Singular analysis of the optimizers of the principal eigenvalue in indefinite weighted Neumann problems

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Abstract

We study the minimization of the positive principal eigenvalue associated to a weighted Neumann problem settled in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, within a suitable class of sign-changing weights. This problem arises in the study of the persistence of a species in population dynamics. Denoting with u the optimal eigenfunction and with D its super-level set associated to the optimal weight, we perform the analysis of the singular limit of the optimal eigenvalue as the measure of D tends to zero. We show that, when the measure of D is sufficiently small, u has a unique local maximum point lying on the boundary of Ω and D is connected. Furthermore, the boundary of D intersects the boundary of the box Ω, and more precisely, $\mathcal{H}^{N-1}(\partial D \cap \partial\Omega) \geq C|D|^{(N-1)/N}$ for some universal constant $C > 0$. Though widely expected, these properties are still unknown if the measure of D is arbitrary.

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1 Introduction

A classical model describing the spatial dispersal of a population in an heterogenous environment relies on a reaction-diffusion equation of logistic type. In order to represent the habitat subdivided into patches, one introduces a sign-changing weight m in the equation, so that the favorable and hostile zones respectively correspond to the positivity and negativity sets of m (see [4, 2, 6]). The equation is naturally associated with homogeneous Neumann boundary conditions when considering this problem in a bounded open set $\Omega \subset \mathbb{R}^N$ with $\partial\Omega$ acting as a reflecting barrier.

It is well known (see e.g. [6]) that the survival of the population is guaranteed when the principal eigenvalue of the weighted problem

$$\begin{cases} -\Delta u = \lambda m u & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

is below a certain positive threshold. A principal eigenvalue for (1) is a number λ having a positive eigenfunction. In case m^+ and m^- are both nontrivial, (1) admits two principal eigenvalues, 0 and $\lambda(m)$. Moreover, $\lambda(m) > 0$ if and only if $\int_\Omega m < 0$, in which case

$$\lambda(m) := \min \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega m u^2 \, dx} : u \in H^1(\Omega),\ \int_\Omega m u^2 \, dx > 0 \right\}. \tag{2}$$

Taking into account that the smaller $\lambda(m)$ is, the more chances of survival the population has, a largely studied problem consists in minimizing $\lambda(m)$ with respect to the weight or to other relevant parameters of the model. The literature is quite extensive, and we refer to the recent papers [15, 1, 20, 8] and references therein, for various interesting phenomena ranging from fragmentation to different nonlocal effects.

In this paper, we focus on the minimization of $\lambda(m)$ with respect to the weight. As proved in [14], when the mean $\int_\Omega m$ is fixed, as well as lower and upper bounds $-\beta \leq m \leq 1$, the infimum of $\lambda(m)$ is achieved by a bang-bang (i.e. piecewise constant) optimal weight $m = \mathbb{1}_D - \beta\mathbb{1}_{\Omega\setminus D}$, for some measurable set $D \subset \Omega$. Therefore, one can equivalently consider the minimization over the class of bang-bang weights $\mathbb{1}_D - \beta\mathbb{1}_{\Omega\setminus D}$, under a volume constraint on D in order to fix the average of m.

Definition 1.1. Let $\beta > 0$. For any $D \subset \Omega$ such that $|D| < \dfrac{\beta|\Omega|}{\beta+1}$ we define the eigenvalue of the corresponding bang-bang weight as

$$\lambda(D) := \lambda(\mathbb{1}_D - \beta\mathbb{1}_{\Omega\setminus D}) = \min\left\{ \frac{\int_\Omega |\nabla u|^2\,dx}{\int_D u^2\,dx - \beta\int_{\Omega\setminus D} u^2\,dx} : \int_D u^2\,dx > \beta\int_{\Omega\setminus D} u^2\,dx \right\}. \tag{3}$$

For $0 < \delta < \dfrac{\beta|\Omega|}{\beta+1}$, the optimal design problem for the survival threshold is

$$\Lambda(\delta) = \min\Big\{\lambda(D) : D \subset \Omega,\ \text{measurable},\ |D| = \delta\Big\}. \tag{4}$$

It follows that $\Lambda(\delta)$ is achieved by a set D_δ (see [4, 14]), which turns out to be the superlevel set of a corresponding principal (positive) eigenfunction u_δ. In particular, D_δ contains the maximum points of u_δ. As a consequence, both for modeling reasons, and from the mathematical point of view, natural questions arise about the location and the shape of the optimal set D and of the free boundary $\partial D \cap \Omega$.

This issue is mostly open in its generality, and it is completely understood only in dimension one: if $\Omega = (0,1)$, then u_δ has a unique maximum point, located at the boundary, and D_δ is either the interval $(0,\delta)$ or $(1-\delta,1)$ (see [5, 14, 11], also for the related problem with different boundary conditions). In particular, in this case the optimal set is connected and its boundary intersects the one of Ω. While these properties are expected also in higher dimensions (see e.g. [15, Open Problem 1]), their current understanding, up to our knowledge, is confined to the case in which $\Omega = \Pi_{i=1}^N(0,a_i)$ is an orthotope, a situation of particular interest because of its natural relation with the so-called periodically fragmented environment model (see [2]).

This case is investigated in [11], where Steiner symmetrization arguments are exploited to show that both the principal eigenfunction and the optimal weight are non-increasing along each coordinate direction. It follows that the maximum of u_δ is located at one of the vertices of Ω, D_δ is connected, and $\overline{D}_\delta \cap \partial\Omega$ has positive $N-1$ Hausdorff measure.

The main goal of this paper is to study the location and shape of the optimal set for general domains, when the measure δ is small. Our main result concerning the properties of D_δ and of the corresponding eigenfunction u_δ is the following.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with boundary of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$. For every $0 < \varepsilon < 1$ there exists $\delta_0 > 0$ such that, for every $\delta \in (0,\delta_0)$:*

1. *u_δ has a unique local maximum point $P_\delta \in \partial\Omega$;*

2. *D_δ is connected;*

3. *defining $r_\pm(\delta)$ in such a way that $|B_{r_\pm(\delta)}| = 2\delta(1 \pm \varepsilon)$, it results*

$$B_{r_-}(P_\delta) \cap \Omega \subset D_\delta \subset B_{r_+}(P_\delta) \cap \Omega.$$

In particular, $\mathcal{H}^{N-1}(\partial D_\delta \cap \partial\Omega) \geq C\delta^{(N-1)/N}$ for some universal constant $C > 0$.

Theorem 1.2 provides much information concerning u_δ and D_δ. First of all, the uniqueness of the local maximum point P_δ, for δ sufficiently small, immediately implies that D_δ is connected. Secondly, $P_\delta \in \partial\Omega$ and $D_\delta \subset B_{r_+}(P_\delta)$; as a consequence, as $\delta \to 0$ both D_δ and u_δ concentrate at P_δ. In this line, a more precise description of the decay properties of u_δ is obtained in Proposition 4.11 ahead. In addition, the third conclusion shows that the shape of D_δ, roughly speaking, is approximated by the intersection of a ball with Ω. Hence, the $N-1$ dimensional Hausdorff measure of the intersection $\partial D_\delta \cap \partial\Omega$ is positive for δ sufficiently small (see Remark 4.10).

It has been discussed for a while in the literature whether D_δ has a spherical shape or not, meaning that $D_\delta \cap \Omega = B \cap \Omega$ for some suitable ball B. In particular, this was suggested by some numerical simulations in [21], when $\Omega \subset \mathbb{R}^2$ is a square and δ is small. On the other hand, in [11] it is proved that, for general Ω, $\partial D_\delta \cap \Omega$ cannot contain portions of spheres. Motivated by this result, we devoted some previous papers to analyze the occurrence of spherical shapes for D_δ in some singularly perturbed regime. In particular, in [16, 17] we have shown that, for polyhedral domains Ω and δ fixed, spherical shapes can emerge, centered at the vertex of the smallest solid angle, when the parameter β in Definition 1.1 diverges to $+\infty$. From this point of view, the last part of Theorem 1.2 shows the same phenomenon, for smooth Ω, β fixed and $\delta \to 0$.

Theorem 1.2 is reminiscent of well-known results in the study of singularly perturbed elliptic nonlinear Neumann problems, firstly treated in [18]. Actually, the general strategy adopted here is inspired by the one developed by Ni and Takagi. More precisely, we perform a blow-up analysis near a maximum point of the eigenfuction u_δ, rescaling the problem in order to pass to the limit. On the other hand, our setting is quite different and it requires some new ideas.

Indeed, although our problem is linear, the presence of the sign-changing, discontinuous weight m_δ, depending on the unknown set D_δ, gives rise to several difficulties. First of all, the fact that the weight is only in L^∞ entails that the optimal regularity for the eigenfunction u_δ is merely $C^{1,\alpha}$. This obstructs higher order convergence of the blow-up sequences, preventing the direct deduction of finer qualitative properties. Furthermore, in order to complete our argument, we need to keep track of the behavior of the optimal set during the asymptotical analysis; this delicate information is obtained by showing that the free boundary $\partial D_\delta \cap \Omega$ is trapped in suitable annuli, for δ small enough (see for more details Proposition 4.7). In turn, this property will be fundamental in proving the third conclusion of Theorem 1.2.

As usual, when performing a blow-up analysis it is crucial to detect the natural associated limit problem. In our case, this is provided (up to scaling) by the following one.

Definition 1.3.

$$I_{\mathcal{M}} := \inf\left\{\mu(m) : m \in \mathcal{M}\right\}, \quad \text{where} \quad \mu(m) := \inf\left\{ \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\int_{\mathbb{R}^N} mv^2} : v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} mv^2 > 0 \right\}$$

for m lying in the class

$$\mathcal{M} := \left\{ m \in L^\infty(\mathbb{R}^N) : -\beta \leq m \leq 1 \text{ a.e. in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} (m+\beta) \leq 1+\beta \right\}.$$

Concerning $I_{\mathcal{M}}$ we prove the following result.

Theorem 1.4. *It results*

$$I_{\mathcal{M}} = \mu(\mathbb{1}_B - \beta \mathbb{1}_{B^c})$$

where B is the ball centered at zero and of measure 1 and $\mu(\mathbb{1}_B - \beta \mathbb{1}_{B^c})$ is achieved by a positive radially symmetric eigenfunction $w \in C^{1,1}(\mathbb{R}^N)$ exponentially decaying w.r.t. to $r = |x|$.

Theorem 1.4 allows to conclude the blow-up procedure, in view of the following result.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with boundary of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$. As $\delta \to 0$*

$$\Lambda(\delta) = \frac{1}{4^{1/N}} I_{\mathcal{M}}\, \delta^{-2/N}(1 + o(1)). \tag{5}$$

Moreover

$$\Lambda(\delta) \leq \frac{1}{4^{1/N}} I_{\mathcal{M}}\, \delta^{-2/N} \left(1 - \Gamma \widehat{H} \delta^{1/N} + o(\delta^{1/N})\right), \tag{6}$$

where $\widehat{H}$ denotes the maximum of the mean curvature of $\partial\Omega$, and $\Gamma > 0$ is a universal constant, independent of Ω (see equation (17)).

The proof of Theorem 1.4 relies on the following argument: first, an adaptation of the bathtub principle shows that $I_{\mathcal{M}}$ is achieved by a bang-bang weight $m = \mathbb{1}_A - \beta \mathbb{1}_{A^c}$; then symmetrization arguments yield that $A = B$ and finally we prove that $\mu(\mathbb{1}_B - \beta \mathbb{1}_{B^c})$ is achieved. In demonstrating this last step, it is crucial that the optimal weight is positive only in a bounded set, to guarantee good compactness properties.

On the other hand, to prove (6) in Theorem 1.5, we construct a competitor for $\Lambda(\delta)$: note that this amounts to build a suitable $H^1(\Omega)$ function and a weight of the form $\mathbb{1}_A - \beta \mathbb{1}_{A^c}$ where $|A| = \delta$. This last requirement makes the argument delicate as it forces us to introduce an unknown rescaling factor $r = r(\delta)$. Then, in order to obtain the desired comparison, we need to study the asymptotical behavior of $r(\delta)$.

Remark 1.6. We observe that Theorem 1.2 cannot be directly applied to the relevant case of the orthotope, which is not $C^{2,\alpha}$. Actually, such smoothness is crucial in the blow-up procedure: indeed, as concentration happens near $\partial\Omega$, we need to exploit a suitable diffeomorphism to straighten the boundary and extend the solution by reflection. However, a similar procedure can be used in case $\Omega = (0,1)^N$, by more straightforward reflection arguments. In particular, we can complement the results in [11, Proposition 5], obtaining that

$$\Lambda(\delta) = \frac{1}{4} I_{\mathcal{M}} \cdot \delta^{-2/N} + o(\delta^{-2/N}) \qquad \text{as } \delta \to 0 \tag{7}$$

and that, defining $r_\pm(\delta)$ in such a way that $|B_{r_\pm(\delta)}| = 2^N \delta(1 \pm \varepsilon)$, it results

$$B_{r_-}(0) \cap \Omega \subset D_\delta \subset B_{r_+}(0) \cap \Omega.$$

In the light of the above considerations, one can wonder, when Ω is smooth, where the concentration points P_δ accumulate, as $\delta \to 0$. A natural conjecture is that this should happen at points of maximal mean curvature, and there is a number of clues in this direction. First, as we mentioned, this is the case when Ω is an orthotope; secondly, the maximal mean curvature appears in the upper bound (6); moreover, we obtained strong indications of such behavior in [16], dealing with the double asymptotic $\beta \to +\infty$ and $\delta \to 0$. Actually, in the semilinear case, Ni and Takagi proved this property in [19], by showing that the upper bound analogous to (6) is actually an exact expansion of the critical level. To this aim, the crucial step was a sharp analysis of a linearized equation associated to the problem under study. Other strategies have been proposed in [22, 12, 7], still based on this ingredient. Unfortunately, it is not clear how to extend such ideas in our context, as this should involve a "linearization" of the optimal set D_δ and of the associated free boundary.

The paper is organized as follows. In Section 2 we study the limit problem, proving all the results involving $I_{\mathcal{M}}$ and in particular Theorem 1.4. Section 3 is devoted to the proof of the bound from above in Theorem 1.5 (equation (6)) and in Section 4 we complete the proof of Theorems 1.2 and 1.5.

Notation.

- $|\cdot|$ denotes the Lebesgue N dimensional measure and $\mathcal{H}^{N-1}(\cdot)$ the Hausdorff $N-1$ dimensional measure.
- For a function f, its positive/negative parts are denoted as $f^{\pm}(x) = \max\{\pm f(x), 0\}$.
- The characteristic function of a set E is denoted by $\mathbb{1}_E$.
- $B_r(x)$ denotes the ball of radius $r > 0$ centered at $x \in \mathbb{R}^N$. If $x = 0$, we often write $B_r = B_r(0)$. On the other hand, B is the ball centered at the origin and with $|B| = 1$.
- We call $\omega_N = |B_1|$ the measure of a ball of radius 1.
- $B_r^+ = B_r \cap \mathbb{R}^N_+$.
- H_P denotes the mean curvature of $\partial\Omega$ at $P \in \partial\Omega$, and $\widehat{H} = \max_{P\in\partial\Omega} H_P$.
- For a sequence $(\delta_k)_k$, we write $P_k = P_{\delta_k}$, $w_k = w_{\delta_k}$, and so on.
- $C, C_1, C', \ldots$ denote any (non-negative) universal constant, which may also change from line to line.

2 A spectral optimal design problem in $\mathbb{R}^N$

In this section we consider the minimization problem

$$I_{\mathcal{M}_k} := \inf\Big\{\mu(m) : m \in \mathcal{M}_k\Big\}, \tag{8}$$

for the weighted eigenvalue

$$\mu(m) := \inf\left\{\frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\int_{\mathbb{R}^N} m v^2} : v \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} m v^2 > 0\right\}, \tag{9}$$

with m a sign-changing weight belonging to the class

$$\mathcal{M}_k := \left\{ m \in L^\infty(\mathbb{R}^N) : \begin{array}{l} -\beta \le m \le 1 \text{ a.e. in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} (m+\beta) \le k(1+\beta) \end{array} \right\}, \tag{10}$$

where $k > 0$, $\beta > 0$ are fixed constants. Notice that $m \le 0$ a.e. implies $\mu(m) = +\infty$, thus the minimization can be restricted to the weights satisfying $|\{x \in \mathbb{R}^N : m > 0\}| > 0$. Morever, in general, if $m \in \mathcal{M}$, then $m \notin L^1(\mathbb{R}^N)$. Nonetheless, the auxiliary nonnegative weight $\widetilde{m} = m + \beta$ belongs to $L^1(\mathbb{R}^N)$. If $m = \mathbb{1}_E - \beta \mathbb{1}_{E^c}$ is bang-bang, then with a slight abuse of notation we write $\mu(m) = \mu(E)$.

Actually, the parameter k can be easily scaled out, as the following remark shows.

Remark 2.1. We notice that

$$m \in \mathcal{M}_k \qquad \Longleftrightarrow \qquad m_t(x) := m(t^{-\frac{1}{N}} x) \in \mathcal{M}_{tk}.$$

Furthermore, for $v \in H^1(\mathbb{R}^N)$ and $v_t(x) := v(t^{-1/N}x)$,

$$\frac{\int_{\mathbb{R}^N} |\nabla v_t|^2}{\int_{\mathbb{R}^N} m_t v_t^2} = t^{-\frac{2}{N}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\int_{\mathbb{R}^N} m v^2}.$$

We deduce that

$$I_{\mathcal{M}_{k_2}} = \left(\frac{k_2}{k_1}\right)^{-\frac{2}{N}} I_{\mathcal{M}_{k_1}}.$$

Moreover, in case such values are achieved, the optimal weights and eigenfunctions scale as well. More precisely,

$$w_{[1]} \text{ achieves } I_{\mathcal{M}_1} \qquad \Longleftrightarrow \qquad w_{[k]}(x) := w_{[1]}(k^{-\frac{1}{N}} x) \text{ achieves } I_{\mathcal{M}_k}.$$

We notice that, with this notation, equations (5) and (7) entail respectively

$$\Lambda(\delta) \sim I_{\mathcal{M}_2} \delta^{-2/N} \qquad \text{and} \quad \Lambda(\delta) \sim I_{\mathcal{M}_{2N}} \delta^{-2/N}.$$

In view of the previous remark, in the whole paper, when $k = 1$ we drop the dependence on k in the notation, as we mostly work with $I_{\mathcal{M}_1} = I_{\mathcal{M}}$. Our goal is to prove the following result. Analogously, we write $w = w_{[1]}$.

Theorem 2.2. *The value $I_{\mathcal{M}}$ is achieved, uniquely up to translations, by the weight*

$$m(x) = \mathbb{1}_B - \beta \mathbb{1}_{B^c},$$

where B denotes the ball of unit measure, with an associated positive eigenfunction $w \in C^{1,1}(\mathbb{R}^N)$ solving $-\Delta w = I_{\mathcal{M}} m w$. Namely, $I_{\mathcal{M}} = \mu(B)$. Moreover, w is radially symmetric, decreasing in $r = |x|$, and such that

$$w(r) = C_1 r^{-\frac{N-1}{2}} e^{-\sqrt{\mu\beta} r} \left(1 + O\left(r^{-1}\right)\right), \qquad w'(r) = C_2 r^{-\frac{N-1}{2}} e^{-\sqrt{\mu\beta} r} \left(1 + O\left(r^{-1}\right)\right) \tag{11}$$

as $r \to +\infty$, for suitable constants C_1, C_2.

The remaining part of this section is devoted to the proof of Theorem 2.2.

The first step, quite standard in this type of problems, is to reduce to bang-bang weights. To this aim, we use the so called *bathtub principle*, see e.g. [13, Theorem 1.14]. Since here we need a formulation which is slightly different from the usual one, we provide a proof.

Proposition 2.3 (bathtub principle)**.** *Let $f \in L^1(\mathbb{R}^N)$ be a nonnegative function. Then, the problem*

$$\sup_{m \in \mathcal{M}} \int_{\mathbb{R}^N} m f,$$

is solved by a weight $m_o(x) = \mathbb{1}_D(x) - \beta \mathbb{1}_{D^c}(x)$, for a measurable set $\{f > t\} \subset D \subset \{f \geq t\}$, with

$$t := \inf\left\{s \in \mathbb{R} : |\{f > s\}| \leq 1\right\} \qquad \text{and} \qquad |D| = 1.$$

Proof. The intuitive idea of the bathtub principle is to consider a weight of the form $m(x) = \mathbb{1}_{\{f>t\}}(x) - \beta\mathbb{1}_{\{f\le t\}}(x)$, with

$$t := \inf\Big\{s \in \mathbb{R} : |\{f > s\}| \le 1\Big\}.$$

If $|\{f > t\}| < 1$, we need to take a set $A \subset \{f = t\}$ such that

$$|\{f > t\} \cup A| = 1.$$

To check that the choice of such a set A is possible, it is enough to note that, by the definition of t as an infimum, for all $\vartheta > 0$

$$|\{f > t - \vartheta\}| > 1, \qquad \text{hence} \qquad |\{f > t - \vartheta\}| - |\{f > t\}| \ge 1 - |\{f > t\}|,$$

and passing to the limit as $\vartheta \to 0$, we infer that

$$|\{f = t\}| \ge 1 - |\{f > t\}|,$$

so that an appropriate set A exists. On the other hand, if $|\{f > t\}| = 1$, then $\{f > t\}$ is already a good candidate and we choose $A = \emptyset$. In both cases, we define

$$D := \{f > t\} \cup A, \qquad m_o(x) = \mathbb{1}_D(x) - \beta\mathbb{1}_{D^c}(x), \quad x \in \mathbb{R}^N.$$

Recalling (10) it is easy to check that $m_o \in \mathcal{M}$, as the measure constraint on $\{m_o > 0\}$ follows from the definition (this also implies that the weight is sign-changing), as well as the bounds from above and from below. Moreover, the integral constraint

$$\int_{\mathbb{R}^N} (m_o + \beta) = 1 + \beta,$$

is satisfied as well.

Finally, we check that m_o is actually an optimal weight. To do this, we use the layer-cake representation (Talenti formula) and Fubini theorem, to write

$$\int_{\mathbb{R}^N} m(x)f(x)\,dx = \int_0^{+\infty} \left(\int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x)m(x)\,dx \right) ds.$$

Then the claim follows if we prove that for almost all $s > 0$,

$$\int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x)m(x)\,dx \le \int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x)m_o(x)\,dx, \qquad \text{for all } m \in \mathcal{M}. \tag{12}$$

We note that, if $s > t$, it results $\{f > s\} \subset \{f > t\} \subset D$ and, as $m_o = 1$ on D, we get

$$\int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x)m(x)\,dx \le \int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}\,dx = \int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x)m_o(x)\,dx, \qquad \text{for all } m \in \mathcal{M}.$$

On the other hand, if $s < t$, one needs to be more careful. First of all, for every $m \in \mathcal{M}$ we write $\widetilde{m} = m + \beta$, so that $\widetilde{m}$ is a nonnegative function belonging to $L^1(\mathbb{R}^N)$. We observe that, as f is L^1, $|\{f > s\}| < +\infty$ for all $s > 0$. Then proving

$$\int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x)\widetilde{m}_o(x)\,dx \ge \int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x)\widetilde{m}(x)\,dx,$$

is equivalent to prove

$$\int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x) m_o(x)\, dx \geq \int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x) m(x)\, dx,$$

for all $m \in \mathcal{M}$. We first observe that

$$\int_{\mathbb{R}^N} \widetilde{m}_o = (1+\beta)|D| = 1+\beta \geq \int_{\mathbb{R}^N} \widetilde{m}, \qquad \text{for all } \widetilde{m} \in \mathcal{M}+\beta,$$
$$\int_{\mathbb{R}^N} \mathbb{1}_{\{f\leq s\}} \widetilde{m}_o = 0 \leq \int_{\mathbb{R}^N} \mathbb{1}_{\{f\leq s\}} \widetilde{m}, \qquad \text{for all } \widetilde{m} \in \mathcal{M}+\beta.$$

All in all, we have

$$\int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x) \widetilde{m}_o(x)\, dx = \int_{\mathbb{R}^N} \widetilde{m}_o - \int_{\mathbb{R}^N} \mathbb{1}_{\{f\leq s\}} \widetilde{m}_o$$
$$\geq \int_{\mathbb{R}^N} \widetilde{m} - \int_{\mathbb{R}^N} \mathbb{1}_{\{f\leq s\}} \widetilde{m} = \int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x) \widetilde{m}(x)\, dx, \qquad \text{for all } \widetilde{m} \in \mathcal{M}+\beta.$$

Putting all the information above together, we conclude that

$$\int_{\mathbb{R}^N} m(x) f(x)\, dx = \int_0^{+\infty} \left(\int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x) m(x)\, dx \right) ds$$
$$\leq \int_0^{+\infty} \left(\int_{\mathbb{R}^N} \mathbb{1}_{\{f>s\}}(x) m_o(x)\, dx \right) ds = \int_{\mathbb{R}^N} m_o(x) f(x)\, dx,$$

for all $m \in \mathcal{M}$, and the proof is finished. □

We can now show that the minimization in (8) is equivalent to the minimization among bang-bang weights. Introducing the class of admissible favorable sets

$$\mathcal{A} := \left\{ A \subset \mathbb{R}^N : A \text{ is measurable and } 0 < |A| \leq 1 \right\},$$

we note that the optimal set D provided by the bathtub principle is contained in the class $\mathcal{A}$; on the other hand, the weight $\mathbb{1}_A - \beta \mathbb{1}_{A^c} \in \mathcal{M}$ for every $A \in \mathcal{A}$. With a slight abuse of notation, we write

$$\mu(A) = \mu(\mathbb{1}_A - \beta \mathbb{1}_{A^c}), \qquad \text{for all } A \in \mathcal{A}.$$

Lemma 2.4. *We have*

$$I_{\mathcal{M}} = I_{\mathcal{A}} := \inf \left\{ \mu(A) : A \in \mathcal{A} \right\}.$$

Proof. First of all, we notice that the claim can be rewritten as an equality of two *inf-inf*:

$$\inf \left\{ \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\int_{\mathbb{R}^N} m v^2} : v \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} m v^2 > 0 \right\} : m \in \mathcal{M} \right\}$$
$$= \inf \left\{ \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\int_{\mathbb{R}^N} (\mathbb{1}_A - \beta \mathbb{1}_{A^c}) v^2} : v \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} (\mathbb{1}_A - \beta \mathbb{1}_{A^c}) v^2 > 0 \right\} : A \in \mathcal{A} \right\}.$$

Since $\mathbb{1}_A - \beta \mathbb{1}_{A^c} \in \mathcal{M}$ for all $A \in \mathcal{A}$, it is clear that $I_{\mathcal{M}} \leq I_{\mathcal{A}}$, hence we can focus on the opposite inequality. For any $\varepsilon > 0$ we can find $m_\varepsilon \in \mathcal{M}$ and $\psi_\varepsilon \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} m_\varepsilon \psi_\varepsilon^2 > 0$, such that

$$I_{\mathcal{M}} \geq \frac{\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2}{\int_{\mathbb{R}^N} m_\varepsilon \psi_\varepsilon^2} - \varepsilon.$$

Then, thanks to the bathtub principle, we have

$$\frac{\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2}{\int_{\mathbb{R}^N} m_\varepsilon \psi_\varepsilon^2} \geq \frac{\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2}{\sup\limits_{m \in \mathcal{M}} \int_{\mathbb{R}^N} m \psi_\varepsilon^2} = \frac{\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2}{\int_{\mathbb{R}^N} (\mathbb{1}_D - \mathbb{1}_{D^c}) \psi_\varepsilon^2},$$

for some $D \in \mathcal{A}$. Noting that

$$\int_{\mathbb{R}^N} (\mathbb{1}_D - \mathbb{1}_{D^c}) \psi_\varepsilon^2 \geq \int_{\mathbb{R}^N} m_\varepsilon \psi_\varepsilon^2 > 0,$$

we can infer

$$I_{\mathcal{M}} \geq \frac{\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2}{\int_{\mathbb{R}^N} (\mathbb{1}_D - \mathbb{1}_{D^c}) \psi_\varepsilon^2} - \varepsilon \geq \mu(D) - \varepsilon \geq I_{\mathcal{A}} - \varepsilon,$$

and since ε is arbitrary we conclude the proof. □

In order to solve the minimization problem, it is then enough to work on the case of bang-bang weights, where the Schwarz symmetrization comes to our rescue.

Lemma 2.5. *We have*

$$I_{\mathcal{A}} = \mu(B).$$

Proof. The proof of this fact is based on the Schwarz symmetrization. Let $D \in \mathcal{A}$ and $m_D := \mathbb{1}_D - \beta \mathbb{1}_{D^c}$. For any $\varepsilon > 0$ we can find $\psi_\varepsilon \in H^1(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}^N} m_D \psi_\varepsilon^2 > 0 \qquad \text{and} \qquad \mu(D) \geq \frac{\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2}{\int_{\mathbb{R}^N} m_D \psi_\varepsilon^2} - \varepsilon.$$

We denote by $(D^*, \psi_\varepsilon^*)$ the Schwarz rearrangement of (D, ψ_ε). Since m_D is piecewise constant, its Schwarz rearrangement may be defined as

$$m_D^* := \mathbb{1}_{D^*} - \beta \mathbb{1}_{(D^*)^c} = m_{D^*} = (m_D + \beta)^* - \beta \in \mathcal{M}. \tag{13}$$

By the Pólya-Szegö inequality, we have

$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 \geq \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon^*|^2.$$

On the other hand, the denominator in the Rayleigh quotient is a little more complicated to treat. First of all, we apply the Riesz rearrangement inequality [13, Theorem 3.4] to $m_D + \beta$ and ψ_ε^2, which are admissible as they are nonnegative and their positive superlevels have finite measure. This and (13) entail

$$\int_{\mathbb{R}^N} (m_D + \beta) \psi_\varepsilon^2 \leq \int_{\mathbb{R}^N} (m_D + \beta)^* \left(\psi_\varepsilon^2\right)^* = \int_{\mathbb{R}^N} (m_D^* + \beta) \, (\psi_\varepsilon^*)^2,$$

where we used the properties of the Schwarz rearrangement. Since $\|\psi_\varepsilon\|_{L^2} = \|\psi_\varepsilon^*\|_{L^2}$, (13) implies

$$\int_{\mathbb{R}^N} m_D \psi_\varepsilon^2 \leq \int_{\mathbb{R}^N} m_D^* \, (\psi_\varepsilon^*)^2 = \int_{\mathbb{R}^N} m_{D^*} \, (\psi_\varepsilon^*)^2,$$

yielding

$$\mu(D) \geq \frac{\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2}{\int_{\mathbb{R}^N} m_D \psi_\varepsilon^2} - \varepsilon \geq \frac{\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon^*|^2}{\int_{\mathbb{R}^N} m_{D^*} (\psi_\varepsilon^*)^2} - \varepsilon \geq \mu(D^*) - \varepsilon \geq \mu(B) - \varepsilon,$$

and the conclusion follows since $\varepsilon > 0$ and $D \in \mathcal{A}$ are arbitrary. □

Next we show that $\mu(B)$ is achieved. Actually, this can be done for any bounded open set $E \subset \mathbb{R}^N$.

Lemma 2.6. *Let $E \subset \mathbb{R}^N$ be an open and bounded set, $E \in \mathcal{A}$, and $m_E := \mathbb{1}_E - \beta \mathbb{1}_{E^c} \in \mathcal{M}$. There exists an eigenfunction $w \in H^1(\mathbb{R}^N)$ corresponding to the principal eigenvalue $\mu(E) = \mu(m_E)$, that is,*

$$\mu(E) = \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\int_{\mathbb{R}^N} m_E v^2} : v \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} m_E v^2 > 0 \right\} = \frac{\int_{\mathbb{R}^N} |\nabla w|^2}{\int_{\mathbb{R}^N} m_E w^2}. \tag{14}$$

Proof. Taking any $v \in H_0^1(E)$, it is immediate to see that $\mu(E) < +\infty$. Let w_n be a minimizing sequence for (14). Without loss of generality we can suppose that

$$0 < \int_{\mathbb{R}^N} m_E w_n^2 \leq \int_E w_n^2 = 1, \qquad \text{for all } n \in \mathbb{N}.$$

Then it is easy to check that

$$\int_{E^c} w_n^2 \leq \frac{1}{\beta}, \qquad \int_{\mathbb{R}^N} |\nabla w_n|^2 \leq (\mu(E)+1) \int_{\mathbb{R}^N} m_E w_n^2 \leq \mu(E) + 1, \qquad \text{for } n \text{ large.}$$

Hence

$$1 \leq \|w_n\|^2_{L^2(\mathbb{R}^N)} \leq 1 + \frac{1}{\beta} \qquad \text{and} \qquad \|w_n\|^2_{H^1(\mathbb{R}^N)} \leq C, \qquad \text{for all } n \in \mathbb{N},$$

for some constant $C > 0$ independent of n. Therefore, passing to a (nonrelabeled) subsequence, we have

$$w_n \rightharpoonup w, \qquad \text{weakly in } H^1(\mathbb{R}^N), \text{ and strongly in } L^2(E).$$

Hence, $w \not\equiv 0$, as

$$1 = \lim_{n \to +\infty} \int_E w_n^2 = \int_E w^2.$$

On the other hand, by lower semicontinuity of the norm with respect to the weak convergence,

$$\int_{E^c} w^2 \leq \liminf_{n \to +\infty} \int_{E^c} w_n^2.$$

All in all,

$$\int_{\mathbb{R}^N} m_E w^2 \geq \lim_{n \to +\infty} \int_E w_n^2 - \beta \liminf_{n \to +\infty} \int_{E^c} w_n^2 \geq \limsup_{n \to +\infty} \int_{\mathbb{R}^N} m_E w_n^2.$$

The weak lower semicontinuity of the L^2 norm of the gradient allows us to conclude

$$\frac{\int_{\mathbb{R}^N} |\nabla w|^2}{\int_{\mathbb{R}^N} m_E w^2} \leq \liminf_{n \to +\infty} \frac{\int_{\mathbb{R}^N} |\nabla w_n|^2}{\int_{\mathbb{R}^N} m_E w_n^2} = \mu(E),$$

so the claim is proved. □

Remark 2.7. Notice that, by the equation, the critical set of any optimal eigenfunction has zero measure. Taking into account the characterization of the equality cases in the Pólya-Szegö inequality [3], one deduces that the ball is the unique minimizer for $I_{\mathcal{M}}$, and its principal eigenfunction is radial.

To conclude the proof of Theorem 2.2, we study the decay of the optimal eigenfunction at infinity.

Lemma 2.8. *Let $w = w(r)$ be the principal eigenfunction associated to $\mu(B)$. Then there exist C such that (11) is satisfied.*

Proof. Since m_B is piecewise constant and $w = w(r)$ is a radial $H^1(\mathbb{R}^N)$-function, we have that w solves

$$\begin{cases} r^2 w_{rr} + (N-1) r w_r - \mu\beta r^2 w = 0 & \text{for } r > r_0, \\ w(+\infty) = 0, \end{cases}$$

where r_0 is the radius of B. Writing

$$w(r) = r^{-\frac{N}{2}+1} \widetilde{w}(r\sqrt{\mu\beta}),$$

we have that $\widetilde{w}$ solves

$$\begin{cases} r^2 \widetilde{w}_{rr} + r\widetilde{w}_r - \left(\left(\frac{N}{2}-1\right)^2 + r^2\right)\widetilde{w} = 0 & \text{for } r > r_0, \\ \widetilde{w}(+\infty) = 0. \end{cases}$$

We deduce that

$$\widetilde{w}(r) = C K_{\frac{N}{2}-1}(r),$$

where K_ν is the modified Bessel function of the second kind, with parameter ν. The lemma follows by well known decay properties of K_ν, see e.g. [9, p. 5,9,23–24]. □

3 Bound from above

We aim to prove the following result.

Theorem 3.1. *For any $P \in \partial\Omega$, we have that*

$$\Lambda(\delta) \le 2^{-2/N} I_{\mathcal{M}}\, \delta^{-2/N} \left(1 - 2^{1/N} \frac{2\alpha\gamma}{\int_{\mathbb{R}^N_+} |\nabla w|^2} \delta^{1/N} + o(\delta^{1/N})\right) \tag{15}$$

where w achieves $I_{\mathcal{M}}$ (see Theorem 2.2 and Remark 2.1) and

$$\alpha = (N-1)H_P, \qquad \gamma := \frac{1}{N+1} \int_{\mathbb{R}^N_+} |\nabla w|^2 z_N \, dz \tag{16}$$

and H_P denotes the mean curvature of $\partial\Omega$ at the point P.

Remark 3.2. Since $\widehat{H} = \max_{P\in\partial\Omega} H_P$, the bound from above (6) in Theorem 1.5 follows at once, with

$$\Gamma = \frac{2^{1+1/N}(N-1)}{N+1} \frac{\int_{\mathbb{R}^N_+} w'(|z|)^2 z_N \, dz}{\int_{\mathbb{R}^N_+} w'(|z|)^2 \, dz} \tag{17}$$

(recall that w is radial).

Remark 3.3. Recalling Remark 2.1, Theorem 3.1 yields

$$\limsup_{\delta\to 0} \Lambda(\delta) \cdot \delta^{2/N} \le 2^{-2/N} I_{\mathcal{M}} = I_{\mathcal{M}_2}.$$

To prove Theorem 3.1, in the spirit of [18, Sect. 3] we use a diffeomorphism to flatten the boundary of Ω near a suitable point. To this aim, we introduce some notation which will be used in the following.

Let $\Omega \subset \mathbb{R}^N$ be a $C^{2,\alpha}$ domain. Up to an affine change of variables, we can assume that $P = 0 \in \partial\Omega$ and that the outer unit normal to the boundary of Ω is $-e_N$. Then, using the notation

$$x' = (x_1, \dots, x_{N-1}),$$

there is $\delta_0 > 0$, a $C^{2,\alpha}$ function

$$\psi \colon \left\{ x' \in \mathbb{R}^{N-1} : |x'| < \delta_0 \right\} \to \mathbb{R},$$

and a neighborhood of the origin $\mathcal{N}$ such that

i) $\psi(0) = 0$, $\nabla\psi(0) = 0$, $\Delta\psi(0) = (N-1)H_0 = \alpha$,

ii) $\partial\Omega \cap \mathcal{N} = \left\{ (x', x_N) : x_N = \psi(x') \right\}, \qquad \Omega \cap \mathcal{N} = \left\{ (x', x_N) : x_N > \psi(x') \right\}.$

For a certain $\delta_1 > 0$, we define a diffeomorphism

$$\Phi \colon \left\{ y \in \mathbb{R}^N : |y| \le \delta_1 \right\} \to \mathbb{R}^N, \qquad x = \Phi(y) = (\Phi_1(y), \dots, \Phi_N(y)),$$

as

$$\Phi_j(y) = \begin{cases} y_j - y_N \frac{\partial\psi}{\partial x_j}(y'), & \text{for } j = 1, \dots, N-1, \\ y_N + \psi(y'), & \text{for } j = N. \end{cases}$$

We note that $D\Phi(0) = \mathrm{Id}$, due to the properties of ψ, and therefore Φ is locally invertible in, say, $B_{3\ell}$ for some $\ell > 0$. We define, for $j = 1, 2, 3$,

$$D_j = \Phi(B^+_{j\ell}) \subset \Omega, \quad \text{and} \quad \Psi \colon D_3 \to B^+_{3\ell},\ \Psi(x) := \Phi^{-1}(x). \tag{18}$$

The map Ψ can be seen as a local diffeomorphism straightening the boundary around $0 \in \partial\Omega$. For future reference, we remark that

$$\begin{aligned} \det D\Phi(y) &= 1 - \alpha y_N + O(|y|^2), \\ \left| \frac{y}{|y|} D\Psi(\Phi(y)) \right|^2 &= 1 + 2y_N \sum_{i,j=1}^{N-1} \psi_{ij}(0) \frac{y_i y_j}{|y|^2} + O(|y|^2), \end{aligned} \qquad \text{as } y \to 0, \tag{19}$$

see [18, Lemma A.1].

To prove Theorem 3.1, we build a competitor for $\Lambda(\delta)$ by composing the diffeomorphism Ψ with a suitable dilation of the weight

$$m(x) = \mathbb{1}_B - \beta \mathbb{1}_{\mathbb{R}^N \setminus B},$$

and of the corresponding eigenfunction w obtained in Theorem 2.2. A main difference with respect to [18] is that we have to keep track of the measure of the positivity set of the weight. Let us define

$$m_\delta(x) = \begin{cases} m(\Psi(x)/r(\delta)), & \text{if } x \in D_2, \\ -\beta, & \text{if } x \in \Omega \setminus D_2, \end{cases}$$

and $r(\delta)$ in such a way that the weight m_δ is admissible, that is,

$$|\{x \in \Omega : m_\delta(x) = 1\}| = \delta.$$

For δ small, the asymptotic relation between δ and $r(\delta)$ is explicit.

Lemma 3.4. *It holds $r(\delta) \to 0$ and*

$$\delta = r^N(\delta)\left(\frac{1}{2} - \frac{1}{N+1}\omega_{N-1}\omega_N^{-\frac{N+1}{N}}\alpha\, r(\delta) + O(r^2(\delta))\right), \qquad \text{as } \delta \to 0. \tag{20}$$

Proof. We write $r = r(\delta)$. We have

$$\{x \in \Omega : m_\delta(x) = 1\} = \left\{x \in D_2 : \frac{\Psi(x)}{r} \in B\right\} = \left\{\Phi(y) : y \in B^+_{2\ell} \cap rB\right\}.$$

In particular, since this set has measure δ, we have that $B^+_{2\ell} \cap rB = rB^+$ for δ sufficiently small. As consequence, $r \to 0$ as $\delta \to 0$. Using also (19), we compute

$$\begin{aligned}\delta = |\{x \in \Omega : m_\delta(x) = 1\}| &= \int_{rB^+} \det D\Phi(y)\, dy = \int_{rB^+}\left(1 - \alpha y_N + O(|y|^2)\right) dy \\ &= r^N \int_{B^+}\left(1 - \alpha r z_N + O(r^2|z|^2)\right) dz = r^N\left(\frac{1}{2} - \alpha\frac{\omega_{N-1}}{N+1}\omega_N^{-\frac{N+1}{N}} r + O(r^2)\right),\end{aligned}$$

where $\alpha = \Delta\psi(0)$ and we have also used the fact that

$$\int_{B^+} z_N\, dz = \frac{\omega_{N-1}}{N+1}\omega_N^{-\frac{N+1}{N}}. \qquad \square$$

Turning to the eigenfunction w, to build a competitor φ_δ after rescaling we also need to cut-off. We define

$$\varphi_\delta(x) = \zeta_\ell(|\Psi(x)|)w\left(\frac{\Psi(x)}{r(\delta)}\right), \qquad \text{where} \quad \zeta_\rho(t) = \begin{cases} 1 & 0 \le t \le \rho \\ 2 - \rho^{-1}t & \rho < t \le 2\rho \\ 0 & t > 2\rho. \end{cases}$$

For easier notation, it is convenient to introduce also the function $w_*(z) := \zeta_{\ell/r(\delta)}(|z|)w(z)$, in such a way that

$$\varphi_\delta(x) = w_*\left(\frac{\Psi(x)}{r(\delta)}\right).$$

Notice that, in principle, both φ_δ and w_* are not defined in the whole $\mathbb{R}^N$; nonetheless, by trivial extension, we can assume that they are Lipschitz and compactly supported on $\mathbb{R}^N$.

Proposition 3.5. *It holds, as $\delta \to 0$,*

$$\begin{aligned}\int_\Omega |\nabla\varphi_\delta|^2 &= r^{N-2}(\delta)\left\{\frac{1}{2}\int_{\mathbb{R}^N}|\nabla w|^2 - (N-1)\alpha\gamma r(\delta) + O(r^2(\delta))\right\}, \\ \int_\Omega m_\delta \varphi_\delta^2 &= r^N(\delta)\left\{\frac{1}{2}\int_{\mathbb{R}^N} m w^2 - \alpha\gamma_1 r(\delta) + O(r^2(\delta))\right\},\end{aligned}$$

where α and γ are defined in (16) and

$$\gamma_1 = \int_{\mathbb{R}^N_+} m(z)w(z)^2 z_N\, dz.$$

Proof. We write $r = r(\delta)$ and assume that δ and r are small enough.

Step 1. We start from the gradient term. Using the natural change of variable, we obtain

$$\int_\Omega |\nabla\varphi_\delta|^2 = \int_{D_2} |\nabla_x w_*(\Psi(x)/r)|^2\,dx = r^{-2}\int_{D_2}\left|w_*'(\Psi(x)/r)\frac{\Psi(x)}{|\Psi(x)|}D\Psi(x)\right|^2 dx$$
$$= r^{-2}\int_{B^+_{2\ell}}\left|w_*'(y/r)\frac{y}{|y|}D\Psi(\Phi(y))\right|^2 \det D\Phi(y)\,dy.$$

We now use (19) and write

$$y = rz \qquad \text{and} \qquad R = \frac{\ell}{r}.$$

We obtain, calling from now on $\psi_{ij} = \psi_{ij}(0)$,

$$\int_\Omega |\nabla\varphi_\delta|^2 = r^{N-2}\int_{B^+_{2R}} |w_*'(z)|^2\left(1 + rz_N\left[2\sum_{i,j=1}^{N-1}\psi_{ij}\frac{z_iz_j}{|z|^2} - \alpha\right] + O(r^2|z|^2)\right)dz. \tag{21}$$

By the exponential decay of w and w' (see Lemma 2.8) we have, for all $z \in B^+_{2R}$,

$$|w_*'(z)|^2 = \Big[w'(z)\zeta_R(|z|) + w(z)\zeta_R'(|z|)\Big]^2 \le 2\Big[w'(z)^2 + w(z)^2R^{-2}\Big] \le Ce^{-\vartheta|z|},$$

for a suitable $\vartheta > 0$. On the other hand, it is easy to check that there is a constant C_0, independent of r (and thus also of δ), such that

$$\left(1 + rz_N\left[2\sum_{i,j=1}^{N-1}\psi_{ij}\frac{z_iz_j}{|z|^2} - \alpha\right] + O(r^2|z|^2)\right) \le C_0, \qquad \text{for all } z \in B^+_{2R}.$$

Hence,

$$\int_{B^+_{2R}\setminus B^+_R} |w_*'(z)|^2\left(1 + rz_N\left[2\sum_{i,j=1}^{N-1}\psi_{ij}\frac{z_iz_j}{|z|^2} - \alpha\right] + O(r^2|z|^2)\right)dz \le C_0\int_{B^+_{2R}\setminus B^+_R} e^{-\vartheta|z|}\,dz = O(r^2).$$

On the other hand, in B_R we have that that $w_* = w$ and

$$\int_{B^+_R} |w_*'(z)|^2 O(r^2|z|^2)\,dz \le Cr^2\int_{\mathbb{R}^N_+} |w'(z)|^2|z|^2\,dz = O(r^2),$$

again by exponential decay of w. Plugging this information into (21) we have

$$\int_\Omega |\nabla\varphi_\delta|^2 = r^{N-2}\int_{B^+_R} |w'(z)|^2\left(1 + rz_N\left[2\sum_{i,j=1}^{N-1}\psi_{ij}\frac{z_iz_j}{|z|^2} - \alpha\right]\right)dz + O(r^N). \tag{22}$$

To pass to integrals in the half-space, we notice that, by exponential decay,

$$\int_{\mathbb{R}^N_+} w'(z)^2|z|^2\,dz < +\infty \qquad \Longrightarrow \qquad \int_{\mathbb{R}^N_+\setminus B^+_R} w'(z)^2(1+|z|)\,dz = O(r^N).$$

In conclusion, we have proved that

$$\int_\Omega |\nabla\varphi_\delta|^2 = r^{N-2}\int_{\mathbb{R}^N_+} |w'(z)|^2\left(1 + rz_N\left[2\sum_{i,j=1}^{N-1}\psi_{ij}\frac{z_iz_j}{|z|^2} - \alpha\right]\right)dz + O(r^N).$$

We now observe that the radiality of w entails

$$\int_{\mathbb{R}^N_+} w'(z)^2 \frac{z_i z_j}{|z|^2} z_N \, dz = 0, \qquad \text{if } i \neq j.$$

Then one can compute, using also [18, Lemma 3.3],

$$\sum_{i,j=1}^{N-1} \psi_{ij} \int_{\mathbb{R}^N_+} w'(z)^2 \frac{z_i z_j}{|z|^2} z_N \, dz = \sum_{j=1}^{N-1} \psi_{jj} \int_{\mathbb{R}^N_+} \left| \frac{\partial w}{\partial z_j} \right|^2 z_N \, dz = \alpha\gamma.$$

We have thus concluded the first part of the statement (for δ and $r(\delta)$ small enough):

$$\begin{aligned}\int_\Omega |\nabla \varphi_\delta|^2 &= r^{N-2} \left(\int_{\mathbb{R}^N_+} |\nabla w(z)|^2 + r\Big[2\alpha\gamma - (N+1)\alpha\gamma\Big] + O(r^2) \right) \\ &= r^{N-2} \left(\frac{1}{2} \int_{\mathbb{R}^N} |\nabla w(z)|^2 - r(N-1)\alpha\gamma + O(r^2) \right).\end{aligned}$$

Step 2. We now deal with the second part of the claim. With the same techniques as before (with the change of variables $y = \Psi(x)$ and $z = y/r$), we can compute

$$\begin{aligned}\int_\Omega m_\delta \varphi_\delta^2 \, dx = \int_{D_2} m_\delta \varphi_\delta^2 \, dx &= \int_{B^+_{2\ell}} m(y/r) w_*^2(y/r) \det D\Phi(y) \, dy \\ &= r^N \int_{B^+_{2R}} m(z) w_*^2(z) \left(1 - \alpha z_N r + O(r^2|z|^2)\right) dz,\end{aligned}$$

where $R = \ell/r$ and we have used (19).

The exponential decay of w allows to argue as in the previous step, yielding

$$\begin{aligned}r^{-N} \int_\Omega m_\delta \varphi_\delta^2 \, dx &= \int_{\mathbb{R}^N_+} m(z) w^2(z) \left(1 - \alpha z_N r\right) dz + O(r^2) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} m(z) w^2(z) \, dz - r\alpha \int_{\mathbb{R}^N_+} m(z) w^2(z) z_N \, dz + O(r^2) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} m(z) w^2(z) \, dz - r\alpha\gamma_1 + O(r^2),\end{aligned}$$

where γ_1 is defined in the statement. □

Corollary 3.6. *With the notation of Proposition 3.5, for $\delta \to 0$ (and thus $r(\delta) \to 0$), we have*

$$\Lambda(\delta) \leq r^{-2}(\delta) \left\{ \mu(B) + \frac{\alpha r(\delta)}{\int_{\mathbb{R}^N_+} m w^2} \left[\mu(B)\gamma_1 - (N-1)\gamma\right] + o(r(\delta)) \right\}. \tag{23}$$

Proof. We note that φ_δ is an admissible competitor for $\Lambda(\delta)$, for δ and $r(\delta)$ small enough, thus

$$\Lambda(\delta) \leq \frac{\int_\Omega |\nabla \varphi_\delta|^2}{\int_\Omega m_\delta \varphi_\delta}.$$

Then it is enough to apply the expansions proved in Proposition 3.5, also recalling the elementary expansion

$$\frac{a - c_1\varepsilon + o(\varepsilon)}{b - c_2\varepsilon + o(\varepsilon)} = \left(\frac{a}{b} - \frac{c_1}{b}\varepsilon + o(\varepsilon)\right) \cdot \left(1 + \frac{c_2}{b}\varepsilon + o(\varepsilon)\right) = \frac{a}{b} - \left(\frac{c_1}{b} - \frac{a c_2}{b^2}\right)\varepsilon + o(\varepsilon),$$

with

$$a=\int_{\mathbb{R}^N_+}|\nabla w|^2,\quad b=\int_{\mathbb{R}^N_+}mw^2,\quad c_1=(N-1)\alpha\gamma,\quad c_2=\alpha\gamma_1\quad\text{and }\varepsilon=r(\delta)\to 0.\qquad\square$$

In order to deduce the desired bound from above, we need a technical lemma.

Lemma 3.7. *With the notation above, we have*

$$\mu(B)\gamma_1-(N-1)\gamma=-2\gamma+4\mu(B)\omega_N^{-\frac{N+1}{N}}\frac{\omega_{N-1}}{N(N+1)}\int_{\mathbb{R}^N_+}mw^2. \tag{24}$$

Proof. We write $\mu=\mu(B)$. First of all, we test the equation of w in $\mathbb{R}^N_+$ with $z_N^2\partial_N w$:

$$\int_{\mathbb{R}^N_+}(-\Delta w)z_N^2\partial_N w\,dz=\mu\int_{\mathbb{R}^N_+}mwz_N^2\partial_N w\,dz. \tag{25}$$

Using the divergence theorem, the decay to zero of w at infinity and the relation

$$\gamma=\frac{1}{N+1}\int_{\mathbb{R}^N_+}|\nabla w(z)|^2 z_N\,dz=\frac{1}{2}\int_{\mathbb{R}^N_+}\left(\frac{\partial w}{\partial z_N}\right)^2 z_N\,dz.$$

(see [18, Lemma 3.3]) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N_+}(-\Delta w)z_N^2\partial_N w\,dz=\int_{\mathbb{R}^N_+}\nabla w\cdot\nabla(z_N^2\partial_N w)\,dz\\&=2\int_{\mathbb{R}^N_+}z_N(\partial_N w)^2\,dz+\frac{1}{2}\int_{\mathbb{R}^N_+}z_N^2\partial_N|\nabla w|^2\,dz=4\gamma-\int_{\mathbb{R}^N_+}z_N|\nabla w|^2\,dz=-(N-3)\gamma.\end{aligned}$$

On the other hand, let us denote the radius of B as $\overline{R}=\omega_N^{-1/N}$. Using the divergence theorem, the definition of m and γ_1, and the fact that w is radial, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N_+}mwz_N^2\partial_N w\,dz=\frac{1}{2}\int_{B^+}z_N^2\partial_N w^2\,dz-\frac{\beta}{2}\int_{\mathbb{R}^N_+\setminus B^+}z_N^2\partial_N w^2\,dz\\&=-\int_{\mathbb{R}^N_+}mw^2z_N\,dz+\frac{1+\beta}{2}\int_{\partial B^+}z_N^2w^2\frac{z\cdot e_N}{|z|}\,d\mathcal{H}^{N-1}=-\gamma_1+\frac{1+\beta}{N+1}\omega_{N-1}\overline{R}^{N+1}w^2(\overline{R})\end{aligned}$$

where we evaluated

$$\int_{\partial B^+}z_N^2\frac{z\cdot e_N}{|z|}\,d\mathcal{H}^{N-1}=\int_{B^+}\partial_N(z_N^2)\,dz=\frac{2}{N+1}\omega_{N-1}\overline{R}^{N+1}.$$

As a consequence, (25) is equivalent to

$$\mu\gamma_1-(N-1)\gamma=-2\gamma+\mu\frac{1+\beta}{N+1}\omega_{N-1}\overline{R}^{N+1}w^2(\overline{R}). \tag{26}$$

To get rid of the dependence on $w(\overline{R})$ we use the Pohozaev identity: testing the equation with $\nabla w\cdot z$ we obtain

$$\int_{\mathbb{R}^N_+}(-\Delta w)\nabla w\cdot z\,dz=\mu\int_{\mathbb{R}^N_+}mw\nabla w\cdot z\,dz.$$

On the left hand side, using the divergence theorem and the symmetry of $D^2 w$, we get

$$\int_{\mathbb{R}^N_+} (-\Delta w)\nabla w \cdot z\, dz = \int_{\mathbb{R}^N_+} |\nabla w|^2\, dz + \frac{1}{2}\int_{\mathbb{R}^N_+} \nabla |\nabla w|^2 \cdot z\, dz = \left(1 - \frac{N}{2}\right)\int_{\mathbb{R}^N_+} |\nabla w|^2\, dz.$$

On the other hand, for the right hand side, we use again the divergence theorem and the definition of m. Recalling that ∇w has a jump across ∂B, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N_+} m w \nabla w \cdot z\, dz &= -\frac{N}{2}\int_{\mathbb{R}^N_+} m w^2\, dz + \frac{1+\beta}{2}\int_{\partial B^+} w^2(z) z \cdot \frac{z}{|z|}\, d\mathcal{H}^{N-1} \\ &= -\frac{N}{2}\int_{\mathbb{R}^N_+} m w^2\, dz + \frac{1+\beta}{4} N\omega_N \overline{R}^N w^2(\overline{R}).\end{aligned}$$

All in all,

$$\mu \int_{\mathbb{R}^N_+} m w^2\, dz = \int_{\mathbb{R}^N_+} |\nabla w|^2\, dz = \frac{\mu(1+\beta)}{4} N\omega_N \overline{R}^N w^2(\overline{R}).$$

By plugging it into (26) and recalling that $\overline{R} = \omega_N^{-1/N}$, we finally have the claim (24). □

To conclude the proof, we need the following result.

Lemma 3.8 ([16, Lemma 4.10]). *Assume that, for positive constants a, b, c, d,*

$$\delta = a r^N \left(1 - br + o(r)\right), \qquad \nu = c r^{-2}\left(1 - dr + o(r)\right), \qquad \text{as } r \to 0^+.$$

Then

$$\nu = c a^{2/N} \delta^{-2/N}\left(1 - \frac{a^{-1/N}(2b + Nd)}{N}\delta^{1/N} + o(\delta^{1/N})\right) \qquad \text{as } \delta \to 0^+.$$

Proof of Theorem 3.1. Recalling Lemma 3.4, Corollary 3.6 and Lemma 3.7, up to now we have obtained the following relations:

$$\begin{aligned}\delta &= \frac{1}{2} r^N \left(1 - \frac{2}{N+1}\omega_{N-1}\omega_N^{-\frac{N+1}{N}} \alpha\, r + o(r)\right), \\ \frac{\int_\Omega |\nabla \varphi_\delta|^2}{\int_\Omega m_\delta \varphi_\delta^2} &= \frac{\mu(B)}{r^2}\left\{1 - \alpha r\left[\frac{2\gamma}{\mu(B)\int_{\mathbb{R}^N_+} m w^2} - 4\omega_N^{-\frac{N+1}{N}} \frac{\omega_{N-1}}{N(N+1)}\right] + o(r)\right\}.\end{aligned}$$

To merge them together and deduce the claim, we apply Lemma 3.8 with the obvious choice of the parameters. In particular we obtain

$$\begin{aligned}\frac{2b}{N} + d &= \frac{4}{N(N+1)}\omega_{N-1}\omega_N^{-\frac{N+1}{N}}\alpha + \alpha\left[\frac{2\gamma}{\int_{\mathbb{R}^N_+} |\nabla w|^2} - 4\omega_N^{-\frac{N+1}{N}} \frac{\omega_{N-1}}{N(N+1)}\right] \\ &= \frac{2\alpha\gamma}{\int_{\mathbb{R}^N_+} |\nabla w|^2}.\end{aligned}$$

Since φ_δ is an admissible competitor for $\Lambda(\delta)$, we obtain

$$\Lambda(\delta) \le \frac{\int_\Omega |\nabla \varphi_\delta|^2}{\int_\Omega m_\delta \varphi_\delta^2} = 2^{-2/N}\mu(B)\delta^{-2/N}\left(1 - 2^{1/N}\frac{2\alpha\gamma}{\int_{\mathbb{R}^N_+} |\nabla w|^2}\delta^{1/N} + o(\delta^{1/N})\right).$$ □

4 Blow-up argument

This section is mainly devoted to the proof of Theorem 1.2. We follow the approach introduced in [18, Section 4], based on a blow-up analysis. Differently from [18], we cannot deal directly with local maximum points, and we are forced to consider only global maximizers, to avoid the vanishing of the blow-up sequence.

Let $\Lambda(\delta)$ (introduced in Definition 1.1) be achieved by an open set D_δ, and let $u_\delta \in H^1(\Omega)$ be the L^2-normalized, positive principal eigenfunction, having global maximum at $P_\delta \in \overline{\Omega}$:

$$\Lambda(\delta) = \lambda(D_\delta), \qquad \int_\Omega u_\delta^2 = 1, \qquad u(P_\delta) = \max_{\overline{\Omega}} u.$$

First we show that $\mathrm{dist}(P_\delta, \partial\Omega) = o(\delta^{1/N})$ as $\delta \to 0$ (see Lemma 4.1, 4.3); this allows to prove that

$$\liminf_{\delta \to 0} \Lambda(\delta) \cdot \delta^{-2/N} \geq 2^{-2/N} I_{\mathcal{M}} = I_{\mathcal{M}_2}$$

which, together with Remark 3.3, yields Theorem 1.5. At the same time, this first blow-up procedure allows to obtain a strong non-vanishing property (Proposition 4.7), which in turn allows to deal with local maximizers (Lemma 4.8). Then we show that P_δ actually belongs to $\partial\Omega$ and it is unique. As a consequence the qualitative properties of D_δ stated in Theorem 1.2 follow.

We first show that $\mathrm{dist}(P_\delta, \partial\Omega) = O(\delta^{1/N})$.

Lemma 4.1. *There exists $C > 0$ such that, for δ small enough,*

$$\mathrm{dist}(P_\delta, \partial\Omega) \leq C\delta^{1/N}.$$

Proof. We argue by contradiction assuming that there is a sequence $\delta_j \to 0$ such that

$$\rho_j := \frac{\mathrm{dist}(P_j, \partial\Omega)}{\delta_j^{1/N}} \to +\infty, \qquad \text{as } j \to +\infty, \tag{27}$$

with $P_j := P_{\delta_j}$. We prove the lemma by a blow-up procedure, in several steps.

Step 1: convergence of the blow-up sequence. Introducing the rescaled sets and functions

$$\Omega_j := \frac{\Omega - P_j}{\delta_j^{1/N}}, \quad D_j := \frac{D_{\delta_j} - P_j}{\delta_j^{1/N}}, \quad m_j := \mathbb{1}_{D_j} - \beta \mathbb{1}_{\Omega_\delta \setminus D_j}, \quad v_j(z) := \delta_j^{1/2} u_{\delta_j}(P_j + \delta_j^{1/N} z), \tag{28}$$

it is easy to check that

$$|\Omega_j| = \frac{|\Omega|}{\delta_j}, \quad |D_j| = 1, \quad \begin{cases} -\Delta v_j = \lambda_j m_j v_j & \text{in } \Omega_j \\ \partial_\nu v_j = 0 & \text{on } \partial\Omega_j, \end{cases} \qquad \lambda_j = \frac{\int_{\Omega_j} |\nabla v_j|^2}{\int_{\Omega_j} m_j v_j^2} = \delta_j^{2/N} \Lambda(\delta_j).$$

By definition, $\rho_j = \mathrm{dist}(0, \partial\Omega_j)$, so that the ball B_{ρ_j} centered at the origin is contained in Ω_j for all j. Moreover, by definition of m_j and Remark 3.3 we infer that, up to (not relabeled) subsequences,

$$m_j \rightharpoonup m, \quad \text{weakly} * \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}^N), \qquad \Lambda(\delta_j)\delta_j^{2/N} = \lambda_j \to \lambda \in [0, I_{\mathcal{M}_2}], \quad \text{as } j \to +\infty. \tag{29}$$

On the other hand, thanks to the normalization of u_δ, we have

$$\int_{\Omega_j} v_j^2(z)\,dz = \int_\Omega u_{\delta_j}^2(x)\,dx = 1. \tag{30}$$

Let us now fix $B_r = B_r(0)$: we observe that, for all j sufficiently big, it holds $B_r \subset B_{\rho_j} \subset \Omega_j$, as $\rho_j \to +\infty$ by (27). Then, we can compute

$$\int_{B_r} |\nabla v_j|^2 \le \int_{\Omega_j} |\nabla v_j|^2 = \int_{\Omega_j} \lambda_j m_j v_j^2 \le C \int_{\Omega_j} v_j^2 = C, \tag{31}$$

where we used (29), and the fact that $\|m_j\|_{L^\infty(\Omega_j)} \le 1$.

As a consequence of (31) and of (30), we have a bound on the $H^1(B_r)$ norm of v_j, uniform in r. Thus, there exists $v \in H^1(\mathbb{R}^N)$ such that

$$v_j \rightharpoonup v, \qquad \text{weakly in } H^1_{\mathrm{loc}}(\mathbb{R}^N), \text{ strongly in } L^p_{\mathrm{loc}}(\mathbb{R}^N), \text{ for every } p \in [1, 2^*), \text{ as } j \to +\infty.$$

where 2^* is the usual Sobolev exponent, $v \ge 0$ a.e. in $\mathbb{R}^N$, and

$$\|\lambda_j m_j v_j\|_{L^p(B_r)} \le \lambda_j \|m_j\|_{L^\infty(B_r)} \|v_j\|_{L^p(B_r)} \le C\|v_j\|_{L^p(B_r)}. \tag{32}$$

Classical elliptic estimates, (see e.g. [10, Theorem 9.11]) yield

$$\|v_j\|_{W^{2,p}(R')} \le C\Big(\|v_j\|_{L^p(B_r)} + \|\lambda_j m_j v_j\|_{L^p(B_r)}\Big), \qquad \text{for all } 2 \le p < 2^* \text{ and } \overline{R'} \subset B_r,$$

where C depends only on N, p, B_r, R'. Then (32), the Sobolev embedding and (31) yield

$$\|v_j\|_{W^{2,p}(R')} \le C\|v_j\|_{L^p(B_r)} \le C\|v_j\|_{H^1(B_r)} \le C, \quad \text{for all } 2 \le p < 2^* \text{ and } \overline{R'} \subset B_r,$$

where C again depends on N, p, B_r, R'.

Now, if $N = 2, 3$ there exists a $p \in (N, 2^*)$; then Morrey's Theorem implies that $W^{2,p}(R')$ is continuously embedded in $C^{1,\alpha}(R')$, $\alpha = 1 - N/p$. By Ascoli-Arzelà's Theorem one deduces that the sequence $(v_j)_j$ is precompact in $C^{1,\vartheta}(R')$ for all $\vartheta < \alpha$. Thus, up to subsequences,

$$v_j \to v \qquad \text{in } C^{1,\vartheta}_{\mathrm{loc}}(\mathbb{R}^N) \cap H^1_{\mathrm{loc}}(\mathbb{R}^N), \qquad v \in H^1(\mathbb{R}^N)$$

as $j \to +\infty$, by arbitrarity of B_r and R' and by the uniform bounds obtained above. On the other hand, if $N \ge 4$, then we can use a bootstrap argument to prove in a finite number of steps the same uniform bound as above.

Finally, let $\varphi \in C^\infty_c(\mathbb{R}^N)$ and j sufficiently large so that $K := \mathrm{supp}(\varphi) \subset B_{\rho_j} \subset \Omega_j$. Then

$$\int_K \nabla v_j \cdot \nabla \varphi = \lambda_j \int_K m_j v_j \varphi,$$

and taking into account (29) we obtain that the limit function v is a weak solution to

$$-\Delta v = \lambda m v, \qquad \text{in } \mathbb{R}^N. \tag{33}$$

Step 2: the blow-up limit v is non-trivial and $\lambda > 0$. By the L^2 normalization,

$$0 < \int_{\Omega_j} m_j v_j^2 = \int_{D_j} v_j^2 - \beta \int_{\Omega_j \setminus D_j} v_j^2 = \int_{D_j} v_j^2 - \beta\left(1 - \int_{D_j} v_j^2\right) = (\beta + 1)\int_{D_j} v_j^2 - \beta.$$

Since P_j is a global maximum point we infer

$$0 < \frac{\beta}{\beta+1} < \int_{D_j} v_j^2 \le |D_j| \sup_{D_j} v_j^2 = v_j^2(0) \to v^2(0) \tag{34}$$

by pointwise convergence, thus v is non-trivial. Testing (33) with v we obtain that $\lambda > 0$.

Step 3: the limit weight m is admissible for $I_{\mathcal{M}}$ (see (10)). First, we take $\varphi = \mathbb{1}_E$ with $E := \{x \in \mathbb{R}^N : m < -\beta\} \cap B_r(0)$ for $r > 0$; then, by (29),

$$0 \le \int_{\Omega_j} (m_j + \beta)\mathbb{1}_E \to \int_E m + \beta \le 0,$$

showing that $m \ge -\beta$; in an analogous way it is possible to show that $m \le 1$. In addition, taking as test function v in (33) we obtain

$$0 < \int_{\mathbb{R}^N} |\nabla v|^2 = \lambda \int_{\mathbb{R}^N} mv^2,$$

showing that $|\{x \in \mathbb{R}^N : m > 0\}| > 0$. Finally, for all $r > 0$

$$\int_{B_r} m + \beta = \lim_{j \to +\infty} \int_{\mathbb{R}^N} (m_j + \beta)\mathbb{1}_{B_r} \le (1+\beta)|D_j| = 1 + \beta,$$

hence, taking $r \to +\infty$ and using monotone convergence,

$$\int_{\mathbb{R}^N} m + \beta \le (1+\beta).$$

Step 4: conclusion. By the previous steps and Remark 2.1 we have

$$\lambda = \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\int_{\mathbb{R}^N} mv^2} \ge I_{\mathcal{M}} = 2^{\frac{2}{N}} I_{\mathcal{M}_2},$$

which contradicts (29). □

Remark 4.2. We notice that Step 2 is the only point in the proof of Lemma 4.1 in which we need that P_δ is a *global* maximum point. Consequently, the blow-up argument works also for local maxima, up to the fact that, at this point, we can not exclude that the blow-up limit is trivial.

Next we improve the previous result, showing that $\operatorname{dist}(P_\delta, \partial\Omega) = o(\delta^{1/N})$.

Lemma 4.3. *Let P_δ be a global maximum point for u_δ. Then*

$$\limsup_{\delta \to 0} \frac{\operatorname{dist}(P_\delta, \partial\Omega)}{\delta^{1/N}} = 0.$$

Proof. Again, we use a blow up argument. Since here we work near the boundary $\partial\Omega$, we exploit the diffeomorphism introduced in Section 3. We subdivide the proof into two steps.

Step 1: blow-up procedure. Let us consider any sequence $P_k = P_{\delta_k}$ of global maximum points for u_{δ_k}, $\delta_k \to 0$. W.l.o.g. we can assume that $P_k \in \Omega$, for all k. Then, up to subsequences, thanks to Lemma 4.1, we have that $P_k \to P \in \partial\Omega$, as $k \to +\infty$.

As in Section 3, we assume that $P=0$, that the exterior normal to $\partial\Omega$ at 0 coincides with $-e_N$, and we apply the diffeomorphism $\Psi=\Phi^{-1}$ (see (18)), writing $Q_k=\Psi(P_k)\in B_\ell^+$. The transformed eigenfunction is defined by

$$v_k(y):=u_{\delta_k}(\Phi(y)),\qquad y\in\overline{B}_{2\ell}^+. \tag{35}$$

It is then easy to extend the function v_k by symmetry in the whole $B_{2\ell}$ by defining

$$\widetilde{v}_k(y):=\begin{cases} v_k(y), & \text{if } y_N\geq 0,\\ v_k(y',-y_N), & \text{if } y_N<0.\end{cases} \tag{36}$$

At this point, we can introduce the blow-up sequences, for $\delta_k>0$,

$$\Omega_k=\frac{B_{2\ell}-Q_k}{\delta_k^{1/N}},\qquad w_k(z)=\delta_k^{1/2}\,\widetilde{v}_k\left(Q_k+\delta_k^{1/N}z\right),\qquad z\in\overline{B_{\ell\delta_k^{-1/N}}}. \tag{37}$$

Let us denote

$$Q_k=(q_k',\alpha_k\delta_k^{1/N}),\quad \text{with } q_k'\in\mathbb{R}^{N-1},\ \alpha_k>0 \text{ and } Q_k\to 0. \tag{38}$$

By Lemma 4.1 we deduce that the sequence $(\alpha_k)_k$ is bounded, so that $\alpha_k\to\bar{\alpha}\geq 0$. Then, to prove the lemma we have to show that $\bar{\alpha}=0$. We notice that

$$\frac{\partial\widetilde{v}_k}{\partial y_N}=0 \text{ on } \{y_N=0\},\qquad \text{i.e.}\qquad \frac{\partial w_k}{\partial z_N}=0 \text{ on } \{z_N=-\alpha_k\}, \tag{39}$$

therefore

$$w_k\in C^{1,\vartheta}\left(\overline{B}_{\ell\delta_k^{-1/N}}\setminus\{z_N=-\alpha_k\}\right)\cap C^1\left(\overline{B}_{\ell\delta_k^{-1/N}}\right)\cap H^1\left(B_{\ell\delta_k^{-1/N}}\right),$$

thanks to the smoothness of the diffeomorphism and to the regularity of u_{δ_k}. Moreover the $L^2(\Omega)$ normalization of u_{δ_k} entails

$$\int_{B_{\ell\delta_k^{-1/N}}} w_k^2(z)\,dz\leq C,$$

for some constant $C>0$ independent of k, and w_k solves, for almost all $z\in B_{\ell\delta_k^{-1/N}}\setminus\{z_N=-\alpha_k\}$, the elliptic equation

$$-\sum_{i,j=1}^N a_{ij}^k(z)\frac{\partial^2 w_k}{\partial z_i\partial z_j}(z)-\delta_k^{\frac{1}{N}}\sum_{j=1}^N b_j^k(z)\frac{\partial w_k}{\partial z_j}(z)=\delta_k^{\frac{2}{N}}\,\Lambda(\delta_k)m_k(z)w_k(z), \tag{40}$$

where we denote, for $z\in B_{\ell\delta_k^{-1/N}}$,

$$m_k(z)=\begin{cases} m_{\delta_k}\left(\Phi(Q_k+\delta_k^{1/N}z)\right), & \text{if } z_N\geq-\alpha_k,\\ m_{\delta_k}\left(\Phi(q_k'+\delta_k^{1/N}z',-\delta_k^{1/N}(\alpha_k+z_N))\right), & \text{if } z_N<-\alpha_k.\end{cases} \tag{41}$$

The coefficients a_{ij}^k and b_j^k are defined as follows. First of all, let

$$a_{ij}(y)=\sum_{l=1}^N\frac{\partial\Psi_i}{\partial x_l}(\Phi(y))\frac{\partial\Psi_j}{\partial x_l}(\Phi(y)),\quad b_j(y)=(\Delta\Psi_j)(\Phi(y)),\qquad 1\leq i,j\leq N.$$

Then we set

$$a_{ij}^k(z) = \begin{cases} a_{ij}(Q_k + \delta_k^{1/N} z), & \text{if } z_N \geq -\alpha_k, \\ (-1)^{\delta_{jN}+\delta_{iN}} a_{ij}\left(q_k' + \delta_k^{1/N} z', -(\alpha_k + z_N)\delta_k^{1/N}\right), & \text{if } z_N < -\alpha_k, \end{cases}$$

$$b_j^k(z) = \begin{cases} b_j(Q_k + \delta_k^{1/N} z), & \text{if } z_N \geq -\alpha_k, \\ (-1)^{\delta_{jN}} b_j\left(q_k' + \delta_k^{1/N} z', -(\alpha_k + z_N)\delta_k^{1/N}\right), & \text{if } z_N < -\alpha_k, \end{cases}$$

where δ_{nm} is the Kronecker delta. With elliptic regularity arguments as in Lemma 4.1, we deduce that

$$w_k \to w_\infty, \qquad \text{in } C^{1,\vartheta}_{\text{loc}}(\mathbb{R}^N) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^N),$$

for $\vartheta < 1$, $p < +\infty$. Indeed, all the coefficients a_{ij}^k, b_j^k are Lipschitz continuous with uniform constants with respect to k in $B_{\ell\delta_k^{-1/N}}$, except b_N^k, but the product $b_N^k(z)\partial_{z_N} w_k(z)$ is Lipschitz continuous uniformly in k, thanks to (39), and it can be treated as a right hand side term to obtain the Schauder estimates. Moreover, $w_\infty \in C^{1,\vartheta}(\mathbb{R}^N) \cap W^{2,p}(\mathbb{R}^N)$ is non-trivial (see Step 2 in the proof of Lemma 4.1; again, this is the only part of the proof in which we need P_δ to be a *global* maximum point).

Now we recall that, as $k \to +\infty$, $Q_k \to 0$, $\delta_k^{2/N}\Lambda(\delta_k) \to \lambda \in (0, I_{\mathcal{M}_2}]$, $m_k \overset{*}{\rightharpoonup} m$ weakly $*$ in $L^\infty_{\text{loc}}(\mathbb{R}^N)$; moreover, $A^k \to D\Psi(0) = [D\Phi(0)]^{-1} = \text{Id}$ and $\delta^{1/N} b^k \to 0$. We infer that w_∞ satisfies

$$\begin{cases} -\Delta w_\infty = \lambda m w_\infty & \text{in } \mathbb{R}^N, \\ \max_{\mathbb{R}^N} w_\infty = w_\infty(0) \\ w_\infty(z', -z_N - \bar{\alpha}) = w_\infty(z', z_N + \bar{\alpha}), \end{cases} \tag{42}$$

because the maximum of w_k is at $z = 0$, and $\alpha_k \to \bar{\alpha}$.

Step 2: analysis of the limit problem. First of all, let us show that $m \in \mathcal{M}_2$ (see (10)). As in the proof of Lemma 4.1, from the properties of the weak $*$ $L^\infty_{\text{loc}}(\mathbb{R}^N)$ convergence, one deduces that $-\beta \leq m(z) \leq 1$ for a.e. $z \in \mathbb{R}^N$ and that $|\{x \in \mathbb{R}^N : m > 0\}| > 0$. On the other hand, it is more delicate to check that

$$\int_{\mathbb{R}^N} (m + \beta) \leq 2(1 + \beta).$$

First of all, we introduce:

$$D_{\delta_k} = \{x \in \Omega : m_{\delta_k}(x) = 1\}, \qquad |D_{\delta_k}| = \delta_k, \qquad \widetilde{D}_k := \widetilde{D}_k^+ \cup \widetilde{D}_k^-, \tag{43}$$

where

$$\widetilde{D}_k^+ = \left\{ z \in B_{\ell\delta_k^{-1/N}} : z_N \geq -\alpha_k,\ (z', z_N) \in \frac{\Psi(D_{\delta_k}) - Q_k}{\delta_k^{1/N}} \right\},$$

$$\widetilde{D}_k^- = \left\{ z \in B_{\ell\delta_k^{-1/N}} : z_N \leq -\alpha_k,\ (z', -z_N) \in \frac{\Psi(D_{\delta_k}) - (q_k', -\alpha_k \delta_k^{1/N})}{\delta_k^{1/N}} \right\}.$$

Let us fix a ball B_r centered at the origin and of radius $r > 0$, and δ_k small. We need to estimate the measure of the set $\widetilde{D}_k \cap B_r$. To this aim we notice that, by reflection,

$$|\widetilde{D}_k^- \cap B_r(0)| = |\widetilde{D}_k^+ \cap B_r(0', -2\alpha_k)| \leq |\widetilde{D}_k^+ \cap B_r(0)| :$$

this because $\widetilde{D}_k^+ \subset \{z_N \geq -\alpha_k\}$ and $B_r(0', -2\alpha_k) \cap \{z_N \geq -\alpha_k\} \subset B_r(0)$. On the other hand, we have

$$\delta_k = \int_\Omega \mathbb{1}_{D_{\delta_k}}(x)\,dx \geq \int_{B_r\cap\{z_N\geq-\alpha_k\}} \delta_k \mathbb{1}_{D_{\delta_k}}\left(\Phi(Q_k+\delta_k^{1/N}z)\right)\,\det\left(D\Phi(Q_k+\delta_k^{1/N}z)\right)dz,$$

by making the change of variable $z = \frac{\Psi(x)-Q_k}{\delta_k^{1/N}}$. Then, by definition of $\widetilde{D}_k^+$ and using (19),

$$\delta_k \geq \int_{B_r} \mathbb{1}_{\widetilde{D}_k^+}(z)\delta_k\left(1-\alpha\delta_k^{1/N}(\alpha_k+z_N)+O\left(|Q_k+\delta_k^{1/N}z|^2\right)\right)dz,$$

where α_k is defined in (38) (and α in (16)). All in all, we obtain

$$|\widetilde{D}_k \cap B_r| \leq 2|\widetilde{D}_k^+ \cap B_r| \leq 2\left[1+C_1(r)\delta_k^{1/N}+C_2(r)\left(|Q_k|^2+\delta_k^{2/N}|\right)\right] \leq 2+o(1), \tag{44}$$

as $k \to +\infty$, since

$$C_1(r) = \alpha\int_{\widetilde{D}_k\cap B_r}(\alpha_k+|z_N|)\,dz \leq \alpha\int_{B_r}(\alpha_k+|z_N|)\,dz$$

is uniformly bounded with respect to k, as the same is true for the sequence (α_k), and $Q_k \to 0$.

At this point, exploiting the weak $*$ $L^\infty(B_r)$ convergence of m_k to m, we have

$$\begin{aligned}\int_{B_r}(m+\beta) &= \lim_{k\to+\infty}\int_{B_r}(m_k+\beta) = \lim_{k\to+\infty}\int_{\mathbb{R}^N}(m_k+\beta)\mathbb{1}_{B_r}\\ &\leq \lim_{k\to+\infty}(1+\beta)|\widetilde{D}_k\cap B_r| \leq \lim_{k\to+\infty}2(1+\beta)\,[1+o_k(1)] = 2(1+\beta).\end{aligned}$$

Then, by monotone convergence, we conclude

$$\int_{\mathbb{R}^N}(m+\beta) = \lim_{r\to+\infty}\int_{B_r}(m+\beta) \leq 2(1+\beta).$$

Once $m \in \mathcal{M}_2$, we can argue as in Lemma 4.1, showing that $\int_{\mathbb{R}^N} mw_\infty^2 > 0$ and finally, using Theorem 3.1,

$$I_{\mathcal{M}_2} \geq \lambda = \frac{\int_{\mathbb{R}^N}|\nabla w_\infty|^2}{\int_{\mathbb{R}^N} mw_\infty^2} \geq I_{\mathcal{M}_2}.$$

As a consequence, denoting with $w_{[2]}$ a (positive) eigenfunction associated to $I_{\mathcal{M}_2}$, we have that $w_\infty(z) = cw_{[2]}(z+z_0)$ for some $c > 0$ and $z_0 \in \mathbb{R}^N$. Recalling that $w_{[2]}$ is radially symmetric, with a unique maximum point at 0, we deduce that w_∞ is symmetric with respect to an hyperplane if and only if such hyperplane passes through its maximum point. Recalling (42) we obtain that $\bar{\alpha} = 0$ and the proof is concluded. □

Remark 4.4. Note that in the proof of Lemma 4.3 we also show that, given any sequence $\delta_k \to 0$, then some non-relabelled subsequences $(w_k)_k$ and $(m_k)_k$, as introduced in (37), (41) satisfy

$$w_k \to w_\infty, \quad \text{in } C^{1,\vartheta}_{\text{loc}}(\mathbb{R}^N)\cap W^{2,p}_{\text{loc}}(\mathbb{R}^N), \quad m_k \overset{*}{\rightharpoonup} m \quad \text{weakly * in } L^\infty_{\text{loc}}(\mathbb{R}^N)$$

with

$$w_\infty = cw_{[2]}, \qquad m = \mathbb{1}_{2^{1/N}B} - \beta\mathbb{1}_{2^{1/N}B^c},$$

where $c > 0$ and $w_{[2]}$ is the principal eigenfunction associated with $I_{\mathcal{M}_2}$ (see Theorem 2.2 and Remark 2.1).

Finally, by the bathtub principle applied to u_δ and the reflection argument, we have that there exist $t_k > 0$ such that

$$\widetilde{D}_k = \{z \in B_{\ell\delta_k^{-1/N}} : m_k(z) = 1\} = \{z : w_k(z) > t_k\}, \qquad \partial\widetilde{D}_k = \{z : w_k(z) = t_k\}. \tag{45}$$

In the next lemma, we draw some consequences of equation (44) on $\widetilde{D}_k$ and t_k.

Lemma 4.5. *Under the previous notation:*

1. *Let $A \supseteq 2^{1/N}B$ be a bounded set. We have that*

$$\lim_{k\to+\infty} |\widetilde{D}_k \cap A| = 2. \tag{46}$$

2. $\lim_k t_k = t_\infty := \inf_{2^{1/N}B} w_\infty > 0.$

3. *For all $\varepsilon \in (0,1)$, if k is sufficiently large then there exists a connected component $\widetilde{D}_k^{in}$ of $\widetilde{D}_k$ satisfying*

$$2^{1/N}(1-\varepsilon)B \subset \widetilde{D}_k^{in} \subset 2^{1/N}(1+\varepsilon)B. \tag{47}$$

Proof. We keep the same notation of Lemma 4.3 and Remark 4.4.

Step 1: proof of (46). Let A be as in the statement. Since $A \subset B_r$, for some r, (44) yields

$$\limsup_{k\to+\infty} |\widetilde{D}_k \cap A| \le 2. \tag{48}$$

Taking into account Remark 4.4, the lower semicontinuity of the norm with respect to the weak convergence in $L^2(A)$ and (48), we obtain

$$2(1+\beta)^2 = \int_A (m+\beta)^2 \le \liminf_{k\to+\infty} \int_A (m_k+\beta)^2 \le (1+\beta)^2 \limsup_{k\to+\infty} |\widetilde{D}_k \cap A| \le 2(1+\beta)^2,$$

yielding (46).

Step 2: $\liminf_k t_k \ge t_\infty$. We claim that, for every $\varepsilon > 0$,

$$\partial\widetilde{D}_k \cap 2^{1/N}(1+\varepsilon)B \ne \emptyset,$$

for k large. Indeed, on the one hand $0 \in \widetilde{D}_k \cap 2^{1/N}(1+\varepsilon)B$. On the other hand, recalling (44)

$$|\widetilde{D}_k \cap 2^{1/N}(1+\varepsilon)B| \le 2 + o(1) < 2(1+\varepsilon)^N = |2^{1/N}(1+\varepsilon)B|,$$

for k large, whence $2^{1/N}(1+\varepsilon)B \setminus \widetilde{D}_k \ne \emptyset$ and the claim follows.

Then

$$t_k \equiv w_k|_{\partial\widetilde{D}_k} \ge \inf_{2^{1/N}(1+\varepsilon)B} w_k \ge \inf_{2^{1/N}(1+\varepsilon)B} w_\infty + o(1),$$

for k large. Since ε is arbitrary, step 2 follows.

Step 3: $\limsup_k t_k \le t_\infty$. We choose $A = 2^{1/N}B$ in (46) and we observe that, by uniform convergence to the decreasing function w_∞, for every $\sigma > 0$ small there exists $0 < \varepsilon < 1$ such that

$$\sup_{A\setminus 2^{1/N}(1-\varepsilon)B} w_k \le \inf_{2^{1/N}B} w_\infty + \sigma = t_\infty + \sigma. \tag{49}$$

Let us assume by contradiction that, up to subsequences

$$t_k \to t_\infty + \sigma'$$

for some $\sigma' > 0$. Choosing $\sigma < \sigma'/2$ in (49) we obtain, for k large,

$$\sup_{A \setminus 2^{1/N}(1-\varepsilon)B} w_k \le t_k - \sigma \qquad \Longrightarrow \qquad A \cap \widetilde{D}_k \subset 2^{1/N}(1-\varepsilon)B.$$

Then we find a contradiction with (46), as

$$2 = \lim_k |\widetilde{D}_k \cap A| \le |2^{1/N}(1-\varepsilon)B| = 2(1-\varepsilon)^N.$$

Step 4: proof of (47). Again by uniform convergence, for every $0 < \varepsilon < 1$ there exists $\sigma > 0$ such that

$$\inf_{2^{1/N}(1-\varepsilon)B} w_k \ge \inf_{2^{1/N}B} w_\infty + \sigma = t_\infty + \sigma.$$

By the second conclusion we deduce that, for k large

$$\inf_{2^{1/N}(1-\varepsilon)B} w_k \ge t_k + \frac{\sigma}{2} \qquad \Longrightarrow \qquad 2^{1/N}(1-\varepsilon)B \subset \widetilde{D}_k.$$

In the same way, there exists $\sigma > 0$ such that

$$\sup_{2^{1/N}(1+\varepsilon)\partial B} w_k \le \inf_{2^{1/N}B} w_\infty - \sigma = t_\infty - \sigma \le t_k - \frac{\sigma}{2},$$

for k large, so that

$$\widetilde{D}_k \cap 2^{1/N}(1+\varepsilon)\partial B = \emptyset. \qquad \square$$

Remark 4.6. By (47) we have that, for k sufficiently large, $m_k(z) = 1$ for all $z \in 2^{1/N}(1-\varepsilon)B$, thus

$$w_k \to w_\infty, \qquad \text{in } C^2(2^{1/N}(1-\varepsilon)B), \tag{50}$$

and

$$w_\infty \in C^2(2^{1/N}(1-\varepsilon)B) \cap C^{1,\vartheta}_{\text{loc}}(\mathbb{R}^N).$$

This follows by elliptic regularity, recalling that w_k satisfies (40).

The results of Lemma 4.5 can be easily translated in terms of u_δ, D_δ.

Proposition 4.7. *Let D_δ and u_δ (as above) achieve $\Lambda(\delta)$, and P_δ be a global maximum point for u_δ.*

1. *There exists a universal constant $\sigma > 0$ such that, for δ small enough,*

$$\inf_{D_\delta} u_\delta \ge \sigma \delta^{-1/2}.$$

2. *For every $\varepsilon \in (0,1)$, if δ is sufficiently small then there exists a connected component D_δ^{in} of D_δ satisfying*

$$(2\delta)^{1/N}(1-\varepsilon)B(P_\delta) \cap \Omega \subset D_\delta^{in} \subset (2\delta)^{1/N}(1+\varepsilon)B(P_\delta) \cap \Omega. \tag{51}$$

Proof. For the first conclusion, let us assume by contradiction that $\delta_k \to 0$ and $\delta_k^{1/2} \inf_{D_{\delta_k}} u_{\delta_k} =: \sigma_k \to 0$. Then, taking the corresponding blow-up sequence as in Remark 4.4 we obtain that, possibly up to subsequences, $t_k = \sigma_k \to 0$, in contradiction with Lemma 4.5.

Analogously, for the second conclusion, assume by contradiction that $\delta_k \to 0$ and, say,

$$x_k \in \Omega \setminus D_{\delta_k}, \qquad x_k \in (2\delta_k)^{1/N}(1-\varepsilon)B(P_k).$$

Recalling (43) and the fact that $D\Psi(0)$ is the identity we obtain that

$$\frac{\Psi(x_k) - \Psi(P_k)}{\delta_k^{1/N}} \notin \widetilde{D}_k, \qquad \frac{\Psi(x_k) - \Psi(P_k)}{\delta_k^{1/N}} \in 2^{1/N}(1-\varepsilon')B$$

for some $\varepsilon' > 0$ and k large. This is in contradiction with (47). The other inclusion can be obtained in the same way, considering a sequence $x_k \in D_{\delta_k} \cap (2\delta_k)^{1/N}(1+\varepsilon)\partial B(P_k)$. □

Up to now we considered only global maximum points P_δ. Now we are ready to deal also with local ones.

Lemma 4.8. *Let P'_δ be any local maximum point for u_δ. Then*

$$\limsup_{\delta \to 0} \frac{|P_\delta - P'_\delta|}{\delta^{1/N}} = 0.$$

Equivalently, if $\delta_k \to 0$ is such that $Q_k = \Psi(P_k) \to 0$, then also $Q'_k := \Psi(P'_k) \to 0$.

Proof. We recall that $u_\delta \in C^{1,\alpha}(\overline{\Omega})$ is smooth and superharmonic in D_δ, while it is smooth and subharmonic in D_δ^c. By maximum principle and Hopf's lemma, we obtain that any local maximum point $P'_\delta \in \overline{D}_\delta$. Proposition 4.7 readily implies that

$$u_\delta(P'_\delta) \geq \sigma\delta^{-1/2}. \tag{52}$$

for δ sufficiently small. This condition allows to repeat al the previous blow-up arguments, centered at P'_δ, avoiding the vanishing of the blow-up limit.

Step 1: there exists $C > 0$ such that, for δ small enough, $\text{dist}(P'_\delta, \partial\Omega) \leq C\delta^{1/N}$. The proof follows the lines of Lemma 4.1, which contains the same result for global maximizers. According to Remark 4.2 we only need to show that the limit of the blow-up sequence v'_j, defined as in (28) but centered at P'_j, is non-trivial. This follows by pointwise convergence and as $v'_j(0) \geq \sigma$ for j large, due to (52).

Step 2: $\displaystyle\limsup_{\delta \to 0} \frac{\text{dist}(P'_\delta, \partial\Omega)}{\delta^{1/N}} = 0$. Again, we repeat the same argument of Lemma 4.3, with the blow-up sequence w'_k, defined as in (37) but centered at P'_k, exploiting (52) to deduce that its limit is non-trivial.

Step 3: there exists $C > 0$ such that, for δ small enough, $|P_\delta - P'_\delta| \leq C\delta^{1/N}$. Let us assume by contradiction that, for some sequence $\delta_k \to 0$,

$$\lim_{k \to +\infty} \frac{|P_k - P'_k|}{\delta_k^{1/N}} = +\infty. \tag{53}$$

Up to subsequences, and by step 2, this yields $P_k \to P \in \partial\Omega$, $P'_k \to P' \in \partial\Omega$, with $P' \neq P$. Let w_k as in (37), and let w'_k be defined accordingly, centering the blow-up procedure at P'_k, with

Ψ' a diffeomorphism straightening the boundary near P', and $Q'_k = \Psi'(P'_k)$. Proposition 4.7, Remark 4.4, Lemma 4.5 and equation (51) hold true also for w'_∞, the blow-limit of w'_k.

Now, taking ε sufficiently small in (51), we obtain, for k large enough,

$$(2\delta_k)^{1/N}(1-\varepsilon)B(P_k)\cap\Omega \subset D_{\delta_k}, \qquad |(2\delta_k)^{1/N}(1-\varepsilon)B(P_k)\cap\Omega| \geq \frac{3}{4}\delta_k.$$

Analogously,

$$(2\delta_k)^{1/N}(1-\varepsilon')B(P'_k)\cap\Omega \subset D_{\delta_k}, \qquad |(2\delta_k)^{1/N}(1-\varepsilon')B(P'_k)\cap\Omega| \geq \frac{3}{4}\delta_k.$$

Finally, by (53)

$$(2\delta_k)^{1/N}(1-\varepsilon)B(P_k)\cap(2\delta_k)^{1/N}(1-\varepsilon')B(P'_k) = \emptyset.$$

Summing up we obtain the contradiction

$$|D_{\delta_k}| \geq 2\cdot\frac{3}{4}\delta_k = \frac{3}{2}|D_{\delta_k}|, \qquad \text{for all } k \text{ sufficiently large.}$$

Step 4: conclusion. Again by contradiction, let $\delta_k \to 0$ and

$$0 < \lim_{k\to+\infty}\frac{|P_k - P'_k|}{\delta_k^{1/N}} \leq C$$

by the previous step. Considering the usual blow-up sequence (centered at P_k) we have that the function w_k has two local maxima, one for $z = 0$ and one for $z = z_k = (Q'_k - Q_k)\delta_k^{-1/N} \leq C$. Then, up to subsequences, $z_k \to z_\infty \neq 0$, by the contradiction assumption. This is not possible, since by uniform convergence z_∞ is a local maximum for w_∞, i.e. $z_\infty = 0$. □

To conclude the proof of Theorem 1.2 we prove the following.

Proposition 4.9. *There exists $\delta_0 > 0$ such that for every $\delta \in (0,\delta_0)$:*

1. *u_δ has a unique local maximum point P_δ,*
2. *$P_\delta \in \partial\Omega$,*
3. *D_δ is connected.*

Proof. It suffices to prove that w_k has a unique local maximum point in B_R, for some $R > 0$ and k large. Indeed, in such a case, Lemma 4.8 already excludes the presence of other local maxima for u_δ, outside B_R and 1 follows; on the other hand, in case $P_k \in \Omega$ then w_k would have two distinct maxima in B_R, by reflection (recall Lemma 4.3 and (37)), and therefore 2 follows; finally, u_δ has a local maximum in each connected component of D_δ, and also 3 follows.

To prove the claim, let R be such that $\overline{B}_R \subset 2^{1/N}B$. By Remark 4.6 $w_k \to w_\infty$ in $C^2(\overline{B}_R)$; moreover, since $w''_\infty(0) < 0$ we can take R small so that $w''_\infty < 0$ in B_R. Recalling that w_k achieves its maximum at the origin (hence $\nabla w_k(0) = 0$), and applying [18, Lemma 4.2], we deduce that $\nabla w_k(z) \neq 0$ for all $z \in \overline{B}_R \setminus \{0\}$, thus the origin is the unique maximum point of w_k in B_R for k sufficiently large. □

Remark 4.10. By Proposition 4.9, we have that the second conclusion of Proposition 4.7 holds for D_δ, instead of D_δ^{in}. In turn, recalling that $\partial\Omega$ is regular and $P_\delta \in \partial\Omega$, we deduce that

$$\mathcal{H}^{N-1}(\partial D_\delta \cap \partial\Omega) \geq \mathcal{H}^{N-1}((2\delta)^{1/N}(1-\varepsilon)B(P_\delta)\cap\partial\Omega) \geq C\delta^{(N-1)/N},$$

where

$$0 < C < \omega_{N-1} 2^{(N-1)/N} \omega_N^{-(N-1)/N}$$

and δ is sufficiently small.

Finally, we observe that also the third claim of Lemma 4.5 holds with $\widetilde{D}_k$ instead of $\widetilde{D}_k^{\text{in}}$. As a consequence, one can (a posteriori) prove the stronger convergence as $k \to +\infty$:

$$m_k + \beta \to m + \beta, \qquad \text{strongly in } L^p(\mathbb{R}^N), \text{ for all } p < +\infty.$$

We conclude this section with some estimates about the asymptotical decay of the eigenfunction u_δ. Such estimates can be obtained as a byproduct of the previous blow-up analysis. We refer the reader to [18, Theorem 2.3] for more details.

Proposition 4.11. *With the notation above, for all $\eta > 0$ there exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$, there is a subdomain $\Omega_\delta^{(i)} \subset \Omega$ satisfying:*

(i) $P_\delta \in \partial\Omega_\delta^{(i)}$ *and* $\mathrm{diam}(\Omega_\delta^{(i)}) \leq \widehat{C}\delta^{1/N}$,

(ii) $\|\delta^{1/2} u_\delta(\cdot) - w(\Psi(\cdot)/\delta^{1/N})\|_{C^{1,\vartheta}(\overline{\Omega}_\delta^{(i)})} \leq \eta$,

(iii) $|u_\delta(x)| \leq C_1 \delta^{-\frac{1}{2}} \eta e^{-\mu_1 d(x)/\delta^{1/N}}$, *for all* $x \in \Omega \setminus \Omega_\delta^{(i)}$,

where $d(x) := \min\{\mathrm{dist}(x, \partial\Omega_\delta^{(i)}), \eta_0\}$ *and* $\widehat{C}, C_1, \mu_1, \eta_0$ *are positive constant depending only on* Ω.

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