Euler equations and trace properties of minimizers of a functional for motion compensated inpainting

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Abstract

We compute the Euler equations of a functional useful for simultaneous video inpainting and motion estimation, which was obtained in [17] as the relaxation of a modified version of the functional proposed in [16]. The functional is defined on vectorial functions of bounded variations, therefore we also get the Euler equations holding on the singular sets of minimizers, highlighting in particular the conditions on the jump sets. Such conditions are expressed by means of traces of geometrically meaningful vector fields and characterized as pointwise limits of averages on cylinders with axes parallel to the unit normals to the jump sets.

1 Introduction

In [16] Lauze and Nielsen proposed a variational model for motion compensated video inpainting. This model was also applied to video deinterlacing [14] and video super-resolution [15]. In [17] the authors modified the model in order to get better variational properties, moreover they computed the relaxation of the modified functional and its domain, in such a way to recover existence of minimizers by resorting to the Direct Methods of Calculus of Variations. Other properties of the model were studied in [18].

Given $\Omega_s \subset \mathbb{R}^2$ (which represents the spatial domain of a video sequence and which is assumed bounded, open, connected and with Lipschitz boundary) and $[0,T]$ a temporal domain, we set $\Omega = \Omega_s \times [0,T]$ and (x,t) denotes a point belonging to Ω. Let $D \subset \Omega$ denote a known spatiotemporal region where the video data are lost. We assume that D is open, that both D and $\Omega \setminus \overline{D}$ have Lipschitz boundary, and that $\partial D \cap \Omega$ has positive surface measure. Then a degraded video datum can be represented by means of a function $f \in L^1(\Omega \setminus D)$.

The problem of *video inpainting* consists in looking for a restored video content u defined on Ω, matching f outside D and satisfying in D spatial piecewise smoothness. In *motion compensated video inpainting* it is also required that u has in D coherence (in some suitable sense) with the apparent motion of the video data. This can be estimated through gray-value variations of f in $\Omega \setminus D$ and it is represented by a vector field of velocities $\sigma : \Omega \to \mathbb{R}^2$ called *optical flow*. The field σ is an approximation of the two-dimensional *motion field*, which is the projection onto the image plane of the three-dimensional vector field of velocities of objects moving in the scene (see [8], Section 5.3.2). In order to get the simultaneous estimation inside D of both the gray-value video u and the optical flow field σ, the variational approach consists in minimizing a suitable energy functional depending on u, σ and on an approximation of the spatial gradient of u. In Sections 3 and 5 we recall the definition of

the functional proposed in [17] and its relaxation, which is defined on functions of bounded variation ("BV" for short). Differently from Sobolev spaces, BV functions may have discontinuities along hypersurfaces of codimension one and therefore they can be seen as a natural framework for a model of image processing, where discontinuities can represent the contours of objects in a visual scene. In the present paper we compute the Euler equations of the relaxed functional and we will pay attention to the contribute of the singular sets, which involves the jump discontinuity sets. Some basic definitions and properties of BV functions are quoted in Section 4. We exploit some tools developed by Anzellotti in [4] and [5] for functionals with linear growth defined on scalar functions, like for instance the Total Variation or the Area functional. In our case the functional has a much more complicated form and, therefore, more technical steps are needed. Moreover, our functional is defined on vectorial BV functions and, in order to handle the singular part of distributional derivatives, we have to resort to the Rank one Theorem due to Alberti [1]. Roughly speaking, this theorem permits us to perform also for the Cantor part a suitable computation, similar to that we would have in the simpler case where the singular set is made only of the jump set. In particular, we first compute the directional derivatives of the relaxed functional, then, using a suitable integration by parts formula (consequence of the Divergence Theorem) holding in BV, we are able to get extremal conditions, which involve: i) partial differential equations (in divergence form) for the absolutely continuous part; ii) conditions on the boundary; iii) conditions on the singular sets expressed by means of traces of geometrically meaningful vector fields and characterized as pointwise limits of averages on cylinders having a suitable orientation.
Typically, in the image processing literature the numerical schemes (used to get minimizers) concentrate on the equations obtained for the absolutely continuous part. Though the conditions on the singular sets are difficult to handle from the numerical point of view, nevertheless they can highlight properties about the behavior of minimizers.

2 Notation

For n integer, $\mathcal{L}^n$ is the Lebesgue n-dimensional measure in $\mathbb{R}^n$, and $\mathcal{H}^{n-1}$ is the Hausdorff $n-1$-dimensional measure in $\mathbb{R}^n$. We denote by $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ the canonical basis of $\mathbb{R}^n$, and we denote by $\mathbb{M}^{m\times n}$ the space of the $m \times n$ real matrices. If $M \in \mathbb{M}^{m\times n}$, we denote M^t the transpose matrix and $\mathrm{Tr}(M)$ the trace of M. The symbols $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote both the Euclidean scalar product and norm of vectors, and the Frobenius scalar product and norm of matrices. If $a, b \in \mathbb{R}^n$ with $n > 1$ we denote by $a \otimes b$ their tensor product, that means the matrix whose entries are $(a \otimes b)_{ij} = a_i b_j$.
If $\mathcal{O}_1$ and $\mathcal{O}_2$ are bounded subsets of $\mathbb{R}^n$ and $\mathcal{O}_2$ is open, by $\mathcal{O}_1 \subset\subset \mathcal{O}_2$ we mean that $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$. For $\rho > 0$ and $y \in \mathbb{R}^n$, we set $B_\rho(y) = \{z \in \mathbb{R}^n : |z - y| < \rho\}$ and $\omega_n = \mathcal{L}^n(B_1(0))$. For $\mathcal{O} \subset \mathbb{R}^n$ we denote by $1_{\mathcal{O}}$ the characteristic function of $\mathcal{O}$, i.e., $1_{\mathcal{O}}(y) = 1$ if $y \in \mathcal{O}$ and $1_{\mathcal{O}}(y) = 0$ if $y \notin \mathcal{O}$.
Let $\mathcal{O} \subset \mathbb{R}^n$ be open, and let $\mathcal{B}(\mathcal{O})$ be the σ-algebra of Borel subsets of $\mathcal{O}$. Given a measure $\mu : \mathcal{B}(\mathcal{O}) \to \mathbb{R}^k$, we consider its Lebesgue decomposition $\mu = \mu^a \cdot \mathcal{L}^n + \mu^s$, where μ^a is the density of μ with respect to $\mathcal{L}^n$ and μ^s is the singular part of μ with respect to $\mathcal{L}^n$. If μ is a real or vector valued measure and ν is a positive measure, then $d\mu/d\nu$ denotes the corresponding Radon-Nikodym derivative; if μ is absolutely continuous with respect to ν we write $\mu \ll \nu$. We denote by $|\mu|$ the total variation of μ. If $\mathcal{O}_1 \in \mathcal{B}(\mathcal{O})$ we set $\mu\lfloor\mathcal{O}_1(\mathcal{O}_2) = \mu(\mathcal{O}_1 \cap \mathcal{O}_2)$ for every set $\mathcal{O}_2 \in \mathcal{B}(\mathcal{O})$.

3 The model

In this section we describe the variational model proposed by the authors in [17], and we recall the definition of the functional to be minimized. Let Ω_s, Ω, D, f be as in Section 1.
We denote by $v : \Omega \to \mathbb{R}^2$ a vector field which approximates the spatial gradient $\nabla_x u$. We set

$$u \in W^{1,1}(\Omega), \qquad v = (v_1, v_2) \in [W^{1,1}(\Omega)]^2, \qquad \sigma = (\sigma_1, \sigma_2) \in [W^{1,1}(\Omega)]^2,$$

and $w = (u, v, \sigma) \in V(\Omega)$, with $V(\Omega) = [W^{1,1}(\Omega)]^5$, where $W^{1,1}$ denotes the Sobolev space of L^1 functions having L^1 distributional derivatives. For a given $\rho > 0$ we define

$$\sigma_\rho(y) = \frac{1}{\mathcal{L}^3(\Omega \cap B_\rho(y))} \int_{\Omega \cap B_\rho(y)} \sigma(z) d\mathcal{L}^3, \qquad \text{for any } y \in \overline{\Omega}, \tag{1}$$

where $B_\rho(y) = \{z \in \mathbb{R}^3 : |z - y| < \rho\}$. We set $\Sigma_\rho = (\sigma_{1\rho}, \sigma_{2\rho}, 1)$ and we assume that ρ is a fixed small parameter.
Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be differentiable for all $t > 0$, nondecreasing and such that, for every $k \in \mathbb{N}$, the function $\psi : \mathbb{R}^k \to \mathbb{R}$ defined by means of $\psi(\xi) = \varphi(|\xi|^2)$ is convex, Lipschitz, and has linear growth in ξ for large $|\xi|$ (precise growth conditions for ψ are given in Section 4.3).
Such properties are satisfied for instance if $\varphi(t) = \sqrt{t + \varepsilon}$ with $\varepsilon > 0$. Another example of function φ useful in applications is the following:

$$\varphi(t) = \begin{cases} \frac{1}{2} t & \text{if } t \in [0, 1], \\ \\ \sqrt{t} - \frac{1}{2} & \text{if } t \in (1, +\infty). \end{cases}$$

The variational problem consists in minimizing a functional E depending on the vector w of functions $w = (u, v, \sigma) \in V(\Omega)$ which consists of three terms:

$$E(w) = F(u, v) + G(w) + P(\sigma). \tag{2}$$

The functional F is defined by

$$F(u, v) = \int_{\Omega \setminus D} \varphi(|f - u|^2) d\mathcal{L}^3 + \int_\Omega \varphi(|\nabla_x u - v|^2) d\mathcal{L}^3 + \int_\Omega \varphi(|\nabla_x v|^2) d\mathcal{L}^3,$$

where ∇_x is the *spatial* gradient operator. The first term in F enforces the unknown video u to approximate the datum f in the set $\Omega \setminus D$ where data are available. The second and third term are spatial regularization functionals which enforce the vector field v to approximate the spatial gradient $\nabla_x u$, and enforce spatial piecewise smoothness both of the reconstructed video u and of the vector field v. The linear growth of the function φ allows the presence of discontinuities in the function u which represent the boundaries of moving objects in the video sequence.
The functional G is defined by

$$G(w) = \int_\Omega \varphi\left(\langle \nabla u, \Sigma_\rho \rangle^2 + |(\nabla v)\Sigma_\rho|^2\right) d\mathcal{L}^3,$$

where ∇ is the spatiotemporal gradient. Since σ represents the vector field of velocities of the visible surfaces of moving objects in the video [8], $\sigma = (dx_1/dt, dx_2/dt)$, setting $\Sigma = (\sigma_1, \sigma_2, 1)$ we have

$$\langle \nabla u, \Sigma \rangle = \frac{\partial u}{\partial t} + \langle \nabla_x u, \sigma \rangle = \frac{du}{dt}.$$

Now the requirement that the gray-value intensity u of the video is constant along apparent motion trajectories, at least for a short duration, yields the *optical flow constraint*

$$\frac{du}{dt} = 0,$$

which was proposed by Horn and Schunck [13] to estimate the optical flow σ. Moreover, we have for the matrix by vector product

$$(\nabla v)\Sigma = \frac{\partial v}{\partial t} + (\nabla_x v)\sigma = \frac{dv}{dt}.$$

An extension of the optical flow constraint to the constancy of the spatial gradient $\nabla_x u$ was proposed in [19, 9]. Then, after the replacement of Σ with Σ_ρ for small ρ, the term G penalizes global deviations from the gray-value constancy constraint (optical flow constraint) and from the constancy constraint of the vector field v which approximates the spatial gradient $\nabla_x u$. This term then enforces coherence corresponding to apparent motion between frames of the reconstructed video u (for almost any $t \in [0,T]$ the function $x \to u(x,t)$ is a frame of the video u).
Eventually, the functional P is defined by

$$P(\sigma) = \int_\Omega \varphi(|\nabla\sigma|^2) d\mathcal{L}^3 + c\int_\Omega \varphi(|\sigma|^2) d\mathcal{L}^3,$$

where c is a small positive constant. The integrals in P are, respectively, a spatiotemporal regularization term which enforces the piecewise smoothness of the optical flow σ, and a term which helps to make the functional coercive.
Further details of the variational model can be found in [17]. The function space $W^{1,1}(\Omega)$ is not a reflexive Banach space, hence, in order to achieve information about minimizing sequences that are bounded in $V(\Omega)$, we have to resort to the relaxed functional of E (for details about the relaxation method see for instance [10]). The representation of the relaxed functional in a space of vector valued BV functions has been found in [17] and it will be recalled in Section 5. We remark that the average of the optical flow σ on a ball with fixed radius ρ permits us to obtain a representation formula for the relaxed functional of G such that the density of the jump part of the energy can be explicitly computed (see the end of Section 5 and see [17] for more information).

4 Preliminary results

4.1 Functions of bounded variation

For $\mathcal{O} \subset \mathbb{R}^n$ open, we denote by $BV(\mathcal{O})$ the space of scalar functions of bounded variation in $\mathcal{O}$, i.e., the functions $u \in L^1(\mathcal{O})$ such that the distributional gradient of u is representable as a measure $Du : \mathcal{B}(\mathcal{O}) \to \mathbb{R}^n$ with finite total variation.
For any $u \in BV(\mathcal{O})$ and $y \in \mathcal{O}$, we denote by $u^-(y), u^+(y)$ the approximate lower and upper limit of u at the point y, which satisfy $u^-(y) \le u^+(y)$. We set

$$S_u = \left\{ y \in \mathcal{O} :\, u^-(y) < u^+(y) \right\}.$$

The set S_u will be considered as the discontinuity set of u. We denote by N_u the density of the measure Du with respect to $|Du|$, namely

$$N_u = \frac{dDu}{d|Du|}. \tag{3}$$

If $u \in BV(\mathcal{O})$, then the Lebesgue decomposition of Du is given by

$$Du = D^a u + D^s u, \qquad D^a u = \nabla u \cdot \mathcal{L}^n, \qquad D^s u = Ju + Cu,$$

where $D^a u$ is the absolutely continuous part of Du with respect to $\mathcal{L}^n$, with density denoted by $\nabla u = dD^a u/d\mathcal{L}^n$ (which coincides a.e. with the gradient of u defined in the sense of approximate limits [2]), $D^s u$ is the singular part of Du, Ju and Cu are the *jump part* and the *Cantor part* of $D^s u$, respectively. We set

$$N_u^s = \frac{dD^s u}{d|D^s u|}, \qquad N_u^J = \frac{dJu}{d|Ju|}, \qquad N_u^C = \frac{dCu}{d|Cu|}. \tag{4}$$

By the properties of BV functions [2], there exists a set $\mathcal{K}_u$ contained in the support of the measure $D^s u$, having $\mathcal{L}^n(\mathcal{K}_u) = 0$, such that $D^s u$ is concentrated on $\mathcal{K}_u$ and $\mathcal{K}_u = S_u \cup \mathcal{C}_u$, with the sets S_u

and $\mathcal{C}_u$ disjoint and the measure Cu concentrated on $\mathcal{C}_u$. For $\mathcal{H}^{n-1}$-a.e. $y \in S_u$ a normal unit vector $\nu_u(y)$ can be defined such that

$$\lim_{\rho\to 0} \frac{1}{|B_\rho(y) \cap \{z \in \mathbb{R}^n : \langle z-y, \nu_u(y)\rangle \gtrless 0\}|} \int_{B_\rho(y)\cap\{\langle z-y,\nu_u(y)\rangle \gtrless 0\}} |u(y) - u^\pm(y)| d\mathcal{L}^n = 0,$$

and it coincides with $N_u^J(y)$. Moreover there holds

$$Ju(\hat{\mathcal{O}}) = \int_{S_u \cap \hat{\mathcal{O}}} (u^+ - u^-)\nu_u \, d\mathcal{H}^{n-1},$$

with $\hat{\mathcal{O}} \in \mathcal{B}(\mathcal{O})$.
If $U = (U_1, \ldots, U_m) \in [BV(\mathcal{O})]^m$, we denote by DU the $m \times n$ matrix valued measure whose rows are DU_i for $i = 1, \ldots, m$, and we set $S_U = S_{U_1} \cup \cdots \cup S_{U_m}$. We use the same notation of the scalar case in (3) for the density

$$N_U = \frac{dDU}{d|DU|} : \mathcal{O} \to \mathbb{M}^{m\times n},$$

and the same notation as in (4) for the related singular, jump and Cantor part N_U^s, N_U^J, N_U^C. We set $\mathcal{C}_U = \mathcal{C}_{U_1} \cup \cdots \cup \mathcal{C}_{U_m}$ and $\mathcal{K}_U = S_U \cup \mathcal{C}_U$.
In [1] Alberti proved the following rank one property of the singular parts of distributional derivatives of vector valued BV functions. By Corollary 4.6 in [1] there exist $\alpha : \mathcal{O} \to S^{m-1}$ and $\beta : \mathcal{O} \to S^{n-1}$ such that

$$N_U^s(y) = \alpha(y) \otimes \beta(y) \qquad \text{for } |D^sU| - \text{a.e. } y \in \mathcal{O}. \tag{5}$$

Remark 4.1. *For $\mathcal{H}^{n-1}$-a.e. $y \in S_U$ we have that $\alpha = \frac{U^+(y)-U^-(y)}{|U^+(y)-U^-(y)|}$ and β coincides with the normal unit vector on S_U. Therefore, in the sequel we will denote by ν_U the vector β, when we are on S_U.*

Corollary 4.2. *Remark 4.1 implies that, for $i \neq k$, if $S_{U_i} \cap S_{U_k} \neq \emptyset$, on the intersection $\nu_{U_i} \equiv \nu_{U_k}$.*

Remark 4.3. *In particular Corollary 4.6 in [1] allows to state a property analogous to Corollary 4.2 also for CU, the Cantor part of D^sU. In fact, since the measures CU and JU are mutually singular, by (5) we infer:*

$$(CU)_{i,j} = \alpha_i \beta_j |CU|, \tag{6}$$

hence

$$CU_i = \alpha_i |CU| \beta. \tag{7}$$

Moreover, we know

$$CU_i = N_{U_i}^C |CU_i| = N_{U_i}^C \left| \frac{dCU_i}{d|CU|} \right| \cdot |CU|, \tag{8}$$

so that (7) and (8) imply

$$N_{U_i}^C \left| \frac{dCU_i}{d|CU|} \right| = \alpha_i \beta. \tag{9}$$

This holds for every $i = 1, \ldots, m$, therefore all the vectors $N_{U_i}^C$ are parallel whenever they are not null.

Using Remark 4.1 and Remark 4.3, when we are on $\mathcal{K}_U$ we write

$$\beta = \nu_U^s. \tag{10}$$

If $\mathcal{O} = \Omega \subset \mathbb{R}^3$ (the spatiotemporal domain of a video sequence), $u \in BV(\Omega)$ and $y = (x,t) \in \Omega$, then we denote by $D_x u : \mathcal{B}(\Omega) \to \mathbb{R}^2$ the measure whose components are the distributional derivatives of u with respect to the spatial coordinates. We denote by $\nabla_x u$ the density of the absolutely continuous part of $D_x u$ with respect to $\mathcal{L}^3$, we denote by $D_x^s u$ the singular part of $D_x u$, we denote

by $N_{u,x}$, $N^s_{u,x}$, $N^J_{u,x}$, $N^C_{u,x}$ the corresponding Radon-Nikodym derivatives, and we denote by $\nu_{u,x}$ the orthogonal projection of ν_u on the spatial subset of space-time $\mathbb{R}^3$. Moreover, $J_x u$ and $C_x u$ are the jump part and the Cantor part of $D^s_x u$, respectively. We use analogous notations for the distributional derivative of u with respect to time: $D_t u$, $\partial_t u$ and $D^s_t u$.
Analogous notations are used for vector valued functions $U \in [BV(\mathcal{O})]^m$, moreover, using (5) and (10), we set

$$(N^s_U)_x = \alpha \otimes \nu^s_{U,x},$$

and we observe that $N^s_{U,x} \neq (N^s_U)_x$.

4.2 Traces of vector fields

In [3] Anzellotti introduced a suitable notion of trace (on the discontinuity set of a BV function) for bounded vector fields having distributional divergence in a Lebesgue space (see [3, 5] for more details). Let $\mathcal{O} \subset \mathbb{R}^n$ be bounded, open and with Lipschitz boundary, and denote by $\nu_{\mathcal{O}}$ the unit outward normal vector to $\partial\mathcal{O}$. First we introduce the following function space for bounded vector fields:

$$W(\mathcal{O})_p = \{T \in [L^\infty(\mathcal{O})]^n : \mathrm{div}T \in L^p(\mathcal{O})\}, \qquad 1 \le p \le +\infty.$$

Definition 4.4. *If $T \in W(\mathcal{O})_n$, then there exists a unique function $[\langle T, \nu_{\mathcal{O}}\rangle] \in L^\infty(\partial\mathcal{O})$ such that*

$$\int_{\partial\mathcal{O}} [\langle T, \nu_{\mathcal{O}}\rangle] g \, d\mathcal{H}^{n-1} = \int_{\mathcal{O}} g \, \mathrm{div}T d\mathcal{L}^n + \int_{\mathcal{O}} \langle T, \nabla g\rangle d\mathcal{L}^n \tag{11}$$

for all $g \in C^1(\overline{\mathcal{O}})$ and there holds:

$$\|[\langle T, \nu_{\mathcal{O}}\rangle]\|_{L^\infty(\partial\mathcal{O})} \le \|T\|_{L^\infty(\mathcal{O})}. \tag{12}$$

We remark that, if T is continuous up to the boundary, $[\langle T, \nu_{\mathcal{O}}\rangle]$ coincides with $\langle T, \nu_{\mathcal{O}}\rangle$ and this motivates the above notation.

Definition 4.5. *Let $T \in W(\mathcal{O})_n$; for $\nu \in S^{n-1}$ and $y \in \mathcal{O}$ we set:*

$$[\![\langle T, \nu\rangle]\!](y) = \lim_{\rho\to 0^+} \lim_{r\to 0^+} \frac{1}{2r\omega_{n-1}\rho^{n-1}} \int_{\mathcal{C}_{r,\rho}(y,\nu)} \langle T(y'), \nu\rangle d\mathcal{L}^n(y'), \tag{13}$$

where $\mathcal{C}_{r,\rho}(y,\nu)$ is the cylinder

$$\mathcal{C}_{r,\rho}(y,\nu) = \{\xi \in \mathbb{R}^n : |\langle(\xi - y), \nu\rangle| < r, \quad |(\xi - y) - \langle(\xi - y), \nu\rangle\nu| < \rho\}.$$

The following theorem generalizes to BV functions the classical Integration by parts Formula, which follows by the Divergence Theorem. In particular, we remark that, when the distributional gradient of u has no singular part, the second term at the right hand-side of (14) reduces to $\int_{\mathcal{O}} \langle T, \nabla u\rangle d\mathcal{L}^n$ and, therefore, (14) becomes the usual formula holding in Sobolev spaces.

Theorem 4.6. *Let $u \in BV(\mathcal{O})$, let N_u be defined as in (3), and let $T \in W(\mathcal{O})_n$. Then the function $[\![\langle T, N_u(y)\rangle]\!](y)$ is defined for $|Du|$-a.e. $y \in \mathcal{O}$ and it is $|Du|$-measurable. Moreover, the following formula holds:*

$$\int_{\mathcal{O}} u(y) \, \mathrm{div}T(y) \, d\mathcal{L}^n(y) = \int_{\partial\mathcal{O}} [\langle T, \nu_{\mathcal{O}}\rangle](y) u(y) \, d\mathcal{H}^{n-1}(y) - \int_{\mathcal{O}} [\![\langle T, N_u(y)\rangle]\!](y) d|Du|(y), \tag{14}$$

and

$$[\![\langle T, N_u(y)\rangle]\!](y) = \langle T(y), N_u(y)\rangle \qquad |D^a u| - a.e. \text{ in } \mathcal{O}.$$

Let now $U \in [BV(\mathcal{O})]^m$ and $M = (T_1, \dots, T_m)^t \in \mathbb{M}^{m\times n}$ a matrix such that $T_i \in W(\mathcal{O})_n$ for any $i = 1, \dots, m$. We write $M \in [W(\mathcal{O})_n]^m$ and we denote $\mathrm{div} M \in \mathbb{R}^m$ the vector with components $\mathrm{div} T_i$. The following matrix form of the identity (14), which follows by componentwise summation, will also be useful (we drop the dependence on y):

$$\int_{\mathcal{O}} \langle U, \mathrm{div} M\rangle d\mathcal{L}^n = \int_{\partial\mathcal{O}} \langle U, [M\nu_{\mathcal{O}}]\rangle d\mathcal{H}^{n-1} - \int_{\mathcal{O}} \langle M, \nabla U\rangle d\mathcal{L}^n - \int_{\mathcal{O}} \langle \alpha, [\![M\nu_U^s]\!]\rangle d|D^s U|, \tag{15}$$

where α is the vector field defined in (5), the components of the vector $[\![M\nu_U^s]\!]$ are the functions $[\![\langle T_i, \nu_U^s\rangle]\!]$ defined in (13) by pointwise limits of averages on cylinders, and the components of the vector $[M\nu_{\mathcal{O}}]$ are the functions $[\langle T_i, \nu_{\mathcal{O}}\rangle]$.
Recalling that, for $M \in \mathbb{M}^{m\times n}$ and $\alpha \in \mathbb{R}^m, \beta \in \mathbb{R}^n$, there holds

$$\langle \alpha, M\,\beta\rangle = \langle M, \alpha \otimes \beta\rangle, \tag{16}$$

using (5) and (10) we have

$$\langle \alpha(y), M(y')\nu_U^s(y)\rangle = \langle M(y'), N_U^s(y)\rangle \qquad \text{for } \mathcal{L}^n - \text{a.e. } y' \in \mathcal{O},\ |D^s U| - \text{a.e. } y \in \mathcal{K}_U,$$

so that the function $[\![\langle M, N_U^s(y)\rangle]\!](y)$ is defined for $|D^s U|$-a.e. $y \in \mathcal{K}_U$, and there holds

$$[\![\langle M, N_U^s(y)\rangle]\!](y) = \langle \alpha(y), [\![M\nu_U^s(y)]\!](y)\rangle \qquad \text{for } |D^s U| - \text{a.e. } y \in \mathcal{K}_U. \tag{17}$$

4.3 Directional derivatives of functions of measures with linear growth

Let $\mathcal{O} \subset \mathbb{R}^n$ be open and bounded. Let $k \in \mathbb{N}$ and $\psi : \mathbb{R}^k \to \mathbb{R}$ be a convex function satisfying the following growth conditions

$$\exists\, a_1 > 0 \text{ and } a_2 \geq 0 : \quad a_1|\xi| - a_2 \leq \psi(\xi) \leq a_1|\xi| + a_2 \quad \forall \xi \in \mathbb{R}^k. \tag{18}$$

Then for any $\xi \in \mathbb{R}^k$ there exists the limit

$$\psi_\infty(\xi) = \lim_{t\to+\infty} \frac{\psi(t\xi)}{t} = a_1|\xi|, \tag{19}$$

which is said the recession function of ψ. According to [12], for any measure $\mu : \mathcal{B}(\mathcal{O}) \to \mathbb{R}^k$ the following function of measure can be defined:

$$I(\mu) = \int_{\mathcal{O}} \psi(\mu^a) d\mathcal{L}^n + \int_{\mathcal{O}} \psi_\infty\left(\frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s|. \tag{20}$$

For the properties of functions of measures we refer to [12]. We now recall a result proved by Anzellotti which allows the computation of directional derivatives of functions of measures (see [4, Theorem 2.4]). Let the function ψ satisfy the further properties:

(i) $\psi(\xi)$ is differentiable for all $\xi \in \mathbb{R}^k$ or $\psi(\xi)$ is differentiable for all $\xi \neq 0$ and $\psi(0) = 0$;

(ii) there exists $M > 0$ such that $|\partial_\xi \psi(\xi)| \leq M$ for any $\xi \in \mathbb{R}^k$.

Theorem 4.7. *Let μ and γ be $\mathbb{R}^k$-valued Borel measures on $\mathcal{O}$. Then the function of measure $I(\mu)$ is differentiable at μ in the direction γ if and only if $|\gamma^s| \ll |\mu^s|$ and in this case there holds:*

$$\frac{d}{d\lambda} I(\mu + \lambda\gamma)\bigg|_{\lambda=0} = \int_{\mathcal{O}} \langle \partial_\xi \psi(\mu^a), \gamma^a\rangle d\mathcal{L}^n + \int_{\mathcal{O}} \langle \partial_\xi \psi_\infty\left(\frac{d\mu^s}{d|\mu^s|}\right), \frac{d\gamma^s}{d|\gamma^s|}\rangle d|\gamma^s|. \tag{21}$$

In our case property (ii) of function ψ is satisfied with $M = a_1$. Indeed, by using the definition of convexity one can check that

$$\psi_\infty(\zeta) \geq \langle \partial_\xi \psi(\xi), \zeta \rangle \quad \forall \xi, \zeta \in \mathbb{R}^k, \quad \zeta \neq 0, \tag{22}$$

from which, for $\partial_\xi \psi(\xi) \neq 0$, taking $\zeta = \partial_\xi \psi(\xi)/|\partial_\xi \psi(\xi)|$ and using (19), it follows

$$|\partial_\xi \psi(\xi)| \leq \psi_\infty(\zeta) = a_1,$$

so that we have

$$|\partial_\xi \psi(\xi)| \leq a_1 \qquad \forall \xi \in \mathbb{R}^k. \tag{23}$$

5 The relaxed functional

In this section we recall the results found in [17] regarding the relaxation of the functional E introduced in Section 3. We set $X(\Omega) = [L^1(\Omega)]^5$, and we extend the functional E to $X(\Omega)$ by means of the functional $\mathcal{E} : X(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{E}(w) = \begin{cases} E(w) & \text{if } w \in V(\Omega), \\ +\infty & \text{elsewhere on } X(\Omega). \end{cases}$$

The functionals F, G, P are extended to $\mathcal{F}, \mathcal{G}, \mathcal{P}$ analogously [17]. We denote by $\overline{\mathcal{E}}$ the relaxed functional of $\mathcal{E}$, i.e., the lower semicontinuous envelope of $\mathcal{E}$ with respect to the strong topology of $X(\Omega)$. For every $w \in X(\Omega)$ we have

$$\overline{\mathcal{E}}(w) = \inf \left\{ \liminf_{h \to +\infty} \mathcal{E}(w^h) : \{w^h\} \subset V(\Omega), w^h \to w \text{ in } X(\Omega) \right\}. \tag{24}$$

Denoted by $Y(\Omega) \subset X(\Omega)$ the set where $\overline{\mathcal{E}}$ is finite, in [17] it is proved that $Y(\Omega) = [BV(\Omega)]^5$ and the relaxed functional $\overline{\mathcal{E}}$ can be written in the form

$$\overline{\mathcal{E}}(w) = \overline{\mathcal{F}}(u, v) + \overline{\mathcal{G}}(w) + \overline{\mathcal{P}}(\sigma), \tag{25}$$

where $\overline{\mathcal{F}}, \overline{\mathcal{G}}, \overline{\mathcal{P}}$ are the relaxation of $\mathcal{F}, \mathcal{G}, \mathcal{P}$, respectively (see [17] for details).
In particular, for $w = (u, v, \sigma) \in Y(\Omega)$ we have the following representation formulae:

$$\begin{aligned} \overline{\mathcal{F}}(u, v) \quad &= \quad \int_{\Omega \setminus D} \varphi(|f - u|^2) d\mathcal{L}^3 + \int_\Omega \varphi(|\nabla_x u - v|^2) d\mathcal{L}^3 + \int_\Omega \varphi(|\nabla_x v|^2) d\mathcal{L}^3 \\ &+ \quad M_\varphi \left[|D_x^s u|(\Omega) + |D_x^s v|(\Omega) \right], \end{aligned} \tag{26}$$

where the constant M_φ is given by the recession function of $\varphi(s^2)$ evaluated at $s = 1$,

$$M_\varphi = \lim_{t \to +\infty} \frac{\varphi(t^2)}{t}; \tag{27}$$

$$\overline{\mathcal{G}}(w) = \int_\Omega \varphi \left(\langle \nabla u, \Sigma_\rho \rangle^2 + |(\nabla v) \Sigma_\rho|^2 \right) d\mathcal{L}^3 + M_\varphi |\mu_w^s|(\Omega), \tag{28}$$

where the measure $\mu_w : \mathcal{B}(\Omega) \to \mathbb{R}^3$, with components $\mu_w = (\mu_{w0}, \mu_{w1}, \mu_{w2})$, is defined as follows,

$$\begin{aligned} \mu_{w0}(\mathcal{O}) &= \langle \Sigma_\rho, Du \rangle(\mathcal{O}) = \int_{\mathcal{O}} \langle \Sigma_\rho, dDu \rangle, \\ \mu_{wi}(\mathcal{O}) &= \langle \Sigma_\rho, Dv_i \rangle(\mathcal{O}) = \int_{\mathcal{O}} \langle \Sigma_\rho, dDv_i \rangle, \quad \text{for } i = 1, 2, \end{aligned} \tag{29}$$

for any $\mathcal{O} \in \mathcal{B}(\Omega)$ and the measure μ_w^s is the singular part of μ_w with respect to $\mathcal{L}^3$;

$$\overline{\mathcal{P}}(\sigma) = \int_\Omega \varphi(|\nabla\sigma|^2) d\mathcal{L}^3 + M_\varphi |D^s\sigma|(\Omega) + c\int_\Omega \varphi(|\sigma|^2) d\mathcal{L}^3. \tag{30}$$

Note that, if $\psi : \mathbb{R}^k \to \mathbb{R}$ is a function as in Section 4.3 and such that $\psi(\xi) = \varphi(|\xi|^2)$, taking $\zeta \in \mathbb{R}^k$ such that $|\zeta| = 1$ and using (19), we have

$$M_\varphi = \lim_{t\to+\infty} \frac{\varphi(t^2|\zeta|^2)}{t} = \psi_\infty(\zeta) = a_1. \tag{31}$$

Existence of minimizers of the relaxed functional $\overline{\mathcal{E}}$ in the space $Y(\Omega)$ has been proved in [17].
We conclude this section by recalling the representation of the jump part of the relaxed functional $\overline{\mathcal{G}}$, which yields the contribution to $\overline{\mathcal{G}}$ from discontinuities of functions u and v. This is interesting, since it contains the interaction on the discontinuity set $S_u \cup S_v$ between the optical flow σ, the video u, and the vector field v which approximates $\nabla_x u$. Decomposing the measure μ_w^s into the jump part μ_w^J and the Cantor part μ_w^C, the result found in [17] is the following:

$$|\mu_w^J|(\Omega) = \int_{S_u\cup S_v} \sqrt{\langle\nu_u, \Sigma_\rho\rangle^2 (u^+ - u^-)^2 + \langle\nu_v, \Sigma_\rho\rangle^2 [(v_1^+ - v_1^-)^2 + (v_2^+ - v_2^-)^2]}\, d\mathcal{H}^2 .$$

In [17] the role played by the average of the optical flow σ on a ball with a fixed radius ρ, in obtaining such a representation formula, has been discussed.

6 Main results

The purpose of the present paper is to compute the Euler equations of $\overline{\mathcal{E}}$, highlighting the necessary conditions of minimality holding for the absolutely continuous part and the singular part of the measures Du, Dv and $D\sigma$, respectively, and focusing the attention on the discontinuity sets of u, v, σ.
Let $w = (u, v, \sigma) \in Y(\Omega)$; we set $U = (u, v_1, v_2) \in [BV(\Omega)]^3$ and $\sigma = (\sigma_1, \sigma_2) \in [BV(\Omega)]^2$. First we define suitable vector fields that will be involved in the Euler equations. We denote $\widehat{v} : \Omega \to \mathbb{R}^3$ the vector field with components $(v_1, v_2, 0)$ and, given $g \in BV(\Omega)$, we denote $\widehat{\nabla}_x g : \Omega \to \mathbb{R}^3$ the vector field with components $(\partial_{x_1} g, \partial_{x_2} g, 0)$.
Let $T_1 : \Omega \to \mathbb{R}^3$ be the vector field defined as

$$T_1 = \varphi'\left(|(\nabla U)\Sigma_\rho|^2\right) \langle\nabla u, \Sigma_\rho\rangle \Sigma_\rho, \tag{32}$$

where φ' denotes the derivative of φ, and let $T_2 : \Omega \to \mathbb{M}^{2\times 3}$ be the matrix-valued field defined as

$$T_2 = \varphi'\left(|(\nabla U)\Sigma_\rho|^2\right) ((\nabla v)\Sigma_\rho) \otimes \Sigma_\rho . \tag{33}$$

We collect T_1 and T_2 in the matrix-valued field $T : \Omega \to \mathbb{M}^{3\times 3}$ defined as

$$T = \varphi'\left(|(\nabla U)\Sigma_\rho|^2\right) ((\nabla U)\Sigma_\rho) \otimes \Sigma_\rho ,$$

so that T_1 is the first row of T and T_2 is the submatrix constituted with the second and third row of T. Let now $A : \Omega \to \mathbb{R}^3$ be the vector field defined as

$$A = A_0 + T_1, \qquad A_0 = \varphi'\left(|\nabla_x u - v|^2\right) (\widehat{\nabla}_x u - \widehat{v}), \tag{34}$$

let $B : \Omega \to \mathbb{M}^{2\times 3}$ be the matrix-valued field defined as

$$B = B_0 + T_2, \qquad B_0 = \varphi'\left(|\nabla_x v|^2\right) \widehat{\nabla}_x v, \tag{35}$$

and let $Q : \Omega \to \mathbb{M}^{2\times 3}$ be the matrix-valued field defined as

$$Q = \varphi'\left(|\nabla\sigma|^2\right)\nabla\sigma. \tag{36}$$

We need the following nonlinear operator $\Lambda_{z,\rho} : Y(\Omega) \to \mathbb{R}^2$ built with an average on the ball $B_\rho(z)$, with $z \in \Omega$:

$$\Lambda_{z,\rho}(w) = \int_{\Omega\cap B_\rho(z)} \frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)(\nabla_x U(y))^t(\nabla U(y))\Sigma_\rho(y)\,d\mathcal{L}^3(y), \tag{37}$$

where the matrix by vector product $(\nabla_x U)^t(\nabla U)\Sigma_\rho$ is a vector in $\mathbb{R}^2$. We also need the nonlinear operator $\Theta_{z,\rho} : Y(\Omega) \to \mathbb{R}^2$, built with the singular part D^sU of DU, defined as

$$\Theta_{z,\rho}(w) = \int_{\mathcal{K}_U\cap\{N_U^s\Sigma_\rho\neq 0\}\cap B_\rho(z)} \frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\frac{1}{|N_U^s(y)\Sigma_\rho(y)|}((N_U^s)_x(y))^t N_U^s(y)\Sigma_\rho(y)\,d|D^sU|(y), \tag{38}$$

where the matrix by vector product $(N_U^s)_x^t N_U^s\Sigma_\rho$ is a vector in $\mathbb{R}^2$.

We prove the following result.

Theorem 6.1. (Euler equations and trace properties) *Let $w = (u, v, \sigma) \in Y(\Omega)$ be a minimizer of $\overline{\mathcal{E}}$. Then the following necessary conditions hold.*
Absolutely continuous part*. We have $A \in W(\Omega)_\infty$ and $B, Q \in [W(\Omega)_\infty]^2$, and the vector field w satisfies the following set of Euler equations for $\mathcal{L}^3$-a.e. $y = (x,t) \in \Omega$:*

$$\begin{aligned} \mathrm{div}A &= \chi_{\Omega\setminus D}\varphi'\left(|f-u|^2\right)(u-f), && (39)\\ -\mathrm{div}B &= \varphi'\left(|\nabla_x u - v|^2\right)(\nabla_x u - v), && (40)\\ \mathrm{div}Q &= c\,\varphi'\left(|\sigma|^2\right)\sigma + \Lambda_{y,\rho}(w) + \frac{M_\varphi}{2}\Theta_{y,\rho}(w). && (41)\end{aligned}$$

Conditions on the singular sets $\mathcal{K}_U$ and $\mathcal{K}_\sigma$*. We have*

$$\begin{aligned} [\![\langle A_0, N_u^s\rangle]\!](y) &= \frac{M_\varphi}{2}|\nu_{u,x}^s(y)|, && \text{for } |D^su| - \text{a.e. } y \in \mathcal{K}_u,\\ [\![\langle B_0, N_v^s\rangle]\!](y) &= \frac{M_\varphi}{2}|\nu_{v,x}^s(y)|, && \text{for } |D^sv| - \text{a.e. } y \in \mathcal{K}_v,\\ [\![\langle T, N_U^s\rangle]\!](y) &= \frac{M_\varphi}{2}\left|\langle\nu_U^s(y), \Sigma_\rho(y)\rangle\right|, && \text{for } |D^sU| - \text{a.e. } y \in \mathcal{K}_U,\\ [\![\langle Q, N_\sigma^s\rangle]\!](y) &= \frac{M_\varphi}{2}, && \text{for } |D^s\sigma| - \text{a.e. } y \in \mathcal{K}_\sigma.\end{aligned} \tag{42}$$

Natural boundary conditions on $\partial\Omega$*. For $\mathcal{H}^2$-a.e. $y = (x,t) \in \partial\Omega$ we have*

$$\begin{aligned} [\langle A, \nu_\Omega\rangle](y) &= 0, && (43)\\ [B\nu_\Omega](y) &= 0, && (44)\\ [Q\nu_\Omega](y) &= 0. && (45)\end{aligned}$$

Though from the theoretical point of view, when we deal with models involving BV functions, we have to consider their Cantor part, actually in the case of real images the important objects are the discontinuity sets (of the images or of other quantities related to them, like in our case, for instance, the velocities of moving objects). Therefore, in the following corollary we explicitly write conditions (42) for the jump sets of minimizers.

Corollary 6.2 (Conditions on the jump sets)**.** *Using the identity* (17) *and Remark* 4.1*, on the jump sets conditions* (42) *become:*

$$
\begin{aligned}
[\![\varphi'\left(|\nabla_x u - v|^2\right)\langle \nabla_x u - v, \nu_{u,x}\rangle]\!](y) &= \frac{M_\varphi}{2}|\nu_{u,x}(y)|, \qquad \text{for } \mathcal{H}^2 - \text{a.e. } y \in S_u,\\
[\![\varphi'\left(|\nabla_x v|^2\right)\langle v^+ - v^-, (\nabla_x v)\nu_{v,x}\rangle]\!](y) &= \frac{M_\varphi}{2}|\nu_{v,x}(y)| \cdot |v^+(y) - v^-(y)|,\\
&\quad \text{for } \mathcal{H}^2 - \text{a.e. } y \in S_v,\\
[\![\varphi'\left(|(\nabla U)\Sigma_\rho|^2\right)\langle U^+ - U^-, (\nabla U)\Sigma_\rho\rangle\langle \Sigma_\rho, \nu_U\rangle]\!](y) &= \frac{M_\varphi}{2}\left|\langle \nu_U(y), \Sigma_\rho(y)\rangle\right| \cdot \left|U^+(y) - U^-(y)\right|,\\
&\quad \text{for } \mathcal{H}^2 - \text{a.e. } y \in S_U,\\
[\![\varphi'\left(|\nabla \sigma|^2\right)\langle \sigma^+ - \sigma^-, (\nabla\sigma)\nu_\sigma\rangle]\!](y) &= \frac{M_\varphi}{2}|\sigma^+(y) - \sigma^-(y)|, \quad \text{for } \mathcal{H}^2 - \text{a.e. } y \in S_\sigma.
\end{aligned} \tag{46}
$$

Remark 6.3. *Recalling that* $\Omega = \Omega_s \times [0, T]$*, the natural boundary conditions* (43) *and* (44) *can be written in the form:*

$$
\left[\varphi'(|\nabla_x u - v|^2)\langle \nabla_x u - v, \nu_{\Omega_s}\rangle + \varphi'\left(\left|(\nabla U)\Sigma_\rho\right|^2\right)\langle \nabla u, \Sigma_\rho\rangle\langle \sigma_\rho, \nu_{\Omega_s}\rangle\right](x,t) = 0, \qquad \mathcal{H}^1\text{-a.e. } x \in \partial\Omega_s,\ t \in (0,T),
$$

and

$$
\left[\varphi'(|\nabla_x v|^2)\,(\nabla_x v)\,\nu_{\Omega_s} + \varphi'\left(\left|(\nabla U)\Sigma_\rho\right|^2\right)\langle \sigma_\rho, \nu_{\Omega_s}\rangle\,(\nabla v)\,\Sigma_\rho\right](x,t) = 0, \qquad \mathcal{H}^1\text{-a.e. } x \in \partial\Omega_s,\ t \in (0,T),
$$

where $\nu_{\Omega_s} \in \mathbb{R}^2$ *denotes the unit outward normal vector to* Ω_s*.*

Example 6.4. *Let* $\varphi(s) = \sqrt{s + \varepsilon}$*, then* $M_\varphi = 1$ *and*

$$
\begin{aligned}
A &= \frac{\widehat{\nabla}_x u - \widehat{v}}{2\sqrt{\varepsilon + |\nabla_x u - v|^2}} + \frac{\langle \nabla u, \Sigma_\rho\rangle}{2\sqrt{\varepsilon + |(\nabla U)\Sigma_\rho|^2}}\,\Sigma_\rho,\\
B &= \frac{\widehat{\nabla}_x v}{2\sqrt{\varepsilon + |\nabla_x v|^2}} + \frac{((\nabla v)\Sigma_\rho) \otimes \Sigma_\rho}{2\sqrt{\varepsilon + |(\nabla U)\Sigma_\rho|^2}},\\
Q &= \frac{\nabla \sigma}{2\sqrt{\varepsilon + |\nabla\sigma|^2}}.
\end{aligned}
$$

Moreover, the necessary conditions (46) *become*

$$
\begin{aligned}
[\![\frac{\langle \nabla_x u - v, \nu_{u,x}\rangle}{\sqrt{\varepsilon + |\nabla_x u - v|^2}}]\!](y) &= |\nu_{u,x}(y)|, \qquad \text{for } \mathcal{H}^2 - \text{a.e. } y \in S_u,\\
[\![\frac{\langle v^+ - v^-, (\nabla_x v)\nu_{v,x}\rangle}{\sqrt{\varepsilon + |\nabla_x v|^2}}]\!](y) &= |\nu_{v,x}(y)| \cdot |v^+(y) - v^-(y)|, \qquad \text{for } \mathcal{H}^2 - \text{a.e. } y \in S_v,\\
[\![\frac{\langle U^+ - U^-, (\nabla U)\Sigma_\rho\rangle\langle \Sigma_\rho, \nu_U\rangle}{\sqrt{\varepsilon + |(\nabla U)\Sigma_\rho|^2}}]\!](y) &= \left|\langle \nu_U(y), \Sigma_\rho(y)\rangle\right| \cdot \left|U^+(y) - U^-(y)\right|, \qquad \text{for } \mathcal{H}^2 - \text{a.e. } y \in S_U,\\
[\![\frac{\langle \sigma^+ - \sigma^-, (\nabla\sigma)\nu_\sigma\rangle}{\sqrt{\varepsilon + |\nabla\sigma|^2}}]\!](y) &= |\sigma^+(y) - \sigma^-(y)|, \qquad \text{for } \mathcal{H}^2 - \text{a.e. } y \in S_\sigma.
\end{aligned}
$$

7 Application of Euler equations and traces

In this section we discuss how the results obtained in the present paper could be of some help in the design of numerical algorithms. In order to get a numerical approximation of our model, we observe that a possible strategy can be derived from the nonlinear primal-dual method proposed by Chan, Golub and Mulet in [11] for Total Variation image restoration. At least in principle, such a method could be adapted to the much more complicated problem here considered. The vector field A and the matrix-valued fields B, Q defined in (34)-(36) play the role of the dual variable introduced in the primal-dual method. The equations (39)-(41), together with the definitions of the fields A, B, Q and the boundary conditions (43)-(45), constitute a system of equations in the variables (u, v, σ, A, B, Q), that we assume can be solved. The problem of the solution of such a complicated system of equations is beyond the aims of the present paper, and it is introduced only for the purpose of discussing how the primal-dual method could be applied in principle to the problem at hand.

First we observe that in [11] the authors replace the Euler equation of the Total Variation functional with a system of equations obtained by introducing a dual vector field. Such a system of equations is first derived in the case of regular solutions (without jumps). In the presence of jumps the authors do not use directly such a system of equations, nevertheless, their numerical scheme corresponds to a discrete approximation of the equations for the absolutely continuous part obtained by means of the method used by Anzellotti in [4] for the Total Variation functional. Such equations correspond to equations (39)-(41) for our variational problem.

In the following we show how a solution of the system of equations for the absolutely continuous part, together with the conditions (42) on the singular sets, could be used to locate approximately the jump sets.

Let $\psi_1(\xi) = \varphi(|\xi|^2)$ as in Section 4.3 with $k = 2$; using (23) with $\psi = \psi_1$ and $\xi = \nabla_x u - v$, and using (31), we have

$$|A_0(y)| \leq \frac{M_\varphi}{2} \quad a.e. \text{ in } \Omega.$$

Let $\overline{A_0}(y, \nu_u(y), r, \rho)$ denote the average of the vector field A_0 over the cylinder $\mathcal{C}_{r,\rho}(y, \nu_u(y))$, so that there also holds:

$$|\overline{A_0}(y)| \leq \frac{M_\varphi}{2} \quad a.e. \text{ in } \Omega. \tag{47}$$

For $|D^s u|$-a.e. $y \in S_u$ the first of equations (42) yields

$$\lim_{\rho \to 0^+} \lim_{r \to 0^+} \langle \nu_u(y), \overline{A_0}(y, \nu_u(y), r, \rho) \rangle = \frac{M_\varphi}{2} |\nu_{u,x}(y)|. \tag{48}$$

Since the temporal component of A_0 is null, we have:

$$|\langle \nu_u(y), \overline{A_0}(y, \nu_u(y), r, \rho) \rangle| \leq |\nu_{u,x}(y)| \cdot \left|\overline{A_0}(y, \nu_u(y), r, \rho)\right|. \tag{49}$$

Hence, by (47), (48) and (49) we get

$$\lim_{\rho \to 0^+} \lim_{r \to 0^+} \left|\overline{A_0}(y, \nu_u(y), r, \rho)\right| = \frac{M_\varphi}{2}, \qquad \text{for } |D^s u|\text{-a.e. } y \in S_u, \tag{50}$$

and the average $\overline{A_0}$ tends to be parallel to $\nu_{u,x}$ as $\rho, r \to 0^+$.

Let us now consider a function φ such that inequality (23) is strict, so that $|A_0(y)| < M_\varphi/2$ *a.e.* in Ω: that happens for instance for the function $\varphi(t) = \sqrt{t + \varepsilon}$ with $\varepsilon > 0$. Then, assuming the jump set S_u of u regular enough, for given r, ρ and δ small enough, we can approximate S_u with a neighborhood $\Delta_{r,\rho,\delta}(S_u)$ of S_u defined by

$$\Delta_{r,\rho,\delta}(S_u) = \left\{ y \in \Omega : \exists \nu \in S^2 : \left| \left|\overline{A_0}(y, \nu, r, \rho)\right| - \frac{M_\varphi}{2} \right| < \delta \right\},$$

where the projection ν_x of the unit vector ν gives an approximation to $\nu_{u,x}$.
Let now $\psi_2(\xi) = \varphi(|\xi|^2)$ as in Section 4.3 with $k = 4$; using (23) with $\psi = \psi_2$ and $\xi = \nabla_x v$, and using again (31), we have

$$|B_0(y)| \leq \frac{M_\varphi}{2} \quad a.e. \text{ in } \Omega.$$

For $|D^s v|$-a.e. $y \in S_v$, setting $N_v^s = \alpha_v \otimes \nu_v$ according to (5) and Remark 4.1, and using (17), the second of equations (42) yields

$$\lim_{\rho\to 0^+} \lim_{r\to 0^+} \langle \alpha_v(y), \overline{B_0}(y, \nu_v(y), r, \rho)\nu_v(y)\rangle = \frac{M_\varphi}{2} |\nu_{v,x}(y)|,$$

where $\overline{B_0}(y, \nu_v(y), r, \rho)$ denotes the average of the matrix-valued field B_0 over the cylinder $\mathcal{C}_{r,\rho}(y, \nu_v(y))$. Since the third column of the matrix B_0 is null, then it follows again

$$\lim_{\rho\to 0^+} \lim_{r\to 0^+} \left|\overline{B_0}(y, \nu_v(y), r, \rho)\right| = \frac{M_\varphi}{2}, \qquad \text{for } |D^s v|\text{-a.e. } y \in S_v,$$

and both the row vectors of the average matrix $\overline{B_0}$ tend to be parallel to $\nu_{v,x}$ as $\rho, r \to 0^+$. The neighborhood $\Delta_{r,\rho,\delta}(S_v)$ of S_v, which approximates the jump set S_v, is analogously defined by

$$\Delta_{r,\rho,\delta}(S_v) = \left\{ y \in \Omega : \exists \nu \in S^2 : \left| \left|\overline{B_0}(y, \nu, r, \rho)\right| - \frac{M_\varphi}{2} \right| < \delta \right\},$$

where the projection ν_x of the unit vector ν gives an approximation to $\nu_{v,x}$.
The jump set S_σ of the optical flow field σ can be analogously located by resorting to the average $\overline{Q}$ over cylinders of the matrix valued field Q. In a discrete algorithm both the averages over cylinders and the approximating neighborhoods of the jump sets have to be computed by resorting to the tessellation of the domain Ω into finite elements, which also yields a discrete sets of orientations for the axes of the cylinders.
The above procedure yields the localization not only of the jump sets, but also of the sets where the gradients tend to infinity though the functions being continuous (in the sense of approximate limits). Therefore, such further parts have to be removed by looking at the traces of u, v and σ on the sets selected by the above method. It follows that a detection of the jump sets during the computation is in principle possible, and that would give a further explicit information about the location of such sets with respect to the numerical algorithm used in [11]. Such an information could be used to better handle the evolution of the jump sets during an algorithm, though the study of such a possibility is beyond the aims of the present paper.
Eventually, in order to enforce consistently the natural boundary conditions on $\partial\Omega$, we first observe that the fields A, B, Q depend on quantities, such as the gradients of BV functions, which do not have a trace on the boundary. Nevertheless, in order to compute explicitly the natural boundary conditions, according to Proposition 2.2 in [6] (as stated also in Fact 1.1 in [5]), it is possible to characterize the traces on the boundary of the vector fields A, B, Q by means of the limit of the averages on cylinders centered on $\partial\Omega$, analogously to formula (13). Then, in the typical case of a rectangular spatial domain Ω_s, such averages can be easily computed by means of the tessellation of Ω_s in rectangular elements used in discrete algorithms, since the cylinders can be chosen in such a way to have the axes normal to the boundary and their projections on the spatial domain coincident with the rectangles of the tessellation. Since the natural boundary conditions are necessary conditions obtained by the vanishing of the first variation associated to a solution of the relaxed variational problem, which has been proved to exist in [17], the above procedure shows that a consistent implementation of numerically approximated boundary conditions can be in principle achieved, though somewhat complicated.

8 Directional derivatives

In order to prove Theorem 6.1, first we compute the directional derivatives of $\overline{\mathcal{E}}$ with respect to u, v, σ by resorting to Theorem 4.7. In the computations the main difficulties (due to the singular part of the measures given by the distributional derivatives of u, v, σ) are overcome resorting to fine properties of Geometric Measure Theory, particularly to the rank one property (5) of derivatives due to Alberti [1]. By means of this property we can write the whole singular part of the distributional derivative of a vector valued BV function as a suitable tensor product as well as it happens for the jump part. In particular, when we compute the directional derivative of the functional $\overline{\mathcal{G}}$ at the point $U = (u, v)$ (for fixed σ) in the direction θ, under the condition $|D^s\theta| \ll |D^sU|$ of Theorem 4.7, such a property allows to state that the vectors ν_U^s and ν_θ^s, defined in (5) and (10), are equal (see the proof of Lemma 8.3).

8.1 Directional derivatives of $\overline{\mathcal{E}}$ with respect to u and v

We compute first the directional derivatives of the term $\overline{\mathcal{F}}(u, v)$ in the expression (25) of the relaxed functional $\overline{\mathcal{E}}$, and defined by means of (26).
We denote by $\overline{\mathcal{F}}_1$ the part of $\overline{\mathcal{F}}$ depending on u for a fixed v, that means:

$$\overline{\mathcal{F}}_1(u) = \int_{\Omega\setminus D} \varphi(|f-u|^2)d\mathcal{L}^3 + \int_\Omega \varphi(|\nabla_x u - v|^2)d\mathcal{L}^3 + M_\varphi |D_x^s u|(\Omega).$$

The following lemma yields the directional derivative of $\overline{\mathcal{F}}_1$ with respect to $u \in BV(\Omega)$ for a fixed vector field $v \in [L^1(\Omega)]^2$.

Lemma 8.1. *Let $v \in [L^1(\Omega)]^2$. Then $\overline{\mathcal{F}}_1(u)$ is differentiable at the point $u \in BV(\Omega)$ in the direction $\eta \in BV(\Omega)$ if and only if $|D_x^s\eta| \ll |D_x^s u|$ and in this case there holds:*

$$\begin{aligned} \frac{d}{d\lambda}\overline{\mathcal{F}}_1(u+\lambda\eta)\Big|_{\lambda=0} &= 2\int_{\Omega\setminus D} \varphi'(|f-u|^2)(u-f)\eta \, d\mathcal{L}^3 + 2\int_\Omega \varphi'(|\nabla_x u - v|^2)\langle \nabla_x u - v, \nabla_x \eta\rangle d\mathcal{L}^3 \\ &+ M_\varphi \int_\Omega \langle N_{u,x}^s, N_{\eta,x}^s\rangle d|D_x^s\eta|. \end{aligned} \tag{51}$$

Proof. Let $\psi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the convex and differentiable function satisfying the growth conditions (18) and such that $\psi_1(\xi) = \varphi(|\xi|^2)$. Let $\mu_1 : \mathcal{B}(\Omega) \to \mathbb{R}^2$ be the measure defined by means of

$$\mu_1 = D_x u - v \cdot \mathcal{L}^3.$$

Then, using (20) and taking into account that the Radon-Nikodym derivative $d\mu_1^s/d|\mu_1^s|$ is a vector valued function with unit norm ([2], Corollary 1.29), for the function of measure $I(\mu_1)$ we have

$$I(\mu_1) = \int_\Omega \varphi(|\nabla_x u - v|^2)d\mathcal{L}^3 + M_\varphi |D_x^s u|(\Omega).$$

Now we apply Theorem 4.7 to $I(\mu_1)$. Let $\eta \in BV(\Omega)$ and $\gamma = D_x\eta$. Using Theorem 4.7, if and only if $|\gamma^s| \ll |\mu_1^s|$, then we have

$$\frac{d}{d\lambda}I(\mu_1+\lambda\gamma)\Big|_{\lambda=0} = 2\int_\Omega \varphi'(|\nabla_x u - v|^2)\langle\nabla_x u - v, \nabla_x\eta\rangle d\mathcal{L}^3 + M_\varphi \int_\Omega \langle N_{u,x}^s, N_{\eta,x}^s\rangle d|D_x^s\eta|,$$

from which the statement of the lemma follows. □

Now we denote by $\overline{\mathcal{F}}_2$ the part of $\overline{\mathcal{F}}$ depending on v for a fixed u, that means:

$$\overline{\mathcal{F}}_2(v) = \int_\Omega \varphi(|\nabla_x u - v|^2) d\mathcal{L}^3 + \int_\Omega \varphi(|\nabla_x v|^2) d\mathcal{L}^3 + M_\varphi |D_x^s v|(\Omega).$$

The following lemma yields the directional derivative of $\overline{\mathcal{F}}_2$ with respect to $v \in [BV(\Omega)]^2$ for a fixed function $u \in BV(\Omega)$.

Lemma 8.2. *Let $u \in BV(\Omega)$. Then $\overline{\mathcal{F}}_2(v)$ is differentiable at the point $v \in [BV(\Omega)]^2$ in the direction $\eta \in [BV(\Omega)]^2$ if and only if $|D_x^s \eta| \ll |D_x^s v|$ and in this case there holds:*

$$\begin{aligned} \frac{d}{d\lambda} \overline{\mathcal{F}}_2(v + \lambda\eta)\Big|_{\lambda=0} &= -2 \int_\Omega \varphi'(|\nabla_x u - v|^2) \langle \nabla_x u - v, \eta \rangle d\mathcal{L}^3 + 2 \int_\Omega \varphi'(|\nabla_x v|^2) \langle \nabla_x v, \nabla_x \eta \rangle d\mathcal{L}^3 \\ &+ M_\varphi \int_\Omega \langle N^s_{v,x}, N^s_{\eta,x} \rangle d|D_x^s \eta|. \end{aligned} \tag{52}$$

Proof. Let $\psi_2 : \mathbb{R}^4 \to \mathbb{R}$ be the convex and differentiable function satisfying the growth conditions (18) and such that $\psi_2(\xi) = \varphi(|\xi|^2)$. Let $\mu_2 : \mathcal{B}(\Omega) \to \mathbb{M}^{2\times 2}$ be the measure defined by means of

$$\mu_2 = D_x v.$$

We treat the matrix valued measure μ_2 as a vector valued measure $\mu_2 : \mathcal{B}(\Omega) \to \mathbb{R}^4$ with components $(D_x v_1, D_x v_2)$. Analogously, we treat $\nabla_x v$ as a vector in $\mathbb{R}^4$ with components $(\nabla_x v_1, \nabla_x v_2)$. Then, using (20) and arguing as in the proof of Lemma 8.1, for the function of measure $I(\mu_2)$ we have

$$I(\mu_2) = \int_\Omega \varphi(|\nabla_x v|^2) d\mathcal{L}^3 + M_\varphi |D_x^s v|(\Omega).$$

Now we apply Theorem 4.7 to $I(\mu_2)$. Let $\eta \in [BV(\Omega)]^2$ and $\gamma = D_x \eta$. Using Theorem 4.7, if and only if $|\gamma^s| \ll |\mu_2^s|$, then we have

$$\frac{d}{d\lambda} I(\mu_2 + \lambda\gamma)\Big|_{\lambda=0} = 2 \int_\Omega \varphi'(|\nabla_x v|^2) \langle \nabla_x v, \nabla_x \eta \rangle d\mathcal{L}^3 + M_\varphi \int_\Omega \langle N^s_{v,x}, N^s_{\eta,x} \rangle d|D_x^s \eta|,$$

from which, observing that the scalar products of vectors in $\mathbb{R}^4$ coincides with the Frobenius scalar products of the corresponding matrices in $\mathbb{M}^{2\times 2}$, the statement of the lemma follows. □

In order to achieve the directional derivatives of $\overline{\mathcal{G}}$ with respect to u and v, we first compute the derivative with respect to the variable $U = (u, v)$, then in Corollaries 8.4 and 8.5 we infer the derivatives with respect to u and v separately.
We set $U = (u, v) \in [BV(\Omega)]^3$ and we write $\overline{\mathcal{G}}(U, \sigma) = \overline{\mathcal{G}}(u, v, \sigma)$. Given also $\theta \in [BV(\Omega)]^3$, recalling (5) and (10), using Remark 4.1, Corollary 4.2 and Remark 4.3, we write

$$N_U^s = \alpha_U^s \otimes \nu_U^s, \qquad N_\theta^s = \alpha_\theta^s \otimes \nu_\theta^s.$$

Moreover, we define the sets

$$\widetilde{\mathcal{K}}_U = \{y \in \mathcal{K}_U : N_U^s(y)\Sigma_\rho(y) \neq 0\}, \qquad \widetilde{\mathcal{K}}_\theta = \{y \in \mathcal{K}_\theta : N_\theta^s(y)\Sigma_\rho(y) \neq 0\}. \tag{53}$$

The following lemma yields the directional derivative of $\overline{\mathcal{G}}$ with respect to $U \in [BV(\Omega)]^3$ for a fixed vector field $\sigma \in [L^1(\Omega)]^2$.

Lemma 8.3. *Let $\sigma \in [L^1(\Omega)]^2$. Then $\overline{\mathcal{G}}(U,\sigma)$ is differentiable at the point $U \in [BV(\Omega)]^3$ in the direction $\theta \in [BV(\Omega)]^3$ if and only if $|D^s\theta| \ll |D^sU|$ in $\widetilde{\mathcal{K}}_U$, and in this case there holds:*

$$\begin{aligned}\frac{d}{d\lambda}\overline{\mathcal{G}}(U+\lambda\theta,\sigma)\Big|_{\lambda=0} &= 2\int_\Omega \varphi'\left(|(\nabla U)\Sigma_\rho|^2\right)\langle(\nabla U)\Sigma_\rho,(\nabla\theta)\Sigma_\rho\rangle d\mathcal{L}^3 \\ &+ M_\varphi \int_{\widetilde{\mathcal{K}}_U} |\langle \nu_U^s,\Sigma_\rho\rangle|\,\langle \alpha_U^s,\alpha_\theta^s\rangle d|D^s\theta|. \end{aligned} \tag{54}$$

Proof. Let $\psi_3 : \mathbb{R}^3 \to \mathbb{R}$ be the convex and differentiable function satisfying the growth conditions (18) and such that $\psi_3(\xi) = \varphi(|\xi|^2)$. We set $(DU)\Sigma_\rho = \mu_w$, where $\mu_w : \mathcal{B}(\Omega) \to \mathbb{R}^3$ is the measure defined in (29). In the sequel of the proof we also write $\mu_U = \mu_w$ in order to remind that the vector field σ is kept fixed.
Since $\mu_U^a = (\nabla U)\Sigma_\rho$, using (20) and arguing as in the proof of Lemma 8.1, for the function of measure $I(\mu_U)$ we have

$$I(\mu_U) = \int_\Omega \varphi\left(|(\nabla U)\Sigma_\rho|^2\right) d\mathcal{L}^3 + M_\varphi \int_\Omega \left|\frac{d\mu_U^s}{d|\mu_U^s|}\right| d|\mu_U^s|\,, \tag{55}$$

so that, using (28) and (29), we have $I(\mu_U) = \overline{\mathcal{G}}(U,\sigma)$. Now we remark that we cannot directly apply Theorem 4.7 to the function of measure $I(\mu_U)$, since we need to compute the derivative of $I(\mu_U)$ with respect to the measure DU and not μ_U.
Let $\theta \in [BV(\Omega)]^3$ and $\mu_\theta = (D\theta)\Sigma_\rho$. We have

$$\begin{aligned}\frac{d}{d\lambda}\overline{\mathcal{G}}(U+\lambda\theta,\sigma)\Big|_{\lambda=0} &= \frac{d}{d\lambda} I(\mu_{U+\lambda\theta})\Big|_{\lambda=0} \\ = \frac{d}{d\lambda}\int_\Omega \varphi\left(|(\nabla U)\Sigma_\rho + \lambda(\nabla\theta)\Sigma_\rho|^2\right) d\mathcal{L}^3\Big|_{\lambda=0} &+ M_\varphi \frac{d}{d\lambda}\int_\Omega \left|\frac{d\mu_U^s}{d|\mu_U^s|} + \lambda\frac{d\mu_\theta^s}{d|\mu_U^s|}\right| d|\mu_U^s|\Big|_{\lambda=0}. \end{aligned} \tag{56}$$

Using Lebesgue's dominated convergence Theorem, for the absolutely continuous part we have

$$\frac{d}{d\lambda}\int_\Omega \varphi\left(|(\nabla U)\Sigma_\rho + \lambda(\nabla\theta)\Sigma_\rho|^2\right) d\mathcal{L}^3\Big|_{\lambda=0} = 2\int_\Omega \varphi'\left(|(\nabla U)\Sigma_\rho|^2\right)\langle(\nabla U)\Sigma_\rho,(\nabla\theta)\Sigma_\rho\rangle d\mathcal{L}^3\,. \tag{57}$$

For the singular part we adapt the method of proof of Theorem 2.4 in [4]. We consider the decomposition $\mu_\theta^s = \mu_\theta^{sa} + \mu_\theta^{ss}$, where μ_θ^{sa} and μ_θ^{ss} are the absolutely continuous part and the singular part of μ_θ^s with respect to $|\mu_U^s|$, respectively. Since the measures $\mu_U^s + \lambda\mu_\theta^{sa}$ and μ_θ^{ss} are mutually singular, we have

$$\int_\Omega \left|\frac{d\mu_U^s}{d|\mu_U^s|} + \lambda\frac{d\mu_\theta^s}{d|\mu_U^s|}\right| d|\mu_U^s| = \int_\Omega \left|\frac{d\mu_U^s}{d|\mu_U^s|} + \lambda\frac{d\mu_\theta^{sa}}{d|\mu_U^s|}\right| d|\mu_U^s| + |\lambda|\int_\Omega d|\mu_\theta^{ss}|. \tag{58}$$

Therefore, in order to get differentiability of $\overline{\mathcal{G}}$, the second term in the sum at the righthand side of (58) has to be zero, so that we have $\mu_\theta^{ss} = 0$, which implies $|\mu_\theta^s| \ll |\mu_U^s|$. It follows

$$|D^s\theta|(\widetilde{\mathcal{K}}_\theta \setminus \widetilde{\mathcal{K}}_U) = 0, \tag{59}$$

indeed, since

$$|\mu_U^s|(\mathcal{O}) = \int_{\mathcal{O}\cap\widetilde{\mathcal{K}}_U} |N_U^s\Sigma_\rho|\, d|D^sU|, \qquad |\mu_\theta^s|(\mathcal{O}) = \int_{\mathcal{O}\cap\widetilde{\mathcal{K}}_\theta} |N_\theta^s\Sigma_\rho|\, d|D^s\theta|,$$

setting $\mathcal{O} = \widetilde{\mathcal{K}}_\theta \setminus \widetilde{\mathcal{K}}_U$, if $|D^s\theta|(\mathcal{O}) > 0$, then it follows $|\mu_U^s|(\mathcal{O}) = 0$ and $|\mu_\theta^s|(\mathcal{O}) > 0$ contradicting the relation $|\mu_\theta^s| \ll |\mu_U^s|$. Moreover, by the results in [1] we have $\nu_\theta^s(y) = \nu_U^s(y)$ for $|D^s\theta| - a.e.$ $y \in \mathcal{K}_\theta \cap \widetilde{\mathcal{K}}_U$, so that the condition (59) implies $|D^s\theta| \ll |D^sU|$ in $\widetilde{\mathcal{K}}_U$.

For the Radon-Nikodym derivatives there holds:

$$\frac{d\mu_U^s}{d|D^sU|}(y) = N_U^s(y)\Sigma_\rho(y) \qquad \text{for } |D^sU| - a.e.\ y \in \mathcal{K}_U, \tag{60}$$

from which, using Besicovitch derivation Theorem (Theorem 2.22 in [2]), it follows

$$\frac{d\mu_U^s}{d|\mu_U^s|}(y) = \frac{d|D^sU|}{d|\mu_U^s|}(y)N_U^s(y)\Sigma_\rho(y) \qquad \text{for } |D^sU| - a.e.\ y \in \widetilde{\mathcal{K}}_U. \tag{61}$$

Analogously, there holds

$$\frac{d\mu_\theta^s}{d|\mu_\theta^s|}(y) = \frac{d|D^s\theta|}{d|\mu_\theta^s|}(y)N_\theta^s(y)\Sigma_\rho(y) \qquad \text{for } |D^s\theta| - a.e.\ y \in \widetilde{\mathcal{K}}_\theta, \tag{62}$$

and, for $|D^sU| - a.e.\ y \in \widetilde{\mathcal{K}}_U$, by (60) we get

$$\frac{d|D^sU|}{d|\mu_U^s|}(y) = |N_U^s(y)\Sigma_\rho(y)|^{-1}. \tag{63}$$

Moreover, using (59) and Proposition 4.4 of [1], we have $\nu_\theta^s(y) = \nu_U^s(y)$ for $|D^s\theta| - a.e.\ y \in \widetilde{\mathcal{K}}_\theta$. Then we have

$$|N_U^s(y)\Sigma_\rho(y)| = |(\alpha_U^s(y) \otimes \nu_U^s(y))\Sigma_\rho(y)| = |\langle \nu_U^s(y), \Sigma_\rho(y)\rangle|,$$

and, using (63), for $|D^sU| - a.e.\ y \in \widetilde{\mathcal{K}}_U \cap \widetilde{\mathcal{K}}_\theta$, we infer:

$$\begin{aligned}\langle N_U^s(y)\Sigma_\rho(y), N_\theta^s(y)\Sigma_\rho(y)\rangle \frac{d|D^sU|}{d|\mu_U^s|}(y) &= \frac{\langle (\alpha_U^s(y) \otimes \nu_U^s(y))\Sigma_\rho(y), (\alpha_\theta^s(y) \otimes \nu_\theta^s(y))\Sigma_\rho(y)\rangle}{|N_U^s(y)\Sigma_\rho(y)|} \\ &= |\langle \nu_U^s(y), \Sigma_\rho(y)\rangle| \langle \alpha_U^s(y), \alpha_\theta^s(y)\rangle,\end{aligned} \tag{64}$$

so that for $|D^sU| - a.e.\ y \in (\mathcal{K}_U \setminus \widetilde{\mathcal{K}}_U) \cap (\mathcal{K}_\theta \setminus \widetilde{\mathcal{K}}_\theta)$ we may define by extension:

$$\langle N_U^s(y)\Sigma_\rho(y), N_\theta^s(y)\Sigma_\rho(y)\rangle \frac{d|D^sU|}{d|\mu_U^s|}(y) = 0. \tag{65}$$

Since the vector valued function $d\mu_U^s/d|\mu_U^s|$ has unit norm, using (61), (62), (64) and (65), we obtain:

$$\begin{aligned}\frac{d}{d\lambda}\int_\Omega \left|\frac{d\mu_U^s}{d|\mu_U^s|} + \lambda\frac{d\mu_\theta^s}{d|\mu_U^s|}\right| d|\mu_U^s|\Bigg|_{\lambda=0} &= \int_{\widetilde{\mathcal{K}}_U} \langle \frac{d\mu_U^s}{d|\mu_U^s|}, \frac{d\mu_\theta^s}{d|\mu_U^s|}\rangle d|\mu_U^s| \\ &= \int_{\widetilde{\mathcal{K}}_U} \langle \frac{d\mu_U^s}{d|\mu_U^s|}, \frac{d\mu_\theta^s}{d|\mu_\theta^s|}\rangle d|\mu_\theta^s| \\ &= \int_{\widetilde{\mathcal{K}}_U} \langle N_U^s\Sigma_\rho, N_\theta^s\Sigma_\rho\rangle \frac{d|D^sU|}{d|\mu_U^s|} \cdot \frac{d|D^s\theta|}{d|\mu_\theta^s|} d|\mu_\theta^s| \\ &= \int_{\widetilde{\mathcal{K}}_U} |\langle \nu_U^s, \Sigma_\rho\rangle|\, \langle \alpha_U^s, \alpha_\theta^s\rangle d|D^s\theta|.\end{aligned} \tag{66}$$

Eventually, by (56), (57) and (66) we get the thesis.

□

Now we give the separate directional derivatives of $\overline{\mathcal{G}}(u, v, \sigma)$ with respect to u and v.
Using $U = (u, v_1, v_2) \in \mathbb{R}^3$, we define the following orthogonal unit vectors in $\mathbb{R}^3$:

$$\begin{aligned}\mathbf{e}_u &= \frac{\langle \alpha_U^s, \mathbf{e}_1\rangle}{|\langle \alpha_U^s, \mathbf{e}_1\rangle|}\mathbf{e}_1 \quad |D^su| - a.e. \text{ in } \mathcal{K}_u, \qquad \mathbf{e}_u = \mathbf{e}_1 \quad |D^sv| - a.e. \text{ in } \mathcal{K}_v \setminus \mathcal{K}_u, \\ \mathbf{e}_v &= \frac{\alpha_U^s - \langle \alpha_U^s, \mathbf{e}_1\rangle\mathbf{e}_1}{|\alpha_U^s - \langle \alpha_U^s, \mathbf{e}_1\rangle\mathbf{e}_1|} \quad |D^sv| - a.e. \text{ in } \mathcal{K}_v, \qquad \mathbf{e}_v = \mathbf{e}_2 \quad |D^su| - a.e. \text{ in } \mathcal{K}_u \setminus \mathcal{K}_v,\end{aligned} \tag{67}$$

particularly, $\langle \mathbf{e}_u, \mathbf{e}_v \rangle = 0$, $|D^s U| - a.e.$ in $\mathcal{K}_U$.
Using (5) and (10), we define $\alpha_u^s = \pm 1$ in such a way that

$$N_u^s = \alpha_u^s \nu_u^s \qquad |D^s u| - a.e. \text{ in } \mathcal{K}_u. \tag{68}$$

Corollary 8.4. *Let $w = (u, v, \sigma)$ be such that $v \in [BV(\Omega)]^2$ and $\sigma \in [L^1(\Omega)]^2$. Then $\overline{\mathcal{G}}(u, v, \sigma)$ is differentiable at the point $u \in BV(\Omega)$ in the direction $\eta \in BV(\Omega)$ if and only if $|D^s \eta| \ll |D^s U|$ in $\widetilde{\mathcal{K}}_U$, and in this case there holds:*

$$\begin{aligned} \frac{d}{d\lambda} \overline{\mathcal{G}}(u + \lambda\eta, v, \sigma)\Big|_{\lambda=0} &= 2 \int_\Omega \varphi' \left(|(\nabla U)\Sigma_\rho|^2 \right) \langle \nabla u, \Sigma_\rho \rangle \langle \nabla \eta, \Sigma_\rho \rangle d\mathcal{L}^3 \\ &+ M_\varphi \int_{\mathcal{K}_u} |\langle \nu_U^s, \Sigma_\rho \rangle| \, \langle \alpha_U^s, \mathbf{e}_u \rangle \alpha_u^s \alpha_\eta^s \, d|D^s \eta|. \end{aligned} \tag{69}$$

Proof. Using (5) and (10), we have

$$\frac{dD^s u}{d|D^s U|} = \langle \alpha_U^s, \mathbf{e}_1 \rangle \nu_U^s,$$

and by Besicovitch derivation Theorem:

$$\frac{dD^s u}{d|D^s U|} = N_u^s \frac{d|D^s u|}{d|D^s U|} = |\langle \alpha_U^s, \mathbf{e}_1 \rangle| \, \alpha_u^s \nu_u^s = \langle \alpha_U^s, \mathbf{e}_u \rangle \, \alpha_u^s \nu_u^s. \tag{70}$$

Then, using Proposition 4.4 of [1] we have $\nu_U^s(y) = \nu_u^s(y)$ for $|D^s U| - a.e.$ $y \in \mathcal{K}_u$, so that it follows

$$\langle \alpha_U^s, \mathbf{e}_1 \rangle = \langle \alpha_U^s, \mathbf{e}_u \rangle \alpha_u^s. \tag{71}$$

Let now $\theta \in [BV(\Omega)]^3$ be such that $\theta = (\eta, 0, 0)$. By Lemma 8.3, for fixed v and σ, the directional derivative of $\overline{\mathcal{G}}$ at u in the direction η exists if and only if $|D^s \theta| = |D^s \eta| \ll |D^s U|$ in $\widetilde{\mathcal{K}}_U$. Using (5) we have

$$\frac{dD^s \eta}{d|D^s \theta|} = \frac{dD^s \eta}{d|D^s \eta|} = N_\eta^s = \alpha_\eta^s \nu_\eta^s,$$

from which it follows $\alpha_\theta^s = (\alpha_\eta^s, 0, 0)$.
The statement of the corollary then follows by substituting $\theta = (\eta, 0, 0)$ in the directional derivative (54), taking into account that, using (71), we have

$$\langle \alpha_U^s, \alpha_\theta^s \rangle = \langle \alpha_U^s, \mathbf{e}_1 \rangle \alpha_\eta^s = \langle \alpha_U^s, \mathbf{e}_u \rangle \alpha_u^s \alpha_\eta^s,$$

and that $\langle \alpha_U^s, \mathbf{e}_u \rangle = 0$ outside of $\mathcal{K}_u$. □

In order to compute the directional derivative of $\overline{\mathcal{G}}(u, v, \sigma)$ with respect to v, using (5) and (10), now we set

$$N_v^s = \alpha_v^s \otimes \nu_v^s \in \mathbb{M}^{2\times 3}. \tag{72}$$

Corollary 8.5. *Let $w = (u, v, \sigma)$ be such that $u \in BV(\Omega)$ and $\sigma \in [L^1(\Omega)]^2$. Then $\overline{\mathcal{G}}(u, v, \sigma)$ is differentiable at the point $v \in [BV(\Omega)]^2$ in the direction $\eta \in [BV(\Omega)]^2$ if and only if $|D^s \eta| \ll |D^s U|$ in $\widetilde{\mathcal{K}}_U$, and in this case there holds:*

$$\begin{aligned} \frac{d}{d\lambda} \overline{\mathcal{G}}(u, v + \lambda\eta, \sigma)\Big|_{\lambda=0} &= 2 \int_\Omega \varphi' \left(|(\nabla U)\Sigma_\rho|^2 \right) \langle (\nabla v)\Sigma_\rho, (\nabla \eta)\Sigma_\rho \rangle d\mathcal{L}^3 \\ &+ M_\varphi \int_{\mathcal{K}_v} |\langle \nu_U^s, \Sigma_\rho \rangle| \, \langle \alpha_U^s, \mathbf{e}_v \rangle \langle \alpha_v^s, \alpha_\eta^s \rangle d|D^s \eta|. \end{aligned} \tag{73}$$

Proof. We write $\mathbf{e}_v = (0, \overline{\mathbf{e}}_v)$, where $\overline{\mathbf{e}}_v \in \mathbb{R}^2$ has unit norm, so that we have

$$\frac{dD^s v}{d|D^s U|} = \langle \alpha_U^s, \mathbf{e}_v \rangle \overline{\mathbf{e}}_v \otimes \nu_U^s,$$

and by Besicovitch derivation Theorem:

$$\frac{dD^s v}{d|D^s U|} = N_v^s \frac{d|D^s v|}{d|D^s U|} = \langle \alpha_U^s, \mathbf{e}_v \rangle \alpha_v^s \otimes \nu_v^s. \tag{74}$$

Then, using Proposition 4.4 of [1] we have $\nu_U^s(y) = \nu_v^s(y)$ for $|D^s U| - a.e.\ y \in \mathcal{K}_v$, so that it follows

$$\alpha_v^s = \overline{\mathbf{e}}_v. \tag{75}$$

Let now $\eta = (\eta_1, \eta_2)$ and let $\theta \in [BV(\Omega)]^3$ be such that $\theta = (0, \eta_1, \eta_2)$. By Lemma 8.3, for fixed u and σ, the directional derivative of $\overline{\mathcal{G}}$ at v in the direction η exists if and only if $|D^s\theta| = |D^s\eta| \ll |D^s U|$ in $\widetilde{\mathcal{K}}_U$. Using (5) we have

$$\frac{dD^s \eta}{d|D^s \theta|} = \frac{dD^s \eta}{d|D^s \eta|} = N_\eta^s = \alpha_\eta^s \otimes \nu_\eta^s \in \mathbb{M}^{2\times 3},$$

from which it follows $\alpha_\theta^s = (0, \alpha_\eta^s)$.
The statement of the corollary then follows by substituting $\theta = (0, \eta)$ in the directional derivative (54), taking into account that, using (75), we have

$$\langle \alpha_U^s, \alpha_\theta^s \rangle = \langle \alpha_U^s, \mathbf{e}_v \rangle \langle \overline{\mathbf{e}}_v, \alpha_\eta^s \rangle = \langle \alpha_U^s, \mathbf{e}_v \rangle \langle \alpha_v^s, \alpha_\eta^s \rangle,$$

and that $\langle \alpha_U^s, \mathbf{e}_v \rangle = 0$ outside of $\mathcal{K}_v$. □

Remark 8.6. *Let $w = (u, v, \sigma)$ be such that $v \in [BV(\Omega)]^2$ and $\sigma \in [L^1(\Omega)]^2$. Then $\overline{\mathcal{E}}(u, v, \sigma)$ is differentiable, for fixed v, σ, at the point $u \in BV(\Omega)$ in the direction $\eta \in BV(\Omega)$ if and only if $|D_x^s \eta| \ll |D_x^s u|$ and $|D^s \eta| \ll |D^s U|$ in $\widetilde{\mathcal{K}}_U$, and in this case the corresponding directional derivative is obtained by adding the derivatives* (51) *and* (69).
Let $w = (u, v, \sigma)$ be such that $u \in BV(\Omega)$ and $\sigma \in [L^1(\Omega)]^2$. Then $\overline{\mathcal{E}}(u, v, \sigma)$ is differentiable, for fixed u, σ, at the point $v \in [BV(\Omega)]^2$ in the direction $\eta \in [BV(\Omega)]^2$ if and only if $|D_x^s \eta| \ll |D_x^s v|$ and $|D^s \eta| \ll |D^s U|$ in $\widetilde{\mathcal{K}}_U$, and in this case the corresponding directional derivative is obtained by adding the derivatives (52) *and* (73).

8.2 Directional derivative of $\overline{\mathcal{E}}$ with respect to σ

We compute first the directional derivative of $\overline{\mathcal{G}}(u, v, \sigma)$ with respect to σ for fixed functions $U = (u, v) \in [BV(\Omega)]^3$. Differently from the computations in the previous subsection for u and v, here the variable σ appears in the functional by means of its average σ_ρ, and therefore we need to apply Fubini's Theorem in order to have, in the final expression of the directional derivative, the direction η instead of its average η_ρ. Roughly speaking, it can be seen as a passage of the average from η to the rest of the integrand, which, in some sense, plays a role analogous to an Integration by parts Formula. For applying Fubini's Theorem, we prove the following preliminary lemma, which will be also useful in Section 9 in order to prove that the vector fields A, B, Q have bounded distributional divergence.
We observe that, using (1), the function Σ_ρ is continuous and bounded: $|\Sigma_\rho(y)| \le C$ for any $y \in \Omega$, with $C = C(\sigma, \rho, \Omega)$ positive constant.

Lemma 8.7. *For any $y \in \Omega$ and any $\xi \in \mathbb{M}^{3\times 3}$ the following inequality holds:*

$$\varphi'\left(|\xi \Sigma_\rho(y)|^2\right) |\xi \Sigma_\rho(y)| \le C M_\varphi / 2,$$

where $\xi \Sigma_\rho(y)$ denotes the matrix by vector product of ξ and $\Sigma_\rho(y)$.

Proof. The matrix $\xi \in \mathbb{M}^{3\times 3}$ will also be treated as a vector in $\mathbb{R}^9$. Let $\Psi : \Omega \times \mathbb{R}^9 \to \mathbb{R}$ be the function defined by

$$\Psi(y,\xi) = \varphi\left(|\xi\Sigma_\rho(y)|^2\right), \tag{76}$$

then, by definition of the function φ, the function Ψ is continuous in (y,ξ), and for any $y \in \Omega$ it is convex in ξ and satisfies the growth condition (18) from above. Using (27), the recession function of Ψ is given by

$$\Psi_\infty(y,\xi) = M_\varphi\,|\xi\Sigma_\rho(y)|, \tag{77}$$

moreover, $\Psi(y,\xi)$ is differentiable in ξ for all $(y,\xi) \in \Omega \times \mathbb{R}^9$, and we have

$$\partial_\xi \Psi(y,\xi) = 2\varphi'\left(|\xi\Sigma_\rho(y)|^2\right)[\xi\Sigma_\rho(y)] \otimes \Sigma_\rho(y). \tag{78}$$

Let (y,ξ) be such that $\partial_\xi\Psi(y,\xi) \neq 0$ and let $\zeta = \partial_\xi\Psi(y,\xi)/|\partial_\xi\Psi(y,\xi)|$. Using inequality (22) we have

$$|\partial_\xi\Psi(y,\xi)| \le \Psi_\infty(y,\zeta) = M_\varphi\,|\zeta\Sigma_\rho(y)| \le M_\varphi|\zeta|\,|\Sigma_\rho(y)| \le CM_\varphi. \tag{79}$$

Recalling now that, given vectors $a, c \in \mathbb{R}^m$ and $b, d \in \mathbb{R}^n$, there holds

$$\langle a \otimes b, c \otimes d\rangle = \langle a, c\rangle\langle b, d\rangle, \tag{80}$$

using (78) and (79), and taking into account that φ is nondecreasing, we have

$$|\partial_\xi\Psi(y,\xi)| = 2\varphi'\left(|\xi\Sigma_\rho(y)|^2\right)|\xi\Sigma_\rho(y)| \cdot |\Sigma_\rho(y)| \le CM_\varphi.$$

Since $\Sigma_\rho = (\sigma_{1\rho}, \sigma_{2\rho}, 1)$, then we have $|\Sigma_\rho(y)| \ge 1$ and the statement of the lemma follows. □

Now we compute the directional derivative of $\overline{\mathcal{G}}(u,v,\sigma)$ with respect to σ. We remark that in the following lemma we may assume σ to belong to $[L^1(\Omega)]^2$, because in the definition of $\overline{\mathcal{G}}$ its derivatives do not appear. In the subsequent lemma, instead, we have to restrict to $[BV(\Omega)]^2$.

Lemma 8.8. *Let $U = (u,v) \in [BV(\Omega)]^3$. Then $\overline{\mathcal{G}}(U,\sigma)$ is differentiable at the point $\sigma \in [L^1(\Omega)]^2$ in the direction $\eta \in [L^1(\Omega)]^2$ and there holds:*

$$\frac{d}{d\lambda}\overline{\mathcal{G}}(U,\sigma+\lambda\eta)\Big|_{\lambda=0} = \int_\Omega \langle 2\Lambda_{z,\rho}(w) + M_\varphi\Theta_{z,\rho}(w), \eta(z)\rangle\, d\mathcal{L}^3(z), \tag{81}$$

where the nonlinear operators $\Lambda_{z,\rho}$ and $\Theta_{z,\rho}$ are defined in (37) and (38), respectively.

Proof. Using (55) and (60), we write the relaxed functional $\overline{\mathcal{G}}(U,\sigma)$ in the form

$$\overline{\mathcal{G}}(U,\sigma) = \int_\Omega \varphi\left(|(\nabla U)\Sigma_\rho|^2\right)d\mathcal{L}^3 + M_\varphi\int_\Omega |N_U^s\Sigma_\rho|\,d|D^sU|. \tag{82}$$

Let $\eta \in [L^1(\Omega)]^2$ and $\widetilde{\eta} = (\eta, 0) \in [L^1(\Omega)]^3$.

Step 1. Directional derivative of the absolutely continuous part of $\overline{\mathcal{G}}$.

Using Lebesgue's dominated convergence Theorem, we have

$$\frac{d}{d\lambda}\int_\Omega \varphi\left(|(\nabla U)(\Sigma_\rho + \lambda\widetilde{\eta}_\rho)|^2\right)d\mathcal{L}^3\Big|_{\lambda=0} = 2\int_\Omega \varphi'\left(|(\nabla U)\Sigma_\rho|^2\right)\langle(\nabla U)\Sigma_\rho, (\nabla_x U)\eta_\rho\rangle d\mathcal{L}^3, \tag{83}$$

where η_ρ is the average of η on the ball of radius ρ defined as in (1). We set $\chi_\rho(y,z) = 1_{\Omega\cap B_\rho(y)}(z)$, so that we get

$$\begin{aligned}
&\int_\Omega \varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)\langle(\nabla U(y))\Sigma_\rho(y), (\nabla_x U(y))\eta_\rho(y)\rangle\, d\mathcal{L}^3(y)\\
=\ &\int_\Omega \varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)\left\langle(\nabla U(y))\Sigma_\rho(y), \frac{\nabla U_x(y)}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\int_{\Omega\cap B_\rho(y)}\eta(z)\, d\mathcal{L}^3(z)\right\rangle d\mathcal{L}^3(y)\\
=\ &\int_\Omega \frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)\left\langle(\nabla U(y))\Sigma_\rho(y), \nabla_x U(y)\int_\Omega \chi_\rho(y,z)\eta(z)\, d\mathcal{L}^3(z)\right\rangle d\mathcal{L}^3(y).
\end{aligned}$$

Using Lemma 8.7 with $\xi = \nabla U(y)$ we have

$$\varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)|(\nabla U(y))\Sigma_\rho(y)| \le CM_\varphi/2.$$

Then we have

$$\begin{aligned}
&\left|\frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)\langle(\nabla U(y))\Sigma_\rho(y),(\nabla_x U(y))\eta(z)\rangle\chi_\rho(y,z)\right| \\
\le\ & \frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)|(\nabla U(y))\Sigma_\rho(y)|\cdot|\nabla_x U(y)|\cdot|\eta(z)| \\
\le\ & \frac{CM_\varphi}{2\mathcal{L}^3(\Omega\cap B_\rho(y))}|\nabla U(y)|\cdot|\eta(z)| = g(y,z),
\end{aligned}$$

from which, since $\nabla U \in [L^1(\Omega)]^3$, $\eta \in [L^1(\Omega)]^2$, and Ω is a set with Lipschitz boundary, it follows that the function $g(y,z)$ is summable with respect to the product measure $\mathcal{L}^3(y)\times\mathcal{L}^3(z)$. Then we may apply Fubini's Theorem and, using (83) and the following equality for $(y,z)\in\Omega\times\Omega$:

$$\chi_\rho(y,z) = 1_{\Omega\cap B_\rho(y)}(z) = 1_\Omega(z)1_{B_\rho(y)}(z) = 1_\Omega(y)1_{B_\rho(z)}(y) = 1_{\Omega\cap B_\rho(z)}(y) = \chi_\rho(z,y),$$

we find

$$\begin{aligned}
&\frac{d}{d\lambda}\int_\Omega \varphi\left(|(\nabla U)(\Sigma_\rho+\lambda\widetilde{\eta}_\rho)|^2\right)d\mathcal{L}^3\Big|_{\lambda=0} \\
=\ & 2\int_\Omega\left[\int_\Omega \frac{\chi_\rho(z,y)}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)\Big\langle(\nabla U(y))\Sigma_\rho(y),\nabla_x U(y)\,d\mathcal{L}^3(y)\right]\eta(z)\Big\rangle\,d\mathcal{L}^3(z) \\
=\ & 2\int_\Omega\ \Big\langle\left[\int_{\Omega\cap B_\rho(z)}\frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)(\nabla_x U(y))^t(\nabla U(y))\Sigma_\rho(y)\,d\mathcal{L}^3(y)\right],\eta(z)\Big\rangle\,d\mathcal{L}^3(z),
\end{aligned}$$

from which we get

$$\frac{d}{d\lambda}\int_\Omega \varphi\left(|(\nabla U)(\Sigma_\rho+\lambda\widetilde{\eta}_\rho)|^2\right)d\mathcal{L}^3\Big|_{\lambda=0} = 2\int_\Omega \langle\Lambda_{z,\rho}(w),\eta(z)\rangle\,d\mathcal{L}^3(z). \tag{84}$$

Step 2. Directional derivative of the singular part of $\overline{\mathcal{G}}$.
We have:

$$\int_\Omega |N_U^s\Sigma_\rho|\,d|D^sU| = \int_{\widetilde{\mathcal{K}}_U}|N_U^s\Sigma_\rho|\,d|D^sU|,$$

from which, using Lebesgue's dominated convergence Theorem, it follows

$$\frac{d}{d\lambda}\int_\Omega |N_U^s(\Sigma_\rho+\lambda\widetilde{\eta}_\rho)|\,d|D^sU|\Big|_{\lambda=0} = \int_{\widetilde{\mathcal{K}}_U}\frac{1}{|N_U^s\Sigma_\rho|}\langle N_U^s\Sigma_\rho,(N_U^s)_x\eta_\rho\rangle d|D^sU|. \tag{85}$$

Arguing as in Step 1, we write the integral on the right in the form

$$\int_{\widetilde{\mathcal{K}}_U}\frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\frac{1}{|N_U^s(y)\Sigma_\rho(y)|}\Big\langle N_U^s(y)\Sigma_\rho(y),(N_U^s)_x(y)\int_\Omega \chi_\rho(y,z)\eta(z)\,d\mathcal{L}^3(z)\Big\rangle\,d|D^sU|(y).$$

Now for $(y,z)\in\widetilde{\mathcal{K}}_U\times\Omega$ we estimate

$$\begin{aligned}
&\left|\frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\frac{1}{|N_U^s(y)\Sigma_\rho(y)|}\langle N_U^s(y)\Sigma_\rho(y),(N_U^s)_x(y)\eta(z)\rangle\chi_\rho(y,z)\right| \\
\le\ & \frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}|N_U^s(y)|\cdot|\eta(z)| = h(y,z),
\end{aligned}$$

from which, since $|N_U^s|$ is summable on $\mathcal{K}_U$ with respect to the measure $|D^sU|$, $\eta \in [L^1(\Omega)]^2$, and Ω is a set with Lipschitz boundary, it follows that the function $h(y,z)$ is summable with respect to the product measure $|D^sU|(y) \times \mathcal{L}^3(z)$. Then we may apply Fubini's Theorem and, using (85) and the following equality for $(y,z) \in \mathcal{K}_U \times \Omega$:

$$\chi_\rho(y,z) = 1_{\Omega\cap B_\rho(y)}(z) = 1_\Omega(z)1_{B_\rho(y)}(z) = 1_{\mathcal{K}_U}(y)1_{B_\rho(z)}(y) = 1_{\mathcal{K}_U\cap B_\rho(z)}(y),$$

we find

$$\begin{aligned}
&\frac{d}{d\lambda}\int_\Omega |N_U^s(\Sigma_\rho + \lambda\widetilde{\eta}_\rho)|\, d|D^sU|\Big|_{\lambda=0} \\
&= \int_\Omega \Big\langle \left[\int_{\widetilde{\mathcal{K}}_U\cap B_\rho(z)} \frac{((N_U^s)_x(y))^t N_U^s(y)\Sigma_\rho(y)}{\mathcal{L}^3(\Omega\cap B_\rho(y))\,|N_U^s(y)\Sigma_\rho(y)|}\, d|D^sU|(y)\right], \eta(z)\Big\rangle\, d\mathcal{L}^3(z),
\end{aligned}$$

from which we get

$$\frac{d}{d\lambda}\int_\Omega |N_U^s(\Sigma_\rho + \lambda\widetilde{\eta}_\rho)|\, d|D^sU|\Big|_{\lambda=0} = \int_\Omega \langle \Theta_{\,z,\rho}(w), \eta(z)\rangle\, d\mathcal{L}^3(z). \tag{86}$$

Collecting (82), (84) and (86) we obtain the statement of the lemma. □

Now we compute the directional derivative of $\overline{\mathcal{P}}(\sigma)$ with respect to $\sigma \in [BV(\Omega)]^2$.

Lemma 8.9. *The functional $\overline{\mathcal{P}}(\sigma)$ is differentiable at the point $\sigma \in [BV(\Omega)]^2$ in the direction $\eta \in [BV(\Omega)]^2$ if and only if $|D^s\eta| \ll |D^s\sigma|$ and in this case there holds:*

$$\frac{d}{d\lambda}\overline{\mathcal{P}}(\sigma+\lambda\eta)\Big|_{\lambda=0} = 2\int_\Omega \varphi'(|\nabla\sigma|^2)\langle\nabla\sigma,\nabla\eta\rangle d\mathcal{L}^3 + M_\varphi\int_\Omega \langle N_\sigma^s, N_\eta^s\rangle d|D^s\eta| + 2c\int_\Omega \varphi'\left(|\sigma|^2\right)\langle\sigma,\eta\rangle d\mathcal{L}^3. \tag{87}$$

Proof. Let $\psi_4 : \mathbb{R}^6 \to \mathbb{R}$ be the convex and differentiable function satisfying the growth conditions (18) and such that $\psi_4(\xi) = \varphi(|\xi|^2)$. Let $\mu_4 : \mathcal{B}(\Omega) \to \mathbb{M}^{2\times 3}$ be the measure defined by means of

$$\mu_4 = D\sigma.$$

We treat the matrix valued measure μ_4 as a vector valued measure $\mu_4 : \mathcal{B}(\Omega) \to \mathbb{R}^6$ with components $(D\sigma_1, D\sigma_2)$. Analogously, we treat $\nabla\sigma$ as a vector in $\mathbb{R}^6$ with components $(\nabla\sigma_1, \nabla\sigma_2)$. Then, using (20) and arguing as in the proof of Lemma 8.1, for the function of measure $I(\mu_4)$ we have

$$I(\mu_4) = \int_\Omega \varphi(|\nabla\sigma|^2)d\mathcal{L}^3 + M_\varphi|D^s\sigma|(\Omega),$$

and

$$\overline{\mathcal{P}}(\sigma) = I(\mu_4) + c\int_\Omega \varphi(|\sigma|^2)d\mathcal{L}^3. \tag{88}$$

Now we apply Theorem 4.7 to $I(\mu_4)$. Let $\eta \in [BV(\Omega)]^2$ and $\gamma = D\eta$. Using Theorem 4.7, if and only if $|\gamma^s| \ll |\mu_4^s|$, then we have

$$\frac{d}{d\lambda}I(\mu_4+\lambda\gamma)\Big|_{\lambda=0} = 2\int_\Omega \varphi'(|\nabla\sigma|^2)\langle\nabla\sigma,\nabla\eta\rangle d\mathcal{L}^3 + M_\varphi\int_\Omega \langle N_\sigma^s, N_\eta^s\rangle d|D^s\eta|,$$

from which, observing that the scalar products of vectors in $\mathbb{R}^6$ coincides with the Frobenius scalar products of the corresponding matrices in $\mathbb{M}^{2\times 3}$, the statement of the lemma follows using (88) and observing that

$$\frac{d}{d\lambda}\int_\Omega \varphi\left(|\sigma+\lambda\eta|^2\right)d\mathcal{L}^3\Big|_{\lambda=0} = 2\int_\Omega \varphi'\left(|\sigma|^2\right)\langle\sigma,\eta\rangle d\mathcal{L}^3.$$

□

Remark 8.10. *Let $w = (u, v, \sigma)$ be such that $u \in BV(\Omega)$ and $v \in [BV(\Omega)]^2$. Then $\overline{\mathcal{E}}(u, v, \sigma)$ is differentiable, for fixed u, v, at the point $\sigma \in [BV(\Omega)]^2$ in the direction $\eta \in [BV(\Omega)]^2$ if and only if $|D^s\eta| \ll |D^s\sigma|$, and in this case the corresponding directional derivative is obtained by adding the derivatives* (81) *and* (87).

9 Euler equations and trace properties

Here we use the results of the previous section in order to compute the Euler equations of $\overline{\mathcal{E}}$, distinguishing the contributes of the absolutely continuous part, of the conditions on the singular sets, and of the natural boundary conditions on $\partial\Omega$. In particular, we collect the contributes given by the directional derivatives of the several terms defining the functional, and we use the Integration by parts Formula and the definitions related to traces of vector fields given in Subsection 4.2. This procedure was used by Anzellotti in [5] for functionals defined on scalar BV functions, and it reproduces in the BV framework the same strategy useful to compute the Euler equations for functionals defined in Sobolev spaces. The results are summarized in Theorem 6.1 that we now prove.

Proof of Theorem 6.1

Let $w = (u, v, \sigma) \in Y(\Omega)$ be a minimizer of $\overline{\mathcal{E}}$.

Absolutely continuous part

Step 1. We first consider the necessary condition on w which follows deriving $\overline{\mathcal{E}}$ with respect to u, namely:

$$\frac{d}{d\lambda}\overline{\mathcal{E}}(u + \lambda\eta, v, \sigma)\Big|_{\lambda=0} = 0, \tag{89}$$

with $\eta \in BV(\Omega)$. Using Lemma 8.1, Corollary 8.4 and Remark 8.6, and taking $\eta \in \mathcal{C}_0^\infty(\Omega)$, we get

$$\begin{aligned} &\int_{\Omega\setminus D} \varphi'(|f-u|^2)(u-f)\eta \, d\mathcal{L}^3 + \int_\Omega \varphi'(|\nabla_x u - v|^2)\langle \nabla_x u - v, \nabla_x \eta\rangle d\mathcal{L}^3 \\ + \quad &\int_\Omega \varphi'\left(|(\nabla U)\Sigma_\rho|^2\right)\langle \nabla u, \Sigma_\rho\rangle\langle \nabla\eta, \Sigma_\rho\rangle d\mathcal{L}^3 = 0. \end{aligned} \tag{90}$$

Using the vector fields A, A_0 and T_1 defined in (32) and (34), the identity (90) becomes

$$\int_\Omega \langle A, \nabla\eta\rangle d\mathcal{L}^3 = -\int_\Omega \chi_{\Omega\setminus D}\, \varphi'(|f-u|^2)(u-f)\eta \, d\mathcal{L}^3, \qquad \text{for any } \eta \in \mathcal{C}_0^\infty(\Omega). \tag{91}$$

Let $\psi_1(\xi) = \varphi(|\xi|^2)$ as in Section 4.3 with $k = 2$; using (23) with $\psi = \psi_1$ and $\xi = \nabla_x u - v$, and using (31), we have

$$|A_0(y)| = \varphi'(|\nabla_x u(y) - v(y)|^2)\, |\nabla_x u(y) - v(y)| \le M_\varphi/2, \qquad \textit{a.e. } \text{in } \Omega, \tag{92}$$

and, using Lemma 8.7 with $\xi = \nabla U(y)$, we have

$$|T_1(y)| \le \varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)|(\nabla U(y))\Sigma_\rho(y)| \cdot |\Sigma_\rho(y)| \le C^2 M_\varphi/2, \qquad \textit{a.e. } \text{in } \Omega,$$

so that it follows $A = A_0 + T_1 \in [L^\infty(\Omega)]^3$. By the properties of the function φ we also have

$$\|\varphi'(|f-u|^2)(u-f)\|_{L^\infty(\Omega)} \le M_\varphi/2. \tag{93}$$

Using (91), the vector field A has a divergence in the sense of distributions, inequality (93) yields $\mathrm{div}A \in L^\infty(\Omega)$, so that $A \in W(\Omega)_\infty$ and we have

$$\mathrm{div}A = \chi_{\Omega\setminus D}\, \varphi'(|f-u|^2)(u-f), \qquad a.e. \text{ in } \Omega, \tag{94}$$

which coincides with Equation (39).
Step 2. We now consider the necessary condition on w which follows deriving $\overline{\mathcal{E}}$ with respect to v, namely:

$$\frac{d}{d\lambda}\overline{\mathcal{E}}(u, v+\lambda\eta, \sigma)\Big|_{\lambda=0} = 0, \tag{95}$$

with $\eta \in [BV(\Omega)]^2$. Using Lemma 8.2, Corollary 8.5 and Remark 8.6, and taking $\eta \in [\mathcal{C}_0^\infty(\Omega)]^2$, we get

$$\begin{aligned} &-\int_\Omega \varphi'(|\nabla_x u - v|^2)\langle \nabla_x u - v, \eta\rangle d\mathcal{L}^3 + \int_\Omega \varphi'(|\nabla_x v|^2)\langle \nabla_x v, \nabla_x \eta\rangle d\mathcal{L}^3 \\ &+ \int_\Omega \varphi'\left(|(\nabla U)\Sigma_\rho|^2\right)\langle (\nabla v)\Sigma_\rho, (\nabla \eta)\Sigma_\rho\rangle d\mathcal{L}^3 = 0. \end{aligned} \tag{96}$$

Using the identity (16) with $\alpha = (\nabla v)\Sigma_\rho$, $\beta = \Sigma_\rho$ and $M = \nabla\eta$, and using the matrix-valued fields B, B_0 and T_2 defined in (33) and (35), the identity (96) becomes

$$\int_\Omega \langle B, \nabla\eta\rangle d\mathcal{L}^3 = \int_\Omega \varphi'(|\nabla_x u - v|^2)\langle \nabla_x u - v, \eta\rangle d\mathcal{L}^3, \qquad \text{for any } \eta \in [\mathcal{C}_0^\infty(\Omega)]^2. \tag{97}$$

Let $\psi_2(\xi) = \varphi(|\xi|^2)$ as in Section 4.3 with $k = 4$; using (23) with $\psi = \psi_2$ and $\xi = \nabla_x v$, and using (31), we have

$$|B_0(y)| = \varphi'(|\nabla_x v(y)|^2)\,|\nabla_x v(y)| \le M_\varphi/2, \qquad a.e. \text{ in } \Omega,$$

and, using Lemma 8.7 with $\xi = \nabla U(y)$ and (80), we have

$$|T_2(y)| \le \varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)|(\nabla U(y))\Sigma_\rho(y)|\cdot|\Sigma_\rho(y)| \le C^2 M_\varphi/2, \qquad a.e. \text{ in } \Omega,$$

so that it follows $B = B_0 + T_2 \in L^\infty(\Omega;\mathbb{M}^{2\times 3})$. Using (97), the matrix-valued field B has a divergence in the sense of distributions, inequality (92) yields $\mathrm{div}B \in [L^\infty(\Omega)]^2$, so that $B \in [W(\Omega)_\infty]^2$ and we have

$$-\mathrm{div}B = \varphi'\left(|\nabla_x u - v|^2\right)(\nabla_x u - v), \qquad a.e. \text{ in } \Omega, \tag{98}$$

which coincides with Equation (40).
Step 3. We now consider the necessary condition on w which follows deriving $\overline{\mathcal{E}}$ with respect to σ, namely:

$$\frac{d}{d\lambda}\overline{\mathcal{E}}(u, v, \sigma+\lambda\eta)\Big|_{\lambda=0} = 0, \tag{99}$$

with $\eta \in [BV(\Omega)]^2$. Using Lemma 8.8, Lemma 8.9 and Remark 8.10, and taking $\eta \in [\mathcal{C}_0^\infty(\Omega)]^2$, we get

$$\int_\Omega \langle \Lambda_{z,\rho}(w) + \frac{M_\varphi}{2}\Theta_{z,\rho}(w), \eta\rangle\, d\mathcal{L}^3 + \int_\Omega \varphi'(|\nabla\sigma|^2)\langle\nabla\sigma, \nabla\eta\rangle d\mathcal{L}^3 + c\int_\Omega \varphi'\left(|\sigma|^2\right)\langle\sigma, \eta\rangle d\mathcal{L}^3 = 0. \tag{100}$$

Using the matrix-valued field Q, defined in (36), the identity (100) becomes

$$\int_\Omega \langle Q, \nabla\eta\rangle d\mathcal{L}^3 = -\int_\Omega \langle c\,\varphi'\left(|\sigma|^2\right)\sigma + \Lambda_{z,\rho}(w) + \frac{M_\varphi}{2}\Theta_{z,\rho}(w), \eta\rangle\, d\mathcal{L}^3, \qquad \text{for any } \eta \in [\mathcal{C}_0^\infty(\Omega)]^2. \tag{101}$$

Let $\psi_4(\xi) = \varphi(|\xi|^2)$ as in Section 4.3 with $k = 6$; using (23) with $\psi = \psi_4$ and $\xi = \nabla\sigma$, and using (31), we have

$$|Q(z)| \le M_\varphi/2, \qquad a.e. \text{ in } \Omega,$$

so that it follows $Q \in L^\infty(\Omega; \mathbb{M}^{2\times 3})$. Arguing as in Step 1 of the proof of Lemma 8.8, we have

$$\left|\varphi'\left(|(\nabla U(y))\Sigma_\rho(y)|^2\right)(\nabla_x U(y))^t(\nabla U(y))\Sigma_\rho(y)\right| \le \frac{CM_\varphi}{2}|\nabla U(y)|,$$

from which it follows

$$|\Lambda_{z,\rho}(w)| \le \frac{CM_\varphi}{2}\int_{\Omega\cap B_\rho(z)} \frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}|\nabla U(y)|\, d\mathcal{L}^3(y),$$

from which, since $\nabla U \in [L^1(\Omega)]^3$ and Ω is a set with Lipschitz boundary, it follows $\Lambda_{z,\rho}(w) \in [L^\infty(\Omega)]^2$ as a function of z. Arguing now as in Step 2 of the proof of Lemma 8.8, for $|D^sU| - a.e.\ y \in \widetilde{\mathcal{K}}_U$ we have

$$\frac{1}{|N^s_U(y)\Sigma_\rho(y)|}\left|((N^s_U)_x(y))^t N^s_U(y)\Sigma_\rho(y)\right| \le |N^s_U(y)| = 1,$$

from which it follows

$$|\Theta_{z,\rho}(w)| \le \int_{\widetilde{\mathcal{K}}_U\cap B_\rho(z)} \frac{1}{\mathcal{L}^3(\Omega\cap B_\rho(y))}\, d|D^sU|(y),$$

from which, since $U \in [BV(\Omega)]^3$ and Ω is a set with Lipschitz boundary, it follows $\Theta_{z,\rho}(w) \in [L^\infty(\Omega)]^2$ as a function of z. Using again $\psi_1(\xi) = \varphi(|\xi|^2)$ with $k = 2$, and arguing as before with $\xi = \sigma$, we have

$$c\,\varphi'\left(|\sigma(z)|^2\right)|\sigma(z)| \le cM_\varphi/2, \qquad a.e.\ \text{in } \Omega,$$

so that

$$c\,\varphi'\left(|\sigma|^2\right)\sigma + \Lambda_{z,\rho}(w) + \frac{M_\varphi}{2}\Theta_{z,\rho}(w) \in [L^\infty(\Omega)]^2.$$

Using (101), the matrix-valued field Q has a divergence in the sense of distributions with $\mathrm{div}Q \in [L^\infty(\Omega)]^2$, so that $Q \in [W(\Omega)_\infty]^2$ and we have

$$\mathrm{div}Q = c\,\varphi'\left(|\sigma|^2\right)\sigma + \Lambda_{z,\rho}(w) + \frac{M_\varphi}{2}\Theta_{z,\rho}(w), \tag{102}$$

which coincides with Equation (41).

Natural boundary conditions on $\partial\Omega$

We choose $\eta \in \mathcal{C}^\infty(\overline{\Omega})$ and we use (11) for the vector field A:

$$\int_\Omega \eta\,\mathrm{div}A\, d\mathcal{L}^3 = \int_{\partial\Omega}[\langle A, \nu_\Omega\rangle]\eta\, d\mathcal{H}^2 - \int_\Omega \langle A, \nabla\eta\rangle d\mathcal{L}^3. \tag{103}$$

Then, using (94) and subtracting (91) from (103) we find

$$\int_{\partial\Omega}[\langle A, \nu_\Omega\rangle]\eta\, d\mathcal{H}^2 = 0, \qquad \text{for any } \eta \in \mathcal{C}^\infty(\overline{\Omega}),$$

from which, using Lemma 6.2.1 of [7], the natural boundary condition (43) follows. In [7] Lemma 6.2.1 is stated for sets Ω with $\mathcal{C}^1$ boundary, but one can check that the statement holds true also for sets with Lipschitz boundary.
Now we choose $\eta \in [\mathcal{C}^\infty(\overline{\Omega})]^2$ and by (11) we get the following formula for the matrix-valued field B:

$$\int_\Omega \langle \eta, \mathrm{div}B\rangle\, d\mathcal{L}^3 = \int_{\partial\Omega}\langle [B\nu_\Omega], \eta\rangle\, d\mathcal{H}^2 - \int_\Omega \langle B, \nabla\eta\rangle d\mathcal{L}^3. \tag{104}$$

Then, using (98) and subtracting (97) from (104) we find

$$\int_{\partial\Omega}\langle [B\nu_\Omega], \eta\rangle\, d\mathcal{H}^2 = 0, \qquad \text{for any } \eta \in [\mathcal{C}^\infty(\overline{\Omega})]^2,$$

from which, using Lemma 6.2.1 of [7], the natural boundary condition (44) follows.
The natural boundary condition (45) follows analogously.

Conditions on the singular sets

We first consider the conditions on $\mathcal{K}_U$.
Using Remark 8.6 for the directional derivative of $\overline{\mathcal{E}}$ with respect to u, substituting (51) and (69) into (89), using the definition (32) and (34) of the vector field A, taking $\eta = u$ and using $|N^s_{u,x}| = 1$ and $(\alpha^s_u)^2 = 1$, we get

$$2\int_\Omega \chi_{\Omega\setminus D}\,\varphi'(|f-u|^2)(u-f)u\,d\mathcal{L}^3 + 2\int_\Omega \langle A, \nabla u\rangle d\mathcal{L}^3 \tag{105}$$
$$+ \quad M_\varphi |D^s_x u|(\Omega) + M_\varphi \int_{\widetilde{\mathcal{K}}_U} |\langle \nu^s_U, \Sigma_\rho\rangle|\,\langle \alpha^s_U, \mathbf{e}_u\rangle\, d|D^s u| = 0.$$

We now use the boundary condition (43) and we apply Theorem 4.6 to the vector field A and the function u:

$$\int_\Omega u\,\mathrm{div} A\, d\mathcal{L}^3 = -\int_\Omega \langle A, \nabla u\rangle d\mathcal{L}^3 - \int_\Omega [\![\langle A, N^s_u\rangle]\!] d|D^s u|\,,$$

from which, using (39) and (105), it follows

$$M_\varphi |D^s_x u|(\Omega) + M_\varphi \int_{\widetilde{\mathcal{K}}_U} |\langle \nu^s_U, \Sigma_\rho\rangle|\,\langle \alpha^s_U, \mathbf{e}_u\rangle\, d|D^s u| - 2\int_\Omega [\![\langle A, N^s_u\rangle]\!] d|D^s u| = 0. \tag{106}$$

Using Remark 8.6 for the directional derivative of $\overline{\mathcal{E}}$ with respect to v, substituting (52) and (73) into (95), using the definition (33) and (35) of the matrix-valued field B, taking $\eta = v$ and using (16), $|N^s_{v,x}| = 1$ and $|\alpha^s_v|^2 = 1$, we get

$$-2\int_\Omega \varphi'(|\nabla_x u - v|^2)\langle \nabla_x u - v, v\rangle d\mathcal{L}^3 + 2\int_\Omega \langle B, \nabla v\rangle d\mathcal{L}^3 \tag{107}$$
$$+ \quad M_\varphi |D^s_x v|(\Omega) + M_\varphi \int_{\widetilde{\mathcal{K}}_U} |\langle \nu^s_U, \Sigma_\rho\rangle|\,\langle \alpha^s_U, \mathbf{e}_v\rangle\, d|D^s v| = 0.$$

We now use the boundary condition (44), we apply the identity (15) to the matrix-valued field B and the vector field v, and we use (17):

$$\int_\Omega \langle v, \mathrm{div} B\rangle d\mathcal{L}^3 = -\int_\Omega \langle B, \nabla v\rangle d\mathcal{L}^3 - \int_\Omega [\![\langle B, N^s_v\rangle]\!] d|D^s v|,$$

from which, using (40) and (107), it follows

$$M_\varphi |D^s_x v|(\Omega) + M_\varphi \int_{\widetilde{\mathcal{K}}_U} |\langle \nu^s_U, \Sigma_\rho\rangle|\,\langle \alpha^s_U, \mathbf{e}_v\rangle\, d|D^s v| - 2\int_\Omega [\![\langle B, N^s_v\rangle]\!] d|D^s v| = 0. \tag{108}$$

Now, using (70) and (74) we have

$$\frac{d|D^s u|}{d|D^s U|} = \langle \alpha^s_U, \mathbf{e}_u\rangle, \qquad \frac{d|D^s v|}{d|D^s U|} = \langle \alpha^s_U, \mathbf{e}_v\rangle, \tag{109}$$

and taking into account that, using $\alpha^s_U \in S^2$ and (67),

$$\langle \alpha^s_U, \mathbf{e}_u\rangle^2 + \langle \alpha^s_U, \mathbf{e}_v\rangle^2 = 1,$$

adding (106) and (108), we obtain

$$M_\varphi |D^s_x u|(\Omega) - 2\int_\Omega [\![\langle A, N^s_u\rangle]\!] d|D^s u| + M_\varphi |D^s_x v|(\Omega) - 2\int_\Omega [\![\langle B, N^s_v\rangle]\!] d|D^s v|$$
$$+ \quad M_\varphi \int_{\widetilde{\mathcal{K}}_U} |\langle \nu^s_U, \Sigma_\rho\rangle|\; d|D^s U| = 0. \tag{110}$$

Now we observe that, using (68), we have

$$\frac{dD^s_x u}{d|D^s u|} = \alpha^s_u \nu^s_{u,x}, \qquad |D^s_x u|(\Omega) = \int_\Omega \left|\frac{dD^s_x u}{d|D^s u|}\right| \, d|D^s u| = \int_\Omega |\nu^s_{u,x}| \, d|D^s u|,$$

where we remind that $\nu^s_{u,x}$ denotes the orthogonal projection of ν^s_u on the spatial subset of space-time $\mathbb{R}^3$. Analogously, using (72), we have

$$\frac{dD^s_x v}{d|D^s v|} = \alpha^s_v \otimes \nu^s_{v,x}, \qquad |D^s_x v|(\Omega) = \int_\Omega \left|\frac{dD^s_x v}{d|D^s v|}\right| \, d|D^s v| = \int_\Omega |\nu^s_{v,x}| \, d|D^s v|.$$

Then the equality (110) becomes

$$\begin{aligned} & \int_\Omega \left\{ M_\varphi |\nu^s_{u,x}| - 2[\![\langle A, N^s_u\rangle]\!] \right\} d|D^s u| + \int_\Omega \left\{ M_\varphi |\nu^s_{v,x}| - 2[\![\langle B, N^s_v\rangle]\!] \right\} d|D^s v| \\ + \;& M_\varphi \int_{\widetilde{\mathcal{K}}_U} |\langle \nu^s_U, \Sigma_\rho\rangle| \; d|D^s U| = 0, \end{aligned}$$

from which, by the definition (53) of $\widetilde{\mathcal{K}}_U$, taking into account that $N^s_U \Sigma_\rho = \alpha^s_U \langle \nu^s_U, \Sigma_\rho\rangle$, and using again (109), we get

$$\begin{aligned} & \int_\Omega \left\{ M_\varphi |\nu^s_{u,x}| - 2[\![\langle A, N^s_u\rangle]\!] \right\} \langle \alpha^s_U, \mathbf{e}_u\rangle d|D^s U| + \int_\Omega \left\{ M_\varphi |\nu^s_{v,x}| - 2[\![\langle B, N^s_v\rangle]\!] \right\} \langle \alpha^s_U, \mathbf{e}_v\rangle d|D^s U| \\ + \;& M_\varphi \int_\Omega |\langle \nu^s_U, \Sigma_\rho\rangle| \; d|D^s U| = 0. \end{aligned} \tag{111}$$

In order to get (42), we first need to prove that the overall integrand in (111) is non-negative, so that (111) will implies that it is actually equal to 0 (namely (120)).
Let $\psi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the convex and differentiable function satisfying the growth conditions (18) and such that $\psi_1(\xi) = \varphi(|\xi|^2)$. Applying the inequality (22) to the function ψ_1, using (19), (31) and the definition (34) of the vector field A_0, we have

$$\begin{aligned} \langle A_0, N^s_u\rangle &= \alpha^s_u \varphi'(|\nabla_x u - v|^2)\langle \nabla_x u - v, \nu^s_{u,x}\rangle \\ &= \frac{\alpha^s_u}{2}\langle \partial_\xi \psi_1(\nabla_x u - v), \nu^s_{u,x}\rangle \le \frac{\alpha^s_u}{2}\psi_{1,\infty}(\nu^s_{u,x}) \le \frac{M_\varphi}{2}|\nu^s_{u,x}|, \end{aligned}$$

from which it follows the inequality

$$\langle A_0(y'), N^s_u(y)\rangle \le \frac{M_\varphi}{2}|\nu^s_{u,x}(y)|, \qquad \text{for a.e. } y' = (x',t') \in \Omega \text{ and } |D^s u| - \text{a.e.}\, y = (x,t) \in \mathcal{K}_u. \tag{112}$$

Now we apply the inequality (22) to the convex function $\psi_2 : \mathbb{R}^4 \to \mathbb{R}$ such that $\psi_2(\xi) = \varphi(|\xi|^2)$ and to the matrix-valued field B_0 defined in (35), obtaining

$$\begin{aligned} \langle B_0, N^s_v\rangle &= \varphi'(|\nabla_x v|^2)\langle \nabla_x v, \alpha^s_v \otimes \nu^s_{v,x}\rangle \\ &= \frac{1}{2}\langle \partial_\xi \psi_2(\nabla_x v), \alpha^s_v \otimes \nu^s_{v,x}\rangle \le \frac{1}{2}\psi_{2,\infty}(\alpha^s_v \otimes \nu^s_{v,x}) \le \frac{M_\varphi}{2}|\nu^s_{v,x}|, \end{aligned}$$

from which it follows the inequality

$$\langle B_0(y'), N^s_v(y)\rangle \le \frac{M_\varphi}{2}|\nu^s_{v,x}(y)|, \qquad \text{for a.e. } y' = (x',t') \in \Omega \text{ and } |D^s v| - \text{a.e.}\, y = (x,t) \in \mathcal{K}_v. \tag{113}$$

Using now (109) and the definitions (32-35), we have

$$\langle T_1, N^s_u\rangle\langle \alpha^s_U, \mathbf{e}_u\rangle + \langle T_2, N^s_v\rangle\langle \alpha^s_U, \mathbf{e}_v\rangle = \langle T, N^s_U\rangle. \tag{114}$$

Then we apply the inequality (22) to the function $\Psi(y,\xi)$ defined in (76), which is convex in ξ for any $y\in\Omega$, and to the matrix-valued field T: using (77) and (78) with $\xi=\nabla U$ (we drop the dependence on y) we obtain

$$\begin{aligned}\langle T, N_U^s\rangle &= \varphi'\left(|(\nabla U)\Sigma_\rho|^2\right)\langle((\nabla U)\Sigma_\rho)\otimes\Sigma_\rho, \alpha_U^s\otimes\nu_U^s\rangle\\ &= \frac{1}{2}\langle\partial_\xi\Psi(\nabla U), \alpha_U^s\otimes\nu_U^s\rangle \le \frac{1}{2}\Psi_\infty(\alpha_U^s\otimes\nu_U^s)\\ &= \frac{M_\varphi}{2}\left|(\alpha_U^s\otimes\nu_U^s)\,\Sigma_\rho\right| = \frac{M_\varphi}{2}\left|\langle\nu_U^s,\Sigma_\rho\rangle\right|,\end{aligned}$$

from which it follows the inequality

$$\begin{aligned}&\langle T(y'), N_U^s(y)\rangle \le \frac{M_\varphi}{2}\left|\langle\nu_U^s(y),\Sigma_\rho(y')\rangle\right|,\\ &\text{for a.e. } y'=(x',t')\in\Omega \text{ and } |D^sU|-\text{a.e.}\, y=(x,t)\in\mathcal{K}_U.\end{aligned}\tag{115}$$

According to Definition 4.5, and using the identities (15) and (17), we consider the averages on infinitesimal cylinders $\mathcal{C}_{r,\delta}(y,\nu_U^s)$ of the vector field A_0 and the matrix-valued field B_0. Using inequalities (112) and (113) we have

$$\frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu_u^s(y))}\langle A_0(y'), N_u^s(y)\rangle d\mathcal{L}^3(y') \le \frac{M_\varphi}{2}|\nu_{u,x}^s(y)|, \quad \text{for } |D^su|-\text{a.e.}\, y=(x,t)\in\mathcal{K}_u, \tag{116}$$

$$\frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu_v^s(y))}\langle B_0(y'), N_v^s(y)\rangle d\mathcal{L}^3(y') \le \frac{M_\varphi}{2}|\nu_{v,x}^s(y)|, \quad \text{for } |D^sv|-\text{a.e.}\, y=(x,t)\in\mathcal{K}_v. \tag{117}$$

Using inequality (115), for the matrix-valued field T we get

$$\begin{aligned}&\frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu_U^s(y))}\langle T(y'), N_U^s(y)\rangle d\mathcal{L}^3(y') \le \frac{M_\varphi}{4\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu_U^s(y))}\left|\langle\nu_U^s(y),\Sigma_\rho(y')\rangle\right| d\mathcal{L}^3(y'),\\ &\text{for } |D^sU|-\text{a.e.}\, y=(x,t)\in\mathcal{K}_U.\end{aligned}\tag{118}$$

Since the function Σ_ρ is continuous on $\overline{\Omega}$, then it is uniformly continuous and, for $|D^sU|$-a.e. $y\in\mathcal{K}_U$, we have

$$\begin{aligned}&\lim_{\delta\to0^+}\lim_{r\to0^+}\frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu_U^s(y))}\left|\langle\nu_U^s(y),\Sigma_\rho(y')\rangle\right| d\mathcal{L}^3(y')\\ \le\ &\left|\langle\nu_U^s(y),\Sigma_\rho(y)\rangle\right| + \lim_{\delta\to0^+}\lim_{r\to0^+}\sup_{y'\in\mathcal{C}_{r,\delta}(y,\nu_U^s(y))}|\Sigma_\rho(y')-\Sigma_\rho(y)| = \left|\langle\nu_U^s(y),\Sigma_\rho(y)\rangle\right|.\end{aligned}\tag{119}$$

Using now inequalities (116-119), taking the limit for $r,\delta\to0^+$, and using Definition 4.5, for $|D^sU|$-a.e. $y\in\mathcal{K}_U$, we have

$$\begin{aligned}&2[\![\langle A, N_u^s\rangle]\!](y)\langle\alpha_U^s(y),\mathbf{e}_u(y)\rangle + 2[\![\langle B, N_v^s\rangle]\!](y)\langle\alpha_U^s(y),\mathbf{e}_v(y)\rangle\\ \le\ &M_\varphi\left\{|\nu_{u,x}^s(y)|\langle\alpha_U^s(y),\mathbf{e}_u(y)\rangle + |\nu_{v,x}^s(y)|\langle\alpha_U^s(y),\mathbf{e}_v(y)\rangle + |\langle\nu_U^s(y),\Sigma_\rho(y)\rangle|\right\}.\end{aligned}$$

Hence the overall integrand with respect to the measure $|D^sU|$ in the identity (111) is non-negative, from which it follows

$$\begin{aligned}&2[\![\langle A, N_u^s\rangle]\!](y)\langle\alpha_U^s(y),\mathbf{e}_u(y)\rangle + 2[\![\langle B, N_v^s\rangle]\!](y)\langle\alpha_U^s(y),\mathbf{e}_v(y)\rangle\\ =\ &M_\varphi\left\{|\nu_{u,x}^s(y)|\langle\alpha_U^s(y),\mathbf{e}_u(y)\rangle + |\nu_{v,x}^s(y)|\langle\alpha_U^s(y),\mathbf{e}_v(y)\rangle + |\langle\nu_U^s(y),\Sigma_\rho(y)\rangle|\right\}.\end{aligned}\tag{120}$$

for $|D^sU|$-a.e. $y\in\mathcal{K}_U$.

We now go to conclude the proof of the first three equalities in (42), proceeding in the following steps.
Step 1. For $|D^s u|$-a.e. $y \in \mathcal{K}_u$ we define

$$\begin{aligned}\overline{A}_0^{(\delta,r)}(y) &= \frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu_u^s(y))}\langle A_0(y'), N_u^s(y)\rangle d\mathcal{L}^3(y'),\\ \overline{T}_1^{(\delta,r)}(y) &= \frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu_u^s(y))}\langle T_1(y'), N_u^s(y)\rangle d\mathcal{L}^3(y'),\end{aligned}$$

and for $|D^s v|$-a.e. $y \in \mathcal{K}_v$ we define

$$\overline{B}^{(\delta,r)}(y) = \frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu_v^s(y))}\langle B(y'), N_v^s(y)\rangle d\mathcal{L}^3(y').$$

We rewrite equality (120) as follows (we drop the dependence on y):

$$\begin{aligned}&2\langle\alpha_U^s,\mathbf{e}_u\rangle\lim_{\delta,r\to0^+}\left(\overline{A}_0^{(\delta,r)}+\overline{T}_1^{(\delta,r)}\right)+2\langle\alpha_U^s,\mathbf{e}_v\rangle\lim_{\delta,r\to0^+}\overline{B}^{(\delta,r)}\\ =\;& M_\varphi\left\{|\nu_{u,x}^s|\langle\alpha_U^s,\mathbf{e}_u\rangle+|\nu_{v,x}^s|\langle\alpha_U^s,\mathbf{e}_v\rangle+|\langle\nu_U^s,\Sigma_\rho\rangle|\right\}.\end{aligned}\tag{121}$$

Using (114) and inequalities (116-119), we have for any δ, r:

$$\begin{aligned}\overline{A}_0^{(\delta,r)} &\leq \frac{M_\varphi}{2}|\nu_{u,x}^s|,\\ \langle\alpha_U^s,\mathbf{e}_u\rangle\overline{T}_1^{(\delta,r)}+\langle\alpha_U^s,\mathbf{e}_v\rangle\overline{B}^{(\delta,r)}+O(\delta,r) &\leq \frac{M_\varphi}{2}\left[|\nu_{v,x}^s|\langle\alpha_U^s,\mathbf{e}_v\rangle+|\langle\nu_U^s,\Sigma_\rho\rangle|\right],\end{aligned}$$

where $\lim_{\delta,r\to0^+}O(\delta,r)=0$. It follows the existence of the separate limit

$$\lim_{\delta,r\to0^+}\overline{A}_0^{(\delta,r)}=\frac{M_\varphi}{2}|\nu_{u,x}^s|,\tag{122}$$

otherwise, on a subsequence (not relabeled) of (δ, r), from the previous inequalities we have

$$\lim_{\delta,r\to0^+}\overline{A}_0^{(\delta,r)}<\frac{M_\varphi}{2}|\nu_{u,x}^s|,$$

and equality (121) would not be satisfied. Then, for $|D^s u|$-a.e. $y \in \mathcal{K}_u$, there exists the separate trace $[\![\langle A_0, N_u^s\rangle]\!](y)$ as a limit of the averages on cylinders, and from equality (122) it follows

$$[\![\langle A_0,N_u^s\rangle]\!](y)=\frac{M_\varphi}{2}|\nu_{u,x}^s(y)|,\qquad\text{for } |D^s u|-\text{a.e. } y\in\mathcal{K}_u,$$

which is the first equality of (42) in the main results.
Step 2. As above, for $|D^s v|$-a.e. $y \in \mathcal{K}_v$ we define

$$\begin{aligned}\overline{B}_0^{(\delta,r)}(y) &= \frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu_v^s(y))}\langle B_0(y'), N_v^s(y)\rangle d\mathcal{L}^3(y'),\\ \overline{T}_2^{(\delta,r)}(y) &= \frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu_v^s(y))}\langle T_2(y'), N_v^s(y)\rangle d\mathcal{L}^3(y'),\end{aligned}$$

and, using (122), we rewrite equality (120) as follows

$$2\langle\alpha_U^s,\mathbf{e}_u\rangle\lim_{\delta,r\to0^+}\overline{T}_1^{(\delta,r)}+2\langle\alpha_U^s,\mathbf{e}_v\rangle\lim_{\delta,r\to0^+}\left(\overline{B}_0^{(\delta,r)}+\overline{T}_2^{(\delta,r)}\right)=M_\varphi\left\{|\nu_{v,x}^s|\langle\alpha_U^s,\mathbf{e}_v\rangle+|\langle\nu_U^s,\Sigma_\rho\rangle|\right\}.$$

Using now (114) and inequalities (117-119), we have for any δ, r:

$$\begin{aligned} \overline{B}_0^{(\delta,r)} &\leq \frac{M_\varphi}{2}|\nu^s_{v,x}|, \\ \langle\alpha^s_U, \mathbf{e}_u\rangle\overline{T}_1^{(\delta,r)} + \langle\alpha^s_U, \mathbf{e}_v\rangle\overline{T}_2^{(\delta,r)} + O(\delta, r) &\leq \frac{M_\varphi}{2}\left|\langle\nu^s_U, \Sigma_\rho\rangle\right|. \end{aligned}$$

Arguing as before, the existence of the separate limit

$$\lim_{\delta,r\to 0^+} \overline{B}_0^{(\delta,r)} = \frac{M_\varphi}{2}|\nu^s_{v,x}| \tag{123}$$

follows. Then, for $|D^s v|$-a.e. $y \in \mathcal{K}_v$, there exists the separate trace $[\![\langle B_0, N^s_v\rangle]\!](y)$ as a limit of the averages on cylinders, and from equality (123) it follows

$$[\![\langle B_0, N^s_v\rangle]\!](y) = \frac{M_\varphi}{2}|\nu^s_{v,x}(y)|, \qquad \text{for } |D^s v| - \text{a.e. } y \in \mathcal{K}_v,$$

which is the second equality of (42) in the main results.
Step 3. Now, for $|D^s U|$-a.e. $y \in \mathcal{K}_U$ we define

$$\overline{T}^{(\delta,r)}(y) = \frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu^s_U(y))} \langle T(y'), N^s_U(y)\rangle d\mathcal{L}^3(y'),$$

and, using (122), (123) and (114), we rewrite equality (120) as follows

$$2\lim_{\delta,r\to 0^+}\left[\overline{T}_1^{(\delta,r)}\langle\alpha^s_U, \mathbf{e}_u\rangle + \overline{T}_2^{(\delta,r)}\langle\alpha^s_U, \mathbf{e}_v\rangle\right] = 2\lim_{\delta,r\to 0^+}\overline{T}^{(\delta,r)} = M_\varphi\left|\langle\nu^s_U, \Sigma_\rho\rangle\right|.$$

Then, for $|D^s U|$-a.e. $y \in \mathcal{K}_U$, there exists the separate trace $[\![\langle T, N^s_U\rangle]\!](y)$ as a limit of the averages on cylinders, and we have

$$[\![\langle T, N^s_U\rangle]\!](y) = \frac{M_\varphi}{2}\left|\langle\nu^s_U(y), \Sigma_\rho(y)\rangle\right|, \qquad \text{for } |D^s U| - \text{a.e. } y \in \mathcal{K}_U,$$

which is the third equality of (42) in the main results.

We now consider the conditions on the set $\mathcal{K}_\sigma$.
Using now Remark 8.10 for the directional derivative of $\overline{\mathcal{E}}$ with respect to σ, substituting (81) and (87) into (99), using the definition (36) of the matrix-valued field Q, taking $\eta = \sigma$ and using $|N^s_\sigma| = 1$, we get

$$\begin{aligned} &\int_\Omega \langle 2\Lambda_{\,z,\rho}(w) + M_\varphi\Theta_{\,z,\rho}(w), \sigma(z)\rangle\, d\mathcal{L}^3(z) + 2\int_\Omega \varphi'(|\nabla\sigma|^2)|\nabla\sigma|^2 d\mathcal{L}^3 \\ + \quad & M_\varphi|D^s\sigma|(\Omega) + 2c\int_\Omega \varphi'\left(|\sigma|^2\right)|\sigma|^2 d\mathcal{L}^3 = 0. \end{aligned} \tag{124}$$

We now use the boundary condition (45), we apply the identity (15) to the matrix-valued field Q and the vector field σ, and we use (17):

$$\int_\Omega \langle\sigma, \mathrm{div}Q\rangle d\mathcal{L}^3 = -\int_\Omega \langle Q, \nabla\sigma\rangle d\mathcal{L}^3 - \int_\Omega [\![\langle Q, N^s_\sigma\rangle]\!] d|D^s\sigma|,$$

from which, using (41) and (124), it follows

$$M_\varphi|D^s\sigma|(\Omega) - 2\int_\Omega [\![\langle Q, N^s_\sigma\rangle]\!] d|D^s\sigma| = 0,$$

so that

$$\int_\Omega \{M_\varphi - 2[\![\langle Q, N^s_\sigma\rangle]\!]\} d|D^s\sigma| = 0. \tag{125}$$

Now we apply the inequality (22) to the convex function $\psi_4 : \mathbb{R}^6 \to \mathbb{R}$ such that $\psi_4(\xi) = \varphi(|\xi|^2)$ and to the matrix-valued field Q defined in (36), obtaining

$$\begin{aligned}\langle Q, N^s_\sigma\rangle &= \varphi'(|\nabla\sigma|^2)\langle\nabla\sigma, \alpha^s_\sigma \otimes \nu^s_\sigma\rangle \\ &= \frac{1}{2}\langle\partial_\xi\psi_4(\nabla\sigma), \alpha^s_\sigma \otimes \nu^s_\sigma\rangle \le \frac{1}{2}\psi_{4,\infty}(\alpha^s_\sigma \otimes \nu^s_\sigma) = \frac{M_\varphi}{2}|\alpha^s_\sigma \otimes \nu^s_\sigma| = \frac{M_\varphi}{2},\end{aligned}$$

from which it follows the inequality

$$\langle Q(y'), N^s_\sigma(y)\rangle \le \frac{M_\varphi}{2}, \qquad \text{for a.e. } y' = (x', t') \in \Omega \text{ and } |D^s\sigma| - \text{a.e.}\, y = (x,t) \in \mathcal{K}_\sigma. \tag{126}$$

Then we consider the average on infinitesimal cylinders $\mathcal{C}_{r,\delta}(y, \nu^s_\sigma)$ of the vector field Q so that, using inequality (126), we have

$$\frac{1}{2\pi r\delta^2}\int_{\mathcal{C}_{r,\delta}(y,\nu^s_\sigma(y))} \langle Q(y'), N^s_\sigma(y)\rangle d\mathcal{L}^3(y') \le \frac{M_\varphi}{2}, \quad \text{for } |D^s\sigma| - \text{a.e.}\, y = (x,t) \in \mathcal{K}_\sigma.$$

Then, taking the limit for $r, \delta \to 0^+$, for $|D^s\sigma|$-a.e. $y \in \mathcal{K}_\sigma$, we have

$$2[\![\langle Q, N^s_\sigma\rangle]\!](y) \le M_\varphi,$$

hence the integrand with respect to the measure $|D^s\sigma|$ in the identity (125) is non-negative, from which it follows

$$[\![\langle Q, N^s_\sigma\rangle]\!](y) = \frac{M_\varphi}{2}, \qquad \text{for } |D^s\sigma| - \text{a.e. } y \in \mathcal{K}_\sigma,$$

which is the last equality of (42) in the main results.
The proof of Theorem 6.1 is completed. □

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