On backward Euler approximations for systems of conservation laws

Maria Teresa Chiri[(1)], Minyan Zhang[(2)]

[(1)] Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L3N6, Canada.
[(2)] Department of Mathematics, Penn State University, University Park, Pa. 16802, USA.

E-mails: maria.chiri@queensu.ca, muz94@psu.edu.

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Abstract

We study approximate solutions to a hyperbolic system of conservation laws, constructed by a backward Euler scheme, where time is discretized while space is still described by a continuous variable $x \in \mathbb{R}$. We prove the global existence and uniqueness of these approximate solutions, and the invariance of suitable subdomains. Furthermore, given a left and a right state u_l, u_r connected by an entropy-admissible shock, we construct a traveling wave profile for the backward Euler scheme connecting these two asymptotic states in two main cases. Namely: (i) a scalar conservation law with strictly convex flux, where the jump $u_l - u_r$ can be arbitrarily large, and (ii) a strictly hyperbolic system, assuming that the jump $u_l - u_r$ occurs in a genuinely nonlinear family and is sufficiently small.

Keywords: backward Euler approximation, hyperbolic system of conservation laws, invariant set, entropy admissible shock, traveling wave profile, center manifold.

1 Introduction

Consider the hyperbolic system of conservation laws

$$u_t + f(u)_x \;=\; 0, \tag{1.1}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is smooth function with Jacobian matrix $Df(u) = A(u)$. Given an initial datum

$$u(0, x) \;=\; \bar{u}(x), \tag{1.2}$$

with small total variation, it is well know that a unique entropy weak solution exists, globally in time [8]. Several approximations methods have been studied in the literature [6, 12, 16]. For some of them, rigorous convergence results are known [4, 5]. In particular, semi-discrete schemes, where space is discretized while time remains a continuous variable, have been studied in [1, 3]

On the other hand, solutions generated by backward Euler approximations have been relatively less explored. These are also "semidiscrete" approximations, but now it is the time variable that is discretized, while space remains continuous.
Among the reasons why we are interested in backwar Euler approximations, it should be mentioned that in general this scheme, although computationally more expensive than the forward Euler, has greater stability properties. A natural question is whether it is possible to prove uniform bounds on the BV norm of the approximations as done for the upwind scheme [3]. In connection with a scalar conservation law, backward Euler approximations provide a basic tool for the construction of a contractive semigroup [10, 11]. For hyperbolic systems, however, little is known. Aim of the present paper is to establish some results in this direction.

To construct backward Euler approximations, we fix a time step $\varepsilon > 0$, and set $t_k = k\varepsilon$. Then, if $u(t_{k-1}, \cdot)$ is given, an approximate value for $u(t_k, \cdot)$ is computed by solving

$$u(t_k, x) \;=\; u(t_{k-1}, x) - \varepsilon f(u(t_k, \cdot))_x\,. \tag{1.3}$$

Equivalently,

$$\Big[I + \varepsilon Df(u(t_k, x))\Big] u(t_k, x) \;=\; u(t_{k-1}, x). \tag{1.4}$$

If the matrix $I + \varepsilon Df(u)$ has a uniformly bounded inverse on the domain under consideration, then for each $k \geq 1$ the profile $u_k(\cdot) = u(t_k, \cdot)$ is obtained by solving an ODE, with suitable asymptotic conditions at $x \to \pm\infty$.

Throughout the following, we shall assume that the all matrices $A(u)$ have uniformly bounded norm, say

$$\|A(u)\| \;\leq\; M \qquad \text{for all } x \in \mathbb{R}^n. \tag{1.5}$$

Starting from the quasilinear system $u_t + A(u)u_x = 0$ and performing the linear change of coordinates

$$\tau \;=\; Mt, \qquad y \;=\; x + 2Mt,$$

we obtain the system

$$u_\tau + 2u_y + \frac{1}{M} A(u) u_y \;=\; 0.$$

We shall thus work with a system of the form

$$u_t + [2u + f(u)]_x \;=\; 0, \tag{1.6}$$

assuming that the matrix $A(u) \;=\; Df(u)$ satisfies

$$\|A(u)\| \;\leq\; 1, \qquad\qquad \|A(u) - A(v)\| \;\leq\; L|u - v| \qquad \text{for all } \; u, v \in \mathbb{R}^n, \tag{1.7}$$

for some Lipschitz constant L.

Lemma 1.1. *Under the assumptions (1.7), the matrix $2I + A(u)$ has a uniformly bounded, Lipschitz continuous inverse:*

$$\Big\|(2I{+}A(u))^{-1}\Big\| \;\leq\; 1, \qquad\qquad \Big\|(2I{+}A(u))^{-1}{-}(2I{+}A(v))^{-1}\Big\| \;\leq\; L|u{-}v| \qquad \textit{for all } u, v \in \mathbb{R}^n. \tag{1.8}$$

Proof. For every $w \in \mathbb{R}^n$ we have

$$\big|(2I + A(u))w\big| \;\geq\; 2|w| - |A(u)||w| \;\geq\; |w|.$$

Hence the inverse matrix $(2I - A(u))^{-1}$ satisfies the first inequality in (1.8).

Next, for every $w \in \mathbb{R}^n$ we have

$$\begin{aligned}
\big|(2I + A(u))^{-1}w - (2I + A(v))^{-1}w\big| \;&=\; \left|\int_0^1 \frac{d}{ds}(2I + A(su + (1-s)v)^{-1}\, w\, ds\right| \\
\leq\; \int_0^1 & \frac{1}{\left\|(2I + A(su + (1-s)v))^{-1}\right\|^2} \left|\frac{d}{ds}(2I + A(su + (1-s)v))\right| |w|\, ds \\
\leq\; L\, |u-v|\, |w|.&
\end{aligned}$$

This proves the second inequality in (1.8). □

Aim of our analysis is to establish four main results. For a fixed time step $\varepsilon > 0$, we shall prove

- Global existence, uniqueness of backward Euler approximations.
- Positive invariance of suitable domains $S \subset \mathbb{R}^n$.
- Existence of traveling profiles, corresponding to large entropy admissible shocks, for scalar conservation laws
- Existence of traveling profiles, for small, entropy admissible shocks, in the case of genuinely nonlinear hyperbolic systems.

Existence of traveling profiles for scalar conservation laws with nonlocal flux has been studied in [13, 19, 20]. For semidiscrete approximations of genuinely nonlinear systems, traveling profiles connecting the left and right states of a small shock were obtained in [1], by constructing a center manifold on a suitable functional space. Our proof relies on similar techniques.

2 Solving the backward Euler step

Setting $w = u(t_{k-1})$, $u = u(t_k)$, the Backward Euler step for (1.6) amounts to solving the ODE

$$u + \varepsilon(2I + A(u))u_x \;=\; w. \tag{2.1}$$

Equivalently

$$u'(x) \;=\; g(x,u) \;\doteq\; (2I + A(u))^{-1}\frac{w(x) - u}{\varepsilon}\,. \tag{2.2}$$

As usual, by a Carathéodory solution to (2.2) we mean an absolutely continuous function u which satisfies

$$u(b) - u(a) \;=\; \int_a^b g(x, u(x))\, dx, \tag{2.3}$$

for every $a < b$.

We now prove the existence and uniqueness of the solution to the Backward Euler step.

Theorem 2.1. *Consider the system of conservation laws (1.6), where the matrix $A(u) = Df(u)$ satisfies (1.7). Given a step size $\varepsilon > 0$, for every $w \in \mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$ the ODE (2.2) has a unique solution $u = E(w) \in \mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$. Moreover, one has*

$$\|u\|_{\mathbf{L}^1} \;\leq\; 9\,\|w\|_{\mathbf{L}^1}\,, \tag{2.4}$$

$$\|E(w_1) - E(w_2)\|_{\mathbf{L}^1} \;\leq\; \exp\left\{\frac{10L}{\varepsilon}\|w_2\|_{\mathbf{L}^1}\right\}\cdot 9\|w_1 - w_2\|_{\mathbf{L}^1}, \tag{2.5}$$

for every $w_1, w_2 \in \mathbf{L}^1$.

Proof. 1. We first consider the case where $w \in \mathcal{C}^1_c$ is continuously differentiable with compact support. In this case, there exists $x_0 \in \mathbb{R}$ such that $w(x) = 0$ for $x \leq x_0$. We can define $u(x) = 0$ for $x \leq x_0$, and solve the Cauchy problem (2.2) on the half line $[x_0, +\infty[$ with initial data $u(x_0) = 0$. We observe that this Cauchy problem has unique local solution, because the right hand side is locally Lipschitz continuous w.r.t. both variables x and u. The fact that the solution is globally defined follows from the sublinear growth of g. Indeed, the first bound in (1.7) implies

$$|g(x,u)| \;\leq\; \frac{|w(x)| + |u(x)|}{\varepsilon}\,.$$

2. By (1.7), for every $v \in \mathbb{R}^n$ we have

$$|v| \;\leq\; 2|v| - |A(u)v| \;\leq\; \big|(2I + A(u))v\big| \;\leq\; |2v| + |A(u)v| \;\leq\; 3|v|.$$

Hence, for every $w \in \mathbb{R}^n$, we have

$$\frac{1}{3}|w| \;\leq\; |(2I + A(u))^{-1}w| \;\leq\; |w|. \tag{2.6}$$

Furthermore, from the inequalities

$$|v|^2 \;\leq\; \big\langle v, (2I + A(u))v\big\rangle \;\leq\; 3|v|^2,$$

taking $v = (2I + A(u))^{-1}w$ and using (2.6) we obtain

$$\frac{1}{9}|w|^2 \;\leq\; |(2I + A(u))^{-1}w|^2 \;\leq\; \big\langle (2I + A(u))^{-1}w, w\big\rangle \;\leq\; 3|(2I + A(u))^{-1}w|^2 \;\leq\; 3|w|^2.$$

3. To get an a priori estimate on the size of the solution, we observe that

$$\begin{aligned} \frac{d}{dx}|u(x)| \;&=\; \frac{\big\langle u\;,\;(2I + A(u))^{-1}w\big\rangle}{\varepsilon|u|} - \frac{\big\langle u\;,\;(2I + A(u))^{-1}u\big\rangle}{\varepsilon|u|} \\ &\leq\; \frac{1}{\varepsilon}|w(x)| - \frac{1}{9\varepsilon}|u(x)|, \end{aligned} \tag{2.7}$$

This yields

$$\begin{aligned} \int_{x_0}^{+\infty} |u(x)|\,dx \;&\leq\; \int_{x_0}^{+\infty}\left(\int_{x_0}^{x} e^{(y-x)/9\varepsilon}\cdot\frac{|w(y)|}{\varepsilon}\,dy\right)dx \\ &=\; \int_{x_0}^{+\infty}|w(y)|\left(\int_{y}^{+\infty}\frac{1}{\varepsilon}e^{(y-x)/9\varepsilon}dx\right)dy \\ &=\; 9\int_{x_0}^{+\infty}|w(y)|\,dy. \end{aligned} \tag{2.8}$$

This proves that, for $w \in \mathcal{C}_c^1$, the solution $u = E(w)$ of (2.2) satisfies (2.4).

4. Next, let two functions $w_1, w_2 \in \mathcal{C}_c^1$ be given. Choose $x_0 \in \mathbb{R}$ so that $w_1(x) = w_2(x) = 0$ for all $x \le x_0$. Let u_1, u_2 be two corresponding solutions to (2.2).

We now estimate

$$
\begin{aligned}
&\frac{d}{dx}|u_1(x) - u_2(x)| \\
&\quad = \frac{\Big\langle u_1 - u_2\,,\ (2I + A(u_1))^{-1}(w_1 - u_1) - (2I + A(u_2))^{-1}(w_2 - u_2)\Big\rangle}{\varepsilon |u_1 - u_2|} \\
&\quad \le -\frac{\Big\langle u_1 - u_2\,,\ (2I + A(u_1))^{-1}(u_1 - u_2)\Big\rangle}{\varepsilon|u_1 - u_2|} + \frac{1}{\varepsilon}\,\|(2I + A(u_1))^{-1}\|\,|w_1 - w_2| \\
&\qquad + \frac{1}{\varepsilon}\,\Big\|(2I + A(u_1))^{-1} - (2I + A(u_2))^{-1}\Big\|\,\big(|u_2| + |w_2|\big) \\
&\quad \le -\frac{|u_1 - u_2|}{9\varepsilon} + \frac{|w_1 - w_2|}{\varepsilon} + \frac{L}{\varepsilon}\,\big(|u_2| + |w_2|\big)\,|u_1 - u_2|.
\end{aligned}
\tag{2.9}
$$

As in (2.8) we thus obtain

$$
\begin{aligned}
&\int_{x_0}^{+\infty} |u_1(x) - u_2(x)|\,dx \\
&\quad \le \int_{x_0}^{+\infty}\left(\int_{x_0}^{x} \exp\left\{\int_y^x \frac{-(1/9) + L|u_2(z)| + L|w_2(z)|}{\varepsilon}\,dz\right\}\cdot\frac{|w_1(y) - w_2(y)|}{\varepsilon}\,dy\right)dx \\
&\quad \le \exp\left\{\frac{L\|u_2\|_{\mathbf{L}^1} + L\|w_2\|_{\mathbf{L}^1}}{\varepsilon}\right\}\cdot\int_{x_0}^{+\infty}|w_1(y) - w_2(y)|\left(\int_y^{+\infty}\frac{1}{\varepsilon}e^{(y-x)/9\varepsilon}dx\right)dy \\
&\quad \le \exp\left\{\frac{10L}{\varepsilon}\|w_2\|_{\mathbf{L}^1}\right\}\cdot 9\int_{x_0}^{+\infty}|w_1(y) - w_2(y)|\,dy.
\end{aligned}
\tag{2.10}
$$

This establishes the Lipschitz estimate (2.5), for functions $w_1, w_2 \in \mathcal{C}_c^1$.

5. It now remains to extend the map $w \mapsto u = E(w)$ by continuity, for all $w \in \mathbf{L}^1$. Given $w \in \mathbf{L}^1$, take any sequence $w_n \in \mathcal{C}_c^1$ such that $\|w_n - w\|_{\mathbf{L}^1} \to 0$. Let $u_n \in \mathbf{L}^1$ be the corresponding solutions to (2.2).

By (2.5) it follows

$$
\|u_m - u_n\|_{\mathbf{L}^1} \ \le\ \exp\left\{\frac{10L}{\varepsilon}\|w_n\|_{\mathbf{L}^1}\right\}\cdot 9\|w_m - w_n\|_{\mathbf{L}^1} \ \le\ C\,\|w_m - w_n\|_{\mathbf{L}^1},
\tag{2.11}
$$

for every m, n. This implies that the sequence $(u_n)_{n\ge 1}$ is a Cauchy sequence in $\mathbf{L}^1$. Hence it converges to a unique limit $u \in \mathbf{L}^1(\mathbb{R}; \mathbb{R}^n)$. By possibly selecting a subsequence, we can assume the pointwise convergence $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}$.

6. We claim that the limit u is absolutely continuous (after possibly modifying its values on a set of measure zero), and satisfies (2.2). Using (2.2) and (1.8), for every $m, n \ge 1$ one obtains

$$
\|u_m' - u_n'\|_{\mathbf{L}^1} \ \le\ \frac{1}{\varepsilon}\|w_m - w_n\|_{\mathbf{L}^1} + \frac{1}{\varepsilon}\big(1 + L\|w_n\|_{\mathbf{L}^1} + L\|u_n\|_{\mathbf{L}^1}\big)\|u_m - u_n\|_{\mathbf{L}^1}.
$$

Therefore, the sequence of derivatives $(u'_n)_{n\geq 1}$ is a Cauchy sequence in $\mathbf{L}^1$ as well, and converges to some limit is $v \in \mathbf{L}^1(\mathbb{R};\, \mathbb{R}^n)$. Consider the integral

$$\hat{u}(x) \;=\; \int_{-\infty}^{x} v(y)dy.$$

The definition of $v, \hat{u}$ implies that, for any $x \in \mathbb{R}$,

$$|u_n(x) - \hat{u}(x)| \;\leq\; \int_{-\infty}^{x} |u'_n(y) - v(y)|dy \;\leq\; \|u'_n - v\|_{\mathbf{L}^1} \to 0, \text{ as } n \to \infty.$$

This proves the convergence $u_n(x) \to \hat{u}(x)$, uniformly for $x \in \mathbb{R}$. In particular, this implies $\hat{u}(x) = u(x)$ for a.e. $x \in \mathbb{R}$. This shows that the function $u = E(w)$ constructed in step **5** is absolutely continuous (up to redefining its values on a set of measure zero), and provides a solution to (2.2).

7. In this last step, we prove that the equation (2.2) has a unique Carathéodory solution $u \in \mathbf{L}^1$.

For a given $w \in \mathbf{L}^1(\mathbb{R})$, suppose that there exists two solutions: u_1 and u_2.

Given $\epsilon_0 > 0$, there exists x_0 such that

$$\int_{-\infty}^{x_0} |w(x)|\, dx \;<\; \epsilon_0, \qquad \int_{-\infty}^{x_0} |u_1(x)|\, dx \;<\; \epsilon_0, \qquad \int_{-\infty}^{x_0} |u_2(x)|\, dx \;<\; \epsilon_0, \tag{2.12}$$

$$|u_1(x_0)| \;<\; \epsilon_0, \qquad\qquad |u_2(x_0)| \;<\; \epsilon_0.$$

since $u \in \mathbf{L}^1(\mathbb{R})$, which implies $\liminf_{|x|\to\infty} |u(x)| = 0$.

We observe that u_1, u_2 satisfy the same equation (2.2) on $[x_0, +\infty[$ and

$$|u_1(x_0) - u_2(x_0)| \;<\; 2\epsilon_0\,.$$

By (2.9), u_1, u_2 satisfy

$$\frac{d}{dx}\big|u_1(x) - u_2(x)\big| \;\leq\; \Big(-\frac{1}{9\varepsilon} + \frac{L}{\varepsilon}(|u_2| + |w|)\Big)\big|u_1(x) - u_2(x)\big|.$$

This yields

$$\begin{aligned}
\big|u_1(x) - u_2(x)\big| \;&\leq\; \exp\left\{-\frac{1}{9\varepsilon}(x - x_0) + \int_{x_0}^{x} \frac{L}{\varepsilon}\big(|u_2| + |w|\big)dx\right\}\big|u_1(x_0) - u_2(x_0)\big| \\
&\leq\; \exp\left\{-\frac{1}{9\varepsilon}(x - x_0) + \frac{L}{\varepsilon}\big(\|u_2\|_{\mathbf{L}^1} + \|w\|_{\mathbf{L}^1}\big)\right\} 2\epsilon_0\,,
\end{aligned}$$

$$\begin{aligned}
\int_{x_0}^{+\infty} \big|u_1(x) - u_2(x)\big|\, dx \;&\leq\; \int_{x_0}^{+\infty} \exp\left\{-\frac{1}{9\varepsilon}(x - x_0) + \frac{L}{\varepsilon}\big(\|u_2\|_{\mathbf{L}^1} + \|w\|_{\mathbf{L}^1}\big)\right\} 2\epsilon_0\, dx \\
&\leq\; 9\varepsilon\, \exp\left\{\frac{L}{\varepsilon}\big(\|u_2\|_{\mathbf{L}^1} + \|w\|_{\mathbf{L}^1}\big)\right\} 2\epsilon_0.
\end{aligned}$$

Combining the above estimate with (2.12), we obtain

$$\|u_1-u_2\|_{\mathbf{L}^1} \;=\; \left(\int_{-\infty}^{x_0}+\int_{x_0}^{+\infty}\right)\big|u_1(x)-u_2(x)\big|\,dx \;\leq\; 2\epsilon_0+9\varepsilon\,\exp\left\{\frac{L}{\varepsilon}\big(\|u_2\|_{\mathbf{L}^1}+\|w\|_{\mathbf{L}^1}\big)\right\}2\epsilon_0.$$

Since the above inequality holds for every $\epsilon_0 > 0$, we conclude that $u_1 = u_2$, hence the Carathéodory solution $u \in \mathbf{L}^1$ is unique. □

3 An invariance property

In the remainder of this paper we focus on the case where the system (1.1) is strictly hyperbolic [6, 12, 16]. More precisely, we shall assume

(A) *The system (1.1) is strictly hyperbolic, with $\mathcal{C}^2$ coefficients. For every $u \in \mathbb{R}^n$ the Jacobian matrix $Df(u)$ has positive and distinct eigenvalues:*

$$0 \;<\; \lambda_1(u) \;<\; \cdots \;<\; \lambda_n(u). \tag{3.1}$$

We underline that, up to performing a suitable change of variable, condition (3.1) on the eigenvalues is not restrictive. As customary, we shall denote by

$$\big\{r_i(u),\ldots,r_n(u)\big\}, \qquad \big\{\ell_i(u),\ldots,\ell_n(u)\big\}. \tag{3.2}$$

dual bases of right and left eigenvectors of the matrix $Df(u)$.

Given $w \in \mathbf{L}^1(\mathbb{R};\,\mathbb{R}^n)$, the backward Euler step amounts to finding $u \in \mathbf{L}^1(\mathbb{R};\,\mathbb{R}^n)$ such that

$$u'(x) \;=\; g(x,u) \;\doteq\; \big(Df(u)\big)^{-1}\frac{w(x)-u}{\varepsilon}\,. \tag{3.3}$$

Notice that here the matrix $Df(u)$ is invertible, because all of its eigenvalues are strictly positive.

In this section we prove an invariance property of Backward Euler approximations, in the same spirit as [15]. Let $S \subset \mathbb{R}^n$ be a closed domain. Assuming that the initial datum $\bar u$ takes values inside S, we seek conditions on S ensuring that all approximations constructed by the backward Euler scheme still take values inside S.

Theorem 3.1. *Let the assumptions* **(A)** *hold. Let $S \subset \mathbb{R}^n$ be a closed, convex set. Assume that the boundary ∂S is contained in the union of finitely many $\mathcal{C}^1$ hypersurfaces*

$$\Sigma_k \;=\; \big\{u \in \mathbb{R}^n\,;\;\; \varphi_k(u) = 0\big\}, \qquad k = 1,\ldots,N,$$

such that, for each $u \in \Sigma_k$, the gradient $\nabla\varphi_k(u)$ is a left eigenvector of $Df(u)$. Then S is positively invariant for the backward Euler scheme.

Proof. 1. Given $w \in \mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$ with $w(x) \in S$ for all x, we need to show that the solution to (2.2) remains inside S as well.

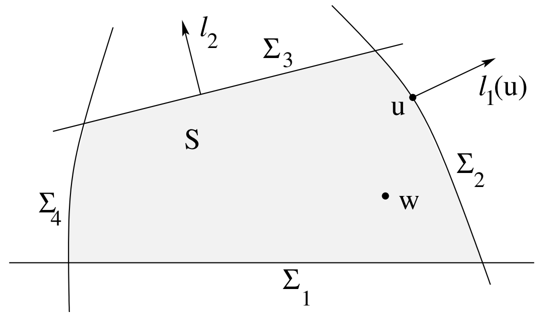

Figure 1: A positively invariant domain. Each boundary Σ_k is perpendicular to one of the left eigenvectors ℓ_i of the Jacobian matrix $Df(u)$.

As a first step, we show that this is true when $w \in \mathcal{C}^1_c$. As remarked in the proof of Theorem 2.1, in this case there exists $x_0 \in \mathbb{R}$ such that $w(x) = 0$ for $|x| \leq x_0$. We can thus define $u(x) = 0$ for $x \leq x_0$, and solve the ODE (3.3) on the half line $[x_0, +\infty[$ with initial data $u(x_0) = 0$.

By a well known invariance property for solutions to ODEs (see for example [18]), it suffices to prove that

$$u(x) \in \partial S \qquad \Longrightarrow \qquad u'(x) = g(x, u) \in T_{u(x)}(S) \tag{3.4}$$

where $T_u(S)$ denotes the tangent cone to the set S at the point u.

2. To fix ideas, consider a point $u \in \partial S$, say with

$$\begin{cases} \varphi_k(u) &= 0 \qquad k \in \mathcal{I}, \\ \varphi_k(u) &> 0 \qquad k \notin \mathcal{I}, \end{cases} \tag{3.5}$$

for some subset $\mathcal{I} \subseteq \{1, \ldots, N\}$. By assumption, for every $k \in \{1, \ldots, N\}$ there exists an index $i = i(k) \in \{1, \ldots n\}$ such that the gradient $\nabla \varphi_k(u)$ is parallel to the left eigenvector $\ell_i(u)$. As shown in Fig. 1, we assume that the vector ℓ_i points outward from the set S. The tangent cone to the set S at the point u is now given by

$$T_u(S) \;=\; \Big\{ \mathbf{v} \in \mathbb{R}^n\,;\;\; \big\langle \ell_i(u), \mathbf{v} \big\rangle \leq 0 \quad \text{for all } i \in I \Big\}, \tag{3.6}$$

where

$$I \;=\; \big\{ i(k)\,;\;\; k \in \mathcal{I} \big\}. \tag{3.7}$$

3. Now let $u = u(x)$ be the solution of (3.3), with $w \in \mathcal{C}^1_c$. Consider any point x where $u(x) \in \partial S$, and let $I \subset \{1, \ldots n\}$ be the corresponding set of indices constructed as in (3.5)–(3.7). According to (3.6) we need to show that

$$\Big\langle \ell_i(u(x)), u'(x) \Big\rangle \;\leq\; 0 \qquad\qquad \text{for all } i \in I. \tag{3.8}$$

The convexity of S implies

$$S \;\subseteq\; \Big\{ w \in \mathbb{R}^n\,;\;\; \Big\langle \ell_i\big(u(x)\big)\,,\; w - u(x) \Big\rangle \;\leq\; 0 \Big\}. \tag{3.9}$$

Using (3.3), (3.9), and the fact that $\ell_i(u)$ is an eigenvector of $Df(u)$ with eigenvalue $\lambda_i(u) > 0$, we obtain

$$\begin{aligned}\Big\langle \ell_i(u(x)), u'(x)\Big\rangle &= \Big\langle \ell_i(u(x)),\ \big(Df(u(x))\big)^{-1}\frac{w(x)-u(x)}{\epsilon}\Big\rangle \\ &= \Big\langle \ell_i(u(x)),\ \frac{1}{\lambda_i(u(x))}\,\frac{w(x)-u(x)}{\epsilon}\Big\rangle \ \leq\ 0.\end{aligned}$$

By the invariance principle for solutions of ODEs, this implies that $u(x) \in S$ for all $x \in \mathbb{R}$.

4. The previous analysis shows that the backward Euler step u takes values inside S for every $w \in \mathcal{C}^1_c$. We claim that the conclusion remains valid also if $w \in \mathbf{L}^1$. Indeed, consider a sequence of functions $w_m \in \mathcal{C}^1_c$, $m \geq 1$, converging to w in $\mathbf{L}^1$. By the previous steps, the corresponding solutions u_m of (3.3) satisfy $u_m(x) \in S$ for all $x \in \mathbb{R}$. As in the proof of Theorem 2.1, by (2.11) the sequence $(u_m)_{m\geq 1}$ is Cauchy, and converges to a unique limit $u \in \mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$, which satisfies (3.3). Since S is closed, this implies $u(x) \in S$ for all $x \in \mathbb{R}$, completing the proof. □

4 Traveling wave profiles

In the remainder of the paper, always under the assumptions **(A)**, we study solutions to the backward Euler scheme in the form of traveling waves, so that

$$u(t,x) \;=\; w(x-c\,t). \tag{4.1}$$

By (1.3), it follows that w should satisfy

$$f(w(x))_x \;=\; \frac{w(x+c\,\varepsilon)-w(x)}{\varepsilon}, \tag{4.2}$$

which leads to the ODE with non-local argument

$$w'(x) \;=\; (Df(w(x))^{-1}\frac{w(x+c\,\varepsilon)-w(x)}{\varepsilon}\,. \tag{4.3}$$

It is of interest to construct solutions such that

$$\lim_{x\to-\infty} w(x) \;=\; w^-, \qquad\qquad \lim_{x\to+\infty} w(x) \;=\; w^+, \tag{4.4}$$

for some constant states $w^-, w^+ \in \mathbb{R}^n$.

The equation (4.2) can be written as

$$\frac{d}{dx}\left(f(w(x)) - \frac{1}{\varepsilon}\int_x^{x+c\varepsilon} w(y)\,dy\right) \;=\; 0. \tag{4.5}$$

Integrating (4.5) we obtain

$$f(w(x)) - \frac{1}{\varepsilon}\int_x^{x+c\varepsilon} w(y)\,dy \;=\; \mathbf{v}, \tag{4.6}$$

for some constant vector $\mathbf{v} \in \mathbb{R}^n$. Letting $x \to \pm\infty$ and assuming (4.4) we obtain

$$f(w(x)) - \frac{1}{\varepsilon}\int_x^{x+c\varepsilon} w(y)\,dy \;=\; f(w^+) - cw^+ \;=\; f(w^-) - cw^-. \tag{4.7}$$

In particular, (4.7) yields the Rankine-Hugoniot jump conditions

$$f(w^+) - f(w^-) \;=\; c(w^+ - w^-). \tag{4.8}$$

Remark 4.1. A second order Taylor approximation of the right hand side of (4.2) yields

$$\varepsilon f\big(w(x)\big)_x \;=\; c\varepsilon w_x(x) + \frac{(c\varepsilon)^2}{2} w_{xx}(x).$$

Hence

$$\frac{c^2\varepsilon}{2}\, w'' \;=\; \big[f - cw\big]_x .$$

Notice that this is the same equation satisfied by a viscous traveling wave, with viscosity coefficient $c^2\varepsilon/2$. In first approximation, we thus expect that the solution to (4.2) will satisfy

$$w'(x) \;=\; \frac{2}{\varepsilon c^2}\big[f(w(x)) - cw(x) - C\big],$$

for some integration constant C.

Remark 4.2. In (4.2), it is not restrictive to assume $\varepsilon = 1$. Indeed, if w satisfies

$$f(w(x))_x \;=\; w(x+c) - w(x), \tag{4.9}$$

then $w_\varepsilon(x) = w(x/\varepsilon)$ provides a solution to (4.2).

5 Traveling profiles for a scalar conservation law

In this section we consider a scalar conservation law

$$u_t + f(u)_x \;=\; 0, \tag{5.1}$$

and assume that the left and right states $u^- > u^+$ are connected by an entropy admissible shock with Rankine-Hugoniot speed

$$c \;=\; \frac{f(u^+) - f(u^-)}{u^+ - u^-}\,. \tag{5.2}$$

More precisely, we shall assume the stability conditions

$$f'(u^-) \;>\; c \;>\; f'(u^+), \tag{5.3}$$

$$f\big(\theta u^+ + (1-\theta)u^-\big) \;<\; \theta f(u^+) + (1-\theta) f(u^-) \qquad \text{for all } 0 < \theta < 1, \tag{5.4}$$

together with

$$M \;\geq\; f'(u) \;\geq\; c_0 \;>\; 0 \qquad \text{for all } u \in \mathbb{R}. \tag{5.5}$$

By (5.5) the characteristic speed remains uniformly positive. Taking $\varepsilon = 1$, the delay differential equation (4.3) describing a traveling wave profile takes the form

$$z'(x) \;=\; \frac{1}{f'\big(z(x)\big)}\big[z(x+c) - z(x)\big]\,. \tag{5.6}$$

The goal of this section is to prove the existence of a traveling wave profile for the Backward Euler scheme, connecting the states u^-, u^+.

Theorem 5.1. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a $\mathcal{C}^1$ flux function satisfying (5.2)–(5.5), for some states $u^+ < u^-$. Then there exists a solution to the equation (5.6), with asymptotic conditions*

$$\lim_{x\to-\infty} z(x) \;=\; u^-, \qquad\qquad \lim_{x\to+\infty} z(x) \;=\; u^+. \tag{5.7}$$

Proof. 1. We construct a sequence of approximate traveling profiles

$$z_n : \,]-\infty, n+c] \;\mapsto\; [u^+, u^-],$$

defined as follows. We start by setting

$$z_n(x) \;=\; u^+ + e^{-x} \qquad \text{for } x \in [n, n+c]. \tag{5.8}$$

Then we solve the delay differential equation (5.6) backwards, and construct the values of $z_n(x)$ for $x \in]-\infty, n]$.

2. Let $z_n(x) = u^+ + e^{-x}$ on $[n, n+c]$, we observe that it is monotonically decreasing on this interval. We claim that z_n is also monotonically decreasing on each interval $[n-kc, n-(k-1)c]$ for $k \geq 1$. Since $f'(z) \geq c_0 > 0$, by (5.6) the derivative $z'(x) < 0$ for all $x \in]n-c, n]$. By induction on k, we conclude that this derivative remains negative, and z_n is monotone decreasing on the entire domain $]-\infty, n+c]$.

3. This step will establish an upper bound for z_n. By (4.8) we can introduce the constant

$$C \;\doteq\; f(u^+) - cu^+ \;=\; f(u^-) - cu^-. \tag{5.9}$$

Moreover, the same argument used at (4.6) now yields

$$f\big(z_n(x)\big) - \int_x^{x+c} z_n(y)\,dy \;=\; f(z_n(n)) - \int_n^{n+c} z_n(y)\,dy \;\doteq\; C_n\,. \tag{5.10}$$

Letting $n \to +\infty$ we obtain

$$\lim_{n\to+\infty} C_n \;=\; \lim_{n\to+\infty} f\big(u^+ + e^{-n}\big) - \int_n^{n+c} \big(u^+ + e^{-y}\big)\,dy \;=\; f(u^+) - cu^+ \;=\; C. \tag{5.11}$$

Next, since $z_n(x)$ is decreasing w.r.t. x, we have

$$f\big(z_n(x)\big) - cz_n(x) \;\leq\; f\big(z_n(x)\big) - \int_x^{x+c} z_n(y)\,dy \;=\; C_n. \tag{5.12}$$

Thanks to (5.3), we can find a sequence of points $\widetilde{u}_n > u^-$, with

$$\lim_{n\to\infty} \widetilde{u}_n = u^-$$

and such that

$$f(\widetilde{u}_n) - c\widetilde{u}_n \;>\; C_n \tag{5.13}$$

for all n sufficiently large. Combining (5.12) with (5.13), by continuity we conclude that

$$z_n(x) \;\leq\; \widetilde{u}_n \tag{5.14}$$

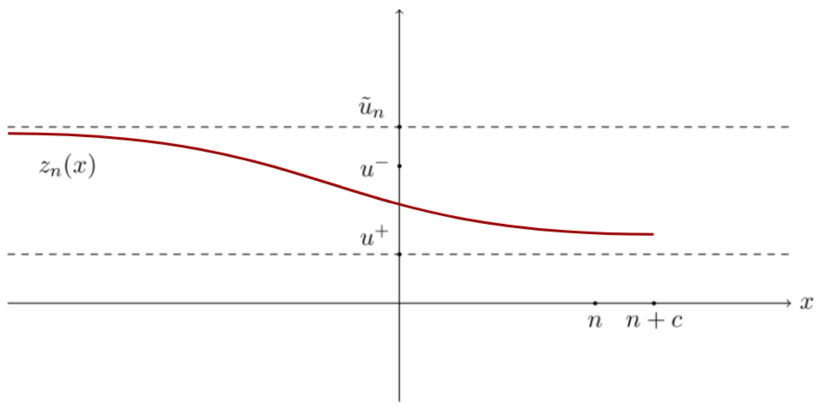

Figure 2: $z_n :]-\infty, x+c] \to [u^+, \widetilde{u}_n]$ is decreasing and $\widetilde{u}_n \to u^-$ as $n \to \infty$.

for all $x \leq n$ and $n \geq 1$ sufficiently large.

4. By (5.14) it follows that every function z_n is decreasing and takes values within an interval $[u^+, \widetilde{u}_n]$, where $\widetilde{u}_n \to u^-$ (Fig. 2) . As a consequence, the limit

$$z_n^- \doteq \lim_{x \to -\infty} z_n(x) \tag{5.15}$$

is well defined. We claim that

$$\lim_{n \to \infty} z_n^- = u^-. \tag{5.16}$$

Indeed, by possibly taking a subsequence, we can assume

$$z_n^- \to z^- \in [u^+, u^-]. \tag{5.17}$$

Since

$$f(z_n^-) - cz_n^- = C_n \to C \qquad \text{as} \quad n \to \infty, \tag{5.18}$$

this already implies

$$z^- = u^- \qquad \text{or} \qquad z^- = u^+.$$

To rule out the second alternative, we argue as follows. By (5.3), there exist $\varepsilon, \delta > 0$ such that

$$\frac{d}{du}\big[f(u) - cu\big] \leq -\varepsilon \qquad\qquad \text{for} \;\; u \in [u^+, \, u^+ + \delta].$$

However, as long as $z_n(x) \in [u^+, u^+ + \delta]$,

$$\begin{aligned} f\big(z_n(x)\big) - cz_n(x) &= f(u^+ + e^{-n}) - c(u^+ + e^{-n}) + \int_{u^+ + e^n}^{z_n(x)} \big[f'(u) - c\big]\, du \\ &\leq f(u^+ + e^{-n}) - \int_n^{n+c} (u^+ + e^{-x})\, dx - \varepsilon\big[z_n(x) - u^+ - e^{-n}\big] \\ &= C_n - \varepsilon\big[z_n(x) - u^+ - e^{-n}\big]. \end{aligned} \tag{5.19}$$

We now observe that the right hand side of (5.19) remains strictly smaller that C_n as long as

$$u^+ - e^{-n} < z_n(x) \leq u^+ + \delta.$$

By (5.18) this implies $z_n^- \geq u^+ + \delta$, for every n large enough, ruling out the possibility that $z_n^- \to u^-$. Hence (5.16) must hold.

5. Next, we claim that, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$z_n(x) \in [u^+ + \varepsilon, u^- - \varepsilon] \qquad \Longrightarrow \qquad z_n'(x) \ \leq \ -\delta, \tag{5.20}$$

for all n sufficiently large.

If not, in view of (5.6) and the fact that f' is uniformly positive, we could find a sequence of points x_n such that

$$z_n(x_n) \ \in \ [u^+ + \varepsilon, u^- - \varepsilon], \qquad\qquad z_n(x_n + c) - z_n(x_n) \ \to \ 0.$$

Since $z_n(\cdot)$ is monotone decreasing, we have $z_n(x) - z_n(y) \to 0$ for any $x, y \in [x_n, x_n + c]$ and hence

$$\int_{x_n}^{x_n+c} z_n(y)dy \ \to \ cz_n(x_n).$$

Taking a subsequence, we can assume

$$z_n(x_n) \to \overline{z} \ \in \ [u^+ + \varepsilon, u^- - \varepsilon],$$

By (5.10) this implies

$$f(\overline{z}) - c\,\overline{z} \ = \ C.$$

However, this equation does not have solutions within the interval $[u^+ + \varepsilon, u^- - \varepsilon]$. This contradiction shows that (5.20) must hold.

6. By possibly performing a horizontal shift, and consider the functions

$$u_n(x) \ = \ z_n(x - a_n),$$

where a_n is chosen so that

$$u_n(0) \ = \ \frac{u^+ + u^-}{2}\,.$$

For any $\varepsilon > 0$, let $\delta > 0$ be as in (5.20). Setting

$$M_\varepsilon \ \doteq \ \frac{u^- - u^+}{\delta}\,,$$

for every $n \geq 1$ sufficiently large we achieve

$$\begin{cases} u_n(x) - u^+ < \varepsilon & \text{for } \ x > M_\varepsilon\,, \\ u_n(x) - u^- > -\varepsilon & \text{for } \ x < -M_\varepsilon\,. \end{cases} \tag{5.21}$$

By the Ascoli-Arzela theorem, by possibly taking a subsequence we obtain the uniform convergence $u_n(x) \to u(x)$ on the interval $[-M_\varepsilon, M_\varepsilon]$. Outside this interval, thanks to (5.21) we have

$$\limsup_{m,n\to+\infty} \ \sup_{x\in\mathbb{R}} \ \big|u_m(x) - u_n(x)\big| \ \leq \ 2\varepsilon. \tag{5.22}$$

To complete the construction, we take a sequence $\varepsilon_\nu \downarrow 0$. Repeating the previous construction, we obtain a subsequence $(u_n^{(\nu)})_{n\geq 1}$ which satisfies (5.22) with ε replaced by ε_ν. By a standard diagonal procedure, this yields a subsequence uniformly converging on the whole real line.

7. It remains to prove that the limit $u(x) = \lim_{n\to\infty} u_n(x)$ provides a traveling wave solution. From the identity (5.10), letting $n \to \infty$ we obtain

$$f\big(z(x)\big) - \int_x^{x+c} z(y)\,dy \;=\; C \qquad\qquad \text{for all } x \in \mathbb{R}. \tag{5.23}$$

Since z is continuous and $f \in \mathcal{C}^1$, this integral equation yields (5.6). □

6 Traveling profiles for hyperbolic systems

Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $f : \Omega \mapsto \mathbb{R}^n$ be a smooth flux function such that, for every $u \in \Omega$, the Jacobian $Df(u)$ has n real distinct, positive eigenvalues $0 < \lambda_1(u) < \cdots < \lambda_n(u)$. If the k-th characteristic field is genuinely nonlinear, for every right state $u_r \in \Omega$ there exists a 1-parameter family of left states u_l which are joined to u_r by an entropy-admissible shock [6, 17, 21].

At a fixed state u_r, let $\{r_1, \ldots, r_n\}$ and $\{\ell_1, \ldots, \ell_n\}$ be bases of right and left eigenvectors for the Jacobian matrix $A = Df(u_r)$, normalized so that

$$|r_k| \;=\; 1, \qquad \nabla\lambda_k(u_r)\cdot r_k \;>\; 0, \qquad \ell_i \cdot r_j \;=\; \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The main result of this section is the existence of traveling profiles for the backward Euler approximation scheme.

Theorem 6.1. *In the above setting, every $u_r \in \Omega$ has a neighborhood $\mathcal{V}$ with the following property. If $u_l \in \mathcal{V}$ is a left state connected to u_r by an admissible shock of the genuinely nonlinear k-th family, then there exists a traveling profile for the backward Euler scheme having u_l, u_r as asymptotic limits.*

Namely, there exists a smooth function $u : \mathbb{R} \mapsto \Omega$ and a speed c such that

$$u'(x) \;=\; \big(Df(u(x))\big)^{-1} \frac{u(x + c\,\varepsilon) - u(x)}{\varepsilon}\,, \tag{6.1}$$

$$\lim_{x\to-\infty} u(x) \;=\; u_l, \qquad\qquad \lim_{x\to+\infty} u(x) \;=\; u_r\,. \tag{6.2}$$

Remark 6.2. A similar result was proved in [1], in the case of semidiscrete approximations, where space is discretized but time remains continuous. In such case, (6.1) is replaced by

$$u'(x) \;=\; \mu\Big(f\big(u(x)\big) - f\big(u(x-1)\big)\Big)\,. \tag{6.3}$$

We shall follow the same steps of the proof in [1], with the appropriate modifications.

Most of the theory of delay differential equations is stated for positive delays (see [14]), hence for convenience we apply the rescaling $x \to -\frac{x}{c\varepsilon}$ which leads to the following equation

$$u' \;=\; c\big(Df(u(x))\big)^{-1}\big(u(x) - u(x-1)\big), \tag{6.4}$$

to be solved in $\mathbb{R}^n$, with the asymptotic conditions

$$\lim_{x\to-\infty} u(x) = u_r, \qquad \lim_{x\to+\infty} u(x) = u_l. \tag{6.5}$$

Here u should be considered as a $\frac{1}{c\varepsilon}$-stretching and reflection of the original solution for (4.3).

We write (6.4) as a system of $n+1$ delay differential equations

$$\begin{cases} u'(x) = c(x)\big(Df(u(x))\big)^{-1}\big(u(x) - u(x-1)\big), \\ c'(x) = 0, \end{cases} \tag{6.6}$$

with the additional asymptotic condition $\lim_{x\to\pm\infty} c(x) = c$, where the shock speed c is determined by the Rankine-Hugoniot equations.

It will be convenient to introduce the space $\mathcal{C} \doteq \mathcal{C}^0\big([-1,0]\,;\ \mathbb{R}^{n+1}\big)$ endowed with norm

$$\left\|\begin{pmatrix}\phi\\ e\end{pmatrix}\right\|_{\mathcal{C}} \;\doteq\; \sup_{-1\le\theta\le 0}\left|\begin{pmatrix}\phi(\theta)\\ e(\theta)\end{pmatrix}\right|.$$

The system (6.6) can now be rewritten as a functional differential equation on $\mathcal{C}$, namely

$$\frac{d}{dx}D\left(\begin{pmatrix}u\\ c\end{pmatrix}_x\right) = F\left(\begin{pmatrix}u\\ c\end{pmatrix}_x\right) \tag{6.7}$$

where

$$D\left(\begin{pmatrix}\phi\\ e\end{pmatrix}\right) = \begin{pmatrix}\phi(0)\\ e(0)\end{pmatrix} \quad \text{and} \quad F\left(\begin{pmatrix}\phi\\ e\end{pmatrix}\right) = \begin{pmatrix} e(0)\big(Df(\phi(0))\big)^{-1}\big(\phi(0)-\phi(-1)\big)\\ 0\end{pmatrix}.$$

Notice that F is bounded and Lipschitz continuous operator on $\mathcal{C}$. To find the profile of (6.7), we need to define the solution operator $T(x)$ of (6.7). According to Lemma 7.1 in [14], we have the followig result.

Theorem 6.3. *For any initial data* $\begin{pmatrix}u\\ c\end{pmatrix}_0(\theta) = \begin{pmatrix}\phi\\ e\end{pmatrix}(\theta)$, *with* $\begin{pmatrix}\phi\\ e\end{pmatrix} \in \mathcal{C}$ *and* $\theta \in [-1,0]$, *there exists a unique solution of the functional differential equation* (6.7) *and the associated strongly continuous solution operator* $T(x) : \mathcal{C} \to \mathcal{C}$, *with* $x > 0$, *satisfies*

$$T(x)\begin{pmatrix}\phi\\ e\end{pmatrix}(\theta) = \begin{pmatrix}u\\ c\end{pmatrix}_x(\theta) = \begin{pmatrix}u\\ c\end{pmatrix}(x+\theta). \tag{6.8}$$

The infinitesimal generator $\mathcal{A}$ *of* $T(x)$, *defined as*

$$\mathcal{A}\begin{pmatrix}u\\ c\end{pmatrix} \;=\; \lim_{x\to 0}\frac{1}{x}\left[T(x)\begin{pmatrix}u\\ c\end{pmatrix} - \begin{pmatrix}u\\ c\end{pmatrix}\right],$$

is given by

$$\mathcal{A}\begin{pmatrix}\phi\\ e\end{pmatrix} \;=\; \begin{pmatrix}\phi\\ e\end{pmatrix}'$$

on the domain

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} \phi \\ e \end{pmatrix} \in \mathcal{C} : \begin{pmatrix} \phi \\ e \end{pmatrix}'(0) = \begin{pmatrix} e(0)\big(Df(\phi(0))\big)^{-1}\big(\phi(0)-\phi(-1)\big) \\ 0 \end{pmatrix} \right\}. \tag{6.9}$$

In order to construct the traveling profile for (6.7), we first need to construct the center manifold of the linearized system. We thus linearize this system around the constant solution $\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} \in \mathcal{C}$ and obtain

$$\begin{pmatrix} u \\ c \end{pmatrix}'(x) = \begin{pmatrix} u \\ c \end{pmatrix}_x'(0) = \int_{-1}^{0} d\rho(\theta) \begin{pmatrix} u \\ c \end{pmatrix}_x(\theta), \tag{6.10}$$

$$\rho(\theta) = \begin{pmatrix} \lambda_k(u_r)\big(Df(u_r)\big)^{-1}\big(H(\theta)-H(\theta+1)\big) & 0 \\ 0 & 0 \end{pmatrix},$$

where H is the Heaviside function. Following the definition in Section 7.1 in [14], we can introduce the transposed of the linear system (6.10) on the space $\mathcal{C}^* \doteq \mathcal{C}([0,1],(\mathbb{R}^{n+1})^*)$, namely

$$(\alpha,\upsilon)'(0) \;=\; -\int_{-1}^{0} (\alpha,\upsilon)(-\theta)d\rho(\theta) \;=\; \lambda_k(u_r)\big(Df(u_r)\big)^{-1}\cdot\big(\alpha(1)-\alpha(0),0\big). \tag{6.11}$$

Here the dual product is defined as

$$\left\langle (\alpha,\upsilon), \begin{pmatrix} \phi \\ e \end{pmatrix} \right\rangle \;\doteq\; \alpha(0)\phi(0)+\upsilon(0)e(0) - \int_{-1}^{0} \lambda_k(u_r)\alpha(\theta+1)(Df(u_r))^{-1}\phi(\theta)d\theta. \tag{6.12}$$

The characteristic equation for the linear system is given by

$$\det\Big(zI - \int_{-1}^{0} e^{z\theta}d\rho(\theta)\Big) \;=\; z\prod_{i=1}^{n}\big(z+\lambda_k(u_r)\lambda_i(u_r)^{-1}(e^{-z}-1)\big) \;=\; 0. \tag{6.13}$$

Since $\frac{1-e^{-z}}{z}$ is a decreasing function with a removable singularity at $z=0$, the characteristic equation has a zero of order $n+2$ at $z=0$, and $n-1$ additional zeros of order one.

The center manifold of (6.10) is the eigenspace of the eigenvalue 0. Using ^{t} to denote a transposition, the normalized basis for the center manifold and its adjoint basis can be written as

$$\Phi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \Psi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^t$$

$$\Phi_i = \begin{pmatrix} r_i(u_r) \\ 0 \end{pmatrix},\ i\in\{1,\ldots,n\}\backslash\{k\} \qquad \Psi_i = \frac{\lambda_i(u_r)}{\lambda_i(u_r)-\lambda_k(u_r)}\begin{pmatrix} l_i(u_r) \\ 0 \end{pmatrix}^t,\ i\in\{1,\ldots,n\}\backslash\{k\}$$

$$\Phi_k = \begin{pmatrix} r_k(u_r) \\ 0 \end{pmatrix} \qquad \Psi_k = \frac{2}{3}\begin{pmatrix} l_k(u_r) \\ 0 \end{pmatrix}^t - 2\begin{pmatrix} l_k(u_r)\theta \\ 0 \end{pmatrix}^t$$

$$\Phi_{n+1} = 2\begin{pmatrix} r_k(u_r)\theta \\ 0 \end{pmatrix} \qquad \Psi_{n+1} = \begin{pmatrix} l_k(u_r) \\ 0 \end{pmatrix}^t$$

We now consider the center manifold for the nonlinear system (6.7) and its solution operator $T(x)$ defined in (6.8):

Theorem 6.4. *There exists a neighborhood $\mathcal{U}$ of $\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix}$ in $\mathcal{C}$ and a center manifold $\mathcal{N}$ such that the following holds.*

1. *Every $T(x)$-orbit starting in $\mathcal{N}$ remains in $\mathcal{N}$ as long as it stays in $\mathcal{U}$.*
2. *Invariant sets under $T(x)$ in $\mathcal{U}$ are also in $\mathcal{N}$.*
3. *$\mathcal{N} \cap \mathcal{U} = \left\{ \begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} + (\Phi_0, \dots, \Phi_{n+1})y + h(y); y \in \mathcal{Z} \right\}$, where $\mathcal{Z}$ is a neighborhood of 0 in $\mathbb{R}^{n+2}$ and $h : \mathcal{Z} \to \mathcal{Q}$ is a smooth function with $h(0) = 0, dh(0) = 0$. The codomain $\mathcal{Q}$ is the complement space of the generalized eigenspace for the eigenvalue 0, which is spanned by $(\Phi_0, \dots, \Phi_{n+1})$, in $\mathcal{C}$.*

Remark 6.5. *We observe here few facts relevant in the next steps of the proof of Theorem 6.1.*

i. *The basis $\{\Phi_0, \dots, \Phi_{n+1}\}$ consists of functions contained in the domain $D(\mathcal{A})$.*

ii. *There exists an $(n+2) \times (n+2)$ matrix B such that*

$$\mathcal{A}(\Phi_0, \dots, \Phi_{n+1}) = (\Phi_0, \dots, \Phi_{n+1})B\,.$$

It is immediate to check that such a matrix has the rows $(b_i)_{i \in \{0,\dots,n+1\}}$ all equal to 0 except for $b_k = (0, \dots, 0, 2)$.

iii. *The product $\left\langle \begin{pmatrix} \Psi_0 \\ \vdots \\ \Psi_{n+1} \end{pmatrix}, h(y) \right\rangle$ is equal to 0 for any $y \in \mathcal{Z}$. Therefore we can define a diffeomorphism $L : \mathcal{N} \cap \mathcal{U} \to \mathcal{Z}$ by setting*

$$L\left(\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} + (\Phi_0, \dots, \Phi_{n+1})y + h(y) \right) = \left\langle \begin{pmatrix} \Psi_0 \\ \vdots \\ \Psi_{n+1} \end{pmatrix}, (\Phi_0, \dots, \Phi_{n+1})y + h(y) \right\rangle = y.$$

6.1 The flow on the center manifold $\mathcal{N}$

In the next step, we construct a flow $\left\{ T(x) \begin{pmatrix} \phi \\ e \end{pmatrix}, x \geq 0 \right\}$ on $\mathcal{N} \subset \mathcal{C}$ such that

$$T(x) \begin{pmatrix} \phi \\ e \end{pmatrix} (\theta) = \begin{pmatrix} u \\ c \end{pmatrix} (x + \theta)$$

with the initial data $\begin{pmatrix} u \\ c \end{pmatrix}_0 = \begin{pmatrix} \phi \\ e \end{pmatrix}$. By Theorem 6.4, we can turn this problem into a Cauchy problem on $\mathbb{R}^{n+2}$.

Lemma 6.6. *Let* $\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} + \begin{pmatrix} \phi \\ e \end{pmatrix} \in \mathcal{N} \cap \mathcal{U}$ *and* $x \geq 0$ *be fixed. If* $T(x)\Big(\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} + \begin{pmatrix} \phi \\ e \end{pmatrix}\Big) \in \mathcal{N} \cap \mathcal{U}$, *then*

$$T(x)\Big(\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} + \begin{pmatrix} \phi \\ e \end{pmatrix}\Big) = \begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} + (\Phi_0, \ldots, \Phi_{n+1})y(x) + h\big(y(x)\big)\,,$$

where $y(x)$ *is the unique solution of the following Cauchy problem:*

$$\begin{cases} y'(x) = By + (\Psi_0^t, \ldots, \Psi_{n+1}^t)^t(0)F(y)\,, \\ y(0) \;\; = \;\; \Big\langle (\Psi_0^t, \ldots, \Psi_{n+1}^t)^t, \begin{pmatrix} \phi \\ e \end{pmatrix} \Big\rangle, \end{cases} \tag{6.14}$$

with B *as defined in ii. of Remark 6.5,* $F(y) := \begin{pmatrix} 0 \\ \widehat{F}\big((\Phi_0, \ldots, \Phi_{n+1})y + h(y)\big) \\ 0 \end{pmatrix}$ *and*

$$\widehat{F}\Big(\begin{pmatrix} \phi \\ e \end{pmatrix}\Big) = \big((\lambda_k(u_r) + e(0))(Df(u_r + \phi(0)))^{-1} - \lambda_k(u_r)(Df(u_r))^{-1}\big)(\phi(0) - \phi(-1)) \in \mathbb{R}^n$$

for $\begin{pmatrix} \phi \\ e \end{pmatrix} \in \mathcal{C}$.

Proof: We can find a similar proof in Section 10.2 of [14]. □

6.2 A smaller submanifold $\mathcal{M}$ of the center manifold $\mathcal{N}$

The flow we want to find on $\mathcal{N}$ should satisfy the following conditions:

(a) It has start from $\begin{pmatrix} u_r \\ \sigma \end{pmatrix}$ and end at $\begin{pmatrix} u_l \\ \sigma \end{pmatrix}$ for some constant σ,

(b) For every point $\begin{pmatrix} \phi \\ e \end{pmatrix}$ on this flow, the flow afterwards is given by

$$\Big\{T(x)\begin{pmatrix} \phi \\ e \end{pmatrix};\;\; x \geq 0\Big\}.$$

The flow satisfying these two conditions can be the solution of (6.7) and hence of (6.6) and (6.4). Hence, if we can find this flow on $\mathcal{N}$, we can get one solution to (6.4).

We firstly claim that the flow satisfying these two conditions is on $\mathcal{N}$. We can see that $\begin{pmatrix} u_r \\ \sigma \end{pmatrix}$ and $\begin{pmatrix} u_l \\ \sigma \end{pmatrix}$ belong to $\mathcal{N}$. Hence, by Theorem (6.4), when u_l is close to u_r and σ is close to $\lambda_k(u_r)$, the flow satisfying conditions (a) and (b) should be contained in $\mathcal{N}$.

However, the $(n+2)$-dim manifold $\mathcal{N}$ is too large. We need to restrict the flow to a smaller manifold by reconsidering the exact form of (6.7).

Lemma 6.7. *Let $\mathcal{M}$ be defined as*

$$\mathcal{M} := \left\{ \begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} + \begin{pmatrix} \phi \\ e \end{pmatrix} \in \mathcal{N};\ f(u_r + \phi(0)) - f(u_r) = (\lambda_k(u_r) + e(0)) \int_{-1}^{0} \phi(\theta) d\theta \right\}.$$

There exists a neighborhood $\widetilde{\mathcal{U}} \subset \mathcal{U}$ of $\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix}$ such that $\mathcal{M} \cap \widetilde{\mathcal{U}}$ is a 2-dimensional invariant manifold under T. The tangent space of $\mathcal{M}$ at $\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix}$ is spanned by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} r_k(u_r) \\ 0 \end{pmatrix}$.

The proof is similar to the proof of Lemma 7 in [1].

Lemma 6.8. *If σ is close to $\lambda_k(u_r)$ then $\begin{pmatrix} u_r \\ \sigma \end{pmatrix}$ belongs to $\mathcal{M} \cap \widetilde{\mathcal{U}}$ and the flow starting from this point will be contained in $\mathcal{M} \cap \widetilde{\mathcal{U}}$ if it is contained in $\widetilde{\mathcal{U}}$. Hence, the flow solving* (6.7) *should be contained in $\mathcal{M} \cap \widetilde{\mathcal{U}}$ for a suitable σ.*

Proof: Multiply both sides of (6.4) for $\big(Df(u)\big)^{-1}$ and integrate from $-\infty$ to x. The result is exactly the restriction in the definition of $\mathcal{M}$. Thus, if the solution to (6.4) exists and its values are contained in $\widetilde{\mathcal{U}}$, they belong also to $\mathcal{M}$, hence to $\mathcal{M} \cap \widetilde{\mathcal{U}}$.

6.3 The flow from u_r to u_l with speed σ

For $\varepsilon_0 > 0$ small enough, consider the segment

$$R := \left\{ \begin{pmatrix} u_r \\ \sigma \end{pmatrix};\ \big|\sigma - \lambda_k(u_r)\big| \leq \varepsilon_0 \right\} \tag{6.15}$$

and the curve

$$H := \left\{ \begin{pmatrix} u_l \\ \sigma \end{pmatrix}; \sigma(u_l - u_r) = f(u_l) - f(u_r),\ \ \big|\sigma - \lambda_k(u_r)\big| \leq \varepsilon_0 \right\} \tag{6.16}$$

on $\mathcal{M} \cap \widetilde{\mathcal{U}}$. The same argument as in Lemma 6 in [1] now yields

Lemma 6.9. *The point $\begin{pmatrix} \phi \\ e \end{pmatrix} \in \mathcal{N} \cap \mathcal{U}$ is a fixed point if and only if $y_{n+1} = 0$, where y_{n+1} is the last component of $y = L\Big(\begin{pmatrix} \phi \\ e \end{pmatrix}\Big)$.*

Next, we have the following result.

Lemma 6.10. *Let R and H as defined respectively in* (6.15) *and* (6.16)*. Then*

i. *$L(R)$ and $L(H)$ are transverse curves in $\mathcal{Z} \subset \mathbb{R}^{n+2}$, therefore R and H are transverse to each other in $\mathcal{M} \cap \widetilde{\mathcal{U}}$.*

ii. *R, H are both transverse to any flow in $\mathcal{M} \cap \widetilde{\mathcal{U}}$.*

iii. *The set of fixed points of (6.14) on $\mathcal{M} \cap \widetilde{\mathcal{U}}$ is given by $R \cup H$. A point in $R \setminus \begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix}$ is a repelling equilibrium. Moreover, a point in $H \setminus \begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix}$ is an attracting equilibrium if and only if $\lambda_k(u_r) \le \sigma \le \lambda_k(u_l)$.*

Proof: The proof is similar to the one in Section 5 of [1], but we shall give a more detailed analysis of the equilibrium points.

1. For any point $\begin{pmatrix} u_r \\ \sigma \end{pmatrix} \in R$, $y_0 = \sigma - \lambda_k(u_r)$ and $y_k = 0$ where k denotes the class which the shock connecting u_r and u_l belongs to. For any point $\begin{pmatrix} u_l \\ \sigma \end{pmatrix} \in H$, $y_0 = \sigma - \lambda_k(u_r)$, $y_k = l_k(u_r) \cdot (u_l - u_r)$ for the same k and hence $y_0 \approx \frac{y_k}{2}$. Hence, $L(R)$ and $L(H)$ are transverse to each other.

2. The component y_0 is constant along the flow satisfying the two condition in Section **6.2** while y_0 will always change along R, H.

3. For $\begin{pmatrix} u_r \\ \sigma \end{pmatrix} \in R$, $L\Big(\begin{pmatrix} u_r \\ \sigma \end{pmatrix}\Big) = (\sigma - \lambda_k(u_r), 0, \dots, 0)^t$ and $(\Phi_0, \dots, \Phi_{n+1})y + h(y) \approx \big(\sigma - \lambda_k(u_r)\big)\Phi_0$.

Therefore around this point, (6.14) yields

$$y_k' \approx \Big(2 + \frac{4\big(\sigma - \lambda_k(u_r)\big)}{3\lambda_k(u_r)}\Big) y_{n+1}, \quad y_{n+1}' \approx \frac{2\big(\sigma - \lambda_k(u_r)\big)}{\lambda_k(u_r)} y_{n+1}, \quad y_i' \approx 0.$$

The point is a repelling equilibrium if and only if $y_{n+1} \to 0$ as $x \to -\infty \Leftrightarrow \sigma \ge \lambda_k(u_r)$.

4. For $\begin{pmatrix} u_l \\ \sigma \end{pmatrix} \in H$, $L\Big(\begin{pmatrix} u_l \\ \sigma \end{pmatrix}\Big) = (\sigma - \lambda_k(u_r), l_1(u_r) \cdot (u_l - u_r), \dots, l_n(u_r) \cdot (u_l - u_r), 0)^t$ and $(\Phi_0, \dots, \Phi_{n+1})y + h(y) \approx \begin{pmatrix} u_l - u_r \\ \sigma - \lambda_k(u_r) \end{pmatrix}$.

Therefore around this point, (6.14) yields

$$\begin{aligned} y_0' &= 0 \\ y_i' &\approx \frac{2\lambda_i(u_r)}{\lambda_i(u_r) - \lambda_k(u_r)} \sigma l_i(u_r)\big(Df(u_l)\big)^{-1} r_k(u_r) y_{n+1} \\ y_k' &\approx 2y_{n+1} + \frac{4}{3}\Big(\sigma l_k(u_r)\big(Df(u_l)\big)^{-1} r_k(u_r) - 1\Big) y_{n+1} \\ y_{n+1}' &\approx 2\Big(\sigma l_k(u_r)\big(Df(u_l)\big)^{-1} r_k(u_r) - 1\Big) y_{n+1} \end{aligned}$$

Take $u_l = S_k(z), \sigma = \lambda_k(z)$ with $S_k(0) = u_r, \lambda_k(0) = \lambda_k(u_r)$ as Theorem 5.1 in [6]. By analyzing z, $y_{n+1} \to 0$ as $x \to +\infty$ if and only if $\lambda_k(u_r) \le \sigma \le \lambda_k(u_l)$ and $\begin{pmatrix} u_l \\ \sigma \end{pmatrix}$ is an attracting equilibrium point if and only if $y_{n+1} \to 0$ as $x \to +\infty$.

6.4 Proof of Theorem 6.1

By the Lemma 6.10, we can prove that if u_l, u_r are connected by a k-shock with the speed σ, then there exist a profile connecting these two points, $\begin{pmatrix} u_r \\ \sigma \end{pmatrix}$ and $\begin{pmatrix} u_l \\ \sigma \end{pmatrix}$.

This is because if u_l, u_r are connected by a k-shock with speed c, u_l, u_r and c will satisfy (4.8) and hence $\lambda_k(u_r) < c < \lambda_k(u_l)$. Then, by Lemma 6.10, $\begin{pmatrix} u_r \\ c \end{pmatrix}$ is a repelling equilibrium point and $\begin{pmatrix} u_l \\ c \end{pmatrix}$ is an attracting equilibrium point. So, the flow starting from $\begin{pmatrix} u_r \\ c \end{pmatrix}$ will end at $\begin{pmatrix} u_l \\ c \end{pmatrix}$.

By Lemma 6.6, this flow corresponds to a solution of (6.7) and hence of (6.4). By applying a reflection an dilatation we obtain the solution of the original problem (6.1) and (6.2).

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