0, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z}) = \begin{cases} \mathfrak{b}_{\psi_\mathsf{u}}(u', \sigma_\mathsf{u}) + \big(\psi_\mathsf{z}\big)_{\mathrm{ri}}(z') & \text{for } \sigma_\mathsf{z} = 0, \\ \mathfrak{b}_{\psi_\mathsf{z}}(z', \sigma_\mathsf{z}) & \text{for } \sigma_\mathsf{z} > 0 \text{ and } u' = 0, \\ \infty & \text{otherwise,} \end{cases}$$

where $\psi := \psi_\mathsf{u} \oplus \psi_\mathsf{z} : (u', z') \mapsto \psi_\mathsf{u}(u'){+}\psi_\mathsf{z}(z')$. Thus, the functional $\mathfrak{B}_0^\alpha(\cdot,\cdot,\cdot,\sigma_\mathsf{u},\sigma_\mathsf{z})$ is convex and 1-homogeneous for all $(\sigma_\mathsf{u},\sigma_\mathsf{z}) \in [0,\infty)^2$.

Proof. Case $\tau > 0$. Using $\psi_\mathsf{x}(v) \geq \big(\psi_\mathsf{x}\big)_{\mathrm{ri}}(v)$ we have

$$\mathfrak{B}_\varepsilon^\alpha(\tau, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z}) \geq \big(\psi_\mathsf{u}\big)_{\mathrm{ri}}(u') + \frac{\tau}{\varepsilon^\alpha}\,\sigma_\mathsf{u} + \big(\psi_\mathsf{z}\big)_{\mathrm{ri}}(z') + \frac{\tau}{\varepsilon}\,\sigma_\mathsf{z},$$

which easily provides the desired liminf estimate. The limsup estimate follows with the constant recovery sequence $(u'_\varepsilon, z'_\varepsilon, \sigma_{\mathsf{u},\varepsilon}, \sigma_{\mathsf{z},\varepsilon}) = (u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z})$.

Case $\tau = 0$ and $\alpha = 1$. By definition of $\mathfrak{b}_\psi = \mathfrak{b}_{\psi_\mathsf{u}\oplus\psi_\mathsf{z}}$ we have

$$\mathfrak{B}_\varepsilon^1(\tau, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z}) = \frac{\tau}{\varepsilon}\psi\big(\frac{\varepsilon}{\tau}(u', z')\big) + \frac{\tau}{\varepsilon}\,(\sigma_\mathsf{u}{+}\sigma_\mathsf{z}) \geq \mathfrak{b}_\psi((u', z'), \sigma_\mathsf{u}{+}\sigma_\mathsf{z}) \qquad \text{for all } \tau > 0.$$

Hence, the liminf estimate follows from Proposition 3.2.

For the limsup estimate for $\mathfrak{B}_0^1(0, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z})$ we choose λ_ε such that $\lambda_\varepsilon\psi(\frac{1}{\lambda_\varepsilon}(u', z')) + \lambda_\varepsilon(\sigma_\mathsf{u}{+}\sigma_\mathsf{z}) \to \mathfrak{b}_\psi((u', z'), \sigma_\mathsf{u}{+}\sigma_\mathsf{z})$, where we may assume $\lambda_\varepsilon \leq 1/\sqrt{\varepsilon}$. Then, it suffices to set $\tau_\varepsilon = \lambda_\varepsilon\varepsilon \to 0$ to conclude $\mathfrak{B}_\varepsilon^1(\tau, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z}) \to \mathfrak{b}_\psi((u', z'), \sigma_\mathsf{u}{+}\sigma_\mathsf{z}) = \mathfrak{B}_0^1(0, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z})$.

Case $\tau = 0$ and $\alpha > 1$. For the lower bound in the liminf estimate we only need to consider the case $\sigma_\mathsf{u} = 0$ and the case $\sigma_\mathsf{u} > 0$ and $z' = 0$. In the latter situation we may drop the two last terms in the definition of $\mathfrak{B}_\varepsilon^\alpha$ and the lower bound is established by the lower semicontinuity of $\mathfrak{b}_{\psi_\mathsf{u}}$. In the case $\sigma_\mathsf{u} = 0$, we have the lower bound

$$\mathfrak{B}_\varepsilon^\alpha(\tau, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z}) \geq \big(\psi_\mathsf{u}\big)_{\mathrm{ri}}(u') + \mathfrak{b}_{\psi_\mathsf{z}}(z', \sigma_\mathsf{z})$$

and the liminf again follows by the lsc.

For the limsup estimates we use the recovery sequence $(\tau_\varepsilon, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z})$ converging strongly with $\tau_\varepsilon \to 0$, as in the previous case. For $\sigma = 0$ we choose $\tau_\varepsilon = \lambda_\varepsilon\varepsilon$ where λ_ε realizes the infimum in $\mathfrak{b}_{\psi_\mathsf{z}}(z', \sigma_\mathsf{z})$. In the case $\sigma_\mathsf{u} > 0$ and $z' = 0$ we choose $\tau_\varepsilon = \hat\lambda_\varepsilon\varepsilon^\alpha$, where $\hat\lambda_\varepsilon$ realizes the infimum in $\mathfrak{b}_{\psi_\mathsf{u}}(u', \sigma)$. In the remaining case, which has $\sigma > 0$, we may choose $\tau_\varepsilon = \varepsilon$.

Case $\tau = 0$ and $\alpha < 1$. This case is similar to the case $\alpha > 1$ if we interchange the role of u' and z'. Thus, Proposition 3.6 is proved. □

3.3. Lower bounds for the B-function $\mathfrak{B}_\varepsilon^\alpha$.

In the subsequent convergence analysis for the vanishing-viscosity limit we will need ε-uniform a priori bounds for the time derivatives of the solutions $q_\varepsilon = (u_\varepsilon, z_\varepsilon)$. They are derived by lower bounds for the B-functions, however, we have already observed in Example 3.4 that the simple lower bound $\mathfrak{b}_\psi(v, \psi^*(\xi)) \geq \|\xi\|_*\|v\|$ in (3.9) cannot be expected. The following result provides suitable surrogates of such estimate. They will play a crucial role in the vanishing-viscosity analysis, specifically in controlling $\|z'\|$ along jump paths, see Lemma 5.4. For this it will be important that the function $\varkappa$ occurring in (3.15) is strictly increasing, which implies $\varkappa(\sigma) > 0$ for $\sigma > 0$.

Lemma 3.7 (Lower bound on $\mathfrak{B}_\varepsilon^\alpha$)**.** *Let ψ_u and ψ_z satisfy* (3.6) *and let $\mathfrak{B}_\varepsilon^\alpha$ be given as in* (3.14)*. Then, there exists a continuous, convex, nondecreasing, and superlinear function $\varphi : [0,\infty) \to [0,\infty)$ such that*

$$\forall\,\alpha > 0\ \forall\,\varepsilon \in [0,1]\ \forall\,(\tau, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z}) \in [0,\infty)\times\mathbf{U}\times\mathbf{Z}\times[0,\infty)^2:$$

$$\psi_\mathsf{u}(u') \geq \varphi(\|u'\|_\mathbf{U}) \ \ \text{and} \ \ \psi_\mathsf{z}(z') \geq \varphi(\|z'\|_\mathbf{Z}), \tag{3.15a}$$

$$\mathfrak{B}_\varepsilon^\alpha(\tau, u', z', \sigma_\mathsf{u}, \sigma_\mathsf{z}) \geq \|u'\|_\mathbf{U}\,\varkappa(\sigma_\mathsf{u}) + \|z'\|_\mathbf{Z}\,\varkappa(\sigma_\mathsf{z}), \tag{3.15b}$$

where $\varkappa \in \mathrm{C}([0,\infty);[0,\infty))$ *is given by* $\varkappa(\sigma) = (\varphi^*)^{-1}(\sigma)$, *is concave, and strictly increasing with* $\varkappa(0)=0$ *and* $\varkappa(\sigma)\to\infty$ *for* $\sigma\to\infty$. *We additionally have*

$$\alpha<1: \quad \mathfrak{B}^\alpha_\varepsilon(\tau,u',z',\sigma_{\mathsf{u}},\sigma_{\mathsf{z}}) \geq \|u'\|_{\mathbf{U}}\,\varkappa\big(\sigma_{\mathsf{u}}+\sigma_{\mathsf{z}}\big), \tag{3.15c}$$

$$\alpha=1: \quad \mathfrak{B}^1_\varepsilon(\tau,u',z',\sigma_{\mathsf{u}},\sigma_{\mathsf{z}}) \geq \big(\|u'\|_{\mathbf{U}}+\|z'\|_{\mathbf{Z}}\big)\,\varkappa\big(\frac{1}{2}(\sigma_{\mathsf{u}}+\sigma_{\mathsf{z}})\big), \tag{3.15d}$$

$$\alpha\geq1: \quad \mathfrak{B}^\alpha_\varepsilon(\tau,u',z',\sigma_{\mathsf{u}},\sigma_{\mathsf{z}}) \geq \|z'\|_{\mathbf{Z}}\,\varkappa\big(\sigma_{\mathsf{u}}+\sigma_{\mathsf{z}}\big). \tag{3.15e}$$

Proof. *Step 1: Construction of φ.* Since ψ_{u} and ψ_{z} are superlinear, for each $K\geq0$ there exists $S_K\geq0$ such that

$$\forall (u',z')\in\mathbf{U}\times\mathbf{Z}: \quad \psi_{\mathsf{u}}(u')\geq K\|u'\|_{\mathbf{U}}-S_K \text{ and } \psi_{\mathsf{z}}(z')\geq K\|z'\|_{\mathbf{Z}}-S_K.$$

Hence, the estimates in (3.15a) hold for the nonnegative, convex function φ given by

$$\varphi(r):=\sup\big\{\,Kr-S_K\,\big|\,K\geq0\,\big\}.$$

From $\varphi(0)=0$ and non-negativity we conclude that φ is nondecreasing. Moreover, it is superlinear by construction.

Step 2: Lower bound on $\mathfrak{b}_{\psi_{\mathsf{x}}}$. In the definition of $\mathfrak{b}_\psi$ the dependence on ψ is monotone (because of $\tau>0$) so that $\psi_1\leq\psi_2$ implies $\mathfrak{b}_{\psi_1}\leq\mathfrak{b}_{\psi_2}$. Setting $\widetilde{\varphi}(v)=\varphi(\|v\|)$ we obtain $\mathfrak{b}_{\psi_{\mathsf{x}}}\geq\mathfrak{b}_{\widetilde{\varphi}}$, and using Lemma 3.3 and the definition of $\varkappa$ yields

$$\mathfrak{b}_{\psi_{\mathsf{x}}}(v,\sigma)\geq\|v\|\,\varkappa(\sigma) \qquad \text{for } \mathsf{x}\in\{\mathsf{u},\mathsf{z}\}.$$

Step 3: Lower bound on $\mathfrak{B}^\alpha_\varepsilon$. The definitions of $\mathfrak{B}^\alpha_\varepsilon$ in (3.13b) and of $\mathfrak{b}_\psi$ give, for $\varepsilon>0$,

$$\begin{aligned}\mathfrak{B}^\alpha_\varepsilon(\tau,u',z',\sigma_{\mathsf{u}},\sigma_{\mathsf{z}}) &= \mathcal{B}_{\psi_{\mathsf{u}}}(\frac{\tau}{\varepsilon^\alpha},u',\sigma_{\mathsf{u}})+\mathcal{B}_{\psi_{\mathsf{z}}}(\frac{\tau}{\varepsilon},z',\sigma_{\mathsf{z}})\\ &\geq \mathfrak{b}_{\psi_{\mathsf{u}}}(u',\sigma_{\mathsf{u}})+\mathfrak{b}_{\psi_{\mathsf{z}}}(z',\sigma_{\mathsf{z}})\geq\|u'\|_{\mathbf{U}}\,\varkappa(\sigma_{\mathsf{u}})+\|z'\|_{\mathbf{Z}}\,\varkappa(\sigma_{\mathsf{z}}),\end{aligned}$$

where Step 2 was invoked for the last estimate. This proves (3.15b).

Estimate (3.15d) follows from the simple observation that, because of $\alpha=1$, the rescaled B-function $\mathfrak{B}^1_\varepsilon$ only depends $\sigma_{\mathsf{u}}+\sigma_{\mathsf{z}}$, such that each of σ_{u} and σ_{z} can be replaced by their arithmetic mean.

For $\alpha\geq1$ and $\varepsilon\in(0,1]$, we have $\tau/\varepsilon^\alpha\geq\tau/\varepsilon$ so that

$$\mathfrak{B}^\alpha_\varepsilon(\tau,u',z',\sigma_{\mathsf{u}},\sigma_{\mathsf{z}})\geq\frac{\tau}{\varepsilon^\alpha}\sigma_{\mathsf{u}}+\mathcal{B}_{\psi_{\mathsf{z}}}(\frac{\tau}{\varepsilon},z',\sigma_{\mathsf{z}})\geq\mathcal{B}_{\psi_{\mathsf{z}}}(\frac{\tau}{\varepsilon},z',\sigma_{\mathsf{u}}+\sigma_{\mathsf{z}})\geq\|z'\|_{\mathbf{Z}}\,\varkappa(\sigma_{\mathsf{u}}+\sigma_{\mathsf{z}}).$$

This shows estimate (3.15e), and (3.15c) follows similarly.

All estimates remain true for $\varepsilon=0$ because $\mathfrak{B}^\alpha_0$ is the Mosco limit of $\mathfrak{B}^\alpha_\varepsilon$. □

4. Setup and existence for the viscous system

In Section 4.1 we will introduce our basic conditions on the ambient spaces, the energy, and the dissipation potentials, collected in Hypotheses 4.1, 4.2, 4.3, and 4.5, which will be assumed throughout the paper. Let us mention in advance that we will often omit to explicitly recall these assumptions in the various intermediate statements, with the exception of our main results in Theorems 5.11, 5.14, 6.8, and 6.12.

Then, in Section 4.4 we will address the existence of solutions to the viscous system (1.4). Its main result, Theorem 4.8 shows that, under two additional conditions on the driving energy functional, the existence result from [MRS13, Thm. 2.2] can be applied to deduce the existence of solutions for the doubly nonlinear system (1.4). It will be crucial to our analysis that we are able to show that these solutions satisfy the (Ψ,Ψ^*) energy-dissipation balance (1.8).

4.1. Function spaces.

Here we state our standing assumptions on the function spaces for the energy functionals and for the dissipation potentials.

Hypothesis 4.1 (Function spaces)**.** *In addition to conditions* (1.1) *on the ambient spaces* $\mathbf{U}$ *and* $\mathbf{Z}$*, our (coercivity) conditions on the energy* $\mathcal{E}$ *will involve two other* reflexive *spaces* $\mathbf{U}_{\mathrm{e}}$ *and* $\mathbf{Z}_{\mathrm{e}}$*, such that*

$$\mathbf{U}_{\mathrm{e}}\subset\mathbf{U} \textit{ continuously and densely, and } \mathbf{Z}_{\mathrm{e}}\Subset\mathbf{Z} \textit{ compactly and densely.}$$

The subscript e *refers to the fact that the latter are 'energy spaces' relating to* $\mathcal{E}$*. Furthermore, the* 1*-homogeneous dissipation potential* $\mathcal{R}$ *will be in fact defined on a (separable) space* $\mathbf{Z}_{\mathrm{ri}}$ *(where the subscript* ri *accordingly refers to rate-independence), such that*

$$\mathbf{Z}\subset\mathbf{Z}_{\mathrm{ri}} \textit{ continuously and densely.}$$

We refer to (4.9) for some examples of relevant ambient spaces. In what follows, we will often use the notation

$$q := (u, z) \in \mathbf{Q} := \mathbf{U}\times\mathbf{Z}. \tag{4.1}$$

4.2. Assumption on the dissipation potentials.

We will develop the general theory under the condition that the viscous dissipation potentials $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$ as well as the 1-homogeneous potential $\mathcal{R}$ take only finite values in $[0,\infty)$ and are thus continuous. Recall that $\mathcal{V}_{\mathsf{x}}^*$ is the Legendre-Fenchel conjugate of $\mathcal{V}_{\mathsf{x}}$, see Definition 3.1.

Hypothesis 4.2 (Conditions on $\mathcal{V}_{\mathsf{u}}$, $\mathcal{V}_{\mathsf{z}}$, $\mathcal{R}$)**.** *Let $\mathcal{V}_{\mathsf{u}} : \mathbf{U} \to [0,\infty)$ and $\mathcal{V}_{\mathsf{z}} : \mathbf{Z} \to [0,\infty)$ be dissipation potentials with the following additional conditions:*

$$\lim_{\|v\|_{\mathbf{U}}\to\infty} \frac{\mathcal{V}_{\mathsf{u}}(v)}{\|v\|_{\mathbf{U}}} = \lim_{\|\mu\|_{\mathbf{U}^*}\to\infty} \frac{\mathcal{V}_{\mathsf{u}}^*(\mu)}{\|\mu\|_{\mathbf{U}^*}} = \infty = \lim_{\|\eta\|_{\mathbf{Z}}\to\infty} \frac{\mathcal{V}_{\mathsf{z}}(\eta)}{\|\eta\|_{\mathbf{Z}}} = \lim_{\|\zeta\|_{\mathbf{Z}^*}\to\infty} \frac{\mathcal{V}_{\mathsf{z}}^*(\zeta)}{\|\zeta\|_{\mathbf{Z}^*}}, \tag{4.2a}$$

$$\lim_{\lambda\to0^+} \frac{1}{\lambda}\mathcal{V}_{\mathsf{u}}(\lambda v) = 0 \ \text{ for all } v \in \mathbf{U}, \quad \text{and} \quad \lim_{\lambda\to0^+} \frac{1}{\lambda}\mathcal{V}_{\mathsf{z}}(\lambda \eta) = 0 \ \text{ for all } \eta \in \mathbf{Z}\,. \tag{4.2b}$$

Let $\mathcal{R} : \mathbf{Z}_{\mathrm{ri}} \to [0,\infty]$ be a 1-homogeneous dissipation potential, i.e.

$$\mathcal{R}(\lambda\eta) = \lambda\mathcal{R}(\eta) \quad \text{for all } \eta \in \mathbf{Z}_{\mathrm{ri}} \text{ and } \lambda > 0, \tag{4.3a}$$

that is additionally $\mathbf{Z}$-bounded and $\mathbf{Z}_{\mathrm{ri}}$-coercive for $\mathbf{Z} \subset \mathbf{Z}_{\mathrm{ri}}$, i.e.

$$\exists\, C_{\mathcal{R}},\, c_{\mathcal{R}} > 0 : \quad \begin{cases} \forall\,\eta \in \mathbf{Z} : & \mathcal{R}(\eta) \le C_{\mathcal{R}}\|\eta\|_{\mathbf{Z}}, \\ \forall\,\eta \in \mathbf{Z}_{\mathrm{ri}} : & \mathcal{R}(\eta) \ge c_{\mathcal{R}}\|\eta\|_{\mathbf{Z}_{\mathrm{ri}}}\,. \end{cases} \tag{4.3b}$$

Due to the superlinear growth of $\mathcal{V}_{\mathsf{x}}$ and $\mathcal{V}_{\mathsf{x}}^*$, $\mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$, both $\partial\mathcal{V}_{\mathsf{x}} : \mathbf{X} \rightrightarrows \mathbf{X}^*$ and $\partial\mathcal{V}_{\mathsf{x}}^* : \mathbf{X}^* \rightrightarrows \mathbf{X}$, $\mathbf{X} \in \{\mathbf{U},\mathbf{Z}\}$, are bounded operators, so that, ultimately, both $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{u}}^*$ are continuous. Likewise, $\mathcal{R}$ is continuous. Indeed, restricting our analysis to the case in which $\mathcal{R}$ takes only finite values in $[0,\infty)$ excludes the direct application of our results to systems modeling unidirectional processes in solids such as damage or delamination. In those cases the existence theory (both for the rate-dependent, 'viscous' system and for BV solutions of the rate-independent process) relies on additional estimates not considered here, see e.g. [KRZ19]. Nevertheless, a broad class of models is still described by *continuous* dissipation functionals. For instance, the coercivity and growth conditions (4.3b) are compatible with the following example of dissipation potential, in the ambient spaces $\mathbf{Z}_{\mathrm{ri}} = \mathrm{L}^1(\Omega)$ and $\mathbf{Z} = \mathrm{L}^2(\Omega)$ (with $\Omega \subset \mathbb{R}^d$ a bounded domain):

$$\mathcal{R} : \mathrm{L}^1(\Omega) \to [0,\infty]; \quad \mathcal{R}(\eta) := \begin{cases} \|\eta^+\|_{\mathrm{L}^2(\Omega)} + \|\eta^-\|_{\mathrm{L}^1(\Omega)} & \text{if } \eta^+ \in \mathrm{L}^2(\Omega), \\ \infty & \text{otherwise.} \end{cases} \tag{4.4}$$

Dissipation potentials with this structure occur, for instance, in models for damage or delamination allowing for possible healing, cf. e.g. [MiR15, Sec. 5.2.7] and Section 8.

Subsequently, $\partial\mathcal{V}_{\mathsf{u}} : \mathbf{U} \rightrightarrows \mathbf{U}^*$, $\partial\mathcal{V}_{\mathsf{u}} : \mathbf{Z} \rightrightarrows \mathbf{Z}^*$, and $\partial\mathcal{R} : \mathbf{Z} \rightrightarrows \mathbf{Z}^*$ will denote the convex subdifferentials of $\mathcal{V}_{\mathsf{u}}$, $\mathcal{V}_{\mathsf{z}}$, and $\mathcal{R}$, respectively. By the 1-homogeneity (4.3a) we have

$$\partial\mathcal{R}(\eta) = \big\{\, \omega \in \partial\mathbf{Z}^* \,\big|\, \forall\, v{\in}\mathbf{Z}\colon \mathcal{R}(v) \ge \mathcal{R}(\eta){+}\langle\omega, v{-}\eta\rangle_{\mathbf{Z}} \,\big\} = \big\{\, \omega \in \partial\mathcal{R}(0) \,\big|\, \mathcal{R}(\eta) = \langle\omega,\eta\rangle \,\big\}. \tag{4.5}$$

Thanks to Hypothesis 4.1, we have $\mathbf{Z}_{\mathrm{ri}}^* \subset \mathbf{Z}^*$ densely and continuously. As a consequence of (4.3b) $\partial\mathcal{R}(0)$ turns out to be a bounded subset in $\mathbf{Z}^*$, viz.

$$\partial\mathcal{R}(\eta) \subset \partial\mathcal{R}(0) \quad \text{and} \quad \overline{B}_{c_{\mathcal{R}}}^{\mathbf{Z}_{\mathrm{ri}}^*}(0) \subset \partial\mathcal{R}(0) \subset \overline{B}_{C_{\mathcal{R}}}^{\mathbf{Z}^*}(0). \tag{4.6}$$

4.3. Assumptions on the energy $\mathcal{E}$.

We now collect our basic requirements on the energy functional $\mathcal{E} : [0,T]\times\mathbf{U}\times\mathbf{Z} \to (-\infty,\infty]$. With slight abuse of notation, we will often write $\mathcal{E}(t,q)$ in place of $\mathcal{E}(t,u,z)$, in accordance with (4.1). Recall the embeddings $\mathbf{U}_{\mathrm{e}} \subset \mathbf{U}$ and $\mathbf{Z}_{\mathrm{e}} \Subset \mathbf{Z} \subset \mathbf{Z}_{\mathrm{ri}}$ and the choice $\mathbf{Q} = \mathbf{U}\times\mathbf{Z}$.

Hypothesis 4.3 (Lower semicontinuity, coercivity, time differentiability of $\mathcal{E}$)**.** *The energy functional $\mathcal{E} : [0,T]\times\mathbf{U}\times\mathbf{Z} \to (-\infty,\infty]$ has the proper domain $\mathrm{dom}(\mathcal{E}) = [0,T]\times\mathrm{D}$ with $\mathrm{D} \subset \mathbf{U}_{\mathrm{e}}\times\mathbf{Z}_{\mathrm{e}}$. Moreover, we require that*

$$\forall\, t \in [0,T] : \quad \text{the map } q \mapsto \mathcal{E}(t,q) \text{ is weakly lower semicontinuous on } \mathbf{Q}, \tag{4.7a}$$

and $\mathcal{E}$ is bounded from below:

$$\exists\, C_0 > 0 \ \ \forall\,(t,q) \in [0,T]\times\mathrm{D} : \qquad \mathcal{E}(t,q) \ge C_0\,. \tag{4.7b}$$

We set $\mathfrak{E}(q) := \sup_{t\in[0,T]} \mathcal{E}(t,q)$ *and require that*

$$\text{the map } q \mapsto \mathfrak{E}(q) + \|q\|_{\mathbf{U}\times\mathbf{Z}_{\mathrm{ri}}} \text{ has sublevels bounded in } \mathbf{U}_{\mathrm{e}}\times\mathbf{Z}_{\mathrm{e}}. \tag{4.7c}$$

Finally, we require that $t \mapsto \mathcal{E}(t,q)$ *is differentiable for all* $q \in \mathrm{D}$ *satisfying the power-control estimate*

$$\exists\, C_\# > 0\ \forall\,(t,q) \in [0,T]\times\mathrm{D}: \quad |\partial_t\mathcal{E}(t,q)| \le C_\#\mathcal{E}(t,q). \tag{4.7d}$$

Concerning our conditions on $\mathrm{dom}(\mathcal{E})$, the crucial requirement is that $\mathrm{dom}(\mathcal{E}(t,\cdot)) \equiv \mathrm{D}$ is independent of time. Let us introduce the energy sublevels

$$\mathcal{S}_E := \{q \in \mathrm{D} : \mathfrak{E}(q) \le E\} \qquad \text{for } E > 0. \tag{4.8}$$

Applying Grönwall's lemma we deduce from (4.7d) that

$$\forall\,(t,q) \in [0,T]\times\mathrm{D}: \qquad \mathfrak{E}(q) \le \mathrm{e}^{C_\# T}\,\mathcal{E}(t,q)\,.$$

Hence, $\mathcal{E}(t,q) \le E$ for some $t \in [0,T]$ and $E > 0$ guarantees $q \in \mathcal{S}_{E'}$ with $E' = \mathrm{e}^{C_\# T} E$. Finally, observe that (4.7c) implies the separate coercivity properties of the functionals $\mathfrak{E}(\cdot,z)$ and $\mathfrak{E}(u,\cdot)$, perturbed by the norm $\|\cdot\|_{\mathbf{U}}$ and $\|\cdot\|_{\mathbf{Z}_{\mathrm{ri}}}$, respectively.

Since we are only requiring that $\mathbf{U}_{\mathrm{e}} \subset \mathbf{U}$ continuously, our analysis allows for the following two cases: (i) the energy $\mathcal{E}(t,\cdot,z)$ and the dissipation potential $\mathcal{V}_{\mathsf{u}}$ have sublevels bounded in the same space and (ii) the energy $\mathcal{E}(t,\cdot,z)$ has sublevels compact in the space $\mathbf{U}$ of the dissipation $\mathcal{V}_{\mathsf{u}}$. To fix ideas, typical examples for the pairs $(\mathbf{U},\mathbf{U}_{\mathrm{e}})$ and the triples $(\mathbf{Z}_{\mathrm{e}},\mathbf{Z},\mathbf{Z}_{\mathrm{ri}})$ are

$$\begin{aligned} &\text{(i)}\ \ \mathbf{U} = \mathbf{U}_{\mathrm{e}} = \mathrm{H}^1(\Omega;\mathbb{R}^d) \quad \text{or} \quad \text{(ii)}\ \ \mathbf{U}_{\mathrm{e}} = \mathrm{H}^1(\Omega;\mathbb{R}^d) \Subset \mathbf{U} = \mathrm{L}^2(\Omega;\mathbb{R}^d),\\ &\text{and}\quad \mathbf{Z}_{\mathrm{e}} = \mathrm{H}^1(\Omega;\mathbb{R}^m) \Subset \mathbf{Z} = \mathrm{L}^2(\Omega;\mathbb{R}^m) \subset \mathbf{Z}_{\mathrm{ri}} = \mathrm{L}^1(\Omega;\mathbb{R}^d). \end{aligned} \tag{4.9}$$

As mentioned in the introduction, in our analysis we aim to allow for nonsmoothness of the energy functional $q = (u,z) \mapsto \mathcal{E}(t,q)$. Accordingly, we will use the Fréchet subdifferential of $\mathcal{E}$ with respect to the variable q, i.e. the multivalued operator $\partial_{\mathsf{q}}\mathcal{E} : [0,T]\times\mathbf{Q} \rightrightarrows \mathbf{Q}^*$ defined for $(t,q) \in [0,T]\times\mathrm{D}$ via

$$\partial_{\mathsf{q}}\mathcal{E}(t,q) := \big\{\, \xi \in \mathbf{Q}^* \,\big|\, \mathcal{E}(t,\hat q) \ge \mathcal{E}(t,q) + \langle \eta, \hat q - q\rangle_{\mathbf{Q}} + o(\|\hat q - q\|_{\mathbf{Q}}) \text{ as } \hat q \to q \text{ in } \mathbf{Q} \,\big\} \tag{4.10}$$

with domain $\mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E}) := \big\{\, (t,q) \in [0,T]\times\mathrm{D} \,\big|\, \partial_{\mathsf{q}}\mathcal{E}(t,q) \ne \emptyset \,\big\}$.

Thus, our aim is to solve the subdifferential inclusion

$$\partial\Psi_{\varepsilon,\alpha}(q'(t)) + \partial_q\mathcal{E}(t,q(t)) \ni 0 \qquad \text{in } \mathbf{Q}^* \text{ for a.a. } t \in (0,T) \tag{4.11}$$

where the scaled dissipation potential $\Psi_{\varepsilon,\alpha}$ is defined in (1.7).

Remark 4.4 (Partial Fréchet subdifferentials)**.** *Observe that*

$$\partial_{\mathsf{q}}\mathcal{E}(t,u,z) \subset \partial_{\mathsf{u}}\mathcal{E}(t,u,z)\times\partial_{\mathsf{z}}\mathcal{E}(t,u,z) \quad \textit{for all } (t,q) = (t,u,z) \in [0,T]\times\mathrm{D}, \tag{4.12}$$

where $\partial_{\mathsf{u}}\mathcal{E}(t,q) \subset \mathbf{U}^*$ *and* $\partial_{\mathsf{z}}\mathcal{E}(t,q) \subset \mathbf{Z}^*$ *are the 'partial' Fréchet subdifferentials of* $\mathcal{E}$ *with respect to the variables* u *and* z*, which are defined as Fréchet subdifferentials of* $\mathcal{E}(t,\cdot,z) : \mathbf{U} \to \mathbb{R}$ *and* $\mathcal{E}(t,u,\cdot) : \mathbf{Z} \to \mathbb{R}$*, respectively. However, equality in* (4.12) *is false, in general, e.g. for* $\mathbf{U} = \mathbf{Z} = \mathbb{R}$ *and* $\mathcal{E}(t,u,z) = |u-z|$.

In view of the inclusion (4.12)*, any curve* $t \mapsto q(t) = (u(t),z(t))$ *solving* (4.11) *also solves the system*

$$\partial\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}(u'(t)) + \partial_{\mathsf{u}}\mathcal{E}(t,u(t),z(t)) \ni 0 \qquad \textit{in } \mathbf{U}^* \qquad \textit{for a.a. } t \in (0,T), \tag{4.13a}$$

$$\partial\mathcal{R}(z'(t)) + \partial\mathcal{V}_{\mathsf{z}}^{\varepsilon}(z'(t)) + \partial_{\mathsf{z}}\mathcal{E}(t,u(t),z(t)) \ni 0 \qquad \textit{in } \mathbf{Z}^* \qquad \textit{for a.a. } t \in (0,T). \tag{4.13b}$$

Nonetheless, let us stress that the 'reference viscous system' for the subsequent discussion will be the one with the smaller solution set, namely (4.11) *or* (4.18a) *below.*

The existence result from [MRS13] can be applied provided that $\mathcal{E}$ fulfills two further conditions, stated in the following Hypotheses 4.5 and 4.7.

Hypothesis 4.5 (Closedness of $(\partial_q\mathcal{E},\partial_t\mathcal{E})$ on sublevels)**.** *For all sequences* $\big((t_n,q_n,\xi_n)\big)_{n\in\mathbb{N}}$ *in the space* $[0,T]\times\mathbf{Q}\times\mathbf{Q}^*$ *with* $t_n \to t$*,* $q_n \rightharpoonup q$ *in* $\mathbf{Q}$*,* $\xi_n \rightharpoonup \xi$ *in* $\mathbf{Q}^*$*,* $\sup_n \mathfrak{E}(q_n) < \infty$*, and* $\xi_n \in \partial_q\mathcal{E}(t,q_n)$*, we have*

$$\xi \in \partial_q\mathcal{E}(t,q) \quad \textit{and} \quad \partial_t\mathcal{E}(t_n,q_n) \to \partial_t\mathcal{E}(t,q). \tag{4.14}$$

Remark 4.6. *For cases in which the energy space* $\mathbf{U}_{\mathrm{e}}$ *is compactly embedded into* $\mathbf{U}$*, the sequences* $(q_n)_n$ *fulfilling the conditions of Hypothesis 4.5 converge strongly in* $\mathbf{Q}$ *in view of the coercivity* (4.7c)*. Therefore, in such cases Hypothesis 4.5 turns out to be a closedness condition on the graph of* $\partial_q\mathcal{E}$ *with respect to the* strong-weak *topology of* $\mathbf{Q}\times\mathbf{Q}^*$.

We also mention that, in contrast to what we did in [MRS13] *(cf. (2.E$_5$) therein), here in Hypothesis 4.5 we omit the requirement of energy convergence* $\mathcal{E}(t_n,q_n) \to \mathcal{E}(t,q)$ *along the sequence* $(t_n,q_n,\xi_n)_n$*. In fact, that additional property was not strictly needed in the proof of the existence result* [MRS13, Thm. 2.2], *to which we will resort later on to conclude the existence of solutions for our viscous system* (4.11). *Rather, in* [MRS13] *the energy-convergence requirement was encompassed in the closedness assumption in order to pave the way for a weakening of the chain-rule condition, cf. the discussion in* [MRS13, Rmk. 4.6]. *Such a weakening is outside the scope of this paper.*

Our final condition on $\mathcal{E}$ is an abstract *chain rule* that has a twofold role: First, it is a crucial ingredient in the proof of Theorem 4.8, and secondly, it ensures the validity of the energy-dissipation balance (4.18). The latter will be the starting point in the derivation of our a priori estimates *uniformly* with respect to the viscosity parameter ε. We refer to Proposition A.1 in Appendix A for a discussion of conditions on $\mathcal{E}$ yielding the validity of Hypothesis 4.7.

Hypothesis 4.7 (Chain rule). *For every absolutely continuous curve $q \in \mathrm{AC}([0,T];\mathbf{Q})$ and all measurable selections $\xi : (0,T) \to \mathbf{Q}^*$ with $\xi(t) \in \partial_q \mathcal{E}(t,q(t))$ for a.a. $t \in (0,T)$,*

$$\sup_{t\in(0,T)} |\mathcal{E}(t,q(t))| < \infty, \quad \text{and} \quad \int_0^T \|\xi(t)\|_{\mathbf{Q}^*} \|q'(t)\|_{\mathbf{Q}} \,\mathrm{d}t < \infty, \tag{4.15}$$

we have the following two properties:

$$\begin{aligned} &\text{the map } t \mapsto \mathcal{E}(t,q(t)) \text{ is absolutely continuous on } [0,T] \text{ and} \\ &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) - \partial_t \mathcal{E}(t,q(t)) = \langle \xi(t), q'(t)\rangle_{\mathbf{Q}} \quad \text{for a.a. } t \in (0,T). \end{aligned} \tag{4.16}$$

4.4. An existence result for the viscous problem.

We are now in the position to state our existence result for the viscous system (4.11). It is based on the (Ψ,Ψ^*)-formulation of the energy-dissipation balance (cf. (1.8) for the case $q \mapsto \mathcal{E}(t,q)$ is smooth), which we now apply to (4.11) using the Fréchet subdifferential $\partial_{\mathsf{q}}\mathcal{E}(t,q)$ and the scaled dissipation potential $\Psi_{\varepsilon,\alpha}$ defined in (1.7). The Legendre-Fenchel conjugate is given by

$$\Psi^*_{\varepsilon,\alpha}(\mu,\zeta) = \frac{1}{\varepsilon^\alpha}\mathcal{V}^*_{\mathsf{u}}(\mu) + \frac{1}{\varepsilon}\mathcal{W}^*_{\mathsf{z}}(\zeta) \quad \text{with} \quad \mathcal{W}^*_{\mathsf{z}}(\zeta) := \min_{\sigma\in\partial\mathcal{R}(0)} \mathcal{V}^*_{\mathsf{z}}(\zeta-\sigma) \qquad \text{for } \zeta \in \mathbf{Z}^*. \tag{4.17}$$

It can be straightforwardly checked that the infimum in the definition of $\mathcal{W}^*_{\mathsf{z}}$ is attained.

Theorem 4.8 (Existence of viscous solutions). *Assume Hypotheses 4.2, 4.3, 4.5, and 4.7. Then, for every $\varepsilon \in (0,1]$ and $q_0 = (u_0,z_0) \in \mathrm{D}$ there exists a curve $q = (u,z) \in \mathrm{AC}([0,T];\mathbf{Q})$ and a function $\xi = (\mu,\zeta) \in \mathrm{L}^1(0,T;\mathbf{U}^*\times\mathbf{Z}^*)$ fulfilling the initial condition $q(0) = q_0$, solving the generalized gradient system* (4.11) *in the sense that for a.a. $r \in (0,T)$*

$$(\mu(r),\zeta(r)) \in \partial_{\mathsf{q}}\mathcal{E}(r,q(r)) \quad \text{and} \quad \begin{cases} -\mu(r) \in \partial\mathcal{V}^{\varepsilon^\alpha}_{\mathsf{u}}(u'(r)), \\ -\zeta(r) \in \partial\mathcal{R}(z'(r)) + \partial\mathcal{V}^{\varepsilon}_{\mathsf{z}}(z'(r)), \end{cases} \tag{4.18a}$$

Moreover, for $0 \le s < t \le T$, these functions satisfy the energy-dissipation balance

$$\begin{aligned} \mathcal{E}(t,q(t)) &+ \int_s^t \Big(\mathcal{V}^{\varepsilon^\alpha}_{\mathsf{u}}(\varepsilon^\alpha u'(r)) + \mathcal{R}(z'(r)) + \mathcal{V}^{\varepsilon}_{\mathsf{z}}(\varepsilon\, z'(r))\Big)\,\mathrm{d}r \\ &+ \int_s^t \Big(\frac{1}{\varepsilon^\alpha}\mathcal{V}^*_{\mathsf{u}}(-\mu(r)) + \frac{1}{\varepsilon}\mathcal{W}^*_{\mathsf{z}}(-\zeta(r))\Big)\,\mathrm{d}r = \mathcal{E}(s,q(s)) + \int_s^t \partial_t\mathcal{E}(r,q(r))\,\mathrm{d}r. \end{aligned} \tag{4.18b}$$

where $\mathcal{V}^\lambda_{\mathsf{x}}$ is defined in (1.5a).

Proof. Since we are in the simple setting of [MRS13, Sec. 2], where the dissipation potential $\Psi_{\varepsilon,\alpha}$ does not depend on the state q, we can appeal to [MRS13, Thm. 2.2]. Thus, it suffices to check the assumptions $(2.\Psi_1)$–$(2.\Psi_3)$, (E_0), and $(2.\mathrm{E}_1)$–$(2.\mathrm{E}_4)$ therein. Our Hypothesis 4.2 clearly implies $(2.\Psi_1)$ and $(2.\Psi_2)$. Hypothesis 4.3 implies the assumptions (E_0) via (4.7a) and (4.7b), and assumption $(2.\mathrm{E}_1)$ follows via (4.7c) and Hypotheses 4.2. Assumption $(2.\mathrm{E}_2)$ follows from Hypothesis 4.5 via [MRS13, Prop. 4.2]. Assumption $(2.\mathrm{E}_3)$ equals (4.7d) in Hypothesis 4.3. Finally, leaving out the energy-convergence requirement assumption $(2.\mathrm{E}_5)$ follows from Hypothesis 4.5.

Thus, all assumptions are satisfied except for $(2.\Psi_3)$ and $(2.\mathrm{E}_4)$. Concerning $(2.\Psi_3)$, we observe that this technical condition was used for the proof of [MRS13, Thm. 2.2] only in one place, namely in the proof of Lemma 6.1 there. In [Bac21, Thm. 3.2.3] or in [MiR21] it is shown that Lemma 6.1, which is also called "*De Giorgi's lemma*", is also valid if the condition [MRS13, Eqn. $(2.\Psi_3)$] is replaced by the condition that the underlying space Banach space $\mathbf{Q}$ is reflexive, but this is true by our Hypothesis 4.1. As for the chain rule [MRS13, $(2.\mathrm{E}_4)$], a close perusal of the proof of [MRS13] shows that our Hypothesis 4.7 can replace it, allowing us to conclude the existence statement. □

Remark 4.9 (Energy-dissipation inequality). *The analysis from* [MRS13] *in fact reveals that, under the chain rule in Hypothesis 4.7, a curve* $q \in \mathrm{AC}([0,T];\mathbf{Q})$ *fulfills* (4.18a) if and only if *the pair* (q,ξ) *satisfies the energy-dissipation balance* (4.18b) *which, again by the chain rule, is in turn equivalent to the upper energy-dissipation estimate* $\leq$. *This characterization of the viscous system will prove handy for the analysis of the delamination system from Section 8.*

4.5. Properties of the generalized slopes.

For the further analysis it is convenient to introduce the *generalized slope functionals* $\mathscr{S}^*_{\mathsf{x}} : [0,T]\times \mathrm{D} \to [0,\infty]$, $\mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$ via

$$\begin{aligned} \mathscr{S}^*_{\mathsf{u}}(t,q) &:= \inf\big\{\, \mathcal{V}^*_{\mathsf{u}}(-\mu) \,\big|\, (\mu,\zeta) \in \partial_{\mathsf{q}}\mathcal{E}(t,q)\,\big\} \quad \text{and} \\ \mathscr{S}^*_{\mathsf{z}}(t,q) &:= \inf\big\{\, \mathcal{W}^*_{\mathsf{z}}(-\zeta) \,\big|\, (\mu,\zeta) \in \partial_{\mathsf{q}}\mathcal{E}(t,q)\,\big\}, \end{aligned} \tag{4.19}$$

where the infimum over the empty set is always $+\infty$. These functionals play the same key role as (the square of) the metric slope for metric gradient systems, hence from now on we shall refer to $\mathscr{S}^*_{\mathsf{u}}$ and $\mathscr{S}^*_{\mathsf{z}}$ as *generalized slopes*. Clearly, energy balance (4.18b) entails the validity of the following energy-dissipation estimate featuring the slopes $\mathscr{S}^*_{\mathsf{u}}$ and $\mathscr{S}^*_{\mathsf{z}}$:

$$\begin{aligned} \mathcal{E}(t,q(t)) + \int_s^t \Big(\mathcal{V}^{\varepsilon^\alpha}_{\mathsf{u}}(u'(r)) + \mathcal{R}(z'(r)) + \mathcal{V}^{\varepsilon}_{\mathsf{z}}(z'(r)) + \frac{\mathscr{S}^*_{\mathsf{u}}(r,q(r))}{\varepsilon^\alpha} + \frac{\mathscr{S}^*_{\mathsf{z}}(r,q(r))}{\varepsilon} \Big)\,\mathrm{d}r \\ \leq \mathcal{E}(s,q(s)) + \int_s^t \partial_t \mathcal{E}(r,q(r))\,\mathrm{d}r \qquad \text{for all } 0 \leq s \leq t \leq T\,. \end{aligned} \tag{4.20}$$

Note that (4.20) is weaker than (4.18b), but it has the advantage that the selections $\xi = (\mu,\zeta)$ in (4.18a) are no longer needed. Moreover, (4.20) will be still strong enough to handle the limit passage $\varepsilon \to 0^+$. For this, we will assume that the infima in (4.19) are attained. We set

$$\begin{aligned} \mathrm{dom}(\partial_q\mathcal{E}) &:= \big\{\, (t,q) \in [0,T]\times\mathbf{Q} \,\big|\, \partial_q\mathcal{E}(t,q) \neq \emptyset \,\big\} \quad \text{and} \\ \mathrm{dom}(\partial_q\mathcal{E}(t,\cdot)) &:= \big\{\, q \in \mathbf{Q} \,\big|\, \partial_q\mathcal{E}(t,q) \neq \emptyset \,\big\}. \end{aligned}$$

In fact, it can be checked (e.g. by resorting to [MRS13, Prop. 4.2]), that $\mathrm{dom}(\partial_q\mathcal{E}(t,\cdot))$ is dense in D.

Hypothesis 4.10 (Attainment and lower semicontinuity). *For every* $(t,q) \in \mathrm{dom}(\partial_q\mathcal{E})$ *the infima in* (4.19) *are attained, namely*

$$\mathfrak{A}^*_{\mathsf{u}}(t,q) := \operatorname*{Argmin}_{(\mu,\zeta)\in\partial_{\mathsf{q}}\mathcal{E}(t,q)} \mathcal{V}^*_{\mathsf{u}}(-\mu) \neq \emptyset \quad \textit{and} \quad \mathfrak{A}^*_{\mathsf{z}}(t,q) := \operatorname*{Argmin}_{(\mu,\zeta)\in\partial_{\mathsf{q}}\mathcal{E}(t,q)} \mathcal{W}^*_{\mathsf{z}}(-\zeta) \neq \emptyset, \tag{4.21}$$

where $\mathcal{W}^*_{\mathsf{z}}$ *is defined in* (4.17). *Furthermore, for all sequences* $(t_n,q_n)_n \subset [0,T]\times\mathbf{Q}$ *with* $t_n \to t$, $q_n \rightharpoonup q$ *in* $\mathbf{Q}$, *and* $\sup_{n\in\mathbb{N}} \mathfrak{E}(q_n) \leq C < \infty$ *there holds*

$$\liminf_{n\to\infty} \mathscr{S}^*_{\mathsf{x}}(t_n,q_n) \geq \mathscr{S}^*_{\mathsf{x}}(t,q) \qquad \textit{for } \mathsf{x} \in \{\mathsf{u},\mathsf{z}\}\,. \tag{4.22}$$

We are going to show in Lemma 4.11 below that a sufficient condition for Hypothesis 4.10 is that (4.12) improves to an equality, namely

$$\partial_{\mathsf{q}}\mathcal{E}(t,q) = \partial_{\mathsf{u}}\mathcal{E}(t,q)\times\partial_{\mathsf{z}}\mathcal{E}(t,q) \quad \text{for all } (t,q) = (t,u,z) \in [0,T]\times\mathrm{D}. \tag{4.23}$$

Observe that (4.23) does hold if, for instance, $\mathcal{E}$ is of the form

$$\begin{aligned} &\mathcal{E}(t,q) := \mathcal{U}(t,u) + \mathcal{Z}(t,z) + \mathcal{F}(t,u,z) \quad \text{for all } (t,q) = (t,u,z) \in [0,T]\times\mathbf{Q} \\ &\text{with } \mathcal{U}(t,\cdot) : \mathbf{U} \to (-\infty,\infty] \text{ and } \mathcal{Z}(t,\cdot) : \mathbf{Z} \to (-\infty,\infty] \text{ proper and lsc,} \\ &\text{and} \quad \mathcal{F}(t,\cdot) : \mathbf{U}\times\mathbf{Z} \to \mathbb{R} \text{ Fréchet differentiable.} \end{aligned}$$

Lemma 4.11. *Assume Hypotheses 4.2, 4.3, 4.5, as well as* (4.23). *Then,*

$$\mathscr{S}^*_{\mathsf{u}}(t,q) = \inf_{\mu\in\partial_{\mathsf{u}}\mathcal{E}(t,q)} \mathcal{V}^*_{\mathsf{u}}(-\mu) \quad \textit{and} \quad \mathscr{S}^*_{\mathsf{z}}(t,q) = \inf_{\zeta\in\partial_{\mathsf{z}}\mathcal{E}(t,q)} \mathcal{W}^*_{\mathsf{z}}(-\zeta) \tag{4.24}$$

for all $(t,q) \in [0,T]\times\mathrm{dom}(\partial_q\mathcal{E})$, *and properties* (4.21) *and* (4.22) *hold.*

Proof. Obviously, for $(t,q) \in \mathrm{dom}(\partial_q\mathcal{E})$ we have (4.24) as a consequence of (4.23). We will just check the attainment (4.21) and the lower semicontinuity (4.22) for $\mathscr{S}^*_{\mathsf{z}}$, as the properties for $\mathscr{S}^*_{\mathsf{u}}$ follow by the same arguments.

Suppose that $(t_n,q_n) \rightharpoonup (t,q)$ and $\liminf_{n\to\infty} \mathscr{S}^*_{\mathsf{z}}(t_n,q_n) < \infty$. Using (4.24), up to a subsequence, there exist $(\zeta_n) \subset \mathbf{Z}^*$ with $\zeta_n \in \partial_{\mathsf{z}}\mathcal{E}(t_n,q_n)$ and $(\sigma_n)_n \subset \partial\mathcal{R}(0) \subset \mathbf{Z}^*$ for all n with

$$\lim_{n\to\infty} \mathcal{V}^*_{\mathsf{z}}(-\zeta_n - \sigma_n) = \lim_{n\to\infty} \mathscr{S}^*_{\mathsf{z}}(t_n,q_n) \leq C\,.$$

It follows from (4.2) that the sequence $(\sigma_n + \zeta_n)_n$ is bounded in $\mathbf{Z}^*$. Since, in view of (4.6), $(\sigma_n)_n$ is bounded in $\mathbf{Z}^*$, $(\zeta_n)_n$ turns out to be bounded in $\mathbf{Z}^*$, too. Then, up to a subsequence we have $\sigma_n \rightharpoonup \sigma$ in

$\mathbf{Z}^*$ and $\zeta_n \rightharpoonup \zeta$ in $\mathbf{Z}^*$. Since $\partial\mathcal{R}(0)$ is sequentially weakly closed in $\mathbf{Z}^*$, we find $\sigma \in \partial\mathcal{R}(0)$. By Hypothesis 4.5 we also have $\zeta \in \partial_{\mathsf{z}}\mathcal{E}(t,q)$, hence

$$\begin{aligned}\lim_{n\to\infty} \mathscr{S}^*_{\mathsf{z}}(t_n,q_n) &= \lim_{n\to\infty} \mathcal{V}^*_{\mathsf{z}}(-\zeta_n-\sigma_n) \geq \mathcal{V}^*_{\mathsf{z}}(-\zeta-\sigma)\\ &\geq \mathcal{W}^*_{\mathsf{z}}(-\zeta) \geq \inf_{\widetilde{\zeta}\in\partial_{\mathsf{z}}\mathcal{E}(t,q)} \mathcal{W}^*_{\mathsf{z}}(-\widetilde{\zeta}) = \mathscr{S}^*_{\mathsf{z}}(t,q),\end{aligned}$$

which is the desired lsc (4.22) for $\mathscr{S}^*_{\mathsf{z}}$.

With similar arguments we deduce the attainment (4.21). □

In the above proof we have used in an essential way that $\partial\mathcal{R}(0)$ is bounded in $\mathbf{Z}^*$ by our assumption (4.3b). Without this property, the argument still goes through provided that, given a sequence $(q_n)_n \subset \mathbf{Q}$ as in Hypothesis 4.10, all sequences $(\zeta_n)_n$ with $\zeta_n \in \mathfrak{A}^*_{\mathsf{z}}(t_n,q_n)$ for all $n\in\mathbb{N}$ happen to be bounded in $\mathbf{U}^*{\times}\mathbf{Z}^*$, which can be, of course, an additional property of the subdifferential $\partial_{\mathsf{z}}\mathcal{E}$.

Throughout the rest of this paper, we will always tacitly assume the validity of Hypotheses 4.1, 4.2, 4.3, 4.5, 4.7, and 4.10 and omit any explicit mentioning of them in most of the upcoming results (with the exception of our main existence theorems).

4.6. A priori estimates for the viscous solutions.

Let $(q_\varepsilon)_\varepsilon$ be a family of solutions to the viscously regularized systems (1.4) in the stricter sense of (4.18), which includes the energy-dissipation balance (4.18b). By Theorem 4.8 the existence of solutions $q_\varepsilon = (u_\varepsilon, z_\varepsilon)$ is guaranteed, and in this subsection we discuss some a priori estimates on $(u_\varepsilon,z_\varepsilon)_\varepsilon$ that are uniform with respect to the parameter ε and that form the core of our vanishing-viscosity analysis.

The starting point is the energy-dissipation estimate (4.20) that follows directly from (4.18b). Recalling the constant $C_\#$ from (4.7d) in Hypothesis 4.3 and $c_\mathcal{R}$ in Hypotheses 4.2, we see that the following *basic a priori estimates*, are valid under the *sole* assumptions of Hypotheses 4.2 and 4.3.

Lemma 4.12 (Basic a priori estimates)**.** *For all* $\varepsilon>0$ *and all solutions* $q_\varepsilon=(u_\varepsilon,z_\varepsilon):[0,T]\to\mathbf{Q}=\mathbf{U}{\times}\mathbf{Z}$ *of* (4.18) *with* $\mathcal{E}(0,q_\varepsilon(0))<\infty$ *we have the a priori estimates*

$$\int_0^T \Big(\frac{1}{\varepsilon^\alpha}\mathcal{V}_{\mathsf{u}}(\varepsilon^\alpha u'_\varepsilon(t)) + \mathcal{R}(z'_\varepsilon(t)) + \frac{1}{\varepsilon}\mathcal{V}_{\mathsf{z}}(\varepsilon z'_\varepsilon(t)) + \frac{\mathscr{S}^*_{\mathsf{u}}(t,q_\varepsilon(t))}{\varepsilon^\alpha} + \frac{\mathscr{S}^*_{\mathsf{z}}(t,q_\varepsilon(t))}{\varepsilon}\Big)\,\mathrm{d}t \leq \mathrm{e}^{C_\# T}\mathcal{E}(0,q_\varepsilon(0)), \tag{4.25a}$$

$$0 \leq \mathcal{E}(t,q_\varepsilon(t)) \leq \mathrm{e}^{C_\# t}\mathcal{E}(0,q_\varepsilon(0)) \text{ for all } t\in[0,T]. \tag{4.25b}$$

whence, in particular,

$$\|z'_\varepsilon\|_{\mathrm{L}^1(0,T;\mathbf{Z}_{\mathrm{ri}})} \leq \frac{\mathrm{e}^{C_\# T}}{c_\mathcal{R}}\,\mathcal{E}(0,q_\varepsilon(0)) \quad \text{and} \quad \sup_{t\in[0,T]} \mathfrak{E}(q_\varepsilon(t)) \leq \mathrm{e}^{2C_\# T}\mathcal{E}(0,q_\varepsilon(0))\,. \tag{4.26}$$

Proof. The proof follows as in the purely rate-independent case treated in [Mie05, Cor 3.3]. We start from (4.18b) and drop the nonnegative dissipation to obtain

$$\mathcal{E}(t,q_\varepsilon(t)) \leq \mathcal{E}(0,q_\varepsilon(0)) + \int_0^t \partial_s\mathcal{E}(s,q_\varepsilon(s))\,\mathrm{d}s \leq \mathcal{E}(0,q_\varepsilon(0)) + \int_0^t C_\#\mathcal{E}(s,q_\varepsilon(s))\,\mathrm{d}s,$$

where we used (4.7d). Thus, Grönwall's estimate gives (4.25b) and this we find

$$\mathcal{E}(0,q_\varepsilon(0)) + \int_0^T \partial_s\mathcal{E}(s,q_\varepsilon(s))\,\mathrm{d}s \leq \mathcal{E}(0,q_\varepsilon(0)) + \int_0^T C_\#\mathrm{e}^{C_\# s}\mathcal{E}(0,q_\varepsilon(0))\,\mathrm{d}s = \mathrm{e}^{C_\# T}\mathcal{E}(0,q_\varepsilon(0))$$

and (4.25a) is established as well, as $\mathcal{E}(T;q_\varepsilon(T)) \geq C_0 > 0$ by (4.7b).

Since $\mathcal{V}_{\mathsf{x}}$ and $\mathscr{S}^*_{\mathsf{x}}$ are nonnegative, assumption (4.3b) leads to the first estimate in (4.26). The last assertion follows from (4.25b) and applying (4.7d) once again. □

Clearly, (4.26) provides a uniform bound on the total variation of the solution component z_ε in the space $\mathbf{Z}_{\mathrm{ri}}$. A similar bound cannot be expected for the components u_ε, unless we add further assumptions. To see the problem consider $\mathbf{U}=\mathbb{R}^2$ and the ordinary differential equation

$$\varepsilon^\alpha u'_\varepsilon(t) + \mathrm{D}\varphi(u_\varepsilon(t)) = z_\varepsilon(t) = a\begin{pmatrix}\cos(\omega t)\\ \sin(\omega t)\end{pmatrix}, \quad \text{where } \varphi(u) = \frac{\lambda}{2}|u|^2 + \frac{1}{2}\max\{|u|{-}1,0\}^2$$

with $\lambda \geq 0$. Note that φ is uniformly coercive for all $\lambda \geq 0$. However, the equation is linear for $|u| \leq 1$ and has an exact periodic solution of the form

$$u(t) = \big(\mathrm{Re}\, U(t), \mathrm{Im}\, U(t)\big) \quad \text{with } U(t) = \frac{a}{\lambda{+}\mathrm{i}\,\omega\varepsilon^\alpha}\,\mathrm{e}^{\mathrm{i}\,\omega t} \in \mathbb{C},$$

as long as $|U(t)| \leq 1$, i.e. $a^2 \leq \lambda^2{+}\omega^2\varepsilon^{2\alpha}$. In this case, the derivatives satisfy the following L^1-estimates

$$\|u_\varepsilon'\|_{\mathrm{L}^1(0,T)} = |\omega| T\, \|u_\varepsilon\|_{\mathrm{L}^\infty} = \Big|\frac{a\omega}{\lambda{+}\mathrm{i}\,\omega\varepsilon^\alpha}\Big|\, T = \frac{1}{(\lambda^2{+}\omega^2\varepsilon^{2\alpha})^{1/2}}\, \|z_\varepsilon'\|_{\mathrm{L}^1(0,T)}.$$

For $\lambda > 0$ we thus obtain a bound on $\|u_\varepsilon'\|_{\mathrm{L}^1(0,T)}$ from a bound on $\|z_\varepsilon'\|_{\mathrm{L}^1(0,T)}$ as in (4.26). However, in the case $\lambda = 0$ the value $\|u_\varepsilon'\|_{\mathrm{L}^1(0,T)}$ may blow up while $\|z_\varepsilon'\|_{\mathrm{L}^1(0,T)}$ remains bounded (or even tends to 0) and $a^2 \leq \omega^2\varepsilon^{2\alpha}$, e.g. choosing $\omega = \varepsilon^{-\alpha/2}$ and $a = \varepsilon^{2\alpha/\alpha}$.

In the main part of this subsection, we provide sufficient conditions for the validity of a uniform bound on $\|u_\varepsilon'\|_{\mathrm{L}^1(0,T;\mathbf{U})}$. In the spirit of the above ODE example we assume that $u \mapsto \mathcal{E}(t,u,z)$ is uniformly convex (i.e. $\lambda > 0$) and that $z \mapsto \mathrm{D}_u\mathcal{E}(t,u,z)$ is Lipschitz. Moreover, we need to assume that $\mathcal{V}_{\mathsf{u}}$ is quadratic. More precisely, we have to confine the discussion to a special setup given by conditions (4.27) and (4.28):
(1) the dissipation potential $\mathcal{V}_{\mathsf{u}}$ is quadratic:

$$\mathbf{U} \text{ is a Hilbert space} \quad \text{and } \mathcal{V}_{\mathsf{u}}(v) := \frac{1}{2}\|v\|_{\mathbf{U}}^2 = \frac{1}{2}\langle \mathbb{V}_{\mathsf{u}} v, v\rangle, \tag{4.27}$$

where $\mathbb{V}_{\mathsf{u}} : \mathbf{U} \to \mathbf{U}^*$ is Riesz' norm isomorphism;
(2) the energy functional $\mathcal{E}$ has domain $\mathrm{D} = \mathrm{D}_{\mathsf{u}}{\times}\mathrm{D}_{\mathsf{z}}$ and admits the decomposition

$$\mathcal{E}(t,u,z) = \mathcal{E}_1(u) + \mathcal{E}_2(t,u,z) \quad \text{with} \tag{4.28a}$$

$$\exists\, \Lambda > 0: \quad \mathcal{E}_1 \text{ is } \Lambda\text{-convex}, \tag{4.28b}$$

$$\forall\,(t,z) \in [0,T]{\times}\mathrm{D}_{\mathsf{z}}: \quad u \mapsto \mathcal{E}_2(t,u,z) \text{ is Fréchet differentiable on } \mathrm{D}_{\mathsf{u}}, \tag{4.28c}$$

$$\begin{aligned} &\exists\, C_{\mathsf{u}} \in (0,\Lambda)\ \forall\, E > 0\ \exists\, C_E > 0\ \forall\, t_1,t_2 \in [0,T]\ \forall\,(u_1,z_1),(u_2,z_2) \in \mathcal{S}_E: \\ &\qquad \|\mathrm{D}_{\mathsf{u}}\mathcal{E}_2(t_1,u_1,z_1) {-} \mathrm{D}_{\mathsf{u}}\mathcal{E}_2(t_2,u_2,z_2)\|_{\mathbf{U}^*} \\ &\qquad\qquad \leq C_E\,(|t_1{-}t_2| + \|z_1{-}z_2\|_{\mathbf{Z}_{\mathrm{ri}}}) + C_{\mathsf{u}}\|u_1{-}u_2\|_{\mathbf{U}} \end{aligned} \tag{4.28d}$$

where $\mathcal{S}_E$ denotes the sublevel of $\mathfrak{E}$, cf. (4.8).

Hence, the possibly nonsmooth, but *uniformly convex* functional $\mathcal{E}_1$ is perturbed by the smooth, but possibly nonconvex, functional $u \mapsto \mathcal{E}_2(t,u,z)$. However, by $C_{\mathsf{u}} < \Lambda$ the mapping $u \mapsto \mathcal{E}(t,u,z)$ is still uniformly convex.

Unfortunately condition (4.28) is rather restrictive, because in concrete examples the driving energy functional features a coupling between the variables u and z that is more complex. Nevertheless the desired a priori estimate derived in Proposition 4.13 may still be valid. Indeed, for our delamination model examined in Section 8 we establish the corresponding estimate via an *ad hoc* approach for the specific system.

The proof of the following results follows the technique for the a priori estimate developed in [Mie11, Prop. 4.17]. We emphasize that the two additional assumptions (4.27) and (4.28) yield that the solution u_ε for $\mathbb{V}_{\mathsf{u}}u' + \partial\mathcal{E}_1(u) + \mathrm{D}\mathcal{E}_1(t,u,z_\varepsilon(t)) \ni 0$ is unique as long as z_ε is kept fixed, since it is a classical Hilbert-space gradient flow for a time-dependent, convex functional.

Proposition 4.13 (L^1 bound on u_ε')**.** *In addition to Hypotheses 4.2 and 4.3 assume* (4.27) *and* (4.28) *and consider initial conditions* $(q_\varepsilon^0)_\varepsilon$ *such that*

$$\exists\, C_{\mathrm{init}} > 0\ \forall\, \varepsilon \in (0,1): \quad \mathcal{E}(0,q_\varepsilon^0) + \varepsilon^{-\alpha}\|\partial_{\mathsf{u}}^0\mathcal{E}(0,q_\varepsilon^0)\|_{\mathbf{U}^*} \leq C_{\mathrm{init}} < \infty,$$

where $\partial_{\mathsf{u}}^0\mathcal{E}(0,q_\varepsilon^0) \subset \mathbf{U}^*$ *denotes the unique element of minimal norm in* $\partial_{\mathsf{u}}\mathcal{E}(0,q_\varepsilon^0) \subset \mathbf{U}^*$*. Then, there exists a* $C > 0$ *such that for all* $\varepsilon \in (0,1)$ *all solutions* $q_\varepsilon = (u_\varepsilon, z_\varepsilon)$ *of system* (4.18) *with* $q_\varepsilon(0) = q_\varepsilon^0$ *satisfy*

$$\begin{aligned} \|u_\varepsilon'\|_{\mathrm{L}^1(0,T;\mathbf{U})} &\leq \frac{1}{\Lambda{-}C_{\mathsf{u}}}\Big(C_{\mathrm{init}} + C_E T + C_E\|z_\varepsilon'\|_{\mathrm{L}^1(0,T;\mathbf{Z}_{\mathrm{ri}})}\Big) \\ &\leq \frac{1}{\Lambda{-}C_{\mathsf{u}}}\Big(C_{\mathrm{init}} + C_E T + \frac{C_E C_{\mathrm{init}}}{c_{\mathcal{R}}}\,\mathrm{e}^{C_\# T}\Big). \end{aligned} \tag{4.29}$$

Proof. By Lemma 4.12 all curves $q_\varepsilon : [0,T] \to \mathbf{Q}$ lie in $\mathcal{S}_E = \big\{\, q \in \mathbf{Q} \,\big|\, \mathfrak{E}(q) \leq E \,\big\}$ for $E = \mathrm{e}^{2C_\# T}C_{\mathrm{init}}$. Throughout the rest of this proof we drop the subscripts ε at $q_\varepsilon = (u_\varepsilon, z_\varepsilon)$, but keep all constant explicitly to emphasize that they do not depend on ε.

Setting $\kappa = \Lambda - C_\mathsf{u} > 0$, the uniform convexity of $\mathcal{E}(t,\cdot,z)$ gives $\langle \mu_1 - \mu_2, u_1 - u_2\rangle \geq \kappa \|u_1 - u_2\|^2_\mathbf{U}$ for all $\mu_j \in \partial_\mathsf{u}\mathcal{E}(t,u_j,z)$. We write the equation for u in the form $0 = \varepsilon^\alpha \mathbb{V}_\mathsf{u} u'(t) + \mu(t)$ with $\mu(t) \in \partial_\mathsf{u}\mathcal{E}(t,u(t),z(t))$. For small $h>0$ and $t \in [0,T{-}h]$ we find

$$\begin{aligned}\frac{\varepsilon^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t{+}h)-u(t)\|^2_\mathbf{U} &= \big\langle \varepsilon^\alpha \mathbb{V}_\mathsf{u}(u'(t{+}h)-u'(t)), u(t{+}h)-u(t)\big\rangle \\ &= -\langle \mu(t{+}h)-\mu(t), u(t{+}h)-u(t)\rangle \\ &\leq -\langle \widetilde\mu_h(t)-\mu(t), u(t{+}h)-u(t)\rangle + \|\widetilde\mu_h(t) - \mu(t{+}h)\|_{\mathbf{U}^*}\|u(t{+}h)-u(t)\|_\mathbf{U},\end{aligned}$$

where $\widetilde\mu_h(t) \in \partial_\mathsf{u}\mathcal{E}(t,u(t{+}h),z(t))$. The uniform convexity and (4.28d) give

$$\frac{\varepsilon^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\varrho_h(t)^2 \leq -\kappa\varrho_h(t)^2 + C_E\big(h + \|z(t{+}h)-z(t)\|_{\mathbf{Z}_\mathrm{ri}}\big)\varrho_h(t),$$

where $\varrho_h(t) := \|u(t{+}h)-u(t)\|_\mathbf{U}$. Choosing $\delta>0$ and setting $\nu_h(t) := \varrho_h(t)^2{+}\delta$ yields

$$\begin{aligned}\varepsilon^\alpha \dot\nu_h = \frac{\varepsilon^\alpha \frac{\mathrm{d}}{\mathrm{d}t}\varrho^2_\mathrm{H}}{2\nu_h} &\leq -\kappa\frac{\nu_n^2-\delta}{\nu_h} + C_E\big(h + \|z(\cdot+h)-z\|_{\mathbf{Z}_\mathrm{ri}}\big)\frac{\varrho_h}{\nu_h} \\ &\leq -\kappa\nu_h + \kappa\delta^{1/2} + C_E\big(h + \|z(\cdot+h)-z\|_{\mathbf{Z}_\mathrm{ri}}\big).\end{aligned}$$

Integrating this inequality in time we arrive at

$$\kappa\int_0^{T-h}\varrho_h(t)\,\mathrm{d}t \leq \kappa\int_0^{T-h}\nu_h(t)\,\mathrm{d}t \leq \varepsilon^\alpha\nu_h(0) + \delta^{1/2}T + C_E hT + C_E\int_0^{T-h}\|z(t{+}h)-z(t)\|_{\mathbf{Z}_\mathrm{ri}}\,\mathrm{d}t.$$

Taking the limit $\delta \to 0^+$, dividing by $h>0$, and using $\|z(t{+}h)-z(t)\|_{\mathbf{Z}_\mathrm{ri}} \leq \int_t^{t+h}\|z'(s)\|_{\mathbf{Z}_\mathrm{ri}}\,\mathrm{d}s$ gives

$$\kappa\int_0^{T-h}\Big\|\frac{1}{h}\big(u(t{+}h)-u(t)\big)\Big\|_\mathbf{U}\,\mathrm{d}t \leq \varepsilon^\alpha\Big\|\frac{1}{h}\big(u(0{+}h)-u(0)\big)\Big\|_\mathbf{U} + C_E T + C_E\int_0^T\|z'(t)\|_{\mathbf{Z}_\mathrm{ri}}\,\mathrm{d}t.$$

Since the equation for u is a Hilbert-space gradient flow we can apply [Bré73, Thm. 3.1], which shows that $\frac{1}{h}(u(h){-}u(0)) \to \partial^0_\mathsf{u}\mathcal{E}(0,u(0),z(0))$ for $h \to 0^+$. Thus, in the limit $\varepsilon \to 0^+$ we find

$$\kappa\int_0^T\|u'(t)\|_\mathbf{U}\,\mathrm{d}t = \lim_{h\to 0^+}\kappa\int_0^{T-h}\big\|\tfrac{1}{h}\big(u(t{+}h)-u(t)\big)\big\|_\mathbf{U}\,\mathrm{d}t \leq C_\mathrm{init} + C_E T + C_E\int_0^T\|z'(t)\|_{\mathbf{Z}_\mathrm{ri}}\,\mathrm{d}t,$$

which is the desired result, when recalling $\kappa = \Lambda - C_\mathsf{u}$. ☐

The above result is valid for all solutions of the viscous system (4.18), but it relies on the rather strong assumptions (4.27) and (4.28). While the uniform convexity of $u \mapsto \mathcal{E}(t,u)z$ in (4.28) seems to be fundamental, it is expected that the rather strong assumption that $\mathcal{V}_\mathsf{u}$ is the square of a Hilbert space norm, see (4.27), can be relaxed, but then the solution u_ε may no longer be uniquely determined for fixed z_ε. In that case it may be helpful to restrict the analysis to specific solution classes satisfying better a priori estimates, e.g. to minimizing movements obtained via time-incremental minimization problems as in [MRS16a, Thm. 3.23] or to solutions obtained as limit of Galerkin approximations as in [MiZ14, Def. 4.3 & Thm. 4.13]. We also refer to our delamination model in Section 8 for a derivation of the additional a priori estimate (4.29) in a more difficult case.

5. Parametrized Balanced-Viscosity solutions

In this section we will give the definition of Balanced-Viscosity solution to the rate-independent system $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{V}_\mathsf{u}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_\mathsf{z}^\varepsilon)_{\varepsilon\downarrow 0}$ in a *parametrized version*. For this, we study instead of the viscous solutions $q_\varepsilon : [0,T] \to \mathbf{Q}$ suitable reparametrizations $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon) : [0,\mathsf{S}_\varepsilon] \to [0,T]{\times}\mathbf{Q}$, i.e., $\mathsf{q}_\varepsilon(s) = q_\varepsilon(\mathsf{t}_\varepsilon(s))$, see Section 5.1. While quite general reparametrizations are possible, we will perform the vanishing-viscosity limit $\varepsilon \to 0^+$ for the one given in terms of the energy-dissipation arclength $s = \mathsf{s}_\varepsilon(t)$ defined in terms of the rescaled joint M-function $\mathfrak{M}^\alpha_\varepsilon$ arising from the rescaled joint B-function $\mathfrak{B}^\alpha_\varepsilon$, see (5.2). The Γ-limit $\mathfrak{M}^\alpha_0$ of $\mathfrak{M}^\alpha_\varepsilon$, which is called the *limiting rescaled joint M-function*, will then be used, to introduce the concept of *admissible parametrized curves*, see Definition 5.6 in Section 5.2. This is the basis of our notion of for *parametrized Balanced-Viscosity* (pBV) *solutions*, defined in Section 5.3. Theorem 5.11 states our main existence result for pBV solutions, which is based on the convergence in the vanishing-viscosity limit $\varepsilon \to 0^+$. However, we emphasize that the notion of 'pBV solutions' is independent of the limiting procedure. Finally, in Section 5.5 we provide a characterization of (enhanced) pBV solutions showing that they are indeed solutions of the time-rescaled generalized gradient system (1.13).

5.1. Reparametrization and rescaled joint M-functions.

This subsection revolves around the concept and the properties of the limiting rescaled joint M-function $\mathfrak{M}_0^\alpha$ that will be introduced in the Definition 5.1. First, we will prove that $\mathfrak{M}_0^\alpha$ is the Γ-limit of the family of M-functions $(\mathfrak{M}_\varepsilon^\alpha)_\varepsilon$ that appear naturally in the reparametrized version of the energy-dissipation estimate (4.20) and that are given by a composition of the rescaled joint B-function $\mathfrak{B}_\varepsilon^\alpha$ and the slopes $\mathscr{S}_{\mathsf{x}}^*$. Namely, the *rescaled joint M-functions* are defined by

$$
\begin{aligned}
&\mathfrak{M}_\varepsilon^\alpha : [0,T]\times \mathrm{D}\times[0,\infty)\times\mathbf{Q} \to [0,\infty],\\
&\mathfrak{M}_\varepsilon^\alpha(t,q,t',q') := \begin{cases} \mathfrak{B}_\varepsilon^\alpha(t',u',z',\mathscr{S}_{\mathsf{u}}^*(t,q),\mathscr{S}_{\mathsf{z}}^*(t,q)) & \text{for } \partial_{\mathsf{q}}\mathcal{E}(t,q)\neq\emptyset,\\ \infty & \text{otherwise;}\end{cases}
\end{aligned}
\tag{5.1}
$$

where $\mathfrak{B}_\varepsilon^\alpha$ is the rescaled joint B-function from (3.14) associated with the dissipation potentials $\psi_{\mathsf{u}} = \mathcal{V}_{\mathsf{u}}$ and $\psi_{\mathsf{z}} = \mathcal{R}+\mathcal{V}_{\mathsf{z}}$.

The basis for the construction of parametrized BV solutions is the reparametrization of the the viscous solutions $q_\varepsilon : [0,T]\to\mathbf{Q}$ in the form $\mathsf{q}_\varepsilon(s) = q_\varepsilon(\mathsf{t}_\varepsilon(s))$ such that the behavior of the function $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon) : [0,\mathsf{S}_\varepsilon]\to[0,T]\times\mathbf{Q}$ is advantageous. In particular, the formation of jumps in q_ε with $\|q_\varepsilon'(t)\|\approx 1/\varepsilon$ can be modeled by a plateau-like behavior of t_ε with $\mathsf{t}_\varepsilon'(s)\approx\varepsilon$ and a soft transition of q_ε with $\|\mathsf{q}_\varepsilon'(s)\|\approx 1$. The first usage of such reparametrizations for the vanishing-viscosity limit goes back to [EfM06], but here we stay close to [MRS16a, Sec. 4.1] in using an 'energy-based time reparametrization'. Hence, for a family $(q_\varepsilon)_\varepsilon = (u_\varepsilon,z_\varepsilon)_\varepsilon$ of solutions to (1.4) for which the estimates from Lemma 4.12 hold, as well as the additional a priori estimate (4.29) on $\int_0^T\|u_\varepsilon'\|_{\mathbf{U}}\,\mathrm{d}t$, we reparametrize the functions q_ε using the *energy-dissipation arclength* $\mathsf{s}_\varepsilon : [0,T]\to[0,\mathsf{S}_\varepsilon]$ with $\mathsf{S}_\varepsilon := \mathsf{s}_\varepsilon(T)$ (cf. [MRS16a, (4.3)]) defined by

$$
\begin{aligned}
\mathsf{s}_\varepsilon(t) := \int_0^t \Big(1 &+ \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}(u_\varepsilon'(t)) + \mathcal{R}(z_\varepsilon'(t)) + \mathcal{V}_{\mathsf{z}}^{\varepsilon}(z_\varepsilon'(t))\\
&+ \frac{\mathscr{S}_{\mathsf{u}}^*(t,q_\varepsilon(t))}{\varepsilon^\alpha} + \frac{\mathscr{S}_{\mathsf{z}}^*(t,q_\varepsilon(t))}{\varepsilon} + \|u_\varepsilon'(t)\|_{\mathbf{U}}\Big)\,\mathrm{d}t\,,
\end{aligned}
\tag{5.2}
$$

such that estimates (4.25) and (4.29) yield that $\sup_{\varepsilon>0}\mathsf{S}_\varepsilon\le C$. Below we consider the reparametrized curves $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon) : [0,\mathsf{S}_\varepsilon]\to[0,T]\times\mathbf{Q}$ defined by $\mathsf{t}_\varepsilon := \mathsf{s}_\varepsilon^{-1}$, $\mathsf{q}_\varepsilon := q_\varepsilon\circ\mathsf{t}_\varepsilon$ and show in Section 5.3 that they have an absolutely continuous limit (t,q), up to choosing a subsequence.

We first remark that the quantities involved in the definition of s_ε rewrite as

$$
\mathcal{R}(z_\varepsilon') + \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}(u_\varepsilon') + \mathcal{V}_{\mathsf{z}}^{\varepsilon}(z_\varepsilon') + \frac{\mathscr{S}_{\mathsf{u}}^*(t,q_\varepsilon)}{\varepsilon^\alpha} + \frac{\mathscr{S}_{\mathsf{z}}^*(t,q_\varepsilon)}{\varepsilon} \;=\; \mathfrak{M}_\varepsilon^\alpha(t,q_\varepsilon,1,q_\varepsilon'). \tag{5.3}
$$

With this, the energy-dissipation estimate (4.20) can be rewritten in terms of the parametrized curves $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon)$ in the form (for all $0\le s_1<s_2\le\mathsf{S}_\varepsilon$)

$$
\begin{aligned}
&\mathcal{E}(\mathsf{t}_\varepsilon(s_2),\mathsf{q}_\varepsilon(s_2)) + \int_{s_1}^{s_2}\mathfrak{M}_\varepsilon^\alpha(\mathsf{t}_\varepsilon(\sigma),\mathsf{q}_\varepsilon(\sigma),\mathsf{t}_\varepsilon'(\sigma),\mathsf{q}_\varepsilon'(\sigma))\,\mathrm{d}\sigma\\
&\le \mathcal{E}(\mathsf{t}_\varepsilon(s_1),\mathsf{q}_\varepsilon(s_1)) + \int_{s_1}^{s_2}\partial_t\mathcal{E}(\mathsf{t}_\varepsilon(\sigma),\mathsf{q}_\varepsilon(\sigma))\,\mathsf{t}_\varepsilon'(\sigma)\,\mathrm{d}\sigma\,.
\end{aligned}
\tag{5.4}
$$

Moreover, the definition of s_ε in (5.2) is equivalent to the normalization condition

$$
\mathsf{t}_\varepsilon'(s) + \mathfrak{M}_\varepsilon^\alpha(\mathsf{t}_\varepsilon(s),\mathsf{q}_\varepsilon(s),\mathsf{t}_\varepsilon'(s),\mathsf{q}_\varepsilon'(s)) + \|\mathsf{u}_\varepsilon'(s)\|_{\mathbf{U}} = 1 \qquad \text{for a.a. } s\in(0,\mathsf{S}_\varepsilon)\,. \tag{5.5}
$$

Of course, the reparametrized solutions q_ε inherit the energy estimate (4.26), namely

$$
\sup_{s\in[0,\mathsf{S}_\varepsilon]}\mathfrak{E}(\mathsf{q}_\varepsilon(s)) \le \mathrm{e}^{2C_\# T}\sup_{\varepsilon\in(0,1)}\mathcal{E}(0,\mathsf{q}_\varepsilon(0))\,. \tag{5.6}
$$

The a priori estimates (5.5) and (5.6) for the reparametrized curves $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon)_\varepsilon$ will be strong enough to ensure their convergence along a subsequence, as $\varepsilon\to0^+$, to a curve $(\mathsf{t},\mathsf{q}) : [0,\mathsf{S}]\to[0,T]\times\mathbf{Q}$, with $\mathsf{S} = \lim_{\varepsilon\to0^+}\mathsf{S}_\varepsilon$. The basic properties of (t,q) are fixed in the concept of *admissible parametrized curve*, see Definition 5.6.

For studying the limit $\varepsilon\to0^+$, we need to bring into play the limiting rescaled joint M-function $\mathfrak{M}_0^\alpha$.

Definition 5.1. *We define* $\mathfrak{M}_0^\alpha : [0,T]\times\mathrm{D}\times[0,\infty)\times\mathbf{Q}\to[0,\infty]$ *via*

$$
\mathfrak{M}_0^\alpha(t,q,t',q') := \begin{cases} \mathfrak{B}_0^\alpha(t',u',z',\mathscr{S}_{\mathsf{u}}^*(t,q),\mathscr{S}_{\mathsf{z}}^*(t,q)) & \textit{for } \partial_{\mathsf{q}}\mathcal{E}(t,q)\neq\emptyset,\\ 0 & \textit{for } t'=0,\ q'=0 \textit{ and}\\ & (t,q)\in\overline{\mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E})}^{\mathrm{w,S}}\backslash\mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E}),\\ \infty & \textit{otherwise,}\end{cases}
\tag{5.7}
$$

where $\mathfrak{B}_0^\alpha$ *is defined in Proposition 3.6 and* $\overline{\mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E})}^{\mathrm{w,S}}$ *is the weak closure of* $\mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E})$ *confined to energy sublevels:*

$$\overline{\mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E})}^{\mathrm{w,S}} := \left\{ (t,q) \,\middle|\, \exists\, (t_n,q_n)_n \subset \mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E})\colon (t_n,q_n) \rightharpoonup (t,q),\ \sup_n \mathfrak{E}(q_n) < \infty \right\}. \tag{5.8}$$

It follows from Proposition 3.6 that

$$(t',q') \mapsto \mathfrak{M}_0^\alpha(t,q,t',q') \text{ is convex and 1-homogeneous for all } (t,q) \in [0,\infty)\times\mathbf{Q}. \tag{5.9}$$

Relying on Proposition 3.6 and Hypothesis 4.10, we are ready to prove the following Γ-convergence result, which straightforwardly gives that $\mathfrak{M}_0^\alpha$ is (sequentially) lower semicontinuous with respect to the weak topology of $\mathbb{R}\times\mathbf{Q}\times\mathbb{R}\times\mathbf{Q}$ along sequences with bounded energy.

Proposition 5.2 (Weak Γ-convergence of M-functions)**.** *The limiting M-function* $\mathfrak{M}_0^\alpha : [0,T]\times\mathrm{D}\times[0,\infty)\times\mathbf{Q} \to [0,\infty]$ *is the* Γ*-limit of the M-functions* $(\mathfrak{M}_\varepsilon^\alpha)_\varepsilon$*, with respect to the weak topology,* along sequences with bounded energy*, namely the following assertions hold:*

(a) Γ-lim inf *estimate:*

$$\begin{aligned}&\Big((t_\varepsilon,q_\varepsilon,t'_\varepsilon,q'_\varepsilon) \rightharpoonup (t,q,t',q') \text{ in } \mathbb{R}\times\mathbf{Q}\times\mathbb{R}\times\mathbf{Q} \text{ as } \varepsilon\to 0^+ \quad \text{with } \sup_{\varepsilon>0}\mathfrak{E}(q_\varepsilon)<\infty\Big)\\ &\quad\Longrightarrow \quad \mathfrak{M}_0^\alpha(t,q,t',q') \le \liminf_{\varepsilon\to0^+} \mathfrak{M}_\varepsilon^\alpha(t_\varepsilon,q_\varepsilon,t'_\varepsilon,q'_\varepsilon);\end{aligned} \tag{5.10a}$$

(b) Γ-lim sup *estimate:*

$$\begin{aligned}&\forall\,(t,q,t',q') \in [0,T]\times\mathrm{D}\times[0,\infty)\times\mathbf{Q}\ \exists\,(t_\varepsilon,q_\varepsilon,t'_\varepsilon,q'_\varepsilon)_\varepsilon \text{ such that}\\ &\quad\text{(i)}\quad (t_\varepsilon,q_\varepsilon,t'_\varepsilon,q'_\varepsilon) \rightharpoonup (t,q,t',q') \text{ in } \mathbb{R}\times\mathbf{Q}\times\mathbb{R}\times\mathbf{Q} \text{ as } \varepsilon\to0^+,\\ &\quad\text{(ii)}\quad \sup\nolimits_{\varepsilon>0}\mathfrak{E}(q_\varepsilon)<\infty, \text{ and}\\ &\quad\text{(iii)}\quad \mathfrak{M}_0^\alpha(t,q,t',q') \ge \limsup\nolimits_{\varepsilon\to0^+}\mathfrak{M}_\varepsilon^\alpha(t_\varepsilon,q_\varepsilon,t'_\varepsilon,q'_\varepsilon)\,.\end{aligned} \tag{5.10b}$$

Proof. Concerning (a), let $(t_\varepsilon,q_\varepsilon,t'_\varepsilon,q'_\varepsilon)_\varepsilon$ be a sequence as in (5.10a). Of course we can suppose that $\liminf_{\varepsilon\to0^+}\mathfrak{M}_\varepsilon^\alpha(t_\varepsilon,q_\varepsilon,t'_\varepsilon,q'_\varepsilon)<\infty$, and thus that $\sup_\varepsilon \mathfrak{M}_\varepsilon^\alpha(t_\varepsilon,q_\varepsilon,t'_\varepsilon,q'_\varepsilon)<\infty$. Then, there exists $\bar\varepsilon>0$ such that for all $\varepsilon\in(0,\bar\varepsilon)$ there holds $t'_\varepsilon>0$, the Fréchet subdifferential $\partial_{\mathsf{q}}\mathcal{E}(t_\varepsilon,q_\varepsilon)$ is non-empty, and $\mathfrak{M}_\varepsilon^\alpha(t_\varepsilon,q_\varepsilon,t'_\varepsilon,q'_\varepsilon) = \mathfrak{B}_\varepsilon^\alpha(t'_\varepsilon,u'_\varepsilon,z'_\varepsilon,\mathscr{S}_{\mathsf{u}}^*(t_\varepsilon,q_\varepsilon),\mathscr{S}_{\mathsf{z}}^*(t_\varepsilon,q_\varepsilon))$. In order to apply Proposition 3.6 we now need to discuss the boundedness of the slopes $(\mathscr{S}_{\mathsf{u}}^*(t_\varepsilon,q_\varepsilon))_\varepsilon$, $(\mathscr{S}_{\mathsf{z}}^*(t_\varepsilon,q_\varepsilon))_\varepsilon$. Indeed, If $t'>0$, then $t'_\varepsilon\ge c>0$ for all $\varepsilon\in(0,\bar\varepsilon)$ (up to choosing a smaller $\bar\varepsilon$), so that, by the definition (3.12) of $\mathfrak{B}_\varepsilon^\alpha$ we infer that $\mathscr{S}_{\mathsf{u}}^*(t_\varepsilon,q_\varepsilon)\le C\varepsilon^\alpha$ and $\mathscr{S}_{\mathsf{z}}^*(t_\varepsilon,q_\varepsilon)\le C\varepsilon$ for all $\varepsilon\in(0,\bar\varepsilon)$. In the case $t'=0$, suppose e.g. that $\liminf_{\varepsilon\to0}\mathscr{S}_{\mathsf{u}}^*(t_\varepsilon,q_\varepsilon)=+\infty$ while $\liminf_{\varepsilon\to0}\mathscr{S}_{\mathsf{z}}^*(t_\varepsilon,q_\varepsilon)<+\infty$. Then, from the coercivity estimate (3.15b) we deduce (up to extracting a not relabeled subsequence) that $u'_\varepsilon\to0$. Thus,

$$\begin{aligned}\liminf_{\varepsilon\to0}\mathfrak{B}_\varepsilon^\alpha(t'_\varepsilon,u'_\varepsilon,z'_\varepsilon,\mathscr{S}_{\mathsf{u}}^*(t_\varepsilon,q_\varepsilon),\mathscr{S}_{\mathsf{z}}^*(t_\varepsilon,q_\varepsilon)) &\ge \liminf_{\varepsilon\to0}\mathcal{B}_{\mathcal{V}_{\mathsf{z}}}\big(\frac{t'_\varepsilon}{\varepsilon},z'_\varepsilon,\mathscr{S}_{\mathsf{z}}^*(t_\varepsilon,q_\varepsilon)\big)\\ &\ge \mathfrak{B}_0^\alpha(0,0,z',\mathscr{S}_{\mathsf{u}}^*(t,q),\mathscr{S}_{\mathsf{z}}^*(t,q))\,,\end{aligned}$$

with the latter estimate due to Proposition 3.6, Hypothesis 4.10, and the monotonicity of $\mathfrak{B}_0^\alpha(\tau,q',\sigma_{\mathsf{u}},\sigma_{\mathsf{z}})$ in σ_{u} and σ_{z}. We may argue similarly in the case $\liminf_{\varepsilon\to0}\mathscr{S}_{\mathsf{u}}^*(t_\varepsilon,q_\varepsilon)<+\infty$ and $\liminf_{\varepsilon\to0}\mathscr{S}_{\mathsf{z}}^*(t_\varepsilon,q_\varepsilon)=+\infty$ and when both limits are finite.

The Γ-lim sup estimate (b) is trivial for all $(t,q,t',q')\in[0,T]\times\mathrm{D}\times[0,\infty)\times\mathbf{Q}$ with $\mathfrak{M}_0^\alpha(t,q,t',q')=\infty$. If $\mathfrak{M}_0^\alpha(t,q,t',q')=\mathfrak{B}_0^\alpha(t',u',z',\mathscr{S}_{\mathsf{u}}^*(t,q),\mathscr{S}_{\mathsf{z}}^*(t,q))<\infty$, then the lim sup estimate immediately follows via the constant recovery sequence $(t_\varepsilon,q_\varepsilon,t'_\varepsilon,q'_\varepsilon)\equiv(t,q,t',q')$ with the same arguments as in the proof of Proposition 3.6. Let us now suppose that $(t',q')=(0,0)$ with $(t,q)\in\overline{\mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E})}^{\mathrm{w,S}}\backslash\mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E})$, so that $\mathfrak{M}_0^\alpha(t,q,t',q')=0$. We observe that there exists a sequence $(t_n,q_n)_n\subset\mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E})$ with $(t_n,q_n)\rightharpoonup(t,q)$ and $\sup_n\mathfrak{E}(q_n)<\infty$. We will show that for every null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ there exists a recovery sequence for $(t,q,0,0)$. For this, we first fix $n\in\mathbb{N}$ and associate with $(t_n,q_n,0,0)$ the recovery sequence $(t_{\varepsilon_k,n},q_{\varepsilon_k,n},t'_{\varepsilon_k,n},q'_{\varepsilon_k,n})=(t_n,q_n,t'_{\varepsilon_k,n},0)$, where we choose $t'_{\varepsilon_k,n}>0$ such that

$$t'_{\varepsilon_k,n}\le\varepsilon_k\,,\quad \frac{t'_{\varepsilon_k,n}}{\varepsilon_k^\alpha}\mathscr{S}_{\mathsf{u}}^*(t_n,q_n)\le\varepsilon_k\,,\quad\text{and}\quad \frac{t'_{\varepsilon_k,n}}{\varepsilon_k}\mathscr{S}_{\mathsf{z}}^*(t_n,q_n)\le\varepsilon_k\,.$$

Setting $n=k$ we obtain the sequence $(\tilde t_{\varepsilon_k},\tilde q_{\varepsilon_k},\tilde t'_{\varepsilon_k},\tilde q'_{\varepsilon_k})=(t_k,q_k,t'_{\varepsilon_k,k},0)\rightharpoonup(t,q,0,0)$, which gives (i). By construction we also have $\sup_{k\in\mathbb{N}}\mathfrak{E}(\tilde q_{\varepsilon_k})\le\sup_{n\in\mathbb{N}}\mathfrak{E}(q_n)<\infty$, which gives (ii). Moreover, because of

$\tilde{t}'_{\varepsilon_k} > 0$ and $\tilde{q}'_{\varepsilon_k} = 0$ we have

$$\begin{aligned}\mathfrak{M}^\alpha_{\varepsilon_k}(\tilde{t}_{\varepsilon_k}, \tilde{q}_{\varepsilon_k}, \tilde{t}'_{\varepsilon_k}, \tilde{q}'_{\varepsilon_k}) &= \mathfrak{B}^\alpha_{\varepsilon_k}(\tilde{t}'_{\varepsilon_k}, 0, \mathscr{S}^*_{\mathsf{u}}(t_k, q_k), \mathscr{S}^*_{\mathsf{z}}(t_k, q_k)) \\ &= \frac{t'_{\varepsilon_k,k}}{\varepsilon_k^\alpha}\mathscr{S}^*_{\mathsf{u}}(t_k, q_k) + \frac{t'_{\varepsilon_k,k}}{\varepsilon_k}\mathscr{S}^*_{\mathsf{z}}(t_k, q_k) \le 2\varepsilon_k \to 0 = \mathfrak{M}^\alpha_0(t, q, 0, 0)\,.\end{aligned}$$

Thus, condition (iii) in (5.10b) holds as well. With this, Proposition 5.2 is established. □

For later use we also introduce the *'reduced' rescaled joint M-function*

$$\mathfrak{M}^{\alpha,\mathrm{red}}_0 : [0,T]\times \mathrm{D}\times[0,\infty)\times\mathbf{Q} \to [0,\infty], \quad \mathfrak{M}^{\alpha,\mathrm{red}}_0(t,q,t',q') := \mathfrak{M}^\alpha_0(t,q,t',q') - \mathcal{R}(z'). \tag{5.11}$$

We observe that the dissipation potentials $\psi_{\mathsf{u}} := \mathcal{V}_{\mathsf{u}}$ and $\psi_{\mathsf{z}} := \mathcal{R}+\mathcal{V}_{\mathsf{z}}$ have rate-independent parts null and equal to $\mathcal{R}$, respectively, and that $\mathfrak{b}_{\psi_{\mathsf{z}}} = \mathcal{R} + \mathfrak{b}_{\mathcal{V}_{\mathsf{z}}}$ thanks to (4.2b). Thus, from (5.7) and Proposition 3.6 we infer that the following representation formula for $\mathfrak{M}^{\alpha,\mathrm{red}}_0$:

for $\partial_{\mathsf{q}}\mathcal{E}(t,q) \neq \emptyset$ we have

$$t' > 0: \quad \mathfrak{M}^{\alpha,\mathrm{red}}_0(t,q,t',q') = \begin{cases} 0 & \text{for } \mathscr{S}^*_{\mathsf{u}}(t,q) = \mathscr{S}^*_{\mathsf{z}}(t,q) = 0, \\ \infty & \text{otherwise;}\end{cases}$$

$$t' = 0,\ \alpha > 1: \ \mathfrak{M}^{\alpha,\mathrm{red}}_0(t,q,0,q') = \begin{cases} \mathfrak{b}_{\mathcal{V}_{\mathsf{z}}}(z', \mathscr{S}^*_{\mathsf{z}}(t,q)) & \text{for } \mathscr{S}^*_{\mathsf{u}}(t,q) = 0, \\ \mathfrak{b}_{\mathcal{V}_{\mathsf{u}}}(u', \mathscr{S}^*_{\mathsf{u}}(t,q)) & \text{for } \mathscr{S}^*_{\mathsf{u}}(t,q) > 0,\ z' = 0, \\ \infty & \text{otherwise;}\end{cases}$$

$$t' = 0,\ \alpha = 1: \ \mathfrak{M}^{1,\mathrm{red}}_0(t,q,0,q') = \mathfrak{b}_{\mathcal{V}_{\mathsf{u}}\oplus\mathcal{V}_{\mathsf{z}}}(q', \mathscr{S}^*_{\mathsf{u}}(t,q)+\mathscr{S}^*_{\mathsf{z}}(t,q)) \tag{5.12a}$$

$$t' = 0,\ \alpha < 1: \ \mathfrak{M}^{\alpha,\mathrm{red}}_0(t,q,0,q') = \begin{cases} \mathfrak{b}_{\mathcal{V}_{\mathsf{u}}}(u', \mathscr{S}^*_{\mathsf{u}}(t,q)) & \text{for } \mathscr{S}^*_{\mathsf{z}}(t,q) = 0, \\ \mathfrak{b}_{\mathcal{V}_{\mathsf{z}}}(z', \mathscr{S}^*_{\mathsf{z}}(t,q)) & \text{for } \mathscr{S}^*_{\mathsf{z}}(t,q) > 0,\ u' = 0, \\ \infty & \text{otherwise,}\end{cases}$$

for $\partial_{\mathsf{q}}\mathcal{E}(t,q) = \emptyset$ we have

$$\mathfrak{M}^{\alpha,\mathrm{red}}_0(t,q,t',q') = \begin{cases} 0 & \text{for } t' = 0,\ q' = 0 \text{ and } (t,q) \in \overline{\mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E})}^{\mathrm{w,S}} \backslash \mathrm{dom}(\partial_{\mathsf{q}}\mathcal{E}), \\ \infty & \text{otherwise.}\end{cases} \tag{5.12b}$$

The expressions in (5.12) reflect the fact that $\mathfrak{M}^\alpha_\varepsilon$ only depends on the three cases given by $\alpha \in (0,1)$, $\alpha = 1$, or $\alpha > 1$.

We emphasize that $\mathfrak{M}^{\alpha,\mathrm{red}}_0$ depends on $\mathcal{R}$ as well, namely through $\mathscr{S}^*_{\mathsf{z}}$ which is defined via $\mathcal{W}^*_{\mathsf{z}}$. In particular, for $t' > 0$ finiteness of $\mathfrak{M}^{\alpha,\mathrm{red}}_0(t,q,t',q')$ enforces that $0 = \mathscr{S}^*_{\mathsf{u}}(t,q) = \mathscr{S}^*_{\mathsf{z}}(t,q)$ and hence, taking into account Hypothesis 4.10,

$$\begin{cases} \text{the stationarity of } u: & \exists\,(\mu,\zeta) \in \partial_{\mathsf{q}}\mathcal{E}(t,q): \quad \mu = 0\,, \\ \text{the local stability of } z: & \exists\,(\widetilde{\mu},\widetilde{\zeta}) \in \partial_{\mathsf{q}}\mathcal{E}(t,q): \quad \widetilde{\zeta} \in \partial\mathcal{R}(0)\,.\end{cases} \tag{5.13}$$

In the specific cases of dissipation potentials $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$ considered in Example 3.4, we even have the explicit expression of the respective contact potentials $\mathfrak{b}_{\mathcal{V}_{\mathsf{u}}}$ and $\mathfrak{b}_{\mathcal{V}_{\mathsf{z}}}$, and thus of the (reduced) rescaled joint M-function $\mathfrak{M}^{\alpha,\mathrm{red}}_0$. In particular, let us revisit the p-homogeneous case:

Example 5.3 (The p-homogeneous case)**.** *Suppose that the dissipation potentials $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$ are positively p-homogeneous with the* same $p \in (1,\infty)$. *Then, combining* (3.10) *with* (5.12) *we conclude that for $t' = 0$ and $\partial_{\mathsf{q}}\mathcal{E}(t,q) \neq \emptyset$ we have (where $\hat{c}_p = p^{1/p}(p')^{1/p'}$)*

$$\alpha > 1: \ \mathfrak{M}^{\alpha,\mathrm{red}}_0(t,q,0,q') = \begin{cases} \hat{c}_p\,(\mathcal{V}_{\mathsf{z}}(z'))^{1/p}\,(\mathscr{S}^*_{\mathsf{z}}(t,q))^{1/p'} & \text{for } \mathscr{S}^*_{\mathsf{u}}(t,q) = 0, \\ \hat{c}_p\,(\mathcal{V}_{\mathsf{u}}(u'))^{1/p}\,(\mathscr{S}^*_{\mathsf{u}}(t,q))^{1/p'} & \text{for } \mathscr{S}^*_{\mathsf{u}}(t,q) > 0,\ z' = 0, \\ \infty & \text{otherwise;}\end{cases}$$

$$\alpha = 1: \ \mathfrak{M}^{\alpha,\mathrm{red}}_0(t,q,0,q') = \hat{c}_p\,(\mathcal{V}_{\mathsf{u}}(u')+\mathcal{V}_{\mathsf{z}}(z'))^{1/p}\,(\mathscr{S}^*_{\mathsf{u}}(t,q)+\mathscr{S}^*_{\mathsf{z}}(t,q))^{1/p'} \tag{5.14}$$

$$\alpha < 1: \ \mathfrak{M}^{\alpha,\mathrm{red}}_0(t,q,0,q') = \begin{cases} \hat{c}_p\,(\mathcal{V}_{\mathsf{u}}(u'))^{1/p}\,(\mathscr{S}^*_{\mathsf{u}}(t,q))^{1/p'} & \text{for } \mathscr{S}^*_{\mathsf{z}}(t,q) = 0, \\ \hat{c}_p\,(\mathcal{V}_{\mathsf{z}}(z'))^{1/p}\,(\mathscr{S}^*_{\mathsf{z}}(t,q))^{1/p'} & \text{for } \mathscr{S}^*_{\mathsf{z}}(t,q) > 0,\ u' = 0, \\ \infty & \text{otherwise.}\end{cases}$$

The M-functions $\mathfrak{M}_\varepsilon^\alpha$ enjoy suitable coercivity properties that will play a key role in the compactness arguments for proving the existence of BV solutions. These estimates are direct consequences of the the lower bounds on $\mathfrak{B}_\varepsilon^\alpha$ derived in Lemma 3.7 and the definition of $\mathfrak{M}_\varepsilon^\alpha$. The importance here is the uniformity in $\varepsilon \in [0,1]$.

We also emphasize that we are stating a result that is focusing on z' and ignoring u', which reflects the fact that we always assume the bound on $\|u_\varepsilon'\|_{\mathrm{L}^1(0,T;\mathbf{U})}$ whereas for z_ε' we only have a bound in $\mathrm{L}^1(0,T;\mathbf{Z}_{\mathrm{ri}})$, but we need the derivative $\mathsf{z}'(s) \in \mathbf{Z}$ at least in points where $\mathscr{S}_{\mathsf{z}}^*(\mathsf{t}(s),\mathsf{q}(s)) > 0$.

Lemma 5.4. *The following estimates hold for all $c > 0$ and $\varepsilon \in [0,1]$ with $\varkappa$ from Lemma 3.7:*

$$\alpha \in (0,1): \quad \mathscr{S}_{\mathsf{z}}^*(t,q) \geq c \qquad \Longrightarrow \quad \|z'\|_{\mathbf{Z}} \leq \frac{\mathfrak{M}_\varepsilon^\alpha(t,q,t',q')}{\varkappa(c)}, \tag{5.15a}$$

$$\alpha \geq 1: \quad \mathscr{S}_{\mathsf{u}}^*(t,q)+\mathscr{S}_{\mathsf{z}}^*(t,q) \geq c \qquad \Longrightarrow \quad \|z'\|_{\mathbf{Z}} \leq \frac{\mathfrak{M}_\varepsilon^\alpha(t,q,t',q')}{\varkappa(c)}. \tag{5.15b}$$

The proof of (5.15a) and (5.15b) follows directly from the definition of $\mathfrak{M}_\varepsilon^\alpha$ and the corresponding estimates (3.15b) and (3.15e) for $\mathfrak{B}_\varepsilon^\alpha$ in Lemma 3.7, respectively.

The following result is an immediate consequence of the definition of $\mathfrak{M}_\varepsilon^\alpha$ and of Proposition 3.2(b5), if we recall the definitions of $\mathfrak{A}_{\mathsf{x}}^{*,0}$ from (4.21).

Lemma 5.5. *For all $\alpha > 0$ and all $\varepsilon \in [0,1]$ we have that*

$$\mathfrak{M}_\varepsilon^\alpha(t,q,t',q') \geq -\langle \mu, u'\rangle_{\mathbf{U}} - \langle \zeta, z'\rangle_{\mathbf{Z}} \tag{5.16}$$

for all $(t,q,t',q') \in [0,T]\times\mathbf{Q}\times[0,\infty)\times\mathbf{Q}$ and all $\xi = (\mu,\zeta) \in \mathfrak{A}_{\mathsf{u}}^*(t,q)$, $\zeta \in \mathfrak{A}_{\mathsf{z}}^*(t,q)$.

5.2. Admissible parametrized curves.

The concept of *admissible parametrized curve* is tailored in such a way that it is able to describe limiting curves $(\mathsf{t},\mathsf{q}) : [a,b] \to [0,T]\times\mathbf{Q}$ of a family of parametrized viscous curves $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon)_\varepsilon$ satisfying

$$\sup_{\varepsilon\in(0,1)} \int_a^b \mathfrak{M}_\varepsilon^\alpha(\mathsf{t}_\varepsilon(s),\mathsf{q}_\varepsilon(s),\mathsf{t}_\varepsilon'(s),\mathsf{q}_\varepsilon'(s))\,\mathrm{d}s \ < \ \infty.$$

Since Proposition 5.2 guarantees that $\mathfrak{M}_0^\alpha$ is the Γ-limit of $\mathfrak{M}_\varepsilon^\alpha$ it seems natural that such curves can be characterized by the condition

$$\int_a^b \mathcal{R}(z'(s))\,\mathrm{d}s + \int_a^b \mathfrak{M}_0^{\alpha,\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))\,\mathrm{d}s < \infty. \tag{5.17}$$

However, this expression is not well-defined, since we are not able to define the derivatives $\mathsf{q}'(s) = (\mathsf{u}'(s),\mathsf{z}'(s))$ almost everywhere. To reformulate (5.17) in a proper way, we take advantage of the special form of $\mathfrak{M}_0^{\alpha,\mathrm{red}}$ given in (5.12) by observing that $\mathsf{z}'(s)$ is only needed on the special sets $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$ to be defined below. Hence, condition (5.17) can be replaced by (5.20) in Definition 5.6 ahead, which relies on the fact that absolutely continuous curves z with values in (the possibly non-reflexive space) $\mathbf{Z}_{\mathrm{ri}}$ need not be differentiable with respect to time. Therefore, the pointwise derivative z' is replaced by a scalar surrogate, cf. (5.18) below, whose definition involves the dissipation potential $\mathcal{R}$ and generalizes the concept of *metric derivative* from the theory of gradient flows in metric spaces [AGS08].

(1) We say that a curve $z : [a,b] \to \mathbf{Z}_{\mathrm{ri}}$ is $\mathcal{R}$-absolutely continuous if there exists a nonnegative function $m \in \mathrm{L}^1(a,b)$ such that

$$\mathcal{R}(z(s_2)-z(s_1)) \leq \int_{s_1}^{s_2} m(s)\,\mathrm{d}s \qquad \text{for every } a \leq s_1 \leq s_2 \leq b,$$

and we denote by $\mathrm{AC}([a,b];\mathbf{Z}_{\mathrm{ri}},\mathcal{R})$ the space of $\mathcal{R}$-absolutely continuous curves.

(2) For a curve $z \in \mathrm{AC}([a,b];\mathbf{Z}_{\mathrm{ri}},\mathcal{R})$ we set

$$\mathcal{R}[z'](s) := \lim_{h\to 0} \mathcal{R}\Big(\frac{1}{h}\big(z(s+h)-z(s)\big)\Big) \qquad \text{for a.a. } s \in (a,b). \tag{5.18}$$

We are now in a position to give our definition of admissible parametrized curve, which adapts [MRS16a, Def. 4.1] to the present multi-rate system. We recall that the slope functions $\mathscr{S}_{\mathsf{x}}^*$ are lsc according to Hypothesis 4.10. Hence, along continuous curves $(\mathsf{t},\mathsf{q}) : [a,b] \to [0,T]\times\mathbf{Q}$ the following sets are relatively open:

$$\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}] := \begin{cases} \big\{\, s \in [a,b] \,\big|\, \mathscr{S}_{\mathsf{u}}^*(\mathsf{t}(s),\mathsf{q}(s))+\mathscr{S}_{\mathsf{z}}^*(\mathsf{t}(s),\mathsf{q}(s)) > 0 \,\big\} & \text{for } \alpha \geq 1, \\ \big\{\, s \in [a,b] \,\big|\, \mathscr{S}_{\mathsf{z}}^*(\mathsf{t}(s),\mathsf{q}(s)) > 0 \,\big\} & \text{for } \alpha \in (0,1). \end{cases} \tag{5.19}$$

The difference between the cases $\alpha > 1$ and $\alpha \in (0,1)$ in the definition of the set $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$ is commented after the following definition.

Definition 5.6 ($\mathscr{A}([a,b];[0,T]{\times}\mathbf{Q})$)**.** *A curve* $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z}) : [a,b] \to [0,T]{\times}\mathbf{Q}$ *is called an* admissible parametrized curve *if*

(1) t *is non-decreasing,* $\mathsf{t} \in \mathrm{AC}([a,b];\mathbb{R})$*,* $\mathsf{u} \in \mathrm{AC}([a,b];\mathbf{U})$ *and* $\mathsf{z} \in \mathrm{AC}([a,b];\mathbf{Z}_{\mathrm{ri}},\mathcal{R})$*;*

(2) $\mathscr{S}_{\mathsf{u}}^*(\mathsf{t},\mathsf{q}) = 0$ *and* $\mathscr{S}_{\mathsf{z}}^*(\mathsf{t},\mathsf{q}) = 0$ *on the set* $\{\, s \in (a,b) \,|\, \mathsf{t}'(s) > 0 \,\}$*;*

(3) z *is locally* $\mathbf{Z}$*-absolutely continuous on the open set* $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$*, and* t *is constant on every connected component of* $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$*;*

(4) $\sup_{s\in[a,b]} \mathfrak{E}(\mathsf{q}(s)) \le E$ *for some* $E > 0$*;*

(5) there holds

$$\int_a^b \mathcal{R}[\mathsf{z}'](s)\,\mathrm{d}s + \int_{\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]} \mathfrak{M}_0^{\alpha,\mathrm{red}}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s),0,\mathsf{u}'(s),\mathsf{z}'(s))\,\mathrm{d}s < \infty\,. \tag{5.20}$$

We will denote by $\mathscr{A}([a,b];[0,T]{\times}\mathbf{Q})$ *the collection of all admissible parametrized curves from* $[a,b]$ *to* $[0,T]{\times}\mathbf{Q}$*. Furthermore, we say that* $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([a,b];[0,T]{\times}\mathbf{Q})$ *is*

- non-degenerate*, if*

$$\mathsf{t}'(s) + \mathcal{R}[\mathsf{z}'](s) + \|\mathsf{u}'(s)\|_{\mathbf{U}} > 0 \quad \textit{for a.a. } s \in (a,b); \tag{5.21}$$

- surjective*, if* $\mathsf{t}(a) = 0$ *and* $\mathsf{t}(b) = T$*.*

Finally, in the case in which the function t*, defined on the canonical interval* $[0,1]$*, is constant with* $\mathsf{t}(s) \equiv t$ *for some* $t \in [0,T]$*, we call* q *an* admissible transition curve between $q_0 := q(0)$ and $q_1 := q(1)$ at time t*, and we will use the notation*

$$\mathcal{A}_t(q_0,q_1) := \{\mathsf{q} : [0,1] \to \mathbf{Q} \;:\; (\mathsf{t},\mathsf{q}) \in \mathscr{A}([0,1];[0,T]{\times}\mathbf{Q}),\ \mathsf{t}(s) \equiv t,\ \mathsf{q}(0) = q_0,\ \mathsf{q}(1) = q_1\}.$$

The requirement that z has to be locally $\mathbf{Z}$-absolutely continuous on the set $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$, and the different definition of $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$ in the cases $\alpha \ge 1$ and $\alpha \in (0,1)$, are clearly motivated by properties (5.15) (which, in turn, derive from Lemma 3.7). Indeed, in the case $\alpha \in (0,1)$, in view of (5.15a), once $\mathfrak{M}_0^\alpha(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}')$ is estimated and $\mathscr{S}_{\mathsf{z}}^*(\mathsf{t},\mathsf{q})$ is strictly positive, then $\mathfrak{M}_0^\alpha(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}')$ provides a control on $\|\mathsf{z}'\|_{\mathbf{Z}}$. Because of this, parametrized curves are required to be absolutely continuous on the set $\mathscr{S}_{\mathsf{z}}^*(\mathsf{t},\mathsf{q}) > 0$. In the case $\alpha \ge 1$, in view of estimate (5.15b), the z-component of admissible parametrized curves is expected to be absolutely continuous on the larger set where $\mathscr{S}_{\mathsf{u}}^*(\mathsf{t},\mathsf{q}) + \mathscr{S}_{\mathsf{z}}^*(\mathsf{t},\mathsf{q}) > 0$.

Hence, on the one hand, in (5.20) we integrate only over the set $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$, because it is in $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$ where the pointwise derivative $\mathsf{z}' \in \mathbf{Z}$ exists, which makes the term $\mathfrak{M}_0^{\alpha,\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),0,\mathsf{q}'(s))$ well defined. On the other hand, the specific form of $\mathfrak{M}_0^{\alpha,\mathrm{red}}$ in (5.12) and the fact that $\mathfrak{b}_\psi(v,0) = 0$ for all v show that $z' \in \mathbf{Z}$ is only needed on the set $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$.

Hereafter, along an admissible parametrized curve (t,q) we shall use the notation

$$\mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](s) := \mathcal{R}[\mathsf{z}'](s) + \mathbf{1}_{\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]}(s)\mathfrak{M}_0^{\alpha,\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),0,\mathsf{q}'(s)) \tag{5.22}$$

with $\mathcal{R}[\mathsf{z}']$ from (5.18), and $\mathbf{1}_{\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]}$ is the indicator function of the set $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$. Let us stress that the above notation makes sense only along an admissible curve. If the admissible curve (t,q) has the additional property $\mathsf{z} \in \mathrm{AC}([a,b];\mathbf{Z})$ and thus $\mathsf{z}'(s)$ is well defined as an element of $\mathbf{Z} \subset \mathbf{Z}_{\mathrm{ri}}$ for almost all $s \in (a,b)$, then $\mathcal{R}[z'](s) = \mathcal{R}(z'(s))$ a.e. in (a,b). Hence, for an admissible curve with $\mathsf{z} \in \mathrm{AC}([a,b];\mathbf{Z})$ we have $\mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](s) = \mathfrak{M}_0^\alpha(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))$ for a.a. $s \in (a,b)$.

5.3. Definition of parametrized Balanced-Viscosity solutions.

We are now in a position to precisely define parametrized Balanced-Viscosity (pBV) solutions to the rate-independent system $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow0}$, see Definition 5.9. At the core of this concept there lies a (parametrized) chain-rule inequality, cf. Hypothesis 5.7 that will be imposed as an additional property of the rate-independent system, while Proposition 5.16 will provide sufficient conditions for the validity of Hypothesis 5.7.

We will also introduce an *enhanced* version of the pBV concept, in which we additionally require z to be absolutely continuous with values in $\mathbf{Z}$. In [MRS16a, Sec. 4.2] this notion had been already introduced, using a different terminology that might create slight confusion in the present multi-rate context and has thus been changed here. We believe the enhanced concept to be significant as well because, for some examples (cf. e.g. the applications discussed in Section 8), the vanishing-viscosity analysis will directly lead to enhanced BV solutions.

The definition of pBV solutions relies on the validity of the following assumption on the rate-independent system $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow0}$.

Hypothesis 5.7 (Chain rule along admissible parametrized curves). *For every admissible parametrized curve* $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([a,b];[0,T]\times\mathbf{Q})$

$$
\begin{aligned}
&\text{the map } s \mapsto \mathcal{E}(\mathsf{t}(s),\mathsf{q}(s)) \text{ is absolutely continuous on } [a,b] \text{ and} \\
&\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s)) - \partial_t\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))\mathsf{t}'(s) \geq -\mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](s) \text{ for a.a. } s\in(a,b).
\end{aligned}
\tag{5.23}
$$

Remark 5.8. *In general, the chain-rule inequality* (5.23) *along a given admissible parametrized curve* (t,q) *does not follow from the chain rule of Hypothesis 4.7, because for these curves the pointwise derivative* z' *exists as an element in* $\mathbf{Z}$ *only on the set* $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$ *from* (5.19). *That is why, Proposition 5.16 provides a sufficient condition under which Hypothesis 4.7 ensures the validity of Hypothesis 5.7, albeit restricted to admissible curves satisfying additionally* $\mathsf{z}\in \mathrm{AC}([a,b];\mathbf{Z})$.

We are now ready to introduce the exact notion of pBV solutions.

Definition 5.9 (pBV and enhanced pBV solutions). *In addition to Hypotheses 4.1, 4.2, 4.3, 4.5, and 4.10, let the rate-independent system* $(\mathbf{U}\times\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ *satisfy Hypothesis 5.7. We call a curve* $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([a,b];[0,T]\times\mathbf{Q})$ *a* parametrized Balanced-Viscosity *(pBV) solution to the rate-independent system* $(\mathbf{U}\times\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ *if* (t,q) *satisfies the* parametrized energy-dissipation balance

$$
\mathcal{E}(\mathsf{t}(s_2),\mathsf{q}(s_2)) + \int_{s_1}^{s_2}\mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](s)\,\mathrm{d}s = \mathcal{E}(\mathsf{t}(s_1),\mathsf{q}(s_1)) + \int_{s_1}^{s_2}\partial_t\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))\mathsf{t}'(s)\,\mathrm{d}s \tag{5.24}
$$

for every $a\leq s_1\leq s_2\leq b$, *where* $\mathfrak{M}_0^\alpha$ *is defined in* (5.22).

A pBV solution $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z})$ *is called* enhanced pBV solution, *if additionally* $\mathsf{z}\in\mathrm{AC}([a,b];\mathbf{Z})$.

For an enhanced pBV solution (t,q) we have $\mathsf{q}\in\mathrm{AC}([a,b];\mathbf{Q})$, since $\mathsf{q}\in\mathscr{A}([a,b];[0,T]\times\mathbf{Q})$ already implies $\mathsf{u}\in\mathrm{AC}([a,b];\mathbf{U})$. As a consequence of the chain-rule inequality (5.23) from Hypothesis 5.7, we have the following characterization.

Lemma 5.10 (Characterization of pBV solutions). *Let Hypothesis 5.7 hold additionally. Then for an admissible parametrized curve* $(\mathsf{t},\mathsf{q})\in\mathscr{A}([a,b];[0,T]\times\mathbf{Q})$, *the following three properties are equivalent:*

(1) (t,q) *is a pBV solution of the rate-independent system* $(\mathbf{U}\times\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$*;*
(2) (t,q) *fulfills the* upper energy estimate $\leq$ *in* (5.24) *on for* $s_1=a$ *and* $s_2=b$*;*
(3) (t,q) *fulfills the pointwise identity for a.a.* $s\in(a,b)$

$$
\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s)) - \partial_t\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))\mathsf{t}'(s) = -\mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](s)\,. \tag{5.25}
$$

The proof is a simple adaptation of the arguments for [MRS12a, Prop. 5.3] and [MRS16a, Cor 3.5] and is thus omitted.

5.4. Existence results for pBV solutions.

Our first main result states that any family $(\mathsf{t}_\varepsilon,\mathsf{u}_\varepsilon,\mathsf{z}_\varepsilon)_\varepsilon$ obtained by suitably rescaling (cf. Remark 5.15 ahead) a family of solutions to the viscous system (1.4) converges along a subsequence, as $\varepsilon\to 0^+$, to a parametrized solution of the rate-independent system $(\mathbf{U}\times\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$.

Theorem 5.11 (Existence of pBV solutions). *Under Hypotheses 4.1, 4.2, 4.3, 4.5, 4.10, and 5.7, let* $(q_{\varepsilon_k})_k = (u_{\varepsilon_k},z_{\varepsilon_k})_k \subset \mathrm{AC}([0,T];\mathbf{Q})$ *be a sequence of solutions to the generalized viscous gradient system* (1.4) *with* $(\varepsilon_k)_k\subset(0,\infty)$ *a null sequence. Suppose that*

$$
q_{\varepsilon_k}(0)\to q_0 \text{ in } \mathbf{Q} \quad\text{and}\quad \mathcal{E}(0,q_{\varepsilon_k}(0))\to\mathcal{E}(0,q_0) \qquad \text{as } k\to\infty, \tag{5.26}
$$

for some $q_0=(u_0,z_0)\in\mathrm{D}$. *Let* $\mathsf{t}_{\varepsilon_k}:[0,\mathsf{S}]\to[0,T]$ *be non-decreasing surjective time-rescalings such that* $\mathsf{q}_{\varepsilon_k}=(\mathsf{u}_{\varepsilon_k},\mathsf{z}_{\varepsilon_k})$ *defined via* $\mathsf{q}_{\varepsilon_k}(s)=q_{\varepsilon_k}(\mathsf{t}_{\varepsilon_k}(s))$ *satisfies*

$$
\begin{aligned}
&\exists\, C>0\ \forall\, k\in\mathbb{N} \ \text{ for a.a. } s\in(0,\mathsf{S}): \\
&\mathsf{t}_{\varepsilon_k}'(s) + \mathcal{R}(\mathsf{z}_{\varepsilon_k}'(s)) + \mathfrak{M}_{\varepsilon_k}^{\alpha,\mathrm{red}}(\mathsf{t}_{\varepsilon_k}(s),\mathsf{q}_{\varepsilon_k}(s),\mathsf{t}_{\varepsilon_k}'(s),\mathsf{q}_{\varepsilon_k}'(s)) + \|\mathsf{u}_{\varepsilon_k}'(s)\|_{\mathbf{U}} \leq C.
\end{aligned}
\tag{5.27}
$$

Then, there exist a (not relabeled) subsequence and a curve $(\mathsf{t},\mathsf{q})\in\mathscr{A}([0,\mathsf{S}];[0,T]\times\mathbf{Q})$ *such that*

(1)

$$
\begin{aligned}
&\mathsf{t}\in\mathrm{C}^0_{\mathrm{lip}}([0,\mathsf{S}];[0,T]), \quad \mathsf{q}=(\mathsf{u},\mathsf{z})\in\mathrm{C}^0_{\mathrm{weak}}([0,\mathsf{S}];\mathbf{U}_{\mathrm{e}}\times\mathbf{Z}_{\mathrm{e}}), \\
&\mathsf{u}\in\mathrm{C}^0_{\mathrm{lip}}([0,\mathsf{S}];\mathbf{U}), \quad \mathsf{z}\in\mathrm{C}^0_{\mathrm{lip}}([0,\mathsf{S}];\mathbf{Z}_{\mathrm{ri}})\cap\mathrm{C}^0([0,\mathsf{S}];\mathbf{Z});
\end{aligned}
\tag{5.28}
$$

(2) the following convergences hold as $k\to\infty$

$$
\mathsf{t}_{\varepsilon_k}\to\mathsf{t} \text{ in } \mathrm{C}^0([0,\mathsf{S}]), \tag{5.29a}
$$

$$
\mathsf{u}_{\varepsilon_k}\overset{*}{\rightharpoonup}\mathsf{u} \text{ in } W^{1,\infty}(0,\mathsf{S};\mathbf{U}), \qquad \mathsf{z}_{\varepsilon_k}\to\mathsf{z} \text{ in } \mathrm{C}^0([0,\mathsf{S}];\mathbf{Z}), \tag{5.29b}
$$

$$
\mathsf{u}_{\varepsilon_k}(s)\rightharpoonup\mathsf{u}(s) \text{ in } \mathbf{U}_{\mathrm{e}} \text{ and } \qquad \mathsf{z}_{\varepsilon_k}(s)\rightharpoonup\mathsf{z}(s) \text{ in } \mathbf{Z}_{\mathrm{e}} \qquad \text{for all } s\in[0,\mathsf{S}]; \tag{5.29c}
$$

(3) (t,q) *fulfills the upper energy-dissipation estimate* $\leq$ *in* (5.24) *on* $[0,\mathsf{S}]$*, hence* (t,q) *is a pBV solution to the rate-independent system* $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow0}$.

Moreover, $(\mathsf{t},\mathsf{u},\mathsf{z})$ *is surjective and there hold the additional convergences*

$$\mathcal{E}(\mathsf{t}_{\varepsilon_k}(s),\mathsf{q}_{\varepsilon_k}(s))\to\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))\qquad \text{for all } s\in[0,\mathsf{S}], \tag{5.30a}$$

$$\int_{s_1}^{s_2}\mathfrak{M}_{\varepsilon_k}^{\alpha}(\mathsf{t}_{\varepsilon_k}(\sigma),\mathsf{q}_{\varepsilon_k}(\sigma),\mathsf{t}_{\varepsilon_k}'(\sigma),\mathsf{q}_{\varepsilon_k}'(\sigma))\,\mathrm{d}\sigma\to\int_{s_1}^{s_2}\mathfrak{M}_0^{\alpha}[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](\sigma)\,\mathrm{d}\sigma \tag{5.30b}$$

for all $0\leq s_1\leq s_2\leq\mathsf{S}$.

We postpone the proof of Theorem 5.11 to Section 7.2, but point out here that the core of the limit passage in the parametrized energy-dissipation estimate (5.4), leading to (5.24), lies in the following straightforward consequence of Ioffe's theorem [Iof77] (see also [Val90, Thm. 21]). A 'metric version' of Proposition 5.12 below was proved in [MRS09, Lemma 3.1].

Proposition 5.12. *Let* $\mathbf{S}$ *be a weakly closed subset of* $\mathbf{Q}$*, and let* $(\mathscr{M}_\varepsilon)_\varepsilon$, $\mathscr{M}_0:\mathbb{R}{\times}\mathbf{S}{\times}\mathbb{R}{\times}\mathbf{Q}\to[0,\infty]$ *be measurable and weakly lower semicontinuous functionals fulfilling the* Γ-$\liminf$ *estimate*

$$\Big((t_\varepsilon,q_\varepsilon,t_\varepsilon',q_\varepsilon')\rightharpoonup(t,q,t',q') \text{ in } \mathbb{R}{\times}\mathbf{S}{\times}\mathbb{R}{\times}\mathbf{Q} \text{ as } \varepsilon\to0^+\Big)\Longrightarrow\mathscr{M}_0(t,q,t',q')\leq\liminf_{\varepsilon\to0^+}\mathscr{M}_\varepsilon(t_\varepsilon,q_\varepsilon,t_\varepsilon',q_\varepsilon'). \tag{5.31}$$

Suppose that, for $\varepsilon\geq0$*, the functional* $\mathscr{M}_\varepsilon(t,q,\cdot,\cdot)$ *is convex for every* $(t,q)\in\mathbb{R}{\times}\mathbf{S}$*. Let* $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon)$, $(\mathsf{t},\mathsf{q})\in\mathrm{AC}([a,b];\mathbb{R}{\times}\mathbf{S})$ *fulfill*

$$\mathsf{t}_\varepsilon(s)\to\mathsf{t}(s),\quad \mathsf{q}_\varepsilon(s)\rightharpoonup\mathsf{q}(s) \text{ for all } s\in[a,b],\qquad (\mathsf{t}_\varepsilon',\mathsf{q}_\varepsilon')\rightharpoonup(\mathsf{t}',\mathsf{q}') \text{ in } \mathrm{L}^1(a,b;\mathbb{R}{\times}\mathbf{Q}). \tag{5.32}$$

Then,

$$\liminf_{\varepsilon\to0^+}\int_a^b\mathscr{M}_\varepsilon(\mathsf{t}_\varepsilon(s),\mathsf{q}_\varepsilon(s),\mathsf{t}_\varepsilon'(s),\mathsf{q}_\varepsilon'(s))\,\mathrm{d}s\geq\int_a^b\mathscr{M}_0(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))\,\mathrm{d}s. \tag{5.33}$$

Proof. It is sufficient to introduce the functional $\bar{\mathscr{M}}:\mathbb{R}{\times}\mathbf{S}{\times}\mathbb{R}{\times}\mathbf{Q}{\times}[0,\infty]\to[0,\infty]$ defined by

$$\bar{\mathscr{M}}(t,q,t',q',\varepsilon):=\begin{cases}\mathscr{M}_\varepsilon(t,q,t',q') & \text{if } \varepsilon>0,\\ \mathscr{M}_0(t,q,t',q') & \text{if } \varepsilon=0,\end{cases}$$

and to observe that $\mathscr{M}$ is lower semicontinuous with respect to the weak topology of $\mathbb{R}{\times}\mathbf{S}{\times}\mathbb{R}{\times}\mathbf{Q}{\times}[0,\infty]$ and convex for every $(t,q)\in\mathbb{R}{\times}\mathbf{Q}$ and $\varepsilon\geq0$. Then, by Ioffe's theorem we conclude that

$$\liminf_{\varepsilon\to0^+}\int_a^b\bar{\mathscr{M}}(\mathsf{t}_\varepsilon(s),\mathsf{q}_\varepsilon(s),\mathsf{t}_\varepsilon'(s),\mathsf{q}_\varepsilon'(s),\varepsilon)\,\mathrm{d}s\geq\int_a^b\bar{\mathscr{M}}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s),0)\,\mathrm{d}s,$$

i.e., (5.33) is established. □

Remark 5.13. *Theorem 5.11 does not guarantee that the pBV solution is* non-degenerate *even if the quantity in* (5.27) *has a uniform positive lower bound. Nonetheless, any (possibly degenerate) solution* (t,q) *can be reparametrized to a* non-degenerate *one* $(\widetilde{\mathsf{t}},\widetilde{\mathsf{q}}):[0,\widetilde{\mathsf{S}}]\to[0,T]{\times}\mathbf{Q}$*, fulfilling*

$$\widetilde{\mathsf{t}}'(\sigma)+\mathcal{R}[\widetilde{\mathsf{z}}'](\sigma)+\|\widetilde{\mathsf{u}}'(\sigma)\|_{\mathbf{U}}=1\qquad \text{for a.a. } \sigma\in(0,\widetilde{\mathsf{S}}). \tag{5.34}$$

For this, we proceed as in [EfM06] *and associate with* (t,q) *the rescaling function* $\widetilde{\sigma}$ *defined by* $\widetilde{\sigma}(s)=\int_0^s(\mathsf{t}'(r){+}\mathcal{R}[\mathsf{z}'](r){+}\|\mathsf{u}'(r)\|_{\mathbf{U}})\,\mathrm{d}r$ *and set* $\widetilde{\mathsf{S}}=\sigma(\mathsf{S})$*. We then define* $(\widetilde{\mathsf{t}}(\sigma),\widetilde{\mathsf{q}}(\sigma)):=(\mathsf{t}(s),\mathsf{q}(s))$ *for* $\sigma=\widetilde{\sigma}(s)$*. The very same calculations as in* [Mie11, Lem. 4.12] *(or based on the reparametrization result* [AGS08, Lem. 1.1.4]*), yield* (5.34).

Our next result, whose proof is omitted (cf. also Remark 5.15), addresses the existence of *enhanced* pBV solutions.

Theorem 5.14 (Existence of enhanced pBV solutions)**.** *Assume Hypotheses 4.1, 4.2, 4.3, 4.5, 4.10, and 5.7. Suppose that there exist rescaled solutions* $(\mathsf{t}_{\varepsilon_k},\mathsf{q}_{\varepsilon_k})_k$ *to the viscous system* $(1.4)_{\varepsilon_k}$ *such that, in addition to* (5.27)*, there also holds the estimate*

$$\exists\, C>0\ \forall\, k\in\mathbb{N}\quad \text{for a.a. } s\in(0,\mathsf{S}):\qquad \|\mathsf{z}_{\varepsilon_k}'(s)\|_{\mathbf{Z}}\leq C. \tag{5.35}$$

Then, up to a (not relabeled) subsequence the curves $(\mathsf{t}_{\varepsilon_k},\mathsf{q}_{\varepsilon_k})_k$ *converge to an admissible parametrized curve* $(\mathsf{t},\mathsf{q})\in\mathscr{A}([0,\mathsf{S}];[0,T]{\times}\mathbf{Q})$ *such that* (5.28)*,* (5.29)*,* (5.30) *hold and additionally* $\mathsf{z}\in\mathrm{C}^0_{\mathrm{lip}}([0,\mathsf{S}];\mathbf{Z})$*, i.e.,* (t,q) *is an* enhanced *pBV solution to the rate-independent system* $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow0}$.

Remark 5.15. *In the statement of Theorem 5.11, the reparametrization $t = \mathsf{t}_{\varepsilon_k}(s)$ yielding the rescaled solutions $\mathsf{q}_{\varepsilon_k}$ can be chosen arbitrarily, provided it guarantees the Lipschitz bound* (5.27). *Under Hypotheses 4.1, 4.2 and 4.3,* all *viscous solutions $(u_{\varepsilon_k}, z_{\varepsilon_k})$ satisfy the uniform bound $\|z'_{\varepsilon_k}\|_{\mathrm{L}^1(0,T;\mathbf{Z}_{\mathrm{ri}})} \leq C$, see* (4.26). *If, additionally $\|u'_{\varepsilon_k}\|_{\mathrm{L}^1(0,T;\mathbf{Z}_{\mathrm{ri}})} \leq C$ holds (i.e.* (4.29) *from Corollary 4.13), then a reparametrization yielding* (5.27) *is easily obtained, for instance, by using the energy-dissipation arclength in* (5.2).

Similarly, under the stronger a priori estimate

$$\exists\, C > 0\ \forall\, k \in \mathbb{N}: \qquad \big\|z'_{\varepsilon_k}\big\|_{\mathrm{L}^1(0,T;\mathbf{Z})} = \int_0^T \|z'_{\varepsilon_k}(t)\|_{\mathbf{Z}}\, \mathrm{d}t \leq C, \tag{5.36}$$

one easily obtains rescaled solutions satisfying the stronger Lipschitz bound (5.35). *Hence, one gains enhanced compactness information for the sequence $(\mathsf{z}_{\varepsilon_k})_k$, and the proof of Theorem 5.11 immediately yields a proof of Theorem 5.14.*

We conclude this section with some sufficient conditions for the validity of (a stronger form of) the parametrized chain rule in Hypothesis 5.7. It will be derived as a consequence of the non-parametrized chain rule in Hypothesis 4.7.

Proposition 5.16 (Sufficient conditions for parametrized chain rule)**.** *Assume that Hypothesis 4.7 holds and that the vanishing-viscosity contact potentials associated with $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$ satisfy*

$$\exists\, c_{\mathsf{x}} > 0\ \ \forall\,(v,\eta) \in \mathbf{X}\times\mathbf{X}^*: \qquad \mathfrak{b}_{\mathcal{V}_{\mathsf{x}}}(v, \mathcal{V}^*_{\mathsf{x}}(\eta)) \geq c_{\mathsf{x}}\|v\|_{\mathbf{X}}\|\eta\|_{\mathbf{X}^*} \tag{5.37}$$

for $\mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$ and $\mathbf{X} \in \{\mathbf{U},\mathbf{Z}\}$.

Then, the parametrized chain rule (5.23) *holds along all admissible curves $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([a,b];[0,T]{\times}\mathbf{Q})$ with $\mathsf{q} \in \mathrm{AC}([a,b];\mathbf{Q})$. In particular, we have*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathsf{t},\mathsf{q}) - \partial_t\mathcal{E}(\mathsf{t},\mathsf{q})\mathsf{t}' = \langle\mu,\mathsf{u}'\rangle_{\mathbf{U}} + \langle\zeta,\mathsf{z}'\rangle_{\mathbf{Z}} \geq -\mathfrak{M}^\alpha_0(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}') \quad \text{a.e. in } (a,b) \tag{5.38}$$

for all measurable selections $(a,b) \ni s \mapsto (\mu(s),\zeta(s)) \in \mathbf{U}^*{\times}\mathbf{Z}^*$ satisfying for almost all $s \in (a,b)$ $(\mu(s),\zeta(s)) \in \mathfrak{A}^*_{\mathsf{u}}(\mathsf{t}(s),\mathsf{q}(s)){\times}\mathfrak{A}^*_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s))$.

The proof will be carried out in Appendix A.

5.5. Differential characterization of enhanced pBV solutions.

The main result of this section is Theorem 5.20, which provides a further characterization of *enhanced pBV solutions* in terms of solutions of a system of subdifferential inclusions, see (5.46). This differential form has the very same structure as the viscous system (4.13), except that the small parameters ε^α and ε multiplying the viscous terms are replaced by coefficients λ_{u} and λ_{z} satisfying the switching conditions (5.47c). For this, we use the optimality in the energy-dissipation balance.

In Lemma 5.5 we have established the estimate

$$\mathfrak{M}^\alpha_0(t,q,t',q') \geq -\langle\mu,u'\rangle_{\mathbf{U}} - \langle\zeta,z'\rangle_{\mathbf{Z}} \quad \text{for all } (\mu,\zeta) \in \mathfrak{A}^*_{\mathsf{u}}(t,q){\times}\mathfrak{A}^*_{\mathsf{z}}(t,q), \tag{5.39}$$

which is valid for all $(t,q,t',q') \in [0,T]{\times}\mathbf{Q}{\times}[0,\infty){\times}\mathbf{Q}$ and which is a generalization of the classical Young–Fenchel inequality $\psi(v) + \psi^*(-\xi) \geq -\langle\xi,v\rangle$. With the first result of this section we will show that, in analogy to the characterization of generalized gradient-flow equations via the energy-dissipation principle, we are able to characterize pBV solutions via the optimality condition that estimate (5.39) holds as an equality. Thus, we define the *contact set* Σ_α (cf. [MRS13, Def. 3.6]) via

$$\Sigma_\alpha := \Big\{ (t,q,t',q') \in [0,T]{\times}\mathbf{Q}{\times}[0,\infty){\times}\mathbf{Q} \;\Big|\; \exists\,(\mu,\zeta) \in \mathfrak{A}^*_{\mathsf{u}}(t,q){\times}\mathfrak{A}^*_{\mathsf{z}}(t,q): \tag{5.40}$$
$$\mathfrak{M}^\alpha_0(t,q,t',q') = -\langle\mu,u'\rangle_{\mathbf{U}} - \langle\zeta,z'\rangle_{\mathbf{Z}} \Big\}.$$

Proposition 5.17 below makes the relation between enhanced pBV solutions and the contact set Σ_α rigorous. We emphasize here that we need to exploit the stronger version (5.38) of the parametrized chain rule from Hypothesis 5.7, in addition to Hypotheses 4.1, 4.2, 4.3, 4.5, and 4.10, always tacitly assumed. Recall that a sufficient condition for such a chain rule is provided by Proposition 5.16.

Proposition 5.17 (Enhanced pBV solutions lie in Σ_α)**.** *Suppose that the parametrized chain rule* (5.38) *holds along all admissible curves $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([a,b];[0,T]{\times}\mathbf{Q})$ with $\mathsf{q} \in \mathrm{AC}([a,b];\mathbf{Q})$. Then, a curve $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([a,b];[0,T]{\times}\mathbf{Q})$ is an enhanced pBV solution of $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{V}^{\varepsilon^\alpha}_{\mathsf{u}}{+}\mathcal{R}{+}\mathcal{V}^{\varepsilon}_{\mathsf{z}})_{\varepsilon\downarrow0}$ if and only if $\mathsf{q} \in \mathrm{AC}([a,b];\mathbf{Q})$ and $(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}') \in \Sigma_\alpha$ a.e. in (a,b).*

Proof. Let us consider an admissible parametrized curve $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([a,b];[0,T]\times\mathbf{Q})$ with $\mathsf{q} \in \mathrm{AC}([a,b];\mathbf{Q})$. By the characterization provided in Lemma 5.10, (t,q) is a pBV solution if and only if $-\mathfrak{M}_0^\alpha(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}') = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathsf{t},\mathsf{q}) - \partial_t\mathcal{E}(\mathsf{t},\mathsf{q})\mathsf{t}'$ almost everywhere in (a,b). Combining this with the chain-rule inequality (5.38) we in fact conclude that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathsf{t},\mathsf{q}) - \partial_t\mathcal{E}(\mathsf{t},\mathsf{q})\mathsf{t}' = \langle \mu, \mathsf{u}'\rangle_{\mathbf{U}} + \langle \zeta, \mathsf{z}'\rangle_{\mathbf{Z}} = -\mathfrak{M}_0^\alpha(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}') \quad \text{a.e. in } (a,b),$$

for all measurable selections $\xi = (\mu,\zeta) : (a,b) \to \mathfrak{A}_{\mathsf{u}}^*(\mathsf{t}(s),\mathsf{q}(s))\times\mathfrak{A}_{\mathsf{z}}^*(\mathsf{t}(s),\mathsf{q}(s))$, hence $(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}') \in \Sigma_\alpha$ a.e. in (a,b). The converse implication follows by the same argument. □

The final step in relating enhanced pBV solutions to the solutions of the subdifferential system (5.46) is obtained by analyzing the structure of Σ_α. For this, we exploit the exact form on $\mathfrak{M}_0^\alpha$ and use the definition of the set $\mathfrak{A}_{\mathsf{x}}^*(t,q)$ in terms of the Fréchet subdifferential $\partial_{\mathsf{x}}\mathcal{E}(t,q)$, $\mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$. To formulate this properly, we recall the definition of the rescaled viscosity potentials $\mathcal{V}_{\mathsf{x}}^\lambda$ and their subdifferentials $\partial\mathcal{V}_{\mathsf{x}}^\lambda$ from (1.5) for $\lambda \in [0,\infty]$. In particular, we have

$$\partial\mathcal{V}_{\mathsf{x}}^\lambda(v) = \partial\mathcal{V}_{\mathsf{x}}(\lambda v) \text{ for all } \lambda \in [0,\infty), \text{ and } \partial\mathcal{V}_{\mathsf{x}}^\infty(v) = \begin{cases} \mathbf{X}^* & \text{for } v = 0, \\ \emptyset & \text{otherwise.} \end{cases} \tag{5.41}$$

Observe that, thanks to (4.2b) we have $\partial\mathcal{V}_{\mathsf{x}}(0) = \{0\}$ for $\mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$.

We now consider the system of subdifferential inclusions for the quadruple $(t,q,t',q') = (t,u,z,t',u',z')$ including the two parameters $\lambda_{\mathsf{u}},\lambda_{\mathsf{z}} \in [0,\infty]$:

$$\partial\mathcal{V}_{\mathsf{u}}^{\lambda_{\mathsf{u}}}(u') + \partial_{\mathsf{u}}\mathcal{E}(t,q) \ni 0 \qquad \text{in } \mathbf{U}^*, \tag{5.42a}$$

$$\partial\mathcal{R}(z') + \partial\mathcal{V}_{\mathsf{z}}^{\lambda_{\mathsf{z}}}(z') + \partial_{\mathsf{z}}\mathcal{E}(t,q) \ni 0 \qquad \text{in } \mathbf{Z}^*, \tag{5.42b}$$

$$t'\,\frac{\lambda_{\mathsf{u}}}{1+\lambda_{\mathsf{u}}} = t'\,\frac{\lambda_{\mathsf{z}}}{1+\lambda_{\mathsf{z}}} = 0\,. \tag{5.42c}$$

Here we use the usual convention $\infty/(1+\infty) = 1$ and emphasize that, at this stage, system (5.42) is not to be understood as a system of subdifferential inclusions. Instead, $(t',q') \in [0,\infty)\times\mathbf{Q}$ are treated as independent variables. With this we are able to introduce the following subsets of $[0,T]\times\mathbf{Q}\times[0,\infty)\times\mathbf{Q}$, called *evolution regimes*, thus providing a basis for the informal discussion at the end of Section 2:

$$\begin{aligned} \mathrm{E}_{\mathsf{u}} &:= \big\{ (t,q,t',q') \,\big|\, \exists\, \lambda_{\mathsf{z}} \in [0,\infty]\text{: (5.42) holds with } \lambda_{\mathsf{u}} = 0 \big\}, \\ \mathrm{R}_{\mathsf{z}} &:= \big\{ (t,q,t',q') \,\big|\, \exists\, \lambda_{\mathsf{u}} \in [0,\infty]\text{: (5.42) holds with } \lambda_{\mathsf{z}} = 0 \big\}, \\ \mathrm{V}_{\mathsf{u}} &:= \big\{ (t,q,t',q') \,\big|\, \exists\, \lambda_{\mathsf{z}} \in [0,\infty]\text{: (5.42) holds with } \lambda_{\mathsf{u}} \in (0,\infty) \big\}, \\ \mathrm{V}_{\mathsf{z}} &:= \big\{ (t,q,t',q') \,\big|\, \exists\, \lambda_{\mathsf{u}} \in [0,\infty]\text{: (5.42) holds with } \lambda_{\mathsf{z}} \in (0,\infty) \big\}, \\ \mathrm{V}_{\mathsf{uz}} &:= \big\{ (t,q,t',q') \,\big|\, \text{ (5.42) holds with } \lambda_{\mathsf{u}} = \lambda_{\mathsf{z}} \in (0,\infty) \big\}, \\ \mathrm{B}_{\mathsf{u}} &:= \big\{ (t,q,t',q') \,\big|\, \exists\, \lambda_{\mathsf{z}} \in [0,\infty]\text{: (5.42) holds with } \lambda_{\mathsf{u}} = \infty \big\}, \\ \mathrm{B}_{\mathsf{z}} &:= \big\{ (t,q,t',q') \,\big|\, \exists\, \lambda_{\mathsf{u}} \in [0,\infty]\text{: (5.42) holds with } \lambda_{\mathsf{z}} = \infty \big\}. \end{aligned} \tag{5.43}$$

The letters E, R, V, B, stand for *Equilibrated*, *Rate-independent*, *Viscous*, and *Blocked*, respectively. We will discuss the meaning of the names of the evolution regimes below. It will be efficient to use the notation

$$\mathrm{A}_{\mathsf{u}}\mathrm{C}_{\mathsf{z}} := \mathrm{A}_{\mathsf{u}} \cap \mathrm{C}_{\mathsf{z}} \quad \text{ for A} \in \{\mathrm{E},\mathrm{V},\mathrm{B}\} \text{ and C} \in \{\mathrm{R},\mathrm{V},\mathrm{B}\};$$

nonetheless, note that the set V_{uz} is different from (indeed, strictly contained in) $\mathrm{V}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}$. We also remark that any set involving 'V' of 'B' is necessarily restricted to the subspace with $t' = 0$ because of (5.42c). With this, we are now in a position to state our result for the contact sets Σ_α, under the additional condition (4.23) on the product form of the Fréchet subdifferential $\partial_q\mathcal{E}$. Proposition 5.18 below will be proven in Section 7.3.

Proposition 5.18 (Σ_α and evolution regimes)**.** *If, in addition, the Fréchet subdifferential $\partial_q\mathcal{E}$ has the product structure* (4.23)*, then we have the following inclusions for the contact set Σ_α:*

$$\alpha > 1: \quad \Sigma_\alpha \subset \mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \cup \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}} \cup \mathrm{B}_{\mathsf{z}}, \tag{5.44a}$$

$$\alpha = 1: \quad \Sigma_1 \subset \mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \cup \mathrm{V}_{\mathsf{uz}} \cup \mathrm{B}_{\mathsf{u}}\mathrm{B}_{\mathsf{z}}, \tag{5.44b}$$

$$\alpha \in (0,1): \quad \Sigma_\alpha \subset \mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \cup \mathrm{V}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \cup \mathrm{B}_{\mathsf{u}}\,, \tag{5.44c}$$

where in all cases the three sets on the right-hand side are disjoint.

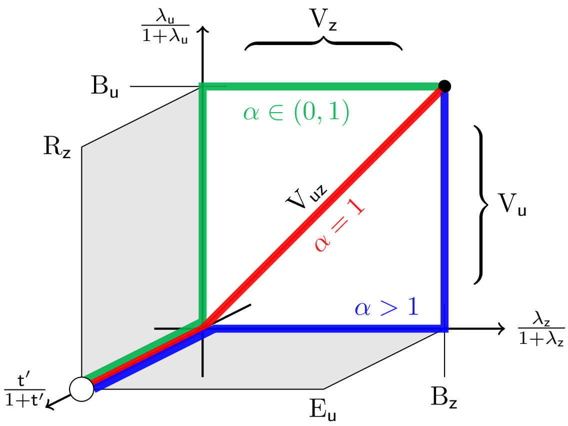

FIGURE 5.1.
The switching conditions and the different regimes are displayed in the space for $(t', \lambda_u, \lambda_z) \in [0,\infty]^3$.
For $t' > 0$ the only admissible regime is given by the intersection
$E_uR_z = E_u \cap R_z$.
For $t' = 0$ the different admissible regimes depend on $\alpha > 0$:
$\alpha > 1$: $E_u \cup B_z$
$\alpha = 1$: $E_uR_z \cup V_{uz} \cup B_uB_z$
$\alpha \in (0,1)$: $V_uR_z \cup B_u$

Remark 5.19. *In the characterization of (enhanced) pBV solution provided by Proposition 5.17, the contact condition* $\mathfrak{M}_0^\alpha(t,q,t',q') = -\langle \mu, u'\rangle_{\mathbf{U}} - \langle \zeta, z'\rangle_{\mathbf{Z}}$ *holds* for all $(\mu,\zeta) \in \mathfrak{A}_u^*(t,q)\times\mathfrak{A}_z^*(t,q)$. *Hence, it seems possible to define a smaller contact set* $\widetilde{\Sigma}_\alpha$ *by replacing "∃" in* (5.40) *by "∀". Because of* $\widetilde{\Sigma}_\alpha \subset \Sigma_\alpha$ *inclusions* (5.44) *would remain true. However, using our larger set* Σ_α *is sufficient to deduce that pBV solutions satisfy the system of subdifferential inclusions* (5.46) *ahead.*

The different evolution regimes characterized by the right-hand sides in (5.44) can be visualized by considering the three real parameters $(t', \lambda_u, \lambda_z) \in [0,\infty)\times[0,\infty]^2$, since the rate-independent regimes E_u and R_z are given by $\lambda_u = 0$ and $\lambda_z = 0$ respectively. Similarly, the viscous regimes V_x, $x \in \{u, z\}$, are defined via $\lambda_x \in (0,\infty)$, and the blocking regime B_x is determined by $\lambda_x = \infty$. The sets on the right-hand sides in (5.44) are then defined in terms of the switching conditions

$$(5.42c) \text{ holds} \quad \text{and} \quad \begin{cases} \lambda_u = 0 \text{ or } \lambda_z = \infty & \text{for } \alpha > 1, \\ \lambda_u = \lambda_z \in [0,\infty] & \text{for } \alpha = 1, \\ \lambda_u = \infty \text{ or } \lambda_z = 0 & \text{for } \alpha \in (0,1). \end{cases} \tag{5.45}$$

The corresponding sets in the $(t', \lambda_u, \lambda_z)$ space are depicted in Figure 5.1.

The inclusions (5.44) that relate the contact sets to the different evolution regimes A_uC_z have a clear and immediate interpretation in terms of the evolutionary behavior of an enhanced pBV solution (t, q):

- E_u encodes the regime where $u = u(s)$ stays in *equilibria*, which may depend on s. Indeed, inserting $\lambda_u(s) = 0$ in (5.42a) leads to the equilibrium condition $0 \in \partial_u\mathcal{E}(t(s), q(s))$. This means that $u(s)$ follows $z(s)$ that may evolve rate-independently when $t' > 0$, and may follow a viscous jump path, or may be blocked, when $t'(s) = 0$.
- R_z denotes the rate-independent evolution for $z(s)$, where $\lambda_z(s) = 0$. The component $u(s)$ either follows staying in equilibria, evolves viscously, or is blocked.
- In the case $t' > 0$ only the rate-independent regime E_uR_z is admissible. This is the regime with $\lambda_u = \lambda_z = 0$ where the viscous dissipation potentials $\mathcal{V}_u$ and $\mathcal{V}_z$ do not come into action.
- In the regime V_x, the variable $x(s)$ evolves viscously with $\lambda_x(s) \in (0,\infty)$, and necessarily $t'(s) = 0$.
- V_{uz} is the special case occurring only for $\alpha = 1$, where $\lambda_u(s) = \lambda_z(s) \in (0,\infty)$, i.e. both components have a synchronous viscous phase.
- The blocked regime B_x, occurring when $t'(s) = 0$, encodes the situation that $\lambda_x(s) = \infty$, which means that on the given time scale the viscosity is so strong that the x-component cannot move, i.e. it is blocked with $x'(s) = 0$.
- $B_{uz} = B_uB_z$ means that both components are blocked, namely $q'(s) = 0$. This can occur, for instance, if we set $(t(s), q(s)) = (t_*, q_*)$ for $s \in (s_1, s_2)$. Then, $\lambda_u(s) = \lambda_z(s) = \infty$ still gives a trivial, constant solution. Such a behavior may occur after taking a limit like $\varepsilon \to 0^+$, but of course the interval can be cut out by defining a pBV solution on $[0, S-s_2+s_1]$.

We are now in a position to prove a characterization of *enhanced* pBV solutions in terms of the following system of subdifferential inclusions

$$\begin{aligned} \partial\mathcal{V}_u^{\lambda_u(s)}(u'(s)) + \partial_u\mathcal{E}(t(s), u(s), z(s)) \ni 0 \quad &\text{in } \mathbf{U}^*, \\ \partial\mathcal{R}(z'(s)) + \partial\mathcal{V}_z^{\lambda_z(s)}(z'(s)) + \partial_z\mathcal{E}(t(s), u(s), z(s)) \ni 0 \quad &\text{in } \mathbf{Z}^*, \end{aligned} \tag{5.46}$$

where the balanced interplay of viscous and rate-independent behavior in the equations for u and z, respectively, is determined by the (arclength-dependent) parameters $\lambda_u(s)$ or $\lambda_z(s)$. We emphasize that the so-called *switching conditions* for $t' \geq 0$ and λ_{u}, $\lambda_{\mathsf{z}} \in [0,\infty]$, cf. (5.47c) below, are different for the three cases $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$.

Theorem 5.20 (Differential characterization of enhanced pBV solutions). *Assume Hypotheses 4.1, 4.2, 4.3, 4.5, and 4.10 and let the parametrized chain rule* (5.38) *hold. In addition, suppose that the Fréchet subdifferential* $\partial_q \mathcal{E}$ *has the product structure from* (4.23)*. Let* $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([0,\mathsf{S}];[0,T]{\times}\mathbf{Q})$ *be an admissible parametrized curve with* $\mathsf{q} \in \mathrm{AC}([0,\mathsf{S}];\mathbf{Q})$.

(1) If $(\mathsf{t},\mathsf{q}) : (0,\mathsf{S}) \to \mathbf{Q}$ *is a enhanced pBV solution of* $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$*, then there exist measurable functions* $(\lambda_{\mathsf{u}},\lambda_{\mathsf{z}}) : (0,\mathsf{S}) \to [0,\infty]^2$ *and* $\xi = (\mu,\zeta) : (0,\mathsf{S}) \to \mathbf{U}^*{\times}\mathbf{Z}^*$ *with*

$$\mu(s) \in \partial_{\mathsf{u}}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s)) \quad \text{and} \quad \zeta(s) \in \partial_{\mathsf{z}}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s)) \quad \text{for a.a. } s \in (0,\mathsf{S}) \tag{5.47a}$$

satisfying for almost all $s \in (0,\mathsf{S})$ *the subdifferential inclusions*

$$\begin{aligned} -\mu(s) &\in \partial\mathcal{V}_{\mathsf{u}}^{\lambda_{\mathsf{u}}(s)}(\mathsf{u}'(s)) && \text{in } \mathbf{U}^*,\\ -\zeta(s) &\in \partial\mathcal{R}(\mathsf{z}'(s)) + \partial\mathcal{V}_{\mathsf{z}}^{\lambda_{\mathsf{z}}(s)}(\mathsf{z}'(s)) && \text{in } \mathbf{Z}^*, \end{aligned} \tag{5.47b}$$

and the switching conditions

$$\mathsf{t}'(s)\,\frac{\lambda_{\mathsf{u}}(s)}{1{+}\lambda_{\mathsf{u}}(s)} = 0 = \mathsf{t}'(s)\,\frac{\lambda_{\mathsf{z}}(s)}{1{+}\lambda_{\mathsf{z}}(s)} \quad \text{and} \quad \begin{cases} \lambda_{\mathsf{u}}(s)\,\frac{1}{1+\lambda_{\mathsf{z}}(s)} = 0 \text{ for } \alpha > 1,\\ \lambda_{\mathsf{u}}(s) = \lambda_{\mathsf{z}}(s) \quad \text{for } \alpha = 1,\\ \frac{1}{1+\lambda_{\mathsf{u}}(s)}\,\lambda_{\mathsf{z}}(s) = 0 \text{ for } \alpha \in (0,1). \end{cases} \tag{5.47c}$$

(2) Conversely, if there exist measurable functions $(\lambda_{\mathsf{u}},\lambda_{\mathsf{z}}) : (0,\mathsf{S}) \to [0,\infty]^2$ *and* $\xi = (\mu,\zeta) : (0,\mathsf{S}) \to \mathbf{U}^*{\times}\mathbf{Z}^*$ *satisfying* (5.47) *and, in addition,*

$$\sup_{s\in(0,\mathsf{S})} |\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))| < \infty, \quad \text{and} \quad \int_0^{\mathsf{S}} \big(\|\mu(s)\|_{\mathbf{U}^*}\|\mathsf{u}'(s)\|_{\mathbf{U}}{+}\|\zeta(s)\|_{\mathbf{Z}^*}\|\mathsf{z}'(s)\|_{\mathbf{Z}}\big)\,\mathrm{d}s < \infty, \tag{5.48}$$

then (t,q) *is an enhanced pBV solution.*

Proof. Part (1) basically follows from combining the characterization of enhanced pBV solutions from Proposition 5.17 in terms of the contact set, with Proposition 5.18. Only the measurability of the coefficients λ_{u}, $\lambda_{\mathsf{z}} : [0,\mathsf{S}] \to [0,\infty]$ and of the selections $\xi = (\mu,\zeta) : (0,\mathsf{S}) \to \mathbf{U}^*{\times}\mathbf{Z}^*$ deserves some discussion that is postponed to Appendix B.

Let us now carry out the proof of Part (2). After cutting out possible intervals where (t,q) may be constant (i.e. in the blocking regime $\mathrm{B_u B_z}$), we may suppose that the admissible parametrized curve (t,q) fulfills the non-degeneracy condition (5.21). In what follows, we will use the short-hand notation

$$(0,\mathsf{S}) \cap \mathrm{A_u C_z} := \{s \in (0,\mathsf{S}) \ : \ (\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) \in \mathrm{A_u C_z}\} \tag{5.49}$$

for $\mathrm{A} \in \{\mathrm{E},\mathrm{V},\mathrm{B}\}$ and $\mathrm{C} \in \{\mathrm{R},\mathrm{V},\mathrm{B}\}$. We will discuss at length the case $\alpha > 1$; the very same arguments yield the thesis also in the cases $\alpha = 1$ and $\alpha \in (0,1)$. It follows from the switching conditions (5.47c) that the integral $I := \int_0^{\mathsf{S}} \big(\langle -\mu,\mathsf{u}'\rangle_{\mathbf{U}} + \langle -\zeta,\mathsf{z}'\rangle_{\mathbf{U}}\big)\,\mathrm{d}s$ decomposes as

$$I = I_1 + I_2 + I_3 \quad \text{with } I_1 := \int_{(0,\mathsf{S})\cap\mathrm{E_u R_z}} \big(\langle -\mu(s),\mathsf{u}'(s)\rangle_{\mathbf{U}} + \langle -\zeta(s),\mathsf{z}'(s)\rangle_{\mathbf{Z}}\big)\,\mathrm{d}s, \tag{5.50}$$

$$I_2 := \int_{(0,\mathsf{S})\cap\mathrm{E_u V_z}} \big(\langle -\mu,\mathsf{u}'\rangle_{\mathbf{U}} + \langle -\zeta,\mathsf{z}'\rangle_{\mathbf{Z}}\big)\,\mathrm{d}s, \text{ and } I_3 := \int_{(0,\mathsf{S})\cap\mathrm{B_z}} \big(\langle -\mu,\mathsf{u}'\rangle_{\mathbf{U}} + \langle -\zeta,\mathsf{z}'\rangle_{\mathbf{Z}}\big)\,\mathrm{d}s,$$

where we use that the three regimes $\mathrm{E_u R_z}$, $\mathrm{E_u V_z}$, and $\mathrm{B_z}$ are disjoint. Now, on $(0,\mathsf{S}) \cap \mathrm{E_u R_z}$ we have that $\mu(s) \equiv 0$, while $\zeta(s) \in \partial\mathcal{R}(\mathsf{z}'(s))$, so that

$$I_1 = \int_{(0,\mathsf{S})\cap\mathrm{E_u R_z}} \mathcal{R}(\mathsf{z}'(s)) = \int_{(0,\mathsf{S})\cap\mathrm{E_u R_z}} \mathfrak{M}_0^\alpha(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))\,\mathrm{d}s$$

where we used (5.11) and (5.12), taking into account $\mathscr{S}_{\mathsf{u}}^*(\mathsf{t}(s),\mathsf{q}(s)) = \mathscr{S}_{\mathsf{z}}^*(\mathsf{t}(s),\mathsf{q}(s)) \equiv 0$ on $(0,\mathsf{S})\cap \mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}}$. On $(0,\mathsf{S}) \cap \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}$ we have $\mathscr{S}_{\mathsf{u}}^*(\mathsf{t}(s),\mathsf{q}(s)) \equiv 0$ and the z-equation in (5.46) holds with $\lambda_{\mathsf{z}}(s) > 0$, so that

$$
\begin{aligned}
I_2 &= \int_{(0,\mathsf{S})\cap \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}} \frac{1}{\lambda_{\mathsf{z}}(s)} \langle -\zeta(s), \lambda_{\mathsf{z}}(s) z'(s)\rangle_{\mathbf{U}} \,\mathrm{d}s \\
&\overset{(1)}{=} \int_{(0,\mathsf{S})\cap \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}} \frac{1}{\lambda_{\mathsf{z}}(s)} \big(\mathcal{R}\big(\lambda_{\mathsf{z}}(s)\mathsf{z}'(s)\big)+\mathcal{V}_{\mathsf{z}}\big(\lambda_{\mathsf{z}}(s)\mathsf{z}'(s)\big)+\mathcal{W}_{\mathsf{z}}^*(-\zeta(s))\big) \,\mathrm{d}s \\
&\overset{(2)}{\geq} \int_{(0,\mathsf{S})\cap \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}} \frac{1}{\lambda_{\mathsf{z}}(s)} \big(\mathcal{R}\big(\lambda_{\mathsf{z}}(s)\mathsf{z}'(s)\big)+\mathcal{V}_{\mathsf{z}}\big(\lambda_{\mathsf{z}}(s)\mathsf{z}'(s)\big)+\mathscr{S}_{\mathsf{z}}^*(\mathsf{t}(s),\mathsf{q}(s))\big) \,\mathrm{d}s \\
&\overset{(3)}{\geq} \int_{(0,\mathsf{S})\cap \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}} \big(\mathcal{R}\big(\mathsf{z}'(s)\big)+\mathfrak{b}_{\mathcal{V}_{\mathsf{z}}}(z'(s), \mathscr{S}_{\mathsf{z}}^*(\mathsf{t}(s),\mathsf{q}(s)))\big) \,\mathrm{d}s \overset{(4)}{=} \int_{(0,\mathsf{S})\cap \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}} \mathfrak{M}_0^\alpha(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) \,\mathrm{d}s,
\end{aligned}
$$

where (1) follows from (5.47b) via Fenchel-Moreau conjugation, (2) is a consequence of the definition of $\mathscr{S}_{\mathsf{z}}^*(\mathsf{t},\mathsf{q})$, (3) is due to the definition of $\mathfrak{b}_{\mathcal{V}_{\mathsf{z}}}$, and (4) again ensues from (5.11) and (5.12). Finally, with the very same arguments we find that

$$
\begin{aligned}
I_3 = \int_{(0,\mathsf{S})\cap \mathrm{B}_{\mathsf{z}}} \langle -\mu(s), \mathsf{u}'(s)\rangle_{\mathbf{U}} &= \int_{(0,\mathsf{S})\cap \mathrm{B}_{\mathsf{z}}} \frac{1}{\lambda_{\mathsf{u}}(s)} \langle -\mu(s), \lambda_{\mathsf{u}}(s)\mathsf{u}'(s)\rangle_{\mathbf{U}} \\
\geq \int_{(0,\mathsf{S})\cap \mathrm{B}_{\mathsf{z}}} \mathfrak{b}_{\mathcal{V}_{\mathsf{u}}}(\mathsf{u}(s), \mathscr{S}_{\mathsf{u}}^*(\mathsf{t}(s),\mathsf{q}(s))) \,\mathrm{d}s &= \int_{(0,\mathsf{S})\cap \mathrm{B}_{\mathsf{z}}} \mathfrak{M}_0^\alpha(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) \,\mathrm{d}s\,.
\end{aligned}
$$

Combining the above estimates with (5.50) and with the chain-rule (4.16) (which applies thanks to (5.48)), we ultimately conclude that

$$
\begin{aligned}
\mathcal{E}(\mathsf{t}(0),\mathsf{q}(0)) + \int_0^{\mathsf{S}} \partial_t\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s)) \,\mathrm{d}s &\geq \mathcal{E}(\mathsf{t}(\mathsf{S}),\mathsf{q}(\mathsf{S})) + \int_0^{\mathsf{S}} \big(\langle -\mu(s), \mathsf{u}'(s)\rangle_{\mathbf{U}} + \langle -\zeta(s), \mathsf{z}'(s)\rangle_{\mathbf{Z}} \big) \,\mathrm{d}s \\
&\geq \mathcal{E}(\mathsf{t}(\mathsf{S}),\mathsf{q}(\mathsf{S})) + \int_0^{\mathsf{S}} \mathfrak{M}_0^\alpha(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) \,\mathrm{d}s\,,
\end{aligned}
$$

namely (t,q) fulfills the upper energy-dissipation estimate. Therefore, by Lemma 5.10 we conclude that (t,q) is an (enhanced) pBV solution. □

6. True Balanced-Viscosity solutions

This section is devoted to the the concept of true Balanced-Viscosity (BV) solutions, i.e. solutions defined on the original time interval $[0,T]$ instead via the artificial arc length $s \in [0,\mathsf{S}]$. This concept will be introduced in Section 6.1 in Definition 6.5. The central ingredient in this notion is a Finsler-type transition cost that measures the energy dissipated at jumps of the curve (u,z), see Definition 6.2. In Section 6.2 we will gain further insight into the fine properties of true BV solutions, while Section 6.3 states our two existence results, Theorems 6.8 and 6.12, in which BV solutions to the rate-independent system $(\mathbf{U}{\times}\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ are obtained by taking the vanishing-viscosity limit of system (1.4) in the real process time, *without reparametrization.* Section 6.4 addresses the non-parametrized counterpart of enhanced pBV solutions called *enhanced BV solutions*, and Section 6.5 provides how parametrized and true BV solutions are related.

We start with some notations for functions having well-defined jumps.

Notation 6.1 (Regulated functions)**.** *Given a Banach space* $\mathbf{B}$*, we denote by*

$$
\begin{aligned}
\mathrm{R}(0,T;\mathbf{B}) := \Big\{ f : [0,T] \to \mathbf{B} \;\Big|\; \forall\, t \in [0,T] : f(t^-) := \lim_{s\to t^-} f(s) \text{ exists in } \mathbf{B}, \\
f(t^+) := \lim_{r\to t^+} f(r) \text{ exists in } \mathbf{B} \Big\}
\end{aligned}
\tag{6.1}
$$

the space of (everywhere defined) regulated functions *on* $[0,T]$ *with values in* $\mathbf{B}$*, where we use* $f(0^-) := f(0)$ *and* $f(T^+) := f(T)$*. The symbol* $\mathrm{BV}([0,T];\mathbf{B})$ *denotes the space of everywhere defined functions of bounded* $\mathbf{B}$*-variation such that* $\mathrm{BV}([0,T];\mathbf{B}) \subset \mathrm{R}(0,T;\mathbf{B})$ *with continuous embedding.*

Note that for $f \in \mathrm{R}(0,T;\mathbf{B})$ the three values $f(t^-)$, $f(t)$, $f(t^+)$ may all be different for $t \in (0,T)$, and that distinguishing these values will be crucial for our notion of BV solutions.

For a given $z \in \mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}})$ we also introduce the $\mathcal{R}$-variation

$$
\mathrm{Var}_{\mathcal{R}}(z;[a,b]) := \sup \Big\{ \sum_{i=1}^N \mathcal{R}(z(t_i){-}z(t_{i-1})) \;\Big|\; N \in \mathbb{N},\ a = t_0 < t_1 < \ldots < t_N = b \Big\} \tag{6.2}
$$

for $[a,b] \subset [0,T]$, and we observe that

$$\mathrm{Var}_{\mathcal{R}}(z;[a,b]) = \int_a^b \mathcal{R}[z'](t)\,\mathrm{d}t \qquad \text{for } z \in \mathrm{AC}([a,b];\mathbf{Z}_{\mathrm{ri}},\mathcal{R}). \tag{6.3}$$

We mention in advance that true BV solutions are curves $q = (u,z)$, with $u \in \mathrm{BV}([0,T];\mathbf{U})$ and $z \in \mathrm{R}(0,T;\mathbf{Z}) \cap \mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}})$. For such $q=(u,z)$ we introduce the *jump set*

$$\mathrm{J}[q] = \mathrm{J}[u] \cup \mathrm{J}[z] \text{ with } \mathrm{J}[w] := \left\{ t \in [0,T] \,\middle|\, w(t^-) \neq w(t) \text{ or } w(t^+) \neq w(t) \right\}; \tag{6.4}$$

we record that $\mathrm{J}[q]$ consists of at most countably many points. Note that for $\mathrm{J}[z]$ the left and the right limits are considered with respect to the norm topology of $\mathbf{Z}$. For later use, we finally observe that

$$\mathrm{L}^\infty(0,T;\mathbf{Z}_{\mathrm{e}}) \cap \mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}}) \subset \mathrm{R}(0,T;\mathbf{Z}), \tag{6.5}$$

which can be easily checked exploiting the (compact) embeddings $\mathbf{Z}_{\mathrm{e}} \Subset \mathbf{Z} \subset \mathbf{Z}_{\mathrm{ri}}$.

6.1. Definition of true BV solution.

The (possibly asymmetric) Finsler cost function is obtained by minimizing an ‘infinitesimal cost’, depending on the fixed process time $t \in [0,T]$ and defined in terms of the rescaled joint M-function $\mathfrak{M}_0^\alpha$, along *admissible transition curves* $\mathsf{q} : [0,1] \to \mathbf{Q}$. From now on, for better clarity we will denote a generic transition curve by Θ in place of q.

Definition 6.2 (Admissible transition curves, Finsler cost). *For given $t \in [0,T]$ and $q_0 = (u_0,z_0), q_1 = (u_1,z_1) \in \mathbf{U}\times\mathbf{Z}$, we define the Finsler cost induced by $\mathfrak{M}_0^\alpha$ by*

$$\mathrm{cost}_{\mathfrak{M}_0^\alpha}(t;q_0,q_1) := \inf_{\Theta \in \mathcal{A}_t(q_0,q_1)} \int_0^1 \mathfrak{M}_0^\alpha[t,\Theta,0,\Theta']\,\mathrm{d}r \tag{6.6}$$

with the short-hand notation $\mathfrak{M}_0^\alpha[\cdot,\cdot,\cdot,\cdot]$ from (5.22) and $\mathcal{A}_t(q_0,q_1)$ the set of all admissible transition curves at time t between q_0 and q_1, see Definition 5.6.

Thanks to the 1-positive homogeneity of the functional $\Theta' \mapsto \mathfrak{M}_0^\alpha[t,\Theta,0,\Theta']$, we observe that it is not restrictive to suppose that all transition curves are defined on $[0,1]$.

We are now ready to define a new variation called the $\mathfrak{M}_0^\alpha$-total variation of a curve $q = (u,z) : [0,T] \to \mathbf{Q}$. It consists, cf. (6.8) below, of the $\mathcal{R}$-variation of z as defined in (6.2) plus extra contributions at jump points $t_* \in \mathrm{J}(q)$ that may arise through rate-independent or viscous transition costs between $q(t_*^-)$, $q(t_*)$, and $q(t_*^+)$. These extra contributions are given by the Finsler cost (6.6), from which the $\mathcal{R}$-variation is subtracted to avoid that it is counted twice in the $\mathfrak{M}_0^\alpha$-variation. The resulting terms are positive because we always have $\mathrm{cost}_{\mathfrak{M}_0^\alpha}(t;(u_0,z_0),(u_1,z_1)) \geq \mathcal{R}(z_1{-}z_0)$ since $\mathfrak{M}_0^\alpha[t,q,0,q'] \geq \mathcal{R}(z')$ (using $\mathfrak{M}_0^{\alpha,\mathrm{red}}(t,q,0,q') \geq 0$).

Definition 6.3 ($\mathfrak{M}_0^\alpha$-variations). *Let $q = (u,z) : [0,T] \to \mathbf{Q}$ with $u \in \mathrm{BV}([0,T];\mathbf{U})$ and $z \in \mathrm{R}([0,T];\mathbf{Z}) \cap \mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}})$ be a curve with $\sup_{t\in[0,T]} \mathfrak{E}(q(t)) \leq E < \infty$ and jump set $\mathrm{J}[q]$. For closed subintervals $[a,b] \subset [0,T]$ we define*

(1) the extra Viscous Jump Variation *of q induced by $\mathfrak{M}_0^\alpha$ on $[a,b]$ via*

$$\begin{aligned} \mathrm{eVJV}_{\mathfrak{M}_0^\alpha}(q;[a,b]) := {}& \big(\mathrm{cost}_{\mathfrak{M}_0^\alpha}(a;q(a),q(a^+)) - \mathcal{R}(z(a^+){-}z(a))\big) \\ &+ \sum_{t \in \mathrm{J}[q]\cap(a,b)} \Big(\mathrm{cost}_{\mathfrak{M}_0^\alpha}(t;q(t^-),q(t)) - \mathcal{R}(z(t){-}z(t^-)) \\ &\qquad + \mathrm{cost}_{\mathfrak{M}_0^\alpha}(t;q(t),q(t^+)) - \mathcal{R}(z(t^+){-}z(t))\Big) \\ &+ \big(\mathrm{cost}_{\mathfrak{M}_0^\alpha}(b;q(b^-),q(b)) - \mathcal{R}(z(b){-}z(b^-))\big)\,; \end{aligned} \tag{6.7}$$

(2) the $\mathfrak{M}_0^\alpha$-total variation

$$\mathrm{Var}_{\mathfrak{M}_0^\alpha}(q;[a,b]) := \mathrm{Var}_{\mathcal{R}}(z;[a,b]) + \mathrm{eVJV}_{\mathfrak{M}_0^\alpha}(q;[a,b])\,. \tag{6.8}$$

With slight abuse of notation, here we will use the symbol $\mathrm{Var}_{\mathfrak{M}_0^\alpha}$ for the $\mathfrak{M}_0^\alpha$-total variation, although this is not a standard form of total variation, cf. [MRS12a, Rem. 3.5].

Just like for its parametrized counterpart, our definition of (true) BV solutions will rely on a suitable chain-rule requirement, enhancing Hypothesis 4.7 to curves $q = (u,z)$ having just a BV-time regularity. For consistency, we will formulate this BV-chain rule as a hypothesis.

Hypothesis 6.4 (Chain rule in BV). *For every curve $q = (u,z) : [0,T] \to \mathbf{Q}$ with $u \in \mathrm{BV}([0,T];\mathbf{U})$ and $z \in \mathrm{R}([0,T];\mathbf{Z}) \cap \mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}})$ and satisfying*

$$\mathscr{S}_{\mathsf{u}}^*(t,q(t)) + \mathscr{S}_{\mathsf{z}}^*(t,q(t)) = 0 \quad \textit{for all} \;\; t \in [0,T] \setminus \mathrm{J}[q]$$

the following chain-rule estimate holds, for all closed subset $[t_0, t_1] \subset [0,T]$*:*

$$\begin{aligned}&\text{the map } t \mapsto \mathcal{E}(t,q(t)) \text{ belongs to } \mathrm{BV}([0,T]) \text{ and}\\ &\mathcal{E}(t_1,q(t_1)) - \mathcal{E}(t_0,q(t_0)) - \int_{t_0}^{t_1} \partial_t \mathcal{E}(s,q(s))\,\mathrm{d}s \geq -\mathrm{Var}_{\mathfrak{M}_0^\alpha}(q;[t_0,t_1]).\end{aligned} \tag{6.9}$$

In Lemma A.2 in Appendix A we to show that the parametrized chain rule from Hypothesis 5.7 also guarantees the validity of Hypothesis 6.4. Hence, subsequently we will directly assume Hypothesis 5.7.

Let us now give our definition of BV solutions $q : [0,T] \to \mathbf{U}\times\mathbf{Z}$, i.e. BV solutions without parametrization. We sometimes use the word 'true BV solution' to distinguish BV solutions from 'parametrized BV solutions', hence there is no difference between BV solutions and true BV solutions. Definition 6.5 below is a natural extension of the concept of BV solutions introduced in [MRS16a, Def. 3.10], now taking care of the equilibrium condition (6.10a) for u corresponding to the regime E_{u}, the local stability condition (6.10b) for z corresponding to the regime R_{z}, and an energy-dissipation balance (6.10c). Hence, all jump behavior is compressed into the definition of the Finsler cost $\mathrm{cost}_{\mathfrak{M}_0^\alpha}$, the total $\mathfrak{M}_0^\alpha$-variation, and the validity of the energy-dissipation balance.

Definition 6.5 (BV solutions)**.** *Let the rate-independent system* $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ *fulfill Hypothesis 6.4. A curve* $q=(u,z) : [0,T] \to \mathbf{Q}$ *is called a* Balanced-Viscosity solution *to* $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ *if satisfies the following conditions:*

- $u \in \mathrm{BV}([0,T];\mathbf{U})$ *and* $z \in \mathrm{R}([0,T];\mathbf{Z}) \cap \mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}})$*;*
- *the stationary equation*

$$\mathscr{S}_{\mathsf{u}}^*(t,q(t)) = 0 \quad \textit{for all } t \in [0,T]\setminus \mathrm{J}[q]; \tag{6.10a}$$

- *the local stability condition*

$$\mathscr{S}_{\mathsf{z}}^*(t,q(t)) = 0 \quad \textit{for all } t \in [0,T]\setminus \mathrm{J}[q]; \tag{6.10b}$$

- *the energy-dissipation balance*

$$\mathcal{E}(t,q(t)) + \mathrm{Var}_{\mathfrak{M}_0^\alpha}(q;[s,t]) = \mathcal{E}(s,q(s)) + \int_s^t \partial_t\mathcal{E}(r,q(r))\,\mathrm{d}r \quad \textit{for } 0 \leq s \leq t \leq T\,. \tag{6.10c}$$

We postpone to Section 6.4 a result comparing parametrized and true BV solutions. With the exception of our existence results Theorems 6.8 and 6.12, in the following statements we will omit to explicitly recall the assumptions of Section 4; we will only invoke the chain rule from Hyp. 5.7.

6.2. Characterization and fine properties of BV solutions.

In the same way as for their parametrized version, thanks to the chain rule (6.9) we have a characterization of BV solutions in terms of the upper energy estimate $\leq$ in (4.18), on the *whole* interval $[0,T]$. We also have a second characterization in terms of a simple energy-dissipation balance like for energetic solutions as in [MiT99, DaT02, Mie05, MiR15], combined with jump conditions that balance the different dissipation mechanics that may be active at a jump point. The proof of Proposition 6.6 follows, with minimal changes, from the arguments for [MRS16a, Cor. 3.14, Thm. 3.15], to which the reader is referred.

Proposition 6.6. *Let the rate-independent system* $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ *fulfill Hypothesis 5.7. For a curve* $q=(u,z) \in \mathrm{BV}([0,T];\mathbf{U})\times(\mathrm{R}([0,T];\mathbf{Z})\cap\mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}}))$ *fulfilling the stationary equation* (6.10a)*, and the local stability* (6.10b)*, the following three assertions are equivalent:*

(1) q *is a BV solution of system* $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$*;*

(2) q *fulfills*

$$\mathcal{E}(T,q(T)) + \mathrm{Var}_{\mathfrak{M}_0^\alpha}(q;[0,T]) \leq \mathcal{E}(0,q(0)) + \int_0^T \partial_t\mathcal{E}(r,q(r))\,\mathrm{d}r; \tag{6.11}$$

(3) q *fulfills the* $\mathcal{R}$*-energy-dissipation inequality*

$$\mathcal{E}(t,q(t)) + \mathrm{Var}_{\mathcal{R}}(q;[s,t]) \leq \mathcal{E}(s,q(s)) + \int_s^t \partial_t\mathcal{E}(r,q(r))\,\mathrm{d}r \tag{6.12}$$

with $\mathrm{Var}_{\mathcal{R}}$ *from* (6.2)*,* and *the jump conditions at every* $t \in \mathrm{J}[q]$:

$$\begin{aligned}\mathcal{E}(t,q(t^-)) - \mathcal{E}(t,q(t)) &= \mathrm{cost}_{\mathfrak{M}_0^\alpha}(t;q(t^-),q(t)),\\ \mathcal{E}(t,q(t)) - \mathcal{E}(t,q(t^+)) &= \mathrm{cost}_{\mathfrak{M}_0^\alpha}(t;q(t),q(t^+)).\end{aligned} \tag{6.13}$$

Conditions (6.13) provide a fine description of the behavior of BV solutions (u, z) at jumps. However, the inf in the definition of $\mathrm{cost}_{\mathfrak{M}_0^\alpha}$ need not be attained, as the functional $\mathfrak{M}_0^\alpha$ does not control the norm of the space where we look for the ϑ_{u}-component of admissible transition curves. Nonetheless, in certain situations (cf. the proof of Theorem 6.15 below) the existence of transitions attaining the optimal cost will play a key role. In fact, it will be sufficient to require the existence of these curves in cases in which the Finsler cost equals the energy release, which happens at the jump points of a true BV solution as in (6.13). That is why, hereafter we will refer to such transitions as *optimal jump transitions*, a notion that will be made precise in Definition 6.7. Therein we restrict to transition curves, defined on $[0, 1]$, connecting points $q_- = (u_-, z_-)$ and $q_+ = (u_+, z_+)$ such that the u-components u_- and u_+ are at equilibrium, and the z-components z_- and z_+ are locally stable.

Definition 6.7. *Given* $t \in [0, T]$ *and* $q_- = (u_-, z_-)$, $q_+ = (u_+, z_+) \in \mathbf{Q}$ *fulfilling* $\mathscr{S}^*_{\mathsf{u}}(t, q_\pm) = \mathscr{S}^*_{\mathsf{z}}(t, q_\pm) = 0$*, we call an admissible curve* $\Theta \in \mathcal{A}_t(q_-, q_+)$ *an* optimal transition *between* q_- *and* q_+ *at time* t *if it fulfills*

$$\mathcal{E}(t, q_-) - \mathcal{E}(t, q_+) = \mathrm{cost}_{\mathfrak{M}_0^\alpha}(t; q_-, q_+) = \mathfrak{M}_0^\alpha[t, \Theta, 0, \Theta'] \quad \text{a.e. in } (0, 1)\,.$$

Furthermore, we say that $\Theta = (\theta_{\mathsf{u}}, \theta_{\mathsf{z}})$ *is of*

- sliding type *if* $\mathscr{S}^*_{\mathsf{u}}(t, \Theta(r)) = \mathscr{S}^*_{\mathsf{z}}(t, \Theta(r)) = 0$ *for all* $r \in [0, 1]$*;*
- viscous type $\mathscr{S}^*_{\mathsf{u}}(t, \Theta(r)) + \mathscr{S}^*_{\mathsf{z}}(t, \Theta(r)) > 0$ *for all* $r \in (0, 1)$*.*

Observe that an optimal transition of *viscous* type can be governed by viscosity either in u, or in z, or in both variables. With the very same argument as for the proof of [MRS16a, Prop. 3.19], to which we refer for all details, we can also show that every optimal transition can be decomposed in a canonical way into an (at most) countable collection of *sliding* and *viscous* transitions. We also refer to [RSV21, Sec. 2.3] for the concept of so-called *two-speed solutions*, which are defined in terms of slow rate-independent parts connected by jumps which themselves a concatenation of at most countable 'jump resolution maps'.

6.3. Existence of BV solutions.

A most interesting feature of BV solutions, already observed in [MRS16a], is that it is possible to prove their existence by directly taking the vanishing-viscosity limit of the viscous system (4.11), *without* reparametrization. In the following result, we take a slightly different viewpoint and in fact prove that every limit point q (in the sense of pointwise weak convergence) of a sequence of viscous solutions $(q_{\varepsilon_k})_k = (u_{\varepsilon_k}, z_{\varepsilon_k})_k$, starting from well-prepared initial data and such that the $\mathrm{BV}([0, T]; \mathbf{U})$-norm of $(u_{\varepsilon_k})_k$ is *a priori bounded* (cf. (6.15) below), is in fact a true BV solution. In fact, the existence of limit points can be proved, based on the energy estimates from Lemma 4.12 and on (6.15), via a standard compactness argument and the Helly Theorem.

The statement of Theorem 6.8 below mirrors that of Theorem 5.11:

- First, (6.15) corresponds exactly to the a priori estimate for $\|\mathsf{u}'_{\varepsilon_k}\|_{\mathbf{U}}$ in (5.27), and to estimate (4.29) established in Proposition 4.13. Sufficient conditions for this estimate have been discussed in Section 4.6; alternatively, in concrete examples this estimate could be verified by direct calculations.
- Secondly, in the same way as with (5.30) for parametrized solutions, with (6.17b)–(6.17c) ahead we are stating the convergence of the left-hand side terms in the viscous energy-dissipation estimate (4.20) - in particular, (6.17c) ensures the convergence for $\varepsilon_k \to 0^+$ of

$$\begin{aligned} &\int_s^t \mathfrak{M}^\alpha_{\varepsilon_k}(r, q_{\varepsilon_k}(r), 1, q'_{\varepsilon_k}(r))\, \mathrm{d}r \\ &= \int_s^t \left(\mathcal{V}_{\mathsf{u}}^{\varepsilon_k^\alpha}(u'_{\varepsilon_k}(r)) + \mathcal{R}(z'_{\varepsilon_k}(r)) + \mathcal{V}_{\mathsf{z}}^{\varepsilon_k}(z'_{\varepsilon_k}(r)) + \frac{\mathscr{S}^*_{\mathsf{u}}(r, q_{\varepsilon_k}(r))}{{\varepsilon_k}^\alpha} + \frac{\mathscr{S}^*_{\mathsf{z}}(r, q_{\varepsilon_k}(r))}{\varepsilon_k} \right) \mathrm{d}r \end{aligned} \tag{6.14}$$

  to the corresponding terms in the energy-dissipation balance (6.10c). We emphasize here that, for (6.17c) to hold it is crucial that the definition of the total variation functional $\mathrm{Var}_{\mathfrak{M}_0^\alpha}$, in the general closed subinterval $[s, t] \subset [0, T]$, takes into account the appropriate contributions at the jump points. In particular, we point out that, by (6.7), also the jumps occurring at the extrema s and t are taken into account exactly, in the sense that $\mathrm{cost}_{\mathfrak{M}_0^\alpha}(s; q(s), q(s^+)) = \lim_{\sigma\to s^+} \lim_{\varepsilon_k \to 0} \int_0^\sigma \mathfrak{M}^\alpha_{\varepsilon_k}(\cdot)\, \mathrm{d}r$.

Theorem 6.8 (Convergence to BV solutions)**.** *Let the rate-independent system* $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha} + \mathcal{R} + \mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ *fulfill Hypotheses 4.1, 4.2, 4.3, 4.5, 4.10, and 5.7. For any null sequence* $(\varepsilon_k)_k$ *let* $(q_{\varepsilon_k})_k = (u_{\varepsilon_k}, z_{\varepsilon_k})_k \subset \mathrm{AC}([0, T]; \mathbf{Q})$ *be a sequence of solutions to the generalized gradient system* (4.11)*, such that convergences* (5.26) *to a pair* $(u_0, z_0) \in \mathrm{D}$ *hold at the initial time* $t = 0$*, and such that, in addition,*

$$\widehat{S} = \sup_k \|u_{\varepsilon_k}\|_{\mathrm{BV}([0,T];\mathbf{U})} < \infty. \tag{6.15}$$

Let $q:[0,T]\to\mathbf{Q}$ *be such that, along a not relabeled subsequence, there holds as* $k\to\infty$

$$q_{\varepsilon_k}(t)\rightharpoonup q(t)\quad \text{in } \mathbf{Q}\quad \text{for all } t\in[0,T] \tag{6.16}$$

(every sequence in the above conditions possesses at least one limit point in the sense of (6.16)*). Then,*

(1) $q=(u,z)\in \mathrm{BV}([0,T];\mathbf{U})\times(\mathrm{R}(0,T;\mathbf{Z})\cap BV([0,T];\mathbf{Z}_{\mathrm{ri}}))$*, and* q *is a true BV solution to the rate-independent system* $(\mathbf{U}\times\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$*;*

(2) there hold the additional convergences as $k\to\infty$

$$u_{\varepsilon_k}(t)\rightharpoonup u(t) \text{ in } \mathbf{U}_{\mathrm{e}},\qquad z_{\varepsilon_k}(t)\rightharpoonup z(t) \text{ in } \mathbf{Z}_{\mathrm{e}}\qquad \text{for all } t\in[0,T], \tag{6.17a}$$

$$\mathcal{E}(t,q_{\varepsilon_k}(t))\to\mathcal{E}(t,q(t))\quad \text{for all } t\in[0,T], \tag{6.17b}$$

$$\lim_{k\to\infty}\int_s^t \mathfrak{M}_{\varepsilon_k}^{\alpha}(r,q_{\varepsilon_k}(r),1,q_{\varepsilon_k}'(r))\,\mathrm{d}r=\mathrm{Var}_{\mathfrak{M}_0^\alpha}(q;[s,t])\qquad \text{for all } 0\le s\le t\le T. \tag{6.17c}$$

The proof will be carried out in Section 7.2.

Remark 6.9 (Vanishing-viscosity approximation versus BV solutions)**.** *We emphasize that the concept of BV solutions enjoys better closedness properties than defining solutions simply as all the limiting points in the vanishing-viscosity approximation. Such solutions are called '*approximable*' in* [Mie11] *and there, in Examples 2.5 and 2.6, it is shown in a simple model with* $\mathbf{Z}=\mathbb{R}$ *that there are more BV solutions than approximable solutions. It is also made apparent that, for systems with* δ*-dependent energy* $\mathcal{E}_\delta$*, approximable solutions* $q^\delta:[0,T]\to\mathbb{R}$ *may have a limit* q^{δ_*}*for* $\delta\to\delta_*$ *that is no longer an approximable solution, but* q^{δ_*} *is still a BV solution. Thus, BV solutions seem to have better stability properties, see e.g.* [MRS13, Thm. 4.8].

Remark 6.10 (Existence of BV solutions by time discretization)**.** *Another interesting features of true BV solutions is that they can be obtained as limits of discrete solutions of the time-incremental scheme*

$$q_{\tau,\varepsilon}^n\in\operatorname*{Argmin}_{q\in\mathbf{Q}}\Big\{\tau\Psi_{\varepsilon,\alpha}\Big(\frac{q-q_{\tau,\varepsilon}^{n-1}}{\tau}\Big)+\mathcal{E}(t_\tau^n,q)\Big\},\quad n=1,\ldots,N_\tau \tag{6.18}$$

with $\Psi_{\varepsilon,\alpha}$ *from* (1.7)*, as the viscosity parameter* ε and *the time-step* τ *jointly tend to* 0*. (Of course,* fixing $\varepsilon>0$ *and letting* $\tau\to0^+$ *in* (6.18) *gives rise to solutions* $q_\varepsilon:[0,T]\to\mathbf{Q}$ *of the generalized gradient system* (1.6)*). This alternative construction of BV solutions in the joint discrete-to-continuous and vanishing-viscosity limit of the time-incremental scheme for viscous solutions was carefully explored in* [MRS12a, Thm. 4.10] *and* [MRS16a, Thm. 3.12]*. Following these lines it is possible to show convergence to BV solutions along (a subsequence of) any sequence* (τ_k,ε_k) *as long as* τ_k *tends to* 0 *faster than the time scales in our system, i.e.*

$$\lim_{k\to\infty}\frac{\tau_k}{\min\{\varepsilon_k^\alpha,\varepsilon_k\}}=0\,. \tag{6.19}$$

To avoid overburdening of the exposition here we refrain from giving a precise convergence statement, but refer to [MRS16a, Thm. 3.12]*, which can adapted to our setup using condition* (6.19)*. The same applies to the convergence of time-discrete solutions to enhanced BV solutions introduced below.*

6.4. Enhanced BV solutions.

This solution concept is to be compared with the notion introduced in [MRS16a, Def. 3.21] and, of course, with enhanced pBV solutions. In particular, recall that for an enhanced pBV solution $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z})$ we required the additional regularity $\mathsf{z}\in\mathrm{AC}([0,\mathsf{S}];\mathbf{Z})$. Accordingly, an *enhanced BV solution* $q=(u,z)$ is required to fulfill $z\in\mathrm{BV}([0,T];\mathbf{Z})$. Moreover, enhanced BV solutions enjoy the additional regularity property that at all jump points the left and right limits are connected by optimal transitions with finite length in $\mathbf{U}\times\mathbf{Z}$, such that the total length of the connecting paths $\vartheta=(\vartheta_{\mathsf{u}},\vartheta_{\mathsf{z}})$ is finite. In contrast, for general BV solutions it is only required that length of the ϑ_{u}-component of an optimal jump transition is finite in $\mathbf{U}$.

Definition 6.11 (Enhanced BV solutions)**.** *A curve* $q:(u,z):[0,T]\to\mathbf{Q}$ *is called an* enhanced BV solution *of* $(\mathbf{U}\times\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow0}$*, if it is a BV solution and it satisfies the following additional properties:*

(i) $q\in\mathrm{BV}([0,T];\mathbf{Q})$*;*

(ii) *for all* $t\in\mathrm{J}[q]$ *there exists an optimal jump transition* $\vartheta^t=(\vartheta_{\mathsf{u}}^t,\vartheta_{\mathsf{z}}^t)\in\mathcal{A}_t(q(t^-),q(t^+))$ *such that* $\vartheta^t\in\mathrm{AC}([0,1];\mathbf{Q})$ *and* $q(t)=\vartheta^t(\hat r_t)$ *for some* $\hat r_t\in[0,1]$*;*

(iii) $\sum_{t\in\mathrm{J}[q]}\int_0^1\|(\vartheta^t)'(r)\|_{\mathbf{Q}}\,\mathrm{d}r=\sum_{t\in\mathrm{J}[q]}\int_0^1\big(\|(\vartheta_{\mathsf{u}}^t)'(r)\|_{\mathbf{U}}+\|(\vartheta_{\mathsf{z}}^t)'(r)\|_{\mathbf{Z}}\big)\,\mathrm{d}r<\infty.$

Our existence result for enhanced BV solutions can be again proved by our vanishing-viscosity approach without reparametrizing the trajectories, by taking the vanishing-viscosity limit of viscous solutions that satisfy an additional estimate on $\sup_{k\in\mathbb{N}}\|z_{\varepsilon_k}\|_{\mathrm{BV}([0,T];\mathbf{Z})}$.

Theorem 6.12 (Convergence of viscous solutions to enhanced BV solutions)**.** *Assume Hypotheses 4.1, 4.2, 4.3, 4.5, 4.10, and 5.7. Let* $(q_{\varepsilon_k})_k \subset \mathrm{AC}([0,T];\mathbf{Q})$ *be a sequence of solutions to the generalized gradient system* (1.4) *such that convergences* (5.26) *hold at* $t=0$*, as well as*

$$\exists\, S>0\ \forall\, k\in\mathbb{N}: \qquad \|q_{\varepsilon_k}\|_{\mathrm{BV}([0,T];\mathbf{Q})} \le \widehat{S}. \tag{6.20}$$

Let $q:[0,T]\to\mathbf{Q}$ *be a limit point for* $(q_{\varepsilon_k})_k$ *in the sense of* (6.16)*. Then, q is an* enhanced BV solution *of* $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$*, and the additional convergences* (6.17) *hold.*

Since the proof of Theorem 6.12 follows from combining the argument for Theorem 6.8 with that developed for [MRS16a, Thm. 3.22], it is omitted.

6.5. Comparing pBV and true BV solutions.

In this final subsection we explore the relations between parametrized and true BV solutions, also in the enhanced case. Indeed, there is a very natural transition between parametrized and true BV solutions. The converse passage will be obtained by 'filling the graph' of a true BV solution at its jump points, by means of an optimal jump transition, under the *additional* assumption that it exists. This condition is codified in the following

Hypothesis 6.13. *For every* $t\in[0,T]$ *and* $q^-,q^+\in\mathbf{Q}$ *such that* $\mathscr{S}_{\mathsf{u}}^*(t,q_\pm)=\mathscr{S}_{\mathsf{z}}^*(t,q_\pm)=0$ *and*

$$\mathcal{E}(t,q^-)-\mathcal{E}(t,q^+) = \mathrm{cost}_{\mathfrak{M}_0^\alpha}(t;q^-,q^+)$$

there exists an optimal jump transition $\Theta^{\mathrm{opt}}\in\mathcal{A}_t(q^-,q^+)$*.*

Remark 6.14. *Let us emphasize that Hypothesis 6.13 plays no role in proving the existence of BV solutions. It only serves the purpose of showing that a true BV solution gives rise to a parametrized one. In this connection, let us mention in advance that, in the statement of Theorem 6.15, Hypothesis 6.13 will not be required for relating* enhanced *BV solutions to their parametrized analogues, as the definition of enhanced BV solutions already encompasses the information that optimal jump transitions exist.*

We are now ready to state the following relations between true and parametrized BV solutions.

Theorem 6.15 (pBV versus true BV solutions)**.** *Let* $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ *fulfill Hypothesis 5.7. Then the following statements are true:*

(1) If $(\mathsf{t},\mathsf{q}):[0,\mathsf{S}]\to[0,T]{\times}\mathbf{Q}$ *is a non-degenerate pBV solution of* $(\mathbf{U}{\times}\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}{+}\mathcal{R}{+}\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ *with* $\mathsf{t}(0)=0$ *and* $\mathsf{t}(\mathsf{S})=T$*, then every* $q:[0,T]\to\mathbf{Q}$ *satisfying*

$$q(t)\in\left\{\,\mathsf{q}(s)\,\middle|\,\mathsf{t}(s)=t\,\right\} \tag{6.21}$$

is a (true) BV solution that enjoys, moreover, the following property: for every $t\in \mathrm{J}[q]$ *there exists an optimal jump transition* $\Theta^{\mathrm{opt}}\in\mathcal{A}_t(q(t^-),q(t^+))$ *such that* $q(t)=\Theta^{\mathrm{opt}}(\hat r)$ *for some* $\hat r\in[0,1]$*. Furthermore, there holds*

$$\mathrm{Var}_{\mathfrak{M}_0^\alpha}(q;[t_0,t_1]) = \int_{\mathsf{s}(t_0)}^{\mathsf{s}(t_1)} \mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](s)\,\mathrm{d}s \quad \text{for all } 0\le t_0\le t_1\le T\,. \tag{6.22}$$

(2) Conversely, assume additionally Hypothesis 6.13. Then, for every BV solution $q:[0,T]\to\mathbf{Q}$*, there exists a non-degenerate, surjective pBV solution* $(\mathsf{t},\mathsf{q})\in\mathscr{A}([0,\mathsf{S}];[0,T]{\times}\mathbf{Q})$ *such that* (6.21) *and* (6.22) *hold.*

(3) If $(\mathsf{t},\mathsf{q}):[0,\mathsf{S}]\to[0,T]{\times}\mathbf{Q}$ *is a (non-degenerate) enhanced pBV solution with* $\mathsf{t}(0)=0$ *and* $\mathsf{t}(\mathsf{S})=T$*, then every* $q:[0,T]\to\mathbf{Q}$ *given by* (6.21) *is an enhanced BV solution, and* (6.22) *holds.*

(4) Conversely, for any enhanced BV solution $q:[0,T]\to\mathbf{Q}$*, there exists a (non-degenerate, surjective) enhanced pBV solution* $(\mathsf{t},\mathsf{q})\in\mathscr{A}([0,\mathsf{S}];[0,T]{\times}\mathbf{Q})$ *such that* (6.21) *and* (6.22) *hold.*

Remark 6.16 (Greater generality of true BV solutions)**.** *Theorem 6.15 seems to suggest that true BV solutions are more general than their parametrized analogues. Indeed, while, under the standing assumptions of Section 4, parametrized solutions always give rise to true BV ones, the converse passage is possible under the additional Hypothesis 6.13. Hence, the set of true BV solutions is apparently bigger.*

To emphasize this, we have chosen to prove that any limit curve q for a sequence $(q_{\varepsilon_k})_k$ *of (non-parametrized) viscous solutions is a true BV solution, as stated in Theorem 6.8, by resorting to Theorem 5.11 for parametrized solutions. Namely, in Sec. 7.2 we will use that the graphs of a sequence* $(q_{\varepsilon_k})_k$ *of viscous solutions are contained in the image sets of their parametrized counterparts* $(\mathsf{t}_{\varepsilon_k},\mathsf{q}_{\varepsilon_k})_k$ *and apply Theorem 5.11 to the latter curves, guaranteeing their convergence to a pBV solution* (t,q)*. We will then proceed to showing that q and* (t,q) *are related by* (6.21) *and thus conclude, by Thm. 6.15(1), that q is a true BV solution.*

Proof. Step 1: From pBV to BV solutions. First, we show that, given a pBV solution $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z})$, formula (6.21) defines a curve $q = (u,z) \in \mathrm{BV}([0,T];\mathbf{U}){\times}(\mathrm{R}(0,T;\mathbf{Z}){\cap}\mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}})$. Indeed, let $\mathsf{s} : [0,T] \to [0,\mathsf{S}]$ be any inverse of t, with jump set $\mathrm{J}[\mathsf{s}]$. It can be easily checked that, since $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z})$ is non-degenerate,

$$t \in \mathrm{J}[q] = \mathrm{J}[u] \cup \mathrm{J}[z] \qquad \Longleftrightarrow \qquad t \in \mathrm{J}[\mathsf{s}] \text{ and } \mathsf{t}(s) \equiv t \text{ for all } s \in [\mathsf{s}(t^-),\mathsf{s}(t^+)]\,.$$

If for $t \in \mathrm{J}[\mathsf{s}]$ we have $q(t) = \mathsf{q}(s_*)$ for some $s_* \in [\mathsf{s}(t^-),\mathsf{s}(t^+)]$, then defining $\mathsf{s}(t) := s_*$ gives the identity

$$q(t) = (u(t),z(t)) = \mathsf{q}(\mathsf{s}(t)) = (\mathsf{u}(\mathsf{s}(t)),\mathsf{z}(\mathsf{s}(t))) \qquad \text{for all } t \in [0,T]. \tag{6.23}$$

From this, we deduce $u \in \mathrm{BV}([0,T];\mathbf{U})$ and $z \in \mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}})$. Moreover, since $\sup_{t\in[0,T]} \mathfrak{E}(q(t)) \le E$ for some $E > 0$ and the functional $\mathfrak{E} + \|\cdot\|_{\mathbf{U}} + \|\cdot\|_{\mathbf{Z}_{\mathrm{ri}}}$ has sublevels bounded in $\mathbf{U}_{\mathrm{e}}{\times}\mathbf{Z}_{\mathrm{e}}$, we also have $z \in \mathrm{L}^\infty(0,T;\mathbf{Z}_{\mathrm{e}})$, which gives $z \in \mathrm{R}(0,T;\mathbf{Z})$ thanks to (6.5).

From (6.23) we easily deduce that

$$\mathrm{Var}_{\mathcal{R}}(z;[t_0,t_1]) = \int_{\mathsf{s}(t_0)}^{\mathsf{s}(t_1)} \mathcal{R}[\mathsf{u}'](s)\,\mathrm{d}s \qquad \text{for all } 0 \le t_0 \le t_1 \le T\,. \tag{6.24}$$

Furthermore, we mimic the argument from the proof of [MRS16a, Prop. 4.7] and observe that for every $t \in \mathrm{J}[q]$ the curve $\mathsf{q} = (\mathsf{u},\mathsf{z}) : [\mathsf{s}(t^-),\mathsf{s}(t^+)] \to \mathbf{U}{\times}\mathbf{Z}$, reparametrized in such a way that it is defined on the interval $[0,1]$, is an admissible transition curve between $q(t^-)$ and $q(t^+)$. Hence,

$$\mathrm{cost}_{\mathfrak{M}_0^\alpha}(t;q(t^-),q(t)) \le \int_{\mathsf{s}(t^-)}^{\mathsf{s}(t)} \mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},0,\mathsf{q}'](s)\,\mathrm{d}s, \quad \mathrm{cost}_{\mathfrak{M}_0^\alpha}(t;q(t),q(t^+)) \le \int_{\mathsf{s}(t)}^{\mathsf{s}(t^+)} \mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},0,\mathsf{q}'](s)\,\mathrm{d}s\,.$$

Combining this with (6.24) we conclude that

$$\mathrm{Var}_{\mathfrak{M}_0^\alpha}(q;[t_0,t_1]) \le \int_{\mathsf{s}(t_0)}^{\mathsf{s}(t_1)} \mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](s)\,\mathrm{d}s \tag{6.25}$$

for all $[t_0,t_1] \subset [0,T]$. Ultimately, we infer that q fulfills the energy-dissipation estimate (6.11).

In order to show that q complies with the stationary equation (6.10a) and the local stability condition (6.10b), we argue in the following way. Recalling the definition of the sets $\mathscr{G}^\alpha$ from (5.19), we introduce

$$\mathscr{H}^\alpha[q] := \begin{cases} \{t \in [0,T] \,:\, \mathscr{S}_{\mathsf{u}}^*(t,q(t)) = \mathscr{S}_{\mathsf{z}}^*(t,q(t)) = 0\} & \text{if } \alpha \ge 1, \\ \qquad \{t \in [0,T] \,:\, \mathscr{S}_{\mathsf{z}}^*(t,q(t)) = 0\} & \text{if } \alpha \in (0,1). \end{cases}$$

Observe that the set $\mathscr{H}^\alpha[q]$ is dense in $[0,T]$. Indeed, its complement $[0,T]\setminus\mathscr{H}^\alpha[q] = \mathsf{t}(\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}])$ has null Lebesgue measure, since t is constant on each connected component of the open set $\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]$. Therefore, by the lower semicontinuity properties of $\mathscr{S}_{\mathsf{u}}^*$ and $\mathscr{S}_{\mathsf{z}}^*$ ensured by Hypothesis 4.10, in the case $\alpha \ge 1$ we immediately conclude (6.10a) and (6.10b). For $\alpha \in (0,1)$, the above argument only yields (6.10b), and for the validity of (6.10a), we observe that for any $t \notin \mathrm{J}[q]$, then

$$t = \mathsf{t}(\bar{s}) \text{ and } q = \mathsf{q}(\bar{s}) \quad \text{for } \bar{s} \in \overline{\{s \in [0,\mathsf{S}] \,:\, \mathsf{t}'(s) > 0\}}\,.$$

Then, since $\mathscr{S}_{\mathsf{u}}^*(\mathsf{t},\mathsf{q}) \equiv 0$ on the set $\{s \in (0,\mathsf{S}) \,:\, \mathsf{t}'(s) > 0\}$ as prescribed by Definition 5.6, we conclude that $\mathscr{S}_{\mathsf{u}}^*(t,q(t)) = 0$.

Since q complies with (6.10a), (6.10b), and (6.11), by Proposition 6.6 we conclude that it is a true BV solution. In order to conclude (6.22), we observe that, for all $0 \le t_0 \le t_1 \le T$ and $s_0 \le s_1 \in [0,\mathsf{S}]$ such that $\mathsf{t}(s_i) = t_i$ for $i \in \{0,1\}$, there holds

$$\begin{aligned} &\mathrm{Var}_{\mathfrak{M}_0^\alpha}(q;[t_0,t_1]) \overset{(6.10c)}{=} \mathcal{E}(t_0,q(t_0)) - \mathcal{E}(t_1,q(t_1)) + \int_{t_0}^{t_1} \partial_t\mathcal{E}(r,q(r))\,\mathrm{d}r \\ &= \mathcal{E}(\mathsf{t}(s_0),\mathsf{q}(s_0)) - \mathcal{E}(\mathsf{t}(s_1),\mathsf{q}(s_1)) + \int_{s_0}^{s_1} \partial_t\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))\mathsf{t}'(s)\,\mathrm{d}s \overset{(5.24)}{=} \int_{s_0}^{s_1} \mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](s)\,\mathrm{d}s\,. \end{aligned} \tag{6.26}$$

It is immediate to see that the above arguments also yield an enhanced BV solution from any enhanced pBV solution. Hence, assertions (1) and (3) are proved.

Step 2: From BV to pBV solutions First of all, we show that, under the additional Hypothesis 6.13, with any true BV solution $q \in \mathrm{BV}([0,T];\mathbf{U}){\times}(\mathrm{R}(0,T;\mathbf{Z}){\cap}\mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}}))$ we can associate a non-degenerate, surjective curve $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z}) \in \mathscr{A}([0,\mathsf{S}];[0,T]{\times}\mathbf{Q})$ such that (6.21) holds and

$$\mathrm{Var}_{\mathfrak{M}_0^\alpha}(q;[0,T]) = \int_0^{\mathsf{S}} \mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](s)\,\mathrm{d}s\,. \tag{6.27}$$

Indeed, along the lines of [MRS16a, Prop. 4.7] we introduce the parametrization s, defined on $[0,T]$ by

$$\mathsf{s}(t) := t + \mathrm{Var}_{\mathfrak{M}_0^\alpha}(q;[0,t]), \qquad \mathsf{S} := \mathsf{s}(T) \quad \text{with}$$
$$\mathrm{J}[\mathsf{s}] = \mathrm{J}[u] \cup \mathrm{J}[z] = (t_m)_{m\in M} \text{ and } M \text{ a countable set.}$$

We set $I := \cup_{m\in M} I_m$ with $I_m = (\mathsf{s}(t_m^-), \mathsf{s}(t_m^+))$. Hence, we define $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z})$ on $[0,\mathsf{S}] \setminus I$ by $\mathsf{t} := \mathsf{s}^{-1} : [0,\mathsf{S}] \setminus I \to [0,T]$ and $\mathsf{q} := q \circ \mathsf{t}$. In order to extend t and q to I, we need to use the fact that, by Hypothesis 6.13, for every $m \in M$ there exists an optimal jump transition jump transition $\Theta_m^{\mathrm{opt}} \in \mathcal{A}_t(q(t_m^-), q(t_m^+))$, defined on the canonical interval $[0,1]$ and such that $\Theta_m^{\mathrm{opt}}(\hat{r}_m) = q(t_m)$ for some $\hat{r}_m \in [0,1]$. We may then define t and q on $I = \cup_{m\in M} I_m$ by

$$\mathsf{t}(s) \equiv t_m, \qquad \mathsf{q}(s) := \Theta_m^{\mathrm{opt}}(\mathsf{r}_m(s)) \text{ for } s \in I_m, \quad \text{where } \mathsf{r}_m(s) = \frac{s - \mathsf{s}(t_m^-)}{\mathsf{s}(t_m^+) - \mathsf{s}(t_m^-)}\,.$$

It can be easily checked that $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([0,\mathsf{S}];[0,T]\times\mathbf{Q})$. By construction, the curves q and (t,q) satisfy (6.21). Furthermore, recalling (6.24) and the fact that $\Theta_m^{\mathrm{opt}} \in \mathcal{A}_t(q(t_m^-), q(t_m^+))$, it is not difficult to check that (6.27) holds. Therefore, since q is a BV solution, we infer that (t,q) is a pBV solution, and we obtain (6.22) by repeating the argument in (6.26).

This argument also allows us to prove that any enhanced BV solution gives rise to an enhanced pBV solution. Hence, the proof of Theorem 6.15 is finished. □

7. Proof of major results

This section focuses on the proofs of our main existence results for pBV and true BV solutions, i.e. Theorems 5.11 and 6.8. They will be carried out in Sections 7.1 and 7.2, respectively. Moreover, Section 7.3 provides the proof of Proposition 5.18.

Throughout this section and, in particular, in the statement of the various auxiliary results, we will always tacitly assume the validity of Hypotheses 4.1, 4.2, 4.3, 4.5, 4.10, and of the parametrized chain rule from Hyp. 5.7: recall that, by Lemma A.2 it implies the BV-chain rule (6.4).

7.1. Proof of Theorem 5.11.

Our first result lays the ground for the vanishing-viscosity analysis of Theorem 5.11 by settling the compactness properties of a sequence of parametrized curves enjoying the a priori estimates (5.27). We have chosen to extrapolate such properties from the proof of Theorem 5.11, since we believe them to be of independent interest.

Prior to stating Proposition 7.1, let us specify the meaning of the third convergence in (7.2b) below. Indeed, the sequence $(\mathsf{u}_k)_k$ is contained in a closed ball $\overline{B}_R \subset \mathbf{U}$ by virtue of estimate (7.1) (cf. Hypothesis 4.7). Now, since $\mathbf{U}$ is reflexive and separable, it is possible to introduce a distance d_{weak} inducing the weak topology on $\overline{B}_R$. Hence, convergence in $\mathrm{C}^0([0,\mathsf{S}];\mathbf{U}_{\mathrm{weak}})$ means convergence in $\mathrm{C}^0([0,\mathsf{S}];(\mathbf{U},d_{\mathrm{weak}}))$.

Proposition 7.1. *Let $(\mathsf{t}_k,\mathsf{q}_k)_k \subset \mathrm{AC}([0,\mathsf{S}];[0,T]\times\mathbf{Q})$, with t_k non-decreasing and $\mathsf{q}_k = (\mathsf{u}_k,\mathsf{z}_k)$, enjoy the following bounds, along a null sequence $(\varepsilon_k)_k$:*

$$\exists\, C_* \geq 1 \quad \forall\, k \in \mathbb{N} : \quad \begin{cases} \sup_{s\in[0,\mathsf{S}]} \mathfrak{E}(\mathsf{q}_k(s)) \leq C_*, \\ \mathsf{t}_k'(s) + \mathcal{R}(\mathsf{z}_k'(s)) + \mathfrak{M}_{\varepsilon_k}^{\alpha,\mathrm{red}}(\mathsf{t}_k(s),\mathsf{q}_k(s),\mathsf{t}_k'(s),\mathsf{q}_k'(s)) \\ \qquad\qquad + \|\mathsf{u}_k'(s)\|_{\mathbf{U}} \leq C_* \text{ for a.a. } s \in (0,\mathsf{S}). \end{cases} \tag{7.1}$$

Then, there exist an admissible parametrized curve $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z}) \in \mathscr{A}([0,\mathsf{S}];[0,T]\times\mathbf{Q})$ with

$$\mathsf{t} \in \mathrm{C}^0_{\mathrm{lip}}([0,\mathsf{S}];[0,T]), \quad \mathsf{u} \in \mathrm{C}^0_{\mathrm{lip}}([0,\mathsf{S}];\mathbf{U}), \quad \text{and } \mathsf{z} \in \mathrm{C}^0_{\mathrm{lip}}([0,\mathsf{S}];\mathbf{Z}_{\mathrm{ri}}) \cap \mathrm{C}^0([0,\mathsf{S}];\mathbf{Z}), \tag{7.2a}$$

and a (not relabeled) subsequence such that the following convergences hold as $k \to \infty$:

$$\begin{cases} \mathsf{t}_k \to \mathsf{t} \text{ in } \mathrm{C}^0([0,\mathsf{S}]) \quad \text{and} \quad \mathsf{t}_k' \overset{*}{\rightharpoonup} \mathsf{t}' \text{ in } \mathrm{L}^\infty(0,\mathsf{S}), \\ \mathsf{u}_k \overset{*}{\rightharpoonup} \mathsf{u} \text{ in } W^{1,\infty}(0,\mathsf{S};\mathbf{U}), \\ \mathsf{u}_k \to \mathsf{u} \text{ in } \mathrm{C}^0([0,\mathsf{S}];\mathbf{U}_{\mathrm{weak}}) \\ \mathsf{z}_k \to \mathsf{z} \text{ in } \mathrm{C}^0([0,\mathsf{S}];\mathbf{Z}), \\ \mathsf{u}_k(s) \rightharpoonup \mathsf{u}(s) \text{ in } \mathbf{U}_{\mathrm{e}} \text{ and } \mathsf{z}_k(s) \rightharpoonup \mathsf{z}(s) \text{ in } \mathbf{Z}_{\mathrm{e}} \quad \text{for all } s \in [0,\mathsf{S}], \end{cases} \tag{7.2b}$$

$$\int_0^{\mathsf{S}} \mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](\sigma)\,\mathrm{d}\sigma \leq \liminf_{k\to\infty} \int_0^{\mathsf{S}} \mathfrak{M}_{\varepsilon_k}^\alpha(\mathsf{t}_k(\sigma),\mathsf{q}_k(\sigma),\mathsf{t}_k'(\sigma),\mathsf{q}_k'(\sigma))\,\mathrm{d}\sigma\,. \tag{7.2c}$$

Proof. We split the proof in three steps.

Step 1. Compactness: From (7.1) we infer the following compactness information.

(1.A) By the Ascoli-Arzelà Theorem, there exists a non-decreasing $\mathsf{t} \in W^{1,\infty}(0, \mathsf{S})$ such that $\mathsf{t}_k \to \mathsf{t}$ uniformly in $[0, \mathsf{S}]$ and weakly* in $W^{1,\infty}(0, \mathsf{S})$.

(1.B) Since the sequence $(\mathsf{u}_k)_k$ is bounded in $W^{1,\infty}(0, \mathsf{S}; \mathbf{U})$ we conclude that there exists u with the regularity from (7.2a) such that, along a not relabeled subsequence, the second convergence in (7.2b) hold for $(\mathsf{u}_k)_k$. The convergence in $\mathrm{C}^0([0, \mathsf{S}]; \mathbf{U}_{\mathrm{weak}})$ follows from an Ascoli-Arzelà type theorem, see e.g. [AGS08, Prop. 3.3.1]).

(1.C) From $\sup_{s\in[0,\mathsf{S}]} \mathfrak{E}(\mathsf{q}_k(s)) \le C$ we deduce that there exists a ball

$$\overline{B}_M^{\mathbf{Z}_\mathrm{e}} \subset \mathbf{Z}_\mathrm{e} \Subset \mathbf{Z} \text{ such that } \mathsf{z}_k(s) \in \overline{B}_M^{\mathbf{Z}_\mathrm{e}} \text{ for all } s \in [0, \mathsf{S}] \text{ and all } k \in \mathbb{N}. \tag{7.3a}$$

Using $\mathbf{Z}_\mathrm{e} \Subset \mathbf{Z} \subset \mathbf{Z}_\mathrm{ri}$ and the coercivity (4.3b) of $\mathcal{R}$, Ehrling's lemma gives that

$$\forall\, \omega > 0\ \exists\, C_\omega > 0\ \forall\, z \in \overline{B}_M^{\mathbf{Z}_\mathrm{e}} \quad \|z\|_{\mathbf{Z}} \le \omega + C_\omega \mathcal{R}(z).$$

Hence, defining $\Omega_M(r) := \inf_{\omega>0}(\omega{+}C_\omega r)$ and noting that $\Omega_M(\lambda r) \le \lambda\Omega_M(r)$ for all $\lambda \ge 1$, we find

$$\|\mathsf{z}_k(s_1){-}\mathsf{z}_k(s_2)\|_{\mathbf{Z}} \le \Omega_M(\mathcal{R}(\mathsf{z}(s_1){-}\mathsf{z}_k(s_2))) \le C_*\Omega_M(|s_1{-}s_2|) \text{ for all } 0 \le s_1 \le s_2 \le \mathsf{S}, \tag{7.3b}$$

where the last estimate follows from the bound for $\mathcal{R}(\mathsf{z}_k')$ in (7.1). We combine the compactness information provided by (7.3a) with the equicontinuity estimate (7.3b) and again apply, [AGS08, Prop. 3.3.1] to deduce that there exists $\mathsf{z} \in \mathrm{C}^0([0, \mathsf{S}]; \mathbf{Z})$ such that, along a not relabeled subsequence, $(\mathsf{z}_k)_k$ converges to z in the sense of (7.2b).

Let us denote by q the curve (u, z).

Step 2. q is an admissible parametrized curve: Combining the previously found convergences with the first estimate in (7.1), we obtain $\sup_{s\in[0,\mathsf{S}]} \mathfrak{E}(\mathsf{q}(s)) \le C$. Using the second estimate in (7.1) and (4.3b) we have $\|\mathsf{z}(s_2){-}\mathsf{z}(s_1)\|_{\mathbf{Z}_\mathrm{ri}} \le C_*|s_2{-}s_1|/c_{\mathcal{R}}$. With (7.3b) we also infer that $\mathsf{z} \in \mathrm{C}^0_{\mathrm{lip}}([0, \mathsf{S}]; \mathbf{Z}_\mathrm{ri})$.

We will now show that z is locally absolutely continuous in the set $\mathscr{G}^\alpha[t, \mathsf{q}]$ from (5.19). Let us first examine the case $\alpha \in (0, 1)$. Since the function $s \mapsto \mathscr{S}^*_\mathsf{z}(t, \mathsf{q}(s))$ is lower semicontinuous thanks to Hypothesis 4.10, for every $[\varsigma, \beta] \subset \mathscr{G}^\alpha[t, \mathsf{q}]$ there exists $c > 0$ such that $\mathscr{S}^*_\mathsf{z}(t, \mathsf{q}(s)) \ge c$ for all $s \in [\varsigma, \beta]$. This estimate bears two consequences:

(1) Exploiting the *uniform* convergence of z_k to z and again relying on Hypothesis 4.10,

$$\exists\, \bar{k} \in \mathbb{N}\ \forall\, k \ge \bar{k}\ \ \forall\, s \in [\varsigma, \beta]: \qquad \mathscr{S}^*_\mathsf{z}(t, \mathsf{q}_k(s)) \ge \frac{c}{2}. \tag{7.4}$$

This implies that, for $k \ge \bar{k}$, the sets $\mathscr{G}^\alpha[t, \mathsf{q}_k] = \{s : \mathscr{S}^*_\mathsf{z}(t, \mathsf{q}_k(s)) > 0\}$ contain the interval $[\varsigma, \beta]$.

(2) Since, by (7.1), $C_* \ge \mathfrak{M}^{\alpha,\mathrm{red}}_{\varepsilon_k}(\mathsf{t}_k(s), \mathsf{q}_k(s), \mathsf{t}_k'(s), \mathsf{q}_k'(s))$ for almost all $s \in (\varsigma, \beta)$, we are in a position to apply estimate (5.15a) from Lemma 5.4 and deduce that

$$\exists\, \overline{C} > 0\ \exists\, \bar{k} \in \mathbb{N}\ \forall\, k \ge \bar{k} \text{ for a.a. } s \in (\varsigma, \beta): \qquad \|\mathsf{z}_k'(s)\|_{\mathbf{Z}} \le \overline{C}\,. \tag{7.5}$$

The discussion of the case $\alpha \ge 1$ follows the very same lines: for every $[\varsigma, \beta] \subset \mathscr{G}^\alpha[t, \mathsf{q}]$ we find $\tilde{c} > 0$ and $\tilde{k} \in \mathbb{N}$ such that for every $k \ge \tilde{k}$ we have $\mathscr{S}^*_\mathsf{u}(t, \mathsf{q}_k(s)){+}\mathscr{S}^*_\mathsf{z}(t, \mathsf{q}_k(s)) \ge \frac{\tilde{c}}{2}$ for every $s \in [\varsigma, \beta]$. Then, estimate (7.5) follows from (5.15b) in Lemma 5.4.

All in all, for all $\alpha > 0$ the curves z_k are uniformly $\mathbf{Z}$-Lipschitz on $[\varsigma, \beta]$. This entails that z is ultimately $\mathbf{Z}$-Lipschitz on any subinterval $[s_1, s_2] \subset \mathscr{G}^\alpha[t, \mathsf{q}]$, and reflexivity of $\mathbf{Z}$ gives us

$$\mathsf{z}_k \overset{*}{\rightharpoonup} \mathsf{z} \text{ in } W^{1,\infty}(\varsigma, \beta; \mathbf{Z}) \quad \text{for all } [s_1, s_2] \subset \mathscr{G}^\alpha[t, \mathsf{q}]. \tag{7.6}$$

Step 3. Proof of (7.2c): In order to conclude that $(\mathsf{t}, \mathsf{q}) \in \mathscr{A}([0, \mathsf{S}]; [0, T]{\times}\mathbf{Q})$, it remains to show that it fulfills property (5.20), which will a consequence of (7.2c). By the lower semicontinuity we have

$$\liminf_{k\to\infty} \int_0^\mathsf{S} \mathcal{R}(\mathsf{z}_k'(s))\,\mathrm{d}s \overset{(1)}{=} \liminf_{k\to\infty} \mathrm{Var}_{\mathcal{R}}(\mathsf{z}_k; [0, \mathsf{S}]) \ge \mathrm{Var}_{\mathcal{R}}(\mathsf{z}; [0, \mathsf{S}]) \overset{(2)}{=} \int_0^\mathsf{S} \mathcal{R}[\mathsf{z}'](s)\,\mathrm{d}s, \tag{7.7}$$

with $\overset{(1)}{=}$ and $\overset{(2)}{=}$ due to (6.3). Furthermore, we have

$$\begin{aligned} \liminf_{k\to\infty} \int_0^\mathsf{S} \mathfrak{M}^{\alpha,\mathrm{red}}_{\varepsilon_k}(\mathsf{t}_k, \mathsf{q}_k, \mathsf{t}_k', \mathsf{q}_k')\,\mathrm{d}s &\ge \liminf_{k\to\infty} \int_{(0,\mathsf{S})\cap\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]} \mathfrak{M}^{\alpha,\mathrm{red}}_{\varepsilon_k}(\mathsf{t}_k, \mathsf{q}_k, \mathsf{t}_k', \mathsf{q}_k')\,\mathrm{d}s \\ &\overset{(3)}{\ge} \int_{(0,\mathsf{S})\cap\mathscr{G}^\alpha[\mathsf{t},\mathsf{q}]} \mathfrak{M}^{\alpha,\mathrm{red}}_0(\mathsf{t}, \mathsf{q}, 0, \mathsf{q}')\,\mathrm{d}s\,. \end{aligned} \tag{7.8}$$

Here, $\overset{(3)}{\geq}$ follows from Proposition 5.12, applied to the functionals $\mathfrak{M}_{\varepsilon_k}^{\alpha,\mathrm{red}}$ and $\mathfrak{M}_0^{\alpha,\mathrm{red}}$, which we consider restricted to the (weakly closed, by assumption (4.7a)) energy sublevel $\mathbf{S} = \{q \in \mathbf{Q} : \mathfrak{E}(q) \leq C\}$. Combining (7.7) and (7.8), we infer (7.2c) and thus conclude the proof of Proposition 7.1. □

We are now in the position to conclude the

Proof of Theorem 5.11. Let $(\mathsf{t}_{\varepsilon_k}, \mathsf{q}_{\varepsilon_k})_k$ be a sequence of rescaled viscous trajectories satisfying (5.27). We apply Proposition 7.1 and conclude that there exist a limit parametrized curve $(\mathsf{t}, \mathsf{q}) \in \mathscr{A}([0, \mathsf{S}]; [0, T]\times\mathbf{Q})$, fulfilling (5.28), and a (not relabeled) subsequence along which convergences (5.29) hold.

We now show that the curves (t, q) fulfill the upper energy-dissipation estimate $\leq$ in (5.24) by passing to the limit as $\varepsilon_k \to 0^+$ in (5.4) for $s_1 = 0$ and $s_2 = s \in (0, \mathsf{S}]$. The key lower semicontinuity estimate

$$\int_0^s \mathfrak{M}_0^\alpha[\mathsf{t}, \mathsf{q}, \mathsf{t}', \mathsf{q}'](\sigma)\, \mathrm{d}\sigma \leq \liminf_{k\to\infty} \int_0^s \mathcal{M}_\varepsilon^\alpha(\mathsf{t}_{\varepsilon_k}(\sigma), \mathsf{q}_{\varepsilon_k}(\sigma), \mathsf{t}'_{\varepsilon_k}(\sigma), \mathsf{q}'_{\varepsilon_k}(\sigma))\, \mathrm{d}\sigma \quad \text{for all } s \in [0, \mathsf{S}] \tag{7.9}$$

follows from (7.2c) in Proposition 7.1. Convergences (5.29), the lower semicontinuity (4.7b) of $\mathcal{E}$, and the continuity (4.14) of $\partial_t\mathcal{E}$ give for all $s \in [0, \mathsf{S}]$ that

$$\liminf_{k\to\infty} \mathcal{E}(\mathsf{t}_{\varepsilon_k}(s), \mathsf{q}_{\varepsilon_k}(s)) \geq \mathcal{E}(\mathsf{t}(s), \mathsf{q}(s)) \quad \text{and} \quad \int_0^s \partial_t\mathcal{E}(\mathsf{t}_{\varepsilon_k}, \mathsf{q}_{\varepsilon_k})\mathsf{t}'_{\varepsilon_k}\, \mathrm{d}\sigma \to \int_0^s \partial_t\mathcal{E}(\mathsf{t}, \mathsf{q})\mathsf{t}'\, \mathrm{d}\sigma\,. \tag{7.10}$$

For the last convergence we use $\mathsf{t}'_k \overset{*}{\rightharpoonup} \mathsf{t}'$ in $\mathrm{L}^\infty(0, \mathsf{S})$ and $|\partial_t\mathcal{E}(\mathsf{t}_{\varepsilon_k}(\sigma), \mathsf{q}_{\varepsilon_k}(\sigma))| \leq C_\#\mathcal{E}(\mathsf{t}_{\varepsilon_k}(\sigma), \mathsf{q}_{\varepsilon_k}(\sigma)) \leq C$ by (4.7d) and (4.26), which together with (4.14) gives $\partial_t\mathcal{E}(\mathsf{t}_{\varepsilon_k}, \mathsf{q}_{\varepsilon_k}) \to \partial_t\mathcal{E}(\mathsf{t}, \mathsf{q})$ strongly in $\mathrm{L}^2(0, \mathsf{S})$.

Taking into account the convergence of the initial energies guaranteed by (5.26), we complete the limit passage in (5.4). Thanks to Lemma 5.10, the validity of the upper energy-dissipation estimate in (5.24) ensures that $(\mathsf{t}, \mathsf{q}) = (\mathsf{t}, \mathsf{u}, \mathsf{z})$ is a pBV solution.

The enhanced convergences (5.30a) and (5.30b) are a by-product of this limiting procedure. Although the argument is standard, we recap it for the reader's convenience and later use, and introduce the following place-holders for every $s \in [0, \mathsf{S}]$:

$$\begin{cases} E_{\varepsilon_k}^s := \mathcal{E}(\mathsf{t}_{\varepsilon_k}(s), \mathsf{q}_{\varepsilon_k}(s)), & E_0^s := \mathcal{E}(\mathsf{t}(s), \mathsf{q}(s)) \\ M_{\varepsilon_k}^s := \int_0^s \mathfrak{M}_{\varepsilon_k}^\alpha(\mathsf{t}_{\varepsilon_k}(\sigma), \mathsf{q}_{\varepsilon_k}(\sigma), \mathsf{t}'_{\varepsilon_k}(\sigma), \mathsf{q}'_{\varepsilon_k}(\sigma))\ \mathrm{d}\sigma & M_0^s := \int_0^s \mathfrak{M}_0^\alpha[\mathsf{t}, \mathsf{q}, \mathsf{t}', \mathsf{q}'](\sigma)\ \mathrm{d}\sigma \\ E_{\varepsilon_k}^0 := \mathcal{E}(\mathsf{t}_{\varepsilon_k}(0), \mathsf{q}_{\varepsilon_k}(0)) & E_0^0 := \mathcal{E}(\mathsf{t}(0), \mathsf{q}(0)) \\ P_{\varepsilon_k}^s := \int_0^s \partial_t\mathcal{E}(\mathsf{t}_{\varepsilon_k}(\sigma), \mathsf{q}_{\varepsilon_k}(\sigma))\, \mathsf{t}'_{\varepsilon_k}(\sigma)\ \mathrm{d}\sigma & P_0^s := \int_0^s \partial_t\mathcal{E}(\mathsf{t}(\sigma), \mathsf{q}(\sigma))\, \mathsf{t}'(\sigma)\ \mathrm{d}\sigma\,. \end{cases}$$

Hence, the parametrized energy-dissipation estimate (5.4) rephrases as $E_{\varepsilon_k}^s + M_{\varepsilon_k}^s \leq E_{\varepsilon_k}^0 + P_{\varepsilon_k}^s$, and the limiting energy-dissipation balance rewrites as $E_0^s + M_0 = E_0^0 + P_0^s$. So far, we have shown that

$$E_0^s + M_0^s \leq \liminf_{k\to\infty} \left(E_{\varepsilon_k}^s + M_{\varepsilon_k}^s\right) \leq \limsup_{k\to\infty}(E_{\varepsilon_k}^s + M_{\varepsilon_k}^s) \leq \limsup_{k\to\infty}(E_{\varepsilon_k}^0 + P_{\varepsilon_k}^s) = E_0^0 + P_0^s = E_0^s + M_0^s\,.$$

Since we have $\liminf_{k\to\infty} E_{\varepsilon_k}^s \geq E_0^s$ and $\liminf_{k\to\infty} M_{\varepsilon_k}^s \geq M_0^s$, we thus conclude that $\liminf_{k\to\infty} E_{\varepsilon_k}^s = E_0^s$ and $\lim_{k\to\infty} M_{\varepsilon_k}^s = M_0^s$ for all $s \in [0, \mathsf{S}]$, which means (5.30). Thus, Theorem 5.11 is established. □

7.2. Proof of Theorem 6.8.

Proof. We split the argument in three steps.

Step 1. Construction of a suitable pBV solution. Let $(q_{\varepsilon_k})_k$, q be as in the statement of Theorem 6.8. Lemma 4.12 ensures the validity of the basic energy estimates (4.25) and (4.26) for the sequence $(q_k)_k = (u_{\varepsilon_k}, z_{\varepsilon_k})_k$. The additional estimate for $(u_{\varepsilon_k})_k$ in $\mathrm{BV}([0, T]; \mathbf{U})$ is assumed in (6.15), such that the arc-length functions $\mathsf{s}_{\varepsilon_k}$ from (5.2) fulfill $\sup_{k\in\mathbb{N}} \mathsf{s}_{\varepsilon_k}(T) \leq C$. We reparametrize the curves q_{ε_k} by means of the rescaling functions $\mathsf{t}_{\varepsilon_k} := \mathsf{s}_{\varepsilon_k}^{-1}$ by setting $\mathsf{q}_{\varepsilon_k} := q_{\varepsilon_k} \circ \mathsf{t}_{\varepsilon_k}$. Without loss of generality we may suppose that $(\mathsf{t}_{\varepsilon_k}, \mathsf{q}_{\varepsilon_k})$ is surjective and defined on a fixed interval $[0, \mathsf{S}]$.

Now, for the sequence $(\mathsf{t}_{\varepsilon_k}, \mathsf{q}_{\varepsilon_k})_k$ the a priori estimate (5.27) holds. Hence, we are in a position to apply Thm. 5.11 to the curves $(\mathsf{t}_{\varepsilon_k}, \mathsf{q}_{\varepsilon_k})_k$ and we conclude that $(\mathsf{t}_{\varepsilon_k}, \mathsf{q}_{\varepsilon_k})_k$ convergence along a (not relabeled) subsequence to a pBV solution $(\mathsf{t}, \mathsf{q}) : [0, \mathsf{S}] \to [0, T]\times\mathbf{Q}$. In what follows, we will prove that q is related to the parametrized curve (t, q) via (6.21).

Step 2. Every limit point q is a true BV solution: We first prove that

$$q(t) \in \{\mathsf{q}(s) : \mathsf{t}(s) = t\} \text{ for all } t \in [0, T]. \tag{7.11}$$

For this we fix $t_* \in [0, T]$ and choose $s_k \in [0, \mathsf{S}]$ such that $\mathsf{t}_{\varepsilon_k}(s_k) = t_*$. After choosing a (not relabeled) subsequence we may assume $s_k \to s_*$. As $(\mathsf{t}_{\varepsilon_k}, \mathsf{q}_{\varepsilon_k})_k$ converges uniformly to (t, q) in $\mathrm{C}^0([0, \mathsf{S}]; \mathbb{R}\times\mathbf{Q}_{\mathrm{weak}})$ we obtain

$$\mathsf{t}_{\varepsilon_k}(s_k) \to \mathsf{t}(s_*) \quad \text{and} \quad \mathsf{q}_{\varepsilon_k}(s_k) \rightharpoonup \mathsf{q}(s_*).$$

However, by construction we have

$$\mathsf{t}_{\varepsilon_k}(s_k) = t_* \quad \text{and} \quad \mathsf{q}_{\varepsilon_k}(s_k) \overset{\text{Step 1}}{=} q_{\varepsilon_k}(\mathsf{t}_{\varepsilon_k}(s_k)) = q_{\varepsilon_k}(t_*) \rightharpoonup q(t_*),$$

where the last convergence is the assumption in (6.16). Hence we conclude $\mathsf{t}(s_*) = t_*$ and $\mathsf{q}(s_*) = q(t_*)$ which is the desired relation (7.11).

Thanks to (7.11), we can apply Theorem 6.15(1), which ensures that q is a true BV solution.

Step 3. Proof of convergences (6.17): Since $(q_{\varepsilon_k})_k$ is bounded in $\mathrm{L}^\infty(0,T;\mathbf{U}_\mathrm{e}{\times}\mathbf{Z}_\mathrm{e})$ by estimate (4.26) and Hypothesis 4.3, the pointwise weak convergence in $\mathbf{Q}$ improves to convergences (6.17a). Next, we observe that for every $0 \le s_0 \le s_1 \le \mathsf{S}$ there holds

$$\begin{aligned}\lim_{k\to\infty}\int_{\mathsf{t}(s_0)}^{\mathsf{t}(s_1)} \mathfrak{M}^\alpha_{\varepsilon_k}(r, q_{\varepsilon_k}(r), 1, q'_{\varepsilon_k}(r))\,\mathrm{d}r &= \lim_{k\to\infty}\int_{s_0}^{s_1} \mathfrak{M}^\alpha_{\varepsilon_k}(\mathsf{t}_{\varepsilon_k}(\sigma), \mathsf{q}_{\varepsilon_k}(\sigma), \mathsf{t}'_{\varepsilon_k}(\sigma), \mathsf{q}'_{\varepsilon_k}(\sigma))\,\mathrm{d}\sigma \\ &\overset{(1)}{=} \int_{s_0}^{s_1} \mathfrak{M}^\alpha_0[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'](\sigma)\,\mathrm{d}\sigma \overset{(2)}{=} \mathrm{Var}_{\mathfrak{M}^\alpha_0}(q;[\mathsf{t}(s_0),\mathsf{t}(s_1)]),\end{aligned}$$

with (1) due to (5.30b) and (2) due to (6.22). Hence, (6.17c) follows.

Finally, the lower semicontinuity of $\mathcal{E}$, the continuity (4.14) of $\partial_t\mathcal{E}$, give that

$$\liminf_{k\to\infty} \mathcal{E}(t, q_k(t)) \ge \mathcal{E}(t, q(t)) \quad \text{for all } t\in[0,T] \quad \text{and} \quad \int_0^t \partial_t\mathcal{E}(s, q_k(s))\,\mathrm{d}s \to \int_0^t \partial_t\mathcal{E}(s,q(s))\,\mathrm{d}s.$$

Hence, with similar arguments as in the proof of Theorem 5.11 (cf. the end of Section 7.1), we conclude

$$\mathcal{E}(t,q(t)) + \mathrm{Var}_{\mathfrak{M}^\alpha_0}(q;[0,t]) = \lim_{k\to\infty}\mathcal{E}(t, q_{\varepsilon_k}(t)) + \lim_{k\to\infty}\int_0^t \mathfrak{M}^\alpha_{\varepsilon_k}(r, q_{\varepsilon_k}(r), 1, q'_{\varepsilon_k}(r))\,\mathrm{d}r \quad \text{for all } t\in[0,T],$$

and (6.17b) ensues from the previously obtained (6.17c). This finishes the proof of Theorem 6.8. □

7.3. Proof of Proposition 5.18.

Our task is to show inclusions (5.44) for the contact sets Σ_α and the flow regimes $\mathrm{A}_\mathsf{u}\mathrm{C}_\mathsf{z}$ for the three different cases for α. We rely on the explicit form of $\mathfrak{M}^\alpha_0 = \mathcal{R} + \mathfrak{M}^{\alpha,\mathrm{red}}_0$ from (5.12).

Proof of Proposition 5.18. *Step 1: The case $t' > 0$.* We start by showing that for all $\alpha > 0$ we have

$$\Sigma^{>0}_\alpha := \big\{\, (t,q,t',q') \in \Sigma_\alpha \,\big|\, t' > 0 \,\big\} = \mathrm{E}_\mathsf{u}\mathrm{R}_\mathsf{z} = \mathrm{E}_\mathsf{u} \cap \mathrm{R}_\mathsf{z}.$$

Indeed, in the case $t' > 0$ we have $\mathfrak{M}^\alpha_0(t,q,t',q') < \infty$ if and only if $\mathscr{S}^*_\mathsf{u}(t,q) = \mathscr{S}^*_\mathsf{z}(t,q) = 0$ and then $\mathfrak{M}^\alpha_0(t,q,t',q') = \mathcal{R}(z')$. From the former we obtain that, in fact, every $(\mu,\zeta) \in \mathfrak{A}^*_\mathsf{u}(t,q){\times}\mathfrak{A}^*_\mathsf{z}(t,q)$ satisfies $\mu = 0$ and $-\zeta \in \partial\mathcal{R}(0)$. From the contact condition $\mathcal{R}(z') = \mathfrak{M}^\alpha_0(t,q,t',q') = -\langle\zeta, z'\rangle_\mathbf{Z}$ and the 1-homogeneity of $\mathcal{R}$ we infer that $-\zeta \in \partial\mathcal{R}(z')$, see (4.5). Taking into account that $\mathfrak{A}^*_\mathsf{x}(t,q) \subset \partial_\mathsf{x}\mathcal{E}(t,q)$ for $\mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$, we ultimately infer

$$\partial_\mathsf{u}\mathcal{E}(t,q) \ni 0, \qquad \partial\mathcal{R}(z') + \partial_\mathsf{z}\mathcal{E}(t,q) \ni 0, \tag{7.12}$$

namely system (5.42) holds with $\lambda_\mathsf{u} = \lambda_\mathsf{z} = 0$, i.e. $(t,q,t',q') \in \mathrm{E}_\mathsf{u}\mathrm{R}_\mathsf{z}$. Hence, we have shown $\Sigma^{>0}_\alpha \subset \mathrm{E}_\mathsf{u}\mathrm{R}_\mathsf{z}$. In fact, reverting the arguments the opposite inclusion holds as well.

Step 2. The case $t' = 0$. We define $\Sigma^0_\alpha := \big\{\, (t,q,t',q') \in \Sigma_\alpha \,\big|\, t' = 0 \,\big\}$ and treat the three cases $\alpha = 1$, $\alpha > 1$, and $\alpha \in (0,1)$, separately.

Step 2.A. $t' = 0$ and $\alpha = 1$. We want to show the inclusion

$$\Sigma^0_{\alpha=1} \subset \big(\mathrm{E}_\mathsf{u}\mathrm{R}_\mathsf{z} \cap \{t'=0\}\big) \cup \mathrm{V}_\mathsf{uz} \cup \mathrm{B}_\mathsf{u}\mathrm{B}_\mathsf{z}\,. \tag{7.13}$$

From (5.12) we have $\mathfrak{M}^\alpha_0(t,q,0,q') = \mathcal{R}(z') + \mathfrak{b}_{\mathcal{V}_\mathsf{u}\oplus\mathcal{V}_\mathsf{z}}(q', \mathscr{S}^*_\mathsf{u}(t,q){+}\mathscr{S}^*_\mathsf{z}(t,q))$.
Hence, for $\mathscr{S}^*_\mathsf{u}(t,q){+}\mathscr{S}^*_\mathsf{z}(t,q) = 0$ we argue as in Step 1 and obtain $(t,q,0,q') \in \mathrm{E}_\mathsf{u}\mathrm{R}_\mathsf{z} \cap \{t'=0\}$.

We may now suppose that $\mathscr{S}^*_\mathsf{u}(t,q){+}\mathscr{S}^*_\mathsf{z}(t,q) > 0$ and $q' = (u',z') = 0$. Clearly, the contact condition $\mathfrak{M}^{\alpha=1}_0(t,q,t',q') = -\langle\mu,u'\rangle_\mathbf{U} - \langle\zeta,z'\rangle_\mathbf{Z}$ holds for all $(\mu,\zeta) \in \mathfrak{A}^*_\mathsf{u}(t,q){\times}\mathfrak{A}^*_\mathsf{z}(t,q)$. However, $\mathscr{S}^*_\mathsf{u}(t,q){+}\mathscr{S}^*_\mathsf{z}(t,q) > 0$ gives $\big(\{0\}{\times}\partial\mathcal{R}(0)\big) \cap \partial_\mathsf{q}\mathcal{E}(t,q) = \emptyset$, and because of $\mathfrak{A}^*_\mathsf{u}(t,q){\times}\mathfrak{A}^*_\mathsf{z}(t,q) \subset \partial_\mathsf{q}\mathcal{E}(t,q)$ we conclude that $(t,q,0,(0,0))$ fulfills system (5.42) with $\lambda_\mathsf{u} = \lambda_\mathsf{z} = \infty$. Hence, $(t,q,0,q') = (t,q,0,0) \in \mathrm{B}_\mathsf{u}\mathrm{B}_\mathsf{z}$ as desired.

Suppose now $\mathcal{V}_\mathsf{z}(z'){+}\mathcal{V}_\mathsf{u}(u') > 0$ in addition to $\mathscr{S}^*_\mathsf{u}(t,q){+}\mathscr{S}^*_\mathsf{z}(t,q) > 0$. According to Proposition 3.2(b2) there exists $\ell = \ell(t,q,q') > 0$ with

$$\mathfrak{b}_{\mathcal{V}_\mathsf{u}\oplus\mathcal{V}_\mathsf{z}}(q', \mathscr{S}^*_\mathsf{u}(t,q){+}\mathscr{S}^*_\mathsf{z}(t,q)) = \ell\,\Big(\mathcal{V}_\mathsf{u}\big(\frac{1}{\ell}\,u'\big){+}\mathcal{V}_\mathsf{z}\big(\frac{1}{\ell}\,z'\big){+}\mathscr{S}^*_\mathsf{u}(t,q){+}\mathscr{S}^*_\mathsf{z}(t,q)\Big)\,.$$

Now, $(t,q,0,q') \in \Sigma_1^0$ means that there exists $(\mu,\zeta) \in \mathfrak{A}_{\mathsf{u}}^*(t,q) \times \mathfrak{A}_{\mathsf{z}}^*(t,q)$ fulfilling the contact condition

$$\mathfrak{M}_0^1(t,q,0,q') = \mathcal{R}(z') + \mathfrak{b}_{\psi_{\mathsf{u}}\oplus\psi_{\mathsf{z}}}(q', \mathscr{S}_{\mathsf{u}}^*(t,q) + \mathscr{S}_{\mathsf{z}}^*(t,q)) = -\langle \mu, u' \rangle_{\mathbf{U}} - \langle \zeta, z' \rangle_{\mathbf{Z}}.$$

Moreover, the definition of $\mathfrak{A}_{\mathsf{x}}^*(t,q)$ gives $\mathscr{S}_{\mathsf{u}}^*(t,q) = \mathcal{V}_{\mathsf{u}}^*(-\mu)$ and $\mathscr{S}_{\mathsf{z}}^*(t,q) = \mathcal{W}_{\mathsf{z}}^*(-\zeta)$. Together with the definition of ℓ we find the identity

$$\begin{aligned}&\mathcal{V}_{\mathsf{u}}\big(\frac{1}{\ell}\,u'\big) + \mathcal{R}\big(\frac{1}{\ell}\,z'\big) + \mathcal{V}_{\mathsf{z}}\big(\frac{1}{\ell}\,z'\big) + \mathcal{V}_{\mathsf{u}}^*(-\mu) + \mathcal{W}_{\mathsf{z}}^*(-\zeta)\\ &= \frac{1}{\ell}\,\mathfrak{M}_0^1(t,q,0,q') = -\frac{1}{\ell}\big(\langle \mu, u' \rangle_{\mathbf{U}} + \langle \zeta, z' \rangle_{\mathbf{Z}}\big) = \langle -\mu, \frac{1}{\ell}\,u' \rangle_{\mathbf{U}} + \langle -\zeta, \frac{1}{\ell}\,z' \rangle_{\mathbf{Z}}.\end{aligned}$$

Since $\mathcal{V}_{\mathsf{u}}^* \oplus \mathcal{W}_{\mathsf{z}}^*$ is is the Legendre-Fenchel dual of $\mathcal{V}_{\mathsf{u}} \oplus (\mathcal{R}+\mathcal{V}_{\mathsf{z}})$ we conclude

$$-\mu \in \partial\mathcal{V}_{\mathsf{u}}\big(\frac{1}{\ell}\,u'\big) = \partial\mathcal{V}_{\mathsf{u}}^{1/\ell}(u') \quad\text{and}\quad -\zeta \in \partial\mathcal{R}\big(\frac{1}{\ell}\,z'\big) + \partial\mathcal{V}_{\mathsf{z}}\big(\frac{1}{\ell}\,z'\big) = \partial\mathcal{R}(z') + \partial\mathcal{V}_{\mathsf{z}}^{1/\ell}(z').$$

From this we see that $(t,q,0,q')$ system (5.42) holds with $\lambda_{\mathsf{u}} = \lambda_{\mathsf{z}} = 1/\ell \in (0,\infty)$, i.e. we have $(t,q,0,q') \in \mathrm{V}_{\mathsf{uz}}$, and the inclusion (7.13) is established.

Step 2.B. $t' = 0$ and $\alpha > 1$. Let us now examine the case $\alpha > 1$ and prove that

$$\Sigma_\alpha^0 \subset \big(\mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \cap \{t'=0\}\big) \cup \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}} \cup \mathrm{B}_{\mathsf{z}}. \tag{7.14}$$

Using the explicit expression for $\mathfrak{M}_0^{\alpha,\mathrm{red}}(t,q,0,q')$ in (5.12), we see that $\mathfrak{M}_0^\alpha(t,q,0,q') < \infty$ implies that either (i) $\mathscr{S}_{\mathsf{u}}^*(t,q) = 0$ or (ii) $\big(\mathscr{S}_{\mathsf{u}}^*(t,q) > 0$ and $z' = 0\big)$.

In case (i), which means E_{u}, the contact condition reads

$$\mathscr{S}_{\mathsf{u}}^*(t,q) = 0 \quad\text{and}\quad \exists\,\zeta \in \mathfrak{A}_{\mathsf{z}}^*(t,q):\ \mathcal{R}(z') + \mathfrak{b}_{\psi_{\mathsf{z}}}(z', \mathscr{S}_{\mathsf{z}}^*(t,q)) = -\langle \zeta, z' \rangle_{\mathbf{Z}}.$$

If $\mathscr{S}_{\mathsf{z}}^*(t,q) = 0$, we have $\mathfrak{b}_{\psi_{\mathsf{z}}}(z', \mathscr{S}_{\mathsf{z}}^*(t,q)) = 0$ and infer $\mathcal{R}(z') + \langle \zeta, z' \rangle_{\mathbf{Z}} = 0$. Moreover, $\mathscr{S}_{\mathsf{z}}^*(t,q) = 0$ implies $\zeta \in \mathfrak{A}_{\mathsf{z}}^*(t,q) = \partial\mathcal{R}(0)$, and we conclude $\partial\mathcal{R}(z') + \zeta \ni 0$ by (4.5). Hence, we can choose $\lambda_{\mathsf{z}} = 0$ in (5.42b) and obtain $(t,q,0,q') \in \mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}}$.

If $z' = 0$ holds but not $\mathscr{S}_{\mathsf{z}}^*(t,q) > 0$, then (5.42b) holds for $\lambda_{\mathsf{z}} = \infty$ and $(t,q,0,q') \in \mathrm{B}_{\mathsf{z}}$.

Finally, if $z' \neq 0$ and $\mathscr{S}_{\mathsf{z}}^*(t,q) > 0$, then the very same discussion as in the last part of Step 2.A provides $\lambda_{\mathsf{z}} \in (0,\infty)$ such that $\partial\mathcal{R}(z') + \partial\mathcal{V}_{\mathsf{z}}^{\lambda_{\mathsf{z}}}(z') + \partial_z\mathcal{E}(t,q) \ni 0$, which means $(t,q,0,q') \in \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}$.

The discussion of the case (ii) with $\mathscr{S}_{\mathsf{u}}^*(t,q) > 0$ and $z' = 0$ proceeds along the same lines, relying on the contact condition

$$\exists\,\mu \in \mathfrak{A}_{\mathsf{u}}^*(t,q): \quad \mathfrak{M}_0^\alpha(t,q,0,u',0) = \mathcal{R}(0) + \mathfrak{b}_{\psi_{\mathsf{z}}}(u', \mathscr{S}_{\mathsf{u}}^*(t,q)) = -\langle \mu, u' \rangle_{\mathbf{U}}.$$

For $u' \neq 0$ we find $\lambda_{\mathsf{u}} \in (0,\infty)$ with $\partial\mathcal{V}_{\mathsf{u}}^{\lambda_{\mathsf{u}}}(u') + \mu \ni 0$, which gives $(t,q,0,q') \in \mathrm{V}_{\mathsf{u}}\mathrm{B}_{\mathsf{z}}$. For $u' = 0$ we can choose $\lambda_{\mathsf{u}} = \infty$ such that $\partial\mathcal{V}_{\mathsf{u}}^\infty(0) + \mu = \mathbf{U}^* + \mu \ni 0$. Hence (5.42) holds with $\lambda_{\mathsf{z}} = \infty$ and $\lambda_{\mathsf{u}} \in (0,\infty]$, i.e. $(t,q,0,q') \in \mathrm{B}_{\mathsf{z}}$.

Thus, in both cases, (i) and (ii), we conclude (7.14), and Step 2.B is completed.

Step 2.C. $t' = 0$ and $\alpha \in (0,1)$. This case works similarly as the case $\alpha > 1$ in Step 2.B, but with the roles of u and z interchanged, where E_{u} is interchanged with R_{z}. Thus, in analogy to (7.14) we obtain $\Sigma_\alpha^0 \subset \big(\mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \cap \{t'=0\}\big) \cup \mathrm{V}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \cup \mathrm{B}_{\mathsf{u}}$.

This concludes the proof of Proposition 5.18. □

8. Application to a model for delamination

In this section we discuss the application of our vanishing-viscosity analysis techniques to a PDE system modeling adhesive contact. A previous vanishing-viscosity (and vanishing-inertia, in the momentum balance) analysis was carried out for a delamination model in [Sca17] where, however, an energy balance only featuring defect measures, in place of contributions describing the dissipation of energy at jumps, was obtained in the null-viscosity limit.

After introducing the viscous model and discussing its structure as a generalized gradient system in Section 8.1, we are going to state the existence of *enhanced* BV and parametrized solutions to the corresponding rate-independent system $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha} + \mathcal{R} + \mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ in Theorem 8.1. This result will be proved throughout Sections 8.2–8.5 by showing that the 'abstract' Theorems 5.14 and 6.12 apply.

As we will emphasize later on, our analysis crucially relies on the fact that, in the delamination system, the coupling between the displacements and the delamination parameter only occurs through lower order terms.

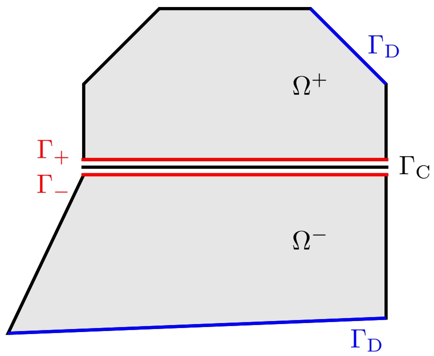

FIGURE 8.1.
The two domains Ω^+ and Ω^- touch along the delamination hypersurface Γ_C.

8.1. THE 'VISCOUS' SYSTEM FOR DELAMINATION. We consider two bodies located in two bounded Lipschitz domains $\Omega^\pm \subset \mathbb{R}^3$ and adhering along a prescribed interface Γ_C, on which some adhesive substance is present. We denote that part of $\Omega^\pm$ that coincides with Γ_C by $\Gamma_\pm$, see Figure 8.1, thus being able to talk about one-sided boundary conditions. In what follows, for simplicity we will assume that Γ_C is a 'flat' interface, i.e., Γ_C is contained in a plane, so that, in particular, $\mathscr{H}^2(\Gamma_C) = \mathscr{L}^2(\Gamma_C) > 0$. While the generalization to a smooth curved interface is standard, this restriction will allow us to avoid resorting to Laplace-Beltrami operators in the flow rule for the delamination parameter.

The state variables in the model are indeed the displacement $u : \Omega \to \mathbb{R}^3$, with $\Omega := \Omega^+ \cup \Omega^-$, and the delamination variable $z : \Gamma_C \to [0,1]$, representing the fraction of fully effective molecular links in the bonding. Therefore, $z(t,x) = 1$ ($z(t,x) = 0$, respectively) means that the bonding is fully intact (completely broken) at a given time instant $t \in [0,T]$ and in a given material point $x \in \Gamma$. We denote by $n^\pm$ the outer unit normal of $\Omega^\pm$ restricted to $\Gamma_\pm$ and by $[\![u]\!]$ the jump of u across Γ_C, namely $[\![u]\!] = u|_{\Gamma_+} - u|_{\Gamma_-}$, but now defined as function on Γ_C.

For simplicity, we impose homogeneous Dirichlet boundary conditions $u = 0$ on the Dirichlet part Γ_D of the boundary $\partial\Omega$, with $\mathscr{H}^2(\Gamma_D) > 0$. We consider a given applied traction f on the Neumann part $\Gamma_N = \partial\Omega \setminus (\Gamma_D \cup \Gamma_C)$.

All in all, we address the following *rate-dependent* PDE system

$$-\operatorname{div}(\varepsilon^\alpha \mathbb{D}e(\dot u) + \mathbb{C}e(u)) = F \qquad \text{in } \Omega\times(0,T), \tag{8.1a}$$

$$u = 0 \qquad \text{on } \Gamma_D\times(0,T), \tag{8.1b}$$

$$(\varepsilon^\alpha \mathbb{D}e(\dot u) + \mathbb{C}e(u))|_{\Gamma_N}\nu = f \qquad \text{on } \Gamma_N\times(0,T), \tag{8.1c}$$

$$(\varepsilon^\alpha \mathbb{D}e(\dot u) + \mathbb{C}e(u))|_{\Gamma_C} n^\pm \pm \gamma(z)\partial\psi([\![u]\!]) \pm \beta([\![u]\!]) \ni 0 \qquad \text{on } \Gamma_\pm\times(0,T), \tag{8.1d}$$

$$\partial \mathrm{R}(\dot z) + \varepsilon \dot z - \Delta z + \tilde\phi(z) + \partial\gamma(z)\psi([\![u]\!]) \ni 0 \qquad \text{on } \Gamma_C\times(0,T), \tag{8.1e}$$

where $\dot u$ and $\dot z$ stand for the partial time derivatives of u and z. Here, F is a volume force, $\mathbb{D}, \mathbb{C} \in \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ the positive definite and symmetric viscosity and elasticity tensors, ν the exterior unit normal to $\partial(\Omega^+ \cup \Gamma_C \cup \Omega^-)$, and R is given by

$$\mathrm{R}(r) = \kappa_+ \max\{r, 0\} + \kappa_- \max\{-r, 0\} \quad \text{ with } \kappa_\pm > 0. \tag{8.2}$$

Hence, healing of the broken molecular links is disfavored, but not totally blocked. Giving up unidirectionality allows for a more straightforward application of our abstract results. Nonetheless, we expect that, at the price of some further technicalities our techniques could be adapted to deal with unidirectionality by means of additional estimates (like for instance in the application of BV solutions to *unidirectional* damage developed in [KRZ13]).

The term $\gamma(z)\partial\psi([\![u]\!])$, with γ and ψ nonnegative functions (we may think of $\gamma(z) = \max\{z, 0\}$) and ψ convex, in (8.1d) derives from the contribution $\gamma(z)\psi([\![u]\!])$ to the surface energy, cf. (8.6d) ahead, which penalizes the constraint $z[\![u]\!] = 0$ a.e. on Γ_C, typical of *brittle* delamination models. Indeed, to our knowledge, existence results for brittle models are available only in the case of a rate-independent evolution for z, cf. e.g. [RSZ09, RoT15a]. In fact, (8.1) is rather a model for *contact with adhesion* and will be accordingly referred to in this way. Our assumptions on the constitutive functions γ, ψ and β, and on the multivalued operator $\tilde\phi$ (indeed, on the mapping $z \mapsto \tilde\phi(z) - z$), will be specified in (8.5) ahead.

We define the operators $\boldsymbol{C}, \boldsymbol{D} : \mathrm{H}^1(\Omega;\mathbb{R}^3) \to \mathrm{H}^1(\Omega;\mathbb{R}^3)^*$ via

$$\langle \boldsymbol{C}u, v\rangle_{\mathrm{H}^1(\Omega)} := \int_\Omega \mathbb{C}e(u) : e(v)\,\mathrm{d}x, \qquad \langle \boldsymbol{D}u, v\rangle_{\mathrm{H}^1(\Omega)} := \int_\Omega \mathbb{D}e(u) : e(v)\,\mathrm{d}x,$$

while we denote by $\boldsymbol{J} : \mathrm{H}^1(\Omega;\mathbb{R}^3) \to \mathrm{L}^4(\Gamma_\mathrm{C};\mathbb{R}^3)$; $\mathrm{u} \mapsto [\![\mathrm{u}]\!]$ the jump operator, by $\|\boldsymbol{J}\|$ its operator norm, and by $\boldsymbol{J}^*$ its adjoint. We denote by $\boldsymbol{A}$ the Laplacian with homogeneous Neumann boundary conditions

$$\boldsymbol{A} : \mathrm{H}^1(\Gamma_\mathrm{C}) \to \mathrm{H}^1(\Gamma_\mathrm{C})^* \qquad \langle \boldsymbol{A}z, \omega\rangle_{\mathrm{H}^1(\Gamma_\mathrm{C})} := \int_{\Gamma_\mathrm{C}} \big(\nabla z \nabla \omega + z\omega\big)\,\mathrm{d}x\,.$$

In particular, we have $\|z\|^2_{\mathrm{H}^1(\Gamma_\mathrm{C})} = \langle \boldsymbol{A}z, z\rangle_{\mathrm{H}^1(\Gamma_\mathrm{C})}$. Finally, we denote by $\ell_u : (0,T) \to \mathbf{U}^*$ the functional encompassing the volume and surface forces F and f, namely

$$\langle \ell_u(t), u\rangle_{\mathrm{H}^1(\Omega)} := \int_\Omega F(t)u\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} f(t)u\,\mathrm{d}S\,.$$

Throughout, we will assume that

$$\ell_u \in \mathrm{C}^1([0,T];\mathrm{H}^1(\Omega;\mathbb{R}^3)^*)\,. \tag{8.3}$$

Hence, system (8.1) takes the form

$$0 \in \varepsilon^\alpha \boldsymbol{D}\dot u + \boldsymbol{C}u + \boldsymbol{J}^*\big(\beta([\![u]\!]) + \gamma(z)\partial\psi([\![u]\!])\big) - \ell_u \qquad \text{in } \mathrm{H}^1(\Omega;\mathbb{R}^3)^* \tag{8.4a}$$

$$0 \in \partial\mathrm{R}(\dot z) + \varepsilon \dot z + \boldsymbol{A}z + \partial\widehat\phi(z) + \partial\gamma(z)\psi([\![u]\!]) \qquad \text{a.e. in } \Gamma_\mathrm{C} \tag{8.4b}$$

almost everywhere in $(0,T)$. In (8.4), $\widehat\beta$ a primitive for β and $\widehat\phi$ a primitive for the multivalued operator $z \mapsto \tilde\phi(z) - z$.

Structure as a (generalized) gradient system. First of all, let us specify our assumptions on the constitutive functions $\widehat\beta$, γ, $\widehat\phi$, and ψ:

$$\left.\begin{array}{l}
\psi,\ \widehat\beta : \mathbb{R}^3 \to [0,\infty) \text{ are lsc and convex with } \psi(0) = \widehat\beta(0) = 0,\\
\exists\, C_\psi > 0\ \forall\, a \in \mathbb{R}^3 : \quad \psi(a) \le C_\psi(1+|a|^2),\\
\widehat\beta \in \mathrm{C}^1(\mathbb{R}^3) \text{ and } \beta = \mathrm{D}\widehat\beta \text{ is globally Lipschitz}\\
\gamma \text{ is convex, non-decreasing and 1-Lipschitz, with } \gamma(0) = 0,\\
\widehat\phi : \mathbb{R} \to [0,\infty] \text{ is lsc and } (-\Lambda_\phi)\text{-convex for some } \Lambda_\phi > 0, \text{ with } \widehat\phi(z) = \infty \text{ for } z \notin [0,1].
\end{array}\right\} \tag{8.5}$$

Hence, $\partial\gamma$ and $\partial\psi$ in (8.4) are convex analysis subdifferentials, while $\partial\widehat\phi : \mathbb{R} \rightrightarrows \mathbb{R}$ is the Fréchet subdifferential of $\widehat\phi$.

To fix ideas, prototypical choices for $\widehat\beta$, γ, $\widehat\phi$, and ψ would be:

(i) $\widehat\beta$ the Yosida regularization of the indicator function of the cone $C = \{v \in \mathbb{R}^3 : v\cdot n^+ \le 0\}$ (cf. also Remark 8.3);
(ii) $\gamma(z) = \max\{z, 0\}$;
(iii) $\widehat\phi$ encompassing the indicator function $I_{[0,1]}$, which would ensure that $z \in [0,1]$;
(iv) $\psi([\![u]\!]) = \frac{k}{2}|[\![u]\!]|^2$ with $k > 0$.

Observe that (8.4) falls into the class of gradient systems (4.11), with the ambient spaces

$$\mathbf{U} = \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3), \quad \mathbf{Z} = \mathrm{L}^2(\Gamma_\mathrm{C}), \quad \mathbf{Z}_\mathrm{ri} = \mathrm{L}^1(\Gamma_\mathrm{C}) \tag{8.6a}$$

where $\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)$ the space of H^1-functions on Ω fulfilling a homogeneous Dirichlet boundary condition on Γ_D. By Korn's inequality, the quadratic form associated with the operator $\boldsymbol{D}$ induces on $\mathbf{U}$ a norm equivalent to the H^1-norm; hereafter, we will in fact use that

$$\|u\|^2_\mathbf{U} := \langle \boldsymbol{D}u, u\rangle_{\mathrm{H}^1(\Omega)}, \qquad \langle \boldsymbol{C}u, u\rangle_{\mathrm{H}^1(\Omega)} \ge c_{\boldsymbol{C}}\|u\|^2_\mathbf{U}.$$

The 1-homogeneous dissipation potential $\mathcal{R} : \mathbf{Z}_\mathrm{ri} \to [0,\infty)$ is defined by

$$\mathcal{R}(\dot z) := \int_{\Gamma_\mathrm{C}} \mathrm{R}(\dot z)\,\mathrm{d}x \qquad \text{with R from (8.2).} \tag{8.6b}$$

The viscous dissipation potentials $\mathcal{V}_\mathsf{u} : \mathbf{U} \to [0,\infty)$ and $\mathcal{V}_\mathsf{z} : \mathbf{U} \to [0,\infty)$ are

$$\mathcal{V}_\mathsf{u}(\dot u) := \frac12\,\langle \boldsymbol{D}\dot u, \dot u\rangle_{\mathrm{H}^1(\Omega)} \qquad \mathcal{V}_\mathsf{z}(\dot z) := \int_{\Gamma_\mathrm{C}} \tfrac12|\dot z|^2\,\mathrm{d}x. \tag{8.6c}$$

The driving energy functional $\mathcal{E} : [0,T]\times\mathbf{U}\times\mathbf{Z} \to (\infty, +\infty]$ is given by

$$\mathcal{E}(t,u,z) := \frac12\,\langle \boldsymbol{C}u, u\rangle_{\mathrm{H}^1(\Omega)} - \langle \ell_u(t), u\rangle_{\mathrm{H}^1(\Omega)} + \frac12\,\langle \boldsymbol{A}z, z\rangle_{\mathrm{H}^1(\Gamma_\mathrm{C})} + \int_{\Gamma_\mathrm{C}} \big(\widehat\beta([\![u]\!]) + \gamma(z)\psi([\![u]\!]) + \widehat\phi(z)\big)\,\mathrm{d}x$$
$$\text{if } z \in \mathrm{H}^1(\Gamma_\mathrm{C}) \text{ and } \widehat\phi(z) \in \mathrm{L}^1(\Gamma_\mathrm{C}), \tag{8.6d}$$

and ∞ otherwise.

As we will see in Proposition 8.2, under the conditions on $\widehat{\beta}$, γ, $\widehat{\phi}$, and ψ specified in (8.5), $\mathcal{E}$ complies with the coercivity conditions from Hyp. 4.3 with the spaces

$$\mathbf{U}_{\mathrm{e}} = \mathbf{U} = \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3) \quad \text{and} \quad \mathbf{Z}_{\mathrm{e}} = \mathrm{H}^1(\Gamma_{\mathrm{C}}) \Subset \mathbf{Z}, \tag{8.7}$$

and its Fréchet subdifferential $\partial_q \mathcal{E} : [0,T]\times\mathbf{U}\times\mathbf{Z} \rightrightarrows \mathbf{U}^*\times\mathbf{Z}^*$ is given by

$$(\mu,\zeta) \in \partial_q \mathcal{E}(t,u,z) \qquad \text{if and only if}$$

$$\begin{cases} \mu = \boldsymbol{C}u + \boldsymbol{J}^*\big(\beta([\![u]\!]) + \gamma(z)\varrho\big) - \ell_u(t) \text{ for some selection } \Gamma_{\mathrm{C}} \ni x \mapsto \varrho(x) \in \partial\psi([\![u(x)]\!]) \\ \zeta = \boldsymbol{A}z + \omega\psi([\![u]\!]) + \phi \text{ for selections } \Gamma_{\mathrm{C}} \ni x \mapsto \omega(x) \in \partial\gamma(z(x)) \text{ and } \Gamma_{\mathrm{C}} \ni x \mapsto \phi(x) \in \partial\widehat{\phi}(z(x)) \text{ s.t.} \\ \qquad \boldsymbol{A}z + \phi \in \mathrm{L}^2(\Gamma_{\mathrm{C}}) \end{cases} \tag{8.8}$$

(indeed, observe that, by the growth properties of γ and ψ, the term $\omega\psi([\![u]\!])$ is in $\mathrm{L}^2(\Gamma_{\mathrm{C}})$ for any selection $\omega \in \partial\gamma$). In particular, here we have used that the Fréchet subdifferential of the (($-\Lambda_\phi$)-convex) functional $\mathcal{F} : \mathbf{Z} \to [0,\infty]$

$$\mathcal{F}(z) := \begin{cases} \frac{1}{2}\langle \boldsymbol{A}z, z\rangle_{\mathrm{H}^1(\Gamma_{\mathrm{C}})} + \int_{\Gamma_{\mathrm{C}}} \widehat{\phi}(z)\,\mathrm{d}x & \text{if } z \in \mathrm{H}^1(\Gamma_{\mathrm{C}}) \text{ and } \widehat{\phi}(z) \in \mathrm{L}^1(\Gamma_{\mathrm{C}}), \\ \infty & \text{else,} \end{cases} \tag{8.9a}$$

is given by

$$\partial\mathcal{F}(z) = \{\boldsymbol{A}z + \tilde{\phi} \,:\, \tilde{\phi}(x) \in \partial\widehat{\phi}(x) \text{ for a.a. } x \in \Gamma_{\mathrm{C}},\ \boldsymbol{A}z + \tilde{\phi} \in \mathrm{L}^2(\Gamma_{\mathrm{C}})\}\,. \tag{8.9b}$$

We also point out for later use that $\partial_{\mathsf{q}}\mathcal{E}$ fulfills the structure condition (4.23), i.e. $\partial_{\mathsf{q}}\mathcal{E}(t,u,z) = \partial_{\mathsf{u}}\mathcal{E}(t,u,z)\times\partial_{\mathsf{z}}\mathcal{E}(t,u,z)$ for every $(t,u,z) \in [0,T]\times\mathbf{U}\times\mathbf{Z}$.

Existence for the viscous system. As we will check in Proposition 8.2 ahead, our general existence result, Theorem 4.8, applies to the viscous delamination system. Hence, for every pair of initial data $(u_0,z_0) \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)\times\mathrm{H}^1(\Gamma_{\mathrm{C}})$ there exists a solution

$$u \in \mathrm{H}^1(0,T;\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)) \text{ and } z \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Gamma_{\mathrm{C}})) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Gamma_{\mathrm{C}})), \tag{8.10}$$

to the Cauchy problem for system (8.4).

8.2. The vanishing-viscosity limit.

We will now address the vanishing-viscosity limit as $\varepsilon \to 0^+$ of system (8.4). Our main result states the convergence of (a selected family of) viscous solutions to an *enhanced* BV solution to the system $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha} + \mathcal{R} + \mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ defined by (8.6), in fact enjoying the additional regularity $z \in \mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{e}})$ (with $\mathbf{Z}_{\mathrm{e}} = \mathrm{H}^1(\Gamma_{\mathrm{C}})$). Analogously, we also obtain the existence of parametrized BV solutions, to which Theorem 5.20 applies, providing a characterization in terms of system (8.15) ahead.

In fact, we will be able to obtain solutions to the viscous system (8.4) enjoying estimates, uniform with respect to the viscosity parameter, suitable for the vanishing-viscosity analysis, only by performing calculations on a version of system (8.4) in which the functions $\widehat{\beta}$, γ, $\widehat{\phi}$, and ψ are suitably smoothened, cf. (8.18). That is why, in Theorem 8.1 below will state:

(i) the existence of *qualified* viscous solutions to (the Cauchy problem for) (8.4), where by 'qualified' we mean enjoying estimates (8.13) below;
(ii) their convergence (up to a subsequence) to an enhanced BV solution (we mention that, since the viscous dissipation potentials from (8.6c) are both 2-homogeneous, the formulas in (5.11) and (5.14) yield an explicit representation formula for the functional $\mathfrak{M}_0^\alpha$ involved in the definition of BV solution);
(iii) the convergence of reparametrized (*qualified*) viscous solutions to an enhanced pBV solution for which the differential characterization from Theorem 5.20 holds.

For simplicity, in Theorem 8.1 we shall not consider a sequence of initial data $(u_0^\varepsilon, z_0^\varepsilon)_\varepsilon$ but confine the statement to the case of *fixed* data (u_0,z_0). We will impose that (u_0,z_0) fulfill the additional 'compatibility condition' (8.11).

Theorem 8.1. *Assume conditions* (8.3) *and* (8.5)*. Let* $(u_0,z_0) \in \mathbf{U}\times\mathbf{Z}_{\mathrm{e}}$ *fulfill*

$$u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3), \qquad \Delta z_0 \in \mathrm{L}^2(\Gamma_{\mathrm{C}}), \quad \partial\widehat{\phi}(z_0) \cap \mathrm{L}^2(\Gamma_{\mathrm{C}}) \neq \emptyset. \tag{8.11}$$

Then, there exists a family

$$(u_\varepsilon, z_\varepsilon)_\varepsilon \subset \mathrm{H}^1(0,T;\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3))\times\mathrm{H}^1(0,T;\mathrm{H}^1(\Gamma_{\mathrm{C}})) \tag{8.12}$$

solving the Cauchy problem for the viscous delamination system (8.10) *with the initial data* (u_0, z_0)*, and enjoying the following estimate*

$$\sup_{\varepsilon>0} \int_0^T \left\{ \|\dot{u}_\varepsilon\|_{\mathrm{H}^1(\Omega)} + \|\dot{z}_\varepsilon\|_{\mathrm{H}^1(\Gamma_{\mathrm{C}})} \right\} \mathrm{d}t \leq C\,. \tag{8.13}$$

Moreover, for any null sequence $(\varepsilon_k)_k$ *the sequence* $(u_{\varepsilon_k}, z_{\varepsilon_k})_k$ *admits a (not relabeled) subsequence, and there exists a pair*

$$(u, z) \in \mathrm{BV}([0,T]; \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)) \times BV([0,T]; \mathrm{H}^1(\Gamma_{\mathrm{C}})),$$

such that

(1) the following convergences hold as $k \to \infty$

$$u_{\varepsilon_k}(t) \rightharpoonup u(t) \text{ in } \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3), \quad z_{\varepsilon_k}(t) \rightharpoonup z(t) \text{ in } \mathrm{H}^1(\Gamma_{\mathrm{C}}) \qquad \text{for all } t \in [0,T]; \tag{8.14}$$

(2) (u, z) *is an* enhanced *BV solution to the delamination system* $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha} + \mathcal{R} + \mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ *from* (8.6)*.*

Finally, reparametrizing the sequence $(u_{\varepsilon_k}, z_{\varepsilon_k})_k$ *in such a way that the rescaled curves* $(\mathsf{t}_{\varepsilon_k}, \mathsf{u}_{\varepsilon_k}, \mathsf{z}_{\varepsilon_k})_k$ *enjoy estimates* (5.27) *and* (5.35)*, up to a subsequence we have convergence of* $(\mathsf{t}_{\varepsilon_k}, \mathsf{u}_{\varepsilon_k}, \mathsf{z}_{\varepsilon_k})_k$*, in the sense of* (5.29)*, to an enhanced pBV solution* $(\mathsf{t}, \mathsf{u}, \mathsf{z}) : [0, \mathsf{S}] \to [0,T] \times \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3) \times \mathrm{H}^1(\Gamma_{\mathrm{C}})$ *for which the differential characterization from Theorem 5.20 holds. Namely, there exist measurable functions* $\lambda_{\mathsf{u}}, \lambda_{\mathsf{z}} : (0, \mathsf{S}) \to [0, \infty]$ *satisfying for almost all* $s \in (0, \mathsf{S})$ *the switching conditions* (5.47c) *and the subdifferential inclusions*

$$0 \in \lambda_{\mathsf{u}}(s)\boldsymbol{D}\dot{\mathsf{u}}(s) + \boldsymbol{C}\mathsf{u}(s) + \boldsymbol{J}^*\big(\beta(\llbracket \mathsf{u}(s) \rrbracket) + \gamma(\mathsf{z}(s))\partial\psi(\llbracket \mathsf{u}(s) \rrbracket)\big) - \ell_u(\mathsf{t}(s)) \qquad \text{in } \mathrm{H}^1(\Omega;\mathbb{R}^3)^* \tag{8.15a}$$

$$0 \in \partial\mathrm{R}(\dot{\mathsf{z}}(s)) + \lambda_{\mathsf{z}}(s)\dot{\mathsf{z}}(s) + \boldsymbol{A}\mathsf{z}(s) + \partial\widehat{\phi}(\mathsf{z}(s)) + \partial\gamma(\mathsf{z}(s))\psi(\llbracket \mathsf{u}(s) \rrbracket) \qquad \text{a.e. in } \Gamma_{\mathrm{C}} \tag{8.15b}$$

(with convention (5.41) *in the case* $\lambda_{\mathsf{x}}(s) = \infty$*).*

Proof. It is sufficient to check that the rate-independent system $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha} + \mathcal{R} + \mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ from (8.6) complies with the assumptions of Theorems 5.14, 5.20, and 6.12, and that there exist 'qualified' viscous solutions enjoying estimates (8.13). More precisely,

(1) In Proposition 8.2 ahead we will check that the rate-independent delamination system $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha} + \mathcal{R} + \mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ complies with Hypotheses 4.1, 4.2, 4.3, 4.5, 4.10, and 5.7 (in fact, the parametrized chain rule (5.38) holds).

(2) We will obtain the existence of viscous solutions enjoying estimates (8.13) by working on a smoothened version of system (8.4), introduced in Section 8.4 ahead. Therein, we will obtain estimates for the solutions to the regularized viscous system *uniform* with respect to the regularization parameter. Hence, with Proposition 8.5 in Section 8.5 we will conclude the existence of 'qualified' solutions for which (8.13) holds, and thereby conclude the proof of Theorem 8.1.

□

In what follows, we will most often use the place-holders **U**, **Z**, ... (cf. (8.6a) and (8.7)) for the involved function spaces.

8.3. Properties of the rate-independent system for delamination.

This section is centered around Proposition 8.2 below, in which we check the rate-independent system $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha} + \mathcal{R} + \mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ from (8.6). complies with the 'abstract' Hypotheses from Section 4. In particular, from the following result we gather that Theorem 4.8 is applicable, yielding the existence of solutions as in (8.10) to the viscous delamination system.

Proposition 8.2. *Assume* (8.3) *and* (8.5)*. Then, the delamination system* $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha} + \mathcal{R} + \mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow 0}$ *from* (8.6) *fulfills Hypotheses 4.1, 4.2, 4.3, 4.5, 4.10, and 5.7.*

Proof. The proof consists of three steps.
Step 1. Hypotheses 4.1, 4.2, 4.3, and 4.10: The validity of Hypothesis 4.1, 4.2, and 4.3 is obvious. A straightforward calculation shows that the Fréchet subdifferential of $\mathcal{E}$ is given by (8.8), so that the structure condition $\partial_{\mathsf{q}}\mathcal{E}(t,q) = \partial_{\mathsf{u}}\mathcal{E}(t,q) \times \partial_{\mathsf{z}}\mathcal{E}(t,q)$ holds at every $q = (u,z) \in \mathbf{U} \times \mathbf{Z}$. Therefore, by Lemma 4.11, Hypothesis 4.10 will be ensured by the validity of Hypothesis 4.5, which we now check.
Step 2. Hypothesis 4.5: Let $(t_n)_n \subset [0,T]$ and $(u_n, z_n)_n \subset \mathbf{U}\times\mathbf{Z}$ be in the conditions of Hypothesis 4.5, and let $(\mu_n, \zeta_n)_n$, with $\mu_n \in \partial_{\mathsf{u}}\mathcal{E}(t_n, u_n, z_n)$ and $\zeta_n \in \partial_{\mathsf{z}}\mathcal{E}(t_n, u_n, z_n)$, fulfill $\mu_n \rightharpoonup \mu$ in $\mathbf{U}^*$ and $\zeta_n \rightharpoonup \zeta$ in $\mathbf{Z}^*$. Hence,

$$\mu_n = \boldsymbol{C}u_n + \boldsymbol{J}^*(\beta(\llbracket u_n \rrbracket) + \gamma(z_n)\varrho_n) - \ell_u(t_n) \qquad \text{with } \varrho_n \in \partial\psi(\llbracket u_n \rrbracket) \text{ a.e. in } \Gamma_{\mathrm{C}},$$

$$\zeta_n = \boldsymbol{A}z_n + \omega_n\psi(\llbracket u_n \rrbracket) + \phi_n \text{ for some } \omega_n \in \partial\gamma(z_n) \text{ and } \phi_n \in \partial\widehat{\phi}(z_n)\,.$$

We observe that, by Sobolev embeddings and trace theorems, from the convergences $u_n \rightharpoonup u$ in $\mathbf{U}_\mathrm{e}$ and $z_n \rightharpoonup z$ in $\mathbf{Z}_\mathrm{e}$ we infer that $[\![u_n]\!] \to [\![u]\!]$ in $\mathrm{L}^q(\Gamma_\mathrm{C};\mathbb{R}^3)$ for all $1 \le q < 4$, and $z_n \to z$ in $\mathrm{L}^p(\Gamma_\mathrm{C})$ for all $1 \le p < \infty$. Furthermore, since $0 \le z_n \le 1$ a.e. on Γ_C, we even have $z_n \overset{*}{\rightharpoonup} z$ in $\mathrm{L}^\infty(\Gamma_\mathrm{C})$. Since γ is Lipschitz, we gather that $\gamma(z_n) \to \gamma(z)$ in $\mathrm{L}^p(\Gamma_\mathrm{C})$ for all $1 \le p < \infty$, too. By the growth properties of ψ, we have that the sequence $(\varrho_n)_n$ with $\varrho_n \in \partial\psi([\![u_n]\!])$ a.e. in Γ_C is bounded in $\mathrm{L}^4(\Gamma_\mathrm{C})$ and thus, up to a subsequence, it weakly converges in $\mathrm{L}^4(\Gamma_\mathrm{C})$ to some ϱ. By the strong-weak closedness of the graph of $\partial\psi$ (or, rather, of the maximal monotone operator that $\partial\psi : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ induces on $\mathrm{L}^2(\Gamma_\mathrm{C})$), we have that $\varrho \in \partial\psi([\![u]\!])$ a.e. in Γ_C. Moreover, we find that $\gamma(z_n)\varrho_n \rightharpoonup \gamma(z)\varrho$, for instance in $\mathrm{L}^2(\Gamma_\mathrm{C})$. Since β is Lipschitz, we also have $\beta([\![u_n]\!]) \to \beta([\![u]\!])$ in $\mathrm{L}^q(\Gamma_\mathrm{C};\mathbb{R}^3)$ for all $1 \le q < 4$. Also taking into account that $\ell_u \in \mathrm{C}^1([0,T];\mathbf{U}^*)$, we then conclude the weak limit μ of the sequence $(\mu_n)_n$ belongs to $\partial_\mathsf{u}\mathcal{E}(t,q)$.

Let us now discuss the weak $\mathbf{Z}$-limit ζ of the sequence $(\zeta_n)_n$. First of all, from the Lipschitz continuity of γ we gather that the sequence $(\omega_n)_n$ is bounded in $\mathrm{L}^\infty(\Gamma_\mathrm{C})$. Hence, $(\omega_n\psi([\![u_n]\!]))_n$ is bounded in $\mathrm{L}^2(\Gamma_\mathrm{C})$ and, a fortiori, we gather that also the terms $(\boldsymbol{A}z_n+\phi_n)_n$ are bounded in $\mathrm{L}^2(\Gamma_\mathrm{C})$. By the strong-weak closedness of the graph of (the operator induced by) $\partial\gamma$ (on $\mathrm{L}^2(\Gamma_\mathrm{C})$), we infer that $\omega \in \partial\gamma(z)$ a.e. in Γ_C. Since ψ has at most quadratic growth, from $[\![u_n]\!] \to [\![u]\!]$ in $\mathrm{L}^q(\Gamma_\mathrm{C};\mathbb{R}^3)$ for all $1 \le q < 4$ we obtain that $\psi([\![u_n]\!]) \to \psi([\![u]\!])$ in $\mathrm{L}^{q/2}(\Gamma_\mathrm{C};\mathbb{R}^3)$ via the dominated convergence theorem. All in all, we have that $\omega_n\psi([\![u_n]\!]) \rightharpoonup \omega\psi([\![u]\!]) \in \psi([\![u]\!])\partial\gamma(z)$ in $\mathrm{L}^2(\Gamma_\mathrm{C})$.

Now, with arguments similar to those in the previous lines and that in $\mathrm{L}^2(\Gamma_\mathrm{C})$, since $\omega_n \overset{*}{\rightharpoonup} \omega$ in $\mathrm{L}^\infty(\Gamma_\mathrm{C})$ and $\psi([\![u_n]\!]) \to \psi([\![u]\!])$ in $\mathrm{L}^{q/2}(\Gamma_\mathrm{C})$ for all $1 \le q < 4$. In turn, also taking into account that the sequence $(\boldsymbol{A}z_n)_n$ is itself bounded in $\mathrm{H}^1(\Gamma_\mathrm{C})^*$, it is immediate to check that there exists $\phi \in \mathrm{H}^1(\Gamma_\mathrm{C})^*$ such that $\boldsymbol{A}z_n + \phi_n \rightharpoonup \boldsymbol{A}z + \phi$ in $\mathrm{L}^2(\Gamma_\mathrm{C})$. Since the functional $\mathcal{F}$ from (8.9) $\mathcal{F}$ is also $(-\Lambda_\phi)$-convex, its Fréchet subdifferential has a strongly-weakly closed graph in $\mathrm{L}^2(\Gamma_\mathrm{C})\times\mathrm{L}^2(\Gamma_\mathrm{C})$, and we thus infer that $\phi \in \partial\widehat{\phi}(z)$ a.e. in Γ_C. All in all, we conclude that that $\zeta \in \partial_\mathsf{z}\mathcal{E}(t,q)$. This concludes the proof of Hypothesis 4.5.

Step 3. Hypothesis 5.7: Let us now turn to the parametrized chain rule from Hypothesis 5.7. Since the viscous dissipation potentials are 2-homogeneous, the associated vanishing-viscosity contact potentials are given by (3.10) (cf. Example 3.4) so that, in particular, the coercivity condition (5.37) holds, and Proposition 5.16 is applicable. Therefore, Hypothesis 5.7 follows from the chain rule of Hyp. 4.7. The latter chain-rule property can be verified by resorting to Proposition A.1 ahead. Hence, we need to show $\mathcal{E}$ complies with condition (A.1), which states that the Fréchet subdifferential $\partial_q\mathcal{E}$ can be characterized by a *global* inequality akin to that defining the convex analysis subdifferential: for every $E > 0$, and energy sublevel $\mathcal{S}_E$, there exists an upper semicontinuous function $\varpi^E : [0,T]\times\mathcal{S}_E\times\mathcal{S}_E \to [0,\infty]$, with $\varpi^E(t,q,q) = 0$ for every $t \in [0,T]$ and $q \in \mathcal{S}_E$, such that

$$\mathcal{E}(t,\hat{q}) - \mathcal{E}(t,q) \ge \langle \xi, \hat{q}-q\rangle_\mathbf{Q} - \varpi^E(t,q,\hat{q})\|\hat{q}-q\|_\mathbf{Q} \text{ for all } t \in [0,T],\ q,\ \hat{q} \in \mathcal{S}_E \text{ and all } \xi \in \partial_q\mathcal{E}(t,q)\,. \tag{8.16}$$

In order to check (8.16), we will resort to a decomposition for the energy functional from (8.6d) as

$$\mathcal{E}(t,u,z) = \mathcal{E}_\mathrm{elast}(t,u) + \mathcal{F}(z) + \mathcal{E}_\mathrm{coupl}(u,z) \tag{8.17a}$$

with $\mathcal{F}$ from (8.9),

$$\mathcal{E}_\mathrm{elast}(t,u) := \tfrac{1}{2}\,\langle \boldsymbol{C}u, u\rangle_{\mathrm{H}^1(\Omega)} - \langle \ell_u(t), u\rangle_{\mathrm{H}^1(\Omega)}, \tag{8.17b}$$

while, for later convenience, we encompass the surface term $\int_{\Gamma_\mathrm{C}} \widehat{\beta}([\![u]\!])\,\mathrm{d}x$ in the coupling energy

$$\mathcal{E}_\mathrm{coupl}(u,z) := \int_{\Gamma_\mathrm{C}} \Big(\widehat{\beta}([\![u]\!]) + \gamma(z)\psi([\![u]\!])\Big)\,\mathrm{d}x\,. \tag{8.17c}$$

Now, $\mathcal{E}_\mathrm{elast}(t,\cdot)$ is convex while $\mathcal{F}$ is $(-\Lambda_\phi)$-convex. Hence, they both comply with (8.16). Hence, it is sufficient to check its validity for $\mathcal{E}_\mathrm{coupl}$, and indeed for its second contribution, only, since $\widehat{\beta}$ is also convex. Indeed, for every $\hat{u}$, $u \in \mathbf{U}$ and $\hat{z}$, $z \in \mathbf{Z}$ and for all selections $\Gamma_\mathrm{C} \ni x \mapsto \varrho(x) \in \partial\psi([\![u(x)]\!])$ and

$\Gamma_C \ni x \mapsto \omega(x) \in \partial\gamma(z(x))$ there holds

$$\begin{aligned}
&\int_{\Gamma_C} \big(\gamma(\hat z)\psi(\llbracket \hat u\rrbracket) - \gamma(z)\psi(\llbracket u\rrbracket)\big)\,\mathrm{d}x - \int_{\Gamma_C} \gamma(z)\varrho\llbracket \hat u - \hat u\rrbracket\,\mathrm{d}x - \int_{\Gamma_C} \omega\psi(\llbracket u\rrbracket)(\hat z - z)\,\mathrm{d}x \\
&= \int_{\Gamma_C} \gamma(\hat z)\{\psi(\llbracket \hat u\rrbracket) - \psi(\llbracket u\rrbracket)\}\,\mathrm{d}x - \int_{\Gamma_C} \gamma(z)\varrho\llbracket \hat u - \hat u\rrbracket\,\mathrm{d}x + \int_{\Gamma_C} \{\gamma(\hat z) - \gamma(z) - \omega(\hat z - z)\}\psi(\llbracket u\rrbracket)\,\mathrm{d}x \\
&\overset{(1)}{\geq} \int_{\Gamma_C} \big(\gamma(\hat z) - \gamma(z)\big)\big(\psi(\llbracket \hat u\rrbracket) - \psi(\llbracket u\rrbracket)\big)\,\mathrm{d}x + \int_{\Gamma_C} \gamma(z)\{\psi(\llbracket \hat u\rrbracket) - \psi(\llbracket u\rrbracket) - \varrho\llbracket \hat u - u\rrbracket\}\,\mathrm{d}x \\
&\overset{(2)}{\geq} \int_{\Gamma_C} \big(\gamma(\hat z) - \gamma(z)\big)\big(\psi(\llbracket \hat u\rrbracket) - \psi(\llbracket u\rrbracket)\big)\,\mathrm{d}x \\
&\overset{(3)}{\geq} -\|\hat z - z\|_{\mathrm{L}^2(\Gamma_C)}\|\psi(\llbracket \hat u\rrbracket) - \psi(\llbracket u\rrbracket)\|_{\mathrm{L}^2(\Gamma_C)}
\end{aligned}$$

where (1) & (2) follow from the convexity of γ and ψ, respectively, whereas (3) is due to the 1-Lipschitz continuity of γ. Then, estimate (8.16) follows with the function $\varpi_t^E(\hat q, q) := \|\psi(\llbracket \hat u\rrbracket) - \psi(\llbracket u\rrbracket)\|_{\mathrm{L}^2(\Gamma_C)}$. We have thus checked the validity of (8.16) and, a fortiori, of Hypothesis 5.7. This concludes the proof. □

Remark 8.3. *The Lipschitz continuity of β has played a key role in the proof that $\mathcal{E}$ complies with the closedness condition from Hyp. 4.5. In fact, we could allow for a nonsmooth $\widehat\beta$, but with a suitable polynomial growth condition, that would still ensure that the maximal monotone operator induced by $\beta = \partial\widehat\beta$ on $\mathrm{L}^2(\Gamma_C)$ is strongly-weakly closed. However, it would not be possible to check Hypothesis 4.5 in the case β is an unbounded maximal monotone operator, such as the subdifferential of an indicator function. That is why, we are not in a position to encompass in our analysis the non-interpenetration constraint between the two bodies Ω^+ and Ω^-.*

8.4. A priori estimates for the smooth semilinear system.

In this section we address a version of the viscous system (8.4) in which the functions $\widehat\beta$, γ, $\widehat\phi$, and ψ, complying with (8.5), are also smoothened. Namely, we will additionally suppose that they fulfill

$$\left.\begin{array}{l}\gamma,\ \widehat\phi \in \mathrm{C}^2(\mathbb{R};\mathbb{R}), \quad \psi,\ \widehat\beta \in \mathrm{C}^2(\mathbb{R}^3;\mathbb{R}), \\ \gamma'',\ \widehat\phi'',\ \mathrm{D}^2\widehat\beta \text{ are bounded}, \quad |\mathrm{D}\psi(a)| \leq C_\psi^{(1)} \text{ for all } a \in \mathbb{R}^3.\end{array}\right\} \tag{8.18}$$

These conditions will allow us to *rigorously* perform, on the solutions to system (8.4), calculations that will ultimately lead to bounds, uniform with respect to viscosity parameter, suitable for our vanishing-viscosity analysis. Such estimates will however only depend on the constants occurring in (8.5), and not on those in (8.18). For these calculations we will crucially make use of the *semilinear* structure of this regularized system and of the fact that the coupling between the displacement equation and the flow rule for the delamination parameter is weak enough to allow us to treat those equations separately.

As already mentioned, for all $\varepsilon \in (0,1)$ and all initial data $(u_0, z_0) \in \mathbf{U}\times\mathbf{Z}_\mathrm{e}$ system (8.4) admits finite-energy solutions $(u_\varepsilon, z_\varepsilon)$ with the standard time regularity (8.10). We now aim to derive higher order estimates as well, and to show that these estimates are independent of ε. We will make them as explicit as possible. Let us mention in advance that one crucial argument involves the interpolation between the different norms for the time derivative $\dot z$, namely

$$\forall\, \dot z \in \mathbf{Z}_\mathrm{e}: \quad \|\dot z\|_{\mathbf{Z}} \leq C_{\mathrm{GN}}\mathcal{R}(\dot z)^{1/2}\|\dot z\|_{\mathbf{Z}_\mathrm{e}}^{1/2}. \tag{8.19}$$

Indeed, (8.19) follows by combining the lower bound $\mathcal{R}(v) \geq c_R\|v\|_{\mathrm{L}^1}$ with the classical Gagliardo-Nirenberg interpolation $\|v\|_{\mathrm{L}^2}^2 \leq C\|v\|_{\mathrm{L}^1}\|v\|_{\mathrm{H}^1}$. This will allows us to exploit the ε-independent dissipation estimate $\int_0^T \mathcal{R}(\dot z_\varepsilon)\,\mathrm{d}t \leq C$.

<u>*Step 1. Basic energy and dissipation estimates:*</u> The simple energy-dissipation estimate stemming from the energy balance (4.18b) (cf. Lemma 4.12), together with $\ell_u \in \mathrm{C}^1([0,T;\mathbf{U}^*)$ implies that for all E_0 there exists $C_1^{E_0} > 0$ such all solutions $(u_\varepsilon, z_\varepsilon)$ of (8.4) with $\mathcal{E}(0, u_\varepsilon(0), z_\varepsilon(0) \leq E_0$ satisfy the basic energy estimates

$$\int_0^T \big\{\mathcal{R}(\dot z_\varepsilon(t)) + \varepsilon^\alpha\|\dot u_\varepsilon(t)\|_{\mathbf{U}}^2 + \varepsilon\|\dot z_\varepsilon\|_{\mathbf{Z}}^2\big\}\,\mathrm{d}t \leq C_1^{E_0} \quad \text{and} \quad \forall\, t \in [0,T]: \ \|u_\varepsilon(t)\|_{\mathbf{U}} + \|z_\varepsilon(t)\|_{\mathbf{Z}_\mathrm{e}} \leq C_1^{E_0}. \tag{8.20}$$

As a consequence of this a priori bound, of the fact that $\boldsymbol{J} : \mathbf{U} \to \mathrm{L}^4(\Gamma_C;\mathbb{R}^3)$ is a bounded operator, and of upper estimates on ψ via the constants C_ψ and $C_\psi^{(1)}$, we find another constant $C_2^{E_0}$ such that all solutions $(u_\varepsilon, z_\varepsilon)$ of (8.4) with $\mathcal{E}(0, u_\varepsilon(0), z_\varepsilon(0)) \leq E_0$ satisfy

$$\|\psi(\llbracket u_\varepsilon(t)\rrbracket)\|_{\mathrm{L}^2} \leq C_\psi C_2^{E_0}, \quad \|\mathrm{D}\psi(\llbracket u_\varepsilon(t)\rrbracket)\|_{\mathrm{L}^4} \leq C_\psi C_2^{E_0}, \tag{8.21a}$$

$$\|\psi(\llbracket u_\varepsilon(t)\rrbracket)\|_{\mathrm{L}^4} \leq C_\psi^{(1)} C_2^{E_0}, \quad \|\mathrm{D}\psi(\llbracket u_\varepsilon(t)\rrbracket)\|_{\mathrm{L}^\infty} \leq C_\psi^{(1)} C_2^{E_0}. \tag{8.21b}$$

Estimate (8.21b) will in fact only be used for gaining enhanced regularity of the viscous solutions $(u_\varepsilon, z_\varepsilon)$, and not for the vanishing-viscosity analysis.

<u>*Step 2. Estimate for $\dot u_\varepsilon$:*</u> Because of the smoothness of $\widehat\beta$ and ψ, the displacement equation (8.4a) for u_ε is a semilinear equation with a smooth nonlinearity, if we consider $z_\varepsilon \in \mathrm{H}^1(0,\mathrm{T};\mathbf{Z})$ as *given* datum. Thus, we can use the classical technique of difference quotients to show that $u_\varepsilon \in \mathrm{H}^2(0,\mathrm{T};\mathbf{U})$ provided that $\dot u_\varepsilon(0) = \varepsilon^{-\alpha}\boldsymbol{D}^{-1}\big(\boldsymbol{C}u(0) + \boldsymbol{J}^*(\cdots) - \ell_u(0)\big) \in \mathbf{U}$. Hence, it is possible to differentiate (8.4a) with respect to time, which yields

$$0 = \varepsilon^\alpha \boldsymbol{D}\ddot u_\varepsilon + \boldsymbol{C}\dot u_\varepsilon + \boldsymbol{J}^*\Big(\mathrm{D}^2\widehat\beta(\llbracket u_\varepsilon\rrbracket)\llbracket \dot u_\varepsilon\rrbracket + \gamma(z_\varepsilon)\mathrm{D}^2\psi(\llbracket u_\varepsilon\rrbracket)\llbracket \dot u_\varepsilon\rrbracket + \gamma'(z_\varepsilon)\dot z_\varepsilon \mathrm{D}\psi(\llbracket u_\varepsilon\rrbracket)\Big) - \dot\ell_u(t). \tag{8.22}$$

We can test (8.22) by $\dot u_\varepsilon \in \mathrm{H}^1(0,\mathrm{T};\mathbf{U})$ and obtain

$$\begin{aligned} 0 = {} & \frac{\varepsilon^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{D}\dot u_\varepsilon, \dot u_\varepsilon\rangle_{\mathbf{U}} + \langle \boldsymbol{C}\dot u_\varepsilon, \dot u_\varepsilon\rangle_{\mathbf{U}} + \langle \mathrm{D}^2\widehat\beta(\llbracket u_\varepsilon\rrbracket)\llbracket \dot u_\varepsilon\rrbracket + \gamma(z_\varepsilon)\mathrm{D}^2\psi(\llbracket u_\varepsilon\rrbracket)\llbracket \dot u_\varepsilon\rrbracket, \llbracket \dot u_\varepsilon\rrbracket\rangle_{\mathbf{Z}} \\ & - \langle \dot\ell_u, \dot u_\varepsilon\rangle_{\mathbf{U}} - \langle \gamma'(z_\varepsilon)\dot z_\varepsilon \mathrm{D}\psi(\llbracket u_\varepsilon\rrbracket), \llbracket \dot u_\varepsilon\rrbracket\rangle_{\mathbf{Z}} \end{aligned}$$

Here the last duality product in the first line is nonnegative, because $a \mapsto \widehat\beta(a) + \gamma(z)\psi(a)$ is convex. The last duality product can be estimated using (8.21a). Defining $\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon$, $\boldsymbol{\theta}_{\mathbf{Z}}^\varepsilon$, and $\lambda_{\mathbf{U}^*}$ via

$$\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon(t)^2 := \langle \boldsymbol{D}\dot u_\varepsilon(t), \dot u_\varepsilon(t)\rangle_{\mathbf{U}} \quad \text{and} \quad \boldsymbol{\theta}_{\mathbf{Z}}^\varepsilon(t)^2 := \|\dot z_\varepsilon(t)\|_{\mathbf{Z}}^2, \quad \text{and } \lambda_{\mathbf{U}^*}(t) = \|\dot\ell_u(t)\|_{\mathbf{U}^*},$$

we have established the estimate

$$\frac{\varepsilon^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon)^2 + c_{\boldsymbol{C}}(\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon)^2 \leq \lambda_{\mathbf{U}^*}\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon + 1\, C_\psi C_2^{E_0} C_{\mathrm{H}^1,\mathrm{L}^4}\|\boldsymbol{J}\|\boldsymbol{\theta}_{\mathbf{Z}}^\varepsilon\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon$$

where we have also used that γ is 1-Lipschitz continuous, and $C_{\mathrm{H}^1,\mathrm{L}^4}$ denotes the constant associated with the continuous embedding $\mathbf{U} \subset \mathrm{L}^4(\Gamma_{\mathrm{C}};\mathbb{R}^3)$. Using $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon)^2 = 2\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon\,\dot{\boldsymbol{\theta}}_{\mathbf{U}}^\varepsilon$ we can divide by $\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon \geq 0$ and obtain

$$\varepsilon^\alpha\dot{\boldsymbol{\theta}}_{\mathbf{U}}^\varepsilon + c_{\boldsymbol{C}}\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon \leq \lambda_{\mathbf{U}^*} + C_\psi C_2^{E_0} C_{\mathrm{H}^1,\mathrm{L}^4}\|\boldsymbol{J}\|\,\boldsymbol{\theta}_{\mathbf{Z}}^\varepsilon. \tag{8.23}$$

Let us mention that the above estimate could be rigorously obtained by replacing $\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon$ by $\sqrt{(\boldsymbol{\theta}_{\mathbf{U}}^\varepsilon)^2 + \delta}$, which satisfies the same estimate, and then letting $\delta \downarrow 0$, cf. [Mie11, Sec. 4.4].

<u>*Step 3. Uniqueness and higher regularity of $\dot z_\varepsilon$:*</u> We first observe that *given* $u_\varepsilon \in \mathrm{H}^1([0,\mathrm{T}];\mathbf{U})$ and z_0 there is a unique solution z_ε for (8.4b). Indeed, assuming that z_1 and z_2 are solutions (with $\varrho_j \in \partial\mathrm{R}(\dot z_j)$) we set $w = z_1 - z_2$ and test the difference of the two equations by $\dot w = \dot z_1 - \dot z_2$, which yields

$$0 = \langle \varrho_1 - \varrho_2, \dot z_1 - \dot z_2\rangle_{\mathbf{Z}} + \varepsilon\|\dot w\|_{\mathbf{Z}}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{A}w, w\rangle_{\mathbf{Z}_{\mathrm{e}}} + \langle G(u_\varepsilon, z_1) - G(u_\varepsilon, z_2), \dot w\rangle_{\mathbf{Z}}, \tag{8.24}$$

where we have set $G(u,z) = \widehat\phi'(z) + \gamma'(z)\psi(\llbracket u\rrbracket)$. By our strengthened assumptions (8.18) and Gagliardo-Nirenberg interpolation we have

$$\begin{aligned} \|G(u_\varepsilon, z_1) - G(u_\varepsilon, z_2)\|_{\mathbf{Z}^*} & \leq \|\widehat\phi''\|_\infty\|z_1 - z_2\|_{\mathbf{Z}} + \|\gamma'(z_1) - \gamma'(z_2)\|_{\mathrm{L}^4}\|\psi(\llbracket u\rrbracket_\varepsilon)\|_{\mathrm{L}^4} \\ & \leq \big(\|\widehat\phi''\|_\infty\|z_1 - z_2\|_{\mathbf{Z}} + \|\gamma''\|_\infty\|z_1 - z_2\|_{\mathrm{L}^4} C_\psi^{(1)} C_2^{E_0}\big) \leq C_G\|w\|_{\mathbf{Z}}^{1/2}\|w\|_{\mathbf{Z}_{\mathrm{e}}}^{1/2}. \end{aligned}$$

By using the monotonicity of $\partial\mathrm{R}$, the first term in (8.24) is nonnegative. Using $\|w\|_{\mathbf{Z}_{\mathrm{e}}}^2 = \langle \boldsymbol{A}w, w\rangle_{\mathrm{H}^1(\Gamma_{\mathrm{C}})}$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{\mathbf{Z}_{\mathrm{e}}}^2 + \varepsilon\|\dot w\|_{\mathbf{Z}}^2 \leq C_G\|w\|_{\mathbf{Z}}^{1/2}\|w\|_{\mathbf{Z}_{\mathrm{e}}}^{1/2}\|\dot w\|_{\mathbf{Z}} \leq \frac{C_G^2}{4\varepsilon}\|w\|_{\mathbf{Z}}\|w\|_{\mathbf{Z}_{\mathrm{e}}} + \varepsilon\|\dot w\|_{\mathbf{Z}}^2.$$

Canceling the terms $\varepsilon\|\dot w\|_{\mathbf{Z}}^2$ and using $\|w\|_{\mathbf{Z}} \leq \|w\|_{\mathbf{Z}_{\mathrm{e}}}$ provides the estimate

$$\|z_1(t) - z_2(t)\|_{\mathbf{Z}_{\mathrm{e}}} \leq \mathrm{e}^{C_G{}^2(t-s)/(4\varepsilon)}\|z_1(s) - z_2(s)\|_{\mathbf{Z}_{\mathrm{e}}} \quad \text{for } 0 \leq s \leq t \leq T. \tag{8.25}$$

We emphasize that this uniqueness result is special and relies strongly on the semilinear structure of the flow rule for z under the strengthened assumption (8.18). It is indeed thanks to (8.18) that $G(u,\cdot) : \mathbf{Z}_{\mathrm{e}} \to \mathbf{Z}^*$ is globally Lipschitz, and in fact the constant C_G in (8.25) does depend on $C_\psi^{(1)}$.

This uniqueness is central to derive higher regularity as it is now possible to use suitable regularizations such as Galerkin approximations or replacing the nonsmooth function R by a smoothed version R_δ. We do not go into detail here, but refer to [MiZ14] and [Mie11, Sec. 4.4]. In particular, our problem fits exactly into the abstract setting of [MiZ14, Sec. 3] with $H = \mathbf{Z} = \mathrm{L}^2(\Gamma_\mathrm{C})$, $\mathcal{B} = \boldsymbol{A}$, and $\Phi(t,z) = \int_\Omega \big(\widehat{\phi}(z) + \gamma(z)\psi([\![u(t)]\!])\,\mathrm{d}x$.

Thus, under the additional condition $\boldsymbol{A}z_0 \in \mathbf{Z}$ (or $z_0 \in \mathrm{H}^2(\Gamma_\mathrm{C})$), the unique solution z_ε with $z_\varepsilon(0) = z_0$ satisfies the following higher regularity properties:

$$\dot z_\varepsilon \in \mathrm{L}^\infty(0,T;\mathbf{Z}_\mathrm{e}) \quad \text{and} \quad \sqrt{t}\,\ddot z_\varepsilon \in \mathrm{L}^2(0,T;\mathbf{Z}). \tag{8.26}$$

Of course, at this stage we have no control over the dependence on ε of the corresponding norms.

Step 4. Identities not involving R*:* Surprisingly, there are two identities for the solution z_ε that are completely independent of R, i.e. they look like energy estimates for a semilinear parabolic problem:

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot z_\varepsilon\|^2_{\mathbf{Z}} + \|\dot z_\varepsilon\|^2_{\mathbf{Z}_\mathrm{e}} + \langle \mathrm{D}^2_z\Phi(u_\varepsilon,z_\varepsilon)\dot z_\varepsilon, \dot z_\varepsilon\rangle_{\mathbf{Z}} + \langle \mathrm{D}_z\mathrm{D}_u\Phi(u_\varepsilon,z_\varepsilon)\dot u_\varepsilon, \dot z_\varepsilon\rangle_{\mathbf{Z}} = 0, \tag{8.27a}$$

$$\varepsilon\|\ddot z_\varepsilon\|^2_{\mathbf{Z}} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot z_\varepsilon\|^2_{\mathbf{Z}_\mathrm{e}} + \langle \mathrm{D}^2_z\Phi(u_\varepsilon,z_\varepsilon)\dot z_\varepsilon, \ddot z_\varepsilon\rangle_{\mathbf{Z}} + \langle \mathrm{D}_z\mathrm{D}_u\Phi(u_\varepsilon,z_\varepsilon)\dot u_\varepsilon, \ddot z_\varepsilon\rangle_{\mathbf{Z}} \le 0. \tag{8.27b}$$

We refer to [Mie11, Eqn. (95) and Lem. 4.16] for a rigorous derivation based on the smoothness established in (8.26). Relations (8.27) can be formally derived from equation (8.4b) by forgetting the nonsmooth term ∂R, then differentiating the whole equation with respect to t, and finally testing with $\dot z_\varepsilon$ or $\ddot z_\varepsilon$ respectively. Indeed, (8.27b) will not be used below any more, but its relevance is obvious by comparison with (8.24) and for deriving the (ε-dependent) a priori estimate for $\sqrt{t}\,\ddot z_\varepsilon$ (via Galerkin approximations).

It is the identity (8.27a) that will be crucial for deriving ε-independent a priori estimates. It origin can formally understood by looking at general smooth p-homogeneous dissipation potentials $\boldsymbol{\Psi}$ (i.e. fulfilling $\boldsymbol{\Psi}(\gamma v) = \gamma^p\boldsymbol{\Psi}(v)$ for all v and $\gamma > 0$). Then, Euler's formula gives $\langle \mathrm{D}\boldsymbol{\Psi}(v), v\rangle = p\boldsymbol{\Psi}(v)$, and we find the identity

$$\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{D}\boldsymbol{\Psi}(\dot z)\big), \dot z\Big\rangle = \mathrm{D}^2\boldsymbol{\Psi}(\dot z)[\ddot z, \dot z] = \frac{\mathrm{d}}{\mathrm{d}t}\big(\langle \mathrm{D}\boldsymbol{\Psi}(\dot z), \dot z\rangle - \boldsymbol{\Psi}(\dot z)\big) = (p{-}1)\,\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Psi}(\dot z).$$

The quadratic case $p = 2$ was applied above several times. Of course, in the case $p = 1$ the potential $\mathcal{R}$ is nonsmooth. Hence, the proof in [Mie11, Lem. 4.16] is different and uses simple arguments based on the characterization of $\partial\mathcal{R}$ in the 1-homogeneous case.

Step 5. L^1 *estimates for* $\boldsymbol{\theta}^\varepsilon_{\mathbf{U}}$*,* $\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}}$*, and* $\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}_\mathrm{e}}$*:* In (8.27a) the coupling term $\langle \mathrm{D}_z\mathrm{D}_u\Phi(u_\varepsilon,z_\varepsilon)\dot u_\varepsilon, \dot z_\varepsilon\rangle$ can be estimated via the weaker assumption (8.5), namely

$$\begin{aligned}\langle \mathrm{D}_z\mathrm{D}_u\Phi(u_\varepsilon,z_\varepsilon)\dot u_\varepsilon, \dot z_\varepsilon\rangle_{\mathbf{Z}} &\le 1\|\dot z_\varepsilon\|_{\mathbf{Z}}\|\mathrm{D}\psi([\![u_\varepsilon]\!])\|_{\mathrm{L}^4}\|[\![\dot u_\varepsilon]\!])\|_{\mathrm{L}^4}\\ &\le C_3\,\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}}(t)\,\boldsymbol{\theta}^\varepsilon_{\mathbf{U}}(t) \text{ with } C_3 := C_\psi C_2^{E_0} C_{\mathrm{H}^1,\mathrm{L}^4}\|\boldsymbol{J}\|\,,\end{aligned}$$

where we exploited the 1-Lipschitz continuity of γ and (8.21a). Introducing the short-hand notation $\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}_\mathrm{e}}$ via $(\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}_\mathrm{e}}(t))^2 = \|\dot z_\varepsilon(t)\|^2_{\mathbf{Z}_\mathrm{e}} = \langle \boldsymbol{A}\dot z_\varepsilon(t), \dot z_\varepsilon(t)\rangle_{\mathrm{H}^1(\Gamma_\mathrm{C})}$ and exploiting the Λ_ϕ-convexity of $\widehat{\phi}$ and the convexity of γ, identity (8.27a) yields

$$\varepsilon\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}}\dot{\boldsymbol{\theta}}^\varepsilon_{\mathbf{Z}} + \big(\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}_\mathrm{e}}\big)^2 \le \Lambda_\phi\big(\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}}\big)^2 + C_3\,\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}}\,\boldsymbol{\theta}^\varepsilon_{\mathbf{U}}\,.$$

For the first term on the right-hand side we can now exploit the interpolation (8.19) and after division by $\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}} \ge 0$ (recall $\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}} \le \boldsymbol{\theta}^\varepsilon_{\mathbf{Z}_\mathrm{e}}$) we arrive, together with (8.23), at the differential estimates

$$\varepsilon^\alpha\dot{\boldsymbol{\theta}}^\varepsilon_{\mathbf{U}} + c_{\boldsymbol{C}}\boldsymbol{\theta}^\varepsilon_{\mathbf{U}} \le \lambda_{\mathbf{U}^*} + C_{\mathrm{GN}}C_3\,\big(\mathcal{R}(\dot z_\varepsilon)\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}_\mathrm{e}}\big)^{1/2}, \tag{8.28a}$$

$$\varepsilon\dot{\boldsymbol{\theta}}^\varepsilon_{\mathbf{Z}} + \boldsymbol{\theta}^\varepsilon_{\mathbf{Z}_\mathrm{e}} \le \Lambda_\phi C_{\mathrm{GN}}\mathcal{R}(\dot z_\varepsilon) + C_3\,\boldsymbol{\theta}^\varepsilon_{\mathbf{U}}\,. \tag{8.28b}$$

We emphasize that all the appearing coefficients, except for the leading factors ε^α and ε, are independent of $\varepsilon \in (0,1)$ and indeed depend only on C_ψ. From the first equation we obtain via the constants-of-variation formula (or Grönwall's lemma) the estimate

$$\boldsymbol{\theta}^\varepsilon_{\mathbf{U}}(t) \le K_\varepsilon(t)\varepsilon^\alpha\boldsymbol{\theta}^\varepsilon_{\mathbf{U}}(0) + \int_0^t K_\varepsilon(t{-}s)\big(\lambda_{\mathbf{U}^*}(s){+}C_{\mathrm{GN}}C_3\big(\mathcal{R}(\dot z_\varepsilon(s))\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}_\mathrm{e}}(s)\big)^{1/2}\big)\,\mathrm{d}s \ \text{ with } K_\varepsilon(t) = \frac{\mathrm{e}^{-c_{\boldsymbol{C}}t/\varepsilon^\alpha}}{\varepsilon^\alpha}.$$

Because of $\|K_\varepsilon\|_{\mathrm{L}^1} = \int_0^\infty K_\varepsilon(t)\,\mathrm{d}t = 1/c_{\boldsymbol{C}}$ the L^1-convolution estimate leads to

$$I_U := \int_0^T \boldsymbol{\theta}^\varepsilon_{\mathbf{U}}(t)\,\mathrm{d}t \le \frac{1}{c_{\boldsymbol{C}}}\Big(\varepsilon^\alpha\boldsymbol{\theta}^\varepsilon_{\mathbf{U}}(0) + \int_0^T \lambda_{\mathbf{U}^*}(t)\,\mathrm{d}t + C_{\mathrm{GN}}C_3\int_0^T \big(\mathcal{R}(\dot z_\varepsilon(t))\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}_\mathrm{e}}(t)\big)^{1/2}\,\mathrm{d}t\Big).$$

Applying the Cauchy-Schwarz inequality to the last integral and integrating (8.28b) over $[0,T]$ we obtain the estimates

$$I_{\mathbf{U}} \leq \frac{1}{c_{\boldsymbol{C}}}\Big(\varepsilon^\alpha \boldsymbol{\theta}^\varepsilon_{\mathbf{U}}(0) + \int_0^T \lambda_{\mathbf{U}^*}(t)\,\mathrm{d}t + C_{\mathrm{GN}} C_3\, I_R^{1/2} I_{\mathbf{Z}_\mathrm{e}}^{1/2}\Big),$$
$$I_{\mathbf{Z}_\mathrm{e}} := \int_0^T \boldsymbol{\theta}^\varepsilon_{\mathbf{Z}_\mathrm{e}}(t)\,\mathrm{d}t \leq \varepsilon\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}}(0) + \Lambda_\phi C_{\mathrm{GN}} I_R + C_3 I_{\mathbf{U}}, \quad \text{where } I_R := \int_0^T \mathcal{R}(\dot z_\varepsilon(t))\,\mathrm{d}t.$$

From this it is easy to show that there exists a constant C_*, which only depends on $C_3 = C_\psi C_2^{E_0} C_{\mathrm{H}^1,\mathrm{L}^4}\|\boldsymbol{J}\|$, $c_{\boldsymbol{C}}$, C_{GN}, and Λ_ϕ, such that $I_{\mathbf{U}}+I_{\mathbf{Z}_\mathrm{e}}$ can be estimated by $C_*\big(\varepsilon^\alpha\boldsymbol{\theta}^\varepsilon_{\mathbf{U}}(0) + \varepsilon\boldsymbol{\theta}^\varepsilon_{\mathbf{Z}}(0) + \int_0^T \lambda_{\mathbf{U}^*}\,\mathrm{d}t + I_R\big)$. We have thus proved the following result.

Lemma 8.4 (Rate-independent a priori estimate in the semilinear case)**.** *Assume* (8.3) *and* (8.5). *Additionally, let* $\widehat\beta$, γ, $\widehat\phi$, *and* ψ *satisfy* (8.18) *and let the initial data* $(u_0, z_0) \in \mathbf{U}\times\mathbf{Z}_\mathrm{e}$ *comply with* (8.11). *Then, There exists a constant* $C_* > 0$, *only depending on the initial data and on the constants* Λ_ϕ *and* C_ψ *from* (8.5), *such that the unique solution* $(u_\varepsilon, z_\varepsilon)$ *of* (8.4) *satisfies the a priori estimate*

$$\int_0^T \Big(\|\dot u_\varepsilon\|_{\mathbf{U}} + \|\dot z_\varepsilon\|_{\mathbf{Z}_\mathrm{e}}\Big)\,\mathrm{d}t \leq C_*\Big(\varepsilon^\alpha\|\dot u_\varepsilon(0)\|_{\mathbf{U}} + \varepsilon\|\dot z_\varepsilon\|_{\mathbf{Z}} + \int_0^T \big(\|\dot\ell_u\|_{\mathbf{U}^*} + \mathcal{R}(\dot z_\varepsilon)\big)\,\mathrm{d}t\Big). \tag{8.29}$$

8.5. Existence and a priori estimates in the general case.

We now return to the setup of Sections 8.1 and 8.2, in which the constitutive functions $\widehat\beta$, γ, $\widehat\phi$, and ψ only comply with (8.5). We exhibit approximations of $\widehat\beta$, γ, $\widehat\phi$, and ψ that also satisfy (8.18). For this, we will resort to the following general construction.

Smoothening the Yosida approximation. Following, e.g., the lines of [GiR06, Sec. 3], for a given convex function $\widehat\chi : \mathbb{R}^d \to [0,\infty]$ with subdifferential $\chi = \partial\widehat\chi : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, and for a fixed $\delta \in (0,1)$, we define

$$\chi^\delta := \chi^{\mathrm{Y}}_\delta \star \eta_\delta$$

where χ^{Y}_δ is the Yosida regularization of the maximal monotone operator χ (we refer to, e.g., [Bré73]) and

$$\eta_\delta(x) := \tfrac{1}{\delta^2}\eta\left(\tfrac{x}{\delta^2}\right) \qquad \text{with} \begin{cases} \eta \in \mathrm{C}^\infty(\mathbb{R}^d), \\ \|\eta\|_1 = 1, \\ \mathrm{supp}(\eta) \subset B_1(0). \end{cases} \tag{8.30}$$

Thus, $\chi^\delta \in \mathrm{C}^\infty(\mathbb{R}^d)$ and it has been shown in [GiR06] that

$$\|\mathrm{D}\chi^\delta\|_\infty \leq \frac{1}{\delta}, \qquad |\chi^\delta(x) - \chi^{\mathrm{Y}}_\delta(x)| \leq \delta \text{ for all } x \in \mathbb{R}^d. \tag{8.31a}$$

Taking into account the properties of the Yosida we deduce that

$$|\chi^\delta(x)| \leq |\chi^o(x)| + \delta \qquad \text{with } |\chi^o(x)| = \inf\{|y| \,:\, y \in \chi(x)\}. \tag{8.31b}$$

Furthermore, χ^δ admits a convex potential $\widehat\chi^\delta$ satisfying, as a consequence of (8.31a), (below $\widehat\chi^{\mathrm{Y}}_\delta$ denotes the Yosida approximation of $\widehat\chi$):

$$-\delta|x| \leq \widehat\chi^{\mathrm{Y}}_\delta(x) - \delta|x| \leq \widehat\chi^\delta(x) \leq \widehat\chi^{\mathrm{Y}}_\delta(x) + \delta|x| \leq \widehat\chi(x) + \delta|x| \;\text{ and }\; \widehat\chi^\delta(x) \to \widehat\chi(x) \qquad \text{for all } x \in \mathbb{R}^d. \tag{8.31c}$$

Finally, the following analogue of Minty's trick holds: given $O \subset \mathbb{R}^m$ and sequence $(v_\delta)_\delta\, v$, $\chi \in \mathrm{L}^2(O;\mathbb{R}^d)$ such that $v_\delta \rightharpoonup v$ and $\chi^\delta(v_\delta) \rightharpoonup \eta$ in $\mathrm{L}^2(O;\mathbb{R}^d)$,

$$\limsup_{\delta\to0^+} \int_O \chi^\delta(v_\delta)\cdot v_\delta\,\mathrm{d}x \leq \int_O \eta\cdot v \quad \Longrightarrow \quad \eta \in \partial\widehat\chi(v) \text{ a.e. in } O. \tag{8.31d}$$

We apply this construction to γ, obtaining a smooth approximation γ^δ. The definition of $\widehat\beta^\delta$ clearly simplifies, since we have already required that $\widehat\beta \in \mathrm{C}^1(\mathbb{R})$ with β Lipschitz. As for ϕ, we define

$$\phi^\delta : \mathbb{R} \to \mathbb{R} \qquad \phi^\delta(z) := f^\delta(z) - \frac{\Lambda_\phi}{2}z^2$$

with f^δ the smoothened Yosida approximation of the convex function $z \mapsto f(z) = \widehat\phi(z) + \frac{\Lambda_\phi}{2}z^2$. It follows from (8.31a) that $\widehat\beta^\delta$, γ^δ and ϕ^δ comply with (8.18).

The construction of ψ^δ. In smoothening ψ we also have to take care of the linear growth constraint encompassed in (8.18). Hence, we construct ψ^δ in two steps:
Step 1. Inf-convolution We define $\psi_\delta^{\mathrm{ic}} : \mathbb{R}^3 \to [0,\infty)$ via inf-convolution with the smooth function $h : \mathbb{R}^3 \to [0,\infty)$, $h(a) := \sqrt{1+|a|^2} - 1$ by setting

$$\psi_\delta^{\mathrm{ic}}(a) := \inf_{x\in\mathbb{R}^3} \left(\frac{1}{\delta} h(x-a) + \psi(x)\right) . \tag{8.32}$$

It turns out that $\psi_\delta^{\mathrm{ic}}$ is convex, of class C^1, and since $h(0) = 0$ we have that

$$\psi_\delta^{\mathrm{ic}}(a) \le \psi(a) \qquad \text{for all } a \in \mathbb{R}^3. \tag{8.33a}$$

Since h is even, we also have $\psi_\delta^{\mathrm{ic}}(a) = \inf_{x\in\mathbb{R}^3}\{\frac{1}{\delta}h(x) + \psi(a-x)\}$. Hence, recalling that $\psi(0) = 0$ we find that

$$\psi_\delta^{\mathrm{ic}}(a) \le \frac{1}{\delta} h(a) \qquad \text{for all } a \in \mathbb{R}^3, \tag{8.33b}$$

so that, in particular, $\psi_\delta^{\mathrm{ic}}$ has linear growth. Finally, let $a_\delta \in \operatorname{Argmin}_{x\in\mathbb{R}^3}\{\frac{1}{\delta}h(x-a) + \psi(x)\}$. Then, $\frac{1}{\delta}h(a_\delta-a) \le \psi_\delta^{\mathrm{ic}} \le \psi(a)$, so that $\lim_{\delta\to0^+} h(a_\delta-a) = 0$, hence $|a_\delta-a| = \sqrt{(h(a_\delta-a)+1)^2-1} \longrightarrow 0$ as $\delta \to 0^+$. All in all, we conclude that

$$\psi_\delta^{\mathrm{ic}}(a) = \frac{1}{\delta} h(a_\delta-a) + \psi(a_\delta) \ge \psi(a_\delta) \qquad \text{with } a_\delta \to a \text{ as } \delta \to 0^+ . \tag{8.33c}$$

Step 2. Smoothening We then define $\psi^\delta \in \mathrm{C}^\infty(\mathbb{R}^3;\mathbb{R})$ via

$$\psi^\delta := \psi_\delta^{\mathrm{ic}} \star \eta_\delta \qquad \text{with } \eta_\delta \text{ from (8.30)}. \tag{8.34}$$

Clearly, ψ^δ is also convex. Combining (8.31c) and (8.33a), (8.33b), and (8.33c) we gather that

$$-\delta|a| \le \psi(a_\delta) - \delta|a| \le \psi^\delta(a) \le \min\left\{\frac{1}{\delta}h(a), \psi(a)\right\} + \delta|a| \qquad \text{with } a_\delta \to a \text{ as } \delta \to 0^+ . \tag{8.35a}$$

Thus, ψ^δ has also linear growth. Taking into account that it is convex, from (8.35a) we easily deduce that

$$|\mathrm{D}\psi^\delta(a)| \le |\partial\psi^\circ(a)| + \delta \qquad \text{for all } a \in \mathbb{R}^3, \tag{8.35b}$$

(where we have again used the notation $|\partial\psi^\circ(a)| = \inf\{|\eta| \,:\, \eta \in \partial\psi(a)\}$. Finally,

$$\lim_{\delta\to0^+} \psi^\delta(a) = \psi(a) \qquad \text{for all } a \in \mathbb{R}^3 . \tag{8.36}$$

The delamination system (8.4) featuring $\widehat{\beta}^\delta$, γ^δ, $\widehat{\phi}^\delta$ and ψ^δ obviously has a gradient structure in the ambient spaces (8.6a), with the dissipation potentials from (8.6b) and (8.6c), and with the driving energy (cf. (8.17))

$$\mathcal{E}^\delta(t,u,z) := \mathcal{E}_{\mathrm{elast}}(t,u) + \mathcal{F}^\delta(z) + \mathcal{E}^\delta_{\mathrm{coupl}}(u,z) \tag{8.37a}$$

with $\mathcal{E}_{\mathrm{elast}}$ from (8.17), and

$$\mathcal{F}^\delta(z) := \frac{1}{2}\langle \boldsymbol{A}z, z\rangle_{\mathrm{H}^1(\Gamma_{\mathrm{C}})} + \int_{\Gamma_{\mathrm{C}}} \widehat{\phi}^\delta(z)\,\mathrm{d}x \quad \text{if } z \in \mathrm{H}^1(\Gamma_{\mathrm{C}}), \text{ and } \infty \text{ else}, \tag{8.37b}$$

$$\mathcal{E}^\delta_{\mathrm{coupl}}(u,z) := \int_{\Gamma_{\mathrm{C}}} \left(\widehat{\beta}^\delta(\llbracket u \rrbracket) + \gamma^\delta(z)\psi^\delta(\llbracket u \rrbracket)\right)\mathrm{d}x\,. \tag{8.37c}$$

which indeed Mosco converges as $\delta \to 0^+$, with respect to the topology of $\mathbf{U}\times\mathbf{Z}$, to the energy functional $\mathcal{E}$ from (8.6d). We will pass to the limit, as $\delta \to 0^+$, in the corresponding energy-dissipation balance (4.18b) to prove that the solutions $(u_\delta^\varepsilon, z_\delta^\varepsilon)_\delta$ to the regularized delamination system converge to a solution of the original system (8.4), satisfying the basic energy estimate (8.20) as well as the rate-independent a priori estimate (8.29).

Proposition 8.5 (Existence of viscous solutions with improved estimates)**.** *Under assumptions* (8.5) *for* $\widehat{\beta}$, ψ, γ, *and* $\widehat{\phi}$ *and the compatibility conditions* (8.11) *on the initial data* (u_0, z_0)*, there exists a constant* $C_* > 0$ *such that for all* $\varepsilon > 0$ *there exist a solution* $(u_\varepsilon, z_\varepsilon) \in \mathrm{H}^1(0,T;\mathbf{U})\times\mathrm{H}^1(0,T;\mathbf{Z}_{\mathrm{e}})$ *satisfying the energy estimate* (8.20) *with* $C_1^{E_0} = C_*$*, as well as the improved estimate*

$$\int_0^T \left(\|\dot u_\varepsilon\|_{\mathbf{U}} + \|\dot z_\varepsilon\|_{\mathbf{Z}_{\mathrm{e}}}\right)\mathrm{d}t \le C_*.$$

Proof. Let $(\delta_k)_k$ be a null sequence and, for $\varepsilon > 0$ fixed, let $(q^\varepsilon_{\delta_k})_k$ be the corresponding sequence of solutions to the regularized system (8.4); from now on, we will simply write $(q_k)_k$. Our starting point is the energy-dissipation balance

$$\mathcal{E}^{\delta_k}(t, q_k(t)) + \int_s^t \Big(\mathcal{V}^{\varepsilon^\alpha}_{\mathsf{u}}(\varepsilon^\alpha u_k'(r)) + \mathcal{R}(z_k'(r)) + \mathcal{V}^\varepsilon_{\mathsf{z}}(\varepsilon\, z_k'(r))\Big)\,\mathrm{d}r \tag{8.38}$$

$$+ \int_s^t \Big(\frac{1}{\varepsilon^\alpha}\mathcal{V}^*_{\mathsf{u}}(-\mu_k(r)) + \frac{1}{\varepsilon}\mathcal{W}^*_{\mathsf{z}}(-\zeta_k(r))\Big)\,\mathrm{d}r = \mathcal{E}^{\delta_k}(s, q_k(s)) + \int_s^t \partial_t \mathcal{E}^{\delta_k}(r, q_k(r))\,\mathrm{d}r \text{ for all } [s,t] \subset [0,T]$$

with

$$\mu_k(t) = \boldsymbol{C}u_k(t) + \boldsymbol{J}^*\big(\beta^{\delta_k}([\![u_k(t)]\!]) + \gamma^{\delta_k}(z_k(t))\mathrm{D}\psi^{\delta_k}([\![u_k(t)]\!])\big) - \ell_u(t),$$
$$\zeta_k(t) = \boldsymbol{A}z_k(t) + (\gamma^{\delta_k})'(z_k(t))\psi^\delta([\![u_k(t)]\!]) + \phi^{\delta_k}(z_k(t))\,.$$

Relying on the energy estimate (8.20) and on well known compactness results, we infer that there exists $q_\varepsilon = (u_\varepsilon, z_\varepsilon)$ such that, along a not relabeled subsequence,

$$q_k \rightharpoonup q_\varepsilon \text{ in } \mathrm{H}^1(0,T;\mathbf{U}{\times}\mathbf{Z}) \quad \text{and} \quad q_k(t) \rightharpoonup q_\varepsilon(t) \text{ in } \mathbf{U}{\times}\mathbf{Z}_{\mathrm{e}} \text{ for all } t \in [0,T]\,. \tag{8.39}$$

It also follows from estimate (4.25a) in Lemma 4.12 that there exist μ_ε and ζ_ε such that, up to a further subsequence,

$$\mu_k \rightharpoonup \mu_\varepsilon \text{ in } \mathrm{L}^2(0,T;\mathbf{U}^*) \quad \text{and} \quad \zeta_k \rightharpoonup \zeta_\varepsilon \text{ in } \mathrm{L}^2(0,T;\mathbf{Z}^*)\,.$$

In order to identify the weak limit $\zeta_\varepsilon(t)$ as an element of $\partial_{\mathsf{z}}\mathcal{E}(tt, u_\varepsilon(t), z_\varepsilon(t))$ for almost all $t \in (0,T)$, we observe that, by (8.31b), $|(\gamma^{\delta_k})'(z_k)| \le \delta + |\partial\gamma^o(z_k)| \le \delta + 1$, taking into account that $\gamma(z) = \max\{z, 0\}$. Therefore,

$$\|(\gamma^{\delta_k})'(z_k)\psi^\delta([\![u_k]\!])\|_{\mathrm{L}^2} \overset{(1)}{\le} (\delta{+}1)\left(\|\psi([\![u_k]\!])\|_{\mathrm{L}^2}{+}\delta\|[\![u_k]\!]\|_{\mathrm{L}^2}\right) \overset{(2)}{\le} (\delta{+}1)\left(C^{(2)}_\psi\|[\![u_k]\!]\|^2_{\mathrm{L}^4}{+}\delta\|[\![u_k]\!]\|_{\mathrm{L}^2}{+}C\right)$$

with (1) due to (8.35a) and (2) to (8.5). Since $(u_k)_k$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega;\mathbb{R}^3))$, we immediately deduce that $((\gamma^{\delta_k})'(z_k)\psi^\delta([\![u_k]\!]))_k$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Gamma_{\mathrm{C}}))$. A standard argument based on the fact that $z \mapsto \phi^{\delta_k}(z){+}\Lambda_\phi z$ is a non-decreasing function then yields a separate estimate in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma_{\mathrm{C}}))$ for both $(\boldsymbol{A}z_k)_k$ and $(\phi^{\delta_k}(z_k))_k$ so that, up to a subsequence, $\phi^{\delta_k}(z_k) \rightharpoonup \phi$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma_{\mathrm{C}}))$ for some ϕ. Combining this with the fact that $z_k \to z_\varepsilon$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma_{\mathrm{C}}))$ we immediately conclude by (8.31d) that $\phi \in \partial\widehat{\phi}(z_\varepsilon)$ a.e. in $(0,T){\times}\Gamma_{\mathrm{C}}$. With the same arguments we find that $(\gamma^{\delta_k})'(z_k) \overset{*}{\rightharpoonup} \omega$ in $\mathrm{L}^\infty((0,T){\times}\Gamma_{\mathrm{C}})$ with $\omega \in \partial\gamma(z_\varepsilon)$ a.e. in $(0,T){\times}\Gamma_{\mathrm{C}}$. Finally, again applying (8.35a) to estimate $|\psi^\delta([\![u_k]\!])|$ and taking into account that $[\![u_k]\!] \to [\![u]\!]$ strongly in $\mathrm{L}^\infty(0,T;\mathrm{L}^q(\Gamma_{\mathrm{C}}))$ for every $1 \le q < 4$, with the dominated convergence theorem we conclude that $\psi^\delta([\![u_k]\!]) \to \psi([\![u_\varepsilon]\!])$, for instance, in $\mathrm{L}^{3/2}((0,T){\times}\Gamma_{\mathrm{C}})$. All in all, we find that $(\gamma^{\delta_k})'(z_k)\psi^\delta([\![u_k]\!]) \rightharpoonup \omega\psi([\![u_\varepsilon]\!])$ in $\mathrm{L}^{3/2}((0,T){\times}\Gamma_{\mathrm{C}})$. We have thus proved that

$$\zeta_\varepsilon = \boldsymbol{A}z + \omega\psi([\![u_\varepsilon]\!]) + \phi \qquad \text{with } \omega \in \partial\gamma(z_\varepsilon),\ \phi \in \partial\widehat{\phi}(z_\varepsilon) \text{ a.e. in } (0,T){\times}\Gamma_{\mathrm{C}}\,,$$

and thus $\zeta_\varepsilon(t) \in \partial_{\mathsf{z}}\mathcal{E}(t, u_\varepsilon(t), z_\varepsilon(t))$.

The identification of μ_ε as an element of $\partial_{\mathsf{u}}\mathcal{E}(\cdot, u_\varepsilon(\cdot), z_\varepsilon(\cdot))$ first of all follows from observing that, by (8.39), $\boldsymbol{C}u_k \overset{*}{\rightharpoonup} \boldsymbol{C}u$ in $\mathrm{L}^\infty(0,T;\mathbf{U}^*)$. Moreover, with similar arguments as in the above lines, based on properties (8.31), we find that $\gamma^{\delta_k}(z_k) \to \gamma(z_\varepsilon)$ in $\mathrm{L}^q((0,T){\times}\Gamma_{\mathrm{C}})$ for all $1 \le q < \infty$ and, recalling that β is Lipschitz, that there exists $\widetilde{\beta} \in \mathrm{L}^\infty(0,T;\mathrm{L}^4(\Gamma_{\mathrm{C}}))$ such that $\beta^{\delta_k}([\![u_k]\!]) \rightharpoonup \widetilde{\beta}$ in $\mathrm{L}^\infty(0,T;\mathrm{L}^4(\Gamma_{\mathrm{C}}))$. Finally, taking into account (8.35b) and the fact that ψ has quadratic growth we conclude that there exists $\varrho \in \mathrm{L}^\infty(0,T;\mathrm{L}^4(\Gamma_{\mathrm{C}}))$ such that $\mathrm{D}\psi^{\delta_k}([\![u_k]\!]) \overset{*}{\rightharpoonup} \varrho$ in $\mathrm{L}^\infty(0,T;\mathrm{L}^4(\Gamma_{\mathrm{C}}))$. All in all, we find that

$$\boldsymbol{J}^*(\beta^{\delta_k}([\![u_k]\!]) + \gamma^{\delta_k}(z_k)\mathrm{D}\psi^{\delta_k}([\![u_k]\!])) \rightharpoonup \eta = \widetilde{\beta} + \gamma(z_\varepsilon)\varrho \quad \text{in } \mathrm{L}^2(0,T;\mathbf{U}^*),$$

and it remains to show that $\eta = \boldsymbol{J}^*(\beta([\![u]\!]) + \gamma(z)\mathrm{D}\psi([\![u]\!]))$. For this, we observe that the functionals $\mathcal{J}^{\delta_k} : \mathrm{L}^2(0,T;\mathbf{U}{\times}\mathbf{Z}) \to \mathbb{R}$ defined by $\mathcal{J}^{\delta_k}(u,z) := \int_0^T \int_{\Gamma_{\mathrm{C}}} \big(\widehat{\beta}^{\delta_k}([\![u]\!]){+}\gamma^{\delta_k}(z)\psi^{\delta_k}([\![u]\!])\big)\mathrm{d}x\mathrm{d}t$, clearly fulfilling

$$\mathrm{D}_u\mathcal{J}^{\delta_k}(u,z) = \boldsymbol{J}^*(\beta^{\delta_k}([\![u]\!]) + \gamma^{\delta_k}(z)\mathrm{D}\psi^{\delta_k}([\![u]\!])) \qquad \text{for every } (u,z) \in \mathrm{L}^2(0,T;\mathbf{U}{\times}\mathbf{Z}),$$

enjoy the following property:

$$\left\{\begin{array}{l} (u_k, z_k) \rightharpoonup (u,z) \text{ in } \mathrm{L}^2(0,T;\mathbf{U}{\times}\mathbf{Z}), \\ \mathrm{D}_u\mathcal{J}^{\delta_k}(u_k, z_k) \rightharpoonup \eta \text{ in } \mathrm{L}^2(0,T;\mathbf{U}^*{\times}\mathbf{Z}^*), \\ \limsup_{k\to\infty}\int_0^T \langle \mathrm{D}_u\mathcal{J}^{\delta_k}(u_k,z_k), u_k\rangle_{\mathbf{U}}\,\mathrm{d}t \le \int_0^T \langle \eta, u\rangle_{\mathbf{U}}\,\mathrm{d}t \end{array}\right. \quad \Longrightarrow \quad \eta \in \boldsymbol{J}^*\big(\beta([\![u]\!]) + \gamma(z)\partial\psi([\![u]\!])\big)\,.$$

Hence, we need to prove that

$$\limsup_{k\to\infty}\int_0^T\int_{\Gamma_{\mathrm{C}}} \{\beta^{\delta_k}([\![u_k]\!])[\![u_k]\!]{+}\gamma^{\delta_k}(z_k)\mathrm{D}\psi^{\delta_k}([\![u_k]\!])[\![u_k]\!]\}\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \langle \eta, u\rangle_{\mathrm{H}^1(\Omega)}\,\mathrm{d}t\,.$$

This follows from testing the u-equation (8.4a) at the level δ_k by u_k, taking the limit as $k \to \infty$, and using that, by the convergence arguments in the above lines, the quadruple $(u, z, \tilde{\beta}, \varrho)$ fulfills the limit equation $0 = \varepsilon^\alpha \boldsymbol{D}\dot{u}_\varepsilon + \boldsymbol{C}u_\varepsilon + +\boldsymbol{J}^*(\tilde{\beta}+\gamma(z_\varepsilon)\varrho) - \ell_u$ in $\mathbf{U}^*$ a.e. in $(0,T)$. All in all, we conclude that $\boldsymbol{J}^*(\tilde{\beta}+\gamma(z_\varepsilon)\varrho) \in \boldsymbol{J}^*(\beta(z_\varepsilon)+\gamma(z_\varepsilon)\partial\psi([\![u_\varepsilon]\!]))$, so that

$$\mu_\varepsilon \in \boldsymbol{C}u_\varepsilon + \boldsymbol{J}^*\big(\beta([\![u_\varepsilon]\!]) + \gamma(z_\varepsilon)\partial\psi([\![u_\varepsilon]\!])\big) - \ell_u(t) = \partial_{\mathsf{u}}\mathcal{E}(t, u_\varepsilon, z_\varepsilon)\,.$$

Therefore, passing to the limit as $k \to \infty$ in (4.18b) we infer that the quadruple $(u_\varepsilon, z_\varepsilon, \mu_\varepsilon, \zeta_\varepsilon)$ fulfills $(\mu_\varepsilon(t), \zeta_\varepsilon(t)) \in \partial_{\mathsf{q}}\mathcal{E}(t, q_\varepsilon(t))$ for almost all $t \in (0,T)$, joint with the energy-dissipation upper estimate in (4.18b). Now, by Proposition 8.2 the energy functional $\mathcal{E}$ from (8.6d) complies with the chain rule of Hypothesis 4.7. Hence, by Remark 4.9 the validity of the energy-dissipation upper estimate is sufficient to conclude that $(u_\varepsilon, z_\varepsilon)$ solve the Cauchy problem for the delamination system (8.4).

By lower semicontinuity arguments, the a priori estimate (8.29) is inherited by $(u_\varepsilon, z_\varepsilon)$. This concludes the proof of Proposition 8.5 and, ultimately, of Theorem 8.1. □

Appendix A. Chain rules

In this section we first of all provide a sufficient condition for the chain-rule Hypothesis 4.7 (and, in fact, for the closedness Hypothesis 4.5 as well). More precisely, we will show that its validity is guaranteed by a sort of *uniform subdifferentiability* property of the energy $\mathcal{E}$ on its sublevels, cf. (A.1) below, which we borrow from [MRS13]; as already observed in the proof of Proposition 8.2, (A.1) for instance holds for λ-convex functionals. The proof of the following result combines the argument for [MRS13, Prop. 2.4] with results from [AGS08].

Proposition A.1. *Let $\mathcal{E} : [0,T] \times \mathbf{Q} \to (-\infty, +\infty]$ comply with Hypothesis 4.3. Assume that for every $E > 0$ there exists a modulus of subdifferentiability $\varpi^E : [0,T]\times\mathcal{S}_E\times\mathcal{S}_E \to [0,\infty)$ such that for all $t \in [0,T]$:*

$$\begin{aligned} &\varpi^E(t,q,q) = 0 \qquad \text{for every } q \in \mathcal{S}_E, \\ &\text{the map } (t,q,\hat{q}) \to \varpi^E(t,q,\hat{q}) \text{ is upper semicontinuous, and} \\ &\mathcal{E}(t,\hat{q}) - \mathcal{E}(t,q) \geq \langle \xi, \hat{q}-q \rangle_{\mathbf{Q}} - \varpi^E(t,q,\hat{q})\|\hat{q}-q\|_{\mathbf{Q}} \qquad \text{for all } q,\, \hat{q} \in \mathcal{S}_E \text{ and all } \xi \in \partial_q\mathcal{E}(t,q)\,. \end{aligned} \tag{A.1}$$

Then, $\mathcal{E}$ complies with Hypothesis 4.7.

Proof. In order to show Hypothesis 4.7, let $q \in \mathrm{AC}([0,T];\mathbf{Q})$ and $\xi \in \mathrm{L}^1(0,T;\mathbf{Q}^*)$ fulfill (4.15), and let $E > 0$ be such that $q(t) \in \mathcal{S}_E$ for all $t \in [0,T]$. Preliminarily, let us suppose that $\mathcal{E}$ is independent of time, i.e. $\mathcal{E}(t,q) = \overline{\mathcal{E}}(q)$ (with a modulus of subdifferentiability $\varpi^E(t,\cdot,\cdot) = \varpi^E(\cdot,\cdot)$), and let us prove the absolute continuity of $[0,T] \ni t \mapsto \overline{\mathcal{E}}(q(t))$. For this, as in [MRS13] we resort to [AGS08, Lemma 1.1.4] and reparametrize q to a 1-Lipschitz curve $\tilde{q} : [0,L] \to \mathbf{Q}$, with $L = \int_0^T \|q'(t)\|_{\mathbf{Q}}\,\mathrm{d}t$, $\tilde{q} := q \circ \tilde{t}$, and $\tilde{t} : [0,L] \to [0,T]$ the left-continuous, increasing map

$$\tilde{t}(s) := \min\{t \in [0,T] \,:\, \int_0^t \|q'(r)\|_{\mathbf{Q}}\,\mathrm{d}r = s\}\,.$$

Let us set $\tilde{\xi}(s) := \xi(\tilde{t}(s))$. Then, it follows from (4.15) and a version of the change of variables formula (cf., e.g., [Bog07, Thm. 5.8.30]), that

$$\int_0^L \|\tilde{\xi}(s)\|_{\mathbf{Q}^*}\,\mathrm{d}s = \int_0^L \|\tilde{\xi}(s)\|_{\mathbf{Q}^*}\|\tilde{q}'(s)\|_{\mathbf{Q}}\,\mathrm{d}s = \int_0^T \|\xi(t)\|_{\mathbf{Q}^*}\|q'(t)\|_{\mathbf{Q}}\,\mathrm{d}t < \infty,$$

whence $\tilde{\xi} \in \mathrm{L}^1(0,L)$. Hence, we are in a position to repeat the very same arguments from the proof of [MRS13, Prop. 2.4]. Namely, relying on the validity of (A.1) we obtain that

$$\overline{\mathcal{E}}(\tilde{q}(s_2)) - \overline{\mathcal{E}}(\tilde{q}(s_1)) \geq \langle \tilde{\xi}(s_1), \tilde{q}(s_2)-\tilde{q}(s_1) \rangle_{\mathbf{Q}} - \varpi^E(\tilde{q}(s_1), \tilde{q}(s_2))\|\tilde{q}(s_2)-\tilde{q}(s_1)\|_{\mathbf{Q}}$$

for all $0 \leq s_1 \leq s_2 \leq L$. Exchanging the role of s_1 and s_2 and exploiting the 1-Lipschitz continuity of $\tilde{q}$ leads to

$$\big|\overline{\mathcal{E}}(\tilde{q}(s_2)) - \overline{\mathcal{E}}(\tilde{q}(s_1))\big| \leq \Big(\|\tilde{\xi}(s_1)\|_{\mathbf{Q}^*} + \|\tilde{\xi}(s_2)\|_{\mathbf{Q}^*} + \varpi^E(\tilde{q}(s_1), \tilde{q}(s_2)) + \varpi^E(\tilde{q}(s_2), \tilde{q}(s_1))\Big)\,|s_1 - s_2|\,.$$

From the above estimate the absolute continuity of the function $[0,L] \ni s \mapsto \overline{\mathcal{E}}(\tilde{q}(s))$ follows by repeating the very same arguments as in the proof of [AGS08, Thm. 1.2.5]. Again changing variables in the integral we infer that $[0,T] \ni t \mapsto \overline{\mathcal{E}}(q(t))$ is absolutely continuous.

In the general case, let us first of all check the absolute continuity of $[0,T] \ni t \mapsto \mathcal{E}(t,q(t))$, namely that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every collection of pairwise disjoint intervals $(a_i,b_i) \subset (0,T)$, $i = 1,\ldots,M$, there holds

$$\sum_{i=1}^{M}(b_i{-}a_i) < \delta \implies \sum_{i=1}^{M}|\mathcal{E}(b_i,q(b_i)){-}\mathcal{E}(a_i,q(a_i))| < \varepsilon\,. \tag{A.2}$$

In order to obtain the above estimate, we use that

$$|\mathcal{E}(b_i,q(b_i)){-}\mathcal{E}(a_i,q(a_i))| \le |\mathcal{E}(b_i,q(b_i)){-}\mathcal{E}(a_i,q(b_i))| + |\mathcal{E}(a_i,q(b_i)){-}\mathcal{E}(a_i,q(a_i))| \tag{A.3}$$

and estimate the first term via (4.7d), so that

$$|\mathcal{E}(b_i,q(b_i)){-}\mathcal{E}(a_i,q(b_i))| \le \int_{a_i}^{b_i}|\partial_t\mathcal{E}(r,q(b_i))|\,\mathrm{d}r \le C_\# \int_{a_i}^{b_i}|\mathcal{E}(r,q(b_i))|\,\mathrm{d}r \le C_\# E|b_i-a_i|,$$

Hence, we choose $\delta_1 = \frac{\varepsilon}{2C_1E}$ and get $\sum_{i=1}^{M}|\mathcal{E}(b_i,q(b_i)){-}\mathcal{E}(a_i,q(a_i))| < \frac{\varepsilon}{2}$. As for the second term in (A.3), we rely on the absolute continuity of $\overline{\mathcal{E}}(q) := \mathcal{E}(a_i,q)$. We thus conclude (A.2). Finally, to show the chain-rule formula (4.16), we fix a point $t \in (0,T)$ in which $q'(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t))$ exist, and derive from (A.1) that

$$\begin{aligned}
&\mathcal{E}(t+h,q(t+h)) - \mathcal{E}(t,q(t))\\
&= \mathcal{E}(t+h,q(t+h)) - \mathcal{E}(t,q(t+h)) + \mathcal{E}(t,q(t+h)) - \mathcal{E}(t,q(t))\\
&\ge \int_t^{t+h}\partial_t\mathcal{E}(r,q(t+h))\,\mathrm{d}r + \langle\xi(t),q(t+h){-}q(t)\rangle_{\mathbf{Q}} - \varpi^E(t,q(t),q(t+h))\\
&\ge \int_t^{t+h}(\partial_t\mathcal{E}(r,q(t+h)){-}\partial_t\mathcal{E}(r,q(r)))\,\mathrm{d}r + \int_t^{t+h}\partial_t\mathcal{E}(r,q(r))\,\mathrm{d}r\\
&\qquad + \langle\xi(t),q(t+h){-}q(t)\rangle_{\mathbf{Q}} - \varpi^E(t,q(t),q(t+h))\|q(t+h){-}q(t)\|_{\mathbf{Q}}
\end{aligned} \tag{A.4}$$

We now divide the above estimate by $h > 0$ and take the limit as $h \to 0^+$. Now, recall that $q(t) \in \mathcal{S}_E$ for all $t \in [0,T]$. Therefore, thanks to Hypothesis 4.5 the function $[0,T]{\times}[0,T] \ni (r,s) \mapsto \partial_t\mathcal{E}(r,q(s))$ is uniformly continuous, with a modulus of continuity $\omega : [0,T]\times[0,T] \to [0,\infty)$, so that

$$\left|\frac{1}{h}\int_t^{t+h}(\partial_t\mathcal{E}(r,q(t{+}h)){-}\partial_t\mathcal{E}(r,q(r)))\,\mathrm{d}r\right| \le \frac{1}{h}\int_t^{t+h}\omega(|t{+}h{-}r|)\,\mathrm{d}r \le \omega(h) \to 0 \text{ as } h \to 0^+\,.$$

Taking into account that $\lim_{h\to0^+}\frac{1}{h}\int_t^{t+h}\partial_t\mathcal{E}(r,q(r))\,\mathrm{d}r = \partial_t\mathcal{E}(t,q(t))$, (A.4) thus leads to the estimate $\ge$ in (4.16). The converse inequality follows by dividing (A.4) by $h < 0$ and taking the limit as $h \to 0^-$. This concludes the proof. □

Let us now carry out the proof of Proposition 5.16, which shows the validity of the parametrized chain rule from Hypothesis 5.7 if Hypothesis 4.7 holds and, in addition, the vanishing-viscosity contact potentials associated with $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$ satisfy the coercivity property (5.37).

Proof of Proposition 5.16. Preliminarily, we observe that, if the vanishing-viscosity contact potentials $\mathfrak{b}_{\mathcal{V}_{\mathsf{u}}}$ and $\mathfrak{b}_{\mathcal{V}_{\mathsf{z}}}$ satisfy (5.37), then the 'reduced' rescaled joint M-function $\mathfrak{M}_0^{\alpha,\mathrm{red}}$ enjoys the following coercivity property:

$$\begin{aligned}
\exists\, c>0\ \forall\,(t,q,t',q') \in [0,T]{\times}\mathrm{D}{\times}[0,\infty){\times}\mathbf{Q} :\ & \mathfrak{M}_0^{\alpha,\mathrm{red}}(t,q,t',q') \ge c\,(\|\mu\|_{\mathbf{U}^*}\|u'\|_{\mathbf{U}} + \|\zeta{+}\sigma\|_{\mathbf{U}^*}\|z'\|_{\mathbf{Z}})\\
& \text{for all } (\mu,\zeta) \in \mathfrak{A}_{\mathsf{u}}^*(t,q){\times}\mathfrak{A}_{\mathsf{z}}^*(t,q) \text{ and some } \sigma \in \partial\mathcal{R}(0).
\end{aligned} \tag{A.5}$$

Clearly, the above estimate trivially holds if $t' > 0$, as then $\mathscr{S}_{\mathsf{u}}^*(t,q) = \mathscr{S}_{\mathsf{z}}^*(t,q) = 0$, so that $\mathfrak{A}_{\mathsf{u}}^*(t,q) = \{0\}$ and every $\zeta \in \mathfrak{A}_{\mathsf{z}}^*(t,q)$ fulfills $-\zeta \in \partial\mathcal{R}(0)$. To show it for $t' = 0$, we will separately discuss the cases $\alpha > 1$ (the arguments for $\alpha \in (0,1)$ are indeed specular) and $\alpha = 1$. In the case $\alpha > 1$, we have that, if $\mathscr{S}_{\mathsf{u}}^*(t,q) = 0$, then by (5.12) we have

$$\mathfrak{M}_0^{\alpha,\mathrm{red}}(t,q,0,q') = \mathfrak{b}_{\mathcal{V}_{\mathsf{z}}}(z',\mathscr{S}_{\mathsf{z}}^*(t,q)) \overset{(1)}{\ge} c_{\mathsf{z}}\|z'\|_{\mathbf{Z}}\|\zeta{+}\sigma\|_{\mathbf{Z}^*}$$

for all $\zeta \in \mathfrak{A}_{\mathsf{z}}^*(t,q)$ and all $\sigma \in \partial\mathcal{R}(0)$ with $\mathcal{W}_{\mathsf{z}}^*(\zeta) = \mathcal{V}_{\mathsf{z}}^*(\zeta{-}\sigma)$, where (1) follows from (5.37). Analogously, if $\mathscr{S}_{\mathsf{u}}^*(t,q) > 0$, then $\mathfrak{M}_0^{\alpha,\mathrm{red}}(t,q,t',q') = \mathfrak{b}_{\mathcal{V}_{\mathsf{u}}}(u',\mathscr{S}_{\mathsf{u}}^*(t,q))$ and we have the analogous estimate. If $\alpha = 1$, then by (5.12) and again (3.9) we have

$$\begin{aligned}\mathfrak{M}_0^{\alpha,\mathrm{red}}(t,q,0,q') &= \mathfrak{b}_{\mathcal{V}_{\mathsf{u}}\oplus\mathcal{V}_{\mathsf{z}}}(q',\mathscr{S}_{\mathsf{u}}^*(t,q)+\mathscr{S}_{\mathsf{z}}^*(t,q))\\ &\geq \mathfrak{b}_{\mathcal{V}_{\mathsf{u}}}(u',\mathscr{S}_{\mathsf{u}}^*(t,q))+\mathfrak{b}_{\mathcal{V}_{\mathsf{z}}}(z',\mathscr{S}_{\mathsf{z}}^*(t,q))\\ &\geq c_{\mathsf{u}}\|u'\|_{\mathbf{U}}\|\mu\|_{\mathbf{U}^*}+c_{\mathsf{z}}\|z'\|_{\mathbf{Z}}\|\zeta+\sigma\|_{\mathbf{Z}^*}\end{aligned}$$

for all μ, ζ, and σ as in (A.5).

Let us now consider an admissible curve $(\mathsf{t},\mathsf{q})\in\mathscr{A}([a,b];[0,T]\times\mathbf{Q})$ such that, in addition, $\mathsf{z}\in \mathrm{AC}([a,b];\mathbf{Z})$. Hence, $\mathfrak{M}_0^\alpha[\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}']=\mathfrak{M}_0^\alpha(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}')$ a.e. in (a,b). Then, (5.20) yields that $\mathfrak{M}_0^\alpha(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}')\in \mathrm{L}^1(a,b)$. Let us now choose measurable selections $(a,b)\ni s\mapsto\mu(s)\in\mathfrak{A}_{\mathsf{u}}^*(\mathsf{t}(s),\mathsf{q}(s))$, $(a,b)\ni s\mapsto\zeta(s)\in \mathfrak{A}_{\mathsf{z}}^*(\mathsf{t}(s),\mathsf{q}(s))$, and $(a,b)\ni s\mapsto\sigma(s)\in\partial\mathcal{R}(0)$ such that

$$\|\mu(s)\|_{\mathbf{U}^*}\|\mathsf{u}'(s)\|_{\mathbf{U}}+\|\zeta(s)+\sigma(s)\|_{\mathbf{Z}^*}\|\mathsf{z}'(s)\|_{\mathbf{Z}}\leq\mathfrak{M}_0^{\alpha,\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))\qquad\text{for a.a. } s\in(a,b).$$

Hence, we have $\int_a^b(\|\mu(s)\|_{\mathbf{U}^*}\|\mathsf{u}'(s)\|_{\mathbf{U}}+\|\zeta(s)+\sigma(s)\|_{\mathbf{Z}^*}\|\mathsf{z}'(s)\|_{\mathbf{Z}})\,\mathrm{d}s<\infty$, and using $\int_a^b\|\sigma(s)\|_{\mathbf{Z}^*}\|\mathsf{z}'(s)\|_{\mathbf{Z}}\mathrm{d}s \leq C_{\mathcal{R}}\int_a^b\|\mathsf{z}'(s)\|_{\mathbf{Z}}\,\mathrm{d}s<\infty$ by (4.6), we ultimately deduce that

$$\int_a^b\left(\|\mu(s)\|_{\mathbf{U}^*}\|\mathsf{u}'(s)\|_{\mathbf{U}}+\|\zeta(s)\|_{\mathbf{Z}^*}\|\mathsf{z}'(s)\|_{\mathbf{Z}}\right)\,\mathrm{d}s<\infty\,.$$

We are thus in a position to apply the chain rule from Hypothesis 4.7 and conclude that $s\mapsto\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))$ is absolutely continuous and that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathsf{t},\mathsf{q})-\partial_t\mathcal{E}(\mathsf{t},\mathsf{q})\mathsf{t}'=\langle\mu,\mathsf{u}'\rangle_{\mathbf{U}}+\langle\zeta,\mathsf{z}'\rangle_{\mathbf{Z}}\quad\text{a.e. in } (a,b),$$

Then, the chain-rule estimate (5.38) follows from observing that the right-hand side in the above formula estimates $-\mathfrak{M}_0^\alpha(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}')$ from above thanks to Lemma 5.5. This concludes the proof. ☐

We conclude this section with a result relating parametrized chain rule from Hypothesis 5.7 to that in Hypothesis 6.4.

Lemma A.2. *If the rate-independent system $(\mathbf{U}\times\mathbf{Z},\mathcal{E},\mathcal{V}_{\mathsf{u}}^{\varepsilon^\alpha}+\mathcal{R}+\mathcal{V}_{\mathsf{z}}^{\varepsilon})_{\varepsilon\downarrow0}$ satisfies the chain rule of Hypothesis 5.7, then it also satisfies Hypothesis 6.4.*

Proof. Consider a curve $q\in\mathrm{BV}([0,T];\mathbf{U})\times(\mathrm{R}([0,T];\mathbf{Z})\cap\mathrm{BV}([0,T];\mathbf{Z}_{\mathrm{ri}}))$ fulfilling the stationary equation (6.10a) and the local stability (6.10b), and let us fix $[t_0,t_1]\subset[0,T]$. As in Theorem 6.15(2), we associate with q a parametrized curve $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z})\in\mathscr{A}([\mathsf{S}_0,\mathsf{S}_1];[t_0,t_1]\times\mathbf{Q})$ such that (6.27) holds on the interval $[t_0,t_1]$. The parametrized chain-rule inequality (5.23) reads for a.a. $s\in(\mathsf{S}_0,\mathsf{S}_1)$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))-\partial_t\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))\mathsf{t}'(s)\geq-\mathcal{R}[z'](s)-\mathfrak{M}_0^{\alpha,\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),0,\mathsf{q}'(s))\,.$$

Integrating on the interval $(\mathsf{S}_0,\mathsf{S}_1)$ and using (6.27), we conclude the desired chain-rule inequality (6.9). With this, Lemma A.2 is proved. ☐

Appendix B. Measurability in Theorem 5.20

To prove the statement in Theorem 5.20 concerning the existence of measurable selections $\xi=(\mu,\zeta):(0,\mathsf{S})\to\mathbf{U}^*\times\mathbf{Z}^*$ and $(\lambda_{\mathsf{u}},\lambda_{\mathsf{z}}):(0,\mathsf{S})\to[0,\infty]^2$ satisfying (5.47), we will resort to the following generalization of Filippov's Selection Theorem, proved in [MiR15, Prop. B.1.2].

Proposition B.1. *Let $(O,\mathfrak{O},\mu)$ be a σ-finite complete measure space and X a complete separable metric space. Let $F:O\rightrightarrows X$ be a measurable set-valued mapping with closed non-empty images,* $\mathrm{graph}(F):=\{(s,x)\,:\,x\in F(s)\}$ *its graph, and let $\mathfrak{G}$ be the σ-algebra given by the restriction of $\mathfrak{O}\times\mathfrak{B}(X)$ (with $\mathfrak{B}(X)$ the Borel σ-algebra on X) to* $\mathrm{graph}(F)$. *Let $g:\mathrm{graph}(F)\to\mathbb{R}$ be a $\mathfrak{G}$-measurable mapping such that*

$$\forall\,s\in O:\qquad\begin{cases}\exists\,x\in F(s): & g(s,x)=0,\\ g(s,\cdot):F(s)\to\mathbb{R}\ \textit{is continuous.}\end{cases}\tag{B.1}$$

Then, there exists a measurable selection $f:O\to X$ of F such that $g(s,f(s))=0$ for all $s\in O$.

For the construction of the parameters λ_{u} and λ_{z} introduce the sets $\Lambda_{\mathsf{x}}(v,\sigma)$ via

$$\Lambda_{\mathsf{x}}(v,\sigma):=\operatorname*{Argmin}_{\lambda>0}\mathcal{B}_{\mathcal{V}_{\mathsf{x}}}(\tfrac{1}{\lambda},v,\sigma)=\operatorname*{Argmin}_{\lambda>0}\frac{1}{\lambda}\big(\mathcal{V}_{\mathsf{x}}(\lambda v)+\sigma\big)\qquad\text{for } (v,\sigma)\in\mathbf{X}\times[0,\infty)\tag{B.2}$$

with $\mathsf{x} \in \{\mathsf{u}, \mathsf{z}\}$ and $\mathbf{X} \in \{\mathbf{U}, \mathbf{Z}\}$. Recall that Proposition 3.2 guarantees that $\Lambda_\mathsf{x}(v, \sigma) \neq \emptyset$ for all $\sigma > 0$. Analogously, for $q' = (u', z')$ we will use the notation

$$\Lambda_{\mathsf{uz}}(q', \sigma_\mathsf{u} + \sigma_\mathsf{z}) := \operatorname*{Argmin}_{\lambda > 0} \mathcal{B}_{\mathcal{V}_\mathsf{u} \oplus \mathcal{V}_\mathsf{z}}(\tfrac{1}{\lambda}, q', \sigma_\mathsf{u} + \sigma_\mathsf{z}) = \operatorname*{Argmin}_{\lambda > 0} \frac{1}{\lambda}\big(\mathcal{V}_\mathsf{u}(\lambda u') + \mathcal{V}_\mathsf{z}(\lambda z') + \sigma_\mathsf{u} + \sigma_\mathsf{z}\big)\,.$$

A close perusal of the proof of Proposition 5.18 then reveals that, for a given $(t, q, t', q') \in \Sigma_\alpha$ with $\alpha \neq 1$,

$$(t, q, t', q') \in \mathrm{V}_\mathsf{x} \text{ if and only if system (5.42) holds with } \lambda_\mathsf{x} \in \Lambda_\mathsf{x}(x', \mathscr{S}^*_\mathsf{x}(t, q)),$$

for $\mathsf{x} \in \{\mathsf{u}, \mathsf{z}\}$ and $x' \in \{u', z'\}$. Analogously, in the case $\alpha = 1$, we have that $(t, q, t', q') \in \Sigma_1 \cap \mathrm{V}_{\mathsf{uz}}$ if and only if (5.42) holds with $\lambda \in \Lambda_{\mathsf{uz}}(q', \mathscr{S}^*_\mathsf{u}(t, q) + \mathscr{S}^*_\mathsf{z}(t, q))$.

We are now in a position to the missing part of the proof of Theorem 5.20.

Proof of Part (1) of Theorem 5.20. Let $(\mathsf{t}, \mathsf{q}) \in \mathscr{A}([0, \mathsf{S}]; [0, T] \times \mathbf{Q})$ be a enhanced pBV solution of the rate-independent system $(\mathbf{U} \times \mathbf{Z}, \mathcal{E}, \mathcal{V}^{\varepsilon^\alpha}_\mathsf{u} + \mathcal{R} + \mathcal{V}^\varepsilon_\mathsf{z})_{\varepsilon \downarrow 0}$. We will now prove the existence of measurable λ_u, λ_z : $(0, \mathsf{S}) \to [0, \infty]$, and $\mu : (0, \mathsf{S}) \to \mathbf{U}^*$ and $\zeta : (0, \mathsf{S}) \to \mathbf{Z}^*$ satisfying (5.47) in the case $\alpha > 1$; with similar arguments one can obtain the analogous statement for $\alpha = 1$ and $\alpha \in (0, 1)$. To avoid trivial situations, we also suppose that (t, q) is non-degenerate.

<u>*Step 1: Existence of measurable* $\lambda_\mathsf{u}, \lambda_\mathsf{z} : (0, \mathsf{S}) \to [0, \infty]$.</u> Proposition 5.17 shows that $(\mathsf{t}(s), \mathsf{q}(s), \mathsf{t}'(s), \mathsf{q}'(s))$ belongs to the contact set Σ_α for a.a. $s \in (0, \mathsf{S})$ and that, in turn, $\Sigma_\alpha \subset \mathrm{E}_\mathsf{u}\mathrm{R}_\mathsf{z} \cup \mathrm{E}_\mathsf{u}\mathrm{V}_\mathsf{z} \cup \mathrm{B}_\mathsf{z}$. Recalling the short-hand notation (5.49), we introduce the short-hand $((0, \mathsf{S}) \cap \mathrm{B}_\mathsf{z})^\circ$ for the set $\{(\mathsf{t}, \mathsf{q}, \mathsf{t}', \mathsf{q}') \in \mathrm{B}_\mathsf{z}$ and $\mathscr{S}^*_\mathsf{u}(\mathsf{t}, \mathsf{q}) > 0\}$. Since for $\alpha > 1$ we have $\mathsf{t}' \equiv 0$ and $\mathsf{z}' \equiv 0$ on $((0, \mathsf{S}) \cap \mathrm{B}_\mathsf{z})^\circ$ and since (t, q) is non-degenerate, we have $\mathsf{u}' \neq 0$ on $((0, \mathsf{S}) \cap \mathrm{B}_\mathsf{z})^\circ$. Therefore, thanks to Proposition 3.2, the set $\Lambda_\mathsf{u}(\mathsf{u}'(s), \mathscr{S}^*_\mathsf{u}(\mathsf{t}(s), \mathsf{q}(s))) = \mathrm{Argmin}_{\lambda > 0} \mathcal{B}_{\mathcal{V}_\mathsf{u}}(\frac{1}{\lambda}, \mathsf{u}'(s), \mathscr{S}^*_\mathsf{u}(\mathsf{t}(s), \mathsf{q}(s)))$ is non-empty for every $s \in ((0, \mathsf{S}) \cap \mathrm{B}_\mathsf{z})^\circ$. We consider the multi-valued mapping $\Gamma_\mathsf{u} : ((0, \mathsf{S}) \cap \mathrm{B}_\mathsf{z})^\circ \rightrightarrows [0, \infty)$ defined by $\Gamma_\mathsf{u}(s) := \Lambda_\mathsf{u}(\mathsf{u}'(s), \mathscr{S}^*_\mathsf{u}(\mathsf{t}(s), \mathsf{q}(s)))$ and observe that its graph is a Borel subset of $(0, \infty)^2$. Indeed, Γ_u is given by the composition of the upper semicontinuous multi-valued mapping Λ_u, with the Borel function u' and the lower semicontinuous function $\mathscr{S}^*_\mathsf{u}(\mathsf{t}, \mathsf{q})$.

Hence, we are in a position to apply the von Neumann-Aumann selection theorem (cf. [CaV77, Thm. III.22]) to Γ_u and conclude that

$$\text{there exists a measurable } \widetilde{\lambda}_\mathsf{u} : ((0, \mathsf{S}) \cap \mathrm{B}_\mathsf{z})^\circ \to (0, +\infty) \text{ with } \widetilde{\lambda}_\mathsf{u}(s) \in \Lambda_\mathsf{u}(\mathsf{u}'(s), \mathscr{S}^*_\mathsf{u}(\mathsf{t}(s), \mathsf{q}(s)))\,. \tag{B.3}$$

Let N be the negligible subset of $(0, \mathsf{S})$ on which either t' or q' do not exist, or $(\mathsf{t}, \mathsf{q}, \mathsf{t}', \mathsf{q}') \notin \Sigma_\alpha$: since $\Sigma_\alpha \subset \mathrm{E}_\mathsf{u}\mathrm{R}_\mathsf{z} \cup \mathrm{E}_\mathsf{u}\mathrm{V}_\mathsf{z} \cup \mathrm{B}_\mathsf{z}$, we have that

$$(0, \mathsf{S}) \backslash N = A_1 \cup A_2 \cup A_3 \cup A_4 \text{ with } \begin{cases} A_1 = (0, \mathsf{S}) \cap \mathrm{E}_\mathsf{u}\mathrm{R}_\mathsf{z}, \\ A_2 = (0, \mathsf{S}) \cap \mathrm{E}_\mathsf{u}\mathrm{V}_\mathsf{z}, \\ A_3 = \{s \in (0, \mathsf{S}) \cap \mathrm{B}_\mathsf{z} \ : \ \mathscr{S}^*_\mathsf{u}(\mathsf{t}(s), \mathsf{q}(s)) = 0\}, \\ A_4 = \{s \in (0, \mathsf{S}) \cap \mathrm{B}_\mathsf{z} \ : \ \mathscr{S}^*_\mathsf{u}(\mathsf{t}(s), \mathsf{q}(s)) > 0\}\,. \end{cases}$$

Hence, we define λ_u on $(0, \mathsf{S}) \backslash N$ by

$$\lambda_\mathsf{u}(s) := \begin{cases} \ \ 0 & \text{if } s \in A_1 \cup A_2 \cup A_3, \\ \widetilde{\lambda}_\mathsf{u}(s) & \text{if } s \in A_4\,. \end{cases} \tag{B.4}$$

Analogously, we define λ_z on $(0, \mathsf{S}) \backslash N$ by

$$\lambda_\mathsf{z}(s) := \begin{cases} \ \ 0 & \text{if } s \in A_1, \\ \widetilde{\lambda}_\mathsf{z}(s) & \text{if } s \in A_2, \\ \ \ \infty & \text{if } s \in A_3 \cup A_4\,. \end{cases} \tag{B.5}$$

Here, $\widetilde{\lambda}_\mathsf{z} : (0, \mathsf{S}) \cap \mathrm{E}_\mathsf{u}\mathrm{V}_\mathsf{z} \to (0, +\infty)$ is a measurable selection with $\widetilde{\lambda}_\mathsf{z}(s) \in \Lambda_\mathsf{z}(\mathsf{z}'(s), \mathscr{S}^*_\mathsf{z}(\mathsf{t}(s), \mathsf{q}(s)))$, whose existence is again guaranteed by [CaV77, Thm. III.22].

Clearly, λ_u and λ_z satisfy the switching conditions (5.47c). Also taking into account (B.2), we conclude that (t, q) solve the subdifferential system (5.46) with λ_u and λ_z.

<u>*Step 2: existence of measurable* $\xi = (\mu, \zeta) : (0, \mathsf{S}) \to \mathbf{U}^* \times \mathbf{Z}^*$.</u> We aim to apply Filippov's theorem, in the form of Proposition B.1, with $O = (0, \mathsf{S}) \backslash N$, $X = \mathbf{U}^* \times \mathbf{Z}^*$, and the multi-valued mapping

$$F : (0, \mathsf{S}) \backslash N \rightrightarrows \mathbf{U}^* \times \mathbf{Z}^*, \qquad F(s) := \begin{cases} \partial_\mathsf{q} \mathcal{E}(\mathsf{t}(s), \mathsf{q}(s)) & \text{if } (\mathsf{t}(s), \mathsf{q}(s)) \in \mathrm{dom}(\partial_q \mathcal{E}), \\ \quad \{(0, 0)\} & \text{otherwise.} \end{cases}$$

Observe that F is measurable, with (non-empty) closed images thanks to Hypothesis 4.5. We now consider the mapping $g : \mathrm{graph}(F) \to \mathbb{R}$ given by

$$g(s,\mu,\zeta) := \begin{cases} \mathcal{V}_{\mathsf{u}}^*(-\mu) + \mathcal{W}_{\mathsf{z}}^*(-\zeta) & \text{if } s \in A_1, \\ \mathcal{V}_{\mathsf{u}}^*(-\mu) + \mathcal{R}(\mathsf{z}'(s)) + \frac{1}{\lambda_{\mathsf{z}}(s)}\mathcal{V}_{\mathsf{z}}(\lambda_{\mathsf{z}}(s)\mathsf{z}'(s)) + \frac{1}{\lambda_{\mathsf{z}}(s)}\mathcal{W}_{\mathsf{z}}^*(-\zeta) + \langle \zeta, \mathsf{z}'(s)\rangle_{\mathbf{Z}} & \text{if } s \in A_2, \\ \mathcal{V}_{\mathsf{u}}^*(-\mu) + \mathcal{W}_{\mathsf{z}}^*(-\zeta) & \text{if } s \in A_3, \\ \frac{1}{\lambda_{\mathsf{u}}(s)}\mathcal{V}_{\mathsf{u}}(\lambda_{\mathsf{u}}(s)\mathsf{u}'(s)) + \frac{1}{\lambda_{\mathsf{u}}(s)}\mathcal{V}_{\mathsf{u}}^*(-\mu) + \langle \mu, \mathsf{u}'(s)\rangle_{\mathbf{U}} & \text{if } s \in A_4\,. \end{cases}$$

It turns out that, also in view of the discussion developed in Step 1, g is measurable. Furthermore, for every $s \in (0,\mathsf{S})$ the functional $g(s,\cdot,\cdot)$ is continuous thanks to the continuity of $\mathcal{R}$ and $\mathcal{V}_{\mathsf{x}}$ and $\mathcal{V}_{\mathsf{x}}^*$, $\mathsf{x} \in \{\mathsf{u},\mathsf{z}\}$. Finally, the first of conditions (B.1) holds since, by Step 1, (t,q) solve the subdifferential system (5.46) with λ_{u} and λ_{z}, which exactly means that for every $s \in (0,\mathsf{S})\backslash N$ there exist $(\mu_s,\zeta_s) \in \partial_{\mathsf{q}}\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))$ such that $g(s,\mu_s,\zeta_s) = 0$. Hence, we are in a position to apply Proposition B.1, thus concluding that there exists a measurable selection $(0,\mathsf{S})\backslash N \ni s \mapsto (\mu(s),\zeta(s)) \in F(s)$ such that $g(s,\mu(s),\zeta(s)) \equiv 0$. This yields the desired selection as stated in (5.47a). □

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