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Sobolev W_p^1-spaces on d-thick closed subsets of $\mathbb{R}^n$

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Abstract. Let $S \subset \mathbb{R}^n$ be a nonempty closed set such that for some $d \in [0,n]$ and $\varepsilon > 0$ the d-Hausdorff content $\mathscr{H}^d_\infty(S \cap Q(x,r)) \geqslant \varepsilon r^d$ for all cubes $Q(x,r)$ with centre $x \in S$ and edge length $2r \in (0,2]$. For each $p > \max\{1, n-d\}$ we give an intrinsic characterization of the trace space $W_p^1(\mathbb{R}^n)|_S$ of the Sobolev space $W_p^1(\mathbb{R}^n)$ to the set S. Furthermore, we prove the existence of a bounded linear operator $\operatorname{Ext}\colon W_p^1(\mathbb{R}^n)|_S \to W_p^1(\mathbb{R}^n)$ such that Ext is the right inverse to the standard trace operator. Our results extend those available in the case $p \in (1,n]$ for Ahlfors-regular sets S.

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§ 1. Introduction

Given $m, n \in \mathbb{N}$ let $C^m(\mathbb{R}^n)$ denote the linear space of all functions on $\mathbb{R}^n$ with continuous partial derivatives up to order m equipped with the standard seminorm. The problem considered by Whitney in 1934 in his famous papers [1] and [2] reads as follows.

Classical Whitney Extension Problem. *Let $m, n \in \mathbb{N}$ and let S be an arbitrary nonempty subset of $\mathbb{R}^n$. How can we decide whether a given function $f\colon S \to \mathbb{R}$ extends to a $C^m(\mathbb{R}^n)$-function?*

Whitney [2] solved this problem completely only in the case $n = 1$. Furthermore, he gave a solution of the analogous problem in the context of the Lipschitz spaces $C^{m-1,1}(\mathbb{R}^n)$, $m, n \in \mathbb{N}$ (see [1]). After [1] and [2], great progress was made by many authors (see [3]–[5] and also the references there). Fefferman gave a complete solution (that is, for all $m, n \in \mathbb{N}$) of the Classical Whitney Extension Problem [6]–[9] only recently.

Recall that, according to the classical Sobolev embedding theorem (for example, see [10], Ch. I, § 1.8.2), in the case $m \in \mathbb{N}$, $j \in \{1, \dots, m\}$ and $p > n/(m-j+1)$, for every $F \in W_p^m(\mathbb{R}^n)$ there exists a representative $\widehat{F} \in C^{j-1}(\mathbb{R}^n)$. This fact enables one to identify each element $F \in W_p^m(\mathbb{R}^n)$ with its unique continuous representative. This implies that F has a well-defined restriction to any given nonempty

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subset of $\mathbb{R}^n$. As a result, in the case $p > n$ we can consider the analogue of the Classical Whitney Extension Problem, where $C^m(\mathbb{R}^n)$ is replaced by $W_p^m(\mathbb{R}^n)$. There is extensive literature devoted to these problems (see [11]–[19]).

In the case when $m, n \in \mathbb{N}$, $n \geqslant 2$ and $1 < mp \leqslant n$, functions in the space $W_p^m(\mathbb{R}^n)$ do not (in general) have continuous representatives (see Ch. 5, § 6 in [20]). Nevertheless, every $F \in W_p^m(\mathbb{R}^n)$ has a 'sufficiently nice' representative $\widehat{F}$ which has a well-defined trace $\widehat{F}|_S$ on each set $S \subset \mathbb{R}^n$ with positive $C_{m,p}$-capacity. Unfortunately, by contrast with the case $p > n$, in the case $1 < p \leqslant n$ Whitney-type problems have been posed and solved only in very particular cases. More precisely, either Ahlfors d-regular sets S [21]–[23] or special cusps in $\mathbb{R}^2$ [24] have been considered. Under minimal restrictions on S the corresponding problem is very complicated and has never been considered. In [25], for each $d \in [0, n]$ Rychkov introduced so-called d-thick sets $S \subset \mathbb{R}^n$, which are regular with respect to the d-Hausdorff content, that is, $\mathscr{H}^d_\infty(Q(x, r) \cap S) \approx r^d$ for all $x \in S$ and $r \in (0, 1]$. For such sets he considered Whitney-type problems in the context of Besov and Lizorkin-Triebel spaces. Given $d \in [0, n]$, the class of Ahlfors d-regular sets is strictly contained in the class of d-thick sets, but the latter is much wider. For example, every path connected subset of $\mathbb{R}^n$ is 1-thick but in general is not Ahlfors 1-regular (see Example 2.1 below).

In this paper we solve the following problem.

Problem A. *Let $d \in [0, n]$, $p \in (\max\{1, n - d\}, \infty]$, and let $S \subset \mathbb{R}^n$ be a d-thick closed set. Given a function $f\colon S \to \mathbb{R}$, how can we decide whether there exists a function $F \in W_p^1(\mathbb{R}^n)$ such that $\widehat{F}|_S(x) = f(x)$ for $C_{1,p}$-quasi-every $x \in S$? Consider the $W_p^1(\mathbb{R}^n)$-norms of all functions $F \in W_p^1(\mathbb{R}^n)$ such that $\widehat{F}|_S(x) = f(x)$ for $C_{1,p}$-q.e. $x \in S$. How small can these norms be?*

Given a set $S \subset \mathbb{R}^n$ with $C_{1,p}(S) > 0$, we denote the usual trace space of the space $W_p^1(\mathbb{R}^n)$ by $W_p^1(\mathbb{R}^n)|_S$ and let $\operatorname{Tr}|_S\colon W_p^1(\mathbb{R}^n) \to W_p^1(\mathbb{R}^n)|_S$ denote the corresponding trace operator. In our paper we obtain also a solution to the following problem.

Problem B. *Let $d \in [0, n]$ and $p \in (\max\{1, n - d\}, \infty]$, and let $S \subset \mathbb{R}^n$ be a closed d-thick set. Does there exist a bounded linear operator $\operatorname{Ext}\colon W_p^1(\mathbb{R}^n)|_S \to W_p^1(\mathbb{R}^n)$ such that $\operatorname{Tr}|_S \circ \operatorname{Ext} = \operatorname{Id}$ on $W_p^1(\mathbb{R}^n)|_S$?*

In § 4 we present solutions to Problems A and B. In § 5 we consider simplified versions of these problems, when the set S has a porous boundary. In this case the corresponding criterion (the solution to Problem A) can be simplified. In § 6 we show that our main results include the corresponding results concerning W_p^1-spaces obtained in [21], [23] and [24] as particular cases.

Finally we would like to underline that the methods in [25] gave a solution to Problem B only in the case $d > n - 1$. To the best of our knowledge, Problem A has never been considered in the literature. We introduce new methods which have never been used before. For example, we introduce the concept of a d-regular sequence of measures and generalized Calderón-type maximal functions with respect to such sequences. Such tools help us to capture the smoothness properties of functions in the trace space and enable us to solve Problems A and B for every $d \in [0, n]$.

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§ 2. Necessary background and statements of the main results

Throughout the paper $C, C_1, C_2, \dots$ will be generic positive constants. These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is shown, for instance, by the notation $C = C(n,p,k)$. We write $A \approx B$ if there is a constant $C \geqslant 1$ such that $A/C \leqslant B \leqslant CA$.

Throughout the paper $x = (x_1, \dots, x_n)$ denotes an element of the space $\mathbb{R}^n$. The symbols α and β will be used to denote multi-indices, that is, elements of the space $\mathbb{N}_0^n$. Following [23] it will often be convenient to measure distances in $\mathbb{R}^n$ in the uniform norm $\|x\| := \|x\|_\infty := \max\{|x_i| \colon i = 1, \dots, n\}$, $x \in \mathbb{R}^n$. Given two subsets A and B of $\mathbb{R}^n$, set $\operatorname{dist}(A,B) := \inf\{\|a-b\|_\infty \colon a \in A,\, b \in B\}$. For any $C \subset \mathbb{R}^n$ we also set $\operatorname{diam} C := \sup\{\|a-a'\|_\infty \colon a, a' \in C\}$.

The symbols $B(x,r)$ and $Q(x,r)$ stand for the closed balls with centre x and radius $r>0$ in the standard Euclidean norm $\|\cdot\|_2$ and in the uniform norm $\|\cdot\|_\infty$, respectively (we will also call $Q = Q(x,r)$ a cube) Given a number $c>0$ we write cB (cQ) to denote the ball $B(x,cr)$ (the cube $Q(x,cr)$, respectively). By a dyadic cube we mean an arbitrary half-open cube $\widetilde{Q}_{k,m} := \prod_{i=1}^n [m_i/2^k, (m_i+1)/2^k)$, $k \in \mathbb{Z}$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$. Given $k \in \mathbb{Z}$, we let $\mathscr{Q}_k$ denote the mesh of all dyadic cubes with edge length 2^{-k}.

For any set $E \subset \mathbb{R}^n$ we let $\overline{E}$ and $\operatorname{int} E$ denote the closure and interior of E in the topology induced by an arbitrary norm in $\mathbb{R}^n$ (recall that all norms in $\mathbb{R}^n$ are equivalent). For $A \subset \mathbb{R}^n$ and $\delta > 0$ we define the δ-neighbourhood of A to be $U_\delta(A) := \bigcup_{x \in A} \operatorname{int} B_\delta(x)$.

Given a Borel measure $\mathfrak{m}$ and a nonempty Borel set $S \subset \mathbb{R}^n$, we define the *restriction* of $\mathfrak{m}$ to S. More precisely, we set $\mathfrak{m}\lfloor_S(G) := \mathfrak{m}(G \cap S)$ for every nonempty Borel set $G \subset \mathbb{R}^n$.

Let $\mathfrak{m}$ be an arbitrary Borel measure on $\mathbb{R}^n$. Given $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n, \mathfrak{m})$, for every Borel set $G \subset \mathbb{R}^n$ with $\mathfrak{m}(G) < +\infty$ we set

$$f_{G,\mathfrak{m}} := \fint_G f(x)\,d\mathfrak{m}(x) := \begin{cases} \dfrac{1}{\mathfrak{m}(G)} \displaystyle\int_G f(x)\,d\mathfrak{m}(x), & \mathfrak{m}(G) > 0, \\ 0, & \mathfrak{m}(G) = 0. \end{cases} \tag{2.1}$$

2.1. Ahlfors *d*-regular sets and *d*-thick sets. Given $S \subset \mathbb{R}^n$, $0 \leqslant d \leqslant n$ and $\delta \in (0, +\infty]$ we set

$$\mathscr{H}^d_\delta(S) := \inf \sum_j r_j^d,$$

where the infimum is taken over all countable covers of S by cubes $Q(x_j, r_j)$ with arbitrary centres x_j and radii $r_j < \delta$. We call the quantity $\mathscr{H}^d_\infty(S)$ the *d-Hausdorff content* of S.

We define the *Hausdorff d-measure* of S by $\mathscr{H}^d(S) := \lim_{\delta \to 0} \mathscr{H}^d_\delta(S)$. Note that our definitions of Hausdorff contents and measures are slightly different from the classical ones (cf. § 5.1 in [26]). We use covers by balls in the $\|\cdot\|_\infty$-norm, that is, by cubes instead of classical balls. Up to some universal constants both approaches return the same values of the corresponding measures. Note also that in our case $\mathscr{H}^n$ coincides with the classical Lebesgue measure $\mathscr{L}_n$ on the Borel σ-algebra of $\mathbb{R}^n$.

Given $m \in \mathbb{N}$ and $p \in (1,\infty)$, recall the concept of $C_{m,p}$-*capacity* (see § 2.1 of [26]). In what follows we say that some property holds (m,p)-*quasi-everywhere* ((m,p)-q.e. for short) if it holds everywhere outside some set E with $C_{m,p}(E)=0$. The following property summarizes the connections between $C_{1,p}$-capacity and Hausdorff measures (see Theorems 5.1.9 and 5.1.13 in [26]).

Proposition 2.1. *Let $p \in (1,n]$ and let $E \subset \mathbb{R}^n$. If $\mathscr{H}^{n-p}(E) < +\infty$, then $C_{1,p}(E)=0$. Conversely, if $C_{1,p}(E)=0$, then $\mathscr{H}^d(E)=0$ for every $d > n-p$.*

We have taken the following definition from [22], Ch. 2, §§ 1.1 and 1.2.

Definition 2.1. Given $d \in [0,n]$, we say that a closed set $S \subset \mathbb{R}^n$ is *Ahlfors d-regular* (or just a *d-set* for short) if there exists a *d-measure* on S, that is, a Borel measure $\mathfrak{m}$ with $\operatorname{supp}\mathfrak{m} = S$ such that for all $x \in S$ and $r \in (0,1]$

$$c_1(\mathfrak{m})r^d \leqslant \mathfrak{m}(Q(x,r)\cap S) \leqslant c_2(\mathfrak{m})r^d, \tag{2.2}$$

where the positive constants $c_1(\mathfrak{m}), c_2(\mathfrak{m})$ depend on $\mathfrak{m}$ but do not depend on x or r.

Remark 2.1. We can show that given a (closed) d-set S the restriction $\mathscr{H}^d\lfloor_S$ is a d-measure on S (see [22], § 1.2, Theorem 1, for details).

Using Remark 2.1 we introduce the following important notation. In what follows, given an Ahlfors d-regular (closed) set S we set $c_i^d(S) := c_i\ (\mathscr{H}^d\lfloor_S)$, $i=1,2$.

The following proposition will be useful for comparing d-sets with d-thick sets below.

Lemma 2.1. *Let $d \in [0,n]$ and let $\mathscr{I}$ be a nonempty index set. Let $\{S_\alpha\}_{\alpha\in\mathscr{I}}$ be a family of d_α-sets with $d_\alpha \in [d,n]$, $\alpha \in \mathscr{I}$, such that the set $S := \bigcup_{\alpha\in\mathscr{I}} S_\alpha$ is closed and Ahlfors d-regular. Then the following holds.*

(1) *$d_\alpha = d$ for every $\alpha \in \mathscr{I}$.*

(2) *Suppose the family $\{S_\alpha\}_{\alpha\in\mathscr{I}}$ is such that $\mathscr{H}^d(S_\alpha \cap S_{\alpha'}) = 0$ for every $\alpha,\alpha' \in \mathscr{I}$, $\alpha \neq \alpha'$, $\inf_{\alpha\in\mathscr{I}} c_1^d(S_\alpha) > 0$ and $\bigcap_{\alpha\in\mathscr{I}} S_\alpha \neq \varnothing$. Then* $\operatorname{card}\mathscr{I} < \infty$.

Proof. (1) If there exists $\alpha_0 \in \mathscr{I}$ for which $d_{\alpha_0} > d$, then Remark 2.1 and (2.2) with $\mathfrak{m} = \mathscr{H}^{d_{\alpha_0}}\lfloor_S$, together with elementary properties of Hausdorff measures, show that $\mathscr{H}^d(Q(x,r)\cap S) = +\infty$ for all $x \in S_{\alpha_0}$ and all $r \in (0,1]$. Hence the right-hand side of (2.2) must be violated for $\mathfrak{m} = \mathscr{H}^d\lfloor_S$, each $x \in S_{\alpha_0}$ and $r \in (0,1]$. This contradicts the Ahlfors d-regularity of S.

(2) Fix a point $x_0 \in \bigcap_{\alpha\in\mathscr{I}} S_\alpha$. By (1), $d_\alpha = d$ for all $\alpha \in \mathscr{I}$. Assume that $\operatorname{card}\mathscr{I} = \infty$. Then it is easy to see from our assumptions that for every $r \in (0,1]$ there is a countable family of different indices $\{\alpha_k\} \subset \mathscr{I}$ such that $\widetilde{S}_{\alpha_k}(r) := Q(x_0,r) \cap (S_{\alpha_k} \setminus \bigcup_{j=1}^{k-1} S_{\alpha_j}) \neq \varnothing$ for all $k \in \mathbb{N}$. Hence there is a sequence of distinct points $\{x_k\}$ such that $x_k \in \widetilde{S}_{\alpha_k}(r/2)$ for every $k \in \mathbb{N}$. Let $c^d(S) := \inf_{\alpha\in\mathscr{I}} c_1^d(S_\alpha)$. It is clear from the construction and Remark 2.1 that $\mathscr{H}^d(\widetilde{S}_{\alpha_k}(r)) \geqslant \mathscr{H}^d(Q(x_k,r/2)\cap\widetilde{S}_{\alpha_k}(r)) \geqslant \mathscr{H}^d(Q(x_k,r/2)\cap S_{\alpha_k}) \geqslant c^d(S)(r/2)^d$ for every $r \in (0,1]$.

Hence the fact that the sets $\widetilde{S}_{\alpha_k}(r)$ are disjoint and measurable, together with the countable additivity of the Hausdorff measures, gives the following estimate, which contradicts (2.2). Namely, for all $r \in (0,1]$

$$\mathscr{H}^d(Q(x_0,r)\cap S) \geqslant \mathscr{H}^d\biggl(\bigcup_{j=1}^{\infty}\widetilde{S}_{\alpha_j}(r)\biggr) \geqslant \sum_{j=1}^{\infty} c^d(S)\Bigl(\frac{r}{2}\Bigr)^d = +\infty \quad \forall\, r \in (0,1].$$

The proof is complete.

To the best of our knowledge the following concept was first introduced in [25].

Definition 2.2. Let $d \in [0,n]$. A set $S \subset \mathbb{R}^n$ is said to be *d-thick* if there exists a constant $c_3^d(S) > 0$ such that for all $x \in S$ and $r \in (0,1]$

$$c_3^d(S) r^d \leqslant \mathscr{H}^d_\infty(Q(x,r)\cap S). \tag{2.3}$$

The following proposition is an immediate consequence of Definition 2.2. We omit the proof.

Proposition 2.2. *Let $S \subset \mathbb{R}^n$ be a d-thick set for some $d \in [0,n]$. Then*

(1) *the closure $\overline{S}$ of S is d-thick;*

(2) *S is d'-thick for every d', $0 \leqslant d' \leqslant d$, and $c_3^{d'}(S) \geqslant c_3^d(S)$.*

The following lemma exhibits an important relation between the concepts of Ahlfors d-regular sets and d-thick sets.

Lemma 2.2. *Let $d \in [0,n]$. Every Ahlfors d-regular set $S \subset \mathbb{R}^n$ is d-thick. Furthermore,*

$$c_3^d(S) \geqslant \frac{c_1^d(S)}{c_2^d(S)2^{d+1}}. \tag{2.4}$$

Proof. Suppose that $S \subset \mathbb{R}^n$ is an Ahlfors d-regular set. Fix a cube $Q = Q(x,r)$ with $x \in S$ and $0 < r \leqslant 1$. Let $\{Q_j\}_{j\in\mathbb{N}} = \{Q(x_j,r_j)\}_{j\in\mathbb{N}}$ be a covering of $Q \cap S$ such that $\mathscr{H}^d_\infty(Q\cap S) \geqslant (1/2)\sum_{j\in\mathbb{N}}(r_j)^d$. Clearly, we can assume that $Q_j \cap S \neq \varnothing$ for all $j \in \mathbb{N}$. For every $j \in \mathbb{N}$ fix a point $\widetilde{x}_j \in Q_j \cap S$. Using Remark 2.1, estimate (2.2), and the subadditivity of $\mathscr{H}^d_\infty$ we obtain the required estimate:

$$\begin{aligned} \mathscr{H}^d_\infty(Q\cap S) &\geqslant \frac{1}{2^{d+1}}\sum_{j\in\mathbb{N}}(2r_j)^d \geqslant \frac{1}{c_2^d(S)2^{d+1}}\sum_{j\in\mathbb{N}}\mathscr{H}^d(Q(\widetilde{x}_j,2r_j)\cap S) \\ &\geqslant \frac{1}{c_2^d(S)2^{d+1}}\mathscr{H}^d(Q\cap S) \geqslant \frac{c_1^d(S)}{c_2^d(S)2^{d+1}} r^d. \end{aligned} \tag{2.5}$$

The proof is complete.

The next result is a direct consequence of Lemma 2.2, Proposition 2.2, (2), and the monotonicity of $\mathscr{H}^d_\infty$.

Lemma 2.3. *Let $\mathscr{I}$ be an arbitrary nonempty index set. Let $0 \leqslant d \leqslant d_\alpha \leqslant n$ for every $\alpha \in \mathscr{I}$. Let $\{S_\alpha\}_{\alpha\in\mathscr{I}}$ be a family of Ahlfors d_α-regular sets and let $S := \bigcup_{\alpha\in\mathscr{I}} S_\alpha$. Then S is d-thick and*

$$c_3(S) \geqslant \sup_{\alpha\in\mathscr{I}} c_3^d(S_\alpha). \tag{2.6}$$

Now that we have Lemmas 2.1 and 2.3 at our disposal, we can present useful examples which illustrate the huge difference between Definition 2.1 and Definition 2.2.

Example 2.1. Let Ω be a path connected subset of $\mathbb{R}^n$. Then Ω and $\overline{\Omega}$ are 1-thick. In fact, fix a point $x \in \Omega$. Let $Q = Q(x,r)$ be a cube with edge length $0 < 2r \leqslant 2$. Consider two cases.

In the first case there is a point $y \in \Omega \setminus Q$. Hence there is a curve $\gamma_{x,y}$ which connects x and y. Let $\{Q_j\}_{j\in\mathbb{N}} = \{Q(x_j,r_j)\}_{j\in\mathbb{N}}$ be an arbitrary covering of $Q \cap \Omega$ for which

$$\sum_{j\in\mathbb{N}} r_j \leqslant 2\mathscr{H}^1_\infty(\Omega \cap Q). \tag{2.7}$$

We choose an index set $\mathscr{A} \subset \mathbb{N}$ such that $\gamma_{x,y} \cap Q_j \neq \varnothing$ for every $j \in \mathscr{A}$ and $\gamma_{x,y} \subset \bigcup_{j\in\mathscr{A}} Q_j$. Consider the projections $\gamma^i_{x,y}$, $i = 1,\dots,n$, of our curve and the projections Q^i_j of cubes in the cover onto the ith coordinate axes. Since we measure distances in the $\|\cdot\|_\infty$-norm, there exists $i_0 \in \{1,\dots,n\}$ for which $\mathscr{H}^1(\gamma^{i_0}_{x,y}) \geqslant r$. By construction the family of closed intervals $\{Q^{i_0}_j\}_{j\in\mathscr{A}}$ covers $\gamma^{i_0}_{x,y}$. Hence from (2.7) we derive

$$\mathscr{H}^1_\infty(\Omega \cap Q) \geqslant \frac{1}{2}\sum_{j\in\mathbb{N}} r_j \geqslant \frac{1}{2}\sum_{j\in\mathscr{A}} r_j \geqslant \frac{\mathscr{H}^1(\gamma^{i_0}_{x,y})}{2} \geqslant \frac{r}{2}. \tag{2.8}$$

In the second case $\Omega \subset Q(x,r)$. Since $r \leqslant 1$, we have

$$\mathscr{H}^1_\infty(Q(x,r) \cap \Omega) \geqslant \mathscr{H}^1_\infty(\Omega) \geqslant \mathscr{H}^1_\infty(\Omega) r. \tag{2.9}$$

Combining (2.8) and (2.9) shows that Ω is 1-thick, and we can take $c^1_3(\Omega) = \min\{\mathscr{H}^1_\infty(\Omega), 1/2\}$.

It is obvious that this set cannot be Ahlfors 1-regular for $n \geqslant 2$. In addition, elementary computations show that for each $s > 1$ and $n \geqslant 2$ the cusp $\Omega_s := \{x = (x', x_n) \in \mathbb{R}^n \colon x_n > 0,\ \|x'\|_\infty \leqslant x_n^s\}$ cannot be Ahlfors d-regular for $d \in [0,n]$.

Example 2.2. Let $S := \overline{Q}(0,1) \cup ([1,2] \times \{0\}) \subset \mathbb{R}^2$. This set is 1-thick as a union of a 2-thick set (a square) and a 1-thick set (a line interval). From Lemma 2.1, (1), it follows that S is not Ahlfors 1-regular.

Example 2.3. Let $E := \bigcup_{n\in\mathbb{N}} \{(r,\varphi) \colon r \in [0,1],\ \varphi = 2^{-n}\} \cup [0,1] \times \{0\} \subset \mathbb{R}^2$. From Lemma 2.3 it follows that E is 1-thick. It is not Ahlfors 1-regular by Lemma 2.1, (2).

Example 2.4. We can show that any (ε,δ)-domain Ω is n-thick. We present only a sketch of the proof. Fix $x \in \Omega$ and $r < \min\{\operatorname{diam}\Omega, \delta\}/4$. Choose an arbitrary $y \in \Omega$ so that $\|x - y\| \geqslant r$. Then it easily follows from formulae (1.1) and (1.2) in [27] that there exists a curve $\gamma_{x,y}$ and a point $z \in \gamma_{x,y} \cap \partial Q(x, r/3)$ such that $B(z, c(\varepsilon,\delta,n)r) \subset \Omega$ for some positive constant $c(\varepsilon,\delta,n)$. The case $r \geqslant \min\{\operatorname{diam}\Omega, \delta\}/4$ can be considered similarly to the second case in Example 2.1.

2.2. Regular sequences of measures and Calderón-type maximal functions. The following concept is one of the cornerstones which enable us to solve Problem A.

Definition 2.3. Let S be a closed d-thick set for some $d \in [0,n]$. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a sequence of Borel measures such that $\operatorname{supp}\mu_k = S$, $k \in \mathbb{N}_0$. We say that $\{\mu_k\}_{k\in\mathbb{N}_0}$ is a *d-regular sequence of measures* on S if and only if for some $C > 0$ the following properties hold for every $k \in \mathbb{N}_0$:

$$(1) \quad \mu_k(Q(x,r)) \leqslant r^d \text{ for every } x \in \mathbb{R}^n \text{ and every } r \in (0, 2^{-k}]; \tag{2.10}$$

$$(2) \quad \mu_k(Q(x,2^{-k})) \geqslant C2^{-dk} \text{ for every } x \in S; \tag{2.11}$$

$$(3) \quad \mu_k = \gamma_k\mu_0 \text{ for } \gamma_k \in L_\infty(S,\mu_0) \text{ and}$$

$$2^{d-n}\gamma_{k+1}(x) \leqslant \gamma_k(x) \leqslant \gamma_{k+1}(x) \quad \text{for } \mu_0\text{-a.e. } x \in S. \tag{2.12}$$

Remark 2.2. It is clear that there exists a largest positive constant C for which (2.11) holds. We denote it by $C_{\{\mu_k\}}$.

Remark 2.3. We show in § 3 below that Definition 2.3 is consistent: for every d-thick closed set S there exists a d-regular sequence of measures on S.

Definition 2.4. Let $\mathfrak{m}$ be an arbitrary nonzero Radon measure. Let $Q = Q(x,r)$ be a closed cube. Given a function $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n,\mathfrak{m})$, the *best approximation to f by constants on Q, normalized with respect to* $\mathfrak{m}$, is defined by $\mathscr{E}_{\mathfrak{m}}(f,Q) := \inf_{c\in\mathbb{R}} \fint_Q |f(y)-c|\,d\mathfrak{m}(y)$.

Remark 2.4. Elementary computations give

$$\mathscr{E}_{\mathfrak{m}}(f,Q) \leqslant \widetilde{\mathscr{E}}_{\mathfrak{m}}(f,Q) := \fint_Q |f(y)-f_{Q,\mathfrak{m}}|\,d\mathfrak{m}(y) \leqslant 2\mathscr{E}_{\mathfrak{m}}(f,Q) \tag{2.13}$$

(recall (2.1)).

Here and in the sequel we use the following notation. Given a number $r \in (0,1]$ we set $k(r) := \bigl|[\log_2 r]\bigr|$, so that this is the *unique integer* such that $r \in [2^{-k(r)}, 2^{-k(r)+1})$.

The following definition is a far-reaching generalization of the classical concept of a maximal function measuring smoothness first introduced by Calderón [28].

Definition 2.5. Let $S \subset \mathbb{R}^n$ be a d-thick closed set for some $d \in [0,n]$. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n,\mu_k)$ for every $k \in \mathbb{N}_0$. Given $t \in [0,1]$, we consider the *Calderón-type maximal function with respect to* $\{\mu_k\}_{k\in\mathbb{N}_0}$. For every $x \in \mathbb{R}^n$

$$f^{\sharp}_{\{\mu_k\}}(x,t) := \begin{cases} \sup_{r\in(t,1]} r^{-1}\mathscr{E}_{\mu_{k(r)}}(f,Q(x,r)), & t \in [0,1), \\ \mathscr{E}_{\mu_{k(r)}}(f,Q(x,1)), & t = 1. \end{cases}$$

Remark 2.5. We set $f^{\sharp}_{\{\mu_k\}}(x) := f^{\sharp}_{\{\mu_k\}}(x,0)$ for brevity. If the set S is Ahlfors n-regular, we can take $\mu_k = \mathscr{H}^n\lfloor_S$ for every $k \in \mathbb{N}_0$. Hence, in this case our maximal function $f^{\sharp}_{\{\mu_k\}}$ is similar to that introduced by Shvartsman [23]. In particular, if $S = \mathbb{R}^n$, we obtain the *Calderón-type maximal function* (see [28]).

2.3. Porous sets.

Definition 2.6. Let S be a closed nonempty subset of $\mathbb{R}^n$ and let $\lambda \in (0,1)$. For every $j \in \mathbb{N}_0$ we define

$$S_j(\lambda) := \{x \in S\colon \text{there exists } y \in Q(x,2^{-j}) \text{ such that } \operatorname{int} Q(y,\lambda 2^{-j}) \subset \mathbb{R}^n \setminus S\}$$

and call $S_j(\lambda)$ the *maximal* 2^{-j}*-porous subset of* S. We say that S is *porous* if there exists $\lambda \in (0,1)$ such that $S_j(\lambda) = S$ for every $j \in \mathbb{N}_0$.

We gather some useful facts about porous sets. The second (see item (2) below) is a special case of Proposition 9.18 in [29].

Proposition 2.3. *Let S be a closed nonempty subset of $\mathbb{R}^n$ and let $\lambda \in (0,1)$. Then*

(1) *$S_j(\lambda)$ is closed for every $j \in \mathbb{N}_0$;*
(2) *if S is Ahlfors d-regular for some $d \in [0,n)$, then S is porous.*

Example 2.5. Let $\beta\colon [0,+\infty) \to [0,+\infty)$ be a continuous function that is strictly increasing and such that $\beta(0) = 0$ and $\beta(t) > 0$, $t > 0$. Consider the closed single cusp

$$G^{\beta} := \{x = (x', x_n) \in \mathbb{R}^n\colon x_n \in [0,\infty),\ \|x'\|_{\infty} \leqslant \beta(x_n)\}.$$

It is easy to see that the boundary ∂G^{β} of G^{β} is porous.

2.4. Trace spaces of Sobolev spaces. Recall that given $p \in [1,\infty]$, $n \in \mathbb{N}$ and an open set $G \subset \mathbb{R}^n$, the Sobolev space $W_p^1(G)$ is the linear space of all (equivalence classes of) real functions $F \in L_p(G)$ whose generalized partial derivatives on G $D^{\alpha}F$, $|\alpha| \leqslant 1$, belong to $L_p(G)$. This space is equipped with the norm

$$\|F|W_p^1(G)\| := \sum_{|\alpha| \leqslant 1} \|D^{\alpha}F|L_p(G)\|. \tag{2.14}$$

The next result, which is a very special case of Theorem 6.2.1 in [26], will help us to define the trace of a Sobolev function F consistently on a given 'sufficiently massive' set S.

Proposition 2.4. *Let $p \in (1,\infty]$ and $F \in W_p^1(\mathbb{R}^n)$. If $p \in (1,n]$, then there exists a set $E_F \subset \mathbb{R}^n$ with $C_{1,p}(E_F) = 0$ and a representative $\widehat{F}$ of the element F such that every point $x \in \mathbb{R}^n \setminus E_F$ is a Lebesgue point of $\widehat{F}$. If $p > n$, then there exists a continuous representative $\widehat{F}$ of F.*

In the sequel we call the representative $\widehat{F}$ constructed in Proposition 2.4 a *good representative* of F. Recall that, given $p \in (1,\infty)$, a property of F is said to hold $(1,p)$-quasi-everywhere ($(1,p)$-q.e. for short) if it holds everywhere except on a set of $C_{1,p}$-capacity zero.

Definition 2.7. Let $p \in (1,n]$ and $F \in W_p^1(\mathbb{R}^n)$. Let $\widehat{F}$ be a good representative of F. Given a set $S \subset \mathbb{R}^n$ with $C_{1,p}(S) > 0$, the *trace* $F|_S$ *of* F *on* S is the class of equivalent (modulo sets of $C_{1,p}$-capacity zero) functions $f\colon S \to \mathbb{R}$ such that $\widehat{F}(x) = f(x)$ for $(1,p)$-q.e. $x \in S$.

Definition 2.8. Let $p \in (n, \infty]$ and $F \in W_p^1(\mathbb{R}^n)$. Let $\widehat{F}$ be a continuous representative of F. Given a nonempty set $S \subset \mathbb{R}^n$ the *trace* $F|_S$ *of* F *on the set* S is the pointwise restriction of $\widehat{F}$ to S.

Below we identify a function $f\colon S \to \mathbb{R}$ and the class of functions each of which coincides $(1,p)$-quasi-everywhere with f on S.

Now using Definitions 2.7 and 2.8 we introduce the following.

Definition 2.9. Let $p \in (1, \infty]$. Given a nonempty set $S \subset \mathbb{R}^n$ with $C_{1,p}(S) > 0$, we define the *trace space* $W_p^1(\mathbb{R}^n)|_S$ of the space $W_p^1(\mathbb{R}^n)$ as follows:

$$W_p^1(\mathbb{R}^n)|_S := \{f\colon S \to \mathbb{R}\colon \text{there exists } F \in W_p^1(\mathbb{R}^n) \text{ such that } F|_S = f\}.$$

We equip this space with the usual trace norm

$$\|f|W_p^1(\mathbb{R}^n)|_S\| := \inf \|F|W_p^1(\mathbb{R}^n)\|,$$

where the infimum is taken over all $F \in W_p^1(\mathbb{R}^n)$ such that $F|_S = f$. Furthermore, we define the *trace operator* $\operatorname{Tr}|_S\colon W_p^1(\mathbb{R}^n) \to W_p^1(\mathbb{R}^n)|_S$ which takes $F \in W_p^1(\mathbb{R}^n)$ and gives back $F|_S$.

Definition 2.10. Let $p \in (1, \infty]$. Let $S \subset \mathbb{R}^n$ be a nonempty set. Assume that $C_{1,p}(S) > 0$ whenever $p \in (1, n]$. We say that a map $\operatorname{Ext}\colon W_p^1(\mathbb{R}^n)|_S \to W_p^1(\mathbb{R}^n)$ is an *extension operator* if it is the right inverse for the trace operator, so that $\operatorname{Tr}|_S \circ \operatorname{Ext} = \operatorname{Id}$ on $W_p^1(\mathbb{R}^n)|_S$.

Remark 2.6. Let $d \in [0, n]$, and let $S \subset \mathbb{R}^n$ be a d-thick set. It is important to underline that Proposition 2.1 and Definition 2.9 clearly imply that for every $p \in (\max\{1, n-d\}, \infty)$ the trace space $W_p^1(\mathbb{R}^n)|_S$ is well defined. Furthermore, our definitions immediately implies that the trace operator $\operatorname{Tr}|_S\colon W_p^1(\mathbb{R}^n) \to W_p^1(\mathbb{R}^n)|_S$ *is linear and bounded.*

Remark 2.7. In the case $p = \infty$ the Sobolev space $W_\infty^1(\mathbb{R}^n)$ can be identified with the space $\operatorname{LIP}(\mathbb{R}^n)$ of Lipschitz functions, and it is known that the restriction $\operatorname{LIP}(\mathbb{R}^n)|_S$ of the latter coincides with the space $\operatorname{LIP}(S)$ of Lipschitz continuous functions on S and that, furthermore, the classical Whitney extension operator maps $\operatorname{LIP}(S)$ linearly and continuously into $\operatorname{LIP}(\mathbb{R}^n)$ (for instance, see [20], Ch. 6). Hence in the sequel we will only deal with the case $1 < p < \infty$.

2.5. Statements of the main results. As we said above, without loss of generality we can work with the case $p \neq \infty$.

Given a closed set $S \subset \mathbb{R}^n$ and $k \in \mathbb{N}_0$, we define

$$\Sigma_k := \Sigma_k(S) := \{x \in S\colon \operatorname{dist}(x, \partial S) \leqslant 2^{-k}\}.$$

Definition 2.11. Let $p \in (1, \infty)$, $d \in [0, n]$ and $\lambda \in (0, 1)$. Let $S \subset \mathbb{R}^n$ be a d-thick closed set. Let $\{\mu_k\} = \{\mu_k\}_{k \in \mathbb{N}_0}$ be a d-regular sequence of measures on S. For every $p \in (1,\infty)$ we define the following nonnegative functionals (with values in $[0,+\infty]$) on the space $W_p^1(\mathbb{R}^n)|_S$:

$$\begin{aligned}
\mathscr{S}\mathscr{N}_{S,p}[f] &:= \|f^{\sharp}_{\{\mu_k\}}|L_p(S,\mathscr{H}^n)\|,\\
\widetilde{\mathscr{B}\mathscr{N}}_{S,p,\lambda}[f] &:= \Big(\sum_{k=1}^{\infty} 2^{k(d-n)} \int_{S_k(\lambda)} \big(f^{\sharp}_{\{\mu_k\}}(x,2^{-k})\big)^p \, d\mu_k(x)\Big)^{1/p},\\
\mathscr{B}\mathscr{N}_{S,p}[f] &:= \Big(\sum_{k=1}^{\infty} 2^{kp(1-(n-d)/p)} \int_{\Sigma_k} \big(\mathscr{E}_{\mu_k}(f,Q(x,2^{-k}))\big)^p \, d\mu_k(x)\Big)^{1/p},\\
\widetilde{\mathscr{N}}_{S,p,\lambda}[f] &:= \|f|L_p(\mu_0)\| + \mathscr{S}\mathscr{N}_{S,p}[f] + \widetilde{\mathscr{B}\mathscr{N}}_{S,p,\lambda}[f],\\
\mathscr{N}_{S,p}[f] &:= \|f|L_p(\mu_0)\| + \mathscr{S}\mathscr{N}_{S,p}[f] + \mathscr{B}\mathscr{N}_{S,p}[f].
\end{aligned} \tag{2.15}$$

Remark 2.8. From Proposition 2.1 and Lemma 3.6 it follows that all the functionals in (2.15) are well defined on the trace space $W_p^1(\mathbb{R}^n)|_S$. More precisely their values remain the same after changing the function f on a set of $C_{1,p}$-capacity zero.

Remark 2.9. The symbols $\mathscr{S}\mathscr{N}_{S,p}$, $\widetilde{\mathscr{B}\mathscr{N}}_{S,p,\lambda}$ and $\mathscr{B}\mathscr{N}_{S,p}$ have not been picked at random. Informally speaking, $\mathscr{S}\mathscr{N}_p$ is the 'Sobolev part' of the trace norm, while we may regard the functionals $\mathscr{B}\mathscr{N}_{S,p}$ and $\widetilde{\mathscr{B}\mathscr{N}}_{S,p,\lambda}$ as possible variants for the role of a Besov-type seminorm in the trace space. We clarify this in Examples 6.1 and 6.2, respectively.

Now we are ready to formulate our main result, which solves Problems A and B.

Theorem 2.1. *Let $d \in [0,n]$ and $p \in (\max\{1,n-d\},\infty)$. Let $S \subset \mathbb{R}^n$ be a d-thick closed set. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Then a function $f\colon S \to \mathbb{R}$ belongs to the trace space $W_p^1(\mathbb{R}^n)|_S$ if and only if for $(1,p)$-q.e. $x \in S$*

$$\lim_{r\to 0} \fint_{Q(x,r)\cap S} |f(x)-f(z)|\, d\mu_{k(r)}(z) = 0 \tag{2.16}$$

and $\widetilde{\mathscr{N}}_{S,p,\lambda}[f] < \infty$ for some $\lambda \in (0,1)$. Furthermore,

$$\|f|W_p^1(\mathbb{R}^n)|_S\| \approx \widetilde{\mathscr{N}}_{S,p,\lambda}[f] \tag{2.17}$$

and there exists a bounded linear extension operator $\operatorname{Ext}\colon W_p^1(\mathbb{R}^n)|_S \to W_p^1(\mathbb{R}^n)$.

Remark 2.10. Recall Example 2.1. Consider a path connected closed set $S \subset \mathbb{R}^2$. It is obvious that using Theorem 2.1 we obtain an intrinsic description of the trace space of the Sobolev space $W_p^1(\mathbb{R}^n)$ on S in the full range of parameters $p \in (1,\infty)$. We would like to underline that even this particular case of Theorem 2.1 was never considered in the literature.

The results in Theorem 2.1 can be simplified in the case when either S or $\mathbb{R}^n \setminus S$ possesses a certain 'plumpness'.

Theorem 2.2. *Let $d \in [0,n]$ and $p \in (\max\{1,n-d\},\infty)$. Let $S \subset \mathbb{R}^n$ be a d-thick closed set. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Assume that ∂S is porous. Then a function $f\colon S \to \mathbb{R}$ belongs to the trace space $W_p^1(\mathbb{R}^n)|_S$ if and only if for $(1,p)$-q.e. $x \in S$*

$$\lim_{r\to 0} \fint_{Q(x,r)\cap S} |f(x)-f(z)|\,d\mu_{k(r)}(z) = 0 \tag{2.18}$$

and $\mathscr{N}_{S,p}[f] < \infty$. Furthermore,

$$\|f|W_p^1(\mathbb{R}^n)|_S\| \approx \mathscr{N}_{S,p}[f] \tag{2.19}$$

and there exists a linear bounded extension operator $\operatorname{Ext}\colon W_p^1(\mathbb{R}^n)|_S \to W_p^1(\mathbb{R}^n)$.

§ 3. Main technical tools

The aim of this section is to bring together all the necessary technical results which are essential to the proofs of Theorems 2.1 and 2.2. As well as some very well-known facts, the section contains some new results. We split the section into several subsections for the reader's convenience.

3.1. Maximal functions and potentials. Let $F \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $\alpha \in [0,n)$. Given $t,s \in [0,\infty]$, we introduce the *fractional maximal operator*

$$\mathrm{M}^{<s}_{>t}[F,\alpha](x) := \sup_{r\in(t,s)} r^{\alpha} \fint_{Q(x,r)} |F(y)|\,d\mathscr{H}^n(y), \qquad x \in \mathbb{R}^n.$$

We use the notation $\mathrm{M}_{>t}[\cdot,\alpha] := \mathrm{M}^{<\infty}_{>t}[\cdot,\alpha]$, $\mathrm{M}[\cdot,\alpha] := \mathrm{M}_{>0}[\cdot,\alpha]$ and $\mathrm{M}^{<s}_{>t}[F] := \mathrm{M}^{<s}_{>t}[F,0]$.

Remark 3.1. Assume that $0 < t' \leqslant t < s \leqslant s' \leqslant +\infty$. Then it is easy to see that for every $x \in \mathbb{R}^n$ and $y \in Q(x,t)$

$$\mathrm{M}^{<s}_{>t}[F,\alpha](x) \leqslant \mathrm{M}^{<s'}_{>t'}[F,\alpha](x), \qquad \mathrm{M}^{<s}_{>t}[F,\alpha](x) \leqslant C(n)\,\mathrm{M}^{<2s}_{>t}[F,\alpha](y). \tag{3.1}$$

The following result is a very particular case of Theorem B in [30].

Theorem A. *Let $d \in [0,n]$, $\alpha \in [0,n)$, $s \in (0,+\infty]$ and $\gamma \in (1,\infty)$. Let $\mathfrak{m}$ be a Radon measure on $\mathbb{R}^n$ such that for some (universal) positive constant C*

$$\mathfrak{m}(Q(x,r)) \leqslant Cr^d, \qquad x \in \mathbb{R}^n, \quad r \in (0,s). \tag{3.2}$$

If $\gamma\alpha \geqslant n-d$, then the operator $\mathrm{M}^{<s}[\cdot,\alpha]$ is bounded from $L_\gamma(\mathbb{R}^n)$ into $L_\gamma(\mathbb{R}^n,\mathfrak{m})$.

The following simple fact will be of use in what follows (see [31], § 2.4.3, for instance, for the proof).

Proposition 3.1. *Suppose that $d \in [0,n)$. Then given a function $F \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, there exists a set $E_F \subset \mathbb{R}^n$ with $\mathscr{H}^d(E_F) = 0$ such that for every $x \in \mathbb{R}^n \setminus E_F$*

$$\lim_{r\to 0} \frac{1}{r^d} \int_{Q(x,r)} |F(y)|\,d\mathscr{H}^n(y) = 0.$$

Let $\alpha \in [0,n)$. Given a function $g \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, for every cube Q we define the *reduced Riesz potential* by

$$\mathrm{I}^Q_\alpha[g](x) := \int_Q \frac{g(y)}{\|x-y\|_2^{n-\alpha}}\, d\mathscr{H}^n(y), \qquad x \in \mathbb{R}^n.$$

Remark 3.2. It is useful to note a simple relation between Riesz potentials and fractional maximal operators. Suppose that $\alpha \in (0,n)$. Then for every $R > 0$ and $\delta \in (0,\alpha)$

$$\mathrm{M}^{<R}[g,\alpha](x) \leqslant \mathrm{I}^{Q(x,R)}_\alpha[g](x) \leqslant C(\delta,R)\,\mathrm{M}^{<2R}[g,(\alpha-\delta)](x), \qquad x \in \mathbb{R}^n. \tag{3.3}$$

Now we formulate an important beautiful estimate, which is a special case of one of the implications in Theorem 2.1 in [32]. In fact, in [32] classical (nonreduced) Riesz potentials were considered. Nevertheless, the corresponding proof for reduced Riesz potentials is similar (at any rate, in the case of interest to us).

Given a (nonnegative) Radon measure μ and parameters $q \in (1,\infty)$ and $\alpha \in (0,n/q)$, we define the *Wolf potential at the scale* $R > 0$ by

$$\mathscr{W}^R_{\alpha,q}[\mu](x) := \int_0^R \left(\frac{\mu(Q(x,r))}{r^{n-q\alpha}}\right)^{q'-1} \frac{dr}{r}, \qquad x \in \mathbb{R}^n. \tag{3.4}$$

Theorem B. *Let $R > 0$, $q \in (1,\infty)$, $q\alpha \in (0,n)$, and let μ be a positive Radon measure on $\mathbb{R}^n$. Assume that $\mathscr{W}^{2R}_{\alpha,q}[\mu] \in L_1(\mathbb{R}^n,\mu)$. Then there exists a positive constant C (independent of g) such that*

$$\int_{\mathbb{R}^n} \mathrm{I}^{Q(x,R)}_\alpha[g](x)\, d\mu(x) \leqslant C\|g|L_q(\mathbb{R}^n)\| \tag{3.5}$$

for every $g \in L_q(\mathbb{R}^n)$. Moreover, the least possible constant C in (3.5) satisfies the inequality

$$C \leqslant b\|\mathscr{W}^{2R}_{\alpha,q}[\mu]|L_1(\mathbb{R}^n,\mu)\|^{1/q'}, \tag{3.6}$$

where the positive constant b does not depend on μ.

Recall a classical Poincaré-type inequality (see formula (7.45) in [33]).

Proposition 3.2. *Assume that $F \in W_1^{1,\mathrm{loc}}(\mathbb{R}^n)$. Then for every cube $Q = Q(x,r)$, $r > 0$,*

$$\fint_Q \left| F(y) - \fint_Q F(z)\, d\mathscr{H}^n(z)\right| d\mathscr{H}^n(y) \leqslant C(n) r \fint_Q |\nabla F(y)|\, d\mathscr{H}^n(y). \tag{3.7}$$

The following estimate is well known.

Proposition 3.3. *Let $p \in (1,\infty)$ and $F \in W^1_p(\mathbb{R}^n)$. Then for $(1,p)$-q.e. points $x \in \mathbb{R}^n$ (for every point in the case $p > n$) and every cube $Q = Q(y,r) \ni x$*

$$\left|\widehat{F}(x) - \fint_Q F(z)\, d\mathscr{H}^n(z)\right| \leqslant C\, \mathrm{I}^Q_1[|\nabla F|](x), \tag{3.8}$$

where the positive constant C is independent of F, x and r.

To prove this we use Propositions 2.4 and 3.2, and then repeat the simple arguments in the proof of Theorem 5.2 in [34] almost verbatim, with minor modifications. We omit the elementary details.

3.2. Overlappings of sets. Given a nonempty family $\{E_\alpha\}_{\alpha\in\mathscr{I}}$ of nonempty subsets of $\mathbb{R}^n$, we say that the *multiplicity of overlapping* of the sets E_α is finite if there exists $C > 0$ such that $\sum_{\alpha\in\mathscr{I}} \chi_{E_\alpha}(x) \leqslant C$ for every $x \in \mathbb{R}^n$.

Definition 3.1. Let E be a nonempty set in $\mathbb{R}^n$. Let $\varepsilon > 0$, and let $\{x_j\}_{j\in\mathscr{I}}$, $\mathscr{I} \subset \mathbb{N}$, be a subset of E with the following properties:

(i) $\|x_i - x_j\|_\infty \geqslant \varepsilon$ for every $i, j \in \mathscr{I}$, $i \neq j$;

(ii) for every $x \in E \setminus \{x_j\}_{j\in\mathscr{I}}$ there is a point x_j such that $\|x - x_j\|_\infty < \varepsilon$.

We call the set $\{x_j\}_{j\in\mathscr{I}}$ a *maximal ε-separated subset* of E.

The following propositions will be used often in what follows. We omit the elementary proofs.

Proposition 3.4. *Let $\{Q_j\}_{j\in\mathscr{I}}$ be a family of pairwise disjoint cubes with the same edge length. Then for every $c \geqslant 1$ there is a positive constant $C = C(n, c)$ such that the multiplicity of overlapping of the cubes cQ_j is finite and bounded above by C.*

Proposition 3.5. *Let $\mathfrak{m}$ be a finite Borel measure on $\mathbb{R}^n$. Let $\{E_j\}_{j\in\mathscr{I}}$ be a family of Borel subsets of $\mathbb{R}^n$ such that the multiplicity of overlapping of the sets E_j is finite and bounded above by some constant $N \in \mathbb{N}$. Then*

$$\sum_{j\in\mathscr{I}} \mathfrak{m}(E_j) \leqslant N\mathfrak{m}(\mathbb{R}^n). \tag{3.9}$$

The following elementary observation is a direct consequence of Definition 3.1.

Lemma 3.1. *Let E be a nonempty subset of $\mathbb{R}^n$. Let $\varepsilon > 0$ and let $\{x_j\}_{j\in\mathscr{I}}$ be a maximal ε-separated subset of E. Then*

(1) $E \subset \bigcup_{j\in\mathscr{I}} Q(x_j, \varepsilon)$;

(2) *the family $\{Q(x_j, \varepsilon/2)\}_{j\in\mathscr{I}}$ is pairwise disjoint*;

(3) *every point $x \in E$ belongs to at most 3^n cubes in $\{Q(x_j, \varepsilon)\}_{j\in\mathscr{I}}$.*

3.3. The Whitney decomposition. Recall that we measure the distances in $\mathbb{R}^n$ in the uniform norm $\|\cdot\|_\infty$. For a cube $Q \subset \mathbb{R}^n$ we set $Q^* := (9/8)Q$. Recall that, unless otherwise stated, all cubes are assumed to be closed.

The following result is a slight modification of the Classical Whitney Decomposition Lemma. Its proof repeats the proof of Theorem 1 in [29], Ch. 6, with minor changes.

Lemma 3.2. *For each closed nonempty set $S \subset \mathbb{R}^n$ there exists a family of closed dyadic cubes $W_S = \{Q_\varkappa\}_{\varkappa\in I} = \{Q(x_\varkappa, r_\varkappa)\}_{\varkappa\in I}$ with the following properties*:

(1) $\mathbb{R}^n \setminus S = \bigcup_{\varkappa\in I} Q_\varkappa$;

(2) *for each $\varkappa \in I$*

$$\operatorname{diam}(Q_\varkappa) \leqslant \operatorname{dist}(Q_\varkappa, S) \leqslant 4\operatorname{diam}(Q_\varkappa); \tag{3.10}$$

(3) *the following inequalities hold*:

$$\frac{1}{4}\operatorname{diam}(Q_\varkappa) \leqslant \operatorname{diam}(Q_{\varkappa'}) \leqslant 4\operatorname{diam}(Q_\varkappa), \quad \textit{if } Q^*_\varkappa \cap Q^*_{\varkappa'} \neq \varnothing; \tag{3.11}$$

(4) *for each index* $\varkappa \in I$ *there exist at most* $C(n)$ *indices* $\varkappa'$ *such that* $Q^*_{\varkappa} \cap Q^*_{\varkappa'} \neq \varnothing$;

(5) $\operatorname{int} Q_{\varkappa} \cap \operatorname{int} Q_{\varkappa'} = \varnothing$ *for every* $\varkappa, \varkappa' \in I$, $\varkappa \neq \varkappa'$, *and* $Q^*_{\varkappa} \cap Q^*_{\varkappa'} \neq \varnothing$ *if and only if* $Q_{\varkappa} \cap Q_{\varkappa'} \neq \varnothing$.

The family of cubes $W_S = \{Q_{\varkappa}\}_{\varkappa \in I} = \{Q(x_{\varkappa}, r_{\varkappa})\}_{\varkappa \in I}$, constructed in Lemma 3.2 is called a *Whitney decomposition* of the open set $\mathbb{R}^n \setminus S$, and the cubes $Q_{\varkappa}$ are called *Whitney cubes*. In what follows we also need the part of the Whitney decomposition comprised of the cubes of small edge length. More precisely, we set $\mathscr{W}_S = \{Q_{\varkappa}\}_{\varkappa \in \mathscr{I}}$, where $\mathscr{I} := \{\varkappa \in I \colon r_{\varkappa} \leqslant 1\}$.

The following notation is useful below. Given a closed set S, for every $\varkappa \in I$ set

$$b(Q_{\varkappa}) := b(\varkappa) := \{\varkappa' \in I \colon Q_{\varkappa} \cap Q_{\varkappa'} \neq \varnothing\} = \{\varkappa' \in I \colon Q^*_{\varkappa} \cap Q^*_{\varkappa'} \neq \varnothing\}. \tag{3.12}$$

We call a cube $Q_{\varkappa'}$ *neighbouring* to a cube $Q_{\varkappa}$ if $\varkappa' \in b(Q_{\varkappa})$. Similarly, set $b(x) := \{\varkappa \in I \colon Q^*_{\varkappa} \ni x\}$ for every $x \in \mathbb{R}^n \setminus S$.

To construct our extension operator we use the following (see [20], Ch. 6, § 1.3, for details).

Proposition 3.6. *Let* $S \subset \mathbb{R}^n$ *be a closed nonempty set and let* $\{Q_{\varkappa}\}_{\varkappa \in I}$ *be the Whitney decomposition of the open set* $\mathbb{R}^n \setminus S$ *constructed in Lemma* 3.2. *Then there exists a family of functions* $\{\varphi_{\varkappa}\}_{\varkappa \in I}$ *with the following properties*:

(1) $\varphi_{\varkappa} \in C_0^{\infty}(\mathbb{R}^n \setminus S)$ *for every* $\varkappa \in I$;

(2) $0 \leqslant \varphi_{\varkappa} \leqslant 1$ *and* $\operatorname{supp} \varphi_{\varkappa} \subset (Q_{\varkappa})^* := (9/8)Q_{\varkappa}$ *for every* $\varkappa \in I$;

(3) $\sum_{\varkappa \in I} \varphi_{\varkappa}(x) = 1$ *for all* $x \in \mathbb{R}^n \setminus S$;

(4) $\|D^{\alpha}\varphi_{\varkappa} | L_{\infty}(\mathbb{R}^n)\| \leqslant C(\operatorname{diam} Q_{\varkappa})^{-|\alpha|}$ *for every multi-index* $\alpha \in \mathbb{N}_0^n$ *and every* $\varkappa \in I$, *where the positive constant* C *depends only on* n.

Definition 3.2. Given a closed nonempty set $S \subset \mathbb{R}^n$ and $x \notin S$, we say that $\widetilde{x}$ is a *nearest point to* x or a *metric projection of* x onto S whenever $\operatorname{dist}(x, S) = \operatorname{dist}(x, \widetilde{x})$.

Remark 3.3. Let $\widetilde{x}$ be a metric projection of $x \in \mathbb{R}^n \setminus S$ onto S. Consider the line interval

$$[x, \widetilde{x}] := \{y = x + t(\widetilde{x} - x) \colon t \in [0, 1]\}.$$

Consider an arbitrary $r \in (0, \|x - \widetilde{x}\|)$ and a point $y_r = \partial Q(\widetilde{x}, r) \cap [x, \widetilde{x}]$. We show that $\operatorname{dist}(y_r, S) = \|y_r - \widetilde{x}\|_{\infty} = r$.

Clearly, $\operatorname{dist}(y_r, S) \leqslant r$ because $y_r \in \partial Q(\widetilde{x}, r)$. Assume that $\operatorname{dist}(y_r, S) < r$. Then there is a point $y' \in S$ such that $\|y_r - y'\|_{\infty} < r = \|y_r - \widetilde{x}\|_{\infty}$. Using this and the equality $\|x - \widetilde{x}\|_{\infty} = \|x - y_r\|_{\infty} + \|y_r - \widetilde{x}\|_{\infty}$ we obtain $\operatorname{dist}(x, S) \leqslant \|x - y'\|_{\infty} \leqslant \|x - y_r\|_{\infty} + \|y_r - y'\|_{\infty} < \|x - \widetilde{x}\|_{\infty}$. This contradicts the fact that $\|\widetilde{x} - x\|_{\infty} = \operatorname{dist}(x, S)$.

Definition 3.3. Fix a closed nonempty set S. For a cube $Q = Q(x, r) \subset \mathbb{R}^n$ with $x \notin S$ we call $\widetilde{Q} = Q(\widetilde{x}, r)$ a *reflected cube*, where $\widetilde{x}$ is a metric projection of x onto S.

Remark 3.4. Clearly, a metric projection onto a closed nonempty set exists. It is not unique in general. We will specify an algorithm for choosing $\widetilde{x}$ only when our constructions require this. Otherwise, given a cube $Q(x, r)$, we fix any metric projection $\widetilde{x}$ and the cube $\widetilde{Q}(\widetilde{x}, r)$.

Lemma 3.3. *Let $S \subset \mathbb{R}^n$ be a closed nonempty set and let $W_S = \{Q_\varkappa\}_{\varkappa \in I}$ be a Whitney decomposition of $\mathbb{R}^n \setminus S$. Then for every $c > 0$ there exists a positive constant $C = C(n,c)$ such that*

$$\sup_{r>0} \sup_{x \in \mathbb{R}^n} \sum_{\substack{\varkappa \in I \\ r_\varkappa = r}} \chi_{Q(\widetilde{x}_\varkappa, cr_\varkappa)}(x) \leqslant C(n,c).$$

Proof. Suppose that $Q(\widetilde{x}_\varkappa, cr_\varkappa) \cap Q(\widetilde{x}_{\varkappa'}, cr_{\varkappa'}) \neq \varnothing$ for some $\varkappa, \varkappa' \in I$ with $r_\varkappa = r_{\varkappa'}$. In view of (3.10) we have $\operatorname{dist}(Q_\varkappa, \widetilde{x}_\varkappa) \leqslant 4 \operatorname{diam}(Q_\varkappa)$ and $\operatorname{dist}(Q_{\varkappa'}, \widetilde{x}_{\varkappa'}) \leqslant 4 \operatorname{diam}(Q_{\varkappa'})$. Hence $\operatorname{dist}(Q_\varkappa, Q_{\varkappa'}) \leqslant (8+c) \operatorname{diam}(Q_\varkappa)$. This implies that $Q_{\varkappa'} \subset (19+2c)Q_\varkappa$. Then Lemma 3.2, (5), and arguments based on volume estimates give

$$\begin{aligned} \sup_{r>0} \sup_{x \in \mathbb{R}^n} \sum_{\substack{\varkappa \in I \\ r_\varkappa = r}} \chi_{Q(\widetilde{x}_\varkappa, cr_\varkappa)}(x) &\leqslant \sup_{\varkappa \in I} \operatorname{card}\{\varkappa' \in I \colon r_{\varkappa'} = r_\varkappa \text{ and } Q_{\varkappa'} \subset (19+2c)Q_\varkappa\} \\ &\leqslant \frac{\mathscr{H}^n((19+2c)Q_\varkappa)}{\mathscr{H}^n(Q_\varkappa)} = (19+2c)^n. \end{aligned}$$

The proof is complete.

Lemma 3.4. *Let $S \subset \mathbb{R}^n$ be a closed nonempty set and let $W_S = \{Q_\varkappa\}_{\varkappa \in I}$ be a Whitney decomposition of $\mathbb{R}^n \setminus S$. Let $\mathfrak{m}$ be a finite Borel measure with* $\operatorname{supp} \mathfrak{m} \subset S$. *Then for every $c \geqslant 1$*

$$\sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(Q(\widetilde{x}_\varkappa, r_\varkappa)) \mathfrak{m}(Q(\widetilde{x}_\varkappa, c)) \leqslant C \mathfrak{m}(S),$$

where the positive constant C depends only on c and n.

Proof. Consider the family of cubes $\{Q(\widetilde{x}_\varkappa, c)\}_{\varkappa \in \mathscr{I}}$. Using Vitali's covering theorem (see § 1.5 of [3] for details) we find an index set $\widehat{\mathscr{I}} \subset \mathscr{I}$ such that all cubes in the family $\{Q(\widetilde{x}_\varkappa, c)\}_{\varkappa \in \widehat{\mathscr{I}}}$ are mutually disjoint and

$$\bigcup_{\varkappa \in \mathscr{I}} Q(\widetilde{x}_\varkappa, c) \subset \bigcup_{\varkappa \in \widehat{\mathscr{I}}} Q(\widetilde{x}_\varkappa, 5c). \tag{3.13}$$

Note that if $Q(\widetilde{x}_{\varkappa'}, r_{\varkappa'}) \cap Q(\widetilde{x}_\varkappa, 5c) \neq \varnothing$ for some $\varkappa, \varkappa' \in \mathscr{I}$, then

$$Q(\widetilde{x}_{\varkappa'}, r_{\varkappa'}) \subset Q(\widetilde{x}_{\varkappa'}, c) \subset Q(\widetilde{x}_\varkappa, 7c), \tag{3.14}$$

because $c \geqslant 1$ and $r_{\varkappa'} \leqslant 1$. From this and (3.10) it follows that $Q_{\varkappa'} \subset Q(x_\varkappa, 20c)$. Hence, using Lemma 3.2, (5), we obtain

$$\begin{aligned} &\sum_{\substack{\varkappa' \in \mathscr{I} \\ Q(\widetilde{x}_{\varkappa'}, r_{\varkappa'}) \cap Q(\widetilde{x}_\varkappa, 5c) \neq \varnothing}} \mathscr{H}^n(Q(\widetilde{x}_{\varkappa'}, r_{\varkappa'})) \\ &\qquad \leqslant \sum_{Q_{\varkappa'} \subset Q(\widetilde{x}_\varkappa, 20c)} \mathscr{H}^n(Q(\widetilde{x}_{\varkappa'}, r_{\varkappa'})) \leqslant \mathscr{H}^n(Q(x_\varkappa, 20c)) \leqslant (20c)^n. \end{aligned}$$

Using this fact, (3.13), (3.14) and Propositions 3.4 and 3.5 we obtain the required estimate

$$\begin{aligned}&\sum_{\varkappa\in\mathscr{I}}\mathscr{H}^n(Q(\widetilde{x}_\varkappa,r_\varkappa))\mathfrak{m}(Q(\widetilde{x}_\varkappa,c))\\&\qquad\leqslant\sum_{\varkappa\in\widehat{\mathscr{I}}}\sum_{\substack{\varkappa'\in\mathscr{I}\\Q(\widetilde{x}_{\varkappa'},r_{\varkappa'})\cap Q(\widetilde{x}_\varkappa,5c)\neq\varnothing}}\mathscr{H}^n(Q(\widetilde{x}_{\varkappa'},r_{\varkappa'}))\mathfrak{m}(Q(\widetilde{x}_{\varkappa'},c))\\&\qquad\leqslant\sum_{\varkappa\in\widehat{\mathscr{I}}}\mathfrak{m}(Q(\widetilde{x}_\varkappa,7c))\sum_{\substack{\varkappa'\in\mathscr{I}\\Q(\widetilde{x}_{\varkappa'},r_\varkappa)\cap Q(\widetilde{x}_\varkappa,5c)\neq\varnothing}}\mathscr{H}^n(Q(\widetilde{x}_{\varkappa'},r_{\varkappa'}))\\&\qquad\leqslant(20c)^n\sum_{\varkappa\in\widehat{\mathscr{I}}}\mathfrak{m}(Q(\widetilde{x}_\varkappa,7c))\leqslant C(c,n)\mathfrak{m}(S).\end{aligned}$$

The proof is complete.

Recall Definition 2.1 and Remark 2.1. The following result is a minor modification of Theorem 2.4 in [23] and can be proved analogously.

Theorem C. *Let S be a closed Ahlfors n-regular set in $\mathbb{R}^n$, and let $W_S=\{Q_\varkappa\}_{\varkappa\in I}$ be a Whitney decomposition of $\mathbb{R}^n\setminus S$. Then there exists a family $\mathfrak{U}:=\{\mathscr{U}_\varkappa\colon \varkappa\in\mathscr{I}\}$ of Borel sets with the following properties*:

(1) $\mathscr{U}_\varkappa\subset(10Q_\varkappa)\cap S$ *for all* $\varkappa\in\mathscr{I}$;

(2) $\mathscr{H}^n(Q_\varkappa)\leqslant\kappa_1\mathscr{H}^n(\mathscr{U}_\varkappa)$ *for all* $\varkappa\in\mathscr{I}$;

(3) $\sum_{\varkappa\in\mathscr{I}}\chi_{\mathscr{U}_\varkappa}(x)\leqslant\kappa_2$ *for* $x\in S$.

The positive constants κ_1 and κ_2 depend only on n, and the constants $c_1^n(S)$ and $c_2^n(S)$.

3.4. *d*-regular sequences of measures. The following result is a version of Frostman's theorem adapted for our purposes (cf. Theorem 5.1.12 in [26]). For the reader's convenience we present a detailed proof.

Theorem 3.1. *Let S be a closed nonempty subset of $\mathbb{R}^n$. Then given $d\in[0,n]$, there exists a sequence of Borel measures $\{\nu_k\}_{k\in\mathbb{N}_0}$ with* $\operatorname{supp}\nu_k=S$, $k\in\mathbb{N}_0$, *such that for every $k\in\mathbb{N}_0$ the following properties hold*:

(1)
$$\nu_k(Q(x,r))\leqslant 15^n r^d,\qquad x\in\mathbb{R}^n,\quad r\in(0,2^k];\tag{3.15}$$

(2) *for every finite index set $\mathscr{A}\subset\mathbb{Z}^n$ let $V^k_{\mathscr{A}}:=\bigcup_{m\in\mathscr{A}}Q_{k,m}$; then*

$$\nu_k(\overline{V^k_{\mathscr{A}}\cap S})\geqslant\mathscr{H}^d_\infty(V^k_{\mathscr{A}}\cap S);\tag{3.16}$$

(3) *there exists a function $\gamma_k\in L_\infty(S,\nu_0)$ such that $\nu_k=\gamma_k\nu_0$ and*

$$2^{d-n}\gamma_{k+1}(x)\leqslant\gamma_k(x)\leqslant\gamma_{k+1}(x),\qquad \nu_0\text{-a.e. } x\in S.\tag{3.17}$$

Proof. Fix a nonnegative integer k and let $\nu^{k,0}:=\nu^k$ be a measure with constant density that has mass 2^{-kd} on each $Q_{k,m}$ that intersects S. We now modify ν^k in the following way. If $\nu^k(Q_{k-1,m})>2^{-(k-1)d}$ for some $Q_{k-1,m}\in\mathscr{Q}_{k-1}$, we reduce its mass uniformly on $Q_{k-1,m}$ until it becomes $2^{-(k-1)d}$. On the other hand, if $\nu^k(Q_{k-1,m}) \leqslant 2^{-(k-1)d}$, we leave ν^k unchanged on $Q_{k-1,m}$. In this way we obtain a new measure $\nu^{k,1}$. Using the fact that every cube $Q_{k-1,m}$ which has nonempty intersection with S contains $\leqslant 2^n$ cubes $Q_{k,m'}$ with the property $Q_{k,m'} \cap S \neq \varnothing$ we have

$$\nu^{k,1}(Q_{k,m}) \leqslant \nu^{k,0}(Q_{k,m}) \leqslant 2^{n-d}\nu^{k,1}(Q_{k,m}).$$

We repeat this procedure for $\nu^{k,1}$, obtaining $\nu^{k,2}$, and after k steps we obtain $\nu^{k,k}$. It follows from this construction that

$$\nu^{k,k-j}(Q_{i,m}) \leqslant 2^{-id} \tag{3.18}$$

for every $j = 0, 1, \dots, k$ and every dyadic cube $Q_{i,m} \in \mathscr{Q}_i$, where $i = j, \dots, k$. Furthermore, it is clear that

$$\nu^{k,j+1}(Q_{k,m}) \leqslant \nu^{k,j}(Q_{k,m}) \leqslant 2^{n-d}\nu^{k,j+1}(Q_{k,m}), \qquad j = 0, 1, \dots, k-1. \tag{3.19}$$

Using (3.18) it is easy to see that for every $j \in \mathbb{N}_0$ the sequence $\{\nu^{k,k-j}(E)\}_{k\geqslant j}$ is bounded for every compact subset E of S. Then $\{\nu^{k,k-j}\}_{k\geqslant j}$ has a subsequence that converges weakly to ν_j (see [31], § 1.9, Theorem 2), and clearly $\operatorname{supp}\nu_j \subset S$ (recall that S is closed).

Fix an arbitrary $j \in \mathbb{N}$ and an arbitrary Borel set $G \subset S$. We compare $\nu_j(G)$ and $\nu_{j-1}(G)$. First note that, according to our construction, for every dyadic cube $Q_{k,m}$ we have

$$\nu^{k,k-j+1}(Q_{k,m}) \leqslant \nu^{k,k-j}(Q_{k,m}) \leqslant 2^{n-d}\nu^{k,k-j+1}(Q_{k,m}), \qquad k \geqslant j.$$

Let $C_c(\mathbb{R}^n)$ be the set of continuous functions $f\colon \mathbb{R}^n \to \mathbb{R}$ with compact support. For every nonnegative function $f \in C_c(\mathbb{R}^n)$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\,d\nu^{k,k-j+1}(x) &\leqslant \int_{\mathbb{R}^n} f(x)\,d\nu^{k,k-j}(x)\\ &\leqslant 2^{n-d}\int_{\mathbb{R}^n} f(x)\,d\nu^{k-j+1}(x), \qquad k \geqslant j.\end{aligned} \tag{3.20}$$

Fix an arbitrary nonnegative $f \in C_c(\mathbb{R}^n)$. Choosing an appropriate subsequence if necessary and passing to the limit in (3.20) we obtain

$$\int_{\mathbb{R}^n} f(x)\,d\nu_{j-1}(x) \leqslant \int_{\mathbb{R}^n} f(x)\,d\nu_j(x) \leqslant 2^{n-d}\int_{\mathbb{R}^n} f(x)\,d\nu_{j-1}(x), \qquad j \in \mathbb{N}. \tag{3.21}$$

Using the Borel regularity of the measures ν_j and the Radon-Nikodym theorem and taking (3.21) into account we obtain (3.17).

We show that $\nu_j(Q_{i,m}) \leqslant 3^n 2^{-id}$ for every $i, j \in \mathbb{N}_0$, $i \geqslant j$, and every dyadic cube $Q_{i,m} \in \mathscr{Q}_i$. Indeed, if $f_{i,m} \in C_0^\infty(\mathbb{R}^n)$ is such that $\chi_{Q_{i,m}} \leqslant f_{i,m} \leqslant \chi_{3Q_{i,m}}$, then (3.18) yields

$$\nu_j(Q_{i,m}) \leqslant \int_{\mathbb{R}^n} f_{i,m}(x)\,d\nu_j(x) = \lim_{l\to\infty}\int_{\mathbb{R}^n} f_{i,m}(x)\,d\nu^{k_l,k_l-j}(x) \leqslant 3^n 2^{-id}.$$

Hence, using the fact that every closed cube $Q(x,r)$ with $x \in \mathbb{R}^n$ and $r \in (0, 2^{-k}]$ has nonempty intersection with $\leqslant 5^n$ dyadic cubes $Q_{k(r),m}$, where $k(r) := |[\log_2 r]|$, we obtain (3.15).

Fix an arbitrary nonempty index set $\mathscr{A} \subset \mathbb{Z}^n$ and $k \in \mathbb{N}_0$ and fix an arbitrary $l \in \mathbb{N}$, $l \geqslant k$. The key observation, which follows directly from our construction, is that every $x \in V^k_{\mathscr{A}} \cap S$ belongs to some dyadic cube $Q^{(j)} \in \mathscr{Q}_{n_j}$, $k \leqslant n_j \leqslant l$, (or several cubes) such that $\nu^{l,l-k}(Q^{(j)}) = 2^{-n_j d}$. We can choose a disjoint covering consisting of maximal dyadic cubes with this property, so that $S \cap V^k_{\mathscr{A}} \subset \bigcup_j Q^{(j)}$. This gives

$$\nu^{l,l-k}(V^k_{\mathscr{A}} \cap S) = \sum_j \nu^{l,l-k}(Q^{(j)}) = \sum_j 2^{-n_j d} \geqslant \inf \sum_i 2^{-n_i d},$$

where the infimum is taken over all finite or countable coverings of $V^k_{\mathscr{A}} \cap S$ with dyadic cubes $Q^{(i)} \in \bigcup_{l \geqslant k} \mathscr{Q}_l$. The right-hand side is independent of l. Combining this with the definition of the d-Hausdorff content we note that $\overline{V^k_{\mathscr{A}}}$ is a compact set. This gives

$$\nu_k(\overline{V^k_{\mathscr{A}} \cap S}) \geqslant \varlimsup_{s\to\infty} \nu^{l_s,l_s-k}(V^k_{\mathscr{A}} \cap S) \geqslant \inf \sum_i 2^{-n_i d} \geqslant \mathscr{H}^d_\infty(V^k_{\mathscr{A}} \cap S). \tag{3.22}$$

This completes the proof.

The following result shows that Definition 2.3 is consistent.

Corollary 3.1. *Let $d \in [0,n]$ and let $S \subset \mathbb{R}^n$ be a d-thick closed set. Then there exists a d-regular sequence of measures on S.*

Proof. We apply Theorem 3.1 to the set S. This gives a sequence of Borel measures $\{\nu_k\}_{k\in\mathbb{N}_0}$ with $\operatorname{supp}\nu_k = S$ satisfying (3.15)–(3.17). We set $\mu_k := 15^{-n}\nu_k$ for every $k \in \mathbb{N}_0$.

It is sufficient to verify (2.11). Fix some $x \in S$, and let $\mathscr{A} \subset \mathbb{Z}^n$ be the index set such that $m \in \mathscr{A}$ if and only if $Q_{k+2,m} \cap Q(x, 2^{-k-2}) \neq \varnothing$. It is clear that $\overline{V^{k+2}_{\mathscr{A}}} := \bigcup_{m\in\mathscr{A}} \overline{Q}_{k+2,m} \subset Q(x, 2^{-k})$ and $Q(x, 2^{-k-2}) \subset V^{k+2}_{\mathscr{A}}$. Hence, using Definition 2.2, estimates (3.16) and (3.17) and the monotonicity of the $\mathscr{H}^d_\infty$-content we obtain

$$\begin{aligned} 2^{2(n-d)}\mu_k(Q(x,2^{-k})) &\geqslant \mu_{k+2}(Q(x,2^{-k})) \geqslant \mu_{k+2}(\overline{V^{k+2}_{\mathscr{A}}} \cap Q(x, 2^{-k-2})) \\ &\geqslant 15^{-n}\mathscr{H}^d_\infty(V^{k+2}_{\mathscr{A}} \cap S) \geqslant 15^{-n}\mathscr{H}^d_\infty(Q(x,2^{-k-2}) \cap S) \geqslant 15^{-n}c_3^d(S)4^{-d}2^{-kd}. \end{aligned}$$

This completes the verification of the corollary.

The following lemma gives some asymptotic estimates for measures in a fixed d-regular sequence. Recall Remark 2.2.

Lemma 3.5. *Let $d \in [0,n]$ and let S be a d-thick closed set. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Then for each $c \geqslant 1$ and every $k > \log_2 c$*

$$\frac{C_{\{\mu_k\}}}{(2c)^n}\mu_k(Q(x,2^{-k})) \leqslant \mu_k\biggl(Q\biggl(x, \frac{2^{-k}}{c}\biggr)\biggr) \leqslant \mu_k(Q(x, c2^{-k})) \leqslant \frac{(2c)^n}{C_{\{\mu_k\}}}\mu_k(Q(x,2^{-k})). \tag{3.23}$$

Proof. Fix $j \in \mathbb{N}_0$ such that $c \in [2^j, 2^{j+1})$. It follows from estimates (2.10)–(2.12) that for every $k > \log_2 c$ and every $x \in S$

$$
\begin{aligned}
\mu_k\biggl(Q\biggl(x, \frac{2^{-k}}{c}\biggr)\biggr) &\geqslant 2^{(j+1)(d-n)}\mu_{k+j+1}\biggl(Q\biggl(x, \frac{2^{-k}}{c}\biggr)\biggr)\\
&\geqslant 2^{(d-n)(j+1)}\mu_{k+j+1}\biggl(Q\biggl(x, \frac{2^{-k}}{2^{j+1}}\biggr)\biggr)\\
&\geqslant \frac{C}{2^{n(j+1)}}2^{-dk} \geqslant \frac{C_{\{\mu_k\}}}{(2c)^n}\mu_k(Q(x, 2^{-k})).
\end{aligned}
\tag{3.24}
$$

Similarly,

$$
\begin{aligned}
&\mu_k(Q(x, c2^{-k})) \leqslant 2^{(j+1)(n-d)}\mu_{k-j-1}(Q(x, 2^{j+1-k})) \leqslant 2^{(j+1)(n-d)}2^{(j+1-k)d}\\
&\leqslant \frac{(2c)^n}{C_{\{\mu_k\}}}\mu_k(Q(x, 2^{-k})).
\end{aligned}
\tag{3.25}
$$

The required estimate (3.23) follows from (3.24) and (3.25), which completes the proof.

Remark 3.5. Recall that a Borel measure μ on a metric space (X, d) is called a *doubling measure* if there exists a constant $C_\mu \geqslant 1$ such that $\mu(B(x, 2r)) \leqslant C_\mu \mu(B(x, r))$ for all $x \in X$ and $r > 0$. It is very important to note that the estimate (3.23) does not imply the doubling property of the measures μ_k, $k \in \mathbb{N}_0$. Roughly speaking, the point is that, given $k \in \mathbb{N}_0$, in Lemma 3.5 we compare $\mu_k(Q(x, r))$, where $x \in S$ and $r \in (0, 1]$, only with $\mu_k(Q(x, 2^{-k}))$. If we try to compare $\mu_k(Q(x, r))$ and $\mu_k(Q(x, r/2))$ for $k \gg |\log_2 r|$, then we obtain a bad estimate, with the corresponding positive constant C depending heavily on k.

Lemma 3.6. *Let $d \in [0, n]$ and let S be a d-thick closed set. Let $\{\mu_k\}_{k \in \mathbb{N}_0}$ be a d-regular sequence of measures on S. Let E be a Borel subset of S. If $\mathscr{H}^d(E) = 0$, then $\mu_k(E) = 0$ for every $k \in \mathbb{N}_0$. The converse is false.*

Proof. Fix $E \subset S$ with $\mathscr{H}^d(E) = 0$. Using (2.10) and the definition of Hausdorff measure it is easy to see that $\mu_k(E) = 0$ for every $k \in \mathbb{N}_0$.

To prove that the converse is false we use the construction from Example 6.3 below. More precisely, in Example 6.3 we build a 1-thick path-connected set $S \subset \mathbb{R}^n$ with $\dim_H S = n$ and a 1-regular sequence of measures $\{\mu_k\}_{k \in \mathbb{N}_0}$ on S such that every μ_k is absolutely continuous with respect to $\mathscr{H}^n$. Hence for every smooth curve $\gamma \in S$ with $\mathscr{H}^1(\gamma) > 0$ we obtain $\mu_k(\gamma) = 0$ for all $k \in \mathbb{N}$. The proof is complete.

Recall that for every $r > 0$ we set $k(r) := |[\log_2 r]|$. The following theorem will be an important technical tool in the sequel. Recall Remark 2.2.

Theorem 3.2. *Let $d \in [0, n]$ and let S be a d-thick closed set. Let $\{\mu_k\}_{k \in \mathbb{N}_0}$ be a d-regular sequence of measures on S. Then for every $r \in (0, 1)$, $x \in S$ and every Borel set $G \subset Q(x, r) \cap S$*

$$
\frac{\mathscr{H}^n(G)}{\mathscr{H}^n(Q(x, r))} \leqslant C\frac{\mu_{k(r)}(G)}{\mu_{k(r)}(Q(x, r) \cap S)}.
\tag{3.26}
$$

The positive constant C depends only on n and $C_{\{\mu_k\}}$ in Remark 2.2.

Proof. Fix $x, y \in S$, and t and r, $0 < t < r < 1$, such that $Q(y,t) \subset Q(x,r)$. It is clear that

$$\begin{aligned}\frac{\mathscr{H}^n(Q(y,t)\cap S)}{\mathscr{H}^n(Q(x,r))} &\leqslant \frac{\mathscr{H}^n(Q(y,t)\cap S)}{\mathscr{H}^n(Q(y,t))}\,\frac{\mathscr{H}^n(Q(y,t))}{\mathscr{H}^n(Q(x,r))}\\ &\leqslant 2^{n(k(r)-k(t)+1)}\,\frac{\mathscr{H}^n(Q(y,t)\cap S)}{\mathscr{H}^n(Q(y,t))}.\end{aligned} \tag{3.27}$$

On the other hand, using (2.10)–(2.12) (we can use these estimates because $x, y \in S$) we have

$$\frac{\mu_{k(r)}(Q(y,t)\cap S)}{\mu_{k(r)}(Q(x,r)\cap S)} \geqslant 2^{(d-n)(k(t)-k(r))}\,\frac{\mu_{k(t)}(Q(y,t)\cap S)}{\mu_{k(r)}(Q(x,r)\cap S)} \geqslant C_{\{\mu_k\}} 2^{n(k(r)-k(t))}. \tag{3.28}$$

Combining (3.27) and (3.28) we obtain

$$\begin{aligned}\frac{\mathscr{H}^n(Q(y,t)\cap S)}{\mathscr{H}^n(Q(x,r))} &\leqslant \frac{2^n}{C_{\{\mu_k\}}}\,\frac{\mu_{k(r)}(Q(y,t)\cap S)}{\mu_{k(r)}(Q(x,r)\cap S)}\,\frac{\mathscr{H}^n(Q(y,t)\cap S)}{\mathscr{H}^n(Q(y,t))}\\ &\leqslant \frac{2^n}{C_{\{\mu_k\}}}\,\frac{\mu_{k(r)}(Q(y,t)\cap S)}{\mu_{k(r)}(Q(x,r)\cap S)}.\end{aligned} \tag{3.29}$$

Hence we obtain (3.26) for $G = Q(y,t)$.

Fix $r \in (0,1)$. For every $j \in \mathbb{N}$, $j > j_0 := [2r/(1-r)]+1$, let $\{x_i^j\}_{i=1}^{N(j)}$ be a maximal (r/j)-separated subset of $Q(x,r)\cap S$. Clearly, $Q(x,r)\cap S \subset \bigcup_i \operatorname{int} Q(x_i^j, 2r/j)$ and the cubes $Q(x_i^j, r/(2j))$ are pairwise disjoint. For every $j \in \mathbb{N}$ take an arbitrary nonempty set $\mathscr{A}^j \subset \{1,\dots,N(j)\}$ and consider the set $U_j := \bigcup_{i\in\mathscr{A}^j} \operatorname{int} Q(x_i^j, 2r/j)$. It is clear that $U_j \subset Q(x, r+2r/j_0) \subset Q(x,1)$ for every $j > j_0$. We use this inclusion, (3.29), Propositions 3.4 and 3.5, and (2.12); then for every $j > j_0$ we obtain

$$\begin{aligned}\mathscr{H}^n(U_j\cap S) &\leqslant \sum_{i\in\mathscr{A}^j} \mathscr{H}^n\Big(Q\Big(x_i^j, \frac{2r}{j}\Big)\cap S\Big)\\ &\leqslant C\mathscr{H}^n(Q(x,r)) \sum_{i\in\mathscr{A}^j} \frac{\mu_{k(r)}(Q(x_i^j, 2r/j)\cap S)}{\mu_{k(r)}(Q(x,r)\cap S)}\\ &\leqslant C\mathscr{H}^n(Q(x,r))\frac{\mu_{k(r)}(U_j\cap S)}{\mu_{k(r)}(Q(x,r)\cap S)} \leqslant C\mathscr{H}^n(Q(x,r))\frac{\mu_{k(r)}(U_j\cap S)}{\mu_{k(r)}(Q(x,r)\cap S)}.\end{aligned} \tag{3.30}$$

Fix an arbitrary compact set $K \subset Q(x,r)\cap S$. Now using the σ-additivity of the measures $\mathscr{H}^n$ and $\mu_{k(r)}$ we have

$$\mathscr{H}^n(K) = \lim_{j\to\infty} \mathscr{H}^n(U_j\cap S) \quad \text{and} \quad \mu_{k(r)}(K) = \lim_{j\to\infty} \mu_{k(r)}(U_j\cap S). \tag{3.31}$$

Combining (3.31) and (3.30), we obtain (3.26) for every compact set $K \subset Q(x,r)\cap S$. To establish (3.26) for a general Borel set $G \subset S$ it remains to recall that the measures μ_k, $k \in \mathbb{N}$, and $\mathscr{H}^n$ are Radon measures. The proof is complete.

Corollary 3.2. *Let $d \in [0,n]$ and let S be a d-thick closed set. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Assume that $f \in L_1^{\mathrm{loc}}(S,\mu_k)$ for some (and hence every) $k \in \mathbb{N}_0$. Then there exists $C > 0$ (independent of f, x and r) such that*

$$\frac{1}{\mathscr{H}^n(Q(x,r))}\int_{Q(x,r)\cap S}|f(y)|\,d\mathscr{H}^n(y) \leqslant C \fint_{Q(x,r)\cap S}|f(y)|\,d\mu_{k(r)}(y) \quad \text{for all } x\in S, r\in(0,1]. \tag{3.32}$$

Proof. The estimate (3.32) clearly holds for a simple function $f\colon S \to \mathbb{R}$ due to Theorem 3.2. In the general case we have to construct an increasing sequence of simple functions converging to $|f|$ and use the monotone convergence theorem for integrals (see § 1.3 of [31]). This completes the verification of the corollary.

Lemma 3.7. *Let $d \in [0,n]$ and let S be a d-thick closed set. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Assume that $g \in L_1^{\mathrm{loc}}(\mathbb{R}^n,\mu_k)$ for some (and hence every) $k \in \mathbb{N}_0$. Let $c \geqslant 1$, $x \in S$, $x' \in \mathbb{R}^n$ and $r \in (0,1)$ satisfy $cr \in (0,1)$ and $Q(x,r) \subset Q(x',cr)$. Then*

$$\begin{aligned}\fint_{Q(x,r)}|g(z)|\,d\mu_{k(r)}(z) &\leqslant C\fint_{Q(x',cr)}|g(z)|\,d\mu_{k(r)}(z),\\ \fint_{Q(x,r)}|g(z)|\,d\mu_{k(r)}(z) &\approx \fint_{Q(x,r)}|g(z)|\,d\mu_{k(r)\pm 1}(z),\end{aligned} \tag{3.33}$$

where the positive constant C does not depend on x, x', r or g.

Proof. Clearly, $Q(x,r) \subset Q(x',cr) \subset Q(x,2cr)$. Suppose $cr \in (0,1/2)$. Using this we obtain $\mu_{k(r)}(Q(x',cr)) \leqslant \mu_{k(r)}(Q(x,2cr)) \leqslant C^{-1}_{\{\mu_k\}}(4c)^n\mu_{k(r)}(Q(x,r))$ by Lemma 3.5, which obviously implies the first inequality in (3.33). When $cr \in (1/2,1)$ the corresponding estimate easily follows from (2.11) and (2.12).

The second estimate in (3.33) follows immediately from (2.10)–(2.12). The proof is complete.

3.5. Calderón-type maximal functions. Recall Definition 2.5 and also that $k(r) := -[\log_2 r]$ for every $r > 0$.

Lemma 3.8. *Let $d \in [0,n]$ and let S be a d-thick closed set. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Assume that $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n,\mu_k)$ for every $k \in \mathbb{N}_0$. Let $c \geqslant 1$, $x \in S$, $x' \in \mathbb{R}^n$ and $r \in (0,1)$ be such that $Q(x,r) \subset Q(x',cr)$. Then*

$$f^{\sharp}_{\{\mu_k\}}(x,r) \leqslant C\biggl(f^{\sharp}_{\{\mu_k\}}(x',r) + \fint_{Q(x',c)}|f(y)|\,d\mu_0(y)\biggr). \tag{3.34}$$

The positive constant C in (3.34) depends on the constant c but does not depend on x, x', r or f.

Proof. Fix $t \in (r,1]$. If $tc < 1$, then we use the inclusion $Q(x,t) \subset Q(x',ct)$, Lemma 3.7, Remark 2.4 and the monotonicity of $f^{\sharp}_S(x',t)$ with respect to t.

We obtain

$$\begin{aligned}
\mathscr{E}_{\mu_{k(t)}}(f,Q(x,t)) &\leqslant \fint_{Q(x,t)} \Big| f(y) - \fint_{Q(x,t)} f(z)\,d\mu_{k(t)}(z)\Big|\,d\mu_{k(t)}(y)\\
&\leqslant \fint_{Q(x,t)} \fint_{Q(x,t)} |f(y)-f(z)|\,d\mu_{k(t)}(z)\,d\mu_{k(t)}(y)\\
&\leqslant C \fint_{Q(x',ct)} \fint_{Q(x',ct)} \Big| f(y) \pm \fint_{Q(x',ct)} f(z')\,d\mu_{k(ct)}(z') - f(z)\Big|\,d\mu_{k(ct)}(z)\,d\mu_{k(ct)}(y)\\
&\leqslant C \fint_{Q(x',ct)} \Big| f(y) - \fint_{Q(x',ct)} f(z)\,d\mu_{k(ct)}(z)\Big|\,d\mu_{k(ct)}(y)\\
&\leqslant Ctf^{\sharp}_{\{\mu_k\}}(x',cr) \leqslant Ctf^{\sharp}_{\{\mu_k\}}(x',r). \qquad (3.35)
\end{aligned}$$

Now consider the case $tc \geqslant 1$. We use Remark 2.4 and Lemma 3.7 and note that $k(t) \leqslant |[\log_2 c]| + 1$. This gives

$$\begin{aligned}
\frac{1}{t}\mathscr{E}_{\mu_{k(t)}}(f,Q(x,t)) &\leqslant \frac{1}{t} \fint_{Q(x,t)} \Big| f(y) - \fint_{Q(x,t)} f(z)\,d\mu_{k(t)}(z)\Big|\,d\mu_{k(t)}(y)\\
&\leqslant 2c \fint_{Q(x,t)} |f(y)|\,d\mu_{k(t)}(y) \leqslant C \fint_{Q(x',c)} |f(y)|\,d\mu_{k(t)}(y)\\
&\leqslant C \fint_{Q(x',c)} |f(y)|\,d\mu_0(y). \qquad (3.36)
\end{aligned}$$

Now (3.34) follows directly from Definition 2.5 and estimates (3.35) and (3.36). The proof is complete.

Lemma 3.9. *Let $d \in [0,n]$ and let S be a d-thick closed set. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Assume that $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n,\mu_k)$ for some (and hence every) $k \in \mathbb{N}_0$. Let $0 < r < t < 1$, $x \in S$ and $x' \in \mathbb{R}^n$ be such that $Q(x,r) \subset Q(x',t)$ and $R = \|x-x'\| + t < 1$. Then*

$$\begin{aligned}
&\Big| \fint_{Q(x,r)} f(y)\,d\mu_{k(r)}(y) - \fint_{Q(x',t)} f(z)\,d\mu_{k(t)}(z)\Big|\\
&\quad\leqslant C\min\{\mathscr{E}_{\mu_{k(R)}}(f,Q(x',R)),\mathscr{E}_{\mu_{k(R)}}(f,Q(x,R))\} + Ct \sup_{t'\in(r,t]} \frac{1}{t'}\mathscr{E}_{\mu_{k(t')}}(f,Q(x,t')),
\end{aligned} \qquad (3.37)$$

where the positive constant $C > 0$ does not depend on x, x', r, t or f.

Proof. Clearly, $Q(x,t) \subset Q(x',R) =: Q'$ and $Q(x',t) \subset Q(x,R) = Q$. Since $Q(x,r) \subset Q(x',t)$ we have $\|x-x'\| + r < t$ and hence $R \leqslant 2t$. Arguing as in (3.35) and using Remark 2.4 we obtain

$$\begin{aligned}
&\Big| \fint_{Q(x,t)} f(y)\,d\mu_{k(t)}(y) - \fint_{Q(x',t)} f(z)\,d\mu_{k(t)}(z)\Big|\\
&\qquad\leqslant \fint_{Q(x,t)} \fint_{Q(x',t)} |f(y)-f(z)|\,d\mu_{k(t)}(y)\,d\mu_{k(t)}(z)\\
&\qquad\leqslant C \fint_{Q'} \Big| f(y) - \fint_{Q'} f(z)\,d\mu_{k(R)}(z)\Big|\,d\mu_{k(R)}(y) \leqslant C\mathscr{E}_{\mu_{k(R)}}(f,Q')
\end{aligned}$$

(note that $R \leqslant 2t$). Clearly, similar inequalities hold true with Q' replaced by Q. Hence, we get

$$\left| \fint_{Q(x,t)} f(y)\, d\mu_{k(t)}(y) - \fint_{Q(x',t)} f(z)\, d\mu_{k(t)}(z) \right| \leqslant C \min\{\mathscr{E}_{\mu_{k(R)}}(f, Q'), \mathscr{E}_{\mu_{k(R)}}(f, Q)\}. \tag{3.38}$$

Let $j_0 := [\log_2(t/r)]$. Arguing as in (3.38) and using Remark 2.4 we have

$$\begin{aligned}
&\left| \fint_{Q(x,r)} f(y)\, d\mu_{k(r)}(y) - \fint_{Q(x,t)} f(z)\, d\mu_{k(t)}(z) \right| \\
&\quad \leqslant \left| \fint_{Q(x,r)} f(y)\, d\mu_{k(r)}(y) - \fint_{Q(x,t/2^{j_0})} f(z)\, d\mu_{k(t/2^{j_0})}(z) \right| \\
&\qquad + \sum_{j=0}^{j_0-1} \frac{t}{2^j} \frac{2^j}{t} \left| \fint_{Q(x_0,t/2^j)} f(z)\, d\mu_{k(t/2^j)}(z) - \fint_{Q(x_0,t/2^{j+1})} f(z')\, d\mu_{k(t/2^{j+1})}(z') \right| \\
&\quad \leqslant \frac{t}{2^{j_0}} C \frac{2^{j_0}}{t} \mathscr{E}_{\mu_{k(t/2^{j_0})}}\left(f, Q\left(x, \frac{t}{2^{j_0}}\right)\right) + C \sum_{j=0}^{j_0} \frac{t}{2^j} \left(\frac{2^j}{t} \mathscr{E}_{\mu_{k(t/2^j)}}\left(f, Q\left(x, \frac{t}{2^j}\right)\right)\right) \\
&\quad \leqslant Ct \sup_{t' \in (r,t]} \frac{1}{t'} \mathscr{E}_{\mu_{k(t')}}(f, Q(x,t')).
\end{aligned} \tag{3.39}$$

Combining (3.38) and (3.39) we obtain (3.37), which completes the proof.

Theorem 3.3. *Let $d \in [0,n]$ and let S be a d-thick closed set. Let $\{\mu_k\}_{k \in \mathbb{N}_0}$ be a d-regular sequence of measures on S. Let $f \in L_1^{\mathrm{loc}}(S, \mu_k)$ for some (and hence every) $k \in \mathbb{N}_0$. Then there exists a positive constant C such that for any $x, x' \in S$, $r, t \in (0,1)$ with $\widetilde{R} := \max\{r,t\} + \|x - x'\| < 1/2$*

$$\left| \fint_{Q(x,r)} f(z)\, d\mu_{k(r)}(z) - \fint_{Q(x',t)} f(z')\, d\mu_{k(r)}(z') \right| \leqslant C\widetilde{R}\big(f^{\sharp}_{\{\mu_k\}}(x,r) + f^{\sharp}_{\{\mu_k\}}(x',t)\big). \tag{3.40}$$

Proof. Note that $Q(x,r) \subset Q(x',\widetilde{R})$ and $Q(x',t) \subset Q(x',\widetilde{R})$. Using the triangle inequality, (3.37) and the monotonicity of $f^{\sharp}_{\{\mu_k\}}(x,t)$ with respect to t we see that the left-hand side of (3.40) is bounded above by

$$\begin{aligned}
&\left| \fint_{Q(x,r)} f(z)\, d\mu_{k(r)}(z) - \fint_{Q(x',\widetilde{R})} f(z')\, d\mu_{k(\widetilde{R})}(z') \right| \\
&\qquad + \left| \fint_{Q(x',\widetilde{R})} f(z)\, d\mu_{k(\widetilde{R})}(z) - \fint_{Q(x',t)} f(z')\, d\mu_{k(r)}(z') \right| \\
&\quad \leqslant CR\left(\frac{1}{R} \min\{\mathscr{E}_{\mu_{k(R)}}(f, Q(x,R)), \mathscr{E}_{\mu_{k(R)}}(f, Q(x',R))\}\right) \\
&\qquad + C\widetilde{R} f^{\sharp}_{\{\mu_k\}}(x,r) + C\widetilde{R} f^{\sharp}_{\{\mu_k\}}(x',t) \leqslant C\widetilde{R}\big(f^{\sharp}_{\{\mu_k\}}(x,r) + f^{\sharp}_{\{\mu_k\}}(x',t)\big);
\end{aligned} \tag{3.41}$$

here we have set $R = \widetilde{R} + \|x - x'\| \leqslant 2\widetilde{R} < 1$. This completes the proof.

Corollary 3.3. *Let $d \in [0,n]$ and let S be a d-thick closed set. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Let $f \in L_1^{\mathrm{loc}}(S,\mu_k)$ for some (and hence every) $k \in \mathbb{N}_0$. Let $x, x' \in S$ be such that $\|x - x'\| < 1/2$ and let*

$$f(x) = \lim_{r\to 0} \fint_{Q(x,r)} f(z)\, d\mu_{k(r)}(z) \quad \textit{and} \quad f(x') = \lim_{r\to 0} \fint_{Q(x',r)} f(z')\, d\mu_{k(r)}(z'). \tag{3.42}$$

Then there exists $C > 0$ (independent of x, x' and f) such that

$$|f(x) - f(x')| \leqslant C\|x - x'\|\big(f^{\sharp}_{\{\mu_k\}}(x) + f^{\sharp}_{\{\mu_k\}}(x')\big). \tag{3.43}$$

Proof. From Theorem 3.3, using the monotonicity of $f^{\sharp}_{\{\mu_k\}}(x,t)$ with respect to t we obtain the required estimate

$$\begin{aligned} |f(x) - f(x')| &\leqslant \varlimsup_{r\to 0} \Big| f(x) - \fint_{Q(x,r)} f(z)\, d\mu_{k(r)}(z) \Big| \\ &\quad + \varlimsup_{r\to 0} \Big| \fint_{Q(x,r)} f(z)\, d\mu_{k(r)}(z) - \fint_{Q(x',r)} f(z')\, d\mu_{k(r)}(z') \Big| \\ &\quad + \varlimsup_{r\to 0} \Big| f(x') - \fint_{Q(x',r)} f(z')\, d\mu_{k(r)}(z') \Big| \\ &\leqslant C \varlimsup_{r\to 0} (2r + \|x - x'\|)\big(f^{\sharp}_{\{\mu_k\}}(x,r) + f^{\sharp}_{\{\mu_k\}}(x',r)\big) \\ &\leqslant C\|x - x'\|\big(f^{\sharp}_{\{\mu_k\}}(x) + f^{\sharp}_{\{\mu_k\}}(x')\big). \end{aligned} \tag{3.44}$$

This completes the verification of the corollary.

The following result is crucial in proving the 'direct trace theorem'. Recall Proposition 2.4 and Definitions 2.7 and 2.8.

Theorem 3.4. *Let $d \in [0,n]$ and let S be a d-thick closed set. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Let $q \in (\max\{1, n-d\}, \infty)$ and $F \in W_q^1(\mathbb{R}^n)$. Then for every cube $Q = Q(x,r)$ with $x \in S$ and $r \in (0,1]$*

$$\fint_{Q\cap S} \Big| F|_S(y) - \fint_Q F(z)\, d\mathscr{H}^n(z) \Big|\, d\mu_{k(r)}(y) \leqslant Cr\Big(\fint_Q \sum_{|\alpha|=1} |D^\alpha F(t)|^q\, d\mathscr{H}^n(t) \Big)^{1/q}, \tag{3.45}$$

where the positive constant C does not depend on x, r or F.

Proof. Fix a cube $Q = Q(x,r)$ with $x \in S$ and $r \in (0,1]$. We consider several cases.

Case 1. Assume that $q > n$. Using the well-known Sobolev embedding theorem (see [10], § 1.8.2) we obtain the estimate required in this case:

$$\begin{aligned}&\fint_{Q\cap S}\left|F|_S(y) - \fint_Q F(z)\,d\mathscr{H}^n(z)\right| d\mu_{k(r)}(y)\\&\qquad\leqslant \fint_{Q\cap S}\fint_Q |\widehat{F}(y) - \widehat{F}(z)|\,d\mathscr{H}^n(z)\,d\mu_{k(r)}(y)\\&\qquad\leqslant \sup_{x,y\in Q}|\widehat{F}(x) - \widehat{F}(y)| \leqslant Cr\biggl(\fint_Q \sum_{|\alpha|=1}|D^\alpha F(t)|^q\,d\mathscr{H}^n(t)\biggr)^{1/q}.\end{aligned}$$

Here $\widehat{F}$ is the continuous representative of F and $C > 0$ depends only on n and q.
Case 2. Now we consider the most complicated case, when $d > 0$ and $q \in (\max\{1, n-d\}, n)$. We set $g := \chi_Q|\nabla F|$. It is clear that $Q \cap S \subset Q(y, 2r)$ for every $y \in Q \cap S$. We can rewrite (3.8) as follows. For $(1,p)$-quasi-every (and hence for $\mu_{k(r)}$-almost every) $y \in Q \cap S$

$$\left|\widehat{F}(y) - \fint_Q F(z)\,d\mathscr{H}^n(z)\right| \leqslant C\,\mathrm{I}_1^Q[g](y) \leqslant C\,\mathrm{I}_1^{Q(y,2r)}[g](y). \tag{3.46}$$

Set $\mu_Q := \mu_{k(r)}\lfloor_{Q\cap S}$. Since $Q = Q(x,r)$ lies in a union of at most 5^n cubes with edge length $2^{-k(r)}$, it follows from (2.10) that $\mu_Q(Q(y,t)) \leqslant 5^n t^d$ for $t \in (0, 4r)$ and $y \in Q$. This gives (recall the definition of the Wolf potential (3.4))

$$\int_{\mathbb{R}^n} \mathscr{W}^{4r}_{1,q}[\mu_Q](y)\,d\mu_Q(y) \leqslant Cr^d \int_0^{4r} t^{(q+d-n)(q'-1)-1}\,dt \leqslant Cr^{d+(q+d-n)q'/q}. \tag{3.47}$$

The positive constant C in (3.47) does not depend on x or r.

Now we recall Lemma 3.6 and apply Theorem B with measure μ_Q (instead of μ) and with $\alpha = 1$. Hence, using (2.11), (3.46) and (3.47) we obtain

$$\begin{aligned}&\fint_{Q\cap S}\left|F|_S(y) - \fint_Q F(z)\,d\mathscr{H}^n(z)\right| d\mu_{k(r)}(y) \leqslant Cr^{-d}\int \mathrm{I}_1^{Q(y,2r)}[g](y)\,d\mu_Q(y)\\&\qquad\leqslant Cr^{-d}r^{d/q'+(q+d-n)/q}\biggl(\int_Q |\nabla F|^q(z)\,d\mathscr{H}^n(z)\biggr)^{1/q}\\&\qquad\leqslant Cr\biggl(\fint_Q |\nabla F|^q(z)\,d\mathscr{H}^n(z)\biggr)^{1/q}.\end{aligned} \tag{3.48}$$

Case 3. In the case $d > 0$, $q = n$ we choose an arbitrary $\widetilde{q} \in (\max\{1, n-d\}, n)$ and use the previous step to obtain (3.45) with $\widetilde{q}$ instead of q. To complete the proof it remains to apply Hölder's inequality.

Corollary 3.4. *Let $d \in [0, n]$ and let S be a d-thick closed set. Assume that $q \in (\max\{1, n-d\}, \infty)$ and $F \in W^1_q(\mathbb{R}^n)$ and set $f := F|_S$. Then for every $r \in [0, 1]$ and every $x \in S$*

$$f^\sharp_{\{\mu_k\}}(x, r) \leqslant C\bigl(\mathrm{M}^{\leq 2}_{>r}[|\nabla F|^q](x)\bigr)^{1/q}. \tag{3.49}$$

Proof. Consider the case $r < 1$. The case $r = 1$ is similar. Using Remark 2.4 and Theorem 3.4, for every $x \in S$ we have the required estimate

$$\begin{aligned} f^{\sharp}_{\{\mu_k\}}(x,r) &\leqslant \sup_{t\in(r,1]} \frac{1}{t} \fint_{Q(x,t)\cap S} \left| f(y) - \fint_{Q(x,t)\cap S} f(z)\, d\mu_{k(t)}(z) \right| d\mu_{k(t)}(y) \\ &\leqslant \sup_{t\in(r,1]} \frac{1}{t} \left[\fint_{Q(x,t)\cap S} \left| f(y) - \fint_{Q(x,t)} f(y')\, d\mathscr{H}^n(y') \right| d\mu_{k(t)}(y) \right. \\ &\qquad \left. + \left| \fint_{Q(x,t)} f(y')\, d\mathscr{H}^n(y') - \fint_{Q(x,t)\cap S} f(z)\, d\mu_{k(t)}(z) \right| \right] \\ &\leqslant \sup_{t\in(r,1]} \frac{2}{t} \fint_{Q(x,t)\cap S} \left| f(y) - \fint_{Q(x,t)} f(y')\, d\mathscr{H}^n(y') \right| d\mu_{k(t)}(y) \\ &\leqslant C \sup_{t\in(r,1]} \left(\fint_{Q(x,t)} |\nabla F(y)|^q \, d\mathscr{H}^n(y) \right)^{1/q} \leqslant C \big(\mathrm{M}^{\leq 1}_{>r}[|\nabla F|^q](x) \big)^{1/q}. \end{aligned} \tag{3.50}$$

The verification of the corollary is complete.

3.6. Porous sets. Recall Lemma 3.2 and Definitions 2.6 and 3.2. Also recall that we let $k(\varkappa)$ denote the unique integer such that $r_{\varkappa} = 2^{-k(\varkappa)}$. We continue to measure distances in $\mathbb{R}^n$ in the $\|\cdot\|_{\infty}$-norm.

Lemma 3.10. *Let S be a closed nonempty set in $\mathbb{R}^n$. Let $Q_{\varkappa} = Q(x_{\varkappa}, r_{\varkappa})$ be a Whitney cube in W_S. Then $\widetilde{x}_{\varkappa} \in S_j(\lambda)$ for every $j \geqslant k(\varkappa)$ and $\lambda \in (0,1)$. Furthermore, $Q(\widetilde{x}_{\varkappa}, r_{\varkappa}(c-1)/c) \cap S \subset S_{k(\varkappa)}(\lambda)$ for every $c > 1$ and every $\lambda \in (0, 1/c]$.*

Proof. Consider the interval $(x_{\varkappa}, \widetilde{x}_{\varkappa}) := \{x = x_{\varkappa} + t(\widetilde{x}_{\varkappa} - x_{\varkappa})\colon t \in (0,1)\}$. Clearly, $S \cap (x_{\varkappa}, \widetilde{x}_{\varkappa}) = \varnothing$ because otherwise there exists a point $x' \in S$ such that $\|x_{\varkappa} - x'\| < \|x_{\varkappa} - \widetilde{x}_{\varkappa}\| = \operatorname{dist}(x_{\varkappa}, S)$. For every $r \in (0, r_{\varkappa}]$ consider the point $y_r := (x_{\varkappa}, \widetilde{x}_{\varkappa}) \cap \partial Q(\widetilde{x}_{\varkappa}, r)$. It follows from Remark 3.3 that $\operatorname{dist}(y_r, S) = r$. Hence for every $\lambda \in (0,1)$ the cube $Q(y_r, \lambda r)$ lies in $\mathbb{R}^n \setminus S$. This proves the first claim in the lemma.

Given a number $c > 1$ we set $r_c := r_{\varkappa}/c$. Then from Remark 3.3 we conclude that $\operatorname{dist}(y_{r_c}, S) = r_{\varkappa}/c$. On the other hand it is clear that $y_{r_c} \in Q(x, r_{\varkappa})$ for every $x \in Q(\widetilde{x}_{\varkappa}, r_{\varkappa}(c-1)/c)$. This proves the second claim.

Lemma 3.11. *Let S be a closed nonempty set in $\mathbb{R}^n$. Let $W_S = \{Q_{\varkappa}\}_{\varkappa \in I}$ be a Whitney decomposition of $\mathbb{R}^n \setminus S$. Let $\lambda \in (0,1)$ and $k \in \mathbb{N}_0$. Then for every $x \in S_k(\lambda)$ there exists a point $y(x) \in Q(x, 2^{-k})$ such that*

$$\frac{\lambda 2^{-k}}{5} \leqslant \operatorname{diam} Q_{\varkappa} \leqslant 2^{-k} \tag{3.51}$$

for every Whitney cube $Q_{\varkappa} \ni y(x)$.

Proof. By Definition 2.6 there exists a point $y \in Q(x, 2^{-k})$ such that $Q(y, \lambda 2^{-k}) \subset \mathbb{R}^n \setminus S$. We set $y(x) := y$. Now we prove (3.51). Consider an arbitrary Whitney cube $Q_{\varkappa} \ni y(x)$. From (3.10) we have

$$\operatorname{diam} Q_{\varkappa} \leqslant \operatorname{dist}(Q_{\varkappa}, S) \leqslant \operatorname{dist}(S, y(x)) \leqslant 2^{-k}.$$

On the other hand, using (3.10) again, we see that

$$\lambda 2^{-k} \leqslant \operatorname{dist}(y(x), S) \leqslant \operatorname{dist}(Q_\varkappa, S) + \operatorname{diam}(Q_\varkappa) \leqslant 5 \operatorname{diam}(Q_\varkappa).$$

Combining the above estimates we complete the proof.

§ 4. The main results

Recall that, given $r > 0$, we set $k(r) := -[\log_2 r]$. For each closed nonempty set $S \subset \mathbb{R}^n$ and a Whitney decomposition $\{Q_\alpha\}_{\alpha \in I}$ of $\mathbb{R}^n \setminus S$ we set $k(\varkappa) := k(r_\varkappa)$ for each $\varkappa \in I$. Throughout this section, unless otherwise stated, we equip $\mathbb{R}^n$ with the uniform norm $\|\cdot\|_\infty$.

4.1. The direct trace theorem. Recall Definitions 2.3 and 2.6 and the notation following Lemma 3.2.

Lemma 4.1. *Let $d \in [0, n]$, $p \in (1, \infty)$ and $\lambda \in (0, 1)$. Let S be a d-thick closed set. Let $\{\mu_k\} = \{\mu_k\}_{k \in \mathbb{N}_0}$ be a d-regular sequence of measures on S. Assume that $f \in L_1^{\mathrm{loc}}(S, \mu_k)$ for some (and hence every) $k \in \mathbb{N}_0$. Then there is a positive constant C independent of f such that*

$$\sum_{k=1}^{\infty} 2^{k(d-n)} \int_{S_k(\lambda)} \bigl(f^\sharp_{\{\mu_k\}}(x, 2^{-k})\bigr)^p \, d\mu_k(x) \leqslant C \sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(Q_\varkappa) \biggl[\bigl(f^\sharp_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\kappa)\bigr)^p + \fint_{Q(\widetilde{x}_\varkappa, 8)} |f(y)|^p \, d\mu_0(y)\biggr]. \tag{4.1}$$

Proof. Given $k \in \mathbb{N}_0$, let $\{x_{k,j}\}_{j \in \mathscr{J}_k}$ be a maximal 2^{-k}-separated subset $S_k(\lambda)$ of S. Using Lemma 3.1, (1), for every $k \in \mathbb{N}_0$ we have

$$\int_{S_k(\lambda)} \bigl(f^\sharp_{\{\mu_k\}}(x, 2^{-k})\bigr)^p \, d\mu_k(x) \leqslant \sum_{j \in \mathscr{J}_k} \int_{S_k(\lambda) \cap Q(x_{k,j}, 2^{-k})} \bigl(f^\sharp_{\{\mu_k\}}(x, 2^{-k})\bigr)^p \, d\mu_k(x). \tag{4.2}$$

Using Lemma 3.11, for every $k \in \mathbb{N}_0$ and $j \in \mathscr{J}_k$ we choose a point $y(x_{k,j}) \in Q(x_{k,j}, 2^{-k})$ and fix an index $\varkappa(k,j) \in \mathscr{I}$ such that $Q_{\varkappa(k,j)} \ni y(x_{k,j})$ and (3.51) holds. We define a map $\Theta\colon \bigcup_k \mathscr{J}_k \to I$ as follows:

$$\Theta(k,j) := \varkappa(k,j), \qquad k \in \mathbb{N}_0, \quad j \in \mathscr{J}_k. \tag{4.3}$$

Since $Q_\varkappa \cap Q(x_{k,j}, 2^{-k}) \neq \varnothing$, it follows from (3.51) that $Q_\varkappa \subset 3Q(x_{k,j}, 2^{-k})$. Using this, Lemma 3.1, (2), and Proposition 3.4 for $c = 6$, it is easy to see that there exists a positive constant $C(n)$ such that for every fixed $k \in \mathbb{N}_0$ and every $\varkappa \in \mathscr{I}$

$$\operatorname{card}\{\Theta^{-1}(\varkappa) \cap \mathscr{J}_k\} \leqslant C(n). \tag{4.4}$$

It follows from (3.51) that the equality $\Theta(k,j) = \varkappa = \Theta(k',j')$ implies that

$$|k' - k| \leqslant \log_2 \frac{5}{\lambda}. \tag{4.5}$$

Combining (3.51), (4.4) and (4.5), for every $\varkappa \in \mathscr{I}$ we obtain

$$\sum_{(k,j)\in\Theta^{-1}(\varkappa)} 2^{-kn} \leqslant C(n,\lambda)\mathscr{H}^n(Q_\varkappa). \tag{4.6}$$

If $\varkappa = \Theta(k,j)$ for some $k \in \mathbb{N}_0$, $j \in \mathscr{J}_k$, then it follows from (3.10) and (3.51) that

$$\begin{aligned}\|x_{k,j} - \widetilde{x}_\varkappa\| &\leqslant \|x_{k,j} - y(x_{k,j})\| + \|y(x_{k,j}) - \widetilde{x}_\varkappa\| \\ &\leqslant 2^{-k} + \operatorname{dist}(Q_\varkappa, S) + \operatorname{diam}(Q_\varkappa) \leqslant 2^{-k} + 5\operatorname{diam}(Q_\varkappa) \leqslant \frac{6}{2^k}.\end{aligned}$$

This gives the inclusion

$$Q\left(\widetilde{x}_{\Theta(k,j)}, \frac{8}{2^k}\right) \supset Q(x, 2^{-k}) \quad \text{for each } x \in Q(x_{k,j}, 2^{-k}). \tag{4.7}$$

Now we use (2.10) and (4.7) and apply Lemma 3.8 for $c = 8$. Then for every $k \in \mathbb{N}$ we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\sum_{j\in\mathscr{J}_k} 2^{k(d-n)} \int_{S_k(\lambda)\cap Q(x_{k,j},2^{-k})} \big(f^{\sharp}_{\{\mu_k\}}(x,2^{-k})\big)^p \, d\mu_k(x) \\ &\qquad \leqslant C\sum_{k=1}^{\infty}\sum_{j\in\mathscr{J}_k} 2^{-kn}\left[\big(f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_{\Theta(k,j)}, 2^{-k})\big)^p + \fint_{Q(\widetilde{x}_{\Theta(k,j)},8)} |f(y)|^p \, d\mu_0(y)\right].\end{aligned} \tag{4.8}$$

From (4.6) it follows that

$$\begin{aligned}S_1 &:= \sum_{k=1}^{\infty}\sum_{j\in\mathscr{J}_k} 2^{-kn}\big(f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_{\Theta(k,j)}, 2^{-k})\big)^p \\ &\leqslant C\sum_{\varkappa\in\mathscr{I}}\sum_{(k,j)\in\Theta^{-1}(\varkappa)} 2^{-kn}\big(f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_\varkappa, 2^{-k})\big)^p \leqslant C\sum_{\varkappa\in\mathscr{I}} \mathscr{H}^n(Q_\varkappa)\big(f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\varkappa)\big)^p.\end{aligned} \tag{4.9}$$

Similarly

$$\begin{aligned}S_2 &:= \sum_{k=1}^{\infty}\sum_{j\in\mathscr{J}_k} 2^{-kn}\fint_{Q(\widetilde{x}_{\Theta(k,j)},8)} |f(y)|^p \, d\mu_0(y) \\ &\leqslant C\sum_{\varkappa\in\mathscr{I}} \mathscr{H}^n(Q_\varkappa)\fint_{Q(\widetilde{x}_\varkappa,8)} |f(y)|^p \, d\mu_0(y).\end{aligned} \tag{4.10}$$

Combining (4.2), (4.8), (4.9) and (4.10) we complete the proof of Lemma 4.1.

Lemma 4.2. *Let $d \in [0, n]$, $p \in (1, \infty)$ and $\lambda \in (0, 1)$. Let S be a d-thick closed set. Let $\{\mu_k\} = \{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures in S. Assume that $f \in L_p(S, \mu_0)$. Then for each $c \geqslant 1$ there is a positive constant C depending only on d, p, n, c and $C_{\{\mu_k\}}$ such that*

$$\sum_{\varkappa\in\mathscr{I}} \mathscr{H}^n(Q_\varkappa)\fint_{Q(\widetilde{x}_\varkappa,c)} |f(y)|^p \, d\mu_0(y) \leqslant C\|f|L_p(S,\mu_0)\|^p. \tag{4.11}$$

Proof. We use (2.11) and then apply Lemma 3.4 with $d\mathfrak{m}(y) = |f(y)|^p\, d\mu_0(y)$. This gives the required estimate

$$\sum_{\varkappa\in\mathscr{I}} \mathscr{H}^n(Q_\varkappa) \fint_{Q(\widetilde{x}_\varkappa,c)} |f(y)|^p\, d\mu_0(y) \leqslant \sum_{\varkappa\in\mathscr{I}} \mathscr{H}^n(Q_\varkappa) \int_{Q(\widetilde{x}_\varkappa,c)} |f(y)|^p\, d\mu_0(y)$$
$$\leqslant C\|f|L_p(S,\mu_0)\|^p. \tag{4.12}$$

The proof is complete.

Recall the notion of a good representative $\widehat{F}$ of a given element $F \in W_p^1(\mathbb{R}^n)$.

Lemma 4.3. *Let $d \in (0,n]$ and $p \in (\max\{1,n-d\},n]$. Let $S \subset \mathbb{R}^n$ be a d-thick closed set. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Then for every $F \in W_p^1(\mathbb{R}^n)$*

$$\lim_{r\to 0} \fint_{Q(x,r)\cap S} |\widehat{F}(x) - \widehat{F}(z)|\, d\mu_{k(r)}(z) = 0 \quad \textit{for } (1,p)\textit{-q.e. } x \in S. \tag{4.13}$$

Proof. Let $S' \subset S$ be the intersection of S with the set of all Lebesgue points of the function $\widehat{F}$. It follows from Proposition 2.4 that $C_{1,p}(S \setminus S') = 0$. For every $x \in S'$ we have

$$\fint_{Q(x,r)\cap S} |\widehat{F}(x) - \widehat{F}(z)|\, d\mu_{k(r)}(z) \leqslant \left|\widehat{F}(x) - \fint_{Q(x,r)} \widehat{F}(y)\, d\mathscr{H}^n(y)\right|$$
$$+ \fint_{Q(x,r)\cap S} \left|\widehat{F}(z) - \fint_{Q(x,r)} \widehat{F}(y)\, d\mathscr{H}^n(y)\right| d\mu_{k(r)}(z) =: J_1(x,r) + J_2(x,r). \tag{4.14}$$

Clearly, $J_1(x,r) \to 0$ as $r \to 0$ according to the construction of S'. Combining this fact with (4.14) and Proposition 2.1 we conclude that it is sufficient to show that $J_2(x,r) \to 0$ as $r \to 0$ for $\mathscr{H}^{n-p}$-a.e. $x \in S$. In fact, applying Theorem 3.4 (for $q = p$) and Proposition 3.1 gives

$$\varlimsup_{r\to 0} (J_2(x,r))^p$$
$$\leqslant C \varlimsup_{r\to 0} \frac{1}{r^{n-p}} \sum_{|\alpha|=1} \int_{Q(x,r)} |D^\alpha \widehat{F}(z)|^p\, d\mathscr{H}^n(z) = 0 \quad \text{for } \mathscr{H}^{n-p}\text{-a.e. } x \in S. \tag{4.15}$$

The lemma is proved.

Now we are ready to prove the main result of this subsection. Recall Definitions 2.7–2.9 and 2.11.

Theorem 4.1. *Let $d \in [0,n]$, $p \in (\max\{1,n-d\},\infty)$ and $\lambda \in (0,1)$. Let $S \subset \mathbb{R}^n$ be a d-thick closed set, and let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Then the functional $\widetilde{\mathscr{N}}_{S,p,\lambda}$ is bounded on the trace space $W_p^1(\mathbb{R}^n)|_S$.*

Proof. It is sufficient to verify that there exists a positive constant C (independent of F) such that the inequality

$$\widetilde{\mathscr{N}}_{S,p,\lambda}[f] \leqslant C\|F|W_p^1(\mathbb{R}^n)\| \tag{4.16}$$

holds for each $F \in W_p^1(\mathbb{R}^n)$ with $F|_S = f$.

We fix some $F \in W_p^1(\mathbb{R}^n)$ with $F|_S = f$ throughout the proof.

Step 1. First of all we estimate $\mathscr{S}\mathscr{N}_{S,p}[f]$ from above. Fix an arbitrary $q \in (\max\{1, n-d\}, p)$. We apply Corollary 3.4, Remark 3.1 and Theorem A for $\mathfrak{m} = \mathscr{H}^n$, $\alpha = 0$, $d = n$ and exponent p/q instead of p. Then we obtain

$$\big(\mathscr{S}\mathscr{N}_{S,p}[f]\big)^p \leqslant C \int_S \big(\mathrm{M}^{<1}[|\nabla F|^q](x)\big)^{p/q}\, d\mathscr{H}^n(x) \leqslant C\|F|W_p^1(\mathbb{R}^n)\|^p. \tag{4.17}$$

Step 2. Fix some $q \in (\max\{1, n-d\}, p)$. From Lemma 3.2 it follows that $Q(\widetilde{x}_\varkappa, r) \subset Q(x, 10r)$ for every $\varkappa \in \mathscr{I}$, $x \in Q_\varkappa$ and $r \in (r_\varkappa, 1)$. Applying Corollary 3.4, Remark 3.1, Theorem A for $\mathfrak{m} = \mathscr{H}^n$, $\alpha = 0$ and $\gamma = p/q$, and the fact that the interiors of different Whitney cubes are mutually disjoint gives

$$\begin{aligned}\sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(Q_\varkappa)\big(f^\sharp_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\kappa)\big)^p &\leqslant C \sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(Q_\varkappa)\big(\mathrm{M}^{\leq 2}_{>r_\varkappa}[|\nabla F|^q](\widetilde{x}_\varkappa)\big)^{p/q} \\ &\leqslant C \sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(Q_\varkappa) \inf_{x \in Q_\varkappa} \big(\mathrm{M}_{>r_\varkappa}[|\nabla F|^q](x)\big)^{p/q} \\ &\leqslant C \int_{\mathbb{R}^n} \big(\mathrm{M}[|\nabla F|^q](x)\big)^{p/q}\, d\mathscr{H}^n(x) \leqslant C\|F|W_p^1(\mathbb{R}^n)\|^p. \end{aligned} \tag{4.18}$$

Step 3. We estimate $\|f|L_p(S,\mu_0)\|$ from above. Let $\{x_j\}_{j \in \mathscr{J}}$ be a maximal 1-separated subset of S. Consider the family of cubes $\{Q_j\}_{j \in \mathscr{J}} := \{Q(x_j, 1)\}_{j \in \mathscr{J}}$. It is clear that $Q(x,1) \subset 2Q_j$ for every $x \in S \cap Q_j$. Using this and Lemma 3.1, (1), we derive the following estimate:

$$\begin{aligned}\int_S |f(x)|^p\, d\mu_0(x) &\leqslant \sum_{j \in \mathscr{J}} \int_{Q_j} |f(x)|^p\, d\mu_0(x) \\ &\leqslant C \sum_{j \in \mathscr{J}} \int_{Q_j} \left| f(x) - \fint_{Q(x,1)} F(y)\, d\mathscr{H}^n(y)\right|^p d\mu_0(x) \\ &\quad + C \sum_{j \in \mathscr{J}} \int_{Q_j} \left(\fint_{2Q_j} |F(y)|\, d\mathscr{H}^n(y)\right)^p d\mu_0(x). \end{aligned} \tag{4.19}$$

Using Hölder's inequality, (2.10), Lemma 3.1, (2), Proposition 3.4 with $c = 4$ and Proposition 3.5 with $d\mathfrak{m}(y) = |F(y)|^p\, d\mathscr{H}^n(y)$, we easily obtain

$$\begin{aligned}&\sum_{j \in \mathscr{J}} \int_{Q_j} \left(\fint_{2Q_j} |F(y)|\, d\mathscr{H}^n(y)\right)^p d\mu_0(x) \\ &\qquad \leqslant \sum_{j \in \mathscr{J}} \int_{2Q_j} |F(y)|^p\, d\mathscr{H}^n(y) \leqslant C(n) \int_{\mathbb{R}^n} |F(y)|^p\, d\mathscr{H}^n(y). \end{aligned} \tag{4.20}$$

Recall Proposition 2.1 and Lemma 3.6 (also recall that $d > n-p$). Then applying Proposition 3.3 and Remark 3.2 shows that there exists $C > 0$ (independent of F) such that for every $\delta \in (0,1)$

$$\left| f(x) - \fint_{Q(x,1)} F(y)\, d\mathscr{H}^n(y)\right| \leqslant C\, \mathrm{M}^{<2}[|\nabla F|, (1-\delta)](x) \quad \text{for } \mu_0\text{-a.e. } x \in \mathbb{R}^n. \tag{4.21}$$

Since $d > n-p$, we can choose δ in such a way that $p(1-\delta) > n-d$. Now we use Lemma 3.1, then Proposition 3.5 with $d\mathfrak{m}(x) = \bigl(\mathrm{M}^{<2}[|\nabla F|,(1-\delta)](x)\bigr)^p\,d\mu_0(x)$, and finally apply Theorem A with $\gamma = p$ and $\alpha = (1-\delta)$. As a result, using (4.21) we obtain

$$\begin{aligned}\sum_{j\in\mathscr{J}}\int_{Q_j}\biggl|f(x) - ⨍_{Q(x,1)} F(y)\,d\mathscr{H}^n(y)\biggr|^p\,d\mu_0(x)\\ \leqslant C\sum_{j\in\mathscr{J}}\int_{Q_j}\bigl(\mathrm{M}^{<2}[|\nabla F|,(1-\delta)](x)\bigr)^p d\mu_0(x)\\ \leqslant C\int_{\mathbb{R}^n}\bigl(\mathrm{M}^{<2}[|\nabla F|,(1-\delta)](x)\bigr)^p d\mu_0(x) \leqslant C\int_{\mathbb{R}^n}|\nabla F(y)|^p\,d\mathscr{H}^n(y).\end{aligned}\tag{4.22}$$

Combining (4.19), (4.20) and (4.22) yields

$$\|f|L_p(S,\mu_0)\| \leqslant C\|F|W_p^1(\mathbb{R}^n)\|.\tag{4.23}$$

Step 4. Combining Lemmas 4.1 and 4.2, and estimates (4.18) and (4.23) we obtain

$$\bigl(\widetilde{\mathscr{B}\mathscr{N}}_{S,p,\lambda}[f]\bigr)^p \leqslant C\|F|W_p^1(\mathbb{R}^n)\|^p.\tag{4.24}$$

Now the required estimate (4.16) follows directly from (4.17), (4.23) and (4.24). The proof is complete.

4.2. The reverse trace theorem. The following pointwise characterization of the functions in the space $W_p^1(\mathbb{R}^n)$ is given in [35].

Theorem D. *Let $p \in (1,\infty]$ and $F \in L_p(\mathbb{R}^n)$. Then $F \in W_p^1(\mathbb{R}^n)$ if and only if there exist a nonnegative function $g \in L_p(\mathbb{R}^n)$, a set E_F with $\mathscr{H}^n(E_F) = 0$ and a constant $\delta > 0$ such that*

$$|F(x) - F(y)| \leqslant \|x-y\|\bigl(g(x) + g(y)\bigr)\tag{4.25}$$

for every $x, y \in \mathbb{R}^n \setminus E_F$ with $\|x-y\| < \delta$. Furthermore,

$$\sum_{|\alpha|=1}\|D^\alpha F|L_p(\mathbb{R}^n)\| \leqslant C\|g|L_p(\mathbb{R}^n)\|,\tag{4.26}$$

where the positive constant C does not depend on g.

Proof. We only sketch the proof. One implication was established in [35] (see the text before Theorem 1 there); see also [36]. For the reverse implication we have to cover $\mathbb{R}^n$ by a countable family of ‘sufficiently nicely overlapping’ balls $\{B_i\}_{i\in\mathbb{N}}$ of diameter $\delta > 0$ and for every $i \in \mathbb{N}$ apply Theorem 1 in [35] for $\Omega = \operatorname{int} B_i$. Then we observe that, given a function $F \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, the fact that $F \in \bigcap_{i=1}^N W_p^1(\operatorname{int} B_i)$ implies $F \in W_p^1\bigl(\bigcup_{i=1}^N \operatorname{int} B_i\bigr)$ for any $N \in \mathbb{N}$. This shows that locally integrable weak derivatives $D^\alpha F$ exist on $\mathbb{R}^n$ and their restrictions to any open ball $\operatorname{int} B_i$ coincide with the corresponding weak derivatives of $F|_{\operatorname{int} B_i}$. It remains to sum the appropriate analogues of estimate (4.26) for all open balls $\operatorname{int} B_i$.

Now we are ready to present our construction of the extension operator.

Definition 4.1. Let $S \subset \mathbb{R}^n$ be a d-thick closed set for some $d \in [0,n]$. Let $\{\mu_k\} = \{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Assume that $f \in L_1^{\mathrm{loc}}(S,\mu_k)$ for every $k \in \mathbb{N}_0$. For the same family of functions $\{\varphi_\varkappa\}_{\varkappa\in I}$ as in Proposition 3.6 we set

$$\operatorname{Ext}[f](x) := \chi_S(x)f(x) + \sum_{\varkappa\in\mathscr{I}} \varphi_\varkappa(x) f_\varkappa, \qquad x \in \mathbb{R}^n, \tag{4.27}$$

where

$$f_\varkappa := \fint_{\widetilde{Q}_\varkappa\cap S} f(x)\,d\mu_{k(\varkappa)}(x), \qquad \varkappa\in\mathscr{I}.$$

Remark 4.1. In fact, (4.27) defines not a single extension operator, but a whole family of operators. The reason is that the choice of the d-regular sequence of measures $\{\mu_k\}$ is not unique. Furthermore, generally speaking, the choice of the reflected cubes $\widetilde{Q}_\varkappa$ is not unique either.

The following result plays a crucial role. It gives a pointwise estimate of the extension constructed in (4.27). Recall the notation $b(\varkappa)$: see (3.12).

Lemma 4.4. *Let $d \in [0,n]$ and let S be a d-thick closed set. Let $\{\mu_k\} = \{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Let $f \in L_1^{\mathrm{loc}}(S,\mu_k)$ for some (and hence every) $k \in \mathbb{N}_0$. Suppose that*

$$\lim_{k\to\infty} \fint_{Q(x,2^{-k})} |f(x)-f(y)|\,d\mu_k(y) = 0 \quad \text{for } \mathscr{H}^n\text{-a.e. } x \in S. \tag{4.28}$$

Then there exists a positive constant C depending only on n, p, d and $C_{\{\mu_k\}}$ such that for each $\delta \in (0,1/150)$, for $(\mathscr{H}^n \times \mathscr{H}^n)$-a.e. $(x,y) \in \mathbb{R}^{2n}$, $\|x-y\| < \delta$, the function $F := \operatorname{Ext}[f]\colon \mathbb{R}^n \to \mathbb{R}$ defined in (4.27) satisfies

$$|F(x)-F(y)| \leqslant C\|x-y\|\bigl(g(x)+g(y)\bigr), \qquad x,y \in \mathbb{R}^n, \tag{4.29}$$

where, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} g(x) &:= \chi_S(x) f^\sharp_{\{\mu_k\}}(x) \\ &\quad + \sum_{\varkappa\in\mathscr{I}} \chi_{Q_\varkappa}(x) \sum_{\substack{\varkappa'\in\mathscr{I}\\ \varkappa'\in b(\varkappa)}} \Bigl(f^\sharp_{\{\mu_k\}}(\widetilde{x}_{\varkappa'}, r_{\varkappa'}) + \fint_{\widetilde{Q}_{\varkappa'}\cap S} |f(z)|\,d\mu_{k(\varkappa')}(z) \Bigr). \end{aligned} \tag{4.30}$$

Proof. Fix an arbitrary $\delta \in (0,1/150)$. It is obvious that we have to consider five cases:

(1) $x,y \in S$ and $\|x-y\| < \delta$;
(2) $x \in S$, $y \in \mathbb{R}^n \setminus S$ and $\|x-y\| < \delta$;
(3) $y \in S$, $x \in \mathbb{R}^n \setminus S$ and $\|x-y\| < \delta$;
(4) $x,y \in U_{1/60}(S) \setminus S$ and $\|x-y\| < \delta$;
(5) $x,y \in \mathbb{R}^n \setminus S$ and, furthermore, $\|x-y\| < \delta$ and either $x \notin U_{1/60}(S)$ or $y \notin U_{1/60}(S)$.

By the symmetry of the left-hand side of (4.29) with respect to x and y, we can identify cases (2) and (3) up to changes in notation.

Case (1). From (4.28) and Corollary 3.3 it follows that for $\mathscr{H}^n \times \mathscr{H}^n$-almost every $(x,y) \in S \times S$

$$|F(x) - F(y)| \leqslant C\|x - y\|\big(f^{\sharp}_{\{\mu_k\}}(x) + f^{\sharp}_{\{\mu_k\}}(y)\big). \tag{4.31}$$

Case (2) (= *case* (3)). Consider the case when $x \in S$, $y \in U_\delta(S) \setminus S$ and $\|x-y\| \leqslant \delta$. Since $\delta \in (0, 1/150)$, estimate (3.11) and Proposition 3.6, (2), give

$$\sum_{\varkappa \in \mathscr{I}} \varphi_\varkappa(y) = 1 \quad \text{for all} \quad y \in U_\delta(S). \tag{4.32}$$

Recall the notation $b(y)$ (given after Lemma 3.2). From Proposition 3.6, (2), it follows that $b(y) = \{\varkappa \in \mathscr{I} : y \in Q^*_\varkappa\}$ for every $y \in \mathbb{R}^n \setminus S$. Hence from (4.27) and (4.32) we have

$$|F(x) - F(y)| = |f(x) - F(y)| \leqslant \sum_{\varkappa \in b(y)} \varphi_\varkappa(y) \bigg| f(x) - \fint_{\widetilde{Q}_\varkappa \cap S} f(z)\, d\mu_{k(\varkappa)}(z) \bigg|. \tag{4.33}$$

Fix $\varkappa \in b(y)$ and set $r = 2\max\{\|x - \widetilde{x}_\varkappa\|, \operatorname{diam} \widetilde{Q}_\varkappa\}$. Now, (3.10) and Proposition 3.6, (2), give

$$\|x - y\| \geqslant \operatorname{dist}(x, Q_\varkappa) - \frac{1}{8} \operatorname{diam} Q_\varkappa \geqslant \frac{1}{2} \operatorname{diam} Q_\varkappa = \frac{1}{2} \operatorname{diam} \widetilde{Q}_\varkappa.$$

Hence, using (3.10) again we obtain

$$\begin{aligned} \|x - \widetilde{x}_\varkappa\| &\leqslant \|x - y\| + \|y - x_\varkappa\| + \|x_\varkappa - \widetilde{x}_\varkappa\| \\ &\leqslant \|x - y\| + 6 \operatorname{diam} \widetilde{Q}_\varkappa \leqslant 13\|x - y\|. \end{aligned} \tag{4.34}$$

We use (4.34) and the fact that $\|x - y\| \leqslant \delta \in (0, 1/120)$. Then we obtain

$$r \leqslant 26\|x - y\| < \frac{1}{2}. \tag{4.35}$$

We note that $r_\varkappa$ can be much smaller than r. Hence we need to estimate with care. Note that $\widetilde{Q}_\varkappa := Q(\widetilde{x}_\varkappa, r_\varkappa) \subset Q(\widetilde{x}_\varkappa, r/2) \subset Q(x, r)$. Using the same arguments as in the proof of Corollary 3.3, for every $\varkappa \in b(y)$ and $\mathscr{H}^n$-a.e. $x \in S$ we have

$$\bigg| f(x) - \fint_{\widetilde{Q}_\varkappa \cap S} f(z)\, d\mu_{k(\varkappa)}(z) \bigg| \leqslant Cr\big(f^{\sharp}_{\{\mu_k\}}(x) + f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\varkappa)\big); \tag{4.36}$$

we have also used the fact that $2r < 1$. As a result, combining (4.33), (4.35) and (4.36), for $\mathscr{H}^n$-a.e. $x \in S$ and all $y \in Q(x, \delta) \setminus S$ we deduce that

$$\begin{aligned} |F(x) - F(y)| &\leqslant C\|x - y\| \bigg(f^{\sharp}_{\{\mu_k\}}(x) + \sum_{\varkappa \in b(y)} f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\varkappa) \bigg) \\ &\leqslant C\|x - y\|\big(g(x) + g(y)\big). \end{aligned} \tag{4.37}$$

Case (4). Fix $\delta \in (0, 1/150)$ and $x, y \in U_{1/60}(S) \setminus S$ with $\|x-y\| \leqslant \delta$. From (3.10) it follows that $\varkappa_0, \varkappa_1 \in \mathscr{I}$ for every $Q^*_{\varkappa_0} \ni x$ and $Q^*_{\varkappa_1} \ni y$. Furthermore, from (3.11) it follows that $b(\varkappa_0) \cup b(\varkappa_1) \subset \mathscr{I}$ in this case. Hence we have

$$\sum_{\varkappa \in \mathscr{I}} \varphi_{\varkappa}(x) = \sum_{\varkappa' \in \mathscr{I}} \varphi_{\varkappa'}(y) = 1. \tag{4.38}$$

There are two subcases here. In the first $Q_{\varkappa} \cap Q_{\varkappa'} = \varnothing$ for any $Q^*_{\varkappa} \ni x$ and $Q^*_{\varkappa'} \ni y$; in the second $Q_{\varkappa_0} \cap Q_{\varkappa_1} \neq \varnothing$ for some cubes $Q^*_{\varkappa_0}$ and $Q^*_{\varkappa_1}$ containing x and y, respectively.

Consider the first subcase. Using (4.38) and arguing as in (4.33) we have

$$|F(x) - F(y)| \leqslant \sum_{\varkappa \in b(x)} \sum_{\varkappa' \in b(y)} \biggl| \fint_{\widetilde{Q}_{\varkappa} \cap S} f(z)\, d\mu_{k(\varkappa)}(z) - \fint_{\widetilde{Q}_{\varkappa'} \cap S} f(z)\, \mu_{k(\varkappa')}(z) \biggr|. \tag{4.39}$$

For fixed $\varkappa \in b(x)$ and $\varkappa' \in b(y)$ we set

$$r := \|\widetilde{x}_{\varkappa} - \widetilde{x}_{\varkappa'}\| + \max\{\operatorname{diam}(\widetilde{Q}_{\varkappa}), \operatorname{diam}(\widetilde{Q}_{\varkappa'})\}.$$

It is clear that $\varkappa' \notin b(\varkappa)$ and $\varkappa \notin b(\varkappa')$ in this subcase. Hence we obtain

$$\|x - y\| \geqslant \frac{3}{16} \max\{\operatorname{diam}(\widetilde{Q}_{\varkappa}), \operatorname{diam}(\widetilde{Q}_{\varkappa'})\}. \tag{4.40}$$

On the other hand, using (3.10) we have

$$\begin{aligned} \|\widetilde{x}_{\varkappa} - \widetilde{x}_{\varkappa'}\| &\leqslant \|\widetilde{x}_{\varkappa} - x_{\varkappa}\| + \|\widetilde{x}_{\varkappa'} - x_{\varkappa'}\| + \|x - y\| + \|x - x_{\varkappa}\| + \|y - x_{\varkappa'}\| \\ &\leqslant \|x - y\| + \frac{91}{16} \operatorname{diam}(\widetilde{Q}_{\varkappa}) + \frac{91}{16} \operatorname{diam}(\widetilde{Q}_{\varkappa'}). \end{aligned} \tag{4.41}$$

Combining (4.40) and (4.41) we easily deduce that

$$r \leqslant 67\|x - y\| < \frac{1}{2}. \tag{4.42}$$

It is clear that $\widetilde{Q}_{\varkappa} \subset Q(\widetilde{x}_{\varkappa}, r)$ and $\widetilde{Q}_{\varkappa'} \subset Q(\widetilde{x}_{\varkappa}, r)$. We take (4.42) into account and apply Theorem 3.3 (which is possible due to (4.42) and the restrictions on δ). Then we obtain

$$\begin{aligned} &\biggl| \fint_{\widetilde{Q}_{\varkappa} \cap S} f(z)\, d\mu_{k(\varkappa)}(z) - \fint_{\widetilde{Q}_{\varkappa'} \cap S} f(z)\, d\mu_{k(\varkappa')}(z) \biggr| \\ &\qquad \leqslant C\|x - y\| \bigl(f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_{\varkappa}, r_{\kappa}) + f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_{\varkappa'}, r_{\kappa'}) \bigr). \end{aligned} \tag{4.43}$$

Combining (4.39) and (4.43) and using (4.30), for this choice of x and y we have

$$|F(x) - F(y)| \leqslant C\|x - y\| \bigl(g(x) + g(y) \bigr). \tag{4.44}$$

Now consider the second subcase. Fix arbitrary $Q^*_{\varkappa_0} \ni x$ and $Q^*_{\varkappa_1} \ni y$ with $Q_{\varkappa_0} \cap Q_{\varkappa_1} \neq \varnothing$. Let $\gamma_{x,y} \colon [0,1] \to \cup_{\varkappa \in b(\varkappa_0) \cup b(\varkappa_1)} Q_{\varkappa}$ be a smooth curve with $|\dot{\gamma}_{x,y}(t)| \leqslant C\|x - y\|$, $t \in [0,1]$. We use (4.27), Lagrange's mean value inequality, (4.38), Proposition 3.6, (4), and estimates (3.11). As a result, we obtain (recall that $\operatorname{supp}\mu_k = S$ for each $k \in \mathbb{N}_0$)

$$
\begin{aligned}
\frac{1}{\|x-y\|}|F(x)-F(y)| &\leqslant C \max_{t\in[0,1]} |\nabla F(\gamma_{x,y}(t))| \\
&\leqslant C \max_{t\in[0,1]} \Bigg| \sum_{\varkappa\in b(\varkappa_0)\cup b(\varkappa_1)} \nabla\varphi_\varkappa(\gamma_{x,y}(t)) \\
&\qquad \times \bigg(\fint_{\widetilde{Q}_\varkappa\cap S} f(z)\,d\mu_{k(\varkappa)}(z) - \fint_{\widetilde{Q}_{\varkappa_0}\cap S} f(z')\,d\mu_{k(\varkappa_0)}(z') \bigg) \Bigg| \\
&\leqslant \frac{C}{\max\{r_{\varkappa_0}, r_{\varkappa_1}\}} \Bigg(\sum_{\varkappa\in b(\varkappa_0)} \bigg| \fint_{\widetilde{Q}_{\varkappa_0}\cap S} f(z')\,d\mu_{k(\varkappa_0)}(z') - \fint_{\widetilde{Q}_\varkappa\cap S} f(z)\,d\mu_{k(\varkappa)}(z) \bigg| \\
&\qquad + \sum_{\varkappa\in b(\varkappa_1)} \bigg| \fint_{\widetilde{Q}_{\varkappa_1}\cap S} f(z')\,d\mu_{k(\varkappa_0)}(z') - \fint_{\widetilde{Q}_\varkappa\cap S} f(z)\,d\mu_{k(\varkappa)}(z) \bigg| \\
&\qquad + \bigg| \fint_{\widetilde{Q}_{\varkappa_1}\cap S} f(z')\,d\mu_{k(\varkappa_1)}(z') - \fint_{\widetilde{Q}_{\varkappa_0}\cap S} f(z)\,d\mu_{k(\varkappa_0)}(z) \bigg| \Bigg).
\end{aligned}
\tag{4.45}
$$

From (3.10) and (3.11) it follows that

$$
\begin{aligned}
\|\widetilde{x}_\varkappa - \widetilde{x}_{\varkappa_0}\| &\leqslant \|x_\varkappa - x_{\varkappa_0}\| + \operatorname{dist}(x_\varkappa, S) + \operatorname{dist}(x_{\varkappa_0}, S) \leqslant 5\operatorname{diam} Q_\varkappa + 5\operatorname{diam} Q_{\varkappa_0} \\
&\leqslant 25\min\{\operatorname{diam} Q_\varkappa, \operatorname{diam} Q_{\varkappa_0}\} \quad \text{for every } \varkappa\in b(\varkappa_0).
\end{aligned}
$$

Similarly, $\|\widetilde{x}_\varkappa - \widetilde{x}_{\varkappa_1}\| \leqslant 25\min\{\operatorname{diam} Q_\varkappa, \operatorname{diam} Q_{\varkappa_1}\}$ for every $\varkappa \in b(\varkappa_1)$. As a result, since $\operatorname{diam} Q_{\varkappa_i} \leqslant \max\{\operatorname{dist}(x,S), \operatorname{dist}(y,S)\} \leqslant 1/60$ for $i = 0,1$ (due to (3.10)) we obtain

$$
\|\widetilde{x}_\varkappa - \widetilde{x}_{\varkappa_i}\| + \max_{\varkappa\in b(\varkappa_i)} \max\{\operatorname{diam} Q_\varkappa, \operatorname{diam} Q_{\varkappa_i}\} \leqslant 30\operatorname{diam} Q_{\varkappa_i} < \frac{1}{2}, \qquad i=0,1.
\tag{4.46}
$$

Using (4.46) we apply Theorem 3.3 and continue with (4.45). This gives

$$
\frac{1}{\|x-y\|}|F(x)-F(y)| \leqslant C\big(g(x)+g(y)\big).
\tag{4.47}
$$

Combining (4.44) and (4.47) we can establish case (4).
Case (5). Fix $\delta \in (0, 1/120)$ and $x, y \in \mathbb{R}^n$ such that $\|x-y\| < \delta$. Without loss of generality assume that $x \in \mathbb{R}^n \setminus U_{1/60}(S)$. Then $y \in \mathbb{R}^n \setminus U_{1/120}(S)$. By (3.10) this implies that for every $Q_\varkappa \ni x$ and $Q_{\varkappa'} \ni y$

$$
\begin{aligned}
60^{-1} &\leqslant \operatorname{dist}(x,S) \leqslant \operatorname{diam} Q_\varkappa + \operatorname{dist}(Q_\varkappa, S) \leqslant 5\operatorname{diam} Q_\varkappa, \\
120^{-1} &\leqslant \operatorname{dist}(y,S) \leqslant \operatorname{diam} Q_{\varkappa'} + \operatorname{dist}(Q_{\varkappa'}, S) \leqslant 5\operatorname{diam} Q_{\varkappa'}.
\end{aligned}
\tag{4.48}
$$

Consider two subcases by analogy with case (4).

In the first subcase $Q_\varkappa \cap Q_{\varkappa'} = \varnothing$ for any $Q^*_\varkappa \ni x$ and $Q^*_{\varkappa'} \ni y$. Then the same arguments as in (4.40) together with (4.48) give

$$
\|x-y\| \geqslant \frac{3}{16}\max\{\operatorname{diam}\widetilde{Q}_\varkappa, \operatorname{diam}\widetilde{Q}_{\varkappa'}\} \geqslant \frac{1}{1600}.
\tag{4.49}
$$

By (4.27) and (4.30) this implies that

$$\frac{1}{\|x-y\|}|F(x)-F(y)| \leqslant C\Big(\sum_{\varkappa\in\mathscr{I}}\chi_{Q_\varkappa}(x)\fint_{\widetilde{Q}_\varkappa}|f(z)|\,d\mu_{k(\varkappa)}(z)$$
$$+\sum_{\varkappa\in\mathscr{I}}\chi_{Q_{\varkappa'}}(y)\fint_{\widetilde{Q}_{\varkappa'}}|f(z')|\,d\mu_{k(\varkappa')}(z')\Big) \leqslant C\big(g(x)+g(y)\big). \qquad (4.50)$$

In the second subcase there are $Q^*_{\varkappa_0}\ni x$ and $Q^*_{\varkappa_1}\ni y$ with $Q_{\varkappa_0}\cap Q_{\varkappa_1}\neq\varnothing$. Arguing as in (4.45)) and taking (4.48) into account we obtain

$$\frac{1}{\|x-y\|}|F(x)-F(y)| \leqslant C\sum_{\substack{\varkappa\in\mathscr{I}\\ \varkappa\in b(\varkappa_0)\cup b(\varkappa_1)}}\fint_{\widetilde{Q}_\varkappa\bigcap S}|f(z)|\,d\mu_{k(\varkappa)}(z) \leqslant C\big(g(x)+g(y)\big). \qquad (4.51)$$

Combining (4.50) and (4.51) we obtain case (5).

The proof of Lemma 4.4 is complete.

Lemma 4.5. *Let $d\in[0,n]$, $p\in(1,\infty)$ and $\lambda\in(0,1)$. Let S be a d-thick closed set. Let $\{\mu_k\}=\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Then there exists a positive constant C, depending only on d, n, p and $C_{\{\mu_k\}}$, such that*

$$\frac{1}{C}\sum_{\varkappa\in\mathscr{I}}\mathscr{H}^n(Q_\varkappa)\big(f^\sharp_{\{\mu_k\}}(\widetilde{x}_\varkappa,r_\kappa)\big)^p$$
$$\leqslant\sum_{k=1}^{\infty}\frac{2^{kd}}{2^{kn}}\int_{S_k(\lambda)}\big(f^\sharp_{\{\mu_k\}}(x,2^{-k})\big)^p\,d\mu_k(x)+\|f|L_p(S,\mu_0)\|^p. \qquad (4.52)$$

Proof. Clearly, $Q(\widetilde{x}_\varkappa,r)\subset Q(x,2r)\subset Q(\widetilde{x}_\varkappa,3)$ for every $x\in Q(\widetilde{x}_\varkappa,r_\varkappa)$ and $r\geqslant r_\varkappa$. Hence, applying Lemma 3.8 (for $c=2$) gives

$$f^\sharp_{\{\mu_k\}}(\widetilde{x}_\varkappa,r_\varkappa)\leqslant C\inf_{x\in Q(\widetilde{x}_\varkappa,r_\varkappa)\cap S}\Big(f^\sharp_{\{\mu_k\}}(x,r_\varkappa)+\fint_{Q(x,2)}|f(y)|\,d\mu_0(y)\Big)$$
$$\leqslant C\inf_{x\in Q(\widetilde{x}_\varkappa,r_\varkappa)\cap S}f^\sharp_{\{\mu_k\}}(x,r_\varkappa)+C\int_{Q(\widetilde{x}_\varkappa,3)}|f(y)|\,d\mu_0(y) \qquad (4.53)$$

for every $\varkappa\in\mathscr{I}$. Note that $\mu_{k(r_\varkappa)}(Q(\widetilde{x}_\varkappa,(1-\lambda)r_\varkappa)\cap S)=\mu_{k(r_\varkappa)}(Q(\widetilde{x}_\varkappa,(1-\lambda)r_\varkappa))$ because $\operatorname{supp}\mu_k=S$. Hence using (2.11) and (3.23) for $c=(1-\lambda)^{-1}$, from (4.53) we derive that

$$\mathscr{H}_n(Q_\varkappa)\big(f^\sharp_{\{\mu_k\}}(\widetilde{x}_\varkappa,r_\kappa)\big)^p=\mathscr{H}_n(Q_\varkappa)\frac{\mu_{k(\varkappa)}(Q(\widetilde{x}_\varkappa,(1-\lambda)r_\varkappa))}{\mu_{k(\varkappa)}(Q(\widetilde{x}_\varkappa,(1-\lambda)r_\varkappa))}\big(f^\sharp_{\{\mu_k\}}(\widetilde{x}_\varkappa,r_\kappa)\big)^p$$
$$\leqslant C(r_\varkappa)^{n-d}\int_{Q(\widetilde{x}_\varkappa,(1-\lambda)r_\varkappa)\cap S}\big(f^\sharp_{\{\mu_k\}}(x,r_\varkappa)\big)^p\,d\mu_{k(\varkappa)}(x)$$
$$+C\mathscr{H}^n(Q_\varkappa)\int_{Q(\widetilde{x}_\varkappa,3)}|f(y)|^p\,d\mu_0(y). \qquad (4.54)$$

It follows from Lemma 3.3 that for every fixed $k\in\mathbb{N}_0$ the multiplicity of overlapping of the sets $Q(\widetilde{x}_\varkappa,(1-\lambda)r_\varkappa)\cap S$ with $r_\varkappa=2^{-k}$ is bounded above by some positive constant $C(n)$. Furthermore, from Lemma 3.10 it follows that $Q(\widetilde{x}_\varkappa,(1-\lambda)r_\varkappa)\cap S\subset S_k(\lambda)$ for each $k\in\mathbb{N}_0$, provided that $r_\varkappa=2^{-k}$. This, (4.54) and Lemma 4.2 (with $c=3$) give the required estimate

$$\begin{aligned}\sum_{\varkappa\in\mathscr{I}}&\mathscr{H}^n(Q_\varkappa)\big(f^\sharp_{\{\mu_k\}}(\widetilde{x}_\varkappa,r_\kappa)\big)^p\\&\leqslant C\sum_{k=0}^{\infty}\sum_{r_\varkappa=2^{-k}}2^{k(d-n)}\int_{Q(\widetilde{x}_\varkappa,(1-\lambda)r_\varkappa)\cap S}\big(f^\sharp_{\{\mu_k\}}(x,r_\varkappa)\big)^p\,d\mu_{k(r_\varkappa)}(x)\\&\qquad+C\sum_{\varkappa\in\mathscr{I}}\mathscr{H}^n(Q_\varkappa)\int_{Q(\widetilde{x}_\varkappa,3)}|f(y)|^p\,d\mu_0(y)\\&\leqslant C\sum_{k=0}^{\infty}2^{k(d-n)}\int_{S_k(\lambda)}\big(f^\sharp_{\{\mu_k\}}(x,2^{-k})\big)^p\,d\mu_k(x)+C\int_S|f(y)|^p\,d\mu_0(y).\end{aligned}\tag{4.55}$$

Combining (4.54) and (4.55) we complete the proof.

Lemma 4.6. *Let $d\in[0,n]$, $p\in(1,\infty)$ and $\lambda\in(0,1)$. Let S be a d-thick closed set. Let $\{\mu_k\}=\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Let $f\colon S\to\mathbb{R}$ be a Borel function such that $\widetilde{\mathscr{N}}_{S,p,\lambda}[f]<\infty$. Let $F:=\operatorname{Ext}[f]$ be the function constructed in (4.27) and g be the function defined in (4.30). Then*

$$\|g|L_p(\mathbb{R}^n)\|+\|F|L_p(\mathbb{R}^n)\|\leqslant C\widetilde{\mathscr{N}}_{S,p,\lambda}[f].\tag{4.56}$$

The positive constant C in (4.56) is independent of f.

Proof. From (4.27) and Proposition 3.6 it is clear that $|F(x)|\leqslant\chi_S(x)|f(x)|+\chi_{\mathbb{R}^n\setminus S}(x)g(x)$. It follows from (4.30) that for some positive constant C independent of f we have

$$\|g|L_p(S)\|\leqslant C\mathscr{S}\mathscr{N}_{S,p}[f].\tag{4.57}$$

Hence it is sufficient to establish that

$$\|g|L_p(\mathbb{R}^n\setminus S)\|\leqslant C\widetilde{\mathscr{N}}_{S,p,\lambda}[f]\quad\text{and}\quad\|f|L_p(S,\mathscr{H}^n)\|\leqslant C\|f|L_p(S,\mu_0)\|,\tag{4.58}$$

with a positive constant C independent of f. It is clear that (4.56) follows from (4.57) and (4.58).

Step 1. We establish the first estimate in (4.58). Using Lemma 3.2, (3)–(5), we obtain

$$\begin{aligned}\|g|L_p(\mathbb{R}^n\setminus S)\|^p\leqslant C\sum_{\varkappa\in\mathscr{I}}\mathscr{H}^n(Q_\varkappa)\bigg[&\big(f^\sharp_{\{\mu_k\}}(\widetilde{x}_\varkappa,r_\varkappa)\big)^p\\&+\bigg(⨍_{Q(\widetilde{x}_\varkappa,r_\kappa)}|f(y)|\,d\mu_{k(\varkappa)}(y)\bigg)^p\bigg]=:S_1+S_2.\end{aligned}\tag{4.59}$$

From Lemma 4.5 and (2.15) we clearly have

$$S_1\leqslant C\big(\widetilde{\mathscr{N}}_{S,p,\lambda}[f]\big)^p.\tag{4.60}$$

Using (2.13) and arguing as in (3.39) we obtain

$$\begin{aligned}
&\Big(\fint_{Q(\widetilde{x}_\varkappa,r_\kappa)} |f(y)|\,d\mu_{k(\varkappa)}(y)\Big)^p\\
&\leqslant C\Big(\fint_{Q(\widetilde{x}_\varkappa,r_\kappa)}\Big|f(y)-\fint_{Q(\widetilde{x}_\varkappa,r_\varkappa)} f(z)\,d\mu_{k(\varkappa)}(z)\Big|\,d\mu_{k(\varkappa)}(y)\Big)^p\\
&\quad+\Big|\fint_{Q(\widetilde{x}_\varkappa,1)} f(z)\,d\mu_0(z)-\fint_{Q(\widetilde{x}_\varkappa,r_\varkappa)} f(z)\,d\mu_{k(\varkappa)}(z)\Big|^p+C\Big|\fint_{Q(\widetilde{x}_\varkappa,1)} f(z)\,d\mu_0(z)\Big|^p\\
&\leqslant C\big(f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_\varkappa,r_\varkappa)\big)^p+C\Big(\fint_{Q(\widetilde{x}_\varkappa,1)}|f(z)|\,d\mu_0(z)\Big)^p
\end{aligned}$$

for every $\varkappa\in\mathscr{I}, r_\varkappa\leqslant 2^{-1}$. (4.61)

Using Hölder's inequality, (4.60) and Lemma 4.2 for $c=1$, from (4.61) we obtain

$$S_2\leqslant CS_1+\sum_{\varkappa\in\mathscr{I}}\mathscr{H}^n(Q_\varkappa)\fint_{Q(\widetilde{x}_\varkappa,1)}|f(y)|^p\,d\mu_{k(\varkappa)}(y)\leqslant C\big(\widetilde{\mathscr{N}}_{S,p,\lambda}[f]\big)^p. \tag{4.62}$$

As a result, combination of (4.59), (4.60) and (4.62) gives the first estimate in (4.58).
Step 2. Let $\{x_j\}_{j\in\mathscr{J}}$ be a maximal (1/2)-separated subset in S. Using Definition 2.3, Lemma 3.1, Corollary 3.2 and Propositions 3.4 and 3.5, we obtain the second estimate in (4.58):

$$\begin{aligned}
\int_S|f(x)|^p\,d\mathscr{H}^n(x)&\leqslant\sum_{j\in\mathscr{J}}\int_{Q(x_j,1/2)}|f(x)|^p\,d\mathscr{H}^n(x)\\
&\leqslant C\sum_{j\in\mathscr{J}}\int_{Q(x_j,1/2)}|f(x)|^p\,d\mu_1(x)\leqslant C\int_S|f(x)|^p\,d\mu_0(x).
\end{aligned}\tag{4.63}$$

The lemma is proved.

Recall the definitions of the trace $F|_S$ of a given element $F\in W_p^1(\mathbb{R}^n)$ on the set S (Definitions 2.7 and 2.8). Also recall Definition 4.1.

Theorem 4.2. *Let $d\in[0,n]$ and $p\in(\max\{1,n-d\},\infty)$. Let $S\subset\mathbb{R}^n$ be a d-thick closed set and $\{\mu_k\}_{k\in\mathbb{N}_0}$ a d-regular sequence of measures on S. If $\widetilde{\mathscr{N}}_{S,p,\lambda}[f]<+\infty$ for some $\lambda>0$ and*

$$\lim_{r\to0}\fint_{Q(x,r)\cap S}|f(x)-f(z)|\,d\mu_{k(r)}(z)=0\quad\textit{for } (1,p)\textit{-q.e. } x\in S, \tag{4.64}$$

then $\operatorname{Ext}[f]|_S=f$.

Proof. Let $S'\subset S$ be the set of all points x where (4.64) holds. Set $F:=\operatorname{Ext}[f]$. We are going to show that $\widehat{F}=F$. Fix a cube $Q(x,r)$ with $x\in S$ and $r\in(0,1/100)$. Using (3.10) and the definition of $Q^*_\varkappa$ we see that

$$\frac{7}{8}\operatorname{diam}Q_\varkappa\leqslant r<\frac{1}{100}\quad\text{provided that } \varkappa\in I,\quad Q^*_\varkappa\cap Q(x,r)\neq\varnothing. \tag{4.65}$$

From (4.65) it follows that $r_\varkappa < 1$ and thus $\varkappa \in \mathscr{I}$. Then it follows from Lemma 3.2, (4), and Proposition 3.6, (2), (3), that

$$\begin{aligned}
\fint_{Q(x,r)} |\widehat{F}(x) - \widehat{F}(z)| \, d\mathscr{H}^n(z) &\leqslant \frac{1}{\mathscr{H}^n(Q(x,r))} \int_{Q(x,r)\cap S} |f(x) - f(z)| \, d\mathscr{H}^n(z) \\
&\quad + \frac{C}{\mathscr{H}^n(Q(x,r))} \sum_{\substack{\varkappa \in \mathscr{I} \\ Q^*_\varkappa \cap Q(x,r) \neq \varnothing}} \mathscr{H}^n(Q_\varkappa) \fint_{\widetilde{Q}_\varkappa \cap S} |f(x) - f(z)| \, d\mu_{k(\varkappa)}(z) \\
&:= J_1(x,r) + J_2(x,r).
\end{aligned} \tag{4.66}$$

Using Corollary 3.2 and (4.64), for $(1,p)$-q.e. $x \in S$ we have

$$J_1(x,r) \leqslant C \fint_{Q(x,r)\cap S} |f(x) - f(z)| \, d\mu_{k(r)}(z) \to 0, \quad \text{as } r \to 0. \tag{4.67}$$

To estimate $J_2(x,r)$ we need some preliminaries.

From (4.65) we deduce the inequality $\|x - x_\varkappa\|_\infty \leqslant r + 9r_\varkappa/8 \leqslant r + 9r/14$. Hence for all $\varkappa \in \mathscr{I}$ such that $Q^*_\varkappa \cap Q(x,r) \neq \varnothing$ we have

$$Q_\varkappa := Q(x_\varkappa, r_\varkappa) \subset Q\left(x_\varkappa, \frac{4r}{7}\right) \subset Q(x, 3r). \tag{4.68}$$

Recall that Whitney cubes have disjoint interiors. Hence, using (4.68) we obtain

$$\sum_{\substack{\varkappa \in \mathscr{I} \\ Q^*_\varkappa \cap Q(x,r) \neq \varnothing}} \mathscr{H}^n(Q_\varkappa) \leqslant \mathscr{H}^n(Q(x,3r)) \leqslant 3^n \mathscr{H}^n(Q(x,r)). \tag{4.69}$$

Now for every $\varkappa \in \mathscr{I}$ with $Q^*_\varkappa \cap Q(x,r) \neq \varnothing$ it follows from (3.10) that

$$\|x - \widetilde{x}_\varkappa\| \leqslant r + \frac{9}{8} \operatorname{diam} Q_\varkappa + 4 \operatorname{diam} Q_\varkappa \leqslant 7r. \tag{4.70}$$

For all such $\varkappa$, from (4.70) it follows that

$$\widetilde{Q}_\varkappa := Q(\widetilde{x}_\varkappa, r_\varkappa) \subset Q(x, 9r) \subset Q(\widetilde{x}_\varkappa, 17r). \tag{4.71}$$

Taking the inclusions (4.71) into account we use (2.13) and then apply Lemma 3.9 for $R = \|x - \widetilde{x}\| + 9r$ (note that $17r < 1/2$). Finally, we use (4.70) and the monotonicity of $f^\sharp_{\{\mu_k\}}(\cdot, r)$ with respect to r. This gives

$$\begin{aligned}
&\fint_{\widetilde{Q}_\varkappa \cap S} \left| f(z) - \fint_{Q(x,9r)\cap S} f(z') \, d\mu_{k(9r)}(z') \right| d\mu_{k(\varkappa)}(z) \\
&\quad \leqslant \left| \fint_{\widetilde{Q}_\varkappa \cap S} f(z) \, d\mu_{k(\varkappa)}(z) - \fint_{Q(x,9r)\cap S} f(z') \, d\mu_{k(9r)}(z') \right| \\
&\qquad + \mathscr{E}_{\mu_{k(\varkappa)}}(\widetilde{Q}_\varkappa, f) \leqslant C \min\{\mathscr{E}_{\mu_{k(R)}}(Q(\widetilde{x}_\varkappa, R), f), \mathscr{E}_{\mu_{k(R)}}(Q(x, R), f)\} \\
&\qquad + \mathscr{E}_{\mu_{k(\varkappa)}}(\widetilde{Q}_\varkappa, f) + Crf^\sharp_{\mu_k}(\widetilde{x}_\varkappa, r_\varkappa) \leqslant Crf^\sharp_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\varkappa).
\end{aligned} \tag{4.72}$$

Combining (4.69) and (4.72) we obtain

$$\begin{aligned}
J_2(x,r) &\leqslant \frac{C}{\mathscr{H}^n(Q(x,r))} \sum_{\substack{\varkappa\in\mathscr{I}\\ Q^*_\varkappa\cap Q(x,r)\neq\varnothing}} \mathscr{H}^n(Q_\varkappa)\left|f(x) - \fint_{Q(x,9r)} f(z')\,d\mu_{k(9r)}(z')\right| \\
&\quad + \frac{C}{\mathscr{H}^n(Q(x,r))} \sum_{\substack{\varkappa\in\mathscr{I}\\ Q^*_\varkappa\cap Q(x,r)\neq\varnothing}} \mathscr{H}^n(Q_\varkappa) \fint_{\widetilde{Q}_\varkappa\cap S}\biggl|f(z) \\
&\quad - \fint_{Q(x,9r)} f(z')\,d\mu_{k(9r)}(z')\biggr|\,d\mu_{k(\varkappa)}(z) \\
&\leqslant C \fint_{Q(x,9r)\cap S} |f(x)-f(z')|\,d\mu_{k(9r)}(z') \\
&\quad + C\frac{r}{\mathscr{H}^n(Q(x,r))} \sum_{\substack{\varkappa\in\mathscr{I}\\ Q^*_\varkappa\cap Q(x,r)\neq\varnothing}} \mathscr{H}^n(Q_\varkappa) f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\varkappa).
\end{aligned} \tag{4.73}$$

From (4.69), by Hölder's inequality for sums with exponents p and p' we see that the second term on the right-hand side of (4.73) is bounded above by

$$C\Biggl(\frac{r^p}{\mathscr{H}^n(Q(x,r))} \sum_{\substack{\varkappa\in\mathscr{I}\\ Q^*_\varkappa\cap Q(x,r)\neq\varnothing}} \mathscr{H}^n(Q_\varkappa)\bigl(f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\varkappa)\bigr)^p\Biggr)^{1/p} =: \bigl(K(x,r)\bigr)^{1/p}. \tag{4.74}$$

Clearly, the first term on the right-hand side of (4.73) tends to zero as $r\to+0$ for $(1,p)$-q.e. points $x\in S$ because of (4.64).

In the case $p>n$ we use (4.59) and (4.60) and obtain

$$(K(x,r))^{1/p} \leqslant C(r^{p-n}S_1)^{1/p} \leqslant Cr^{(p-n)/p}\bigl(\widetilde{\mathscr{N}}_{S,p,\lambda}[f]\bigr)^{1/p} \to 0 \quad \text{as } r\to 0, \tag{4.75}$$

for every $x\in S$ because $\widetilde{\mathscr{N}}_{S,p,\lambda}[f]<\infty$. Hence $J_2(x,r)\to 0$ as $r\to 0$ everywhere in this case. This, with (4.66) and (4.67), shows that $(1,p)$-quasi-every point $x\in S$ where (4.64) holds is a Lebesgue point of F. This fact together with Definition 2.7 proves the claim in the case $p>n$.

Consider *the case* $p\in(1,n]$. In view of Proposition 2.1, to show that $J_2(x,r)\to 0$ as $r\to+0$ for $(1,p)$-q.e. points $x\in S$ it is sufficient to verify that $K(x,r)\to 0$ as $r\to 0$ for $\mathscr{H}^{n-p}$-q.e. points $x\in S$.

In the case $p=n$ this is easy. Indeed, since $\widetilde{\mathscr{N}}_{S,p,\lambda}[f]<\infty$ and (4.60) holds, we can estimate $K(x,r)$ from above by a remainder part of a convergent series:

$$K(x,r) \leqslant C \sum_{\substack{\varkappa\in\mathscr{I}\\ \operatorname{diam} Q_\varkappa\leqslant 10r}} \mathscr{H}^n(Q_\varkappa)\bigl(f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\varkappa)\bigr)^p \to 0 \quad \text{as } r\to 0. \tag{4.76}$$

Consider now *the case* $p\in(1,n)$. Fix $j\in\mathbb{N}$ and define

$$S^j := \Bigl\{x\in S\colon \varlimsup_{r\to 0} K(x,r) > 2^{-j}\Bigr\}. \tag{4.77}$$

Fix some $\delta \in (0, 10^{-3})$. For each point $x \in S^j$ we find $\delta_x \in (0, \delta/50)$ with $K(x, \delta_x) > 2^{-j}$. The family of cubes $\{10Q(x, \delta_x)\}_{x \in S^j}$ covers S^j. Using Vitali's covering theorem (see [31], Ch. 1, § 1.5.1) we find an at most countable family of disjoint cubes $\{10Q^\delta_k\} = \{Q(x_k, 10\delta_{x_k})\}_{k \in \mathscr{J}}$ such that $S^j \subset \bigcup_{k \in \mathscr{J}} 50Q^\delta_k$. Since the cubes $10Q^\delta_k$ are disjoint, the inclusions in (4.68) imply that for each $\varkappa \in \mathscr{I}$ the cube $Q^*_\varkappa$ can have nonempty intersection with at most one cube Q^δ_k. This observation together with (4.74) and (4.77) yields

$$\begin{aligned}
\mathscr{H}^{n-p}_\delta(S^j) &\leqslant \sum_{k \in \mathscr{J}} 50^{n-p} \bigl(\operatorname{diam} Q^\delta_k\bigr)^{n-p} \leqslant C2^j \sum_{k \in \mathscr{J}} \sum_{\substack{\varkappa \in \mathscr{I} \\ Q^*_\varkappa \cap Q^\delta_k \neq \varnothing}} \mathscr{H}^n(Q_\varkappa)\bigl(f^\sharp_S(\widetilde{x}_\varkappa, r_\varkappa)\bigr)^p \\
&\leqslant C2^j \sum_{\substack{\varkappa \in \mathscr{I} \\ \operatorname{diam} Q_\varkappa < 10\delta}} \mathscr{H}^n(Q_\varkappa)\bigl(f^\sharp_S(\widetilde{x}_\varkappa, r_\varkappa)\bigr)^p. \qquad (4.78)
\end{aligned}$$

Recall that according to our assumption $\widetilde{\mathscr{N}}_{S,p,\lambda}[f] < \infty$. This fact, with estimates (4.59) and (4.60), shows that the right-hand side of (4.78) is a remainder part of a convergent series and hence tends to zero as $\delta \to 0$. This clearly implies that $\mathscr{H}^{n-p}(S^j) = \lim_{\delta \to 0} \mathscr{H}^{n-p}_\delta(S^j) = 0$, and hence $\mathscr{H}^{n-p}(\bigcup_j S^j) = 0$. As a result, the arguments above and (4.73) imply that

$$J_2(x, r) \to 0, \quad r \to 0, \quad \text{for } (1,p)\text{-q.e. } x \in S. \qquad (4.79)$$

Combining (4.66), (4.67) and (4.79) we complete the proof in the case $p \in (1, n)$. Theorem 4.2 is proved.

Now we can formulate the reverse trace theorem. Recall our construction of the extension operator (4.27).

Theorem 4.3. *Let $p \in (1, \infty)$, $\lambda \in (0,1)$, $d \in [0, n]$ and $d > n - p$. Let $S \subset \mathbb{R}^n$ be a d-thick closed set. Let $\{\mu_k\}_{k \in \mathbb{N}_0}$ be a d-regular sequence of measures on S. Let $\widetilde{\mathscr{N}}_{S,p,\lambda}[f] < \infty$ and*

$$\lim_{r \to 0} \fint_{Q(x,r) \cap S} |f(x) - f(z)|\, d\mu_{k(r)}(z) = 0 \quad \textit{for } (1,p)\textit{-q.e. } x \in S. \qquad (4.80)$$

Then $f \in W^1_p(\mathbb{R}^n)|_S$ and

$$\|f | W^1_p(\mathbb{R}^n)|_S\| \leqslant C \widetilde{\mathscr{N}}_{S,p,\lambda}[f], \qquad (4.81)$$

where the positive constant C depends only on the parameters p, n, λ, d and $C_{\{\mu_k\}}$.

Proof. Assume that $\widetilde{\mathscr{N}}_{S,p,\lambda}[f] < \infty$. Then it is obvious that $f_\varkappa < \infty$ for all $\varkappa \in \mathscr{I}$. Consequently, (4.27) yields a well-defined function $F := \operatorname{Ext}[f] \in C^\infty(\mathbb{R}^n \setminus S)$, whose pointwise restriction to S coincides with the original function f. Applying Theorem 4.2 gives $F|_S = f$, and hence $\operatorname{Tr}|_S \circ \operatorname{Ext} = \operatorname{Id}$.

From Theorem D and Lemmas 4.4 and 4.6 it follows that $F \in W^1_p(\mathbb{R}^n)$ and

$$\|F | W^1_p(\mathbb{R}^n)\| \leqslant C \widetilde{\mathscr{N}}_{S,p,\lambda}[f], \qquad (4.82)$$

where the positive constant C does not depend on f. Combining Definition 2.9 and (4.82) we obtain (4.81). The theorem is proved.

4.3. The proof of the main result. Now we are ready to prove the main result in this paper.

Proof of Theorem 2.1. Given $f \in W_p^1(\mathbb{R}^n)|_S$, from Theorem 4.1 it follows that

$$\widetilde{\mathscr{N}}_{S,p,\lambda}[f] \leqslant C\|f|W_p^1(\mathbb{R}^n)|_S\|, \tag{4.83}$$

where the positive constant C does not depend on f. Furthermore, from Lemma 4.3 we deduce that (2.16) holds.

Conversely, assume that $\widetilde{\mathscr{N}}_{S,p,\lambda}[f] < +\infty$. Then we deduce from Theorem 4.3 that $f \in W_p^1(\mathbb{R}^n)|_S$ and

$$\|f|W_p^1(\mathbb{R}^n)|_S\| \leqslant C\widetilde{\mathscr{N}}_{S,p,\lambda}[f], \tag{4.84}$$

where the positive constant C does not depend on f.

Clearly operator Ext constructed in (4.27) is linear. Furthermore, it was mentioned in the proof of Theorem 4.3 that $\operatorname{Tr}|_S \circ \operatorname{Ext} = \operatorname{Id}$.

Finally, estimates (4.83) and (4.84) obviously imply (2.17) and boundedness of the operator Ext. The proof is complete.

Remark 4.2. As we noted above, while constructing the extension operator we chose a d-regular sequence of measures. It is remarkable, however, that both the proofs of Theorems 2.1 and 4.3 and the constants in these proofs depend only on the constant $C_{\{\mu_k\}}$ in Remark 2.2 but are independent of the concrete choice of the d-regular sequence of measures.

Remark 4.3. We would like to draw the reader's attention to the fact that (in general) it is impossible to obtain (2.16) from the condition $\widetilde{\mathscr{N}}_{S,p,\lambda}[f] < \infty$ alone. The point is that, given a set $E \subset \mathbb{R}^n$ with $\mu_k(E) = 0$, $k \in \mathbb{N}$, we cannot claim that $C_{1,p}(E) = 0$. Hence, changing the given function $f\colon S \to \mathbb{R}$ on a set E such that $\mu_k(E) = 0$, $k \in \mathbb{N}$, does not affect the value of $\widetilde{\mathscr{N}}_{S,p,\lambda}[f]$, but can violate (2.16).

§ 5. A simplified criterion for sets with porous boundary

In this section we are going to prove Theorem 2.2, which is a simplified version of Theorem 2.1 in the case of sets with porous boundary. Recall Definition 2.6.

Let S be a closed set in $\mathbb{R}^n$ with porous boundary. Given $\lambda > 0$, for every $k \in \mathbb{N}_0$ we set

$$\begin{gathered}\partial S_k^+(\lambda) := \{x \in \partial S\colon \text{there exists } y \in Q(x, 2^{-k}) \text{ for which } Q(y, \lambda 2^{-k}) \subset \mathbb{R}^n \setminus S\},\\ \partial S_k^-(\lambda) := \{x \in \partial S\colon \text{there exists } y' \in Q(x, 2^{-k}) \text{ for which } Q(y', \lambda 2^{-k}) \subset S \setminus \partial S\}.\end{gathered} \tag{5.1}$$

From Definition 2.6 it is clear that if ∂S is porous then there exists a number $\lambda > 0$ such that for all $k \in \mathbb{N}_0$

$$\partial S = \partial S_k^+(\lambda) \cup \partial S_k^-(\lambda). \tag{5.2}$$

Definition 5.1. Let S be a closed nonempty subset of $\mathbb{R}^n$. We set

$$\Sigma_k := \Sigma_k(S) := \{x \in S\colon \operatorname{dist}(x, \partial S) \leqslant 2^{-k}\}, \qquad k \in \mathbb{N}_0. \tag{5.3}$$

Lemma 5.1. *Let $d \in [0,n]$, $p \in (1,\infty)$ and $p > n-d$. Let $S \subset \mathbb{R}^n$ be a d-thick closed set and $\{\mu_k\}_{k\in\mathbb{N}_0}$ a d-regular sequence of measures on S. Assume that ∂S is porous. Then there exists a positive constant C depending only on d, n, p and $C_{\{\mu_k\}}$ such that for every $F \in W_p^1(\mathbb{R}^n)$*

$$\sum_{k=1}^{\infty} 2^{kp(1-(n-d)/p)} \int_{\Sigma_k} \big(\mathscr{E}_{\mu_k}(F|_S, Q(x,2^{-k}))\big)^p \, d\mu_k(x) \leqslant C\|F|W_p^1(\mathbb{R}^n)\|^p. \tag{5.4}$$

Proof. Given $k \in \mathbb{N}$, let $\{x_{k,j}\}_{j\in\mathscr{J}_k}$ be an arbitrary maximal 2^{-k}-separated subset of Σ_k. For every $k \in \mathbb{N}_0$ we set $Q_{k,j} := Q(x_{k,j}, 2^{-k})$, $j \in \mathscr{J}_k$.
Step 1. Given $k \in \mathbb{N}$ and $j \in \mathscr{J}_k$, it is clear that $2Q_{k,j} \supset Q(x,2^{-k})$ for every $x \in Q_{k,j}$. Hence elementary computations (similar to (3.35)) give

$$\widetilde{\mathscr{E}}_{\mu_k}(F|_S, Q(x,2^{-k})) \leqslant C\widetilde{\mathscr{E}}_{\mu_k}(F|_S, 2Q_{k,j}). \tag{5.5}$$

Using Lemma 3.1, (1), estimates (2.10) and (5.5) and Remark 2.4 we obtain

$$\begin{aligned} \int_{\Sigma_k} \big(\mathscr{E}_{\mu_k}(F|_S, Q(x,2^{-k}))\big)^p \, d\mu_k(x) &\leqslant \sum_{j\in\mathscr{J}_k} \int_{Q_{k,j}} \big(\widetilde{\mathscr{E}}_{\mu_k}(F|_S, Q(x,2^{-k}))\big)^p \, d\mu_k(x) \\ &\leqslant C2^{-kd} \sum_{j\in\mathscr{J}_k} \big(\widetilde{\mathscr{E}}_{\mu_k}(F|_S, 2Q_{k,j})\big)^p. \end{aligned} \tag{5.6}$$

Fix some $q \in (\max\{1, n-d\}, p)$. Arguing as in (3.50), from (5.6) we derive the following estimate:

$$\begin{aligned} &\sum_{k=1}^{\infty} 2^{kp(1-(n-d)/p)} \int_{\Sigma_k} \big(\mathscr{E}_{\mu_k}(F|_S, Q(x,2^{-k}))\big)^p \, d\mu_k(x) \\ &\qquad \leqslant C \sum_{k=1}^{\infty} 2^{kp(1-n/p)} \sum_{j\in\mathscr{J}_k} \big(\mathscr{E}_{\mu_k}(F|_S, 2Q_{k,j})\big)^p \\ &\qquad \leqslant C \sum_{k=1}^{\infty} \sum_{j\in\mathscr{J}_k} \mathscr{H}^n(Q_{k,j}) \bigg(\fint_{2Q_{k,j}} |\nabla F(y)|^q \, d\mathscr{H}^n(y) \bigg)^{p/q}. \end{aligned} \tag{5.7}$$

Step 2. Fix some $\lambda > 0$ such that (5.2) holds. Since all the cubes $Q_{k,j}$ are assumed to be closed, $Q_{k,j} \cap \partial S \neq \varnothing$ for all $k \in \mathbb{N}$ and $j \in \mathscr{J}_k$. Let $\mathscr{J}_k^1$ be the set of all $j \in \mathscr{J}_k$ such that $Q_{k,j} \cap \partial S_k^+(\lambda) \neq \varnothing$. Let $\mathscr{J}_k^2$ be the set of all $j \in \mathscr{J}_k$ such that $Q_{k,j} \cap \partial S_k^-(\lambda) \neq \varnothing$. It is clear that $\mathscr{J}_k = \mathscr{J}_k^1 \cup \mathscr{J}_k^2$ for every $k \in \mathbb{N}$.

Let W^1 and W^2 be Whitney decompositions of $\mathbb{R}^n \backslash S$ and $S \backslash \partial S$, respectively. Let $\mathscr{I}^1$ and $\mathscr{I}^2$ be the sets of indices corresponding to the cubes with edge length $\leqslant 1$ in W^1 and W^2, respectively.

For every $k \in \mathbb{N}$ and $j \in \mathscr{J}_k^1$ we choose a point $x'_{k,j} \in Q_{k,j} \cap \partial S_k^+(\lambda)$, and for every $j \in \mathscr{J}_k^2$ we choose a point $x'_{k,j} \in Q_{k,j} \cap \partial S_k^-(\lambda)$. Since $Q(x'_{k,j}, 2^{-k}) \subset 2Q_{k,j}$ for every k and j, we apply Lemma 3.11 and for each $j \in \mathscr{J}_k^1$ we find a point $y(x'_{k,j}) \in 2Q_{k,j} \cap \mathbb{R}^n \setminus S$ such that

$$\frac{\lambda}{5} 2^{-k} \leqslant 2r_{\varkappa} \leqslant 2^{-k} \quad \text{for every } W^1 \ni Q_{\varkappa} \ni y(x'_{k,j}). \tag{5.8}$$

Similarly, for each $j \in \mathscr{J}_k^2$ we find a point $z(x'_{k,j}) \in \operatorname{int} S \cap 2Q_{k,j}$ such that

$$\frac{\lambda}{5} 2^{-k} \leqslant 2r_{\varkappa'} \leqslant 2^{-k} \quad \text{for every } W^2 \ni Q_{\varkappa'} \ni z(x'_{k,j}). \tag{5.9}$$

Consider a map Θ^1 that takes a pair (k,j) ($k \in \mathbb{N}, j \in \mathscr{J}_k^1$) and returns an arbitrary index $\varkappa = \Theta^1(k,j) \in \mathscr{I}^1$ such that (5.8) holds. Similarly, we build a map Θ^2 that takes (k,j) and returns an arbitrary $\varkappa' = \Theta^2(k,j) \in \mathscr{I}^2$ such that (5.9) holds. Arguing as in (4.6), from (5.8) and (5.9) we derive the existence of a positive constant $C(n,\lambda)$ such that for every $\varkappa \in \mathscr{I}^1$ and $\varkappa' \in \mathscr{I}^2$

$$\begin{aligned} \sum_{(k,j)\in(\Theta^1)^{-1}(\varkappa)} \mathscr{H}^n(Q_{k,j}) &\leqslant C(n,\lambda)\mathscr{H}^n(Q_{\varkappa}), \\ \sum_{(k,j)\in(\Theta^2)^{-1}(\varkappa')} \mathscr{H}^n(Q_{k,j}) &\leqslant C(n,\lambda)\mathscr{H}^n(Q_{\varkappa'}). \end{aligned} \tag{5.10}$$

Step 3. Let $k \in \mathbb{N}$, $j \in \mathscr{J}_k^1$ and $\varkappa = \Theta^1(k,j)$. From (5.8) it follows that $2Q_{k,j} \subset 5Q(x, 2^{-k})$ for every $x \in Q_{\varkappa}$. Using this and (3.1) we obtain

$$\begin{aligned} &⨍_{2Q_{k,j}} |\nabla F(y)|\, d\mathscr{H}^n(y) \leqslant C \inf_{x\in Q_{\varkappa}} ⨍_{5Q(x,2^{-k})} |\nabla F(y)|^q\, d\mathscr{H}^n(y) \\ &\qquad \leqslant C \inf_{x\in Q_{\varkappa}} \mathrm{M}_{>\frac{5}{2^k}}[|\nabla F|^q](x) \leqslant C \inf_{x\in Q_{\varkappa}} \mathrm{M}[|\nabla F|^q](x). \end{aligned} \tag{5.11}$$

Similarly, if $k \in \mathbb{N}$, $j \in \mathscr{J}_k^2$ and $\varkappa' = \Theta^2(k,j)$, then

$$⨍_{2Q_{k,j}} |\nabla F(y)|^q\, d\mathscr{H}^n(y) \leqslant C \inf_{x\in Q_{\varkappa'}} \mathrm{M}[|\nabla F|^q](x). \tag{5.12}$$

Combining (5.10) and (5.11) we have

$$\begin{aligned} &\sum_{k\in\mathbb{N}} \sum_{j\in\mathscr{J}_k^1} \mathscr{H}^n(Q_{k,j}) \Big(⨍_{2Q_{k,j}} |\nabla F(y)|^q\, d\mathscr{H}^n(y) \Big)^{p/q} \\ &\qquad \leqslant C \sum_{\varkappa\in\mathscr{I}^1} \sum_{(k,j)\in(\Theta^1)^{-1}(\varkappa)} \mathscr{H}^n(Q_{k,j}) \inf_{x\in Q_{\varkappa}} \big(\mathrm{M}[|\nabla F|^q](x)\big)^{p/q} \\ &\qquad \leqslant C \sum_{\varkappa\in\mathscr{I}^1} \mathscr{H}^n(Q_{\varkappa}) \inf_{x\in Q_{\varkappa}} \big(\mathrm{M}[|\nabla F|^q](x)\big)^{p/q} \\ &\qquad \leqslant C \int_{\mathbb{R}^n\setminus S} \big(\mathrm{M}[|\nabla F|^q](x)\big)^{p/q}\, d\mathscr{H}^n(x). \end{aligned} \tag{5.13}$$

Similarly, from (5.10) and (5.12) we obtain

$$\begin{aligned} &\sum_{k\in\mathbb{N}} \sum_{j\in\mathscr{J}_k^2} \mathscr{H}^n(Q_{k,j}) \Big(⨍_{2Q_{k,j}} |\nabla F(y)|^q\, d\mathscr{H}^n(y) \Big)^{p/q} \\ &\qquad \leqslant C \int_{\operatorname{int} S} \big(\mathrm{M}[|\nabla F|^q](x)\big)^{p/q}\, d\mathscr{H}^n(x). \end{aligned} \tag{5.14}$$

We combine estimates (5.7), (5.13) and (5.14) and apply Theorem A with $\alpha=0$, $d=n$ and $\mathfrak{m}=\mathscr{H}^n$. This gives

$$\sum_{k=1}^{\infty} 2^{kp(1-(n-d)/p)} \int_{\Sigma_k} \big(\mathscr{E}_{\mu_k}(F|_S, Q(x,2^{-k}))\big)^p \, d\mu_k(x) \leqslant C \int_{\mathbb{R}^n} \big(\mathrm{M}[|\nabla F|^q](x)\big)^{p/q} \, d\mathscr{H}^n(x) \leqslant C\|F|W^1_p(\mathbb{R}^n)\|^p. \tag{5.15}$$

Lemma 5.1 is proved.

Lemma 5.2. *Let $d\in[0,n]$, $p\in(1,\infty)$ and $p>n-d$. Let $S\subset\mathbb{R}^n$ be a d-thick closed set with porous boundary ∂S. Let $\{\mu_k\}=\{\mu_k\}_{k\in\mathbb{N}_0}$ be a d-regular sequence of measures on S. Assume that $f\in L_1^{\mathrm{loc}}(\mathbb{R}^n,\mu_k)$ for some (and hence every) $k\in\mathbb{N}_0$. Then*

$$\sum_{\varkappa\in\mathscr{I}} \mathscr{H}^n(Q_\varkappa) 2^{kp} \big(\mathscr{E}_{\mu_k}(f, Q(\widetilde{x}_\varkappa, 2^{-k}))\big)^p \leqslant C\big(\mathscr{BN}_{S,p}[f]\big)^p + C\|f|L_p(S,\mu_0)\|^p. \tag{5.16}$$

The positive constant C in (5.16) does not depend on f.

Proof. Fix an arbitrary positive number $k\geqslant 5$. We set $\mathscr{I}_k:=\{\varkappa\in\mathscr{I}: r_\varkappa\leqslant 2^{-k}\}$. Using Vitali's covering theorem (see [31], § 1.5.1) we find an index set $\widehat{\mathscr{I}_k}\subset\mathscr{I}_k$ such that the cubes belonging to the family $\{Q_\varkappa(\widetilde{x}_\varkappa,2^{-k})\}_{\varkappa\in\widehat{\mathscr{I}_k}}$ are mutually disjoint and

$$\bigcup_{\varkappa\in\mathscr{I}_k} Q(\widetilde{x}_\varkappa, 2^{-k}) \subset \bigcup_{\varkappa\in\widehat{\mathscr{I}_k}} Q\left(\widetilde{x}_\varkappa, \frac{5}{2^k}\right). \tag{5.17}$$

Note that if $Q(\widetilde{x}_{\varkappa'},1/2^k)\cap Q(\widetilde{x}_\varkappa,5/2^k)\neq\varnothing$ for some $\varkappa'\in\mathscr{I}_k$ and $\varkappa\in\widehat{\mathscr{I}_k}$, then $Q(\widetilde{x}_{\varkappa'},2^{-k})\subset Q(\widetilde{x}_\varkappa,7/2^k)$. Using Remark 2.4 and reasoning as in (3.35) it is easy to show that for such $\varkappa$ and $\varkappa'$

$$\mathscr{E}_{\mu_k}(f, Q(\widetilde{x}_{\varkappa'}, 2^{-k})) \leqslant C\mathscr{E}_{\mu_k}\left(f, Q\left(\widetilde{x}_\varkappa, \frac{7}{2^k}\right)\right). \tag{5.18}$$

Since different Whitney cubes have disjoint interiors, we find that for every $\varkappa\in\widehat{\mathscr{I}_k}$

$$\sum_{\substack{\varkappa'\in\mathscr{I}_k \\ Q(\widetilde{x}_{\varkappa'},2^{-k})\cap Q(\widetilde{x}_\varkappa,5/2^k)\neq\varnothing}} \mathscr{H}^n(Q_{\varkappa'}) \leqslant \sum_{\substack{\varkappa'\in\mathscr{I} \\ Q(\widetilde{x}_{\varkappa'},2^{-k})\subset Q(\widetilde{x}_\varkappa,7/2^k)}} \mathscr{H}^n(Q_{\varkappa'}) \leqslant C2^{-kn}. \tag{5.19}$$

Combining (5.17), (5.18) and (5.19) we obtain

$$\sum_{\varkappa\in\mathscr{I}_k} \mathscr{H}^n(Q_\varkappa)\big(\mathscr{E}_{\mu_k}(f, Q(\widetilde{x}_\varkappa, 2^{-k}))\big)^p \leqslant C2^{-kn} \sum_{\varkappa\in\widehat{\mathscr{I}_k}} \left(\mathscr{E}_{\mu_k}\left(f, Q\left(\widetilde{x}_\varkappa, \frac{7}{2^k}\right)\right)\right)^p. \tag{5.20}$$

It is clear that $Q(\widetilde{x}_\varkappa, 7/2^k) \subset Q(x, 15/2^k)$ for every $x \in S \cap Q(\widetilde{x}_\varkappa, 7/2^k)$. Using this observation, Remark 2.4, (2.11) and (2.12) and arguing as in (3.35), for every $\varkappa \in \mathscr{I}_k$ we have

$$\begin{aligned}\Big(\mathscr{E}_{\mu_k}\Big(f, Q\Big(\widetilde{x}_\varkappa, \frac{7}{2^k}\Big)\Big)\Big)^p &\leqslant C \inf_{x \in Q(\widetilde{x}_\varkappa, 7/2^k) \cap S} \big(\mathscr{E}_{\mu_k}(f, Q(x, 2^{4-k}))\big)^p \\ &\leqslant C \inf_{x \in Q(\widetilde{x}_\varkappa, 7/2^k) \cap S} \big(\mathscr{E}_{\mu_{k-4}}(f, Q(x, 2^{-(k-4)}))\big)^p \\ &\leqslant C 2^{(k-4)d} \int_{Q(\widetilde{x}_\varkappa, 7/2^k) \cap S} \big(\mathscr{E}_{\mu_{k-4}}(f, Q(x, 2^{-(k-4)}))\big)^p \, d\mu_{k-4}(x). \end{aligned} \tag{5.21}$$

It is clear that $Q(\widetilde{x}_\varkappa, 7/2^k) \cap S \subset \Sigma_{k-4}$. Furthermore, according to our construction of $\widehat{\mathscr{I}_k}$ the multiplicity of overlapping of the sets $Q(\widetilde{x}_\varkappa, 7/2^k) \cap S$, $\varkappa \in \widehat{\mathscr{I}_k}$ is finite and independent of k. Hence, substituting (5.21) into (5.20) and using Proposition 3.5 we obtain

$$\sum_{k=5}^{\infty} \sum_{\varkappa \in \mathscr{I}_k} \mathscr{H}^n(Q_\varkappa) 2^{kp} \big(\mathscr{E}_{\mu_k}(f, Q(\widetilde{x}_\varkappa, 2^{-k}))\big)^p \leqslant C \big(\mathscr{BN}_{S,p}[f]\big)^p. \tag{5.22}$$

Now we use Remark 2.4 and Hölder's inequality. Then we use Lemma 3.7 and Lemma 4.2 for $c = 1$. This gives

$$\begin{aligned}&\sum_{k=1}^{5} \sum_{\varkappa \in \mathscr{I}_k} \mathscr{H}^n(Q_\varkappa) 2^{kp} \big(\mathscr{E}_{\mu_k}(f, Q(\widetilde{x}_\varkappa, 2^{-k}))\big)^p \\ &\qquad \leqslant C \sum_{k=1}^{5} \sum_{\varkappa \in \mathscr{I}_k} \mathscr{H}^n(Q_\varkappa) \fint_{Q(\widetilde{x}_\varkappa, 2^{-k})} |f(y)|^p \, d\mu_k(y) \\ &\qquad \leqslant C \sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(Q_\varkappa) \fint_{Q(\widetilde{x}_\varkappa, 1)} |f(y)|^p \, d\mu_0(y) \leqslant C \int_S |f(y)|^p \, d\mu_0(y). \end{aligned} \tag{5.23}$$

To complete the proof it is sufficient to put (5.22) and (5.23) together.

Lemma 5.3. *Let $\lambda \in (0,1)$, $d \in [0,n]$, $p \in (1,\infty)$ and $p > n-d$. Let S be a d-thick closed set with porous boundary ∂S. Let $\{\mu_k\} = \{\mu_k\}_{k \in \mathbb{N}_0}$ be a d-regular sequence of measures on S. Assume that $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n, \mu_k)$ for every $k \in \mathbb{N}_0$. Then for each $\lambda \in (0,1)$*

$$\big(\widetilde{\mathscr{BN}}_{S,p,\lambda}[f]\big)^p \leqslant C\big[(\mathscr{BN}_{S,p}[f])^p + \|f|L_p(S,\mu_0)\|^p\big]. \tag{5.24}$$

The positive constant C in (5.24) is independent of f.

Proof. It is clear from Lemma 3.5 and Remark 2.4 that for every $\varkappa \in \mathscr{I}$ (recall that $\operatorname{diam} Q_\varkappa \leqslant 1$ for such $\varkappa$) we can choose $j_\varkappa \in \mathbb{N}_0$ so that $0 \leqslant j_\varkappa < |\log_2 r_\varkappa|$ and

$$f^\sharp_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\varkappa) \leqslant C 2^{j_\varkappa} \widetilde{\mathscr{E}}_{\mu_{j_\varkappa}}(f, Q(\widetilde{x}_\varkappa, 2^{-j_\varkappa})) \leqslant C 2^{j_\varkappa} \mathscr{E}_{\mu_{j_\varkappa}}(f, Q(\widetilde{x}_\varkappa, 2^{-j_\varkappa})). \tag{5.25}$$

Given $k \in \mathbb{N}$, let $\mathscr{I}_k := \{\varkappa \in \mathscr{I} \colon j_\varkappa = k\}$. Estimate (5.25) together with Lemmas 4.1, 4.2 and 5.2 allow us to deduce that

$$
\begin{aligned}
\sum_{k=1}^{\infty} 2^{k(d-n)} \int_{S_k(\lambda)} \bigl(f^{\sharp}_{\{\mu_k\}}(x,2^{-k})\bigr)^p \, d\mu_k(x) &\leqslant \sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(Q_\varkappa) \biggl[(f^{\sharp}_{\{\mu_k\}}(\widetilde{x}_\varkappa, r_\varkappa))^p \\
&\qquad + \fint_{Q(\widetilde{x}_\varkappa,15)} |f(y)|^p \, d\mu_0(y)\biggr] \\
&\leqslant C \sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(Q_\varkappa) 2^{kp} \bigl(\mathscr{E}_{\mu_k}(f, Q(\widetilde{x}_\varkappa, 2^{-k}))\bigr)^p \\
&\qquad + \sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(Q_\varkappa) \fint_{Q(\widetilde{x}_\varkappa,15)} |f(y)|^p \, d\mu_0(y) \\
&\leqslant C\bigl[(\mathscr{B}\mathscr{N}_{S,p}[f])^p + \|f|L_p(S,\mu_0)\|^p\bigr].
\end{aligned} \tag{5.26}
$$

The proof is complete.

Proof of Theorem 2.2. Let $f \in W_p^1(\mathbb{R}^n)|_S$. It follows from Lemma 5.1 and (4.16) that

$$\mathscr{N}_{S,p}[f] \leqslant C\|f|W_p^1(\mathbb{R}^n)|_S\|, \tag{5.27}$$

where the positive constant C does not depend on f. Furthermore, from Lemma 4.3 we deduce that (2.18) holds.

Conversely, let $\mathscr{N}_{S,p}[f] < +\infty$. Then, from Lemma 5.3 we deduce that $\widetilde{\mathscr{N}}_{S,p,\lambda}[f] \leqslant C\mathscr{N}_{S,p}[f]$ for some $\lambda \in (0,1)$, with a positive constant C independent of f. Hence, from Theorem 4.3 we deduce that $f \in W_p^1(\mathbb{R}^n)|_S$ and

$$\|f|W_p^1(\mathbb{R}^n)|_S\| \leqslant C\mathscr{N}_{S,p}[f], \tag{5.28}$$

where the positive constant C does not depend on f.

Finally, estimates (5.27) and (5.28) obviously imply (2.19). The proof is complete.

§ 6. Example

The aim of this section is to present several useful examples, which show the power of our main results.

Example 6.1. Let S be an Ahlfors n-regular closed subset of $\mathbb{R}^n$, and let $p > 1$. In this case we can take $\mu_k = \mathscr{H}^n\lfloor S$ for every $k \in \mathbb{N}_0$ to obtain an n-regular sequence of measures on S. Hence, taking Remark 2.4 into account, for every $t \in [0,1)$ we have

$$f^{\sharp}_{\{\mu_k\}}(x,t) \approx \sup_{r \in (t,1)} \frac{1}{r} \fint_{Q(x,r)\cap S} \biggl| f(y) - \fint_{Q(x,r)\cap S} f(z) \, d\mathscr{H}^n(z) \biggr| \, d\mathscr{H}^n(y), \quad x \in S. \tag{6.1}$$

To simplify our notation we set $f^{\sharp}_S := f^{\sharp}_{\{\mu_k\}}$ in this case. This notation was used in [23].

We establish the following key estimate:

$$\widetilde{\mathscr{B}\mathscr{N}}_{S,p,\lambda}[f] \leqslant C(\mathscr{S}\mathscr{N}_{S,p}[f] + \|f|L_p(S)\|), \tag{6.2}$$

where the positive constant C does not depend on f.

We combine Theorem C, (2), and Lemmas 4.1 and 4.2. This gives

$$(\widetilde{\mathscr{B}\mathscr{N}}_{S,p,\lambda}[f])^p \leqslant C \sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(\mathscr{U}_\varkappa)\big(f^\sharp_S(\widetilde{x}_\varkappa, r_\varkappa)\big)^p + C\int_S |f(x)|^p \, d\mathscr{H}^n(x). \tag{6.3}$$

Adapting the arguments in the proof of Lemma 4.5 to this case (and also using the monotonicity of $f^\sharp_S(\cdot,t)$ with respect to t) we easily obtain

$$\begin{aligned}\sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(\mathscr{U}_\varkappa)\big(f^\sharp_S(\widetilde{x}_\varkappa, r_\varkappa)\big)^p &\leqslant \sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(\mathscr{U}_\varkappa) \inf_{x \in \mathscr{U}_\varkappa} \big(f^\sharp_S(x, r_\varkappa)\big)^p \\ &\quad + \sum_{\varkappa \in \mathscr{I}} \mathscr{H}^n(\mathscr{U}_\varkappa) \int_{Q(\widetilde{x}_\varkappa,3)} |f(x)|^p \, d\mathscr{H}^n(x) \\ &\leqslant C\bigg(\sum_{\varkappa \in \mathscr{I}} \int_{\mathscr{U}_\varkappa} \big(f^\sharp_S(x)\big)^p \, d\mathscr{H}^n(x) + \|f|L_p(S,\mathscr{H}^n)\|^p\bigg) \\ &\leqslant C\bigg(\int_S \big(f^\sharp_S(x)\big)^p \, d\mathscr{H}^n(x) + \|f|L_p(S,\mathscr{H}^n)\|^p\bigg) = C(\mathscr{S}\mathscr{N}_{S,p}[f])^p. \end{aligned} \tag{6.4}$$

Now (6.2) clearly follows from (6.3) and (6.4).

Recall that $\mathscr{H}^n$-a.e. points $x \in S$ are Lebesgue points of a function $f \in L_p(S)$. If we relax the notion of the trace of a given $F \in W^1_p(\mathbb{R}^n)$ and identify $F|_S$ with the class of functions equivalent modulo coincidence $\mathscr{H}^n$-a.e. on S, then from (6.2) we deduce the following simplified version of Theorem 2.1.

Let $S \subset \mathbb{R}^n$ be an Ahlfors n-regular set. Then a function $f\colon S \to \mathbb{R}$ belongs to the trace space $W^1_p(\mathbb{R}^n)|_S$ if and only if

$$\|f|L_p(S,\mathscr{H}^n)\| + \|f^\sharp_S|L_p(S,\mathscr{H}^n)\| < +\infty. \tag{6.5}$$

Moreover, the operator Ext *constructed in* (4.27) *is a bounded linear extension operator* $\operatorname{Ext}\colon W^1_p(\mathbb{R}^n)|_S \to W^1_p(\mathbb{R}^n)$ *and*

$$\|f|L_p(S,\mathscr{H}^n)\| + \|f^\sharp_S|L_p(S,\mathscr{H}^n)\| \approx \|f|W^1_p(\mathbb{R}^n)|_S\|.$$

This result is a slight modification of the corresponding result obtained by Shvartsman [23] in the context of first-order Sobolev spaces.

Example 6.2. Let $d \in [0,n)$ and $p \in (\max\{1, n-d\}, \infty)$. Let S be an Ahlfors d-regular subset of $\mathbb{R}^n$. In this case there exists a simple d-regular sequence of measures on S. More precisely, set $\mu_k = \mathscr{H}^d\lfloor S$ for every $k \in \mathbb{N}_0$. Clearly, $\mathscr{H}^n(S) = 0$. Furthermore, $\operatorname{int} S = \varnothing$ and ∂S is porous (see Proposition 2.3).

Note that the measure $\mathscr{H}^d\lfloor S$ is Radon. Hence, from Theorem 1 in [31], § 1.7.1, we conclude that if $f \in L^{\mathrm{loc}}_1(S, \mathscr{H}^d\lfloor S)$, then

$$\fint_{Q(x,r)\cap S} |f(x) - f(y)| \, d\mathscr{H}^d(y) = 0$$

for $\mathscr{H}^d$-almost every $x \in S$.

Now we apply Theorem 2.2 and take the above facts into account. If we relax the notion of the trace of a given element $F \in W_p^1(\mathbb{R}^n)$ and identify $F|_S$ with the class of functions equivalent modulo coincidence $\mathscr{H}^d$-a.e. on S, we obtain the following simplified version of Theorem 2.2.

Given an Ahlfors d-regular set $S \subset \mathbb{R}^n$, for some $d \in [0,n)$, let $p \in (\max\{1, n-d\}, \infty)$. Then a function $f\colon S \to \mathbb{R}$ belongs to the trace space $W_p^1(\mathbb{R}^n)|_S$ if and only if

$$\|f|L_p(S, \mathscr{H}^d\lfloor S)\| + \biggl(\sum_{k=1}^{\infty} 2^{kp(1-(n-d)/p)} \int_S \bigl(\mathscr{E}_{\mathscr{H}^d\lfloor S}(f, Q(x,2^{-k}))\bigr)^p \, d\mathscr{H}^d(x)\biggr)^{1/p} < \infty.$$

Moreover, the operator Ext *constructed in* (4.27) *is a bounded linear extension operator from $W_p^1(\mathbb{R}^n)|_S$ to $W_p^1(\mathbb{R}^n)$* and

$$\|f|W_p^1(\mathbb{R}^n)|_S\| \approx \|f|L_p(S, \mathscr{H}^d\lfloor S)\| + \biggl(\sum_{k=0}^{\infty} 2^{kp(1-(n-d)/p)} \int_S \bigl(\mathscr{E}_{\mathscr{H}^d\lfloor S}(f, Q(x,2^{-k}))\bigr)^p \, d\mathscr{H}^d(x)\biggr)^{1/p}.$$

Note that this result coincides with that obtained in [21] in the context of first-order Sobolev spaces.

In the simplest case when $S = \mathbb{R}^d \subset \mathbb{R}^n$, $d = 1, \dots, n-1$, this is a classical result. Namely, $W_p^1(\mathbb{R}^n)|_{\mathbb{R}^d} = B_{p,p}^{1-(n-d)/p}(\mathbb{R}^d)$. This fact together with Theorem 2.2 implies that $\mathscr{BN}_{\mathbb{R}^d,p,\lambda}[f] \approx \|f|B_{p,p}^{1-(n-d)/p}(\mathbb{R}^d)\|$. This equivalence has motivated us to call $\mathscr{BN}_{S,p}[f]$ a 'Besov-type seminorm'.

Example 6.3. Let $\beta\colon [0,+\infty) \to [0,+\infty)$ be a strictly increasing continuous function such that $\beta(0) = 0$ and $\beta(t) > 0$ for every $t > 0$. Let β^{-1} denote the inverse function, so that $\beta^{-1} \circ \beta = \mathrm{id}$ on $[0,+\infty)$. Consider the closed single cusp $G^\beta := \{x = (x', x_n)\colon \max_{i=1,\dots,n-1} |x_i| \leqslant \beta(x_n)\}$. For each $k \in \mathbb{N}_0$ we also consider the sets

$$\begin{aligned} G_k^\beta := &\Bigl\{x = (x', x_n)\colon \max_{i=1,\dots,\,n-1} |x_i| \leqslant \beta(x_n),\, 0 \leqslant x_n \leqslant \beta^{-1}(2^{-k})\Bigr\} \\ &\cup \Bigl\{x = (x', x_n)\colon \beta(x_n) \geqslant \max_{i=1,\dots,\,n-1} |x_i| > \beta(x_n) - 2^{-k},\, x_n > \beta^{-1}(2^{-k})\Bigr\}. \end{aligned} \tag{6.6}$$

Recall Definition 5.1. It is clear that G_k^β coincides with $\Sigma_k(G^\beta)$.

For every $k \in \mathbb{N}_0$ consider the measure $d\mu_k(x) = w_k^\beta(x)\, d\mathscr{H}^n(x)$, where

$$w_k^\beta(x) := w_k^\beta(x', x_n) := \begin{cases} (\beta(x_n))^{1-n}, & x_n \in [0, \beta^{-1}(2^{-k})], \\ 2^{k(n-1)}, & x_n \geqslant \beta^{-1}(2^{-k}), \\ 0, & x \notin G^\beta. \end{cases} \tag{6.7}$$

It is clear from (6.7) (recall that β is strictly increasing) that

$$w_k^\beta(x) \leqslant w_{k+1}^\beta(x) \leqslant 2^{n-1} w_k^\beta(x) \quad \text{for all} \quad x \in G^\beta. \tag{6.8}$$

Using the monotonicity of β and elementary geometric observations, it is easy to see that for every point $x=(x',x_n)\in G^\beta$ and $r\in(0,1)$

$$\mu_k(Q((0,x_n),r))\geqslant \mu_k(Q((x',x_n),r)). \tag{6.9}$$

On the other hand, using (6.7) and the monotonicity and continuity properties of β, it is easy to show that for every $x=(x',x_n)\in G^\beta$

$$\mu_k(Q((0,x_n),2^{-k}))\leqslant C(\beta)\mu_k(Q(x,2^{-k})). \tag{6.10}$$

Direct computations give

$$\mu_k(Q((0,x_n),r)=c(n)\int_{\max\{0,x_n-r\}}^{x_n+r}(\beta(t))^{n-1}\frac{1}{(\beta(t))^{n-1}}\,dt\approx c(n)r \tag{6.11}$$

for every $x=(x',x_n)\in G^\beta$.

Combining (6.8)–(6.11) we see that the sequence of measures $\{\mu_k\}_{k\in\mathbb{N}_0}$, possibly after multiplying by a fixed constant (depending on n and β), becomes 1-regular on G^β.

Recall Example 2.1, (2), and Example 2.5. Thus we see that the set G^β is 1-thick and has a porous boundary. Consider a slightly relaxed definition of the trace of $F\in W_p^1(\mathbb{R}^n)$ on the set G^β. Namely, we write $F|_{G^\beta}=f$ if $F(x)=f(x)$ for $\mathscr{H}^n$-a.e. $x\in G^\beta$. Then from Theorem 2.2 we clearly derive the following criterion.

Let $p>n-1$. Then a function $f\colon G^\beta\to\mathbb{R}^n$ lies in the trace space $W_p^1(\mathbb{R}^n)|_{G^\beta}$ if and only if

$$\begin{aligned}\mathscr{N}[f]:=\biggl(\int_{G^\beta}(f^{\sharp}_{\{\mu_k\}}(x))^p\,d\mathscr{H}^n(x)\biggr)^{1/p}+\biggl(\int_{G^\beta}\omega_0^\beta(x)|f(x)|^p\,d\mathscr{H}^n(x)\biggr)^{1/p}\\+\biggl(\sum_{k=1}^{\infty}2^{kp(1-(n-1)/p)}\int_{G_k^\beta}\omega_k^\beta(x)\bigl(\mathscr{E}_{\mu_k}(f,Q(x,2^{-k}))\bigr)^p\,d\mathscr{H}^n(x)\biggr)^{1/p}<\infty.\end{aligned} \tag{6.12}$$

Furthermore, the functional $\mathscr{N}$ gives an equivalent norm in the trace space $W_p^1(\mathbb{R}^n)|_{G^\beta}$ and the operator Ext *in (4.27) is a bounded linear extension operator from $W_p^1(\mathbb{R}^n)|_{G^\beta}$ to $W_p^1(\mathbb{R}^n)$.*

Remark 6.1. To the best of our knowledge the results in Example 6.3 are new and could not be obtained using the techniques previously known. However, we have to mention [24], where a similar example was considered under certain additional assumptions on β. More precisely, it was assumed there that β is Lipschitz. Furthermore, in [24] precise statements were only formulated in the case $n=2$.

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