HEAT KERNEL BOUNDS AND RICCI CURVATURE FOR LIPSCHITZ MANIFOLDS

MATHIAS BRAUN AND CHIARA RIGONI

ABSTRACT. Given any d-dimensional Lipschitz Riemannian manifold (M, g) with heat kernel p, we establish uniform upper bounds on p which can always be decoupled in space and time. More precisely, we prove the existence of a constant $C > 0$ and a bounded Lipschitz function $R \colon M \to (0, \infty)$ such that for every $x \in M$ and every $t > 0$,

$$\sup_{y \in M} \mathsf{p}(t, x, y) \leq C \min\{t, R^2(x)\}^{-d/2}.$$

This allows us to identify suitable weighted Lebesgue spaces w.r.t. the given volume measure as subsets of the Kato class induced by (M, g). In the case $\partial M \neq \emptyset$, we also provide an analogous inclusion for Lebesgue spaces w.r.t. the surface measure on ∂M.

We use these insights to give sufficient conditions for a possibly noncomplete Lipschitz Riemannian manifold to be tamed, i.e. to admit a measure-valued lower bound on the Ricci curvature, formulated in a synthetic sense.

CONTENTS

1. INTRODUCTION

Background. A *Lipschitz manifold* is a topological manifold, possibly with boundary, with locally Lipschitz transition maps, cf. Chapter 2 for details and basic notions. Every smooth, C^1, or piecewise linear manifold has a Lipschitz structure. Even

better, by Sullivan's theorem [48] *every* topological manifold, apart from dimension 4, can be endowed with a Lipschitz structure which is unique up to locally bi-Lipschitz homeomorphisms isotopic to the identity. (Both the existence and the uniqueness statements are false in dimension 4, as proven in [17].) While this result follows from smoothability of all topological manifolds of dimension less than 4 [39], it is a key feature in higher dimensions, where C^1 structures might not exist at all [33]. An additional strength of Sullivan's result comes to light in the context of *Lipschitz Riemannian manifolds*, briefly LR manifolds, which are pairs (M, g) of a Lipschitz manifold M equipped with a compatible Riemannian metric g. As summarized in Chapter 2, such an (M, g) automatically induces a length metric space (M, d) [14, 15, 16, 37] and a Dirichlet form $(\mathscr{E}, W^{1,2}(M))$ [10, 13] whose correspondence, however, is not one-to-one in general [54]. (Unless otherwise specified, all Lebesgue and Sobolev spaces are considered w.r.t. the volume measure $\mathfrak{v}$ on M induced by g.) In particular, these notions allow for canonical definitions of Lipschitz functions, geodesics, heat flow and Brownian motion. A further feature of LR manifolds is their well-behavedness under bi-Lipschitz homeomorphisms. Even more geometrically, every compact Alexandrov space of finite Hausdorff dimension contains an open convex subset of full Hausdorff measure which has an LR structure [41]. There is thus high evidence in investigating LR manifolds from probabilistic, analytic, and geometric perspectives.

Remarkably, however, few attention has been devoted to this setting so far (even in cases where the considered LR manifolds are "almost smooth" in a suitable sense, compare e.g. with Definition 4.8 below, but whose regular parts then typically fail to be e.g. complete). Relevant works study differential forms [19, 48, 60], Hodge–de Rham's theorem and index theory [31, 57], eigenvalue estimates [36], boundary-value problems [20], smoothability [29, 59], or Varadhan short-time asymptotics for the heat kernel [40].

In view of the above mentioned diversity of examples as well as recent breakthroughs in metric geometry for nonuniform curvature bounds [5, 8, 18, 26, 56], we predict the class of LR manifolds — regarded as prototypes of Riemannian spaces with singularities — to have great potential for near future research, which this article aims to initiate.

Upper heat kernel bounds. The heat flow $(\mathsf{P}_t)_{t \geq 0}$ corresponding to $\mathscr{E}$ can be represented as an integral operator by a jointly locally Hölder integral kernel $\mathsf{p} \colon (0, \infty) \times M^2 \to (0, \infty)$ [52, 53], as we recall in Section 2.5. It is reluctantly accepted, even in the smooth setting [23], that p is explicitly computable only in few cases. Nevertheless, in practice it typically suffices to know upper bounds on p. Usual estimates of this sort are Gaussian and read as

$$\mathsf{p}(t, x, y) \leq C \, \mathfrak{v}\big[\mathsf{B}_{\sqrt{t}}(x)\big]^{-1/2} \mathfrak{v}\big[\mathsf{B}_{\sqrt{t}}(y)\big]^{-1/2} \exp\Big[-\frac{\mathsf{d}^2(x, y)}{(4 + \varepsilon)t}\Big] \qquad (1.1)$$

for every $\varepsilon > 0$ (see Proposition 2.12 and the references in Subsection 2.5.4).

This is still unsatisfactory for at least three reasons, which to address is of fundamental importance for our later applications, as detailed in the next paragraph.

- First, the constant $C > 0$ and the admissible range of $t > 0$ do in general depend on $x, y \in M$, in which case (1.1) becomes qualitatively useless when, for example, one wants to derive globally or locally uniform upper bounds, both in space and time, on the heat kernel p.
- Second, the dependency of the r.h.s. of (1.1) on t, x, and y can usually not be decoupled.
- Third, one would often like upper bounds on $\mathsf{p}(t, x, y)$ to be independent of y, which cannot be guaranteed by (1.1) either.

The argument for (1.1) from [52] works for general Dirichlet spaces satisfying a local doubling condition [52, p. 293] and a local Sobolev inequality [52, p. 294]. However, our given LR structure of (M, g) is a priori respected only by the fact that these two properties transfer from Euclidean space back to M locally in charts, usually with nonuniform constants, cf. Remark 2.4 and Subsection 2.5.4. Settings in which the respective constants are uniform and yield versions of (1.1) addressing all above mentioned issues, e.g. as described in Section 3.3 (following standard arguments, see e.g. [23, 45]), are too restrictive to be satisfied by all LR manifolds.

The following main result of our work, detailed in Theorem 3.13, establishes a slightly different upper bound on p for *general* LR manifolds which rules out all three problems described above. Let us set $d := \dim M$.

Theorem 1.1. *There exist a constant $C > 0$, depending only on d, and a bounded Lipschitz function $R \colon M \to (0, \infty)$ such that for every $x, y \in M$ and every $t > 0$,*

$$\mathsf{p}(t, x, y) \leq C \min\{t, R^2(x)\}^{-d/2}.$$

To the best of our knowledge, so far Theorem 1.1 has only been proven for smooth Riemannian manifolds without boundary [25]. In particular, Theorem 1.1 seems to be new even in the setting of smooth manifolds with boundary.

The proof of Theorem 1.1, presented in Chapter 3, and its idea have been inspired by [23, Thm. 15.4] and [24, Thm. IV.14]. We first prove the existence of some $a > 0$ such that, given any $x \in M$, we have the *Faber–Krahn inequality*

$$\lambda_1(O) \geq a\, \mathfrak{v}[O]^{-2/d} \tag{1.2}$$

for every open, relatively compact $O \subset M$ contained in a suitable neighborhood of x. (Here, $\lambda_1(O)$ denotes the first Dirichlet eigenvalue on O.) Indeed, given any fixed $b > 0$, modulo a reflection argument in the case $\partial M \neq \emptyset$, we prove that (M, g) locally looks like a Euclidean space $(\mathbf{R}^d, h)$ up to a rescaling of g, cf. Lemma 3.12. Here h is an arbitrary Riemannian metric equivalent to the standard Euclidean metric tensor g^{E}, i.e. $b^{-1} g^{\mathrm{E}} \leq h \leq b\, g^{\mathrm{E}}$, w.r.t. which (1.2) is well-known to hold on $(\mathbf{R}^d, h)$ [23, 45]. Of course, the size of the domain itself in which O is allowed to be situated may vary over M. However, it is controllable in terms of the *Euclidean radius* r_{E}, cf. Definition 3.7: given $x \in M$, $r_{\mathrm{E}}(x, b)$ is naturally defined as the largest possible radius $r > 0$ w.r.t. which, roughly speaking, $\mathsf{B}_r(x)$ is quasi-isometric, with constants b^{-1} and b, to a subset of $(\mathbf{R}^d, h)$. (As shown in Lemma 3.10, truncated versions R of r_{E} obey good regularity properties.) This allows us to pull (1.2) locally back to M while retaining a certain uniformity in a, which is possible by the fact that b is fixed. Then, to get a (Li–Yau-type) heat kernel estimate which takes this spacial variation into account, we use an L^1-mean value inequality for local subsolutions to the heat equation (see Theorem 3.5) derived from its L^2-counterpart (see Proposition 3.3) by Li–Wang's L^2-to-L^1 reduction [25, 35].

We point out that these arguments crucially exploit the LR structure of (M, g), whence this setting seems to be as general as possible to prove Theorem 1.1. Furthermore, Theorem 1.1 is quite concrete and well-suited for applications, in the sense that r_{E} — hence R — is in general easy to bound from below explicitly.

Kato class and heat kernel control pairs. One of our main motivations to establish Theorem 1.1 and to address the mentioned nonuniformities in (1.1) is to understand better the Kato class of LR manifolds (M, g).

Definition 1.2. *The* Kato class $\mathcal{K}(M)$ *consists of all signed Borel measures κ on M which do not charge $\mathscr{E}$-polar sets and such that*

$$\lim_{t \to 0} \sup_{x \in M} \int_0^t \int_M \mathsf{p}(s, x, \cdot)\, \mathrm{d}|\kappa|\, \mathrm{d}s = 0.$$

The prominent role of the Kato class in mathematical physics has been well established in the past decades in a series of works [2, 32, 49]. Recently, its interest in the context of singular Ricci bounds has grown as well [5, 6, 7, 9, 18, 24, 26, 43] (also with applications to Ricci flow and general relativity). The study of the class of LR manifolds in this latter setting constitutes the last part of our work.

It is easy to see that $L^\infty(M)\,\mathfrak{v} \subset \mathcal{K}(M)$. To verify from Definition 1.2 that a given κ does belong to $\mathcal{K}(M)$ beyond this trivial case, the importance of having appropriate upper bounds for p according to the criteria raised above is intuitively clear. In concrete words, in Theorem 4.2 we identify an explicit subspace of $\mathcal{K}(M)$ for a given *heat kernel control pair* (Ξ, Σ), i.e. an ordered pair of suitable functions $\Xi\colon M \to (0,\infty)$ and $\Sigma\colon (0,1] \to (0,\infty)$ for which there exists some $C > 0$ such that

$$\mathsf{p}(t,x,y) \le C\,\Sigma(t)\,\Xi(x)$$

for every $x, y \in M$ and every $t \in (0,1]$, see Definition 3.6.

Theorem 1.3. *Suppose that $d \ge 2$, and assume the existence of some heat kernel control pair (Ξ, Σ). Then $L^p(M, \Xi\,\mathfrak{v})\,\mathfrak{v} + L^\infty(M)\,\mathfrak{v} \subset \mathcal{K}(M)$ for every $p \in (d/2, \infty)$.*

By an evident choice of (Ξ, Σ), this result recovers the well-known fact that $L^p(\mathbf{R}^d)\,\mathscr{L}^d \subset \mathcal{K}(\mathbf{R}^d)$ for every $p \in (d/2, \infty]$ [2].

As an immediate consequence of our Theorem 1.1, the second hypothesis of Theorem 1.3 is *always* satisfied, cf. Corollary 3.15: this in turn implies that, up to the evident identification, $\mathcal{K}(M)$ always contains certain weighted L^p-spaces.

Corollary 1.4. *Every LR manifold admits a canonical heat kernel control pair.*

Taming for LR manifolds. The notion of *tamed spaces*, summarized in Section 4.1, has recently been introduced in [18]. It offers a synthetic way of speaking about the Ricci curvature of a Dirichlet space being bounded from below by a distribution κ. This machinery provides a far-reaching generalization of the Bakry–Émery theory for diffusion operators [3] and its Eulerian counterpart for RCD spaces [1], formulated in terms of Schrödinger operators by a weak form of the 1-Bochner inequality

$$(\Delta - \kappa)|\mathrm{d}f| - g^*(\mathrm{d}f, \mathrm{d}\Delta f)\,|\mathrm{d}f|^{-1} \ge 0. \tag{1.3}$$

Since elements in $\mathcal{K}(M)$ provide special cases of the relevant distributions considered in [18] and since Ricci bounds have powerful probabilistic, analytic and geometric consequences, we believe our following condition for taming of LR manifolds, cf. Theorem 4.9 below, to be of high interest.

Theorem 1.5. *Assume that $d \ge 2$ and that (M, g) is almost smooth, i.e. it contains a $\mathfrak{v}$-conegligible, open, smooth Riemannian manifold $(M^\infty, g\big|_{M^\infty})$. Let $\mathscr{k} \in L^p(M, \Xi\,\mathfrak{v}) + L^\infty(M)$, where $p \in (d/2, \infty)$ and (Ξ, Σ) is given a heat kernel control pair, be a lower bound of the Ricci curvature on M^∞, i.e.*

$$\mathrm{Ric} \ge \mathscr{k} \quad on\ M^\infty.$$

Lastly, suppose that the Laplacian Δ is essentially self-adjoint on $\mathrm{C}_c^\infty(M^\infty)$. Then the Dirichlet space $(M, \mathscr{E}, \mathfrak{v})$ is tamed by

$$\kappa := \mathscr{k}\,\mathfrak{m}.$$

We briefly comment on the assumptions of Theorem 1.5 as well as on its proof, given in Section 4.2 below. The almost smoothness of (M, g) is needed to set up a $\mathfrak{v}$-a.e. Bochner *identity* on its smooth part, which is the starting point to derive (1.3). To integrate the inherent Schrödinger term "$(\Delta - 2\mathscr{k})\,|\mathrm{d}f|^2/2$" by parts and to simultaneously apply the chain rule to pass over to "$(\Delta - \mathscr{k})\,|\mathrm{d}f|$" we will *a priori* require the $W^{1,2}$-regularity of $|\mathrm{d}f|$ for every $f \in \mathscr{D}(\Delta)$, as discussed in Lemma 4.10. The essential self-adjointness of Δ on $\mathrm{C}_c^\infty(M^\infty)$, which ensures

the previous regularity as shown in [30], is then used to extend this statement to arbitrary elements of $\mathscr{D}(\Delta)$, through an integrated form of (1.3).

The hypothesis on essential self-adjointness in Theorem 1.5 is less restrictive than it might seem. Indeed, Theorem 1.5 already applies to *regular almost Riemannian structures* [42], i.e. manifolds which admit a smooth structure, but are endowed with a possibly singular metric. Actually already in the smooth case, interesting tamed situations are covered: indeed, essential self-adjointness holds for example in the case in which M arises from a smooth Riemannian manifold from which a closed set of codimension at least 4 is removed [38] (see Example 4.11 and Example 4.12 for details). In particular, we cover smooth examples of tamed spaces with constant lower Ricci bounds *beyond* complete manifolds, while classical examples in Bakry–Émery's theory typically require completeness (which is also always part of the assumptions in the correspondent framework of RCD spaces), a property implying essential self-adjointness [50].

Lastly, Section 4.3 deals with taming of certain LR manifolds with boundary, see Theorem 4.21. Here, compared to the $\mathfrak{v}$-absolutely continuous case from Theorem 1.3, it is even more sophisticated to figure out when certain L^q-spaces w.r.t. the surface measure σ on ∂M belong to $\mathcal{K}(M)$. This is addressed in Theorem 4.19.

Organization. In Chapter 2, we give a concise overview of LR manifolds and all induced analytic objects needed for our purposes. Chapter 3 introduces the concept of control pairs for the given heat kernel and shows the general existence of such a pair. Lastly, in Chapter 4 we briefly recapitulate basic notions about tamed spaces from [18] and provide sufficient conditions for taming of LR manifolds.

2. Calculus on Lipschitz manifolds

This preliminary chapter is a survey over basic facts, considered as standard and thus mostly stated without proof, about the framework of LR manifolds we work in all over the paper.

In this article, we do mostly not keep track of involved uniform constants (albeit we always try to be clear about their dependencies) and, by convention, allow them to change from line to line without reflecting this change in the notation.

2.1. **LR manifolds.** Following [14, 15, 16, 37, 44, 48, 57], we first introduce relevant notions of *LR manifolds*.

2.1.1. *Lipschitz manifolds.*

Definition 2.1. *A topological Hausdorff space M is called* (d-dimensional) Lipschitz manifold, *$d \in \mathbf{N}$, if its topology is second countable and if it is endowed with an atlas $\mathscr{U} := (U_\alpha, \Phi_\alpha)_{\alpha \in A}$ such that*

    a. *$(U_\alpha)_{\alpha \in A}$ is an open cover of M,*
    b. *$\Phi_\alpha \colon U_\alpha \to \Phi_\alpha(U_\alpha)$ is a homeomorphism between U_α and an open subset $\Phi_\alpha(U_\alpha)$ of $\mathbf{R}^d$ or $\mathbf{R}^{d-1} \times [0, \infty)$ for every $\alpha \in A$, and*
    c. *the transition map $\Phi_{\alpha,\alpha'} \colon \Phi_\alpha(U_\alpha \cap U_{\alpha'}) \to \Phi_{\alpha'}(U_\alpha \cap U_{\alpha'})$, where*

$$\Phi_{\alpha,\alpha'} := \Phi_{\alpha'} \circ \Phi_\alpha^{-1},$$

is a locally Lipschitz map for every $\alpha, \alpha' \in A$.

Item b. allows M to possess vertices, edges or non-isolated conical points. In particular, Lipschitz manifolds are prototypes of topological manifolds and the presence of a boundary or some corners is *not* excluded. When M is compact, the transition functions in c. can be taken to be globally bi-Lipschitz.

Any Lipschitz manifold is henceforth assumed to be connected and paracompact. As the topology of M is second countable, $\mathscr{U}$ may – and thus will always – be chosen to be locally finite with countable sub-atlas. On the other hand, following [40], completeness of M is *not* assumed (cf. Section 2.2 below).

Every Lipschitz manifold of dimension d is Lipschitz homeomorphic to a subset of $\mathbf{R}^{d(d+1)}$ that admits locally bi-Lipschitz parametrizations, cf. Subsection 2.3.2 below, by $\mathbf{R}^d$ [37, Thm. 4.2, Thm. 4.5].

2.1.2. *Differential forms.* Let M denote a Lipschitz manifold with an atlas $\mathscr{U} := (U_\alpha, \Phi_\alpha)_{\alpha \in A}$. A *$k$-form*, $k \in \mathbf{N}$, on M is a family $\omega := (\omega_\alpha)_{\alpha \in A}$ consisting of maps $\omega_\alpha \colon \Phi_\alpha(U_\alpha) \to \bigsqcup_{z \in \Phi_\alpha(U_\alpha)} \Lambda^k(\mathbf{R}^d)^*$ with Borel coefficients such that

$$\Phi_{\alpha,\alpha'}^* \, \omega_{\alpha'} = \omega_\alpha \quad \mathscr{L}^d\text{-a.e.} \quad \text{on } \Phi_\alpha(U_\alpha \cap U_{\alpha'})$$

for every $\alpha, \alpha' \in A$. The pull-back $\Phi_{\alpha,\alpha'}^*$ by $\Phi_{\alpha,\alpha'}$ makes sense $\mathscr{L}^d$-a.e. by Rademacher's theorem [60, p. 272].

Remark 2.2. To be very precise, a k-form should be regarded as equivalence class w.r.t. $\mathscr{L}^d$-a.e. equality of its components ω_α, $\alpha \in A$, on every chart. Without further notice, in the sequel this will be the precise interpretation of *all* objects which are defined up to $\mathscr{L}^d$-a.e. equality on charts.

Such an ω is *smooth* if for every $\alpha \in A$, ω_α has smooth coefficients in its local representation w.r.t. Euclidean coordinates on $\Phi_\alpha(U_\alpha)$; if $\Phi_\alpha(U_\alpha)$ is not open in $\mathbf{R}^d$, this means the existence of smooth extensions of all coefficients of ω_α to an open subset of $\mathbf{R}^d$. It is *essentially bounded* if $\sup_{\alpha \in A} \mathscr{L}^d\text{-esssup} \, |\omega_\alpha|_{g^E} \circ \Phi_\alpha(U_\alpha) < \infty$. Here $\left| \cdot \right|_{g^E}^2 := (g^E)^*(\cdot, \cdot)$, where $(g^E)^*$ is the co-metric induced by the standard Euclidean metric tensor g^E.

Multiplication of a k-form ω on M by a Borel function $\psi \colon M \to \mathbf{R}$ is understood chartwise, i.e. $\psi\,\omega := (\psi \circ \Phi_\alpha^{-1} \, \omega_\alpha)_{\alpha \in A}$ is still a k-form on M. In turn, k-forms on M are evidently defined to be *compactly supported* or to *coincide a.e.* (on Borel subsets of M), respectively.

2.1.3. *Riemannian metrics.*

Definition 2.3. *A* Riemannian metric *$g := (g_\alpha)_{\alpha \in A}$ on M consists of a family of scalar products $g_\alpha \colon \Phi_\alpha(U_\alpha) \to \bigsqcup_{z \in \Phi_\alpha(U_\alpha)} (\mathbf{R}^d)^{*\otimes 2}$ with Borel coefficients which is compatible with $\mathscr{U}$, i.e. for every $\alpha, \alpha' \in A$,*

$$(\Phi_{\alpha,\alpha'})_* \, g_\alpha = g_{\alpha'} \quad \mathscr{L}^d\text{-a.e.} \quad \text{on } \Phi_{\alpha'}(U_\alpha \cap U_{\alpha'}). \tag{2.1}$$

Here $(\Phi_{\alpha,\alpha'})_*$ designates the usual push-forward operation by $\Phi_{\alpha,\alpha'}$ which, as in Subsection 2.1.2, is well-defined $\mathscr{L}^d$-a.e.

Let $g := (g_\alpha)_{\alpha \in A}$ be a Riemannian metric on M. Any k-form ω, $k \in \mathbf{N}$, on M comes with a Borel function $|\omega| \colon M \to \mathbf{R}$ such that for every $\alpha \in A$,

$$|\omega| \circ \Phi_\alpha^{-1} = |\omega_\alpha|_{g_\alpha} \quad \mathscr{L}^d\text{-a.e.} \quad \text{on } \Phi_\alpha(U_\alpha). \tag{2.2}$$

Here $\left| \cdot \right|_{g_\alpha}^2 := g_\alpha^*(\cdot, \cdot)$, where g_α^* is the co-metric associated with g_α.

If a smooth k-form ω on M has compact support in U_α, $\alpha \in A$, we set

$$\left\| \omega \right\|_{g^E}^2 := \int_{\Phi_\alpha(U_\alpha)} \omega_\alpha \wedge \star\omega_\alpha = \int_{\Phi_\alpha(U_\alpha)} \left| \omega_\alpha \right|_{g^E}^2 \, \mathrm{d}\mathscr{L}^d,$$

$$\left\| \omega \right\|_{g_\alpha}^2 := \int_{\Phi_\alpha(U_\alpha)} \omega_\alpha \wedge \star_\alpha\omega_\alpha = \int_{\Phi_\alpha(U_\alpha)} \left| \omega_\alpha \right|_{g_\alpha}^2 \, \sqrt{\det g_\alpha} \, \mathrm{d}\mathscr{L}^d.$$

Here, $\star$ and $\star_\alpha$ are the Hodge operators of the Euclidean metric and of g_α, respectively. Following [16, Sec. 1.4], we always assume the following *local Lipschitz* condition on g: for every $\alpha \in A$, there exists a constant $c_\alpha \geq 1$ such that for every smooth 1-form ω on M with compact support in U_α,

$$c_\alpha^{-1} \, \|\omega\|_{g^{\mathrm{E}}} \leq \|\omega\|_{g_\alpha} \leq c_\alpha \, \|\omega\|_{g^{\mathrm{E}}}.$$

Remark 2.4. Under the previous hypothesis, by [14, Thm. 3.1], for every $\alpha \in A$ there exists a constant $\lambda_\alpha \geq 1$ such that for every $\xi \in \mathbf{R}^d$,

$$\lambda_\alpha^{-1} \, |\xi|^2 \leq g_\alpha(\xi, \xi) \leq \lambda_\alpha \, |\xi|^2 \quad \mathscr{L}^d\text{-a.e.} \quad \text{on } \Phi_\alpha(U_\alpha).$$

Definition 2.5. *A pair (M, g) is called* Lipschitz Riemannian manifold, *briefly* LR manifold, *if M is a Lipschitz manifold and g is a locally Lipschitz Riemannian metric, briefly LR metric, on it.*

2.1.4. *Volume measure.* Given any bounded Borel function $f\colon M \to [0, \infty)$ with support in U_α, $\alpha \in A$, we set

$$\int_M f \, \mathrm{d}\mathfrak{v}_g := \int_{\Phi_\alpha(U_\alpha)} f \circ \Phi_\alpha^{-1} \, \sqrt{\det g_\alpha} \, \mathrm{d}\mathscr{L}^d. \qquad (2.3)$$

By (2.1) and a change of variables, if f is supported in $U_{\alpha'}$ as well, $\alpha' \in A$, then the r.h.s.'s of (2.3) for α and α' agree. Using a partition of unity, we extend (2.3) to all bounded, nonnegative Borelian f. This procedure yields a unique fully supported, σ-finite Radon measure $\mathfrak{v}_g$ on M which we will call *volume measure*. Whenever the metric g is understood, we simply write $\mathfrak{v}$ instead of $\mathfrak{v}_g$.

At various occasions, instead of $\mathfrak{v}$ one could also consider a more general Radon measure $\mathfrak{m}$ on M which is *locally equivalent* to $\mathfrak{v}$, i.e. $\mathfrak{m}$ is of the form $\mathrm{e}^{-2\phi} \, \mathfrak{v}$ for some locally bounded Borel function $\phi\colon M \to \mathbf{R}$. It will always be mentioned explicitly if a result applies to such an $\mathfrak{m}$ (or certain quantities defined in terms of $\mathfrak{m}$).

Given any $p \in [1, \infty]$, we particularly have distinguished (local) Lebesgue spaces $L^p(M, \mathfrak{m})$ and $L^p_{\mathrm{loc}}(M, \mathfrak{m})$ w.r.t. $\mathfrak{m}$. If $\mathfrak{m} = \mathfrak{v}$, these will be abbreviated by $L^p(M)$ and $L^p_{\mathrm{loc}}(M)$, respectively.

2.2. **Metric structure.** (M, g) comes with a natural length distance function d_g, abbreviated by d when the dependence on g is clear, which turns (M, d_g) into a metric space. Here we review the cornerstones of its construction from [14, 15, 16, 40], to which we refer for proofs and technical details. See also Subsection 2.5.4 below.

A map $\gamma\colon [0, 1] \to M$ is *Lipschitz* if for every $\alpha \in A$ and every $a, b \in [0, 1]$ with $\gamma([a, b]) \subset U_\alpha$, $\gamma_\alpha\colon [a, b] \to \Phi_\alpha(U_\alpha)$ is Lipschitz, where

$$\gamma_\alpha := \Phi_\alpha \circ \gamma.$$

As usual, the length of γ_α is defined by

$$L(\gamma_\alpha) := \int_a^b \sqrt{g_\alpha\big(\dot{\gamma}_\alpha(t), \dot{\gamma}_\alpha(t)\big) \circ \gamma_\alpha(t)} \, \mathrm{d}t. \qquad (2.4)$$

However, if $\gamma([a, b]) \subset U_{\alpha'}$ for some $\alpha' \in A \setminus \{\alpha\}$, the quantities $L(\gamma_\alpha)$ and $L(\gamma_{\alpha'})$ do not coincide in general. For instance, albeit the set of non-differentiability points of the transition function $\Phi_{\alpha, \alpha'}$ is $\mathscr{L}^d$-negligible, the set of times at which γ passes through the latter might be charged by $\mathscr{L}^1$ [14, Sec. 2.5].

To overcome this issue, given $x, y \in M$ and $B \subset M$ with $\mathfrak{v}[B] = 0$, we set

$$\mathrm{Lip}_B(x, y; M) := \big\{\gamma\colon [0, 1] \to M : \gamma \text{ Lipschitz},$$
$$\gamma(0) = x, \ \gamma(1) = y, \ \gamma_\sharp \mathscr{L}^1[B] = 0\big\},$$

i.e. the class of Lipschitz curves in M from x to y which are *transversal* to B. Then $\mathrm{Lip}_B(x, y; M) \neq \emptyset$ [14, Lem. 2.4], and there exists a Borel set $B^* \subset M$ with

$\mathfrak{v}[B^*] = 0$ such that the *length* $L(\gamma)$, constructed from (2.4) in the evident way [14, pp. 165–166], is well-defined for every $\gamma \in \mathrm{Lip}_{B^*}(x, y; M)$ [14, p. 166]. Therefore, for x, y and B as above the quantities

$$\mathsf{d}_B(x, y) := \inf\big\{L(\gamma) : \gamma \in \mathrm{Lip}_{B \cup B^*}(x, y; M)\big\},$$
$$\mathsf{d}(x, y) := \sup\big\{\mathsf{d}_B(x, y) : B \subset M, \ \mathfrak{v}[B] = 0\big\}$$

are meaningful. In fact, d is a distance on M, independent of $\mathscr{U}$, which induces the initial topology on M [14, Thm. 4.4, Thm. 4.5, Cor. 4.6]. Moreover, it is a length distance [15, Thm. 3.10], and hence geodesic if M is complete. We call d the *path distance* induced by g.

If (M, g) is a smooth Riemannian manifold, then d coincides with the length distance induced by the metric tensor g [14, Thm. 5.1].

We denote by $\mathsf{B}_r^g(x)$, or simply $\mathsf{B}_r(x)$ if g is understood, the open ball with radius $r > 0$ w.r.t. d around $x \in M$.

Remark 2.6. Every Lipschitz atlas $\mathscr{U} := (U_\alpha, \Phi_\alpha)_{\alpha \in A}$ defines a local distance on the chart U_α, $\alpha \in A$, namely the Euclidean distance induced on the chart

$$\mathsf{D}_\alpha(x, y) = |\Phi_\alpha(x) - \Phi_\alpha(y)|,$$

where $x, y \in U_\alpha$. Starting from D_α, it is possible to construct a global (well-defined, since M is connected) distance D on M, as shown in [16, 37]. This distance depends on the fixed atlas $\mathscr{U}$, but turns out to be locally equivalent to d [14, Thm. 6.2].

2.3. Lipschitz continuity.

2.3.1. *Lipschitz functions.* A function $f\colon M \to \mathbf{R}$ is *locally Lipschitz* if every point in M has an open neighborhood $U \subset M$ such that $f\big|_U$ is Lipschitz as a map between the metric spaces $(U, \mathsf{d}\big|_{U^2})$ and $(\mathbf{R}, |\cdot - \cdot|)$ for every $\alpha \in A$. By Remark 2.6, this is equivalent to the possibly more common definition [16, 40] by requiring local Lipschitz continuity of $f_\alpha\colon \Phi_\alpha(U_\alpha) \to \mathbf{R}$ for every $\alpha \in A$, where

$$f_\alpha := f \circ \Phi_\alpha^{-1}.$$

Furthermore, such an f is *Lipschitz* if it is Lipschitz as a map between the metric spaces (M, d) and $(\mathbf{R}, |\cdot - \cdot|)$. (In particular, unlike e.g. [40] Lipschitz continuity is always intended *globally*.) Denote the spaces of all (locally) Lipschitz functions $f\colon M \to \mathbf{R}$ by $\mathrm{Lip}_{\mathrm{loc}}(M)$ and $\mathrm{Lip}(M)$, respectively. In turn, the spaces $\mathrm{Lip}_{\mathrm{loc}}(M; \mathbf{R}^{\mathscr{d}})$ and $\mathrm{Lip}(M; \mathbf{R}^{\mathscr{d}})$, $\mathscr{d} \in \mathbf{N}$, consist of all functions $f\colon M \to \mathbf{R}^{\mathscr{d}}$ whose components are (locally) Lipschitz.

Clearly, by Section 2.2, every element in $\mathrm{Lip}_{\mathrm{loc}}(M)$ is continuous, and if M is compact, then $\mathrm{Lip}_{\mathrm{loc}}(M) = \mathrm{Lip}(M)$.

By Rademacher's theorem and the chain rule, for every $f \in \mathrm{Lip}_{\mathrm{loc}}(M)$ the family $\mathrm{d}f := (\mathrm{d}f_\alpha)_{\alpha \in A}$ is a 1-form according to Subsection 2.1.2, the *differential* of f. If $f \in \mathrm{Lip}(M)$, then $|\mathrm{d}f|$ belongs to $L^\infty(M)$.

Evidently, the induced linear operator d on $\mathrm{Lip}_{\mathrm{loc}}(M)$ obeys all expected locality, chain and Leibniz rules (as usual, in appropriate a.e. senses).

2.3.2. *Lipschitz maps.* Analogously to Subsection 2.1.3 above, a map $F\colon M \to N$ is evidently defined to be (*locally*) *Lipschitz* in the metric sense. Such an F is locally Lipschitz if and only if for every $\alpha \in A$ and every $\beta \in B$ with $F(U_\alpha) \cap V_\beta \neq \emptyset$, the map $F_{\alpha,\beta}\colon \Phi_\alpha(U_\alpha) \to \Psi_\beta(F(U_\alpha) \cap V_\beta)$ is locally Lipschitz as a map between subsets of d- and $\mathscr{d}$-dimensional Euclidean spaces, where

$$F_{\alpha,\beta} := \Psi_\beta \circ F \circ \Phi_\alpha^{-1}.$$

The spaces of all such F are denoted by $\mathrm{Lip}_{\mathrm{loc}}(M; N)$ and $\mathrm{Lip}(M; N)$, respectively. For $N = \mathbf{R}^{\mathscr{d}}$, the preceeding notions of (local) Lipschitz continuity are compatible with those of Subsection 2.3.1.

Finally, a map $F \colon M \to N$ is called (*locally*) *bi-Lipschitz* if $F \in \mathrm{Lip}_{\mathrm{loc}}(M; N)$ or $F \in \mathrm{Lip}(M; N)$, and F admits an inverse map belonging to $\mathrm{Lip}_{\mathrm{loc}}(N; M)$ or $\mathrm{Lip}(N; M)$, respectively.

2.4. **Quasi-isometry.** In the following, (M, g) and (N, h) will denote two LR manifolds of not necessarily equal dimensions d and $\mathscr{d}$, respectively, $d, \mathscr{d} \in \mathbf{N}$, according to the notions introduced in the Section 2.1, cf. Definition 2.5. Let $\mathscr{U} := (U_\alpha, \Phi_\alpha)_{\alpha \in A}$ and $\mathscr{V} := (V_\beta, \Psi_\beta)_{\beta \in B}$ denote their respective atlases.

Definition 2.7. *We say that (M, g) and (N, h) are* quasi-isometric, *briefly $(M, g) \sim (N, h)$ or $(M, g) \sim_F (N, h)$, if there exists a bi-Lipschitz map $F \colon M \to N$.*

Quasi-isometry of Lipschitz manifolds does not depend on the chosen atlases, is an equivalence relation on the totaliy of LR manifolds, and preserves topological dimensions.

By Lemma 2.8 below — whose easy proof employing the notions from Section 2.2 is omitted — quasi-isometry has a convenient counterpart in terms of pull-back metrics, a concept which is shortly outlined now (and which could easily be defined for locally bi-Lipschitz maps with evident changes in the statements below).

The *pull-back metric* of h under a bi-Lipschitz $F \colon M \to N$ is the family $F^*h := (F^*h_\alpha)_{\alpha \in A}$ of scalar products $F^*h_\alpha \colon \Phi_\alpha(U_\alpha) \to \bigsqcup_{z \in \Phi_\alpha(U_\alpha)}(\mathbf{R}^d)^{*\otimes 2}$ given by

$$F^*h_\alpha = F^*_{\alpha, \beta} h_\beta \quad \mathscr{L}^d\text{-a.e.} \quad \text{on } \Phi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \tag{2.5}$$

for every $\alpha \in A$ and $\beta \in B$ with $F(U_\alpha) \cap V_\beta \neq \emptyset$. Indeed, (2.1) for h and the chain rule ensure that F^*h is well-defined in the above way. However, F^*h itself does not necessarily satisfy (2.1).

Lemma 2.8. *The following properties hold.*

(i) *Suppose that $(M, g) \sim_F (N, h)$ with some bi-Lipschitz $F \colon M \to N$. Then there exists a constant $C \geq 1$ such that for every $\alpha \in A$,*

$$C^{-1} g_\alpha \leq F^*h_\alpha \leq C g_\alpha \quad \mathscr{L}^d\text{-a.e.} \quad \text{on } \Phi_\alpha(U_\alpha) \tag{2.6}$$

in the sense of quadratic forms.

(ii) *Conversely, if there exists a locally bi-Lipschitz map $F \colon M \to N$, a constant $C \geq 1$ and a sub-atlas $\mathscr{U}' := (U_{\alpha'}, \Phi_{\alpha'})_{\alpha' \in A'}$ of $\mathscr{U}$ such that (2.6) holds for every $\alpha' \in A'$ instead of α, then $(M, g) \sim_F (N, h)$.*

Remark 2.9. By Lemma 2.8 and (2.3), it follows in particular that if $(M, g) \sim_F (N, h)$, there exists a constant $C \geq 1$ such that for every $x \in M$ and every $r > 0$,

$$C^{-1} \mathfrak{v}_g\big[\mathsf{B}^g_{r/C}(x)\big] \leq \mathfrak{v}_h\big[\mathsf{B}^h_r(F(x))\big] \leq C \mathfrak{v}_g\big[\mathsf{B}^g_{Cr}(x)\big].$$

According to Lemma 2.8, the setting of a smooth Riemannian manifold with two metrics that are uniformly elliptic to each other, studied in [45], falls into the category of quasi-isometry according to Definition 2.7.

2.5. **Some potential theory.** Finally, we outline how Dirichlet spaces over an LR manifold (M, g) can be constructed. We refer the reader to [13, 40, 52, 53] for details, and to [4, 10] for general Dirichlet form theory.

To relax notation, in the sequel we do not make explicit the dependencies of all introduced objects on g, unless required.

Every statement in this section holds with evident modifications for $\mathfrak{v}$ replaced by a measure $\mathfrak{m}$ as in Subsection 2.1.4, except Lemma 2.10, as underlined in Remark 2.11.

2.5.1. *Weak derivatives.* Let $W^{1,2}(M)$ be the space of all $f \in L^2(M)$ whose chartwise distributional differentials $\mathrm{d}_\alpha f$, $\alpha \in A$, are 1-forms, locally L^2 w.r.t. $\mathscr{L}^d$ on $\Phi_\alpha(U_\alpha)$, which in turn give rise to a 1-form $\mathrm{d}f := (\mathrm{d}f_\alpha)_{\alpha \in A}$ on M such that $|\mathrm{d}f| \in L^2(M)$. The space $W^{1,2}(M)$ is complete and separable w.r.t. the norm $\|\cdot\|_{W^{1,2}(M)}$ given by

$$\|f\|_{W^{1,2}(M)}^2 := \|f\|_{L^2(M)}^2 + \||\mathrm{d}f|\|_{L^2(M)}^2,$$

and d, the so-called *differential*, is a closed operator on $L^2(M)$.

The latter object coincides with the one introduced in Subsection 2.3.1 on the space $W^{1,2}(M) \cap \mathrm{Lip}_{\mathrm{loc}}(M)$. In particular, $W^{1,2}(M) \cap \mathrm{Lip}_{\mathrm{loc}}(M)$ is dense in $W^{1,2}(M)$, and so is $W^{1,2}(M) \cap \mathrm{Lip}(M)$ by partition of unity.

2.5.2. *Dirichlet form.* The quadratic form $\mathscr{E} \colon W^{1,2}(M) \to [0, \infty)$ defined by

$$\mathscr{E}(f) := \frac{1}{2} \int_M |\mathrm{d}f|^2 \, \mathrm{d}\mathfrak{v} \tag{2.7}$$

is a strongly local, regular Dirichlet form with domain $\mathscr{F} := W^{1,2}(M)$ and carré du champ $|\mathrm{d} \cdot|^2 \colon W^{1,2}(M) \to L^1(M)$. (The latter is the same quantity for every choice of $\mathfrak{v}$-locally equivalent reference measure $\mathfrak{m}$ according to Subsection 2.1.4.) Both $W^{1,2}(M) \cap \mathrm{Lip}_{\mathrm{loc}}(M)$ and $W^{1,2}(M) \cap \mathrm{Lip}(M)$ are cores for $\mathscr{E}$.

We use the non-relabeled symbol $\mathscr{E}$ for the polarization $\mathscr{E} \colon W^{1,2}(M)^2 \to \mathbf{R}$ of (2.7), and we do so analogously for any other quadratic form in the sequel.

The following lemma will be useful in Section 3.3 below. It easily follows from the definition (2.5) of the pullback metric, Lemma 2.8 and the above mentioned density of Lipschitz functions in energy.

Lemma 2.10. *Suppose that $(M, g) \sim_F (N, h)$ through a bi-Lipschitz map $F \colon M \to N$. Then $f \in \mathscr{F}_g$ if and only if $f \circ F^{-1} \in \mathscr{F}_h$, and there exists a constant $C \geq 1$ such that for every such f,*

$$C^{-1} \mathscr{E}_g(f) \leq \mathscr{E}_h(f \circ F^{-1}) \leq C \mathscr{E}_g(f).$$

Remark 2.11. If one starts with Dirichlet forms $(\mathscr{E}_{g,\mathfrak{m}}, \mathscr{F}_{g,\mathfrak{m}})$ on M and $(\mathscr{E}_{h,\mathfrak{n}}, \mathscr{F}_{h,\mathfrak{n}})$ on N induced in the evident way as above by two Borel measures $\mathfrak{m}$ on M and $\mathfrak{n}$ on N which are locally equivalent to the respective volume measures, to obtain the conclusion of Lemma 2.10 one has to additionally assume that $\mathfrak{m}$ and $(F^{-1})_\sharp \mathfrak{n}$ are mutually equivalent with bounded densities.

2.5.3. *Laplacian and heat flow.* Let $\Delta/2$ be the closed, self-adjoint and nonpositive generator with dense domain $\mathscr{D}(\Delta) \subset L^2(M)$ generating $\mathscr{E}$.

By the spectral theorem, given any $f \in L^2(M)$ the assignment

$$\mathsf{P}_t f := \mathrm{e}^{\Delta t/2} f$$

gives rise to the unique global solution $u(t) := \mathsf{P}_t f$ of the heat equation — or, in other words, of the parabolic operator $\partial/\partial t - \Delta/2$, cf. [52, p. 285] for more precise definitions — on M with initial condition f. The *heat flow* $(\mathsf{P}_t)_{t \geq 0}$ constitutes a strongly continuous semigroup of linear, positivity-preserving and sub-Markovian contraction operators on $L^2(M)$. $(\mathsf{P}_t)_{t \geq 0}$ extends to a semigroup of linear contraction operators on $L^p(M)$, $p \in [1, \infty]$, which is strongly continuous if $p < \infty$ and weakly* continuous provided $p = \infty$.

2.5.4. *Heat kernel.* The heat operator P_t, $t \geq 0$, can be represented as an integral operator w.r.t. $\mathfrak{v}$ on $L^p(M)$, $p \in [1, \infty]$, through a (spacially symmetric) fundamental solution $\mathsf{p} \colon (0, \infty) \times M^2 \to (0, \infty)$, the *heat kernel*, of the parabolic operator $\partial/\partial t - \Delta/2$. Indeed, $\mathfrak{v}$ is locally doubling and the Dirichlet space $(M, \mathscr{E}, \mathfrak{v})$ obeys a local Sobolev inequality, which are both Euclidean properties that transfer chartwise back to M [46, 52, 53], compare with Section 3.3 below. Moreover [40, Thm. 3.6],

the metric d introduced in Section 2.2 coincides with the one generated by the *intrinsic distance* induced by $|\mathsf{d} \cdot|^2$, i.e. for every $x, y \in M$,

$$\mathsf{d}(x, y) = \sup\{\phi(x) - \phi(y) : \phi \in \mathrm{Lip}_{\mathrm{loc}}(M), \ |\mathsf{d}\phi| \leq 1 \ \mathfrak{v}\text{-a.e.}\}.$$

The general result [52, Prop. 2.3] thus applies and leads to the above heat kernel existence. (However, the intrinsic distance does generally *not* determine g, hence $\mathscr{E}$, unless the coefficients of g are a priori known to be continuous [54].)

The map p may and will be chosen to be (jointly) locally Hölder [53]. Denoting by $\lambda \geq 0$ the least eigenvalue of $-\Delta/2$, we have the following Gaussian heat kernel estimate [52, Thm. 2.4] (whose polynomial terms are absorbed by the exponentials up to the errors $\varepsilon, \delta > 0$ according to the remark at [52, p. 309]).

Proposition 2.12. *Given any $x, y \in M$, let $R_x, R_y > 0$ such that the doubling property [52, p. 293] and the Sobolev inequality [52, p. 294] hold on $\mathsf{B}_{R_x}(x)$ and $\mathsf{B}_{R_y}(y)$, respectively. Define $R := \min\{R_x, R_y\}$. Then for every $\varepsilon, \delta > 0$ there exists a constant $C > 0$ such that for every $t \in (0, R^2]$,*

$$\mathsf{p}(t, x, y) \leq C \, \mathfrak{v}\big[\mathsf{B}_{\sqrt{t}}(x)\big]^{-1/2} \mathfrak{v}\big[\mathsf{B}_{\sqrt{t}}(y)\big]^{-1/2} \exp\Big[-\frac{\mathsf{d}^2(x, y)}{(4 + \varepsilon)t} - (1 - \delta)\lambda t\Big].$$

Of course, in general the task is to find conditions under which the constant C in (3.9) is spacially, hence time uniform. See [12, 13, 22, 23] as well as the survey article [47] and the references therein for the vast literature about conditions for the former. For instance, in our work, see Section 3.3, this will be the case as soon as a local Bishop–Gromov-type property (stronger than the doubling property) and the Sobolev inequality hold with uniform constants.

3. Heat kernel control pairs

In this chapter, we introduce the concept of *heat kernel control pairs*, i.e. suitable pairs of functions giving rise to heat kernel upper bounds with decoupled space and time dependencies. In Section 3.2, we prove their existence in the entire generality of LR manifolds. To this aim, we first collect some preliminaries about local solutions to the heat equation on M, and we prove useful mean value inequalities for these.

3.1. **Mean value inequalities.** Every statement in this section, with evident modifications, still makes sense if we replace $\mathfrak{v}$ by a measure $\mathfrak{m}$ as in Subsection 2.1.4.

3.1.1. *Local solutions.* We refer to [52] and the references therein for details about the topics that are described in this subsection.

Let $I \subset \mathbf{R}$ be an open interval. Let $\mathrm{C}_{\mathrm{b}}(\bar{I}; L^2(M))$ be the Banach space of all bounded, continuous functions $u \colon \bar{I} \to L^2(M)$ with the evident norm.

The space $\mathscr{F}(I \times M)$ is defined to consist of all strongly measurable $u \colon I \to \mathscr{F}$ such that $t \mapsto \|u(t)\|_{\mathscr{F}}$ belongs to $L^2(I, \mathscr{L}^1)$, and the distributional time derivative $\dot{u}$ is a strongly measurable map $t \mapsto \dot{u}(t)$ from I to $\mathscr{F}^{-1}$ for which $t \mapsto \|\dot{u}(t)\|_{\mathscr{F}^{-1}}$ belongs to $L^2(I, \mathscr{L}^1)$. Here $\mathscr{F}^{-1}$ is the dual space of $\mathscr{F}$. $\mathscr{F}(I \times M)$ is a Hilbert space w.r.t. the evident norm.

An important fact we frequently use is the following, cf. [52, p. 285].

Lemma 3.1. *Every $u \in \mathscr{F}(I \times M)$ has a continuous $\mathscr{L}^1$-version on I with values in $L^2(M)$ and which, w.r.t. the uniform topology, continuously extends to $\bar{I}$.*

Without further notice, every element in $\mathscr{F}(I \times M)$ will thus be identified with its $\mathscr{L}^1$-version in $\mathrm{C}_{\mathrm{b}}(\bar{I}; L^2(M))$.

Next, we pass over to local spaces. Let $U \subset M$ be open, and let $\mathscr{F}_{\mathrm{loc}}(I \times U)$ denote the space consisting of all strongly measurable $u \colon I \to L^0(U)$ such that

for every relatively compact, open $O \subset\subset U$ and every open $J \subset\subset I$ there exists $u' \in \mathscr{F}(I \times M)$ with $1_O\, u' = 1_O\, u$ on J.

Lastly, $\mathscr{F}_0(I \times U)$ consists of all $u \in \mathscr{F}(I \times M)$ such that for $\mathscr{L}^1$-a.e. $t \in I$, spt $u(t)$ is compact in U. This means that $u(t) = 0$ on ∂U for $\mathscr{L}^1$-a.e. $t \in I$, but not necessarily $u(\inf I) = 0$, provided $\inf I \in \mathbf{R}$, or $u(\sup I) = 0$, provided $\sup I \in \mathbf{R}$.

Definition 3.2. *A function u is termed* local subsolution *of the heat equation on* $I \times U$ *if* $u \in \mathscr{F}_{\mathrm{loc}}(I \times U)$ *and for every* $J \subset\subset I$ *and every nonnegative* $\phi \in \mathscr{F}_0(I \times U)$,

$$\int_J \mathscr{E}\big(u(t), \phi(t)\big)\, \mathrm{d}t + \int_J \big\langle \dot{u}(t) \,\big|\, \phi(t) \big\rangle\, \mathrm{d}t \leq 0.$$

If additionally, the function $-u$ is a local subsolution on $I \times U$ as well, we call u local solution *of the heat equation on $I \times U$.*

3.1.2. L^2- *and* L^1-*mean value inequalities.* With this concept of local subsolutions, we now state and prove two mean value inequalities of L^2- and L^p-type, $p \in [1, 2]$. In their form from Proposition 3.3 and Theorem 3.5 below and apart from global doubling assumptions, we have only been able to find these for smooth Riemannian manifolds in [21, 23, 25]. The proofs in our more general case are similar, but unlike [21, 23, 25] we have to take care that local solutions are not defined to be C^2 in general, but rather in a weak sense recorded in Definition 3.2 below.

Proposition 3.3. *Let* $\mathsf{B}_r(x) \subset M$, $x \in M$ *and* $r > 0$, *be a relatively compact ball. Suppose that for some* $a, \boldsymbol{n} > 0$, *we have the* Faber–Krahn inequality

$$\lambda(O) \geq a\, \mathfrak{v}[O]^{-2/\boldsymbol{n}}$$

for every open set $O \subset \mathsf{B}_r(x)$. *Then there exists a constant* $C > 0$ *depending only on n such that for every* $t > 0$, *every jointly continuous local subsolution u of the heat equation on* $(0, t) \times \mathsf{B}_r(x)$, *and every* $\varepsilon \in (0, t/4)$,

$$u_+^2(t - \varepsilon)(x) \leq \frac{C\, a^{-\boldsymbol{n}/2}}{\min\{\sqrt{t - 2\varepsilon}, r\}^{\boldsymbol{n}+2}} \int_\varepsilon^{t-\varepsilon} \int_{\mathsf{B}_r(x)} u_+^2(s)\, \mathrm{d}\mathfrak{v}\, \mathrm{d}s.$$

Remark 3.4. Note that, unlike the expression $u(t-\varepsilon)$, $\varepsilon \in (0, t/4)$, in Proposition 3.3, the evaluation $u(t)$ does not need to make sense in general (in contrast to the C^2-notion of [23, Thm. 15.1]) by definition of $\mathscr{F}_{\mathrm{loc}}((0, t) \times \mathsf{B}_r(x))$.

For the same reason, it might a priori happen that $\|u_+\|_{L^2(\mathsf{B}_r(x))}$ is not square-integrable on $(\varepsilon, t - \varepsilon)$ even for $\varepsilon = 0$. However, these degeneracy issues can always be circumvented by shrinking the radius r.

Proof of Proposition 3.3. In terms of local subsolutions as in Definition 3.2, the smooth argument for [23, Thm. 15.1] carries over without essential changes in the following setup. Consider the function $v \in \mathscr{F}_{\mathrm{loc}}((-\varepsilon/2, t - 3\varepsilon/2) \times \mathsf{B}_{r-\iota}(x))$ with

$$v(s) := u(s + \varepsilon),$$

where $\iota \in (0, r)$. In fact, the map $s \mapsto \|v(s)\|_{L^2(\mathsf{B}_{r-\iota}(x))}$ belongs to $L^2((0, t - 2\varepsilon), \mathscr{L}^1)$, and v has a well-defined value at $t - 2\varepsilon$ by Lemma 3.1. Clearly, v is a local solution to the heat equation on $(-\varepsilon/2, t - 3\varepsilon/2) \times \mathsf{B}_{r-\iota}(x)$. Even better, the shifting of u by ε allows us to take $J := (0, t - 2\varepsilon)$ as test interval in Definition 3.2. It therefore remains to follow the lines in [23, Sec. 15.1] with u replaced by v, T replaced by $t - 2\varepsilon$, and R replaced by $r - \iota$ therein. This entails the existence of a constant $C > 0$ depending only on $\boldsymbol{n}$ such that

$$v_+^2(t - 2\varepsilon)(x) \leq \frac{C\, a^{-\boldsymbol{n}/2}}{\min\{\sqrt{t - 2\varepsilon}, r - \iota\}^{\boldsymbol{n}+2}} \int_0^{t-2\varepsilon} \int_{\mathsf{B}_{r-\iota}(x)} v_+^2(s)\, \mathrm{d}\mathfrak{v}\, \mathrm{d}s,$$

which directly provides the asserted inequality by sending $\iota \to 0$. $\qquad\square$

Theorem 3.5. *For every $n > 0$, there exists a constant $C > 0$ with the following property. For every $x \in M$ and every $r > 0$ such that $\mathsf{B}_r(x) \subset M$ is relatively compact and for which there exists $a > 0$ such that*

$$\lambda_1(O) \geq a\,\mathfrak{v}[O]^{-2/n}$$

for every open $O \subset \mathsf{B}_r(x)$, for every $\tau, t > 0$ with $\tau \leq r^2$ and $\tau < t$, every jointly continuous nonnegative local subsolution u of the heat equation on $(t - \tau, t) \times \mathsf{B}_{\sqrt{\tau}}(x)$ such that $u \in \mathscr{F}_{\mathrm{loc}}((t - \tau - \zeta, t + \zeta) \times \mathsf{B}_{\sqrt{\tau}+\zeta}(x))$ for some error $\zeta \in (0, t - \tau)$, and every $q \in [1, 2]$,

$$u^q(t)(x) \leq \frac{C\,a^{-n/2}}{\tau^{1+n/2}} \int_{t-\tau}^t \int_{\mathsf{B}_{\sqrt{\tau}}(x)} u^q(s)\,\mathrm{d}\mathfrak{v}\,\mathrm{d}s.$$

Proof. We apply Proposition 3.3 to the radius $r := \sqrt{\tau}$ and to the local subsolution $v \in \mathscr{F}_{\mathrm{loc}}((0, \tau) \times \mathsf{B}_{\sqrt{\tau}-\iota}(x))$ to the heat equation on $(0, \tau) \times \mathsf{B}_{\sqrt{\tau}-\iota}(x)$, with

$$v(s) := u(t - \tau + s),$$

where $\iota \in (0, \sqrt{\tau})$. Given any $\varepsilon \in (0, \tau/4)$, we thus get

$$v^2(\tau - \varepsilon)(x) \leq \frac{C\,a^{-n/2}}{\min\{\sqrt{\tau - 2\varepsilon}, \sqrt{\tau} - \iota\}^{n+2}} \int_{\varepsilon}^{\tau - \varepsilon} \int_{\mathsf{B}_{\sqrt{\tau}-\iota}(x)} v^2(s)\,\mathrm{d}\mathfrak{v}\,\mathrm{d}s$$

for some universal constant $C > 0$ depending only on n. By a change of variables on the r.h.s., employing that by assumption, the function $s \mapsto \|u(s)\|_{L^2(\mathsf{B}_{\sqrt{\tau}}(x))}$ belongs to $L^2((t - \tau, t), \mathscr{L}^1)$ and that $u(t - \varepsilon)(x) \to u(t)(x)$ as $\varepsilon \to 0$, and sending $\varepsilon \to 0$ and $\iota \to 0$, we obtain

$$u^2(t)(x) \leq \frac{C\,a^{-n/2}}{\tau^{1+n/2}} \int_{t-\tau}^t \int_{\mathsf{B}_{\sqrt{\tau}}(x)} u^2(s)\,\mathrm{d}\mathfrak{v}\,\mathrm{d}s.$$

From here on, Li–Wang's L^2-to-L^q reduction procedure [35] from the proofs of [24, Thm. IV.15] and [25, Prop. 2.10] applies *verbatim* and gives the claim. $\square$

3.2. **Control pairs and their general existence.**

3.2.1. *Basic definition.* The following is motivated by [24, Def. IV.16].

Definition 3.6. *An ordered pair (Ξ, Σ) is termed a* heat kernel control pair *for (M, g) if*

    a. $\Xi\colon M \to (0, \infty)$ *is continuous and bounded away from zero,*
    b. $\Sigma\colon (0, 1] \to (0, \infty)$ *is Borel measurable,*
    c. *there exists a constant $C > 0$ such that for every $x \in M$ and every $t \in (0, 1]$,*

$$\sup_{y \in M} \mathsf{p}(t, x, y) \leq C\,\Sigma(t)\,\Xi(x), \qquad (3.1)$$

    d. *for every $p \in [1, \infty)$ provided $d = 1$ and every $p \in (d/2, \infty)$ provided $d \geq 2$, there exists a constant $C > 0$ such that*

$$\int_0^\infty \Sigma^{1/p}(t)\,\mathrm{e}^{-Ct}\,\mathrm{d}t < \infty.$$

Definition 3.6 is still meaningful on any quasi-regular, strongly local Dirichlet space $(M, \mathscr{E}, \mathfrak{m})$ with heat kernel — in this case, we will usually speak about heat kernel control pairs for $(M, \mathscr{E}, \mathfrak{m})$. There are many known sufficient conditions for these to admit control pairs, compare with Section 3.3. On the other hand, our main result from Theorem 3.13 below — namely that M *always* admits a canonical heat kernel control pair — really uses the (Lipschitz) manifold structure.

3.2.2. *The Euclidean radius.* We introduce the following concept which extends the smooth treatise from [24, Def. IV.12]. The idea behind it, in view of Theorem 3.13, is to transfer the global constants from the Euclidean Faber–Krahn inequality to M, paying the price of possibly non-uniformity of the size of the considered subsets of M on which the latter holds.

Compared to [24], additional care has to be taken to the possible presence of a boundary of M.

Definition 3.7. *Given any $x \in M$ and $b > 1$, we define $r_{\mathrm{E}}(x, b)$ as the supremum over all radii $r > 0$ such that*

   a. $\mathsf{B}_r(x) \subset M$ *is relatively compact, and*
   b. *there exists a chart (U, Φ) around x such that $\mathsf{B}_r(x) \subset U$, $\Phi\big|_{\mathsf{B}_r(x)}$ is a homeomorphism to an open subset $\Phi(\mathsf{B}_r(x))$ of $\mathbf{R}^d$ or $\mathbf{R}^{d-1} \times [0, \infty)$, and for every $\xi \in \mathbf{R}^d$,*

$$b^{-1}\,|\xi|^2 \le g^U(\xi, \xi) \le b\,|\xi|^2 \quad \mathscr{L}^d\text{-a.e.} \quad on\ \Phi(\mathsf{B}_r(x)), \tag{3.2}$$

   where $g^U \colon \Phi(U) \to \bigsqcup_{z \in \Phi(U)} (\mathbf{R}^d)^{*\otimes 2}$ is the Riemannian metric on the image of the chart (U, Φ) according to Definition 2.3.

Remark 3.8. Note that either $r_{\mathrm{E}}(M, b) \subset (0, \infty)$ or $r_{\mathrm{E}}(M, b) = \{\infty\}$ for every $b > 1$. This can easily be seen from the definition, or alternatively using the 1-Lipschitz property from Lemma 3.10 below.

Example 3.9. Let us consider the particular case of spherically symmetric manifolds (see e.g. [11, Ch. 8]). Let $d \ge 2$, fix a point $o \in \mathbf{R}^d$ and a positive bi-Lipschitz function $\psi \colon [0, \infty) \to [0, \infty)$ which is smooth at 0 and satisfies

$$\begin{aligned} \psi(0) &= 0, \\ \psi'(0) &= \mathrm{Lip}(\psi). \end{aligned} \tag{3.3}$$

We define a spherically symmetric LR manifold (M_ψ, g_ψ) in the following way.

   a. As a set of points, M_ψ is $\mathbf{R}^d$.
   b. In polar coordinates $(r, \theta) \in (0, \infty) \times \mathbf{S}^{d-1}$ at o the Riemannian metric g_ψ on $M_\psi \setminus \{o\}$ is defined as

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \psi^2(r)\,\mathrm{d}\theta^2, \tag{3.4}$$

   where $\mathrm{d}\theta^2$ denotes the standard Riemannian metric on $\mathbf{S}^{d-1}$.
   c. The Riemannian metric g_ψ at o is a smooth extension of (3.4), whose existence is ensured by (3.3).

Since ψ is bi-Lipschitz, we have in particular that

$$\mathrm{d}r^2 + \mathrm{Lip}(\psi)^{-2}\,r^2\,\mathrm{d}\theta^2 \le \mathrm{d}r^2 + \psi^2(r)\,\mathrm{d}\theta^2 \le \mathrm{d}r^2 + \mathrm{Lip}(\psi)^2\,r^2\,\mathrm{d}\theta^2.$$

Recalling that $\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$ is the standard Euclidean metric of $\mathbf{R}^d$, the above chain of inequalities ensures that, for any $b > \mathrm{Lip}(\psi)$, it holds $r_{\mathrm{E}}(o, b) = \infty$. Therefore, by the previous Remark 3.8 we have $r_{\mathrm{E}}(M_\psi, b) = \{\infty\}$ for every $b > \mathrm{Lip}(\psi)$.

Lemma 3.10. *For every $x \in M$ and for every $b > 1$, the following properties hold.*

   (i) *We have $r_{\mathrm{E}}(x, b) \in (0, \infty]$.*
   (ii) *For every $\varepsilon > 0$, the function $x \mapsto \min\{r_{\mathrm{E}}(x, b), \varepsilon\}$ on M is 1-Lipschitz w.r.t. d. In particular, for every compact set $K \subset M$,*

$$\inf_{x \in K} r_{\mathrm{E}}(x, b) > 0.$$

Proof. To prove (i), we first choose $r > 0$ such that $\mathsf{B}_r(x) \subset U_\alpha$ is relatively compact for some fixed chart (U_α, Φ_α), $\alpha \in A$. According to Remark 2.4, for this coordinate system we have the validity of (3.2), where b is replaced by a suitable $\lambda_\alpha \ge 1$. If

$\lambda_\alpha \leq b$, (3.2) is automatically satisfied also for the given b, while in the case $\lambda_\alpha \geq b$ we have to rescale the coordinate system. This is done by scaling $g_\alpha^{-1/2}$ in such a way that its eigenvalues range between $\sqrt{b/\lambda_\alpha}$ and $\sqrt{\lambda_\alpha/b}$, namely multiplying those eigenvalues of $g_\alpha^{-1/2}$ which are greater than 1 by the factor $1/\sqrt{b}$, and those which are smaller than 1 by the factor $\sqrt{b}$. In such a way, we obtain a coordinate system for which (3.2) is satisfied, thus $0 < r \leq r_{\mathrm{E}}(x, b)$.

Next, we prove (ii). To simplify notation, let us set $r(x) := r_{\mathrm{E}}(x, b)$ and $r_\varepsilon(x) := \min\{r(x), \varepsilon\}$, $x \in M$ and $\varepsilon > 0$. It clearly suffices to prove the 1-Lipschitz continuity of the function $x \mapsto r_\varepsilon(x)$ w.r.t. the distance d defined in Section 2.2.

Let us first assume that $y \in \mathsf{B}_{r_\varepsilon(x)}(x)$. By definition of $r(y)$, we have

$$r(y) \geq r(x) - \mathsf{d}(x, y) \geq r_\varepsilon(x) - \mathsf{d}(x, y).$$

Since $\mathsf{d}(x, y) < r_\varepsilon(x) \leq \varepsilon$, it also follows that

$$0 < r_\varepsilon(x) - \mathsf{d}(x, y) \leq \varepsilon,$$

and in particular $r_\varepsilon(y) \geq r_\varepsilon(x) - \mathsf{d}(x, y)$. In the case $r_\varepsilon(x) \geq r_\varepsilon(y)$, we already get

$$|r_\varepsilon(x) - r_\varepsilon(y)| \leq \mathsf{d}(x, y). \tag{3.5}$$

Otherwise, if $r_\varepsilon(x) < r_\varepsilon(y)$ we have $x \in \mathsf{B}_{r_\varepsilon(y)}(y)$ and, as above, we derive the inequalities $r(x) \geq r_\varepsilon(y) - \mathsf{d}(x, y)$ and hence $r_\varepsilon(x) \geq r_\varepsilon(y) - \mathsf{d}(x, y)$, which is the remaining part to prove (3.5).

Now, let us assume that $y \notin \mathsf{B}_{r_\varepsilon(x)}(x)$, so that

$$\mathsf{d}(x, y) \geq r_\varepsilon(x) \geq r_\varepsilon(x) - r_\varepsilon(y).$$

From this inequality, (3.5) directly follows if $x \notin \mathsf{B}_{r_\varepsilon(y)}(y)$ as well. Otherwise, if $x \in \mathsf{B}_{r_\varepsilon(y)}(y)$ we can argue as above to prove that $r_\varepsilon(x) \geq r_\varepsilon(y) - \mathsf{d}(x, y)$. Since in this case $r_\varepsilon(y) > r_\varepsilon(x)$, we finally obtain (3.5). $\qquad\square$

Remark 3.11. With some technical effort employing the notions from Section 2.2, following the lines of [24, pp. 59–60] one can show the following. Given any $x \in M$, $b > 1$ and $r \in (0, r_{\mathrm{E}}(x, b))$, we have the inclusions

$$B_{r/\sqrt{b}}(0) \subset \Phi(\mathsf{B}_r(x)) \subset B_{r\sqrt{b}}(0),$$

where B denotes the Euclidean ball in $\mathbf{R}^d$ or $\mathbf{R}^{d-1} \times [0, \infty)$ (depending on whether $\mathsf{B}_r(x) \cap \partial M = \emptyset$ or not), and (Φ, U) is a chart witnessing item b. w.r.t. the data x and b in Definition 3.7. Moreover, we have

$$\mathsf{d}(x, z) \leq \sqrt{b}\,|\Phi(z) - \Phi(x)|$$

for every $z \in \Phi^{-1}(B_{r/\sqrt{b}}(0))$, while for every $z \in \mathsf{B}_r(x)$,

$$|\Phi(z) - \Phi(x)| \leq \sqrt{b}\,\mathsf{d}(x, z).$$

These properties, however, are not needed in the proof of Theorem 3.13 below.

3.2.3. *Main result and construction of the control pair.* Based upon the Euclidean radius r_{E} from Definition 3.7, we now construct the desired general control pair.

We first establish the following local Faber–Krahn inequality with uniform constants. Let $\lambda_1(O)$ and $\mu_1(\Omega)$ denote the first eigenvalues of the Dirichlet Laplacians on given relatively compact, open domains $O \subset M$ and $\Omega \subset \mathbf{R}^d$, respectively. Recall that by the Euclidean Faber–Krahn theorem [23, p. 367], there exists a constant $c > 0$ depending only on d such that for every nonempty Ω as above,

$$\mu_1(\Omega) \geq c\,\mathscr{L}^d[\Omega]^{-2/d}. \tag{3.6}$$

Lemma 3.12. *Given any $b > 1$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 1$, define $R\colon M \to (0, \varepsilon_1/\varepsilon_2)$ by*

$$R(y) := \min\{r_{\mathrm{E}}(y, b), \varepsilon_1\}/\varepsilon_2. \tag{3.7}$$

Then there exists a constant $a > 0$ depending only on b and d such that for every $x \in M$, $\mathsf{B}_{R(x)}(x)$ is relatively compact, and for every nonempty open $O \subset \mathsf{B}_{R(x)}(x)$,

$$\lambda_1(O) \geq a\,\mathfrak{v}[O]^{-2/d}.$$

Proof. We first assume that $O \cap \partial M \neq \emptyset$. Given any $\varepsilon > 0$, let $f \in \mathrm{Lip}(O)$ be compactly supported in O with $0 < \|f\|_{L^2(O)} \leq 1$ and

$$\lambda_1(O) \geq \int_O |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{v} - \varepsilon.$$

Let (U, Φ) be a chart witnessing item b. in Definition 3.7 according to the definition of R and r_{E}. The parametrization $f^\diamond := f \circ \Phi^{-1}\big|_{\Phi(O)}\colon \Phi(O) \to \mathbf{R}$ is Lipschitz and has compact support in $\Phi(O) \subset \mathbf{R}^{d-1} \times [0, \infty)$ which possibly includes $\mathbf{R}^{d-1} \times \{0\}$. Therefore, successively using (2.3), (2.2) and (3.2), we obtain the existence of a constant $\eta > 0$ depending only on b and d such that

$$\lambda_1(O) \geq \eta \int_{\Phi(O)} \big|\mathrm{d}f^\diamond\big|_{g^{\mathrm{E}}}^2 \,\mathrm{d}\mathscr{L}^d - \varepsilon. \tag{3.8}$$

Observe that, however, $\Phi(O)$ is not open in $\mathbf{R}^d$, whence we cannot apply (3.6) to (3.8). We bypass this by a reflection technique. Let $\rho\colon \mathbf{R}^{d-1} \times [0, \infty) \to \mathbf{R}^d$ be the map which precisely flips the sign of the d-th coordinate of its argument, and note that $\Omega := O \cup \rho(O)$ is open and relatively compact. Define the reflection $f_{\mathrm{r}}^\diamond\colon \Omega \to \mathbf{R}$ of the function $f^\diamond$ at $\mathbf{R}^{d-1} \times [0, \infty)$ by $f_{\mathrm{r}}^\diamond(x_1, \ldots, x_{d-1}, x_d) := f^\diamond(x_1, \ldots, x_{d-1}, |x_d|)$. Clearly, $f_{\mathrm{r}}^\diamond$ is Lipschitz and compactly supported in Ω, thus belongs to the form domain of the Dirichlet Laplacian on Ω. Hence, since

$$\big\|f_{\mathrm{r}}^\diamond\big\|_{L^2(\Omega, \mathscr{L}^d)} \asymp \|f\|_{L^2(O)},$$

$$\big\|\big|\mathrm{d}f_{\mathrm{r}}^\diamond\big|_{g^{\mathrm{E}}}\big\|_{L^2(\Omega, \mathscr{L}^d)} \asymp \big\|\big|\mathrm{d}f^\diamond\big|_{g^{\mathrm{E}}}\big\|_{L^2(\Phi(O), \mathscr{L}^d)},$$

up to uniform constants depending only on b and d, (3.8) and (3.6) yield

$$\lambda_1(O) \geq \eta\,\mu_1[\Omega] - \varepsilon \geq \eta\,\mathscr{L}^d[\Omega]^{-2/d} - \varepsilon \geq \eta\,\mathscr{L}^d[\Phi(O)]^{-2/d} - \varepsilon$$

for some $\eta > 0$ depending only on b and d which, as customary, is allowed to change from left to right. Employing (2.3) and (3.2) again, it follows that $\mathscr{L}^d[\Phi(O)]^{-2/d} \geq \eta\,\mathfrak{v}[O]^{-2/d}$ for some constant $\eta > 0$ depending only on b and d, which implies the claimed inequality upon letting $\varepsilon \to 0$.

A similar, more straightforward argument without the above reflection technique readily covers the case $O \cap \partial M = \emptyset$. $\square$

Theorem 3.13. *For every $b > 1$ there exists a constant $C > 0$ depending only on b and d such that for every $\varepsilon_1 > 0$, every $\varepsilon_2 > 1$, every $x, y \in M$ and every $t > 0$*

$$\mathsf{p}(t, x, y) \leq C \min\{t, R^2(x)\}^{-d/2},$$

where R is defined as in (3.7).

Proof. Given any $y \in M$ and any $t > 0$, using Lemma 3.12 we apply Theorem 3.5 to the following setup: $\boldsymbol{n} := d$, $u(s) := \mathsf{p}(s, \cdot, y)$ for every $s > 0$, $r := R(x)$, t replaced by $t + \delta$ for some fixed $\delta > 0$, $\tau := \min\{t, R^2(x)\}$, an appropriate $\zeta \in (0, t + \delta - \tau)$, and $q := 1$. This yields the existence of a constant $C > 0$ depending only on d with

$$\mathsf{p}(t + \delta, x, y) \leq \frac{C\,a^{-d/2}}{\tau^{1+d/2}} \int_{t+\delta-\tau}^{t+\delta} \int_{\mathsf{B}_{\sqrt{\tau}}(x)} \mathsf{p}(s, \cdot, y) \,\mathrm{d}\mathfrak{v} \,\mathrm{d}s \leq \frac{C\,a^{-d/2}}{\tau^{d/2}}.$$

The result follows after letting $\delta \to 0$. $\square$

Example 3.14. Retain the setting of Example 3.9. Employing Theorem 3.13, upon choosing $b > 1$ appropriately we find a constant $C > 0$ depending only on d such that for every $\varepsilon_1 > 0$, every $\varepsilon_2 > 1$, every $x, y \in M_\psi$ and every $t > 0$,

$$\mathsf{p}(t, x, y) \leq C \min\{t, \varepsilon_1^2 \varepsilon_2^{-2}\}^{-d/2}.$$

This resembles a variant of the smooth result [11, Thm. 8.3].

Using these results we can then prove the existence of the claimed heat kernel control pair for an LR manifold.

Corollary 3.15. *Every LR manifold (M, g) admits a heat kernel control pair according to Definition 3.6.*

Proof. Theorem 3.13 ensures that for every $b > 1$ there exists a constant $C > 0$ depending only on b and d such that

$$\mathsf{p}(t, x, y) \leq \frac{C}{t^{d/2}} + \frac{C}{R^d(x)} \leq \frac{C}{R^d(x)} \left[\frac{\varepsilon_1^d}{\varepsilon_2^d \, t^{d/2}} + 1 \right].$$

In the last inequality, we used that $R \leq \varepsilon_1/\varepsilon_2$ on M. Since R is $1/\varepsilon_2$-Lipschitz by Lemma 3.10, the functions $\Xi \colon M \to (0, \infty)$ and $\Sigma \colon (0, 1] \to (0, \infty)$ with

$$\Xi(x) := R^{-d}(x),$$
$$\Sigma(t) := 1 + \varepsilon_1^d \, \varepsilon_2^{-d} \, t^{-d/2}$$

define a heat kernel control pair for (M, g). $\qquad\square$

3.3. **Improvements under uniform constants.** If (M, g) has better geometric properties, it also admits different explicit heat kernel control pairs than the one derived in Corollary 3.15. Deducing these (following standard lines as in [23, 45]) is the goal of this section. In particular, we point out that while Theorem 3.13 and Corollary 3.15 are really restricted to the heat kernel induced by the volume measure $\mathfrak{v}$, Theorem 3.16 below works for any given $\mathfrak{m}$ as in Subsection 2.1.4.

In this section we assume the following conditions.

a. *Uniform local doubling.* There exists a constant $C_{\mathrm{LD}} \geq 1$ such that for every $x \in M$ and every $s, s' > 0$ with $s' \leq s$,

$$\mathfrak{m}[\mathsf{B}_s(x)] \leq C_{\mathrm{LD}} \, \mathfrak{m}[\mathsf{B}_{s'}(x)] \, (s/s')^d \, \mathrm{e}^{C_{\mathrm{LD}} s}.$$

b. *Uniform Sobolev inequality.* There exist constants $C_{\mathrm{S}} \geq 1$ and $N > 2$ such that for every $x \in M$, every $r > 0$ and every $f \in W^{1,2}(M) \cap \mathrm{C}_{\mathrm{c}}(\mathsf{B}_r(x))$,

$$\left[\int_M |f|^{2N/(N-2)} \, \mathrm{d}\mathfrak{m} \right]^{(N-2)/N}$$
$$\leq C_{\mathrm{S}} \, \mathrm{e}^{C_{\mathrm{S}} r} \, \mathfrak{m}[\mathsf{B}_r(x)]^{-2/N} \, r^2 \int_M \left[|\mathrm{d}f|^2 + r^{-2} \, f^2 \right] \mathrm{d}\mathfrak{m}.$$

Theorem 3.16. *Retain the previous assumptions, and let $\varepsilon, \delta > 0$. Then there exists a constant $C > 0$ depending only on d, C_{LD}, C_{S}, ε and δ such that for every $x, y \in M$ and every $t \in (0, 1]$,*

$$\mathsf{p}(t, x, y) \leq C \, \mathfrak{m}[\mathsf{B}_1(x)]^{-1} \, t^{-d/2} \, \exp\left[-\frac{\mathsf{d}^2(x, y)}{(4 + \varepsilon)t} - (1 - \delta)\lambda t \right].$$

Recall Subsection 2.5.4 for the meaning of $\lambda \geq 0$. A *local* estimate similar to Theorem 3.16 can be found in [40, Thm. 2.1].

Combined with Theorem 4.2, Theorem 3.16 then gives the following desired explicit heat kernel control pair.

Corollary 3.17. *Retain the assumptions and the notation of Theorem 3.16, and let $d \geq 2$ as well as $p \in (d/2, \infty)$. Define $\Xi \colon M \to \mathbf{R}$ and $\Sigma \colon (0,1] \to (0,\infty)$ by*

$$\Xi(x) := \mathfrak{m}[\mathsf{B}_1(x)]^{-1},$$
$$\Sigma(t) := t^{-d/2}.$$

Then (Ξ, Σ) is a heat kernel control pair for (M, g).

Proof of Theorem 3.16. By a. applied to $s := 2s'$, $s' \in (0, 1/2)$, the doubling property according to [52, pp. 293–294] holds on every ball $B \subset M$ of radius no larger than 1 with doubling constant $2^{N(B)}$, where $N(B) := d + \log_2(C_{\mathrm{LD}}\, \mathrm{e}^{C_{\mathrm{LD}}})$ in the notation of [52]. By b., the Sobolev inequality according to [52, pp. 294–295] also holds on every ball $B \subset M$ of radius no larger than 1 with constant $C_S(B) := C_{\mathrm{S}}\, \mathrm{e}^{\mathrm{S}}$. Proposition 2.12 ensures that for every $\varepsilon, \delta > 0$ there exists a constant $C > 0$ depending only on $N(B)$ and $C_{\mathrm{S}}(B)$ such that for every $t \in (0,1]$,

$$\mathsf{p}(t,x,y) \leq C\, \mathfrak{m}\big[\mathsf{B}_{\sqrt{t}}(x)\big]^{-1/2} \mathfrak{m}\big[\mathsf{B}_{\sqrt{t}}(y)\big]^{-1/2} \exp\Big[-\frac{\mathsf{d}^2(x,y)}{(4+\varepsilon/2)t} - (1-\delta)\lambda t\Big]. \quad (3.9)$$

Next, we get rid of the term $\mathfrak{m}[\mathsf{B}_{\sqrt{t}}(y)]^{-1/2}$ in (3.9). Using a., we get

$$\mathfrak{m}\big[\mathsf{B}_{\sqrt{t}}(x)\big] \leq \mathfrak{m}\big[\mathsf{B}_{\mathsf{d}(x,y)+\sqrt{t}}(y)\big]$$
$$\leq \mathrm{e}\, C_{\mathrm{LD}}\, \mathfrak{m}\big[\mathsf{B}_{\sqrt{t}}(y)\big] \Big[1 + \frac{\mathsf{d}_g(x,y)}{\sqrt{t}}\Big]^d \mathrm{e}^{C_{\mathrm{LD}}\mathsf{d}(x,y)}$$

for every $x, y \in M$ and every $t \in (0,1]$. After absorbing the latter polynomial and the latter exponential, whose exponent only depends linearly on $\mathsf{d}(x,y)$, into the exponential in (3.9), we infer the existence of a constant $C > 0$ depending only on d, C_{LD}, C_{S}, ε and δ such that for every $x, y \in M$ and every $t \in (0,1]$,

$$\mathsf{p}(t,x,y) \leq C\, \mathfrak{m}\big[\mathsf{B}_{\sqrt{t}}(x)\big]^{-1} \exp\Big[-\frac{\mathsf{d}^2(x,y)}{(4+\varepsilon)t} - (1-\delta)\lambda t\Big]. \quad (3.10)$$

Finally we make the radius of the ball on the r.h.s. of (3.10) independent of t. To this aim, we again use a. and obtain

$$\mathfrak{m}[\mathsf{B}_1(x)] \leq C_{\mathrm{LD}}\, \mathfrak{m}\big[\mathsf{B}_{\sqrt{t}}(x)\big]\, t^{-d/2}\, \mathrm{e}^{C_{\mathrm{LD}}}$$

for every $x \in M$ and every $t \in (0,1]$. With (3.10), this terminates the proof. $\quad\square$

Finally, we discuss an example in which the hypotheses of Theorem 3.16 are satisfied (recall Section 3.3). Example 3.18 follows the smooth treatise [6].

Example 3.18. Let (N, h) be a smooth, complete Riemannian manifold for which there exists $K \geq 0$ such that $\mathrm{Ric}_h \geq -K$ on N. Assume that $(M, g) \sim_F (N, h)$ by some bi-Lipschitz $F \colon M \to N$. Then $(M, \mathscr{E}_{g,\mathfrak{v}_g}, \mathfrak{v}_g)$ satisfies the hypotheses of Theorem 3.16. Indeed, property a. holds true on N for $\mathfrak{m}$ replaced by $\mathfrak{v}_h$ with $C_{\mathrm{LD}} := \max\{1, \sqrt{(d-1)K}\}$ [46, Thm. 5.6.4]. By [53, Thm. 2.6], there exists a constant $C \geq 1$ depending only on d and K such that b. holds for $(N, \mathscr{E}_{g,\mathfrak{v}_h}, \mathfrak{v}_h)$ with $C_{\mathrm{S}} := C$. The claim follows since a. and b. are qualitatively preserved under quasi-isometry thanks to Lemma 2.8, Remark 2.9, Lemma 2.10 and Remark 2.11.

Example 3.19. Suppose that $(M, \mathsf{d}, \mathfrak{m})$, possibly with $\mathfrak{m} \neq \mathfrak{v}$, is bi-Lipschitz equivalent (as a metric measure space) to an $\mathrm{RCD}(K, N)$ space $(N, \mathsf{d}', \mathfrak{n})$, $K \in \mathbf{R}$ and $N \in [1, \infty)$. For instance, $(N, \mathsf{d}', \mathfrak{n})$ might be a smooth, geodesically complete Riemannian manifold with $\mathrm{Ric}_h \geq K$ and convex boundary [28, Thm. 2.4, Cor. 2.6]. Then a. [55, Thm. 2.3] and b. [58, Thm. 30.23] hold on N with uniform constants and, as in

Example 3.18, transfer back qualitatively to M. A heat kernel control pair (Ξ, Σ) for $(M, \mathscr{E}, \mathfrak{m})$ is then given by

$$\Xi(x) := \mathfrak{m}[B_1(x)]^{-1},$$
$$\Sigma(t) := t^{-N/2}.$$

4. Taming for almost smooth LR manifolds

In this last chapter, we show that it is possible to introduce the notion of lower Ricci curvature bounds in the *Kato class* $\mathcal{K}(M)$ (recall Definition 1.2) for suitable LR manifolds (M, g), in the sense of [18]. To do so, we propose a smoothness condition, cf. Definition 4.8 below, on the given LR manifold that leads us to formulate the Bochner identity for this class of spaces.

The next Section 4.1 is devoted to briefly recall the basic notions in the theory of *tamed spaces* from [18] and to adapt them to our setting. We also review important properties of the Feynman–Kac semigroup associated to elements of $\mathcal{K}(M)$ and we prove the important criterion in Theorem 4.2.

4.1. **Tamed spaces.** Various notions presented in this section hold for "quasi-local distributions" κ in much greater generality. Since these are beyond our scope, we refer to [18] for further details.

4.1.1. *The Kato class on LR manifolds.* For convenience, we restate Definition 1.2.

Definition 4.1. *The* Kato class $\mathcal{K}(M)$ *consists of all signed Borel measures κ on M which do not charge $\mathscr{E}$-polar sets and such that*

$$\lim_{t \to 0} \sup_{x \in M} \int_0^t \int_M \mathsf{p}(s, x, \cdot) \, \mathrm{d}|\kappa| \, \mathrm{d}s = 0.$$

Reminiscent of the smooth result [24, Prop. VI.10], we now prove the following.

Theorem 4.2. *Suppose that $\Bbbk \colon M \to \mathbf{R}$ is a Borel function that can be decomposed as $\Bbbk = \Bbbk_1 + \Bbbk_2$ into the sum of two Borel functions $\Bbbk_1, \Bbbk_2 \colon M \to \mathbf{R}$ such that*

  a. *$\Bbbk_2 \in L^\infty(M)$, and*
  b. *there exist $p \in [1, \infty)$ if $d = 1$ or $p \in (d/2, \infty)$ if $d \geq 2$, and a heat kernel control pair (Ξ, Σ) such that $\Bbbk_1 \in L^p(M, \Xi\mathfrak{v})$.*

Then for every $s > 0$ and every $x \in M$,

$$\int_M \mathsf{p}(s, x, \cdot) \, |\Bbbk| \, \mathrm{d}\mathfrak{v} \leq \Sigma^{1/p}(s) \, \|\Bbbk_1\|_{L^p(M, \Xi\mathfrak{v})} + \|\Bbbk_2\|_{L^\infty(M)}. \tag{4.1}$$

In particular, for every choice of (Ξ, Σ) and p as in b. *and every $\Bbbk \in L^p(M, \Xi\mathfrak{v}) + L^\infty(M)$, we have $\Bbbk\mathfrak{v} \in \mathcal{K}(M)$.*

Proof. We first derive the second claim from (4.1). Decompose any $\Bbbk \in L^p(M, \Xi\mathfrak{v}) + L^\infty(M, \mathfrak{v})$ according to a. and b. above. Thus

$$\lim_{t \to 0} \sup_{x \in M} \int_0^t \int_M \mathsf{p}(s, x, \cdot) \, |\Bbbk| \, \mathrm{d}\mathfrak{v} \, \mathrm{d}s$$

$$\leq \|\Bbbk_1\|_{L^p(M, \Xi\mathfrak{v})} \lim_{t \to 0} \int_0^t \Sigma^{1/p}(s) \, \mathrm{d}s + \|\Bbbk_2\|_{L^\infty(M)} \lim_{t \to 0} t = 0.$$

We proceed to show the validity of (4.1). Since $|\Bbbk| \leq |\Bbbk_1| + |\Bbbk_2|$ on M and since the heat kernel $\mathsf{p}(s, x, \cdot)$ is a sub-probability density for every $s > 0$ and every $x \in M$, we may and will assume that $\Bbbk_2$ vanishes identically on M. Then the inequality (4.1) for $p \in [1, \infty)$ if $d = 1$ trivially follows from (3.1), whence we concentrate on

the situation $d \geq 2$, in which case we assume $p \in (d/2, \infty)$. Denote by $q := p/(p-1)$ the dual exponent to p, i.e. $1/p + 1/q = 1$. Using Hölder's inequality and (3.1) yields

$$
\begin{aligned}
\int_M \mathsf{p}(s, x, \cdot) \, |\mathscr{k}_1| \, \mathrm{d}\mathfrak{v} &= \int_M \mathsf{p}^{1/q}(s, x, \cdot) \, \mathsf{p}^{1/p}(s, x, \cdot) \, |\mathscr{k}_1| \, \mathrm{d}\mathfrak{v} \\
&\leq \left[\int_M \mathsf{p}(s, x, \cdot) \, \mathrm{d}\mathfrak{v} \right]^{1/q} \left[\int_M |\mathscr{k}_1|^p \, \mathsf{p}(s, \cdot, x) \, \mathrm{d}\mathfrak{v} \right]^{1/p} \\
&\leq \left[\int_M |\mathscr{k}_1|^p \, \Xi \, \Sigma(s) \, \mathrm{d}\mathfrak{m} \right]^{1/p} = \Sigma^{1/p}(s) \, \|\mathscr{k}_1\|_{L^p(M, \Xi \mathfrak{v})}. \qquad \square
\end{aligned}
$$

Remark 4.3. Since $\inf \Xi > 0$, we trivially have $L^p(M, \Xi \mathfrak{v}) \subset L^p(M)$.

Remark 4.4. Of course, Theorem 4.2 holds true on any (quasi-)regular, strongly local Dirichlet space $(M, \mathscr{E}, \mathfrak{m})$ satisfying the absolute continuity hypothesis, i.e. the existence of a heat kernel for $(\mathsf{P}_t)_{t \geq 0}$ [10, Def. A.2.16].

Example 4.5. In the setting of Example 3.9 and Example 3.14, we have $L^p(M_\psi) \subset \mathcal{K}(M_\psi)$ for every $p \in (d/2, \infty)$.

4.1.2. *Feynman–Kac semigroups induced by Kato class measures.* We denote by $((\mathbf{P}^x)_{x \in M}, (\mathsf{b}_t)_{t \in [0, \zeta)})$ the $\mathfrak{v}$-reversible, continuous, strong Markov process associated with $(\mathscr{E}, W^{1,2}(M))$ and with explosion time ζ. As detailed in [18, Sec. 2.2, Sec. 2.4], every $\kappa \in \mathcal{K}(M)$ induces an $\mathscr{E}$-quasi-local distribution on $W^{1,2}(M)$, and is thus properly associated with a local continuous additive functional (AF, in short) $(\mathsf{a}_t^\kappa)_{t \in [0, \zeta)}$, which is unique up to equivalence of AF's [18, Lem. 2.9]. For instance, if $\kappa = \mathscr{k} \, \mathfrak{v}$ for some nearly Borel function $\mathscr{k} \in L^2(M)$, this AF can be explicitly represented as

$$
\mathsf{a}_t^\kappa = \int_0^t f(\mathsf{b}_s) \, \mathrm{d}s.
$$

We associate to $\kappa \in \mathcal{K}(M)$ the *Feynman–Kac semigroup* $(\mathsf{P}_t^\kappa)_{t \geq 0}$, defined in terms of the Feynman–Kac formula

$$
\mathsf{P}_t^\kappa f := \mathbf{E}^\cdot \left[\mathrm{e}^{-\mathsf{a}_t^\kappa} \, f(\mathsf{b}_t) \, \mathbf{1}_{\{t < \zeta\}} \right] \tag{4.2}
$$

for every Borel function $f \colon M \to [0, \infty)$ and every $t \geq 0$. By Khasminskii's lemma [18, Lem. 2.24], the distribution induced by κ is moderate [18, Def. 2.13], which implies that $(\mathsf{P}_t^\kappa)_{t \geq 0}$ extends to a strongly continuous, exponentially bounded semigroup on $L^p(M)$ for every $p \in [1, \infty]$ [18, Rem. 2.14].

4.1.3. *Singular Bakry–Émery condition.* The key feature of $\mathscr{E}$-quasi-local distributions stemming from $\mathcal{K}(M)$ is that $(\mathsf{P}_t^{q\kappa/2})_{t \geq 0}$, $q \in [1, \infty)$, can be treated with form techniques, which is briefly recapitulated here.

The following result from [18, Cor. 2.25] is crucial for this purpose.

Lemma 4.6. *For every $\rho > 0$, there exists $\alpha \in \mathbf{R}$ such that for every $f \in \mathscr{F}$,*

$$
\int_M \widetilde{f}^2 \, \mathrm{d}|\kappa| \leq \rho \, \mathscr{E}(f) + \alpha \int_M f^2 \, \mathrm{d}\mathfrak{v}.
$$

Hence, by [18, Thm. 2.49] the quadratic form

$$
\mathscr{E}^{q\kappa/2}(f) := \mathscr{E}(f) + \frac{q}{2} \int_M \widetilde{f}^2 \, \mathrm{d}\kappa
$$

is closed, lower semibounded in $L^2(M)$ and associated to $(\mathsf{P}_t^{q\kappa/2})_{t \geq 0}$. Moreover, its domain $\mathscr{D}(\mathscr{E}^{q\kappa/2}) := \{f \in \mathscr{F} : \int_M \widetilde{f}^2 \, \mathrm{d}|\kappa| < \infty\}$ corresponds to the whole $\mathscr{F}$. We call this form the *taming energy*, and we denote by $\mathsf{L}^{q\kappa/2}$ the associated generator. We write $\Delta^{q\kappa} := 2 \, \mathsf{L}^{q\kappa/2}$ and denote its domain by $\mathscr{D}(\Delta^{q\kappa})$.

The following definition follows [18, Def. 3.1].

Definition 4.7. *We call (M, g) tamed* (*through a given $\kappa \in \mathcal{K}(M)$*) *if the Bakry–Émery condition* $\mathrm{BE}_1(\kappa, \infty)$ *holds, that is, if for every $f \in \mathscr{F}$ with $\Delta f \in \mathscr{F}$ and every nonnegative $\phi \in \mathscr{D}(\Delta^\kappa)$, we have*

$$\int_M \Delta^\kappa \phi \, |\mathrm{d}f|^2 \, \mathrm{d}\mathfrak{v} - \int \phi \, g^*(\mathrm{d}f, \mathrm{d}\Delta f) \, |\mathrm{d}f|^{-1} \, \mathrm{d}\mathfrak{v} \geq 0,$$

where the second integral is taken over the set where $|\mathrm{d}f| > 0$.

4.2. **An interior taming criterion.**

Definition 4.8. *We call the LR manifold (M, g) almost smooth if there exists an open, $\mathfrak{v}$-conegligible subset M^∞ of M for which $(M^\infty, g\big|_{M^\infty})$ is a smooth Riemannian manifold.*

Reminiscent of Corollary 3.15, Corollary 3.17 and Theorem 4.2, we now establish our main taming result, Theorem 4.9, for Kato Ricci bounds which are absolutely continuous w.r.t. $\mathfrak{v}$. The crucial step to prove Theorem 4.9 is contained in Lemma 4.10. A similar strategy as in its proof has been pursued for different regularity results within a general second order calculus for tamed spaces, see e.g. [5, Cor. 5.12, Lem. 8.8].

Theorem 4.9. *Let (Ξ, Σ) be a given heat kernel control pair for the given almost smooth LR manifold (M, g), and let $\Bbbk \in L^p(M, \Xi \mathfrak{v}) + L^\infty(M)$ for some $p \in (d/2, \infty)$. Suppose that Δ is essentially self-adjoint on $\mathrm{C}_c^\infty(M^\infty)$. Lastly, assume that the Ricci curvature of M^∞ is bounded from below by $\Bbbk$. Then (M, g) is tamed by*

$$\kappa := \Bbbk \, \mathfrak{v}.$$

Lemma 4.10. *Under the assumptions of Theorem 4.9, if $f \in \mathscr{D}(\Delta)$ then $|\mathrm{d}f| \in W^{1,2}(M)$.*

Proof. The claim is straightforward if $f \in \mathrm{C}_c^\infty(M^\infty)$. Indeed, since the set $M \setminus M^\infty$ is $\mathfrak{v}$-negligible, $|\mathrm{d}f|$ has a differential $\mathrm{d}|\mathrm{d}f|$ which vanishes $\mathfrak{v}$-a.e. on $\{|\mathrm{d}f| = 0\}$, and

$$\big| \mathrm{d}|\mathrm{d}f| \big| \leq \big| \mathrm{Hess}\, f \big|_{\mathrm{HS}} \quad \mathfrak{v}\text{-a.e.} \tag{4.3}$$

by Kato's inequality for the Bochner Laplacian [30, Prop. 2.2], thus $|\mathrm{d}f| \in W^{1,2}(M)$.

To deduce the claim for general elements of $\mathscr{D}(\Delta)$ with Laplacian in $W^{1,2}(M)$, we first collect some preliminary considerations, still assuming that $f \in \mathrm{C}_c^\infty(M^\infty)$. Given any $\varepsilon > 0$, define $\eta_\varepsilon \in \mathrm{C}^\infty([0, \infty))$ through $\eta_\varepsilon(r) := (r + \varepsilon)^{1/2}$. Note that $4\, \eta_\varepsilon''(r)\, r \geq -2\, \eta_\varepsilon'(r)$ and $1/\eta_\varepsilon'(r) = 2\, \eta_\varepsilon(r)$ for every $r \geq 0$. Moreover, since $|\mathrm{d}f|^2 \in \mathrm{C}_c^\infty(M^\infty)$, successively applying the chain rule for the Laplacian and employing Bochner's identity on M^∞ as well as (4.3) yields

$$\begin{aligned}
\Delta\big[\eta_\varepsilon \circ |\mathrm{d}f|^2\big] &= \big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big] \Delta|\mathrm{d}f|^2 + \big[\eta_\varepsilon'' \circ |\mathrm{d}f|^2\big] \big| \mathrm{d}|\mathrm{d}f|^2 \big|^2 \\
&\geq 2\, \big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big] \big[g^*(\mathrm{d}f, \mathrm{d}\Delta f) + \Bbbk\, |\mathrm{d}f|^2 + \big| \mathrm{Hess}\, f \big|_{\mathrm{HS}}^2\big] \\
&\quad + 4\, \big[\eta_\varepsilon'' \circ |\mathrm{d}f|^2\big] |\mathrm{d}f|^2 \big| \mathrm{d}|\mathrm{d}f| \big|^2 \\
&\geq 2\, \big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big] \big[g^*(\mathrm{d}f, \mathrm{d}\Delta f) + \Bbbk\, |\mathrm{d}f|^2 + \big| \mathrm{Hess}\, f \big|_{\mathrm{HS}}^2 - \big| \mathrm{d}|\mathrm{d}f| \big|^2\big] \\
&\geq 2\, \big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big] \big[g^*(\mathrm{d}f, \mathrm{d}\Delta f) - \Bbbk^-\, |\mathrm{d}f|^2\big] \quad \mathfrak{v}\text{-a.e.}
\end{aligned} \tag{4.4}$$

Dividing by $2\, \eta_\varepsilon' \circ |\mathrm{d}f|^2$, integrating the result on M and using that $\Delta[\eta_\varepsilon \circ |\mathrm{d}f|^2]$ has compact support in M^∞ gives

$$\int_M \big| \mathrm{d}\big[\eta_\varepsilon \circ |\mathrm{d}f|^2\big] \big|^2 \, \mathrm{d}\mathfrak{v} \leq -\int_M g^*(\mathrm{d}f, \mathrm{d}\Delta f) \, \mathrm{d}\mathfrak{v} + \int_M \Bbbk^-\, |\mathrm{d}f|^2 \, \mathrm{d}\mathfrak{v}. \tag{4.5}$$

By Theorem 4.2, the measure $\mathscr{k}^- \mathfrak{v}$ belongs to $\mathcal{K}(M)$. Since $|\mathrm{d}f| \in W^{1,2}(M)$, by Lemma 4.6 there exists $\alpha \in \mathbf{R}$ (independent of f) such that

$$\int_M \mathscr{k}^- |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{v} \leq \frac{1}{2} \int_M \big| \mathrm{d}|\mathrm{d}f|^2 \big|^2 \,\mathrm{d}\mathfrak{v} + \alpha \int_M |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{v}.$$

Letting $\varepsilon \to 0$ in (4.5) and employing the lower semicontinuity of $\mathscr{E}$ w.r.t. pointwise $\mathfrak{v}$-a.e. convergence yields

$$\int_M \big| \mathrm{d}|\mathrm{d}f| \big|^2 \,\mathrm{d}\mathfrak{v} \leq 2 \int_M (\Delta f)^2 \,\mathrm{d}\mathfrak{v} + 2\alpha \int_M |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{v}. \tag{4.6}$$

Now we prove the claim for arbitrary $f \in \mathscr{D}(\Delta)$. By assumption, there exists a sequence $(f_n)_{n \in \mathbf{N}}$ in $\mathrm{C}_\mathrm{c}^\infty(M^\infty)$ converging to f in the L^2-graph norm of Δ. We may and will assume without restriction that $|\mathrm{d}f_n| \to |\mathrm{d}f|$ pointwise $\mathfrak{v}$-a.e. as $n \to \infty$. Then, since $(|\mathrm{d}f_n|)_{n \in \mathbf{N}}$ is a bounded sequence in $W^{1,2}(M)$ by (4.6), the claim follows again by lower semicontinuity of $\mathscr{E}$. $\qquad\square$

Proof of Theorem 4.9. As seen in the proof of Lemma 4.10, by Corollary 3.17 it follows that $\mathscr{k}\,\mathfrak{v} \in \mathcal{K}(M)$. To derive $\mathrm{BE}_1(\kappa, \infty)$, let $f \in \bigcup_{t>0} \mathsf{P}_t L^2(M)$. By local elliptic regularity theory, see e.g. [23], we have $f \in \mathrm{C}^\infty(M^\infty)$, whence

$$\Delta\big[\eta_\varepsilon \circ |\mathrm{d}f|^2\big] \geq 2\big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big]\big[g^*(\mathrm{d}f, \mathrm{d}\Delta f) + \mathscr{k}\,|\mathrm{d}f|^2\big] \quad \mathfrak{v}\text{-a.e.}$$

for every $\varepsilon > 0$ by a similar argument as for (4.4), retaining the notations used above. Multiplying this inequality by any given nonnegative, boundedly supported $\phi \in \mathrm{Lip}(M)$ and integrating by parts while using Lemma 4.10 yields

$$0 \leq -\int_M g^*(\mathrm{d}\phi, \mathrm{d}|\mathrm{d}f|)\big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big]|\mathrm{d}f|\,\mathrm{d}\mathfrak{v} - \int_M \mathscr{k}\,\phi\big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big]|\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{v}$$

$$- \int_M \phi\,g^*(\mathrm{d}f, \mathrm{d}\Delta f)\big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big]\,\mathrm{d}\mathfrak{v}$$

$$\leq -\int_M g^*(\mathrm{d}\phi, \mathrm{d}|\mathrm{d}f|)\big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big]|\mathrm{d}f|\,\mathrm{d}\mathfrak{v}$$

$$- \int_M \min\{\mathscr{k}^+, R\}\,\phi\big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big]|\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{v} + \frac{1}{2}\int_M \mathscr{k}^-\,\phi\,|\mathrm{d}f|\,\mathrm{d}\mathfrak{v}$$

$$- \int_M \phi\,g^*(\mathrm{d}f, \mathrm{d}\Delta f)\big[\eta_\varepsilon' \circ |\mathrm{d}f|^2\big]\,\mathrm{d}\mathfrak{v}$$

for any fixed $R > 0$. Setting $\kappa_R := \min\{\mathscr{k}^+, R\}\,\mathfrak{m} - \mathscr{k}^-\,\mathfrak{m} \in \mathcal{K}(M)$, as $\varepsilon \to 0$, by Lebesgue's theorem this reduces to

$$0 \leq -\mathscr{E}^{\kappa_R}(\phi, |\mathrm{d}f|) - \int \phi\,g^*(\mathrm{d}f, \mathrm{d}\Delta f)\,|\mathrm{d}f|^{-1}\,\mathrm{d}\mathfrak{v}$$

and hence, as $R \to \infty$ by Levi's theorem, to

$$0 \leq -\mathscr{E}^\kappa(\phi, |\mathrm{d}f|) - \int \phi\,g^*(\mathrm{d}f, \mathrm{d}\Delta f)\,|\mathrm{d}f|^{-1}\,\mathrm{d}\mathfrak{v}.$$

In both cases, the last integrals are taken over the set where $|\mathrm{d}f| > 0$. By density of boundedly supported Lipschitz functions in $W^{1,2}(M)$ outlined in Subsection 2.5.1, this inequality readily extends to all nonnegative $\phi \in W^{1,2}(M)$. Indeed, the map $g \mapsto \int_M \mathscr{k}\,g\,|\mathrm{d}f|\,\mathrm{d}\mathfrak{v}$ is continuous in $W^{1,2}(M)$ by Cauchy–Schwarz's inequality, Lemma 4.10 and Lemma 4.6.

If in particular $\phi \in \mathscr{D}(\Delta^\kappa)$, which is a subset of $W^{1,2}(M)$ by Subsection 4.1.3, integration by parts again gives

$$\int_M \Delta^\kappa \phi\,|\mathrm{d}f|\,\mathrm{d}\mathfrak{v} - \int \phi\,g^*(\mathrm{d}f, \mathrm{d}\Delta f)\,|\mathrm{d}f|^{-1}\,\mathrm{d}\mathfrak{v} \geq 0.$$

To deduce $\mathrm{BE}_1(\kappa, \infty)$ it remains to follow the proof of [18, Thm. 3.4]. $\qquad\square$

We quickly discuss two situations in which essential self-adjointness of Δ holds. To couple this with lower Ricci bounds, one can e.g. take the space (M, g) under consideration to be (quasi-isometric to) a smooth, geodesically complete Riemannian manifold with uniformly lower bounded Ricci curvature, according to the discussions from Example 3.18 and Example 3.19.

Example 4.11 (Removal of codimension 4 sets)*.* Endow $M := \mathscr{M} \setminus S$, where $\mathscr{M}$ is a smooth manifold of dimension $d \geq 4$ and S is a closed submanifold of codimension at least 4, with a complete Riemannian metric g. Then the induced Laplacian Δ is essentially self-adjoint on $C_c^\infty(M)$ [38, Thm. 3].

In the same spirit, one can take S to be the finite union of compact submanifolds, not necessarily all of the same dimension, with codimension at least 4 [42, Thm. 5]. The finiteness and the compactness condition can be weakened [42, Rem. 3], which allows e.g. for perforated domains such as $\mathbf{R}^5 \setminus \mathbf{Z}^5$ (which obeys $\mathrm{BE}_1(0, \infty)$).

Example 4.12 (Quantum confinements and ARS)*.* Let (M, g) be a smooth, noncomplete Riemannian manifold, and denote by $(\overline{M}, \overline{\mathsf{d}})$ its metric completion. Suppose that the distance function $\delta \colon M \to [0, \infty)$ given by

$$\delta(x) := \inf\{\overline{\mathsf{d}}(x, y) : y \in \overline{M} \setminus M\}$$

has regularity C^2 on $\delta^{-1}((0, \varepsilon])$ for some $\varepsilon > 0$. Lastly, assume that there exists some $\varrho > 0$ such that

$$(\Delta\delta)^2 - 2\left|\mathrm{Hess}\, f\right|_{\mathrm{HS}}^2 - 2\,\mathrm{Ric}(\nabla\delta, \nabla\delta) \geq \frac{3}{\delta^2} - \frac{4\varrho}{\delta} \quad \text{on } \delta^{-1}((0, \varepsilon]).$$

Then Δ is essentially self-adjoint on $C_c^\infty(M)$ [42, Thm. 1].

Examples of spaces satisfying these conditions are cones, metric horns, or anti-cones [42]. In fact [42, Thm. 8], the above hypotheses cover *regular almost Riemannian structures* [42, Def. 7.1, Def. 7.10], briefly regular ARS. These structures, roughly said, consist of a smooth manifold M endowed with a metric g which is singular on an embedded hypersurface, and smooth on its complement.

Let us finally list some corollaries of Theorem 4.9. Corollary 4.13 follows from the proof of Theorem 4.9 and [18, Thm. 3.4]. A short proof of Corollary 4.14 is included for convenience. The integrated Bochner inequality from Corollary 4.15 — according to the notion of Hessian established in [5] — follows from [5, Cor. 8.3].

Corollary 4.13. *In the situation of Theorem 4.9, the heat flow satisfies the following gradient estimate for every $f \in W^{1,2}(M)$ and every $t \geq 0$:*

$$|\mathsf{d}\mathsf{P}_t f| \leq \mathsf{P}_t^{\kappa/2}|\mathsf{d}f| \quad \mathfrak{v}\text{-a.e.}$$

Corollary 4.14. *If in addition to the hypotheses in Theorem 4.9, (M, d) is complete, then its heat flow is conservative.*

Proof. Since the intrinsic distance generated by $\mathscr{E}$ generates the topology of M by Subsection 2.5.4 and since (M, d) is complete, the Dirichlet space induced by (M, g) is intrinsically complete according to [18, Def. 3.8] as proven in [18, Rem. 3.8]. Since every tamed space is weakly tamed [18, Def. 3.10], conservativeness then follows from [18, Thm. 3.11]. $\qquad\square$

Corollary 4.15. *In the situation of Theorem 4.9, every function $f \in \mathscr{D}(\Delta)$ belongs to $\mathscr{D}_{\mathrm{reg}}(\mathrm{Hess})$, and we have*

$$\int_M \left|\mathrm{Hess}\, f\right|_{\mathrm{HS}}^2 \mathrm{d}\mathfrak{v} \leq \int_M (\Delta f)^2 \,\mathrm{d}\mathfrak{v} - \int_M \mathscr{k}\, |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{v}.$$

4.3. **A boundary taming criterion.** Finally, in Theorem 4.19 we establish a condition for weighted surface measures on the boundary of special LR manifolds to belong to $\mathcal{K}(M)$. With this at hand, in Theorem 4.21 we provide a taming condition for LR manifolds with boundary.

Throughout this section, our setting is the following: suppose that (M, g) is quasi-isometric to a smooth, complete Riemannian manifold (N, h) of dimension $d \geq 3$, without or with convex boundary, and with uniformly lower bounded Ricci curvature by a bi-Lipschitz map $F\colon M \to N$. Moreover, let us assume that N has positive injectivity radius. As detailed in Remark 4.17 and Remark 4.20, this assumption will allow us to characterize the Kato class of N in metric terms à la Aizenman–Simon [2, 34], a formulation which is far easier to control in terms of codimension one sets. Using lower bounds for p_h [51], this information is then "pulled back" to M through F.

4.3.1. *Correspondence of the Kato classes.*

Proposition 4.16. *For every $\mu \in \mathcal{K}(N)$, we have $(F^{-1})_\sharp \mu \in \mathcal{K}(M)$.*

Proof. By bi-Lipschitz continuity of F and Lemma 2.10, the signed Borel measure $\kappa := (F^{-1})_\sharp \mu$ on M does not charge $\mathscr{E}_g$-polar sets. Now note that

$$\sup_{x \in M} \int_0^t \int_M \mathsf{p}_g(s, x, \cdot) \, \mathrm{d}|\kappa| \, \mathrm{d}s$$

$$\leq C \sup_{x \in M} \int_0^t \int_M \mathfrak{v}_g \big[\mathsf{B}_{\sqrt{t}}^g(x)\big]^{-1} \exp\Big[-\frac{\mathsf{d}_g^2(x, \cdot)}{5t}\Big] \, \mathrm{d}|\kappa| \, \mathrm{d}s$$

$$\leq C \sup_{z \in N} \int_0^t \int_N \mathfrak{v}_h \big[\mathsf{B}_{\sqrt{t}/C}^h(z)\big]^{-1} \exp\Big[-\frac{\mathsf{d}_h^2(z, \cdot)}{5Ct}\Big] \, \mathrm{d}|\mu| \, \mathrm{d}s$$

$$\leq C \sup_{z \in N} \int_0^{Ct} \int_N \mathsf{p}_h(s, z, \cdot) \, \mathrm{d}|\mu| \, \mathrm{d}s.$$

Here we have used (3.10) in the first inequality, Remark 2.9 and the Lipschitz continuity of F in the second one, and the local doubling property of $\mathfrak{v}_h$ [46, Thm. 5.6.4] and Gaussian lower bounds on p_h [51, Thm. 4.5] in the last step. Moreover, we allow the constant $C > 0$, which does not depend on x, z or t, to change from line to line. These estimates readily yield the claim. $\qquad\square$

Remark 4.17. Proposition 4.16 is a variant of [18, Lem. 2.33]. However, here (M, g) is not assumed to be of Harnack-type, hence to admit Gaussian upper and lower bounds on p. Instead, we use this property on a concrete bi-Lipschitz image of M.

Now let $Y \subset M$ be an open subset having *weakly Lipschitz* boundary, i.e. there exist some $c > 0$, a covering $U_1, \ldots, U_k$, $k \in \mathbf{N}$, of ∂Y by open subsets of M, and c-Lipschitz maps $\varphi_i\colon U_i \to \mathbf{R}^{d-1}$ such that

$$(\varphi_i)_\sharp \sigma_Y \leq c \mathscr{L}^{d-1} \quad \text{on } \varphi_i(U_i \cap \partial Y)$$

for every $i \in \{1, \ldots, k\}$. Here σ_Y is the surface measure of ∂Y.

Example 4.18. A simple framework in which the above hypotheses hold is when $Y = M$, under the assumption of compactness of ∂N (hence ∂M).

Moreover, every Lipschitz domain in $\mathbf{R}^d$ with, say, compact boundary has weakly Lipschitz boundary.

Theorem 4.19. *Retain the previous assumptions and notations, and suppose that $\ell \in L^q(\partial Y, \sigma_Y)$ for some $q \in (d - 1, \infty)$. Then $\kappa := \ell \, \sigma_Y$ belongs to $\mathcal{K}(M)$.*

Proof. We first claim that

$$\limsup_{r \to 0} \sup_{x \in M} \int_{\mathsf{B}_r^g(x)} \mathsf{d}_g^{2-d}(x, \cdot) |\ell| \, \mathrm{d}\sigma_Y = 0. \tag{4.7}$$

Indeed, given any $x \in M$, for $p \in (1, \infty)$ satisfying $1/p + 1/q = 1$ we have

$$\int_{\mathsf{B}_r^g(x)} \mathsf{d}_g^{2-d}(x, \cdot) |\ell| \, \mathrm{d}\sigma_Y \leq \|\ell\|_{L^p(\partial Y, \sigma)} \left[\int_{\mathsf{B}_r^g(x) \cap U} \mathsf{d}_g^{q(2-d)}(x, \cdot) \, \mathrm{d}\sigma_Y \right]^{1/q},$$

where $U := U_1 \cup \cdots \cup U_k$. For every $i \in \{1, \ldots, k\}$, we have

$$\int_{\mathsf{B}_r^g(x) \cap U_i} \mathsf{d}_g^{q(2-d)}(x, \cdot) \, \mathrm{d}\sigma_Y \leq c^{q(d-2)} \int_{V_i} |\varphi_i - \varphi_i(x)|^{q(2-d)} \, \mathrm{d}\sigma_Y$$

$$\leq c^{q(d-2)+1} \int_{B_{cr}(0)} |\cdot|^{q(2-d)} \, \mathrm{d}\mathscr{L}^{d-1}.$$

Here we write $V_i := |\varphi_i - \varphi_i(x)|^{-1}([0, cr)) \cap U_i$, and $B_{cr}(0)$ is intended as open ball in $\mathbf{R}^{d-1}$. As $r \to 0$, the last integral tends to zero precisely by our choice of q. This readily proves the claim (4.7).

Now we finally argue that (4.7) implies $\ell \sigma_Y \in \mathcal{K}(M)$. Indeed, thanks to (4.7), Lemma 2.8 and Remark 2.9,

$$\limsup_{r \to 0} \sup_{z \in N} \int_{\mathsf{B}_r^h(z)} \mathsf{d}_h^{2-d}(z, \cdot) |\ell| \circ F^{-1} \, \mathrm{d}F_\sharp \sigma_Y = 0.$$

By our assumptions on (N, h), the corresponding considerations in [34, Ex. 6.8] and [34, Thm. 3.1], the previous property implies — in fact, is equivalent to — the statement that $[|\ell| \circ F^{-1}] \, F_\sharp \sigma_Y \in \mathcal{K}(N)$ according to Definition 4.1. The assertion follows from Proposition 4.16. □

Remark 4.20. Theorem 1.1 is a variant of [18, Thm. 2.36]. Therein, however, M is supposed to be smooth, and the equivalence of (4.7) with the condition that $\ell \sigma_Y \in \mathcal{K}(M)$ is part of the *assumption* as well. In our setting, we do not know if this equivalence holds (unless p_g satisfies Gaussian lower bounds). Again, instead, we use this correspondence on a concrete bi-Lipschitz image of M.

4.3.2. *Taming of inner uniform domains.* Finally, we study taming of sets Y as above, endowed with the canonical Dirichlet structure $(\mathscr{E}_Y, W^{1,2}(Y))$ inherited by the LR structure of $(Y, g|_Y)$, as outlined in Subsection 2.5.2. In particular, all objects from Chapter 2 associated to $(Y, g|_Y)$ will be assigned (in a notationally evident way) with a sub- or superscript Y.

Partly following the terminology proposed in [27], in addition to the assumptions on Y from Subsection 4.3.1, we make the following hypotheses.

    a. *Inner uniformity.* There exist some constants $c, C > 0$ such that any two points $x, y \in M$ can be connected by a continuous curve $\gamma \colon [0, 1] \to Y$ with length at most $C \mathsf{d}_Y(x, y)$ with the property that for every $z \in \gamma([0, 1])$,

$$\mathsf{d}(z, \partial M) \geq c \min\{\mathsf{d}_Y(z, x), \mathsf{d}_Y(z, y)\}.$$

    b. *Tubular volume.* We have

$$\inf\left\{ \frac{\mathfrak{v}[\mathsf{B}_r(y) \cap Y]}{\mathfrak{v}[\mathsf{B}_r(y)]} : r > 0, \, y \in Y \right\} > 0$$

    c. *Regular exhaustion.* There exists an increasing sequence $(Y_n)_{n \in \mathbf{N}}$ of subsets of Y with the following properties.
- Every Y_n, $n \in \mathbf{N}$, has a smooth structure and smooth boundary such that $g|_{Y_n}$ is smooth.
- The sequence $(\overline{Y}_n)_{n \in \mathbf{N}}$ is an $\mathscr{E}_Y$-nest consisting of compact sets.

- For every compact $K \subset Y$ there exists $N \in \mathbf{N}$ such that $K \subset Y_n$ for every $n \geq N$.

d. *Curvature bounds.* There exist $\mathscr{k} \in L^p(Y, \mathfrak{v}_Y[\mathsf{B}_1^Y(\cdot)]^{-1}\mathfrak{v}_Y) + L^\infty(Y, \mathfrak{v}_Y)$, $p \in (d/2, \infty)$, and $\ell \in L^q(\partial Y, \sigma_Y)$, $q \in (d-1, \infty)$, as well as a sequence $(\ell_n)_{n\in\mathbf{N}}$ of functions $\ell_n \colon \partial Y_n \to \mathbf{R}$, $n \in \mathbf{N}$, such that for every $n \in \mathbf{N}$,

$$\mathrm{Ric}_{Y_n} \geq \mathscr{k} \quad \text{on } Y_n,$$
$$\ell_n = \ell \quad \text{on } \partial Y_n \cap \partial Y,$$
$$\ell_n \geq 0 \quad \text{on } \partial Y_n \setminus \partial Y.$$

Theorem 4.21. *Under the foregoing hypotheses, $(Y, g\big|_Y)$ is tamed by*

$$\kappa := \mathscr{k}\,\mathfrak{v}_Y + \ell\,\sigma_Y.$$

Proof. First, we claim that $\mathscr{k}\,\mathfrak{v}_Y \in \mathcal{K}(Y)$, which will imply that $\kappa \in \mathcal{K}(Y)$ thanks to Theorem 4.19. Indeed, by a. and b. (which grant that distances and volumes of balls in Y are comparable with those in M, cf. also [18, Lem. 2.34]) as well as the considerations from Example 3.18, the assumptions of Theorem 3.16 are satisfied for $(Y, g\big|_Y)$, and the claim follows from Theorem 4.2.

We set $\ell_n^* := \ell_n\,1_{\partial Y_n \cap \partial Y}$, where $n \in \mathbf{N}$. Thanks to d. and Theorem 4.19, we obtain $\ell_n^*\,\sigma_{Y_n} \in \mathcal{K}(Y_n)$. Furthermore, the AF associated with $\mathscr{k}^-\,\mathfrak{v}_{Y_n} + (\ell_n^*)^-\,\sigma_{Y_n}$ is no larger than the AF corresponding to $\mathscr{k}^-\,\mathfrak{v}_Y + \ell^-\,\sigma_Y$ up to the first hitting time of $\partial Y_n \setminus \partial Y$. By Khasminskii's lemma [18, Lem. 2.24], this implies that the family $(\kappa_n)_{n\in\mathbf{N}}$ given by $\kappa_n := \mathscr{k}\,\mathfrak{v}_{Y_n} + \ell_n\,\sigma_{Y_n} \in \mathcal{K}(Y_n)$ is uniformly 1- and 2-moderate according to [18, Thm. 4.5]. Together with c., the same result ensures the taming condition for $(Y, g\big|_Y)$ and hence the claim. $\qquad\square$

References

[1] L. Ambrosio, N. Gigli, G. Savaré. *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds.* Ann. Probab. **43** (2015), no. 1, 339–404.

[2] M. Aizenman, B. Simon. *Brownian motion and Harnack inequality for Schrödinger operators.* Comm. Pure Appl. Math. **35** (1982), no. 2, 209–273.

[3] D. Bakry, M. Émery. *Diffusions hypercontractives.* Séminaire de probabilités, XIX, 1983/84, 177–206, Lecture Notes in Math., 1123, Springer, Berlin, 1985.

[4] N. Bouleau, F. Hirsch. *Dirichlet forms and analysis on Wiener space.* De Gruyter Studies in Mathematics, 14. Walter de Gruyter & Co., Berlin, 1991. x+325 pp.

[5] M. Braun. *Vector calculus for tamed Dirichlet spaces.* Preprint, arXiv:2108.12374, 2021.

[6] M. Braun, B. Güneysu. *Heat flow regularity, Bismut–Elworthy–Li's derivative formula, and pathwise couplings on Riemannian manifolds with Kato bounded Ricci curvature.* Electron. J. Probab. **26** (2021), article no. 1, 1–25.

[7] G. Carron. *Geometric inequalities for manifolds with Ricci curvature in the Kato class.* Ann. Inst. Fourier (Grenoble) **69** (2019), no. 7, 3095–3167.

[8] G. Carron, I. Mondello, D. Tewodrose. *Limits of manifolds with a Kato bound on the Ricci curvature.* Preprint, arXiv:2102.05940, 2021.

[9] G. Carron, C. Rose. *Geometric and spectral estimates based on spectral Ricci curvature assumptions.* J. Reine Angew. Math. **772** (2021), 121–145.

[10] Z.-Q. Chen, M. Fukushima. *Symmetric Markov processes, time change, and boundary theory.* London Mathematical Society Monographs Series, 35. Princeton University Press, Princeton, NJ, 2012. xvi+479 pp.

[11] T. Coulhon, A. A. Grigor'yan. *On-diagonal lower bounds for heat kernels and Markov chains.* Duke Math. J. **89** (1997), no. 1, 133–199.

[12] T. Coulhon, A. Sikora. *Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem.* Proc. Lond. Math. Soc. (3) **96** (2008), no. 2, 507–544.

[13] E. B. Davies. *Heat kernels and spectral theory.* Cambridge Tracts in Mathematics, 92. Cambridge University Press, Cambridge, 1990. x+197 pp.

[14] G. De Cecco, G. Palmieri. *Distanza intrinseca su una varietà Riemanniana di Lipschitz.* Rend. Sem. Mat. Univ. Politec. Torino **46** (1988), no. 2, 157–170 (1990).

[15] ———. *Length of curves on Lip manifolds.* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **1** (1990), no. 3, 215–221.

[16] _____ . *Integral distance on a Lipschitz Riemannian manifold.* Math. Z. **207** (1991), no. 2, 223–243.

[17] S. K. DONALDSON, D. P. SULLIVAN. *Quasiconformal 4-manifolds.* Acta Math. **163** (1989), no. 3–4, 181–252.

[18] M. ERBAR, C. RIGONI, K.-T. STURM, L. TAMANINI. *Tamed spaces — Dirichlet spaces with distribution-valued Ricci bounds.* Preprint, arXiv:2009.03121, 2020.

[19] V. M. GOL'DSHTEIN, V. I. KUZ'MINOV, I. A. SHVEDOV. *Differential forms on a Lipschitz manifold.* Sibirsk. Mat. Zh. **23**(1982), no. 2, 16–30, 215; translation in Siberian Math. J. **23** (1982), no. 2, 151–161.

[20] V. GOL'DSHTEIN, I. MITREA, M. MITREA. *Hodge decompositions with mixed boundary conditions and applications to partial differential equations on Lipschitz manifolds.* Problems in mathematical analysis. No. 52. J. Math. Sci. (N.Y.) **172** (2011), no. 3, 347–400.

[21] A. A. GRIGOR'YAN. *The heat equation on noncompact Riemannian manifolds.* Mat. Sb. **182** (1991), no. 1, 55–87; translation in Math. USSR-Sb. **72** (1992), no. 1, 47–77.

[22] _____ . *Gaussian upper bounds for the heat kernel on arbitrary manifolds.* J. Differential Geom. **45** (1997), no. 1, 33–52.

[23] _____ . *Heat kernel and analysis on manifolds.* AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009. xviii+482 pp.

[24] B. GÜNEYSU. *Covariant Schrödinger semigroups on Riemannian manifolds.* Operator Theory: Advances and Applications, 264. Birkhäuser/Springer, Cham, 2017. xviii+239 pp.

[25] _____ . *Heat kernels in the context of Kato potentials on arbitrary manifolds.* Potential Anal. **46** (2017), no. 1, 119–134.

[26] B. GÜNEYSU, M. VON RENESSE. *Molecules as metric measure spaces with Kato-bounded Ricci curvature.* C. R. Math. Acad. Sci. Paris **358** (2020), no. 5, 595–602.

[27] P. GYRYA, L. SALOFF-COSTE. *Neumann and Dirichlet heat kernels in inner uniform domains.* Astérisque No. **336** (2011), viii+144 pp.

[28] B.-X. HAN. *Measure rigidity of synthetic lower Ricci curvature bound on Riemannian manifolds.* Adv. Math. **373** (2020), 107327, 31 pp.

[29] J. HEINONEN, S. KEITH. *Flat forms, bi-Lipschitz parameterizations, and smoothability of manifolds.* Publ. Math. Inst. Hautes Études Sci. No. **113** (2011), 1–37.

[30] H. HESS, R. SCHRADER, D. A. UHLENBROCK. *Kato's inequality and the spectral distribution of Laplacians on compact Riemannian manifolds.* J. Differential Geometry **15** (1980), no. 1, 27–37 (1981).

[31] M. HILSUM. *Signature operator on Lipschitz manifolds and unbounded Kasparov bimodules.* Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), 254–288, Lecture Notes in Math., 1132, Springer, Berlin, 1985.

[32] T. KATO. *Schrödinger operators with singular potentials.* Israel J. Math. **13** (1972), 135–148 (1973).

[33] M. A. KERVAIRE. *A manifold which does not admit any differentiable structure.* Comment. Math. Helv. **34** (1960), 257–270.

[34] K. KUWAE, T. MORI. L^p-*Kato class measures for symmetric Markov processes under heat kernel estimates.* Preprint, arXiv:2008.10934, 2020.

[35] P. LI, J. WANG. *Mean value inequalities.* Indiana Univ. Math. J. **48** (1999), no. 4, 1257–1283.

[36] J. LOTT. *Eigenvalue estimates and differential form Laplacians on Alexandrov spaces.* Math. Ann. **371** (2018), no. 3–4, 1737–1767.

[37] J. LUUKKAINEN, J. VÄISÄLÄ. *Elements of Lipschitz topology.* Ann. Acad. Sci. Fenn. Ser. A I Math. **3** (1977), no. 1, 85–122.

[38] J. MASAMUNE. *Essential self-adjointness of Laplacians on Riemannian manifolds with fractal boundary.* Comm. Partial Differential Equations **24** (1999), no. 3-4, 749–757.

[39] E. E. MOISE. *Geometric topology in dimensions 2 and 3.* Graduate Texts in Mathematics, Vol. 47. Springer-Verlag, New York-Heidelberg, 1977. x+262 pp.

[40] J. R. NORRIS. *Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds.* Acta Math. **179** (1997), no. 1, 79–103.

[41] G. PERELMAN. *DC structure on Alexandrov space.* Preprint, `http://www.math.psu.edu/petrunin/papers/alexandrov/Cstructure.pdf`.

[42] D. PRANDI, L. RIZZI, M. SERI. *Quantum confinement on non-complete Riemannian manifolds.* J. Spectr. Theory **8** (2018), no. 4, 1221–1280.

[43] C. ROSE. *Li-Yau gradient estimate for compact manifolds with negative part of Ricci curvature in the Kato class.* Ann. Global Anal. Geom. **55** (2019), no. 3, 443–449.

[44] J. ROSENBERG. *Applications of analysis on Lipschitz manifolds.* Miniconferences on harmonic analysis and operator algebras (Canberra, 1987), 269–283, Proc. Centre Math. Anal. Austral. Nat. Univ., 16, Austral. Nat. Univ., Canberra, 1988.

[45] L. SALOFF-COSTE. *Uniformly elliptic operators on Riemannian manifolds.* J. Differential Geom. **36** (1992), no. 2, 417–450.

[46] _____. *Aspects of Sobolev-type inequalities.* London Mathematical Society Lecture Note Series, 289. Cambridge University Press, Cambridge, 2002. x+190 pp.

[47] _____. *The heat kernel and its estimates.* Probabilistic approach to geometry, 405–436, Adv. Stud. Pure Math., 57, Math. Soc. Japan, Tokyo, 2010.

[48] D. SULLIVAN. *Hyperbolic geometry and homeomorphisms.* Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), pp. 543–555, Academic Press, New York-London, 1979.

[49] P. STOLLMANN, J. VOIGT. *Perturbation of Dirichlet forms by measures.* Potential Anal. **5** (1996), no. 2, 109–138.

[50] R. S. STRICHARTZ. *Analysis of the Laplacian on the complete Riemannian manifold.* J. Funct. Anal. **52** (1983), no. 1, 48–79.

[51] K.-T. STURM. *Heat kernel bounds on manifolds.* Math. Ann. **292** (1992), no. 1, 149–162.

[52] _____. *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations.* Osaka J. Math. **32** (1995), no. 2, 275–312.

[53] _____. *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality.* J. Math. Pures Appl. (9) **75** (1996), no. 3, 273–297.

[54] _____. *Is a diffusion process determined by its intrinsic metric?* Chaos Solitons Fractals **8** (1997), no. 11, 1855–1860.

[55] _____. *On the geometry of metric measure spaces. II.* Acta Math. **196** (2006), no. 1, 133–177.

[56] _____. *Distribution-valued Ricci bounds for metric measure spaces, singular time changes, and gradient estimates for Neumann heat flows.* Geom. Funct. Anal. **30** (2020), no. 6, 1648–1711.

[57] N. TELEMAN. *The index of signature operators on Lipschitz manifolds.* Inst. Hautes Études Sci. Publ. Math. No. 58 (1983), 39–78 (1984).

[58] C. VILLANI. *Optimal transport. Old and new.* Grundlehren der mathematischen Wissenschaften, 338. Springer-Verlag, Berlin, 2009. xxii+973 pp.

[59] J. H. C. WHITEHEAD. *Manifolds with transverse fields in euclidean space.* Ann. of Math. (2) **73** (1961), 154–212.

[60] H. WHITNEY. *Geometric integration theory.* Princeton University Press, Princeton, N. J., 1957. xv+387 pp.

UNIVERSITY OF BONN, INSTITUTE FOR APPLIED MATHEMATICS, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

Email address: `braun@iam.uni-bonn.de`

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN, AUSTRIA

Email address: `chiara.rigoni@univie.ac.at`