REGULARITY IMPROVEMENT FOR THE MINIMIZERS OF THE TWO-DIMENSIONAL GRIFFITH ENERGY

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Abstract. In this paper we prove that the singular set of connected minimizers of the planar Griffith functional has Hausdorff dimension strictly less then one, together with the higher integrability of the symetrized gradient.

Contents

1. Introduction

In a planar elasticity setting, the Griffith energy is defined by

$$\mathcal{G}(u,K) := \int_{\Omega\setminus K} \mathbf{A}e(u) : e(u)\,dx + \mathcal{H}^1(K),$$

where $\Omega \subset \mathbb{R}^2$, which is bounded and open, stands for the reference configuration of a linearized elastic body, and

$$\mathbf{A}\xi = \lambda(\mathrm{tr}\xi)I + 2\mu\xi \quad \text{for all } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}},$$

where λ and μ are the Lamé coefficients satisfying $\mu > 0$ and $\lambda + \mu > 0$. Here, $e(u) = (Du + Du^T)/2$ is the symmetric gradient of the displacement $u : \Omega \setminus K \to \mathbb{R}^2$ which is defined outside the crack $K \subset \overline{\Omega}$.

This energy functional is defined on pairs function/set

$$(u,K) \in \mathcal{A}(\Omega) := \{\, K \subset \overline{\Omega} \text{ is closed and } u \in LD(\Omega' \setminus K)\,\},$$

where $\Omega' \supset \overline{\Omega}$ is a bounded open set and LD is the space of functions of Lebesgue deformation for which $e(u) \in L^2$. Note that by definition, K is a compact subset of $\overline{\Omega}$ and u is defined in $\Omega' \setminus K$ but we work in the ambient space Ω so as to build local competitors.

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We say that (u, K) is a minimizer for the Griffith energy if it is a solution to the problem

$$\inf \{ \int_\Omega \mathbf{A}e(v) : e(v)\, dx + \mathcal{H}^1(K) : \ (v, K) \in \mathcal{A}(\Omega),\ v = \psi \text{ a.e. in } \ \Omega' \setminus \overline{\Omega} \},$$

for some datum $\psi \in W^{1,\infty}$.

A lot of attention has been given on the Griffith functional these last years (see [3, 4, 5, 6, 7, 8, 9, 15]), and in particular it has been proved that a global minimizer $(u, K) \in \mathcal{A}(\Omega)$ (with a prescribed Dirichlet boundary condition) does exist and that the crack set K is $\mathcal{H}^1$-rectifiable and locally Ahlfors-regular in Ω. The latter means that there exists $C_0 \geq 1$ and $r_0 > 0$ (depending only on $\mathbf{A}$) such that for all $x \in K$ and all $0 < r \leq r_0$ with $B(x, r) \subset \Omega$,

$$C_0^{-1} r \leq \mathcal{H}^1(K \cap B(x, r)) \leq C_0 r. \tag{1.1}$$

In [3] it was proved that any isolated connected component of the singular set K of a Griffith minimizer is $C^{1,\alpha}$ a.e. It also applies to a connected minimizer K (for e.g. minimizer with connected constraints). In this paper we slightly improve the Hausdorff dimension of the singular set. We also prove some higher integrability property on the symmetrized gradient.

The main results of this paper are the following.

Theorem 1.1. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith energy with K connected. Then*

(1) There exists $\alpha \in (0, 1)$ and a relatively closed set $\Sigma \subset K \cap \Omega$ with $\dim_{\mathcal{H}}(\Sigma) < 1$ such that $K \cap \Omega \setminus \Sigma$ is locally a $\mathcal{C}^{1,\alpha}$ curve.

(2) There exists $r_0 > 0$, $C \geq 1$ and $p > 1$ such that for all ball $B \subset \Omega$ of radius $r \leq r_0$,

$$\int_{\frac{1}{2}B} |e(u)|^{2p}\, \mathrm{d}x \leq C r^{2-p}. \tag{1.2}$$

The proof of our main theorem follows from standard technics that was already used in the scalar context of the Mumford-Shah functional, but adapted to the vectorial Griffith functional in a non trivial manner. In particular, for (1) we follow the approach of David [10] and Rigot [17], based on uniform rectifiability of the singular set and Carleson measure estimates. The idea is to estimate to number of balls in which one can apply the ε-regularity theorem contained in [3]. But the latter needs a topological separating property that one has to control in any initialized balls which is one of the main issue of the present work (Lemma 3.1). We also need to control the 2-energy by a p-energy (Corollary 5.1) which also uses a topological argument (Lemma 5.3). The proof of (2) is based on a strategy similar to what was first introduced by De Philippis and Figalli in [13] and also used in [16], which easily follows from the porosity of the singular set together with elliptic estimates. Since the needed elliptic estimates relatively to the Lamé system are not easy to find in the literature, we have developed an appendix containing the precise results.

Let us stress that the famous Cracktip function that arises as blow-up limits of Mumford-Shah minimizers at the tip of the crack, has a vectorial analogue. This was the purpose of the work in [2]. Since the vectorial Cracktip is homogeneous of degree 1/2 (see [2, Theorem 6.4]), it is natural to conjecture that, akin to the standard Mumford-Shah functional, the integrability exponent of $|e(u)|$ should reach every $p < 4$, as asked by De Giorgi for the Mumford-Shah functional.

2. Preliminaries

Notation. The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the k-dimensional Hausdorff measure by $\mathcal{H}^k$. If E is a measurable set, we will sometimes write $|E|$ instead of $\mathcal{L}^n(E)$. If a and $b \in \mathbb{R}^n$, we write $a \cdot b = \sum_{i=1}^n a_i b_i$ for the Euclidean scalar product, and we denote the norm by $|a| = \sqrt{a \cdot a}$. The open (resp. closed) ball of center x and radius r is denoted by $B(x, r)$ (resp. $\overline{B}(x, r)$).

We write $\mathbb{M}^{2\times 2}$ for the set of real 2×2 matrices, and $\mathbb{M}^{2\times 2}_{\text{sym}}$ for that of all real symmetric 2×2 matrices. Given two matrix $A, B \in \mathbb{M}^{2\times 2}$, we recall the Frobenius inner product $A : B = \text{tr}({}^tAB)$ and the corresponding norm $|A| = \sqrt{\text{tr}(A^T A)}$.

Functions of Lebesgue deformation. Given a weakly differentiable vector field u, the symmetrized gradient of u is denoted by

$$e(u) := \frac{Du + Du^T}{2}.$$

The p-normalized energy. Let $(u, K) \in \mathcal{A}(\Omega)$. Then for any $x_0 \in \Omega$ and $r > 0$ such that $\overline{B}(x_0, r) \subset \Omega$, we define the *normalized elastic energy* by

$$\omega_p(x_0, r) := r^{1-\frac{4}{p}} \left(\int_{B(x_0,r)\setminus K} |e(u)|^p \, dx \right)^{\frac{2}{p}}.$$

The flatness. Let K be a closed set of $\mathbb{R}^2$. For any $x_0 \in K$ and $r > 0$, we define the *(bilateral) flatness* by

$$\beta_K(x_0, r) := \frac{1}{r} \inf_L \max \{ \sup_{y \in K \cap \overline{B}(x_0,r)} \operatorname{dist}(y, L), \sup_{y \in L \cap \overline{B}(x_0,r)} \operatorname{dist}(y, K) \},$$

where L belongs to the set of lines passing through x_0. When a minimizer $(u, K) \in \mathcal{A}(\Omega)$ is given, we write simply $\beta(x_0, r)$ for $\beta_K(x_0, r)$.

Remark 2.1. The flatness $\beta_K(x_0, r)$ only depends on the set $K \cap \overline{B}(x_0, 2r)$. We have that for all $0 < t \leq r$,

$$\beta_K(x_0, t) \leq \frac{r}{t} \beta_K(x_0, r), \tag{2.1}$$

and for $y_0 \in K \cap B(x_0, r/2)$ and $0 < t \leq r/2$

$$\beta_K(y_0, t) \leq \frac{2r}{t} \beta_K(x_0, r). \tag{2.2}$$

If $K' = \frac{1}{r}(K - x_0)$, then

$$\beta_K(x_0, r) = \beta_{K'}(0, 1).$$

In the sequel, we will consider the situation where

$$\beta_K(x_0, r) \leq \varepsilon, \tag{2.3}$$

for $\varepsilon > 0$ small. This implies in particular that $K \cap \overline{B}(x_0, r)$ is contained in a narrow strip of thickness εr passing through the center of the ball.

Let $L(x_0, r)$ be a line passing through x_0 and satisfying

$$\sup_{y \in K \cap \overline{B}(x_0,r)} \operatorname{dist}(y, L) \leq r \beta_K(x_0, r). \tag{2.4}$$

We will often use a local basis (depending on x_0 and r) denoted by (e_1, e_2), where e_1 is a tangent vector to the line $L(x_0, r)$, while e_2 is an orthogonal vector to $L(x_0, r)$. The coordinates of a point y in that basis will be denoted by (y_1, y_2).

Provided (2.3) is satisfied with $\varepsilon \in (0, 1/2)$, we can define two discs $D^+(x_0, r)$ and $D^-(x_0, r)$ of radius $r/4$ and such that $D^\pm(x_0, r) \subset B(x_0, r) \setminus K$. Indeed, using the notation introduced above, setting $x_0^\pm := x_0 \pm \frac{3}{4} r e_2$, we can check that $D^\pm(x_0, r) := B(x_0^\pm, r/4)$ satisfy the above requirements.

A property that will be fundamental in our analysis is the separation in a closed ball.

Definition 2.1. *Let K be a closed set of $\mathbb{R}^2$, $x_0 \in K$ and $r > 0$ be such that $\beta_K(x_0, r) \leq 1/2$. We say that K* separates *$D^\pm(x_0, r)$ in $\overline{B}(x_0, r)$ if the balls $D^\pm(x_0, r)$ are contained into two different connected components of $\overline{B}(x_0, r) \setminus K$.*

The following lemma guarantees that when passing from a ball $B(x_0, r)$ to a smaller one $B(x_0, t)$, and provided that $\beta_K(x, r)$ is relatively small, the property of separating is preserved for t varying in a range depending on $\beta_K(x, r)$.

Lemma 2.1. [3, Lemma 3.1] *Let $\tau \in (0, 1/16)$, let $K \subset \mathbb{R}^2$ a closed set, let $x_0 \in K$, let $r > 0$ be such that $\overline{B}(x_0, r) \subset \Omega$ and $\beta_K(x_0, r) \leq \tau$. If K separates $D^\pm(x_0, r)$ in $\overline{B}(x_0, r)$, then for all $t \in (16\tau r, r)$, we have $\beta_K(x_0, t) \leq 1/2$ and K separates $D^\pm(x_0, t)$ in $\overline{B}(x_0, t)$.*

3. Local separation in many balls in a connected uniformly rectifiable set

The purpose of this section is the following general result on compact connected sets which are locally Ahlfors-regular.

Lemma 3.1. *Let $K \subset \overline{\Omega}$ be a compact connected set which is locally Ahlfors-regular in Ω, that is, there exists $C_0, r_0 \geq 1$ such that for all $x \in K$ and for all $0 < r \leq r_0$ with $B(x,r) \subset \Omega$,*

$$C_0^{-1} r \leq \mathcal{H}^1(K \cap B(x,r)) \leq C_0 r. \tag{3.1}$$

Then for every $0 < \varepsilon \leq \frac{1}{2}$, there exist $a \in (0,1/2)$ small enough (depending on C_0 and ε) such that for all $x \in K$ and $0 < r \leq r_0$ with $B(x,r) \subset \Omega$, one can find $y \in B(x,r/2)$ and $t \in (ar, r/2)$ satisfying:

$$\beta_K(y,t) \leq \varepsilon \text{ and } K \text{ separates } D^{\pm}(y,t) \text{ in } \overline{B}(y,t) \text{ in the sense of Definition 2.1.}$$

Proof. The letter C is a constant ≥ 1 that depends on C_0 and whose value might increase from one line to another but a finite number of times. Let K be as in the statement of the Lemma. For every $\varepsilon > 0$, we denote by $\mathcal{B}(\varepsilon)$ the bad set where β_K is large:

$$\mathcal{B}(\varepsilon) := \{ (z,s) \mid z \in K,\ s > 0 \text{ and } \beta_K(z,s) > \varepsilon \}. \tag{3.2}$$

Let $x \in K$ and let $0 < r \leq r_0$ be such that $B(x,r) \subset \Omega$. It is more convenient to assume $B(x,3r) \subset \Omega$ and $r \leq \frac{1}{6}\mathrm{diam}(K) \wedge \frac{1}{3} r_0$ and we can do this without loss of generality. First, we draw from the local Ahlfors-regularity that $\mathrm{diam}(K) \geq C^{-1} r$. Next, we observe that if (y,t) is a solution for a ball $B(x, C^{-1} r)$ and a certain constant a, it is also a solution for the ball $B(x,r)$ and the constant $C^{-1} a$. We could work with a ball $B(x, C^{-1} r)$ but to simplify the notations, we prefer to assume r smaller directly. So, we assume $B(x,3r) \subset \Omega$ and $r \leq \frac{1}{6}\mathrm{diam}(K) \wedge \frac{1}{3} r_0$.

In the sequel we want to apply the results of [12], which works with sets of infinite diameter. This explains why we need to slightly modify our set K to fit in the definition of [12]. Precisely, given an arbitrary line L passing through x, one can check that the set

$$E = (K \cap B(x,3r)) \cup \partial B(x,3r) \cup (L \setminus B(x,3r)) \tag{3.3}$$

is connected and Ahlfors-regular in the exact sense of [12, Definition 1.13], that is, E is closed and for all $y \in E$ and and for all $\rho > 0$,

$$C^{-1} \rho \leq \mathcal{H}^1(E \cap B(y,\rho)) \leq C \rho. \tag{3.4}$$

As a consequence, it is contained in a (Ahlfors)-regular curve (see [12, (1.63)] and the discussion below) and thus is uniformly rectifiable with constant C ([12, Theorem 1.57 and Definition 1.65]). In particular, it satisfies a geometric characterisation of uniform rectifiability called *Bilateral Weak Geometric Lemma* ([12, Definition 2.2 and Theorem 2.4]). It means that for all $\varepsilon > 0$, there exists $C(\varepsilon) \geq 1$ (depending on C_0 and ε) such that for all $y \in E$ and all $\rho > 0$,

$$\int_{z \in E \cap B(y,\rho)} \int_0^{\rho} \mathbf{1}_{\mathcal{C}(\varepsilon)}(z,s) \frac{\mathrm{d}s}{s} \, \mathrm{d}\mathcal{H}^1(z) \leq C(\varepsilon) \rho, \tag{3.5}$$

where

$$\mathcal{C}(\varepsilon) := \{ (z,s) \mid z \in E,\ s > 0, \text{ and } \beta_E(z,s) > \varepsilon \}. \tag{3.6}$$

We apply this property with $y := x$ and $\rho = r$. We observe that for all $z \in K \cap B(x,r)$ and for all $0 < s < r$, we have $K \cap \overline{B}(z,2s) = E \cap \overline{B}(z,2s)$ and hence $\beta_K(z,s) = \beta_E(z,s)$. Thus, (3.3) simplifies to

$$\int_{z \in K \cap B(x,r)} \int_0^{r} \mathbf{1}_{\mathcal{B}(\varepsilon)}(z,s) \frac{\mathrm{d}s}{s} \, \mathrm{d}\mathcal{H}^1(z) \leq C(\varepsilon) r. \tag{3.7}$$

We now return to the statement of the Lemma. We fix $\varepsilon > 0$. Let $a \in (0,1)$ be a parameters that will be fixed later. Assume by contradiction that for all $y \in K \cap B(x,r)$ and $t \in (ar, r)$, we have

$\beta_K(y,t) > \varepsilon$. This means that for such pairs (y,t), we have $\mathbf{1}_{\mathcal{B}(\varepsilon)}(y,t) = 1$. Moreover we have by local Ahlfors-regularity, $\mathcal{H}^1(K \cap B(x,r)) \geq C_0^{-1} r$,

$$\begin{aligned} \int_{y \in K \cap B(x,r)} \int_{0<t<r} \mathbf{1}_{\mathcal{B}(\varepsilon)}(y,t) \frac{\mathrm{d}t}{t} \, \mathrm{d}\mathcal{H}^1(z) &\geq \mathcal{H}^1(K \cap B(x,r)) \int_{ar}^{r} \frac{dt}{t} \\ &\geq \mathcal{H}^1(K \cap B(x,r)) \ln\left(\frac{1}{a}\right) \geq C_0^{-1} r \ln\left(\frac{1}{a}\right). \end{aligned} \tag{3.8}$$

Using now (3.7), we arrive at a contradiction, provided that a is small enough (depending on C_0 and ε).

At this stage we have proved that, for every $x \in K$ and $0 < r \leq r_0$ with $B(x,r) \subset \Omega$, there exists $y \in B(x, r/2)$ and $t \in (ar, r/2)$ such that $t \leq \operatorname{diam}(K)$ and

$$\beta_K(y,t) \leq \varepsilon.$$

It remains to deal with the separation property. For that purpose we will use the fact that K is arcwise connected and seek for a slightly smaller ball. Let us fix the coordinate system such that $y = (0,0)$ and the line L that realizes the infimum in the definition of $\beta_K(y,t)$ is the x axis: $L = \mathbb{R} \times \{ 0 \}$. This means that

$$K \cap B(y,t) \subset \{ (z_1, z_2) \mid |z_2| \leq \varepsilon t \}.$$

Since $t \leq \operatorname{diam}(K)$, there exists a point $z \in K \setminus B(y,t)$ and a curve $\Gamma \subset K$ from y to z. This curve touches $\partial B(y,t)$ at some point z'. Let $\Gamma' \subset \Gamma$ be the piece of curve from y to z'. The point z' must lie either on $\partial B(y,t) \cap \{ (z_1, z_1) \mid z_1 > 0 \}$ or $\partial B(y,t) \cap \{ (z_1, z_1) \mid z_1 < 0 \}$. Let us assume that the first case occurs (for the second case we can argue similarily). The curve Γ' stays inside the strip $\{ (z_1, z_2) \mid |z_2| \leq \varepsilon t \}$, and runs from y, the center of the ball, to z', on the boundary. Let $y' := y + \frac{t}{2}\mathbf{e}_1$. Then $\beta_K(y', \frac{1}{4}t) \leq 4\varepsilon$ and, assuming $\varepsilon \leq \frac{1}{8}$, it is clear that the curve Γ' separates (in the sense of Definition 2.1) in the ball $B(y', \frac{1}{4}t)$, and so does K. This achieves the proof of the proposition. □

4. Carleson measure estimates on $\omega_p(x,r)$

We define for $r_0 > 0$,

$$\Delta(r_0) := \{ (x,r) \mid x \in K,\ B(x,r) \subset \Omega \text{ and } 0 < r \leq r_0 \}.$$

The purpose of this section is to state the following fact.

Proposition 4.1. *Let $(u,K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional. There exists $r_0 > 0$ (depending on $\mathbf{A}$) such that the following holds. For all $p \in [1,2)$, there exists $C_p \geq 1$ (depending on p and $\mathbf{A}$) such that*

$$\int_{y \in K \cap B(x,r)} \int_{0<t<r} \omega_p(y,t) \frac{\mathrm{d}t}{t} \, \mathrm{d}\mathcal{H}^1(y) \leq C_p r,$$

for all $(x, 2r) \in \Delta(r_0)$.

Proof. The proof was originally performed by David and Semmes in the scalar context of Mumford-Shah minimizers (see [11, Section 23]). It relies on the local Ahlfors-regulary, that is, there exists $C_0 \geq 1$, $r_0 > 0$ (depending on $\mathbf{A}$) such that for all $(x,r) \in \Delta(r_0)$,

$$\int_{B(x,r)} |e(u)|^2 \, dx + \mathcal{H}^1(K \cap B(x,r)) \leq C_0 r \tag{4.1}$$

and

$$\mathcal{H}^1(K \cap B(x,r)) \geq C_0^{-1} r. \tag{4.2}$$

The inequality (4.1) directly follows by taking $(K \setminus B(x,r)) \cup \partial B(x,r)$ and $u\mathbf{1}_{\Omega \setminus B(x,r)}$ as a competitor, and the ellipticity of $\mathbf{A}$. The proof in [11, Section 23] on Mumford-Shah minimizers can be followed verbatim so we prefer to omit the details and refer directly to [11]. □

5. Control of ω_2 by ω_p

The main ε-regularity theorem uses an assumption on the smallness of ω_2. Unfortunately, what we can really control in many balls (thanks to Proposition 4.1) is ω_p for $p < 2$, which is weaker. This is why in this section we prove that ω_2 can be estimated from ω_p, for a minimizer. This strategy was already used in [11] and [17] for the Mumford-Shah functional. The adaptation for the Griffith energy is not straigthforward, but can be done by following a similar approach as the one already used in [3, Section 4.1.], generalized with ω_p instead of only ω_2. Some estimates from the book [11] were also useful.

Lemma 5.1 (Harmonic extension in a ball from an arc of circle)**.** *Let $p \in (1, 2]$, $0 < \delta \leq 1/2$, $x_0 \in \mathbb{R}^2$, $r > 0$ and let $\mathscr{C}_\delta \subset \partial B(x_0, r)$ be the arc of circle defined by*

$$\mathscr{C}_\delta := \{\, (x_1, x_2) \in \partial B(x_0, r) \ : \ (x - x_0)_2 > \delta r \,\}.$$

Then, there exists a constant $C > 0$ (independent of δ, x_0, and r) such that every function $u \in W^{1,p}(\mathscr{C}_\delta; \mathbb{R}^2)$ extends to a function $g \in W^{1,2}(B(x_0, r); \mathbb{R}^2)$ with $g = u$ on $\mathscr{C}_\delta$ and

$$\int_{B(x_0,r)} |\nabla g|^2 \, dx \leq C r^{2-\frac{2}{p}} \left(\int_{\mathscr{C}_\delta} |\partial_\tau u|^p \, d\mathcal{H}^1 \right)^{\frac{2}{p}},$$

where $C = C(p)$.

Proof. Let $\Phi : \mathscr{C}_\delta \to \mathscr{C}_0$ be a bilipschitz mapping with Lipschitz constants independent of $\delta \in (0, 1/2]$, x_0, and $r > 0$. Since $u \circ \Phi^{-1} \in W^{1,p}(\mathscr{C}_0; \mathbb{R}^2)$, we can define the extension by reflection $\tilde{u} \in W^{1,p}(\partial B(x_0, r); \mathbb{R}^2)$ on the whole circle $\partial B(x_0, r)$, that satisfies

$$\int_{\partial B(x_0,r)} |\partial_\tau \tilde{u}|^p d\mathcal{H}^1 \leq C \int_{\mathscr{C}_\delta} |\partial_\tau u|^p \, d\mathcal{H}^1,$$

where $C > 0$ is a constant which is independent of δ. We next define g as the harmonic extension of $\tilde{u}$ in $B(x_0, r)$. Using [11, Lemma 22.16], we obtain

$$\int_{B(x_0,r)} |\nabla g|^2 \, dx \leq C r^{2-\frac{2}{p}} \left(\int_{\partial B(x_0,r)} |\partial_\tau \tilde{u}|^p \, d\mathcal{H}^1 \right)^{\frac{2}{p}} \leq C r^{2-\frac{2}{p}} \left(\int_{\mathscr{C}_\delta} |\partial_\tau u|^2 \, d\mathcal{H}^1 \right)^{\frac{2}{p}},$$

which completes the proof. □

Lemma 5.2. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r > 0$ be such that $B(x_0, r) \subset \Omega$ and $\beta(x_0, r) \leq 1/2$. Let S be the strip defined by*

$$S := \{\, y \in \overline{B}(x_0, r) \mid \operatorname{dist}(y, L) \leq r\beta(x_0, r) \,\},$$

where L is the line passing through x_0 which achieves the infimum in $\beta_K(x_0, r)$. Then there exist a universal constant $C > 0$, $\rho \in (r/2, r)$, and $v^\pm \in H^1(B(x_0, \rho); \mathbb{R}^2)$, such that $v^\pm = u$ on $\mathscr{C}^\pm$, $\mathscr{C}^\pm$ being the connected components of $\partial B(x_0, \rho) \setminus S$, and

$$\int_{B(x_0,\rho)} |e(v^\pm)|^2 \, dx \leq C r^{2-\frac{4}{p}} \left(\int_{B(x_0,r)\setminus K} |e(u)|^p \, dx \right)^{\frac{2}{p}}.$$

Proof. Let $A^\pm$ be the connected components of $B(x_0, r) \setminus S$. Since $K \cap A^\pm = \emptyset$, by Korn inequality there exist two skew-symmetric matrices $R^\pm$ such that the functions $x \mapsto u(x) - R^\pm x$ belong to $W^{1,p}(A^\pm; \mathbb{R}^2)$ and

$$\int_{A^\pm} |\nabla u - R^\pm|^p \, dx \leq C \int_{A^\pm} |e(u)|^p \, dx,$$

where the constant $C > 0$ is universal since the domains $A^\pm$ are all uniformly Lipschitz for all possible values of $\beta(x_0, r) \leq 1/2$. Using the change of variables in polar coordinates, we infer that

$$\int_{A^\pm} |\nabla u - R^\pm|^p \, dx = \int_0^r \left(\int_{\partial B(x_0,\rho)\cap A^\pm} |\nabla u - R^\pm|^p \, d\mathcal{H}^1 \right) d\rho$$

which allows us to choose a radius $\rho \in (r/2, r)$ satisfying

$$\int_{\partial B(x_0,\rho)\cap A^+} |\nabla u - R^+|^p \, d\mathcal{H}^1 + \int_{\partial B(x_0,\rho)\cap A^-} |\nabla u - R^-|^p \, d\mathcal{H}^1$$
$$\leq \frac{2}{r}\int_{A^+} |\nabla u - R^+|^p \, dx + \frac{2}{r}\int_{A^-} |\nabla u - R^-|^p \, dx \leq \frac{C}{r}\int_{B(x_0,r)\setminus K} |e(u)|^p \, dx \,.$$

Setting $\mathscr{C}^\pm := \partial A^\pm \cap \partial B(x_0,\rho)$, in view of Lemma 5.1 applied to the functions $u^\pm : x \mapsto u(x) - R^\pm x$, which belong to $W^{1,p}(\mathscr{C}^\pm;\mathbb{R}^2)$ since they are regular, for $\delta = \beta(x_0,r)$ we get two functions $g^\pm \in W^{1,2}(B(x_0,\rho);\mathbb{R}^2)$ satisfying $g^\pm(x) = u(x) - R^\pm x$ for $\mathcal{H}^1$-a.e. $x \in \mathscr{C}^\pm$ and

$$\int_{B(x_0,\rho)} |\nabla g^\pm|^2 \, dx \leq C\rho^{2-\frac{2}{p}} \left(\int_{\mathscr{C}^\pm} |\partial_\tau u^\pm|^p \, d\mathcal{H}^1\right)^{\frac{2}{p}} \leq Cr^{2-\frac{4}{p}} \left(\int_{B(x_0,r)\setminus K} |e(u)|^p \, dx\right)^{\frac{2}{p}} .$$

Finally, the functions $x \mapsto v^\pm(x) := g^\pm(x) + R^\pm x$ satisfy the required properties. □

Using the competitor above, we can obtain the following.

Proposition 5.1. *Let $(u,K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r > 0$ be such that $B(x_0,r) \subset \Omega$ and $\beta(x_0,r) \leq 1/2$. Then there exist a universal constant $C > 0$ and a radius $\rho \in (r/2, r)$ such that*

$$\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u) \, dx + \mathcal{H}^1(K \cap B(x_0,\rho)) \leq 2\rho + C\rho\big(\omega_p(x_0,r) + \beta(x_0,r)\big).$$

Proof. We keep using the same notation than that used in the proof of Lemma 5.2. Let $\rho \in (r/2, r)$ and $v^\pm \in H^1(B(x_0,\rho);\mathbb{R}^2)$ be given by the conclusion of Lemma 5.2. We now construct a competitor in $B(x_0,\rho)$ as follows. First, we consider a "wall" set $Z \subset \partial B(x_0,\rho)$ defined by

$$Z := \{\, y \in \partial B(x_0,\rho) \mid \operatorname{dist}(y, L(x_0,r)) \leq r\beta(x_0,r) \,\}\,.$$

Note that $K \cap \partial B(x_0,\rho) \subset Z$,

$$\partial B(x_0,\rho) = [\partial A^+ \cap \partial B(x_0,\rho)] \cup [\partial A^- \cap \partial B(x_0,\rho)] \cup Z = \mathscr{C}^+ \cup \mathscr{C}^- \cup Z,$$

and that

$$\mathcal{H}^1(Z) = 4\rho \arcsin\left(\frac{r\beta(x_0,r)}{\rho}\right) \leq 2\pi r\beta(x_0,r).$$

We are now ready to define the competitor (v, K') by setting

$$K' := \big[K \setminus B(x_0,\rho)\big] \cup Z \cup \big[L(x_0,r) \cap B(x_0,\rho)\big],$$

and, denoting by $V^\pm$ the connected components of $B(x_0,\rho) \setminus L(x_0,r)$ which intersect $A^\pm$,

$$v := \begin{cases} v^\pm & \text{in } V^\pm \\ u & \text{otherwise.} \end{cases}$$

Since $\mathcal{H}^1(K' \cap \overline{B}(x_0,\rho)) \leq 2\rho + 2\pi r\beta(x_0,r)$, we deduce that

$$\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u) \, dx + \mathcal{H}^1(K \cap \overline{B}(x_0,\rho))$$
$$\leq \int_{B(x_0,\rho)\setminus K} \mathbf{A}e(v) : e(v) \, dx + \mathcal{H}^1(K' \cap \overline{B}(x_0,\rho)) \tag{5.1}$$
$$\leq Cr^{2-\frac{4}{p}} \left(\int_{B(x_0,r)\setminus K} |e(u)|^p \, dx\right)^{\frac{2}{p}} + \rho(2 + C\beta(x_0,r)) \tag{5.2}$$
$$\leq 2\rho + C\rho\big(\omega_p(x_0,r) + \beta(x_0,r)\big), \tag{5.3}$$

and the proposition follows. □

The next Lemma is of purely topological nature.

Lemma 5.3. *Let $K \subset \mathbb{R}^2$ be a compact connected set with $\mathcal{H}^1(K) < +\infty$. Assume that for some $x \in K$ and $r \in (0, \operatorname{diam}(K))$ we have $\beta_K(x,r) \leq 1/2$. Then*

$$\mathcal{H}^1(K \cap B(x,r)) \geq 2r - 3r\beta_K(x,r).$$

Proof. Let $\varepsilon := \beta_K(x,r)$. We can assume that $x = (0,0)$ and that $L := L(x,r) = \mathbb{R} \times \{0\}$ so that $K \cap B(x,r)$ is contained in the strip S defined by

$$S := B(x,r) \cap \{ (z_1, z_2) \in \mathbb{R}^2 \mid |z_2| \leq \varepsilon r \}.$$

Let $\pi_1 : \mathbb{R}^2 \to L$ be the projection defined by $\pi_1(z_1, z_2) = (z_1, 0)$. As π_1 is 1-Lipschitz, we know that

$$\mathcal{H}^1(K \cap B(x,r)) \geq \mathcal{H}^1(\pi_1(K \cap B(x,r))).$$

Let us denote by $E = \pi_1(K \cap B(x,r))$. Now we define the constant

$$c_\varepsilon := \sqrt{1 - \varepsilon^2},$$

and we use that K is connected to claim that that $L \cap [-rc_\varepsilon, rc_\varepsilon] \setminus E$ is an interval (we identify L with the real axis). Indeed, notice that even if K is connected, it may be that $K \cap B(x,r)$ is not. However, for each $a \in E$, there exists $|t| \leq \varepsilon r$ such that $z_0 := (a,t) \in K$, and since $r < \operatorname{diam}(K)$ there exists a curve Γ that connects z_0 to some point $z_1 \in K \setminus B(x,r)$. But then E has to contain $\pi_1(\Gamma)$, and since $\beta_K(x,r) \leq 1/10$ it means that either $[a, rc_\varepsilon] \subset E$ or $[-rc_\varepsilon, a] \subset E$. Indeed, the curve Γ is contained in the strip S and has to "escape the ball" $B(x,r)$ either from the right or from the left. The projection with minimal length would be when Γ escapes exactly at the corner of $S \cap B(x,r)$ which gives the definition of c_ε (see the picture below).

This holds true for all $a \in E$, which necessarily imply that $[-c_\varepsilon r, c_\varepsilon r] \setminus E$ is an interval, that we denote by I. As $(I \times [-\varepsilon r, \varepsilon r]) \cap K = \emptyset$, we must have $|I| \leq 2\varepsilon r$ otherwise $\beta_K(x,r) > \varepsilon$. All in all we have proved that

$$\mathcal{H}^1(K \cap B(x,r)) \geq \mathcal{H}^1(\pi_1(K \cap B(x,r))) \geq 2rc_\varepsilon - 2\varepsilon r.$$

Now, we estimate $2c_\varepsilon r - 2\varepsilon r$. We have

$$1 - \sqrt{1 - \varepsilon^2} = \frac{\varepsilon^2}{1 + \sqrt{1 - \varepsilon^2}} \leq \varepsilon^2 \leq \tfrac{1}{2}\varepsilon \tag{5.4}$$

whence $2c_\varepsilon \geq 2 - \varepsilon$ and the result follows.

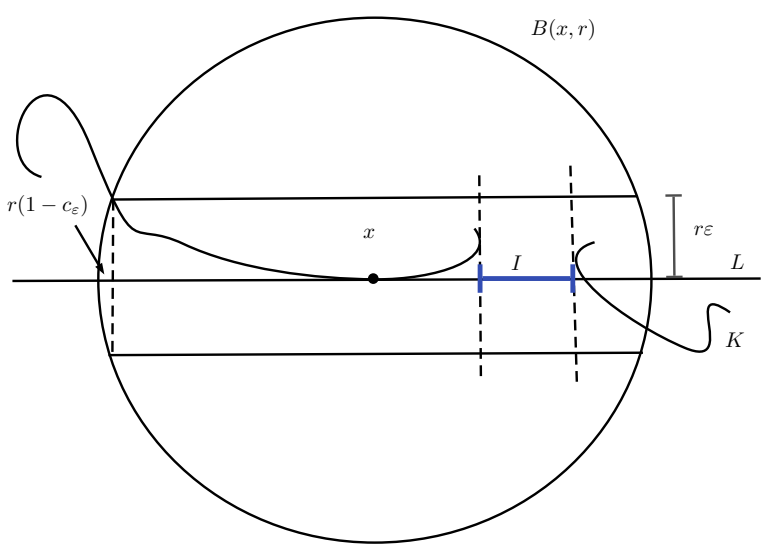

Figure 1: estimating the length of $K \cap B(x,r)$.

□

We now come to the interesting "reverse Hölder" type estimate that will be needed later.

Corollary 5.1. *Let $(u,K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r \in (0, \mathrm{diam}(K))$ be such that $B(x_0,r) \subset \Omega$ and $\beta(x_0,r) \leq 1/2$. Then there exist a universal constant $C > 0$ and a radius $\rho \in (r/2, r)$ such that*

$$\omega_2(x_0,\rho) \leq C\big(\omega_p(x_0,r) + \beta(x_0,r)\big).$$

Proof. By Proposition 5.1, we already know that there exist a universal constant $C > 0$ and a radius $\rho \in (r/2, r)$ such that

$$\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u)\,dx + \mathcal{H}^1(K \cap B(x_0,\rho)) \leq 2\rho + C\rho\big(\omega_p(x_0,r) + \beta(x_0,r)\big).$$

Now noticing that $\beta(x,\rho) \leq 2\beta(x,r) \leq \frac{1}{5}$ we can use Lemma 5.3 in $B(x_0,\rho)$ which yields

$$\mathcal{H}^1(K \cap B(x_0,\rho)) \geq 2\rho - 3\beta(x_0, r\rho)$$

hence,

$$\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u)\,dx \leq C\rho\big(\omega_p(x_0,r) + \beta(x_0,r)\big).$$

Finally, by ellipticity of A we get

$$\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u)\,dx \geq r\omega_2(x_0,\rho),$$

which finishes the proof. □

6. Porosity of the bad set

Given $0 < \alpha < 1$, $x_0 \in K$ and $r > 0$ such that $B(x,r) \subset \Omega$, we say that the crack-set K is $C^{1,\alpha}$-regular in the ball $B(x_0,r)$ if it is the graph of a $C^{1,\alpha}$ function f such that, in a convenient coordinate system it holds $f(0) = x_0$, $f'(0) = 0$ and $r^\alpha \|f'\|_{C^\alpha} \leq 1/4$.

We recall the following ε-regularity theorem coming from [3].

Theorem 6.1. [3] *Let $(u,K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional with K connected and $\mathcal{H}^1(K) > 0$. There exist constants $a, \alpha, \varepsilon_2 \in (0,1)$ depending only on $\mathbf{A}$ such that the following holds. Let $x_0 \in K$ and $r > 0$ be such that $\overline{B}(x_0,r) \subset \Omega$ and*

$$\omega_2(x_0,r) + \beta(x_0,r) \leq \varepsilon_2,$$

and K separates in $B(x_0,r)$. Then K is $C^{1,\alpha}$-regular in $B(x_0,ar)$.

Proof. Unfortunately, the above statement is not explicitely stated in [3], but it directly follows from the proof of [3, Proposition 3.4]. Indeed, in the latter proof, some explicit thresholds $\delta_1 > 0$ and $\delta_2 > 0$ are given so that, provided that

$$\omega_2(x_0,r) \leq \delta_2, \quad \beta(x_0,r) \leq \delta_1,$$

and K separates in $B(x_0,r)$, then

$$\beta(y,tr) \leq Ct^\alpha$$

for all $y \in B(x_0,r/2)$ and $t \in (0,1/2)$, which implies that $B(x_0,a_0r)$ is a $C^{1,\alpha}$ curve as well as a 10^{-2}-Lipschitz graph thanks to [3, Lemma 6.4]. In addition the graph is $C^{1,\alpha}$ with $\|f\|_\infty \leq 10^{-2}r$ and the estimate (6.8) in [3] says moreover $\|f'\|_{C^{0,\alpha}} \leq C$, from which we easily get $(ar)^\alpha \|f'\|_{C^{0,\alpha}} \leq C \leq 1/4$ up to take a smaller radius r. The fact that α and a depend only on $\mathbf{A}$, follows from a careful inspection of the proof in [3]. □

We are now in position to prove the following, which says that the singular set is porus in K.

Proposition 6.1. *Let $(u,K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional with K connected and $\mathcal{H}^1(K) > 0$. There exist constants $a \in (0,1/2)$ and $r_0 > 0$ (depending only on $\mathbf{A}$) such that the following holds. For all $x_0 \in K$ and $0 < r \leq r_0$ such that $B(x_0,r) \subset \Omega$, there exists $y \in K \cap B(x_0,r/2)$ such that $K \cap B(y,ar)$ is $C^{1,\alpha}$-regular (where α is the constant of Theorem 6.1).*

Proof. In view Theorem 6.1, it is enough to prove the following fact: there exists $a \in (0, 1/2)$ and $r_0 > 0$ such that for all $x \in K$ and $0 < r \leq r_0$ with $B(x, r) \subset \Omega$, there exists $y \in K \cap B(x, r/2)$ and $ar < s < r/2$ such that

$$\omega_2(y, s) + \beta(y, s) \leq \varepsilon_2$$

and K separates in $B(y, s)$, where ε_2 is the constant of Theorem 6.1. We already know from Lemma 3.1 how to control β and the separation. We therefore need to add a control on ω_2, and this will be done by applying successively Proposition 4.1 and Corollary 5.1, but we need to fix carefully the constants so that it compiles well.

Let us pick any $p \in (1, 2)$ and let C_p be the constant of Proposition 4.1, and let C_0 be the constant of Corollary 5.1. Then we define

$$b := \frac{1}{4} e^{-\frac{8C_p C_0}{\varepsilon_2}} \quad \text{and} \quad \varepsilon_0 := \frac{b\varepsilon_2}{8C_0} \wedge \frac{1}{2}.$$

We define r_0 as the minimum between the radius of Ahlfors-regularity and the radius given by Proposition 4.1. We fix $x \in K$ and $0 < r \leq r_0$ such that $B(x, r) \subset \Omega$. As noticed at the beginning of the proof of Lemma 3.1, we can assume $r \leq \operatorname{diam}(K)$ without loss of generality. We apply Lemma 3.1 with the previous definition of ε_0 and we get that for some $a \in (0, 1/2)$ (depending on $\mathbf{A}$), there exists $y \in B(x, r/2)$ and $t \in (ar, r/2)$ satisfying:

$$\beta(y, t) \leq \varepsilon_0 \text{ and } K \text{ separates } D^{\pm}(y, t) \text{ in } \overline{B}(y, t) \text{ as in Definition 2.1.}$$

Then we apply Proposition 4.1 in $B(y, t)$ which yields

$$\int_{z \in K \cap B(y,t)} \int_{0<s<t} \omega_p(z, s) \frac{\mathrm{d}s}{s} \, \mathrm{d}\mathcal{H}^1(z) \leq C_p t. \tag{6.1}$$

From this estimate we claim that we obtain the following fact:

$$\text{there exists } z \in B(y, t/2) \text{ and } s \in (bt, t/2) \text{ such that } \omega_p(z, s) \leq \frac{\varepsilon_2}{4C_0}. \tag{6.2}$$

Indeed, remember that K is connected, $y \in K$ and $r \leq \operatorname{diam}(K)$ thus $\mathcal{H}^1(K \cap B(y, t/2)) \geq t/2$. Therefore, if the claim in (6.2) is not true, then

$$\begin{aligned} \int_{z \in K \cap B(y,t/2)} \int_{bt<s<t/2} \omega_p(z, t) \, \frac{\mathrm{d}s}{s} \, \mathrm{d}\mathcal{H}^1(z) &\geq \frac{\varepsilon_2}{4C_0} \mathcal{H}^1(K \cap B(y, t/2)) \int_{bt}^{t/2} \frac{\mathrm{d}t}{t} \\ &\geq \frac{t\varepsilon_2}{8C_0} \ln\left(\frac{1}{2b}\right). \end{aligned} \tag{6.3}$$

Returning back to (6.1), we get

$$\ln\left(\frac{1}{2b}\right) \leq \frac{8C_0}{\varepsilon_2} C_p$$

which contredicts our definition of b. The claim is now proved.

Now let us check what we have got in the ball $B(z, s)$. We already know that $s \geq cr$ for some constant $c \in (0, 1/2)$ (which depending only on $\mathbf{A}$) and $\omega_p(z, s) \leq \frac{\varepsilon_2}{4C_0}$. Concerning β, we have

$$\beta(z, s) \leq \frac{2}{b} \beta(y, t) \leq \frac{2}{b} \varepsilon_0 \leq \frac{\varepsilon_2}{4C_0}.$$

Next, we apply Corollary 5.1 which says that there exists $s' \in (s/2, s)$ such that

$$\omega_2(z, s') \leq C_0 \big(\omega_p(z, s) + \beta(z, s)\big) \leq \frac{\varepsilon_2}{2}.$$

The ball $B(z, s')$ satisfies all the required properties because $\beta(z, s') \leq 2\beta(z, s) \leq \frac{\varepsilon_2}{2}$ so that

$$\omega_2(z, s') + \beta(z, s') \leq \varepsilon_2,$$

as required.

It remains to see that K still separates in the ball $B(z, s')$. But once $\beta(z, s')$ is controlled and knowing that K already separates in $B(y, t)$, it follows from Lemma 2.1. □

We are now ready to state one of our main results about the the Hausdorff dimension of the singular set.

Corollary 6.1. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional with K connected. Then there exists a closed set $\Sigma \subset K$ such that $\dim_{\mathcal{H}}(\Sigma) < 1$ and $K \setminus \Sigma$ is locally a $C^{1,\alpha}$ curve.*

Proof. The proof is standard now that Proposition 6.1 is established. Indeed, we can argue exactly as Rigot in [17, Remark 3.29] which we refer to for more detail. □

7. Higher integrability of $e(u)$

Theorem 7.1. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional with K connected and $\mathcal{H}^1(K) > 0$. There exists $r_0 > 0$, $C \geq 1$ and $p > 1$ (depending only on $\mathbf{A}$) such that the following holds true. For all $x \in \Omega$, for all $0 \leq r \leq r_0$ such that $B(x, r) \subset \Omega$,*

$$\int_{\frac{1}{2}B(x,r)} |e(u)|^{2p} \, \mathrm{d}x \leq C r^{2-p}. \tag{7.1}$$

We rely on a higher integrability lemma ([16, Lemma 4.2]) which is inspired by the technique of [13]. We recall that given $0 < \alpha < 1$, a closed set K, $x_0 \in K$ and $r > 0$, we say that K is $C^{1,\alpha}$-regular in the ball $B(x_0, r)$ if it is the graph of a $C^{1,\alpha}$ function f such that, in a convenient coordinate system it holds $f(0) = x_0$, $\nabla f(0) = 0$ and $r^\alpha \|\nabla f\|_{C^\alpha} \leq 1/4$. We take the convention that the $C^{1,\alpha}$ norm is small enough because we don't want that it interferes with the boundary gradient bound estimates for the Lamé's equations. It is also required by the covering lemma [16, Lemma 4.3] on which [16, Lemma 4.2] is based.

Lemma 7.1. *We fix a radius $R > 0$ and an open ball B_R of radius R in $\mathbb{R}^n$. Let K be a closed subset of B_R and $v \colon B_R \to \mathbb{R}^+$ be a non-negative Borel function. We assume that there exists $C_0 \geq 1$ and $0 < \alpha \leq 1$ such that the following holds true.*

(i) For all ball $B(x, r) \subset B_R$,

$$C_0 r^{n-1} \leq \mathcal{H}^{n-1}(K \cap B(x, r)) \leq C_0 r^{n-1}. \tag{7.2}$$

(ii) For all ball $B(x, r) \subset B_R$ centered in K, there exists a smaller ball $B(y, C_0^{-1} r) \subset B(x, r)$ in which K is $C^{1,\alpha}$-regular.

(iii) For all ball $B(x, r) \subset B_R$ such that K is disjoint from $B(x, r)$ or K is $C^{1,\alpha}$-regular in $B(x, r)$, we have

$$\sup_{\frac{1}{2}B(x,r)} v(x) \leq C_0 \left(\frac{R}{r} \right). \tag{7.3}$$

Then there exists $p > 1$ and $C \geq 1$ (depending on n, C_0) such that

$$\fint_{\frac{1}{2}B_R} v^p \leq C. \tag{7.4}$$

Proof of Theorem 7.1. We apply the previous Lemma. More precisely, there exists $0 < r_0 \leq 1$, such that for all $x \in \Omega$ and all $0 \leq R \leq r_0$ such that $B(x, R) \subset \Omega$, one can applies Lemma [16, Lemma 4.2] in the ball $B(x, R)$ to the function $v := R|e(u)|^2$. The assumption (i) follows from the local Ahlfors-regularity of K. The assumption (ii) follows from the porosity (Proposition 6.1). The assumption (iii) follows from interior/boundary gradient estimates for the Lame's equations and from the local Ahlfors-regularity. In particular, the boundary estimate is detailed in Lemma A.1 in Appendix A. □

Appendices

A. Lamé's equations

We work in the Euclidean space $\mathbb{R}^n$ $(n > 1)$. For $r > 0$, B_r denotes the ball of radius r and centered at 0. We fix a radius $0 < R \leq 1$, an exponent $0 < \alpha \leq 1$, a constant $A > 0$ and a $C^{1,\alpha}$ function $f\colon \mathbb{R}^{n-1} \cap B_R \to \mathbb{R}$ such that $f(0) = 0$, $\nabla f(0) = 0$ and $R^\alpha \left[\nabla f\right]_\alpha \leq A$. We introduce

$$V_R := \{\, x \in B_R \mid x_n > f(x') \,\} \tag{A.1}$$

$$\Gamma_R := \{\, x \in B_R \mid x_n = f(x') \,\}\,. \tag{A.2}$$

We denote by ν the normal vector field to Γ_R going upward. For $0 < t \leq 1$, we write tV_R for $V_R \cap B_t$ and $t\Gamma_R$ for $\Gamma_R \cap B_t$. For $u \in W^{1,2}(V_R;\mathbb{R}^n)$, we denote by u^* the trace of u in $L^2(\partial V_R;\mathbb{R}^n)$. For a function $\xi\colon V_R \to \mathbb{R}^{n\times n}$, we define (formally) $\mathrm{div}(\xi)$ as the vector field whose ith coordinate is given by $\mathrm{div}(\xi)_i = \sum_j \partial_j \xi_{ij}$. We also recall the notation for the linear strain tensor

$$e(u) = \frac{Du + Du^T}{2}. \tag{A.3}$$

and the stress tensor

$$\mathbf{A}e(u) = \lambda \mathrm{div}(u) I_n + 2\mu e(u), \tag{A.4}$$

where λ and μ are the Lamé coefficients satisfying $\mu > 0$ and $\lambda + \mu > 0$. We denote by $W_0^{1,2}(V_R \cup \Gamma_R;\mathbb{R}^n)$ the space of functions $v \in W^{1,2}(V_R;\mathbb{R}^n)$ such that $v^* = 0$ on $\partial V_R \setminus \Gamma_R$.

Our object of study are the functions $u \in W^{1,2}(V_R) \cap L^\infty(V_R)$ which are weak solutions of

$$\begin{cases} \mathrm{div}(\mathbf{A}e(u)) &= 0 \quad \text{in } V_R \\ \mathbf{A}e(u)\cdot\nu &= 0 \quad \text{on } \Gamma_R, \end{cases} \tag{A.5}$$

that is for all $v \in W_0^{1,2}(V_R \cup \Gamma_R;\mathbb{R}^n)$,

$$\int_{V_R} \mathbf{A}e(u) : Dv \,\mathrm{d}x = 0. \tag{A.6}$$

Remark A.1. *As*

$$\mathbf{A}e(u) = (\lambda+\mu)Du^T + \mu Du + \lambda(\mathrm{div}(u)I_n - Du^T) \tag{A.7}$$

and the part $\mathrm{div}(u)I_n - Du^T$ *is divergence free, we can also write formally*

$$\mathrm{div}(\mathbf{A}e(u)) = (\lambda+\mu)\nabla\mathrm{div}(Du) + \mu\Delta u. \tag{A.8}$$

We are going to justify the following estimate.

Lemma A.1. *Let us assume $n = 2$. There exists $C \geq 1$ (depending on α, A, λ, μ) such that*

$$\sup_{\frac{1}{2}V_R} |e(u)| \leq C \left(\fint_{V_R} |e(u)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{A.9}$$

Proof. It suffices to prove that for all solution of A.5, we have

$$\sup_{\frac{1}{2}V_R} |Du| \leq C \left(\fint_{V_R} |Du|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{A.10}$$

Indeed, we observe first that $|e(u)| \leq |Du|$ so (A.10) implies

$$\sup_{\frac{1}{2}V_R} |e(u)| \leq C \left(\fint_{V_R} |Du|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{A.11}$$

By Korn inequality, there exists a skew-symmetric matrix R such that

$$\int_{V_R} |Du - R|^2 \,\mathrm{d}x \leq \int_{V_R} |e(u)|^2 \,\mathrm{d}x \tag{A.12}$$

so it is left to apply (A.11) to $x \mapsto u(x) - Rx$, which also solves Lamé's equations.

From now on, we deal with (A.10). The letter C plays the role of a constant ≥ 1 that depends on λ, μ and α, A. We refer to the proof [14, Theorem 3.18] which itself refers to the proof of [1, Theorem 7.53]. We straighten the boundary Γ_R via the $C^{1,\alpha}$ diffeomorphism $\phi\colon x \mapsto x' + (x_n - f(x'))e_n$. We observe that $\phi(\overline{V_R})$ contains a half-ball ball $B^+ = \overline{B}(0, C_0^{-1}R)^+$, where $C_0 \geq 1$ is a constant that depends on λ, μ, α. The Neumann problem satisfied by u in V_R is transformed into a Neumann problem satisfied by a function v in $\overline{B}(0, C_0^{-1}R)^+$. Then we symmetrize the elliptic system to the whole ball $B = B(0, C_0^{-1}R)$ as in [14, Theorem 3.18]. Following the proof of [1, Theorem 7.53] (in the special case where the right-hand side h is zero), we arrive to the fact there exists $q > n = 2$ (depending on λ, μ, α) such that for all $x_0 \in \frac{1}{2}B$ and $0 < \rho \leq r \leq C^{-1}R$

$$\int_{B_\rho(x_0)} |\nabla v - (\nabla v)_{x_0,\rho}|^2 \,\mathrm{d} \leq C \left(\frac{\rho}{r}\right)^q \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2 + C\rho^q \int_B |\nabla v|^2. \tag{A.13}$$

In particular by Poincaré-Sobolev inequality,

$$\int_{B_\rho(x_0)} |\nabla v - (\nabla v)_{x_0,\rho}|^2 \leq C \left(\frac{\rho}{r}\right)^q \int_B |\nabla v|^2. \tag{A.14}$$

According to the Campanato characterisation of Hölder spaces,

$$[\nabla v]_{C^{0,\sigma}(\frac{1}{2}B)} \leq C \left(R^{-(2+2\sigma)} \int_B |\nabla v|^2\right)^{\frac{1}{2}}, \tag{A.15}$$

where $\sigma = \frac{q-2}{2}$, and this implies

$$\sup_{\frac{1}{2}B} |\nabla v| \leq C \left(\fint_B |\nabla v|^2\right)^2. \tag{A.16}$$

This property is inherited by u via the diffeomorphism ϕ,

$$\sup_{C^{-1}V_R} |\nabla u| \leq C \left(\fint_{V_R} |\nabla u|^2\right)^2. \tag{A.17}$$

We can finally bound the supremum of $|\nabla u|$ on $\frac{1}{2}V_R$ by a covering argument. □

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