A non-parametric Plateau problem with partial free boundary

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Abstract

We consider a Plateau problem in codimension 1 in the non-parametric setting. A Dirichlet boundary datum is given only on part of the boundary $\partial\Omega$ of a bounded convex domain $\Omega \subset \mathbb{R}^2$. Where the Dirichlet datum is not prescribed, we allow a free contact with the horizontal plane. We show existence of a solution, and prove regularity for the corresponding minimal surface. Finally we compare these solutions with the classical minimal surfaces of Meeks and Yau, and show that they are equivalent when the Dirichlet boundary datum is assigned in at most 2 disjoint arcs of $\partial\Omega$.

Key words: Plateau problem, relaxation, Cartesian currents, area functional, minimal surfaces.

AMS (MOS) 2020 Subject Classification: 49J45, 49Q05, 49Q15, 28A75.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set; in this paper we look for an area-minimizing surface which can be written as a graph over a subset of Ω, and spanning a Jordan curve $\Gamma_\sigma = \gamma \cup \sigma \subset \mathbb{R}^2 \times [0, +\infty)$. Here γ is fixed (Dirichlet condition) and is given by a family $\{\gamma_i\}_{i=1}^n \subset \partial\Omega \times [0, +\infty)$ of $n \in \mathbb{N}$ curves each joining pairs of points $\{(p_i, q_i)\}_{i=1}^n$ of $\partial\Omega$. Whereas σ, which represents the *free boundary*, consists of (the image of) n curves $(\sigma_1, \dots, \sigma_n)$ sitting in the plane containing Ω (also called free boundary plane), and joining the endpoints of γ in order that $\gamma \cup \sigma$ forms a Jordan curve Γ_σ in $\mathbb{R}^3$. We assume that each γ_i is Cartesian, i.e., it can be expressed as the graph of a given nonnegative function φ defined on a corresponding portion of $\partial\Omega$. This allows to restrict ourselves to the Cartesian setting, and to assume that the competitors for the Plateau problem are expressed by graphs of functions ψ defined on a suitable subdomain of Ω depending on σ.

Our prototypical example is given by the catenoid. Consider a cylinder in $\mathbb{R}^3$ with a circle of radius r as basis, and height l. Choose a system of Cartesian coordinates in which the x_1x_2-plane contains the cylinder axis, and restrict attention to the half-space $\{x_3 \geq 0\}$ as in Figure 1, where $\Omega = (0, l) \times (-r, r)$ and $n = 2$. Write

$$\partial\Omega = \partial_1^0\Omega \cup \overline{\partial_1^D\Omega} \cup \partial_2^0\Omega \cup \overline{\partial_2^D\Omega},$$

where $\partial_1^0\Omega = (0, l) \times \{r\}$, $\partial_2^0\Omega = (0, l) \times \{-r\}$, $\partial_1^D\Omega = \{0\} \times (-r, r)$, and $\partial_2^D\Omega = \{l\} \times (-r, r)$. On the Dirichlet boundary $\partial^D\Omega = \partial_1^D\Omega \cup \partial_2^D\Omega$ we prescribe a continuous function φ whose graph

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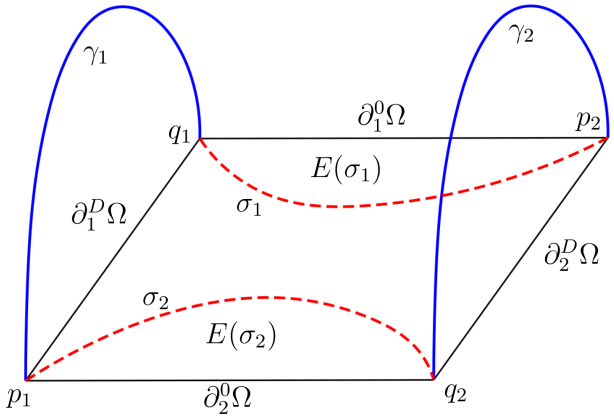

Figure 1: The catenoid: when l is large enough the two dotted curves σ_1 and σ_2 merge and the (generalized) graph of ψ reduces to two vertical half-circles on $\partial^D\Omega = \partial_1^D\Omega \cup \partial_2^D\Omega$. In this case $\partial^D\Omega \subset \partial E(\sigma_1) \cup \partial E(\sigma_2)$.

consists of the two half-circles γ_1 and γ_2. The endpoints of γ_1 and γ_2 live on the free boundary plane (the horizontal plane) and are $p_1 = (0, -r)$, $q_1 = (0, r)$, and $p_2 = (l, r)$, $q_2 = (l, -r)$ respectively. The free boundary σ consists of two curves σ_1 and σ_2 with endpoints q_1, p_2, and q_2, p_1, respectively, constrained to stay in $\overline{\Omega}$. The concatenation of $\gamma = \gamma_1 \cup \gamma_2$ and σ forms a Jordan curve in $\mathbb{R}^3$

$$\Gamma_\sigma = \gamma_1 \cup \sigma_1 \cup \gamma_2 \cup \sigma_2.$$

Therefore we proceed to look for an area-minimizer among all Cartesian surfaces S with boundary Γ_σ *keeping* σ *free*, i.e. we minimize the area among all pairs (σ, S). In this particular case of the catenoid, a minimizing sequence $((\sigma_k, S_k))$ tends (in a suitable way specified in the sequel) to a minimizer (σ, S) which allows for two different possibilities. If l is small, σ_1 and σ_2 remain disjoint and the classical catenoid (half of it, namely the intersection between the catenoid and the half-space $\{x_3 \geq 0\}$) is the surface S, in turn coinciding with the graph of a function ψ defined on the region of Ω "enclosed" by σ. If instead l is large, the two curves σ_1 and σ_2 merge and the region of Ω enclosed by σ tends to become empty (it reduces to the two segments $\partial_1^D\Omega \cup \partial_2^D\Omega$). This describes the solution given by two (half) discs.

A peculiarity of our problem is the presence of a free boundary. The problem of Plateau with partial free boundary has been exhaustively studied (see for instance [10]) but never investigated, to our best knowledge, with the non-parametric approach.

Referring to Section 2 for the precise description of the mathematical framework, here we just describe it with few details. We fix some distinct points $p_1, q_1, p_2, q_2, \ldots, p_n, q_n \in \partial\Omega$ taken in clockwise order. The part of $\partial\Omega$ between the points p_i and q_i is noted by $\partial_i^D\Omega$, and the part between q_i and p_{i+1} by $\partial_i^0\Omega$. We fix a nonnegative continuous function $\varphi\colon \partial\Omega \to [0, +\infty)$ which is positive on $\partial^D\Omega = \cup_{i=1}^n \partial_i^D\Omega$ and vanishes on $\{p_i, q_i\}_{i=1}^n \cup \partial^0\Omega$ with $\partial^0\Omega = \cup_{i=1}^n \partial_i^0\Omega$, and we consider Lipschitz injective and mutually disjoint curves σ_i in $\overline{\Omega}$, $i = 1, \ldots, n$, joining p_i to q_{i+1}. We suppose the graph of φ on $\partial^D\Omega$ to be a Lipschitz curve in $\mathbb{R}^3$. We define $E(\sigma) := \cup_{i=1}^n E(\sigma_i)$, with $E(\sigma_i)$ the planar closed region enclosed between $\partial_i^0\Omega$ and σ_i.

We define the two classes

$$\widehat{\Sigma} := \{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) : [0,1] \to \overline{\Omega}^n \text{ be curves as above}\}, \tag{1.1}$$

$$\mathcal{X}_\varphi := \{(\sigma, \psi) \in \widehat{\Sigma} \times W^{1,1}(\Omega) : \psi = 0 \text{ a.e. in } E(\sigma) \text{ and } \psi = \varphi \text{ on } \partial^D\Omega\}. \tag{1.2}$$

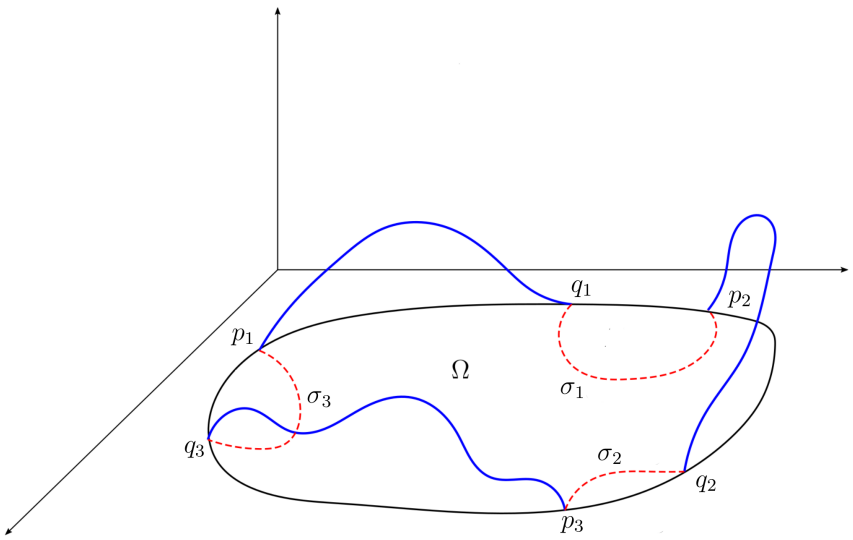

Figure 2: An example of the setting (in 3D), when $n = 3$. On the boundary of the convex set Ω we have fixed the points p_i, q_i; the arc of $\partial\Omega$ joining p_i to q_i is $\partial_i^D\Omega$, while the arc joining q_i to p_{i+1} is $\partial_i^0\Omega$ ($p_4 := p_1$). On $\partial^D\Omega$ the Dirichlet boundary datum φ is imposed, whose graph has been depicted. The dotted arcs are the free planar curves σ_i joining the pairs (q_i, p_{i+1}).

We want to find a solution to the following minimum problem:

$$\inf_{(\sigma,\psi)\in\mathcal{X}_\varphi} \mathbb{A}(\psi; \Omega \setminus E(\sigma)), \tag{1.3}$$

where $\mathbb{A}$ denotes the classical area integral, i.e.,

$$\mathbb{A}(\psi; \Omega \setminus E(\sigma)) := \int_{\Omega\setminus E(\sigma)} \sqrt{1 + |\nabla\psi|^2}dx. \tag{1.4}$$

Since, in general, existence of minimizers is not guaranteed in the class $\mathcal{X}_\varphi$, we need to formulate this problem to a more suited space of admissible pairs. Specifically, a standard relaxation procedure leads one to analyse the problem above for pairs (σ, ψ) belonging to $\Sigma\times BV(\Omega)$, where Σ is a suitable class containing $\widehat{\Sigma}$ but which also allows for partial overlapping of the curves σ_i (a precise definition is given in Section 2.3). Therefore we shall be concerned with the study of the functional $\mathcal{F}_\varphi$ defined as

$$\mathcal{F}_\varphi(\sigma, \psi) := \mathcal{A}(\psi; \Omega) - |E(\sigma)| + \int_{\partial\Omega} |\psi - \varphi| d\mathcal{H}^1, \tag{1.5}$$

where $(\sigma, \psi) \in \mathcal{W} \subset \Sigma \times BV(\Omega)$, $\mathcal{W}$ is the space of pairs $(\sigma, \psi) \in \Sigma \times BV(\Omega)$ such that $\psi = 0$ a.e. on $E(\sigma)$, and $\mathcal{A}(\psi; \Omega)$ is the relaxed area functional defined as in (2.1), which accounts for the area of the generalized graph of the map ψ on Ω. The functional $\mathcal{F}_\varphi$ extends the area integral $\mathbb{A}$ to the larger class $\mathcal{W}$.

We then prove the following result, accounting for existence and regularity of minimizers of $\mathcal{F}_\varphi$.

Theorem 1.1. *There exists a minimizer of $\mathcal{F}_\varphi$ on $\mathcal{W}$. Moreover, any minimizer $(\sigma, \psi) \in \mathcal{W}$ of $\mathcal{F}_\varphi$ satisfies the following regularity properties:*

(1) *The region $E(\sigma)$ consists of a family of closed convex sets. The boundary $\partial E(\sigma)$ is given by the union of the arcs $\partial^0\Omega$ and a family of disjoint Lipschitz curves in $\overline{\Omega}$ (joining the points p_i and q_j, in some order). Moreover, if $\partial_i^D\Omega$ is not a straight segment, then $\partial_i^D\Omega \cap \partial E(\sigma) = \emptyset$. If instead $\partial_i^D\Omega$ is a straight segment, then either $\partial_i^D\Omega \cap \partial E(\sigma) = \emptyset$ or $\partial_i^D\Omega \cap \partial E(\sigma) = \partial_i^D\Omega$.*

(2) *The function ψ is real analytic in $\Omega \setminus E(\sigma)$, and is continuous on $\partial^D\Omega \setminus \partial E(\sigma)$ where it attains the boundary value $\psi = \varphi$.*

(3) *If $\Omega \cap \partial E(\sigma) \neq \emptyset$, there is at least a minimizer (σ, ψ) such that ψ is continuous and null on $\Omega \cap \partial E(\sigma)$, and moreover $\Omega \cap \partial E(\sigma)$ consists of a family of mutually disjoint smooth curves (joining p_i and q_j in some order).*

A comparison with classical solutions of the Plateau problem in parametric form is in order. Denoting by γ_i the graph of the map φ on $\overline{\partial_i^D\Omega}$, we consider also $\mathrm{sym}(\gamma_i)$, namely the graph of $-\varphi$ on $\overline{\partial_i^D\Omega}$, which is symmetric to γ_i with respect to the plane containing Ω. Setting $\Gamma_i := \gamma_i \cup \mathrm{sym}(\gamma_i)$, this turns out to be a simple Jordan curve in $\mathbb{R}^3$, for all $i = 1, \dots, n$. Hence we can consider the classical Plateau problem for the curves Γ_i. In the case $n = 1$ it is intuitive that a disc-type minimal surface S spanning $\Gamma = \Gamma_1$ will be symmetric with respect to the plane containing Ω, and that $S^+ := S \cap \{x_3 \geq 0\}$ will be a minimal disc with partial free boundary on Ω. It is interesting to compare such a minimal disc with the graph of ψ, where $(\sigma, \psi) \in \mathcal{W}$ is a minimizer as in Theorem 1.1. Actually, in this simple case $n = 1$, it is not difficult to see that S^+ is Cartesian, and it is the graph of a function ψ which is positive outside the convex region $E(\sigma)$ enclosed by σ and $\partial^0\Omega$, and further (σ, ψ) is a minimizer as provided by Theorem 1.1. Also the converse is true: Any minimizer (σ, ψ) that satisfies (1)-(3) of Theorem 1.1 has as graph of ψ a disc-type surface S^+ whose double $S = S^+ \cup S^-$ is a classical solution to the Plateau problem for the curve Γ.

This result is rigorously stated in Theorem 6.1 of Section 6.1. In Section 6.2 we instead analyse the case $n = 2$. In this case one might look for minimal surfaces obtained as union of two discs spanning Γ_1 and Γ_2, or else for a catenoid-type surface spanning $\Gamma = \Gamma_1 \cup \Gamma_2$ together. Appealing to an existence result due to Meeks and Yau [18], we are able to show the counterpart of Theorem 6.1: Theorem 6.5, that essentially states that any minimizer $(\sigma, \psi) \in \mathcal{W}$ of $\mathcal{F}_\varphi$ satisfying properties (1)–(3) of Theorem 1.1 is (the nonnegative half of) a Meeks-Yau solution, and vice-versa. In order to prove Theorem 6.5 we will strongly use the convexity of the domain Ω, which implies that the cylinder $\Omega \times \mathbb{R}$, which contains Γ on its boundary, is convex, and so the results of Meeks and Yau are applicable.

Due to the highly nontrivial arguments used to prove this result, we restrict our analysis to the case $n = 2$, since a generalization to the case $n > 2$ probably requires heavy modifications. Indeed, some of the lemmas needed to prove Theorem 6.5 employ crucially the fact that $\partial^0\Omega$ consists of only two connected components. For this reason we leave the case $n > 2$ for future investigations.

Let us now come to the reasons for our study. One motivation is the description of a cluster of soap films which are constrained to wet a given system of wires γ emanating from a given free boundary plane. The soap films are expected to arrange in such a way to form a free boundary on the plane. Therefore, the questions of existence of a minimal configuration and its regularity naturally arise. A second motivation is related to the description of the singular part of the L^1-relaxation of the Cartesian 2-codimensional area functional

$$\int_U \sqrt{1 + |\nabla u_1|^2 + |\nabla u_2|^2 + (\det \nabla u)^2}\, dx, \qquad u = (u_1, u_2) \in C^1(U; \mathbb{R}^2),$$

computed on nonsmooth maps. The L^1-relaxed area functional [1,14], denoted by $\mathcal{A}(\cdot; U)$, is mostly unknown, up to a few exceptions, see [1, 5–7, 20]. One of the remarkable exceptions is the case of

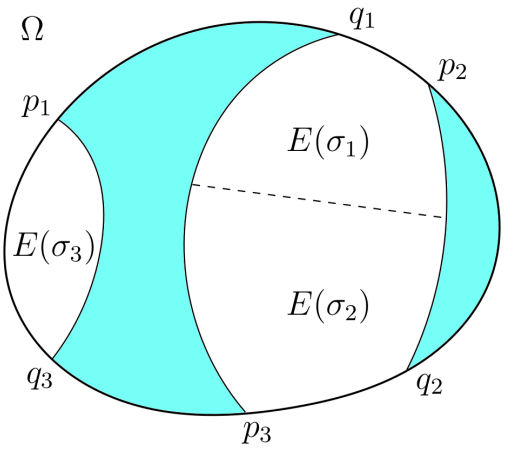

Figure 3: A possible configuration of the sets $E(\sigma_i)$ (in the relaxed problem). Also in this example $n = 3$. The (clockwise oriented) arcs $\widehat{p_1q_1} = \partial_1^D\Omega$, $\widehat{p_2q_2} = \partial_2^D\Omega$, and $\widehat{p_3q_3} = \partial_3^D\Omega$ are the set where φ is prescribed and positive. In the set $\partial^0\Omega = \widehat{q_1p_2} \cup \widehat{q_2p_3} \cup \widehat{q_3p_1}$ and on $E(\sigma) = E(\sigma_1) \cup E(\sigma_2) \cup E(\sigma_3)$ we prescribe $\psi = 0$. The curves σ_i joining q_i to p_{i+1} (with the corresponding set $E(\sigma_i)$) are indicated. On the dotted segment σ_1 and σ_2 overlaps with opposite orientations. The emphasized region $\Omega \setminus E(\sigma)$ is the one where ψ is not necessarily null.

the vortex map $u_V : B_l(0) \setminus \{0\} \subset \mathbb{R}^2 \to \mathbb{R}^2$, defined by $u_V(x) = \frac{x}{|x|}$: in this case it can be proved that

$$\mathcal{A}(u_V; B_l(0)) = \int_{B_l(0)} \sqrt{1 + |\nabla u_V|^2} dx + \inf \mathcal{F}_\varphi(\sigma, \psi), \tag{1.6}$$

where the infimum is taken over all pairs $(\sigma, \psi) \in \Sigma \times BV(R_{2l})$ with $\psi = 0$ a.e. on $E(\sigma)$. Here the setting is the following: $n = 1$, $R_{2l} = (0, 2l) \times (-1, 1)$, $\partial^0 R_{2l} = (0, 2l) \times \{1\}$, $\partial^D R_{2l} = (\{0\} \times (-1, 1)) \cup ([0, 2l] \times \{-1\}) \cup (\{2l\} \times (-1, 1))$, $p = (0, 1)$, $q = (2l, 1)$, and σ is a unique curve in $\overline{R}_{2l}$ joining p to q. The Dirichlet datum $\varphi : \partial^D R_{2l} \to [0, \infty)$ is the function $\varphi(w_1, w_2) = \sqrt{1 - w_2^2}$. This setting is similar to the catenoid case, with the difference that the Dirichlet boundary is here extended to include the basis $(0, 2l) \times \{-1\}$ and the free curve σ is just one simple curve (see Figure 4).

In order to construct a recovery sequence for the relaxed area (1.6) of the vortex map, it is essential to analyse the existence and regularity of minimizers of $\mathcal{F}_\varphi$. In particular, it is necessary to show that there is at least one sufficiently regular[1] minimizer (σ, ψ). The shape of the curve σ and the graph of ψ are related to the vertical part of a Cartesian current in $B_l(0) \times \mathbb{R}^2$ which arises as limit of (the graphs of) a recovery sequence $(v_k) \subset C^1(B_l(0); \mathbb{R}^2)$ for $\mathcal{A}(u_V; B_l(0))$.

According to what happens for the catenoid, also in this case we have a dichotomy for the behaviour of minimizers (σ, ψ). When l is small, the solution (σ, ψ) consists of a curve σ joining p and q whose interior is contained in R_{2l}, and its shape is so that $E(\sigma)$ is convex; at the same time the graph of ψ on $R_{2l} \setminus E(\sigma)$ is a sort of half-catenoid, so that if we double it considering also its symmetric with respect to the plane containing R_{2l}, it becomes a sort of catenoid spanning two radius one circles, and constrained to contain the segment $(0, 2l) \times \{-1\}$. When instead l is larger

[1] Conditions provided by Theorem 1.1 are sufficient.

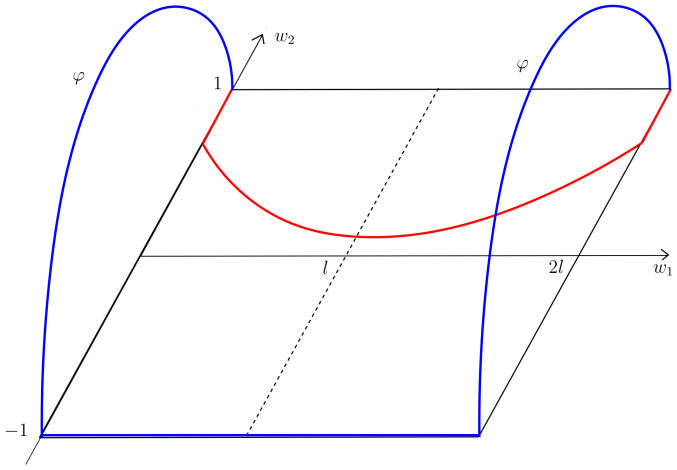

Figure 4: The domain R_{2l} of the vortex map. The graph of φ on $\partial^D R_{2l}$ is emphasized (in particular $\varphi = 0$ on the lower horizontal side), together with an admissible curve σ, which in this specific case partially overlaps the Dirichlet boundary. In this example $n = 1$.

than a certain threshold, then the solution reduces to two circles spanning the two radius one and parallel circles.

The structure of the paper is as follows. In Section 2 we introduce the setting of the problem in detail. In order to prove existence of minimizers of $\mathcal{F}_\varphi$ we first restrict ourselves to prove the result in a smaller class $\mathcal{W}_{\text{conv}} \subset \mathcal{W}$ of admissible pairs (σ, ψ), where compactness is easier and allows to make use of the direct method. Roughly speaking, the class $\mathcal{W}_{\text{conv}}$ accounts only for specific geometries of the free boundary σ, namely, it considers configurations for which each set $E(\sigma_i)$ is convex. In Section 3 we prove the existence of minimizers of $\mathcal{F}_\varphi$ in $\mathcal{W}_{\text{conv}}$. Next, in Section 4, we show the existence of minimizers in the wider class $\mathcal{W}$ where, essentially, σ is not constrained to the previous geometric features; this result is contained in Corollary 4.3. To show this we consider a minimizing sequence in $\mathcal{W}$ and we modify it, by a cut and paste procedure, in order to construct a minimizing sequence in $\mathcal{W}_{\text{conv}}$. In Section 5 we study the regularity properties of minimizers. Specifically, we state and prove Theorem 5.1, which rephrases in a more precise way the results contained in Theorem 1.1. Theorem 1.1 follows from Theorem 3.1, Corollary 4.3, and Theorem 5.1. Eventually in Section 6 we compare the solutions we found with the classical minimal surfaces spanning Γ. Here, as anticipated, we restrict our analysis to the case $n = 1, 2$, the case $n = 2$ essentially giving rise to either a catenoid-type minimal surface, or two disc-type surfaces spanning Γ_1 and Γ_2. The main theorems here are Theorems 6.1 and 6.5. The proof of the former, for the case $n = 1$, is quite simple, whereas Theorem 6.5, for the case $n = 2$, requires a series of lemmas. In particular, if S is a Meeks-Yau catenoid-type minimal surface, at one step, we need to employ a Steiner symmetrization of the 3-dimensional finite perimeter set in $\Omega \times \mathbb{R}$ enclosed by S. In turn, using standard results on the condition of equality for the perimeters of a set and its symmetrization, we are able to show that the starting surface S were already symmetric with respect to the plane containing Ω, and already Cartesian, and the conclusion of the proof of Theorem 6.5 is achieved.

2 Preliminaries

2.1 Area of the graph of a BV function

Let $U \subset \mathbb{R}^2$ be a bounded open set. For any $\psi \in BV(U)$ we denote by $D\psi$ its distributional gradient, so that

$$D\psi = \nabla\psi \mathcal{L}^2 + D^s\psi,$$

where $\nabla\psi$ is the approximate gradient of ψ and $D^s\psi$ denotes the singular part of $D\psi$. We recall that the L^1-relaxed area functional reads as [15]

$$\mathcal{A}(\psi; U) := \int_U \sqrt{1 + |\nabla\psi|^2} dx + |D^s\psi|(U). \tag{2.1}$$

In what follows we denote by $\partial^* A$ the reduced boundary of a set of finite perimeter $A \subset \mathbb{R}^3$ (see [2]). For any $\psi \in BV(U)$ we denote by $R_\psi \subset U$ the set of regular points of ψ, namely the set of points $x \in U$ which are Lebesgue points for ψ, $\psi(x)$ coincides with the Lebesgue value of ψ at x and ψ is approximately differentiable at x. We define the subgraph SG_ψ of ψ as

$$SG_\psi := \{(x, y) \in R_\psi \times \mathbb{R} \colon y < \psi(x)\}.$$

This turns out to be a finite perimeter set in $U \times \mathbb{R}$. Its reduced boundary in $U \times \mathbb{R}$ is the generalised graph $\mathcal{G}_\psi := \{(x, \psi(x)) \colon x \in R_\psi\}$ of ψ, which turns out to be a 2-rectifiable set. If $[\![SG_\psi]\!] \in \mathcal{D}_3(\mathbb{R}^3)$ denotes the integral current given by integration over SG_ψ and $\partial[\![SG_\psi]\!] \in \mathcal{D}_2(\mathbb{R}^3)$ is its boundary in the sense of currents, then

$$[\![\mathcal{G}_\psi]\!] = \partial[\![SG_\psi]\!] \llcorner (U \times \mathbb{R}),$$

with $[\![\mathcal{G}_\psi]\!]$ denoting the integer multiplicity 2-current given by integration over $\mathcal{G}_\psi$ (suitably oriented; see [13] for more details).

2.2 Hausdorff distance

If $A, B \subset \mathbb{R}^2$ are nonempty, the symbol $d_H(A, B)$ stands for the Hausdorff distance between A and B, that is

$$d_H(A, B) := \max\left\{\sup_{a \in A} \mathrm{d}_B(a)\,,\, \sup_{b \in B} \mathrm{d}_A(b)\right\},$$

where $\mathrm{d}_F(\cdot)$ is the distance from the nonempty set $F \subseteq \mathbb{R}^2$. If we restrict d_H to the class of closed sets, then d_H defines a metric. Moreover:

(H1) $\mathrm{d}_A(x) \leq \mathrm{d}_B(x) + d_H(A, B)$ for every $x \in \mathbb{R}^2$;

(H2) $(\mathcal{K}, d_H)$ with $\mathcal{K} := \{K \subset \mathbb{R}^2 \text{ nonempty and compact}\}$ is a complete metric space;

(H3) If $A, B \subset \mathbb{R}^2$ are bounded, closed, nonempty and convex sets, then $d_H(A, B) = d_H(\partial A, \partial B)$;

(H4) If $A \in \mathcal{K}$ is convex, then there exists a sequence $(A_n)_n \subset \mathcal{K}$ of convex sets with boundary of class C^∞ such that $d_H(A_n, A) \to 0$ as $n \to \infty$;

(H5) Let $(A_n)_n$ be a sequence of closed convex sets in $\mathbb{R}^2$, $A \subset \mathbb{R}^2$ and $d_H(A_n, A) \to 0$ as $n \to +\infty$. Then A is convex as well;

(H6) Let $(A_n)_n$ and A be compact convex subsets of $\mathbb{R}^2$ such that $d_H(A_n, A) \to 0$ and let $x \in \mathrm{int}(A)$; then $x \in A_n$ definitely in n;

(H7) Let A and B be closed subsets of $\mathbb{R}^2$ with $d_H(A,B)=\varepsilon$. Then $A\subset B_\varepsilon^+$ and $B\subset A_\varepsilon^+$ where, for all $E\subset\mathbb{R}^2$, we have set $E_\varepsilon^+:=\{x\in\mathbb{R}^2\colon \mathrm{d}_E(x)\le\varepsilon\}$.

Remark 2.1. Property (H1) is straightforward, while (H2) is well-known. Also property (H3) is easily obtained (see, e.g. [21]). Concerning property (H4) we refer to, e.g., [4, Corollary 2]. To see (H5), from (H1) we have that $\mathrm{d}_{A_n}\to\mathrm{d}_A$ pointwise, and therefore since d_{A_n} is convex, also d_A is convex, which implies A convex[2]. Let us now prove (H6) by contradiction; assume that there exists a subsequence (n_k) such that $\mathrm{d}_{A_{n_k}}(x)>0$ for all $k\in\mathbb{N}$; then $x\in\mathbb{R}^2\setminus A_{n_k}$, $\mathrm{d}_{A_{n_k}}(x)=\mathrm{d}_{\partial A_{n_k}}(x)$, and using (H1) twice,

$$\begin{aligned}\mathrm{d}_{\partial A}(x)&\le \mathrm{d}_{\partial A_{n_k}}(x)+d_H(\partial A_{n_k},\partial A)=\mathrm{d}_{A_{n_k}}(x)+d_H(A_{n_k},A)\\&\le \mathrm{d}_A(x)+2d_H(A,A_{n_k})=2d_H(A,A_{n_k})\to 0,\end{aligned}$$

the first equality following from (H3). This implies $x\in\partial A$, a contradiction.

We begin with the following standard result that will be useful later:

Lemma 2.2. *Let $K\subset\mathbb{R}^2$ be a convex compact set with nonempty interior. Then there exists a 1-periodic curve $\widehat\sigma\in\mathrm{Lip}(\mathbb{R};\mathbb{R}^2)$, injective on $[0,1)$, such that $\widehat\sigma([0,1])=\partial K$ and*

$$\widehat\sigma(t)=\widehat\sigma(0)+\ell(\widehat\sigma)\int_0^t\widehat\gamma(s)\,ds,\quad \widehat\gamma(t)=(\cos(\widehat\theta(t))\,,\sin(\widehat\theta(t)))\quad \textit{for all } t\in[0,1],$$

with $\widehat\theta$ a non-decreasing function satisfying $\widehat\theta(t+1)-\widehat\theta(t)=2\pi$ for all $t\in\mathbb{R}$.

Notice that $\widehat\sigma$ is differentiable a.e. in $\mathbb{R}$ and $\widehat\sigma'(t)=\ell(\widehat\sigma)\widehat\gamma(t)$, so that the speed modulus of the curve $|\hat\sigma'(t)|=\ell(\widehat\sigma)$ is constant and coincides with the length of the curve $\ell(\widehat\sigma)=\int_0^1|\widehat\sigma'(s)|ds$.

Proof. We start by approximating K by convex sets with C^∞ boundary. By (H4) for all $n\in\mathbb{N}$ there is a convex compact set $K_n\subset\mathbb{R}^2$ with boundary of class C^∞ and such that $d_H(K_n,K)\to 0$ as $n\to\infty$. For any $n\in\mathbb{N}$ we let $\widehat\sigma_n\in C^\infty(\mathbb{R};\mathbb{R}^2)$ be a 1-periodic function injectively parametrizing ∂K_n on $[0,1)$; therefore $\widehat\sigma_n([0,1])=\partial K_n$, and

$$\widehat\sigma_n(t)=\widehat\sigma_n(0)+\ell(\widehat\sigma_n)\int_0^t\widehat\gamma_n(s)\,ds,\quad \widehat\gamma_n(t)=(\cos(\widehat\theta_n(t))\,,\sin(\widehat\theta_n(t)))\quad \forall t\in[0,1],$$

where $\widehat\theta_n\in C^\infty(\mathbb{R})$ is a non-decreasing function with $\widehat\theta_n(t+1)-\widehat\theta_n(t)=2\pi$, for all $t\in\mathbb{R}$. In view of (H2), by construction we can find $x_0\in K$, $R>r>0$ such that $B_r(x_0)\subset K_n\subset B_R(x_0)$ for all $n\in\mathbb{N}$, and therefore $\mathcal{H}^1(\partial B_r(x_0))\le\ell(\widehat\sigma_n)=\mathcal{H}^1(\partial K_n)\le\mathcal{H}^1(\partial B_R(x_0))$; thus, up to subsequence, $\ell(\widehat\sigma_n)\to\widehat m\in\mathbb{R}^+$ as $n\to\infty$. Moreover, up to subsequence, we might assume $\widehat\sigma_n(0)\to p\in\partial K$. On the other hand observing that

$$\int_t^{t+1}|\widehat\theta_n'(s)|ds=\int_t^{t+1}\widehat\theta_n'(s)ds=2\pi,\ \text{for all } t\in\mathbb{R},$$

we have that, again up to subsequence, $\widehat\theta_n\overset{*}{\rightharpoonup}\widehat\theta\in BV_{\mathrm{loc}}(\mathbb{R})$ and pointwise (by Helly selection principle), with $\widehat\theta$ a non-decreasing function with $\widehat\theta(t+1)-\widehat\theta(t)=2\pi$ for all $t\in\mathbb{R}$. We also have $\widehat\gamma_n\overset{*}{\rightharpoonup}\widehat\gamma$ in $BV_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2)$ where $\widehat\gamma(t)=(\cos(\widehat\theta(t))\,,\sin(\widehat\theta(t)))$.
We let $\widehat\sigma\in\mathrm{Lip}(\mathbb{R};\mathbb{R}^2)$ be the (1-periodic) simple closed curve defined as

$$\widehat\sigma(t):=p+\widehat m\int_0^t\widehat\gamma(s)\,ds\quad\forall t\in\mathbb{R}.\tag{2.2}$$

[2]Since A is closed, it coincides with the sublevel $\{x: d(x,A)\le 0\}$, which is convex.

Note that $\widehat{m} = \ell(\widehat{\sigma})$. Then clearly $\widehat{\sigma}_n \to \widehat{\sigma}$ in $W^{1,1}([0,1];\mathbb{R}^2)$, since

$$\begin{aligned}\|\widehat{\sigma}_n' - \widehat{\sigma}'\|_{L^1([0,1];\mathbb{R}^2)} &= \int_0^1 |\ell(\widehat{\sigma}_n)\widehat{\gamma}_n(t) - \ell(\widehat{\sigma})\widehat{\gamma}(t)|dt \\ &\leq |\ell(\widehat{\sigma}_n) - \ell(\widehat{\sigma})| + \ell(\widehat{\sigma}) \int_0^1 |\widehat{\gamma}_n(t) - \widehat{\gamma}(t)|dt \to 0.\end{aligned}$$

By the continuous embedding $W^{1,1}([0,1];\mathbb{R}^2) \subset C^0([0,1];\mathbb{R}^2)$ (and by 1-periodicity, on $\mathbb{R}$) we also get $\widehat{\sigma}_n \to \widehat{\sigma}$ uniformly on $[0,1]$. This, together with property (H3) gives

$$d_H(\partial K, \widehat{\sigma}([0,1])) \leq d_H(\partial K, \partial K_n) + d_H(\widehat{\sigma}_n([0,1]), \widehat{\sigma}([0,1])) \to 0,$$

which in turn implies $\widehat{\sigma}([0,1]) = \partial K$. The injectivity of $\widehat{\sigma}$ on $[0,1)$ follows from expression (2.2), the fact that $\widehat{m} > 0$, and the fact that K is convex with nonempty interior.

□

Corollary 2.3. *Let $K \subset \mathbb{R}^2$ be a convex compact set with nonempty interior. Let q, p be two distinct points on ∂K, and let $\widehat{pq}$ be the relatively open, connected curve contained in ∂K with endpoints q and p clockwise ordered. Then there exists an injective curve $\sigma \in \mathrm{Lip}([0,1];\mathbb{R}^2)$ such that $\sigma((0,1)) = \widehat{pq}$, $\sigma(0) = q$, $\sigma(1) = p$, and*

$$\sigma(t) = q + \ell(\sigma) \int_0^t \gamma(s)\, ds, \quad \gamma(t) = (\cos(\theta(t)), \sin(\theta(t)) \quad \textit{for all } t \in [0,1],$$

with θ a non-decreasing function satisfying $\theta(1) - \theta(0) \leq 2\pi$.

Proof. Lemma 2.2 provides $\widehat{\sigma} \in \mathrm{Lip}([0,1];\mathbb{R}^2)$ parametrizing ∂K. Then there are two values $t_1, t_2 \in [0,1]$, $t_1 < t_2$, with $q = \sigma(t_1)$ and $p = \sigma(t_2)$. Then the existence of σ follows by reparametrization of the interval $[t_1, t_2]$, and all the properties follows from the corresponding properties of $\widehat{\sigma}$. □

2.3 Setting of the problem

We fix $\Omega \subset \mathbb{R}^2$ to be an open bounded convex set (strict convexity is not required) which will be our reference domain. Given two points $p, q \in \partial\Omega$ in clockwise order, $\widehat{pq}$ stands for the relatively open arc on $\partial\Omega$ joining p and q.

Let $n \in \mathbb{N}$, $n \geq 1$, and let $\{p_i\}_{i=1}^n$ be distinct points on $\partial\Omega$ chosen in clockwise order; we set $p_{n+1} := p_1$. For all $i = 1, \ldots, n$ let q_i be a point in $\widehat{p_i p_{i+1}} \subset \partial\Omega$. We set

$$\partial_i^D\Omega := \widehat{p_i q_i}, \qquad \partial_i^0\Omega := \widehat{q_i p_{i+1}} \qquad \text{for } i = 1, \ldots, n, \tag{2.3}$$

and

$$\partial^D\Omega := \bigcup_{i=1}^n \partial_i^D\Omega, \qquad \partial^0\Omega := \bigcup_{i=1}^n \partial_i^0\Omega. \tag{2.4}$$

Since $\partial_i^D\Omega$ and $\partial_i^0\Omega$ are relatively open in $\partial\Omega$, so are $\partial^D\Omega$ and $\partial^0\Omega$. It follows that $\partial\Omega$ is the disjoint union

$$\partial\Omega = \cup_{i=1}^n \{p_i, q_i\} \cup \partial^D\Omega \cup \partial^0\Omega.$$

We fix a continuous function $\varphi : \partial\Omega \to [0, +\infty)$ such that

$$\varphi = 0 \text{ on } \partial^0\Omega \qquad \text{and} \qquad \varphi > 0 \text{ on } \partial^D\Omega,$$

see Figures 1, 2. We will make a further regularity assumption on φ: we require that the graph $\mathcal{G}_{\varphi \llcorner \partial_i^D\Omega} = \{(x, \varphi(x)) : x \in \partial_i^D\Omega\}$ of φ on $\partial_i^D\Omega$ is a Lipschitz curve in $\mathbb{R}^3$, for all $i = 1, \ldots, n$.

Remark 2.4. The hypothesis $\varphi > 0$ on $\partial^D\Omega$ excludes from our analysis the example in Figure 4 of the introduction. We will further comment on this later on (see Section 5.1); the presence of pieces of $\partial^D\Omega$ where $\varphi = 0$ will bring to some additional technical difficulties that we prefer to avoid here. However, the setting in Figure 4 can be easily achieved by an approximation argument. Namely, one considers a suitable regularization φ_ε of φ on $\partial^D\Omega$ such that $\varphi_\varepsilon > 0$, and then letting $\varepsilon \to 0$ one obtains a solution to the problem with Dirichlet datum φ.

We will analyse the functional $\mathcal{F} = \mathcal{F}_\varphi$ defined in (1.5), namely

$$\mathcal{F}(\sigma, \psi) := \mathcal{A}(\psi; \Omega) - |E(\sigma)| + \int_{\partial\Omega} |\psi - \varphi| d\mathcal{H}^1, \tag{2.5}$$

where the pair (σ, ψ) belongs to the admissible class $\mathcal{W}$, defined as follows:

$$\begin{aligned} \mathcal{W} &:= \Big\{(\sigma, \psi) \in \Sigma \times BV(\Omega) : \psi = 0 \text{ a.e. in } E(\sigma)\Big\}, \\ \Sigma &:= \Big\{\sigma = (\sigma_1, \dots, \sigma_n) \in (\mathrm{Lip}([0,1]; \overline{\Omega}))^n \text{ satisfies (i')-(ii')}\Big\}, \end{aligned} \tag{2.6}$$

where

(i') $\sigma = (\sigma_1, \dots, \sigma_n)$ with σ_i injective, $\sigma_i(0) = q_i$ and $\sigma_i(1) = p_{i+1}$, for all $i = 1, \dots, n$;

(ii') For $i = 1, \dots, n$, denoting by $E(\sigma_i) \subset \overline{\Omega}$ the closed region enclosed between $\partial_i^0\Omega$ and $\sigma_i([0,1])$, we assume $\mathrm{int}(E(\sigma_i)) \cap \mathrm{int}(E(\sigma_j)) = \emptyset$ for $i \neq j$ where int denotes the interior part; we also set

$$E(\sigma) := \bigcup_{i=1}^n E(\sigma_i). \tag{2.7}$$

Remark 2.5. The injectivity property in (i') guarantees that the sets $E(\sigma_i)$ are simply connected (not necessarily connected). The assumption that the interior $\mathrm{int}(E(\sigma_i))$ of the sets $E(\sigma_i)$ are mutually disjoint is an hypothesis on the curves σ_i, which essentially translates into the fact that these curves cannot cross transversally each other, but might overlap. Notice that $\mathrm{int}(E(\sigma_i))$ might be empty, as the case $\partial_i^0\Omega = \sigma_i([0,1])$ is not excluded.

The strategy to show existence and regularity of minimizers of the functional (2.5) (see (3.1)) is to reduce to study the same functional on a restricted class of competitors, more precisely to reduce our analysis to the case where the sets $E(\sigma_i)$ are convex. Specifically, we define:

$$\begin{aligned} \mathcal{W}_{\mathrm{conv}} &:= \Big\{(\sigma, \psi) \in \Sigma_{\mathrm{conv}} \times BV(\Omega) : \ \psi = 0 \text{ a.e. in } E(\sigma)\Big\}, \\ \Sigma_{\mathrm{conv}} &:= \Big\{\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma : \sigma \text{ satisfies (i)}\Big\}, \end{aligned} \tag{2.8}$$

and:

(i) For all $i = 1, \dots, n$ the set $E(\sigma_i)$ is convex.

As we have already said, the sets $\mathrm{int}(E(\sigma_i))$ might also be empty, since from assumption (i') we cannot exclude that σ_i overlaps $\partial_i^0\Omega$: Recalling that Ω is convex, this can happen, by (ii') and (i), only if $\widehat{q_i p_{i+1}}$ is a straight segment[3]. Clearly,

$$\mathcal{W}_{\mathrm{conv}} \subset \mathcal{W}. \tag{2.9}$$

[3]We shall prove that, for a minimizer, $\sigma_i([0,1])$ cannot intersect $\partial^D\Omega$ unless $\partial^D\Omega$ is locally a segment, see Theorem 5.1.

Remark 2.6. Exploiting the characterization of the boundaries of convex sets given in Corollary 2.3, we see that conditions (i'),(ii') and (i) for the curves in $\Sigma_{\rm conv}$ imply the following:

(P) For all $i = 1, \dots, n$ there is a nondecreasing function $\theta_i \colon [0,1] \to \mathbb{R}$ with $\theta_i(1) - \theta_i(0) \leq 2\pi$, and such that, setting $\gamma_i(t) := (\cos(\theta_i(t)), \sin(\theta_i(t)))$ for all $t \in [0,1]$, we have

$$\sigma_i(t) = q_i + \ell(\sigma_i) \int_0^t \gamma_i(s)\, ds \qquad \forall t \in [0,1].$$

Here we have denoted the length of σ_i by $\ell(\sigma_i)$.

3 Existence of minimizers of $\mathcal{F}$ in $\mathcal{W}_{\rm conv}$

The main result of this section reads as follows.

Theorem 3.1 (**Existence of a minimizer of $\mathcal{F}$ in $\mathcal{W}_{\rm conv}$**). *Let $\mathcal{F}$ and $\mathcal{W}_{\rm conv}$ be as in* (2.5) *and* (2.8) *respectively. Then there is a solution to*

$$\min_{(\sigma,\psi)\in\mathcal{W}_{\rm conv}} \mathcal{F}(\sigma,\psi). \tag{3.1}$$

We prove Theorem 3.1 using the direct method. To this aim we need to introduce a notion of convergence in $\mathcal{W}_{\rm conv}$.

Definition 3.2 (**Convergence in $\mathcal{W}_{\rm conv}$**). *We say that the sequence $(((\sigma)_k, \psi_k))_k \subset \mathcal{W}_{\rm conv}$, with $(\sigma)_k = ((\sigma_1)_k, ..., (\sigma_n)_k)$, converges to $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ if:*

(a) *$(\sigma_i)_k$ converges to σ_i uniformly in $[0,1]$ for all $i = 1, \dots, n$;*

(b) *$(\psi_k)_k$ converges to ψ weakly star in $BV(\Omega)$, i.e., $\psi_k \to \psi$ in $L^1(\Omega)$ and $D\psi_k \rightharpoonup D\psi$ weakly star in Ω in the sense of measures as $k \to +\infty$.*

Remark 3.3. For any $i = 1, \dots, n$ we have $\lim_{k\to+\infty} d_H(E((\sigma_i)_k), E(\sigma_i)) = 0$, since by property (H3)

$$d_H(E((\sigma_i)_k), E(\sigma_i)) = d_H(\partial E((\sigma_i)_k), \partial E(\sigma_i)) = d_H((\sigma_i)_k([0,1]), \sigma_i([0,1])) \to 0.$$

Lemma 3.4 (**Compactness of $\mathcal{W}_{\rm conv}$**). *Let $\big(((\sigma)_k, \psi_k)\big)_k \subset \mathcal{W}_{\rm conv}$ be a sequence with $\sup_k \mathcal{F}((\sigma)_k, \psi_k) < +\infty$. Then $\big(((\sigma)_k, \psi_k)\big)_k$ admits a subsequence converging to an element of $\mathcal{W}_{\rm conv}$.*

Proof. We divide the proof in two steps.

Step 1: Compactness of $(\sigma)_k$. For simplicity we use the notation $\sigma_{ik} = (\sigma_i)_k$ for every $k \in \mathbb{N}$ and $i \in \{1, \dots, n\}$. By condition (P) in Remark 2.6, for every $k \in \mathbb{N}$ and $i \in \{1, \dots, n\}$ there exists a non-decreasing function $\theta_{ik} \colon [0,1] \to \mathbb{R}$, $\theta_{ik}(1) - \theta_{ik}(0) \leq 2\pi$, such that

$$\sigma_{ik}(t) = q_i + \ell(\sigma_{ik}) \int_0^t \gamma_{ik}(s) ds, \quad \gamma_{ik}(t) := (\cos\theta_{ik}(t)\,, \sin\theta_{ik}(t)) \quad \forall t \in [0,1],$$

and with $\sigma_{ik}(1) = p_{i+1}$. We observe that

$$\ell(\sigma_{ik}) = \int_0^1 |\sigma'_{ik}(t)| dt \leq \mathcal{H}^1(\partial\Omega),$$

since the orthogonal projection $\Pi_{ki}\colon \partial\Omega \setminus \partial_i^0\Omega \to E(\sigma_{ik})$ is a contraction and $\mathcal{H}^1(\partial\Omega \setminus \partial_i^0\Omega) \leq \mathcal{H}^1(\partial\Omega)$. Hence, up to a (not relabelled) subsequence, $\ell(\sigma_{ik}) \to m_i \in \mathbb{R}^+$ as $k \to +\infty$. The number m_i is positive since, for all k and i, we have $\ell(\sigma_{ik}) \geq |q_i - p_{i+1}| > 0$. Moreover

$$\int_0^1 |\theta'_{ik}(t)|dt = \int_0^1 \theta'_{ik}(t)dt \leq 2\pi;$$

hence, up to a subsequence, $\theta_{ki} \overset{*}{\rightharpoonup} \theta_i$ in $BV([0,1])$ and θ_i is non-decreasing with $\theta_i(1) - \theta_i(0) \leq 2\pi$. Furthermore $\gamma_{ik} \overset{*}{\rightharpoonup} \gamma_i$ in $BV([0,1];\mathbb{R}^2)$ with $\gamma_i(t) = (\cos(\theta_i(t))\,,\, \sin(\theta_i(t)))$.

As a consequence $\sigma_{ik} \to \sigma_i$ in $W^{1,1}([0,1];\mathbb{R}^2)$, where

$$\sigma_i(t) := q_i + m_i \int_0^t \gamma_i(s)ds = q_i + \ell(\sigma_i) \int_0^t \gamma_i(s)ds.$$

Indeed we have

$$\begin{aligned} \|\sigma'_{ik} - \sigma'_i\|_{L^1([0,1];\mathbb{R}^2)} &= \int_0^1 |\ell(\sigma_{ik})\gamma_{ik}(t) - \ell(\sigma_i)\gamma_i(t)|dt \\ &\leq \mathcal{H}^1(\partial\Omega) \int_0^1 |\gamma_{ik}(t) - \gamma_i(t)|dt + |\ell(\sigma_{ik}) - \ell(\sigma_i)|. \end{aligned} \tag{3.2}$$

Now taking the limit as $k \to +\infty$ in (3.2) we conclude. Thus $\lim_{k\to+\infty} \sigma_{ik} = \sigma_i$ uniformly, hence we also conclude that σ_i takes values in $\overline{\Omega}$.
It remains to show that $E(\sigma_i)$ is convex for any $i \in \{1, \dots, n\}$. The uniform convergence of (σ_{ik}) yields

$$\lim_{k\to+\infty} d_H(\partial E(\sigma_{ik}), \partial E(\sigma_i)) = 0.$$

This, together with property (H3), gives for $h \geq k$,

$$\begin{aligned} &d_H(E(\sigma_{ik}), E(\sigma_{ih})) = d_H(\partial E(\sigma_{ik}), \partial E(\sigma_{ih})) \\ \leq &d_H(\partial E(\sigma_{ik}), \partial E(\sigma_i)) + d_H(\partial E(\sigma_{ih}), \partial E(\sigma_i)) \to 0 \quad \text{as} \quad k \to +\infty, \end{aligned}$$

and so $(E(\sigma_{ik}))_{k\in\mathbb{N}}$ is a Cauchy sequence in the space of compact subsets of $\mathbb{R}^2$ endowed with the Hausdorff distance (see (H2)). We find $K \subset \mathbb{R}^2$ convex compact such that $d_H(E(\sigma_{ik}), K) \to 0$. Eventually from (H3) we get

$$\begin{aligned} &d_H(\partial K, \partial E(\sigma_i)) \leq d_H(\partial E(\sigma_{ik}), \partial K) + d_H(\partial E(\sigma_{ik}), \partial E(\sigma_i)) \\ = &d_H(E(\sigma_{ik}), K) + d_H(\partial E(\sigma_{ik}), \partial E(\sigma_i)) \to 0 \quad \text{as} \quad k \to +\infty. \end{aligned}$$

Therefore we conclude that $\partial K = \partial E(\sigma_i)$, so $E(\sigma_{ik}) \to E(\sigma_i)$ in the Hausdorff distance, and $E(\sigma_i)$ is convex by property (H5).

Step 2: Compactness of (ψ_k). Setting $F_k = \cup_{i=1}^n E(\sigma_{ik})$ we have

$$|D\psi_k|(\Omega) \leq \mathcal{A}(\psi_k, \Omega) \leq \mathcal{F}((\sigma)_k, \psi_k) + |F_k| \leq C < +\infty \qquad \forall k > 0,$$

where we used that $|F_k| \leq |\Omega|$. Therefore, up to a subsequence, $\psi_k \overset{*}{\rightharpoonup} \psi$ in $BV(\Omega)$ as $k \to +\infty$. To conclude it remains to show that $\psi = 0$ a.e. in $E(\sigma) = \cup_i E(\sigma_i)$. By $\lim_{k\to+\infty} d_H(F_k, E(\sigma)) = 0$, property (H6) yields

$$\text{if} \quad x \in \text{int}(E(\sigma)) \quad \text{then} \quad x \in F_k \quad \text{definitely in } k,$$

and hence since $\lim_{k\to+\infty} \psi_k = \psi$ a.e. in Ω, we infer $\psi = 0$ a.e. in $E(\sigma)$.

□

Lemma 3.5 (**Lower semicontinuity of $\mathcal{F}$ in $\mathcal{W}_{\rm conv}$**). *Let $\big(((\sigma)_k,\psi_k)\big)_k \subset \mathcal{W}_{\rm conv}$ be a sequence converging to $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$. Then*

$$\mathcal{F}(\sigma,\psi)\le \liminf_{k\to+\infty}\mathcal{F}((\sigma)_k,\psi_k).$$

Proof. By a standard argument [15], the functional

$$\psi\in BV(\Omega)\mapsto \mathcal{A}(\psi;\Omega)+\int_{\partial\Omega}|\psi-\varphi|d\mathcal{H}^1$$

is $L^1(\Omega)$-lower semicontinuous. We now show that the map $\sigma\in\Sigma_{\rm conv}\mapsto|E(\sigma)|$ is continuous. Let $(\sigma)_k\subset\Sigma_{\rm conv}$, $\sigma\in\Sigma_{\rm conv}$, and suppose that $(\sigma_i)_k$ uniformly converges to σ_i for all $i=1,\dots,n$ as $k\to+\infty$. Set $F_k:=\cup_{i=1}^n E((\sigma_i)_k)$ and recall that $E(\sigma)=\cup_{i=1}^n E(\sigma_i)$. By Remark 3.3 $\lim_{h\to+\infty}d_H(E((\sigma_i)_k),E(\sigma_i))=0$ for all $i=1,\dots,n$ and therefore $d_H(F_k,E(\sigma))=:\varepsilon_k\to 0^+$.

By invoking (H7) we have $E(\sigma)\subset(F_k)^+_{\varepsilon_k}$. Moreover, since $d_H((F_k)^+_{\varepsilon_k},E(\sigma))\le 2\varepsilon_k$, we get $(F_k)^+_{\varepsilon_k}\subseteq(E(\sigma))^+_{2\varepsilon_k}$, and so

$$|E(\sigma)|\le|(F_k)^+_{\varepsilon_k}|\le|(E(\sigma))^+_{2\varepsilon_k}|.$$

This implies

$$\limsup_{k\to+\infty}|F_k|\le\limsup_{k\to+\infty}|(F_k)^+_{\varepsilon_k}|\le|E(\sigma)|.$$

The converse inequality is a consequence of Fatou's Lemma and (H6), indeed

$$|E(\sigma)|\le\int_\Omega\liminf_{k\to+\infty}\chi_{F_k}(x)dx\le\liminf_{k\to+\infty}\int_\Omega\chi_{F_k}(x)dx=\liminf_{k\to+\infty}|F_k|.$$

The assertion of the lemma follows. □

Proof of Theorem 3.1. By Lemma 3.4 and Lemma 3.5 we can apply the direct method and conclude. □

4 Existence of a minimizer of $\mathcal{F}$ in $\mathcal{W}$

In this section we extend the previous results to the minimization of $\mathcal{F}$ in the larger class $\mathcal{W}$ of competitors.

One issue we find in minimizing the functional $\mathcal{F}$ on $\mathcal{W}$, is that the class Σ in (2.6) is not closed under uniform convergence, since a uniform limit of elements in Σ needs not be formed by injective curves. To overcome this difficulty, in Theorem 4.1 we prove that the infimum of $\mathcal{F}$ over $\mathcal{W}$ coincides with the infimum of $\mathcal{F}$ over $\mathcal{W}_{\rm conv}$. Thus in particular, by Theorem 3.1, we derive the existence of a minimizer for $\mathcal{F}$ in $\mathcal{W}$ (Corollary 4.3).

Theorem 4.1 (**Existence of a minimizer of $\mathcal{F}$ in $\mathcal{W}$**). *There exists $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ such that*

$$\mathcal{F}(\sigma,\psi)=\inf_{(s,\zeta)\in\mathcal{W}}\mathcal{F}(s,\zeta).$$

Moreover every connected component of $E(\sigma)$ is convex.

Remark 4.2. Since the σ_i's may overlap, the assumption that every $E(\sigma_i)$ is convex does not imply in general that every connected component of $E(\sigma)=\cup_{i=1}^n E(\sigma_i)$ is convex.

As a direct consequence of Theorem 4.1 we have:

Corollary 4.3. *Let $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ be a minimizer as in Theorem 3.1. Then (σ,ψ) is also a minimizer of $\mathcal{F}$ in the class $\mathcal{W}$.*

For the reader convenience we split the proof of Theorem 4.1 into a sequence of intermediate results: Lemmas 4.4, 4.5, 4.6, and the conclusion. First we need to introduce some notation.

Let $(\sigma,\psi)\in\mathcal{W}$. We fix an extension $\widehat{\varphi}\in W^{1,1}(B)$ of φ on an open ball $B\supset\overline{\Omega}$. Extending ψ in $B\setminus\overline{\Omega}$ as $\widehat{\varphi}$ (still denoting by ψ such an extension), we can rewrite $\mathcal{F}(\sigma,\psi)$ as

$$\mathcal{F}(\sigma,\psi)=\mathcal{A}(\psi;B)-|E(\sigma)|-\mathcal{A}(\psi;B\setminus\overline{\Omega}). \tag{4.1}$$

Lemma 4.4. *Let $u\in BV(\mathbb{R}\times(0,+\infty))$ be a nonnegative function with compact support in an open ball B_r. Then*

$$\int_{(\mathbb{R}\times\{0\})\cap B_r} u(s)\,d\mathcal{H}^1(s)\le\mathcal{A}(u;B_r\cap(\mathbb{R}\times(0,+\infty)))-|E_{B_r}|, \tag{4.2}$$

where

$$E_{B_r}:=\{x\in B_r\cap(\mathbb{R}\times(0,+\infty)):u(x)=0\}.$$

Notice that the function u is defined only on the half-plane $\mathbb{R}\times(0,+\infty)$, and in (4.2) the symbol $u(s)$ denotes its trace on the line $\mathbb{R}\times\{0\}$.

Proof. We denote by $x=(x_1,x_2)\in\mathbb{R}^2$ the coordinates in $\mathbb{R}^2$. Set $H^+:=\mathbb{R}\times(0,+\infty)$, $Z:=(B_r\cap H^+)\times\mathbb{R}$. Let

$$L:=\{(x,y)\in Z\colon x\in R_u,\ y\in(-u(x),u(x))\}\subset\mathbb{R}^3,$$

where R_u is the set of regular points of u. We have, recalling the notation in Section 2.1,

$$\begin{aligned}2\mathcal{A}(u;B_r\cap H^+)&=\mathcal{A}(u;B_r\cap H^+)+\mathcal{A}(-u;B_r\cap H^+)\\&=\mathcal{H}^2(\partial^*(Z\cap SG_u))+\mathcal{H}^2(\partial^*(Z\cap SG_{-u}))\\&=\mathcal{H}^2(Z\cap\partial^*L)+2|E_{B_r}|.\end{aligned} \tag{4.3}$$

Suppose $B_r\cap(\mathbb{R}\times\{0\})=(a,b)\times\{0\}$. Then, looking at $\mathcal{G}_u$ as an integral current, a slicing argument yields

$$\begin{aligned}\mathcal{H}^2(Z\cap\partial^*L)&\ge\int_a^b\mathcal{H}^1(Z\cap\{x_1=t\}\cap\partial^*L)dt\\&=\int_a^b\mathcal{H}^1(Z\cap\{x_1=t\}\cap(\mathrm{spt}(\mathcal{G}_u-\mathcal{G}_{-u})))dt\\&\ge\int_a^b 2u(t,0)dt=2\int_{(\mathbb{R}\times\{0\})\cap B_r}u(s)\,d\mathcal{H}^1(s)\,,\end{aligned} \tag{4.4}$$

where the last inequality follows from the following fact: If we denote by $[\![\mathcal{G}_u]\!]_t$ the slice of the current $[\![\mathcal{G}_u]\!]$ on the line $\{x_1=t\}$, then

$$\partial[\![\mathcal{G}_u]\!]_t=\delta_{(t,0,u(t,0))}-\delta_{(t,s_t,0)}\qquad\text{for a.e. } t\in(a,b),$$

where $s_t\ge0$ is such that $(t,s_t)=B_r\cap(\{t\}\times\mathbb{R}^+)$, and in writing $\delta_{(t,s_t,0)}$ we are using that u has compact support in B_r. This can be seen, for instance, by approximation of u by smooth maps[4]. Therefore

$$\partial([\![\mathcal{G}_u]\!]_t-[\![\mathcal{G}_{-u}]\!]_t)=\delta_{(t,0,u(t,0))}-\delta_{(t,0,-u(t,0))}\qquad\text{for a.e. } t\in(a,b).$$

This justifies the last inequality in (4.4), and the proof is achieved. □

[4]With respect to the strict convergence of $BV(B_r\cap(\mathbb{R}\times\{0\}))$, which guarantees the approximation also of the trace of u on $\partial\big(B_r\cap(\mathbb{R}\times\{0\})\big)$.

We now turn to two technical lemmas which are necessary to prove Theorem 4.1. We need to introduce a class of sets whose boundaries are regular enough so that the trace of a BV function on them is well-defined. Precisely we say that an open subset of $\mathbb{R}^2$ is *piecewise Lipschitz* if it can be written as the union of a finite family of (not necessarily disjoint) Lipschitz open sets. Notice that, by (2.1) if $V \subset\subset U$ is a piecewise Lipschitz subset of an open and bounded $U \subset \mathbb{R}^2$, then

$$\mathcal{A}(\psi, \overline{V}) = \mathcal{A}(\psi, V) + \int_{\partial V} |\psi^+ - \psi^-| d\mathcal{H}^1, \tag{4.5}$$

where ψ^+ (respectively ψ^-) denotes the trace of $\psi \llcorner V$ (respectively $\psi \llcorner (U \setminus \overline{V})$) on ∂V.

Lemma 4.5 (**Reduction of energy, I**). *For $N \geq 1$ let $F_1, \dots, F_N$ be nonempty connected subsets of $\overline{\Omega}$, each F_i being the closure of a piecewise Lipschitz set, with $F_i \cap F_j = \emptyset$ for $i, j \in \{1, \dots, N\}$, $i \neq j$. Let $\psi \in BV(B)$ satisfy*

$$\psi = 0 \quad \text{a.e. in} \quad G := \bigcup_{i=1}^{N} F_i \quad \text{and} \quad \psi = \widehat{\varphi} \quad \text{in} \quad B \setminus \Omega\,. \tag{4.6}$$

Then, for any $i \in \{1, \dots, N\}$,

$$\mathcal{A}(\psi_i^\star; B) - |G_i^\star| - \mathcal{A}(\psi_i^\star; B \setminus \overline{\Omega}) \leq \mathcal{A}(\psi; B) - |G| - \mathcal{A}(\psi; B \setminus \overline{\Omega}),$$

where

$$G_i^\star := \bigcup_{j \neq i} F_j \cup \mathrm{conv}(F_i) \quad \text{and} \quad \psi_i^\star := \begin{cases} 0 & \text{in } \mathrm{conv}(F_i) \\ \psi & \text{otherwise.} \end{cases} \tag{4.7}$$

Proof. Fix $i \in \{1, \dots, N\}$. By the convexity of Ω, we have $\psi = \psi_i^\star$ in $B \setminus \overline{\Omega}$, hence it suffices to show that

$$\mathcal{A}(\psi_i^\star; B) - |G_i^\star| \leq \mathcal{A}(\psi; B) - |G|.$$

We start by observing that we may assume F_i to be simply connected. Indeed, if not, we can replace it with the set obtained by filling the holes of F_i, and by setting ψ equal to zero in the holes. This procedure reduces the energy. Indeed, since F_i is piecewise Lipschitz, any hole H of it satisfies $\partial H \subset \cup_{j=1}^n \partial A_j$ where A_j's are the Lipschitz sets whose union is F_i. Hence the trace of $\psi \llcorner H$ on ∂H is well-defined, and the external trace $\psi \llcorner (B \setminus H)$ vanishes.

We have that $(\partial \mathrm{conv}(F_i)) \setminus \partial F_i$ is a countable union of segments. We will next modify ψ by iterating at most countably many operations, setting $\psi = 0$ in the region between each of these segments and ∂F_i.

Step 1: Base case. Let l be one of such segments, and U be the open region enclosed between ∂F_i and l. We define $\psi' \in BV(\Omega)$ as

$$\psi' := \begin{cases} 0 & \text{in } U \\ \psi & \text{otherwise}\,. \end{cases}$$

We claim that

$$\mathcal{A}(\psi'; B) - |G'| \leq \mathcal{A}(\psi; B) - |G|\,, \tag{4.8}$$

where $G' := G \cup \overline{U}$. To prove the claim we introduce the sets

$$H := \mathrm{int}(F_i \cup U), \qquad V := U \cap (\cup_{j \neq i} F_j).$$

Note that H is a piecewise Lipschitz set. By construction

$$|G'| = |H| + |\cup_{j\neq i} F_j| - |V|\,,$$

and (4.8) will follow if we show that

$$\mathcal{A}(\psi'; B) - |H| \leq \mathcal{A}(\psi; B) - |\cup_j F_j| + |\cup_{j\neq i} F_j| - |V| = \mathcal{A}(\psi; B) - |F_i \cup V|\,.$$

Since $|H| = |F_i \cup V| + |U \setminus V|$, this can also be written as

$$\mathcal{A}(\psi'; B) \leq \mathcal{A}(\psi; B) + |U \setminus V|\,.$$

In turn $\mathcal{A}(\psi'; B) = \mathcal{A}(\psi'; \overline{U}) + \mathcal{A}(\psi'; B \setminus \overline{U})$ (and similarly for ψ), so we have reduced ourselves with proving

$$\mathcal{A}(\psi'; \overline{U}) \leq \mathcal{A}(\psi; \overline{U}) + |U \setminus V|\,. \tag{4.9}$$

In view of the definition of ψ' which is zero in U, we have[5] $\mathcal{A}(\psi'; \overline{U}) = \int_l |\psi^+| d\mathcal{H}^1 + |U|$ (ψ^+ denoting the trace of $\psi \llcorner (B \setminus U)$ on the segment l) implying that (4.9) is equivalent to

$$\int_l |\psi^+| d\mathcal{H}^1 \leq \mathcal{A}(\psi; \overline{U}) - |V|\,.$$

Finally, if ψ_U denotes the trace of $\psi \llcorner U$ on l, we write $\mathcal{A}(\psi; \overline{U}) = \mathcal{A}(\psi; \overline{U} \setminus l) + \int_l |\psi^+ - \psi_U| d\mathcal{H}^1$, and the expression above is equivalent to

$$\int_l |\psi^+| d\mathcal{H}^1 \leq \int_l |\psi^+ - \psi_U| d\mathcal{H}^1 + \mathcal{A}(\psi; \overline{U} \setminus l) - |V|\,. \tag{4.10}$$

We now prove (4.10). Fix a Cartesian coordinate system (x_1, x_2) so that l belongs to the x_1-axis and U belongs to the half-plane $\{x_2 > 0\}$. Let u be an extension of ψ in $\mathbb{R} \times (0, +\infty)$ which vanishes outside U. Lemma 4.4, applied to u with the ball $B_r = B$, implies

$$\int_l |\psi_U| d\mathcal{H}^1 = \int_{\{x_2=0\}\cap B} u\, d\mathcal{H}^1 \leq \mathcal{A}(u; B \cap (\mathbb{R} \times (0, +\infty))) - |E_B| \leq \mathcal{A}(\psi; \overline{U} \setminus l) - |V|.$$

Here the last inequality follows by recalling that ψ (and thus u) vanishes on V. From this and the inequality $\int_l |\psi^+| d\mathcal{H}^1 \leq \int_l |\psi^+ - \psi_U| d\mathcal{H}^1 + \int_l |\psi_U| d\mathcal{H}^1$ the proof of (4.10) is achieved, so that (4.8) follows.

Step 2: Iterative case. We set $\partial(\mathrm{conv}(F_i)) \setminus \partial F_i = \cup_{j=1}^{\infty} l_j$ with l_j mutually disjoint segments. For every $h \geq 1$ we define the pair (ψ_h, G_h) as follows:

- if $h = 1$

$$\psi_1 := \begin{cases} 0 & \text{in } U_1 \\ \psi & \text{otherwise,} \end{cases} \qquad \text{and} \quad G_1 := G \cup \overline{U}_1\,,$$

  where U_1 is the open region enclosed between ∂F_i and l_1. We also define $H_1 := \mathrm{int}(\overline{F_i \cup U_1})$.

- if $h \geq 2$

$$\psi_h := \begin{cases} 0 & \text{in } U_h \\ \psi_{h-1} & \text{otherwise,} \end{cases} \qquad \text{and} \quad G_h := G_{h-1} \cup \overline{U}_h\,,$$

  where U_h is the open region enclosed between ∂H_{h-1} and l_h and $H_h := \mathrm{int}(\overline{H_{h-1} \cup U_h})$.

[5] Notice that we use the precise integral formula (4.5) thanks to the boundary regularity of U. More precisely we have $\partial U \setminus l \subset \partial F_i \subset \cup_{j=1}^n \partial A_j$, where A_j's are the Lispchitz sets whose union is F_i.

By construction each H_h is simply connected and piecewise Lipschitz, $H_h \subset H_{h+1}$, $G_h \subset G_{h+1} \subset \overline{\Omega}$ for every $h \geq 1$, and moreover

$$\lim_{h\to+\infty} |H_h| = |\mathrm{conv}(F_i)|\,, \qquad \lim_{h\to+\infty} |G_h| = |G_i^\star|\,, \tag{4.11}$$

where $G_i^\star := \cup_{h=1}^\infty G_h = \cup_{j\neq i} F_j \cup \mathrm{conv}(F_i)$. For any $h \geq 2$ we apply step 1, and after h iterations we get

$$\mathcal{A}(\psi_h; B) - |G_h| \leq \mathcal{A}(\psi_{h-1}; B) - |G_{h-1}| \leq \cdots \leq \mathcal{A}(\psi_1; B) - |G_1| \leq \mathcal{A}(\psi; B) - |G|\,. \tag{4.12}$$

In particular,

$$|D\psi_h|(B) \leq \mathcal{A}(\psi_h; B) \leq \mathcal{A}(\psi; B) + |G_h \setminus G| \leq \mathcal{A}(\psi; B) + |\Omega \setminus G|\,,$$

for all $h \geq 1$, and then we easily see that, up to a subsequence, $\psi_h \overset{*}{\rightharpoonup} \psi_i^\star$ in $BV(B)$, where ψ_i^* is defined as in (4.7). Now the lower semicontinuity of $\mathcal{A}(\cdot; B)$ yields

$$\liminf_{h\to+\infty} \mathcal{A}(\psi_h, B) \geq \mathcal{A}(\psi_i^\star; B)\,. \tag{4.13}$$

Finally, gathering together (4.11)-(4.13) we infer

$$\mathcal{A}(\psi_i^\star; B) - |G_i^\star| \leq \liminf_{h\to+\infty} \mathcal{A}(\psi_h; B) - \lim_{h\to+\infty} |G_h| \leq \mathcal{A}(\psi; B) - |G|\,.$$

This concludes the proof. □

Lemma 4.6 (**Reduction of energy, II**). *Let $N \geq 1$, $F_1, \ldots, F_N, G$ and ψ be as in Lemma 4.5. Then there exist $\tilde{n} \in \{1, \ldots, N\}$ and mutually disjoint closed convex sets $\widetilde{F}_1, \ldots, \widetilde{F}_{\tilde{n}} \subset \overline{\Omega}$ such that*

$$G \subset \bigcup_{i=1}^{\tilde{n}} \widetilde{F}_i =: G^\star\,, \tag{4.14}$$

and

$$\mathcal{A}(\psi^\star; B) - |G^\star| - \mathcal{A}(\psi^\star; B \setminus \overline{\Omega}) \leq \mathcal{A}(\psi; B) - |G| - \mathcal{A}(\psi; B \setminus \overline{\Omega})\,, \tag{4.15}$$

where

$$\psi^\star := \begin{cases} 0 & \text{in } G^\star \\ \psi & \text{otherwise}\,. \end{cases} \tag{4.16}$$

Proof. Base case: ($h = 1$). We take the sets

$$\mathrm{conv}(F_1)\,,\ F_2\,,\ \ldots\,,\ F_N \quad \text{and} \quad G_1^\star := \cup_{i=2}^N F_i \cup \mathrm{conv}(F_1)\,, \tag{4.17}$$

and let

$$\psi_1^\star := \begin{cases} 0 & \text{in } G_1^\star \\ \psi & \text{otherwise}\,. \end{cases}$$

Then by Lemma 4.5,

$$\mathcal{A}(\psi_1^\star; B) - |G_1^\star| - \mathcal{A}(\psi_1^\star; B \setminus \overline{\Omega}) \leq \mathcal{A}(\psi; B) - |G| - \mathcal{A}(\psi; B \setminus \overline{\Omega})\,. \tag{4.18}$$

The next step is not necessary if $N = 1$.

Iterative step: ($h > 1$). Suppose $N > 1$. Let $1 < h \leq m \leq N$ be natural numbers, and let $F_{1,h}, \ldots, F_{m,h}$ be connected closed subsets of $\overline{\Omega}$ with nonempty interior that satisfy the following property: There exists $1 \leq k < h$ such that:

(1) $F_{1,h}, \dots, F_{k,h}$ are convex;

(2) $F_{i,h} \cap F_{j,h} = \emptyset$ for all $i, j \neq k$, $i \neq j$.

Notice that, if $m > 1$, for $h = 2$ the sets

$$F_{1,2} := \mathrm{conv}(F_1)\,,\ F_{2,2} := F_2\,,\ \dots\,,\ F_{N,2} := F_N\,,$$

in the base case satisfy (1), (2) with $m = N$ and $k = 1$. We then set $I_k := \{1 \leq i \leq m,\, i \neq k\colon F_{i,h} \cap F_{k,h} \neq \emptyset\}$ and construct a new family of sets using the following algorithm, distinguishing the two cases (a) and (b):

(a) If $I_k = \emptyset$ we define the sets

$$F_{i,h+1} := \begin{cases} F_{i,h} & \text{for } i \neq k+1 \\ \mathrm{conv}(F_{k+1,h}) & \text{for } i = k+1\,, \end{cases}$$

and

$$G^\star_{h+1} := \cup_{i=1}^m F_{i,h+1}\,;$$

(b) if $I_k \neq \emptyset$, up to relabelling the indices, we may assume that

$$I_k = \{k_1, k_1+1, \dots k_2\} \setminus \{k\},$$

for some $k_1 \neq k_2$ with $1 \leq k_1 \leq k \leq k_2 \leq m$, so that

$$\{1, \dots, m\} \setminus \{k\} \setminus I_k = \{1, \dots, k_1 - 1\} \cup \{k_2 + 1, \dots, m\}.$$

Then we set

$$F_{i,h+1} := \begin{cases} F_{i,h} & \text{for } i = 1, \dots, k_1 - 1 \\ \mathrm{conv}(F_{k,h} \cup (\cup_{j \in I_k} F_{j,h})) & \text{for } i = k_1 \\ F_{i+k_2-k_1,h} & \text{for } i = k_1 + 1, \dots, m - k_2 + k_1\,, \end{cases}$$

and

$$G^\star_{h+1} := \cup_{i=1}^{m-k_2+k_1} F_{i,h+1}\,.$$

In both cases (a) and (b) a direct check shows that the produced sets satisfy properties (1) and (2).

We define also the function

$$\psi^\star_{h+1} := \begin{cases} 0 & \text{in } G^\star_{h+1} \\ \psi^\star_h & \text{otherwise}\,. \end{cases}$$

Then, by induction, for all h we use Lemma 4.5, and in view of (4.18) we infer

$$\begin{aligned} \mathcal{A}(\psi^\star_{h+1}; B) - |G^\star_{h+1}| - \mathcal{A}(\psi^\star_{h+1}; B \setminus \overline{\Omega}) \leq & \mathcal{A}(\psi^\star_h; B) - |G^\star_h| - \mathcal{A}(\psi^\star_h; B \setminus \overline{\Omega}) \\ & \leq \mathcal{A}(\psi; B) - |G| - \mathcal{A}(\psi; B \setminus \overline{\Omega})\,. \end{aligned}$$

Conclusion. If $N = 1$ it is sufficient to apply only the base case. If instead $N > 1$ after a finite number $h^\star \leq N$ of iterations we obtain a collections of mutually disjoint and closed convex sets $F_1 := F_{1,h^\star}, \dots, F_{\tilde{n}} := F_{\tilde{n},h^\star}$ with $1 \leq \tilde{n} \leq n$ such that

$$G \subset \cup_{i=1}^{\tilde{n}} F_i =: G^\star\,,$$

and

$$\mathcal{A}(\psi^\star; B) - |G^\star| - \mathcal{A}(\psi^\star; B \setminus \overline{\Omega}) \leq \mathcal{A}(\psi; B) - |G| - \mathcal{A}(\psi; B \setminus \overline{\Omega}),$$

with

$$\psi^\star := \psi^\star_{h^\star} = \begin{cases} 0 & \text{in } G^\star \\ \psi & \text{otherwise}. \end{cases}$$

□

Proof of Theorem 4.1. By Theorem 3.1 it is enough to show that

$$\inf_{(\sigma,\psi)\in\mathcal{W}} \mathcal{F}(\sigma,\psi) = \inf_{(\sigma,\psi)\in\mathcal{W}_{\rm conv}} \mathcal{F}(\sigma,\psi).$$

Since from (2.9) it follows

$$\inf_{(\sigma,\psi)\in\mathcal{W}} \mathcal{F}(\sigma,\psi) \leq \inf_{(\sigma,\psi)\in\mathcal{W}_{\rm conv}} \mathcal{F}(\sigma,\psi),$$

we only need to show the converse inequality. Take a pair $(\bar\sigma, \bar\psi) \in \mathcal{W}$; we suitably modify $(\bar\sigma, \bar\psi)$ into a new pair $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ satisfying

$$\mathcal{F}(\sigma,\psi) \leq \mathcal{F}(\bar\sigma,\bar\psi),$$

and this will conclude the proof.

Let $E(\bar\sigma_1), \dots, E(\bar\sigma_n)$ be the closed sets with mutually disjoint interiors corresponding to $\bar\sigma$ (as in (ii') of Section 2.3) and let $G := \cup_{i=1}^n E(\bar\sigma_i)$. Consider the (closure of the) connected components $F_1, \dots, F_N$ of G, $N \leq n$. Then by Lemma 4.6 there exist $1 \leq \tilde n \leq N$ and $\widetilde{F}_1, \dots, \widetilde{F}_{\tilde n} \subset \overline{\Omega}$ mutually disjoint closed and convex satisfying (4.14), (4.15) and (4.16). Therefore, by construction, for every $i = 1, \dots, n$, q_i and p_{i+1} belong to $\widetilde{F_j}$ for a unique $j \in \{1, \dots, \tilde n\}$. For every $j = 1, \dots, \tilde n$ we denote by

$$q_{j_1}, p_{j_1+1}, \dots, q_{j_{n_j}}, p_{j_{n_j}+1},$$

the ones that belong to F_j. Then we conclude by taking $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ with $\sigma := (\sigma_1, \dots, \sigma_n)$ and

$$\sigma_{j_k}([0,1]) = \begin{cases} \overline{q_{j_k} p_{j_k+1}} & \text{for} \quad k = 1, \dots, n_j - 1 \\ \partial F_j \setminus \left(\cup_{h=1}^{n_j} \partial^0_{j_h} \Omega \right) \cup \left(\cup_{h=1}^{n_j - 1} \overline{q_{j_h} p_{j_h+1}} \right) & \text{for} \quad k = n_j, \end{cases}$$

for every $j = 1, \dots, \tilde n$ and $\psi := \psi^\star$. □

5 Regularity of minimizers

In this section we investigate regularity properties of minimizers of $\mathcal{F}$. The main result reads as follows.

Theorem 5.1 (**Structure of minimizers**). *Every minimizer $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ of $\mathcal{F}$ in $\mathcal{W}$, namely*

$$\mathcal{F}(\sigma,\psi) = \min_{(s,\zeta)\in\mathcal{W}} \mathcal{F}(s,\zeta),$$

satisfies the following properties:

1 Each connected component of $E(\sigma)$ is convex;

2 ψ is positive and real analytic in $\Omega \setminus E(\sigma)$;

3 *If $\partial_i^D\Omega$ is not a segment for some $i = 1, \dots, n$, then $\partial E(\sigma) \cap \partial_i^D\Omega = \emptyset$, ψ is continuous up to $\partial_i^D\Omega$, and $\psi = \varphi$ on $\partial_i^D\Omega$;*

4 *If $\partial_i^D\Omega$ is a segment for some $i = 1, \dots, n$, then either $\partial E(\sigma) \cap \partial_i^D\Omega = \emptyset$ or $\partial E(\sigma) \cap \partial_i^D\Omega = \partial_i^D\Omega$. In the first case ψ is continuous up to $\partial_i^D\Omega$ and $\psi = \varphi$ on $\partial_i^D\Omega$.*

Moreover, there is a minimizer $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ such that

5 *$\Omega \cap \partial E(\sigma)$ consists of a finite number of disjoint curves of class C^∞, and ψ is continuous and null on $\partial E(\sigma) \setminus \partial^D\Omega$.*

Remark 5.2. If $\partial_i^D\Omega$ is a straight segment nothing ensures that $\partial E(\sigma) \cap \partial_i^D\Omega = \emptyset$. However, if this intersection is nonempty, then it must be $\partial_i^D\Omega \subset \partial E(\sigma)$. The prototypical example is given by the classical catenoid, as explained in the introduction (see also Figure 1) where, if the basis of the rectangle $\Omega = R_{2\ell}$ is large enough, a solution ψ is identically zero, and $\partial^D\Omega \subset \partial E(\sigma)$.

This also explains why in point 5 of Theorem 5.1 we write $\partial E(\sigma) \setminus \partial^D\Omega$, since $\partial^D\Omega$ might be partially included in $\partial E(\sigma)$ if $\partial_i^D\Omega$ is a segment (for some $i = 1, \dots, n$).

For the reader convenience we divide the proof in a number of steps.

Lemma 5.3. *Every minimizer $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ of $\mathcal{F}$ in $\mathcal{W}$ satisfies 1, 2 and $\psi = \varphi$ on $\partial^D\Omega \setminus \partial E(\sigma)$.*

Proof. Item 1 follows by Theorem 4.1. By [15, Theorem 14.13] we also have that ψ is real analytic in $\Omega \setminus E(\sigma)$. Together with the strong maximum principle [15, Theorem C.4], this implies that, in $\Omega \setminus E(\sigma)$, either $\psi > 0$ or $\psi \equiv 0$. On the other hand, since Ω is convex we can apply [15, Theorem 15.9] and get that ψ is continuous up to $\partial^D\Omega \setminus \partial E(\sigma)$; in particular

$$\psi = \varphi > 0 \quad \text{on } \partial^D\Omega \setminus \partial E(\sigma)\,, \tag{5.1}$$

which in turn implies $\psi > 0$ in $\Omega \setminus E(\sigma)$.

□

Lemma 5.4. *Let $\Gamma \subset \mathbb{R}^3$ be a rectifiable, simple, closed and non-planar curve satisfying the following properties:*

(1) *$\Gamma \subset \partial(F \times \mathbb{R})$ for some closed bounded convex set $F \subset \mathbb{R}^2$ with nonempty interior;*

(2) *Γ is symmetric with respect to the horizontal plane $\mathbb{R}^2 \times \{0\}$;*

(3) *There are an arc $\widehat{pq} \subset \partial F$, with endpoints p and q, and $f \in C^0(\widehat{pq} \cup \{p, q\}; [0, +\infty))$ such that f is positive in $\widehat{pq}$ and*

$$\Gamma \cap \{x_3 \geq 0\} = \mathcal{G}_f \cup (\{p\} \times [0, f(p)]) \cup (\{q\} \times [0, f(q)]). \tag{5.2}$$

Let S be a solution to the classical Plateau problem for Γ, i.e., a disc-type area-minimizing surface among all disc-type surfaces spanning Γ. Then:

(1′) *$\beta_{p,q} := S \cap (\mathbb{R}^2 \times \{0\}) \subset F$ is a simple curve of class C^∞ joining p and q such that $\beta_{p,q} \cap \partial F = \{p, q\}$;*

(2′) *S is symmetric with respect to $\mathbb{R}^2 \times \{0\}$;*

(3′) *The surface $S^+ := S \cap \{x_3 \geq 0\}$ is the graph of a function $\widetilde{\psi} \in W^{1,1}(U_{p,q}) \cap C^0(\overline{U}_{p,q})$, where $U_{p,q} \subset \mathrm{int}(F)$ is the open region enclosed between $\widehat{pq}$ and $\beta_{p,q}$. Moreover $\widetilde{\psi}$ is analytic in $U_{p,q}$;*

(4′) *The curve $\beta_{p,q}$ is contained in the closed convex hull of Γ, and $F \setminus U_{p,q}$ is convex.*

Remark 5.5. If the function f in (3) is such that $f(p) = f(q) = 0$ then (5.2) becomes $\Gamma \cap \{x_3 \geq 0\} = \mathcal{G}_f$. For later convenience we prove Lemma 5.4 under the more general assumption (3).

Proof. Even though several arguments are standard, we give the proof for completeness.

Step 1: $\beta_{p,q}$ is a simple curve joining p and q.
Let $B_1 \subset \mathbb{R}^2$ be the open unit disc centred at the origin and let $\Phi = (\Phi_1, \Phi_2, \Phi_3) \colon \overline{B}_1 \to S \subset \mathbb{R}^3$ be a parametrization of S with $\Phi(\partial B_1) = \Gamma$, that is harmonic, conformal, and therefore analytic in B_1, continuous up to ∂B_1. Further by (1) it follows that Φ is an embedding and hence injective (see [18] and also [10, page 343]).
By assumption (5.2) we have $\{w \in \partial B_1 \colon \Phi_3(w) = 0\} = \{\Phi^{-1}(p, 0), \Phi^{-1}(q, 0)\}$, so that Φ_3 changes sign only twice on ∂B_1. By applying Rado's lemma (see e.g. [10, Lemma 2, page 295]) to the harmonic function Φ_3 we deduce that $\nabla \Phi_3 \neq 0$ in B_1 and in particular $\{w \in B_1 \colon \Phi_3(w) > 0\}$ and $\{w \in B_1 \colon \Phi_3(w) < 0\}$ are connected, and $\{w \in B_1 \colon \Phi_3(w) = 0\}$ is a simple smooth curve in B_1 joining $\Phi^{-1}(p, 0)$ and $\Phi^{-1}(q, 0)$. By the injectivity of Φ we have that $S \cap (\mathbb{R}^2 \times \{0\}) = \Phi(\{w \in B_1 \colon \Phi_3(w) = 0\})$ is a simple smooth curve joining p and q.

Step 2: S is symmetric with respect to the horizontal plane $\mathbb{R}^2 \times \{0\}$.
By step 1 the sets $\{w \in \overline{B}_1 \colon \Phi_3(w) \geq 0\}$ and $\{w \in \overline{B}_1 \colon \Phi_3(w) \leq 0\}$ are simply connected and the two surfaces

$$S^+ := \Phi(\{w \in \overline{B}_1 \colon \Phi_3(w) \geq 0\}), \quad S^- := \Phi(\{w \in \overline{B}_1 \colon \Phi_3(w) \leq 0\})$$

have the topology of the disc. We assume without loss of generality that $\mathcal{H}^2(S^+) \leq \mathcal{H}^2(S^-)$. Let

$$\mathrm{Sym}(S^+) := \{(x', x_3) \colon (x', -x_3) \in S^+\}, \quad \widetilde{S} := S^+ \cup \mathrm{Sym}(S^+).$$

Then $\widetilde{S}$ is symmetric surface of disc-type with $\partial \widetilde{S} = \Gamma$ and

$$\mathcal{H}^2(\widetilde{S}) = 2\mathcal{H}^2(S^+) \leq \mathcal{H}^2(S^+) + \mathcal{H}^2(S^-) = \mathcal{H}^2(S).$$

In particular $\widetilde{S}$ is a symmetric solution to the Plateau problem for Γ. Further $S = \widetilde{S}$ on a relatively open subset of S; hence, since they are real analytic surfaces, they must coincide, $S = \widetilde{S}$.

Step 3: S^+ is the graph of a function $\widetilde{\psi} \in W^{1,1}(U_{p,q}) \cap C^0(\overline{U}_{p,q})$.
To show this it is enough to check the validity of the following

Claim: Every vertical plane Π is tangent to $\mathrm{int}(S)$ at most at one point.

In fact by step 2 this readily implies that $\mathrm{int}(S^+)$ has no points with vertical tangent plane and hence we can conclude. We prove the claim arguing by contradiction as in [6, page 97], that is we assume there is a vertical plane Π tangent to $\mathrm{int}(S)$ at x' and x'' with $x' \neq x''$. We define the linear map $d_\nu(x) := (x - x') \cdot \nu$ with ν a unit normal to Π, so that clearly $\Pi = \{x \in \mathbb{R}^3 \colon d_\nu(x) = 0\}$. Since F is convex $\Pi \cap (\partial F \times \{0\})$ contains at most two points. By properties (1)-(3) each of these points is either the projection on the horizontal plane of one or two points of $\Pi \cap \Gamma$, or the projection on the horizontal plane of one of the vertical segments $\{p\} \times [0, f(p)]$ and $\{q\} \times [0, f(q)]$. Hence $\Pi \cap \Gamma$ contains either:

- at most two points and a segment;
- two segments;
- four points.

Without loss of generality we restrict our analysis to the last case (the others are simpler to treat), namely we assume that there are four (clockwise ordered) points $w_1, \dots, w_4 \in \partial B_1$ such that $\Pi \cap \Gamma = \{\Phi(w_1), \dots, \Phi(w_4)\}$, that is $d_\nu \circ \Phi(w_i) = 0$ for $i = 1, \dots, 4$. We may also assume $d_\nu \circ \Phi > 0$ on $\widehat{w_1w_2} \cup \widehat{w_3w_4}$ and $d_\nu \circ \Phi < 0$ on $\widehat{w_2w_3} \cup \widehat{w_4w_1}$. Here $\widehat{w_iw_j}$ denotes the relatively open arc in ∂B_1 joining w_i and w_j for $i, j \in \{1, \dots, 4\}$.
Notice that the function $d_\nu \circ \Phi \colon \overline{B}_1 \to \mathbb{R}$ is harmonic in B_1, continuous up to ∂B_1 and vanishes at $w_1, \dots, w_4$; hence, by classical arguments [19, Section 437] we see that the set $\{w \in B_1 \colon d_\nu \circ \Phi = 0\}$, in a neighbourhood of $w' := \Phi^{-1}(x')$ (respectively $w'' := \Phi^{-1}(x'')$), is the union of a number $m \geq 2$ of analytic curves crossing at w' (respectively w''). Thus near w' and w'' the set $\{w \in B_1 \colon d_\nu \circ \Phi(w) > 0\}$ is the union of at least two disjoint open regions $A_{1,1}$, $A_{1,2}$ and $A_{2,1}$, $A_{2,2}$ respectively such that $\overline{A}_{1,1} \cap \overline{A}_{1,2} = \{w'\}$, $\overline{A}_{2,1} \cap \overline{A}_{2,2} = \{w''\}$. Moreover each $A_{i,j}$ belongs either to the connected component of $\{w \in B_1 \colon d_\nu \circ \Phi(w) > 0\}$ containing $\widehat{w_1w_2}$ or to the one containing $\widehat{w_3w_4}$. Up to relabelling the indices we have two possibilities.

Case 1: $A_{1,1}$ and $A_{1,2}$ belong to the same connected component containing $\widehat{w_1w_2}$. Then we can find two simple curves α_1, α_2 contained in $A_{1,1}$ and $A_{1,2}$ respectively, that connect w' to a point in $\widehat{w_1w_2}$ and such that the region enclosed by the curve $\alpha_1 \cup \alpha_2$ intersects $\{w \in B_1 \colon d_\nu \circ \Phi(w) < 0\}$. Since $d_\nu \circ \Phi > 0$ on $\alpha_1 \cup \alpha_2$ by the maximum principle we have a contradiction.

Case 2: $A_{1,1}$ and $A_{2,1}$ belong to the connected component containing $\widehat{w_1w_2}$ while $A_{1,2}$ and $A_{2,2}$ belong to the connected component containing $\widehat{w_3w_4}$. Then we can find four simple curves $\alpha_{i,j}$ (with $i, j = 1, 2$) contained respectively in $A_{i,j}$, such that $\alpha_{1,1}$ (respectively $\alpha_{2,1}$) connects w' (respectively w'') to a point in $\widehat{w_1w_2}$ and $\alpha_{1,2}$ (respectively $\alpha_{2,2}$) connects w' (respectively w'') to $\widehat{w_3w_4}$. Then the region enclosed by the curve $\cup_{i,j}\alpha_{i,j}$ intersects $\{w \in B_1 \colon d_\nu \circ \Phi(w) < 0\}$, while $d_\nu \circ \Phi > 0$ on $\cup_{i,j}\alpha_{i,j}$, which again by the maximum principle gives a contradiction.

Thus the claim follows.

Step 4: The curve $\beta_{p,q}$ is contained in the closed convex hull of Γ, and the set $F \setminus U_{p,q}$ is convex. Let $\pi(\Gamma) \subset \partial F$ be the projection of Γ onto the plane $\mathbb{R}^2 \times \{0\}$. By [10, Theorem 3, pag. 343] the relative interior of S is strictly contained in the convex hull of Γ, thus in particular the curve $\beta_{p,q}$ (respectively $\beta_{p,q} \setminus \{p, q\}$) is contained (respectively strictly contained) in the same half-plane (with respect to the line $\overline{pq}$) that contains $\pi(\Gamma)$.

Now, assume by contradiction that $F \setminus U_{p,q}$ is not convex. Then there are $p', q' \in \beta_{p,q}$ with the following properties:

- The open region U' enclosed by $\beta_{p,q}$ and the segment $\overline{p'q'}$ is non-empty and contained in $U_{p,q}$;
- the points p and q and the set U' lie on the same side with respect to the line containing $\overline{p'q'}$.

Let then $d_W \colon \mathbb{R}^3 \to \mathbb{R}$ be an affine function that vanishes on the vertical plane containing $\overline{p'q'}$ and is positive on the half-space W^+ containing p, q and U'. We now observe that $\Gamma \cap W^+$ is the union of two connected subcurves Γ_1 and Γ_2, containing p and q respectively. As a consequence $\Phi^{-1}(\Gamma_1) = \widehat{w_1w_2}$ and $\Phi^{-1}(\Gamma_2) = \widehat{w_3w_4}$ for some $w_1, w_2, w_3, w_4 \in \partial B_1$ (clockwise oriented).
On the other hand since $d_W > 0$ on U' we can find $t' \in \partial U' \setminus \overline{p'q'}$ such that $d_W \circ \Phi(\Phi^{-1}(t')) = d_W(t') > 0$ with $\Phi^{-1}(t') \in B_1$.
Once again by the harmonicity of $d_W \circ \Phi \colon \overline{B}_1 \to \mathbb{R}$ we deduce the existence of a curve $\alpha \subset \{w \in B_1 \colon d_W \circ \Phi(w) > 0\}$ joining $\Phi^{-1}(t')$ to one of $\widehat{w_1w_2}$ and $\widehat{w_3w_4}$. Hence $\Phi(\alpha) \subset \Phi(B_1) = \widetilde{\psi}(U_{p,q})$ is a curve joining t' to one of Γ_1 and Γ_2, say Γ_1. This implies that the projection $\pi(\Phi(\alpha))$ of $\Phi(\alpha)$ onto the horizontal plane $\mathbb{R}^2 \times \{0\}$ is a curve contained in $U_{p,q}$ that connects t' to $\pi(\Gamma_1)$. So in particular, the curve $\pi(\Phi(\alpha))$ cannot be included in the half-plane W^+. But this contradicts the fact that $\alpha \subset \{w \in B_1 \colon d_W \circ \Phi(w) > 0\}$ (this is because the values of d_W at a point x and $\pi(x)$ are the same). □

We need also the following technical results on the distance function d_F from a convex set F.

Lemma 5.6. *Let $F \subset \mathbb{R}^2$ be bounded, closed and convex. Then $\Delta \mathrm{d}_F \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus F) \cap L^1(B \setminus F)$ for every ball B with $F \subset\subset B$.*

Proof. By [8, Theorem 3.6.7 pag. 75] it follows that $\mathrm{d}_F \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2 \setminus F)$, hence $\nabla^2 \mathrm{d}_F \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus F; \mathbb{R}^{2\times 2})$. Therefore we only have to check that $\Delta \mathrm{d}_F \in L^1(B \setminus F)$.
Let $\eta > 0$ be fixed sufficiently small. Select $(f_k)_{k\in\mathbb{N}} \subset C^1_c(\mathbb{R}^2;\mathbb{R}^2)$ such that $f_k \to \nabla \mathrm{d}_F$ in $W^{1,1}(B \setminus F^+_{\eta/2})$ as $k \to +\infty$. By the divergence theorem we have

$$\int_{B\setminus F^+_\eta} \mathrm{div} f_k \, dx = \int_{\partial B \cup \partial(F^+_\eta)} f_k \cdot \nu_\eta \, d\mathcal{H}^1, \tag{5.3}$$

with ν_η the outer unit normal to $\partial B \cup \partial(F^+_\eta)$. By taking the limit as $k \to \infty$ we get

$$\lim_{k\to+\infty} \int_{B\setminus F^+_\eta} \mathrm{div} f_k \, dx = \int_{B\setminus F^+_\eta} \Delta \mathrm{d}_F \, dx\,, \tag{5.4}$$

and

$$\lim_{k\to+\infty} \int_{\partial B\cup\partial(F^+_\eta)} f_k \cdot \nu_\eta \, d\mathcal{H}^1 = \int_{\partial B\cup\partial(F^+_\eta)} \nabla \mathrm{d}_F \cdot \nu_\eta \, d\mathcal{H}^1\,, \tag{5.5}$$

where (5.5) follows by using that $\partial(F^+_\eta)$ is of class $C^{1,1}$ and hence $f_k \llcorner (\partial B \cup \partial(F^+_\eta)) \to \nabla \mathrm{d}_F \llcorner (\partial B \cup \partial(F^+_\eta))$ in $L^1(\partial B \cup \partial(F^+_\eta))$. Since d_F is convex we have $\Delta \mathrm{d}_F \geq 0$ a.e. in $\mathbb{R}^2 \setminus F$, moreover $|\nabla \mathrm{d}_F| = 1$ in $\mathbb{R}^2 \setminus F$; then gathering together (5.3), (5.4), (5.5) we have

$$\int_{B\setminus F^+_\eta} |\Delta \mathrm{d}_F| \, dx = \int_{B\setminus F^+_\eta} \Delta \mathrm{d}_F \, dx = \int_{\partial B\cup\partial(F^+_\eta)} \nabla \mathrm{d}_F \cdot \nu_\eta \, d\mathcal{H}^1 \leq \mathcal{H}^1(\partial B \cup \partial(F^+_\eta)) \leq C,$$

with $C > 0$ independent of η. By the arbitrariness of $\eta > 0$, the thesis follows. □

Corollary 5.7. *Let $U \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. Let $F \subset \mathbb{R}^2$ be closed and convex such that $U \cap F = \emptyset$ and let $\psi \in W^{1,1}(U) \cap L^\infty(U) \cap C^0(U)$. Then the following formula holds:*

$$-\int_U \psi \Delta \mathrm{d}_F \, dx = \int_U \nabla\psi \cdot \nabla \mathrm{d}_F \, dx - \int_{\partial U} \psi \, \gamma \, d\mathcal{H}^1,$$

where ν is the outer normal to ∂U and γ denotes the normal trace of $\nabla \mathrm{d}_F$ on ∂U.

Proof. We have $|\nabla \mathrm{d}_F| = 1$ in $\mathbb{R}^2 \setminus F$, moreover since $U \cap F = \emptyset$, by Lemma 5.6 we deduce also $\Delta \mathrm{d}_F \in L^1(U)$. Therefore the thesis readily follows by applying [3, Theorem 1.9]. □

Remark 5.8. The normal trace γ of $\nabla \mathrm{d}_F$ on ∂F equals 1 $\mathcal{H}^1$-a.e. on ∂F. Indeed, from Corollary 5.7 we have that for all $\varphi \in C^1_c(\mathbb{R}^2;\mathbb{R}^2)$ it holds

$$\begin{aligned} -\int_{\mathbb{R}^2\setminus F^+_\eta} \varphi \Delta \mathrm{d}_F \, dx &= \int_{\mathbb{R}^2\setminus F^+_\eta} \nabla\varphi \cdot \nabla \mathrm{d}_F \, dx - \int_{\partial(F^+_\eta)} \varphi \, \gamma \, d\mathcal{H}^1 \\ &= \int_{\mathbb{R}^2\setminus F^+_\eta} \nabla\varphi \cdot \nabla \mathrm{d}_F \, dx - \int_{\partial(F^+_\eta)} \varphi \, d\mathcal{H}^1, \end{aligned}$$

where we have used that $\partial(F^+_\eta)$ being a level set of d_F, it results $\nabla \mathrm{d}_F = \nu_\eta$ on it. Letting $\eta \to 0$ and using that $\Delta \mathrm{d}_F \in L^1(B \setminus F)$ for all balls B, we infer

$$-\int_{\mathbb{R}^2\setminus F} \varphi \Delta \mathrm{d}_F \, dx = \int_{\mathbb{R}^2\setminus F} \nabla\varphi \cdot \nabla \mathrm{d}_F \, dx - \int_{\partial F} \varphi \, d\mathcal{H}^1.$$

By the arbitrariness of φ and again by Corollary 5.7, the claim follows.

Lemma 5.9. *Let $F \subset \overline{\Omega}$ be closed and convex with non-empty interior, and let $\delta > 0$. Let $\psi \in W^{1,1}((F_\delta^+ \setminus F) \cap \Omega) \cap L^\infty((F_\delta^+ \setminus F) \cap \Omega) \cap C^0((F_\delta^+ \setminus F) \cap \Omega)$. Then*

$$\lim_{\varepsilon \to 0^+,\, \varepsilon < \delta} \int_{\Omega \cap \partial(F_\varepsilon^+)} \psi \, d\mathcal{H}^1 = \int_{\Omega \cap \partial F} \psi \, d\mathcal{H}^1 \,. \tag{5.6}$$

Proof. Let $\varepsilon \in (0, \delta)$ and $T_\varepsilon := (F_\varepsilon^+ \setminus F) \cap \Omega$. Since $T_\varepsilon \cap F = \emptyset$, by Corollary 5.7 we get

$$-\int_{T_\varepsilon} \psi \Delta \, \mathrm{d}_F \, dx = \int_{T_\varepsilon} \nabla \psi \cdot \nabla \, \mathrm{d}_F \, dx - \int_{\partial T_\varepsilon} \psi \, \gamma \, d\mathcal{H}^1 \,. \tag{5.7}$$

By Remark 5.8 we have

$$\begin{aligned} -\int_{T_\varepsilon} \psi \Delta \, \mathrm{d}_F \, dx = &\int_{T_\varepsilon} \nabla \psi \cdot \nabla \, \mathrm{d}_F \, dx \\ &+ \int_{\Omega \cap \partial F} \psi \, d\mathcal{H}^1 - \int_{\Omega \cap \partial(F_\varepsilon^+)} \psi \, d\mathcal{H}^1 - \int_{((F_\varepsilon^+) \setminus F) \cap \partial\Omega} \psi \, \gamma \, d\mathcal{H}^1 \,. \end{aligned} \tag{5.8}$$

Now

$$\lim_{\varepsilon \to 0^+} \Big| \int_{T_\varepsilon} \nabla \psi \cdot \nabla \, \mathrm{d}_F \, dx \Big| \le \lim_{\varepsilon \to 0^+} \int_{T_\varepsilon} |\nabla \psi| \, dx = 0 \,, \tag{5.9}$$

and

$$\lim_{\varepsilon \to 0^+} \Big| \int_{(F_\varepsilon^+ \setminus F) \cap \partial\Omega} \psi \, \gamma \, d\mathcal{H}^1 \Big| \le \lim_{\varepsilon \to 0^+} \int_{(F_\varepsilon^+ \setminus F) \cap \partial\Omega} \psi \, d\mathcal{H}^1 = 0 \,. \tag{5.10}$$

Moreover, since $\Delta \, \mathrm{d}_F \in L^1(T_\varepsilon)$ by Lemma 5.6, we deduce also

$$\lim_{\varepsilon \to 0^+} \Big| \int_{T_\varepsilon} -\psi \, \Delta \, \mathrm{d}_F \, dx \Big| \le \|\psi\|_{L^\infty} \lim_{\varepsilon \to 0^+} \int_{T_\varepsilon} |\Delta \, \mathrm{d}_F| \, dx = 0 \,. \tag{5.11}$$

Finally gathering together (5.8)-(5.11) we infer (5.6). □

Remark 5.10. Let F, δ and ψ be as in Lemma 5.9. Let α be any connected component of $\Omega \cap \partial F$, and for every $0 < \varepsilon < \delta$ let α_ε be the corresponding component of $\Omega \cap \partial(F_\varepsilon^+)$; namely, if π_F is the orthogonal projection onto the convex closed set F, setting

$$\widehat{\alpha}_\varepsilon := \{x \in \partial(F_\varepsilon^+) : \pi_F(x) \in \alpha\},$$

then one has $\alpha_\varepsilon := \widehat{\alpha}_\varepsilon \cap \Omega$. Arguing as in Lemma 5.9, we can show that

$$\lim_{\varepsilon \to 0^+} \int_{\alpha_\varepsilon} \psi \, d\mathcal{H}^1 = \int_\alpha \psi \, d\mathcal{H}^1 \,.$$

Lemma 5.11. *Let $(\sigma, \psi) \in \mathcal{W}_{\mathrm{conv}}$ be a minimizer as in Theorem 4.1. Then there is a minimizer $(\widehat{\sigma}, \widehat{\psi}) \in \mathcal{W}_{\mathrm{conv}}$ with the following properties:*

1. *$\partial E(\widehat{\sigma}) \cap \partial\Omega = \partial E(\sigma) \cap \partial\Omega$;*
2. *$\widehat{\psi}$ is continuous and null on $\Omega \cap \partial E(\widehat{\sigma})$.*

The second condition means essentially that $\widehat{\psi}$ vanishes on $\Omega \cap \partial E(\widehat{\sigma})$ when considering its trace from the side of $\Omega \setminus E(\widehat{\sigma})$.

Proof. We know by Lemma 5.3 that (σ, ψ) satisfies the following properties:

- Each connected component of $E(\sigma)$ is convex;
- ψ is positive and real analytic in $\Omega \setminus E(\sigma)$;
- $\psi = \varphi$ on $\partial^D\Omega \setminus \partial E(\sigma)$.

In what follows we are going to modify (σ, ψ) near each arc of $\partial E(\sigma)$ using an iterative argument in order to get a new minimizer $(\widehat{\sigma}, \widehat{\psi}) \in \mathcal{W}_{\rm conv}$ that satisfies 1-2. To this aim we denote by $F_1, \dots, F_k$ with $1 \le k \le n$ the closed connected components of $E(\sigma)$; we also set $\delta_0 := \min_{i \neq j} \mathrm{dist}(F_i, F_j) > 0$. Moreover by the first property we deduce that $\Omega \cap \partial E(\sigma)$ is the union of an at most countable family of pairwise disjoint arcs with endpoints in $\partial\Omega$, i.e.,

$$\Omega \cap \partial E(\sigma) = \bigcup_{i=1}^{k} \bigcup_{j=1}^{\infty} \alpha_{i,j}\,,$$

where $\alpha_{i,j}$ is a connected component of $\Omega \cap \partial F_i$ for $i \in \{1, \dots, k\}$, $j \ge 1$[6].

Step 1: Base case. Let α be one of the connected components of $\Omega \cap \partial F$, with $F := F_i$ for some $i \in \{1, \dots, k\}$. In this step we construct a new minimizer $(\sigma^\alpha, \psi^\alpha) \in \mathcal{W}_{\rm conv}$ such that $\partial E(\sigma^\alpha) \cap \partial\Omega = \partial E(\sigma) \cap \partial\Omega$ and ψ^α is continuous and null on α', where $\alpha' \subset \Omega \cap \partial E(\sigma^\alpha)$ is a suitable curve that replaces α and has the same endpoints as α.

For $\varepsilon \in (0, \delta_0/2)$ we define the stripe

$$\widehat{T}_\varepsilon(\alpha) := \{x \in \Omega \setminus F \colon \mathrm{dist}(x, \alpha) < \varepsilon\} \subset F_\varepsilon^+ \setminus F\,,$$

and consider the planar curve α_ε in $\overline{\Omega}$ defined as in Remark 5.10. Let $T_\varepsilon(\alpha)$ be the connected component of $\widehat{T}_\varepsilon(\alpha)$ whose boundary contains α_ε. Let L_ε be defined as

$$L_\varepsilon := \partial T_\varepsilon(\alpha) \cap \partial\Omega,$$

so that in particular $\partial T_\varepsilon(\alpha) = \alpha \cup \alpha_\varepsilon \cup L_\varepsilon$. Let $p, q \in \partial\Omega$ be the endpoints of α (and then also the endpoints of $\alpha_\varepsilon \cup L_\varepsilon$, which are independent of ε). We define the curves

$$\Gamma_\varepsilon := \Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-\,, \quad \Gamma_\varepsilon^+ := \mathcal{G}_{\psi \llcorner \alpha_\varepsilon} \cup \mathcal{G}_{\varphi \llcorner L_\varepsilon} \cup l^+\,, \quad \Gamma_\varepsilon^- := \mathcal{G}_{-\psi \llcorner \alpha_\varepsilon} \cup \mathcal{G}_{-\varphi \llcorner L_\varepsilon} \cup l^-\,,$$

where

$$l^+ := (\{p\} \times [0, \varphi(p)]) \cup (\{q\} \times [0, \varphi(q)])\,, \quad l^- := (\{p\} \times [-\varphi(p), 0]) \cup (\{q\} \times [-\varphi(q), 0])\,.$$

By observing that $L_\varepsilon \subset \partial^D\Omega \setminus \partial E(\sigma)$ and recalling that $\psi = \varphi$ on $\partial^D\Omega \setminus \partial E(\sigma)$ we deduce that Γ_ε is a closed non-planar curve in $\mathbb{R}^3$ that satisfies assumptions (1)-(3) of Lemma 5.4. In particular a solution S_ε to the classical Plateau problem corresponding to Γ_ε is a disc-type surface such that:

1. $\beta_{p,q}^\varepsilon := S_\varepsilon \cap (\mathbb{R}^2 \times \{0\})$ is a simple curve of class C^∞ joining p and q;
2. S_ε is symmetric with respect to the horizontal plane;
3. the surface $S_\varepsilon^+ := S_\varepsilon \cap \{x_3 \ge 0\}$ is the graph of a function $\psi_{p,q}^\varepsilon \in W^{1,1}(U_{p,q}^\varepsilon) \cap C^0(\overline{U}_{p,q}^\varepsilon)$, where $U_{p,q}^\varepsilon \subset F \cup T_\varepsilon(\alpha)$ is the open region enclosed between $\alpha_\varepsilon \cup L_\varepsilon$ and $\beta_{p,q}^\varepsilon$;
4. the curve $\beta_{p,q}^\varepsilon$ is contained in the closed convex hull of Γ_ε and $(F \cup T_\varepsilon(\alpha)) \setminus U_{p,q}^\varepsilon$ is convex.

We would like to compare the area of S_ε^+ with the area of the generalized graph of ψ on $\overline{T_\varepsilon(\alpha)}$. This is not immediate since, due to the fact that ψ is just BV, we cannot, a priori, conclude that this generalized graph is of disc-type[7]. Hence we proceed as follows. We fix $\bar\varepsilon \in (0, \delta_0/2)$; we claim

[6]Notice that at this stage we do not have any information about the geometry of the set $\partial\Omega \cap \partial E(\sigma)$, and $\Omega \cap \partial F_i$ could a priori be the union of infinitely many connected components.

[7]This is due to the jump of ψ on ∂F which is, in general, not regular enough.

that

$$\mathcal{A}(\psi^{\bar\varepsilon}_{p,q};U^{\bar\varepsilon}_{p,q})\le\mathcal{A}(\psi;T_{\bar\varepsilon}(\alpha))+\int_\alpha\psi\llcorner T_{\bar\varepsilon}(\alpha)\,d\mathcal{H}^1\,. \tag{5.12}$$

Since ψ is analytic in $T_{\bar\varepsilon}(\alpha)\subset\Omega\setminus E(\sigma)$, by Lemma 5.9 and Remark 5.10 it follows that

$$\lim_{\varepsilon\to0^+,\,\varepsilon<\bar\varepsilon}\int_{\alpha_\varepsilon}\psi\llcorner T_{\bar\varepsilon}(\alpha)\,d\mathcal{H}^1=\int_\alpha\psi\llcorner T_{\bar\varepsilon}(\alpha)\,d\mathcal{H}^1\,. \tag{5.13}$$

We take

$$T^{\bar\varepsilon}_\varepsilon(\alpha):=T_{\bar\varepsilon}(\alpha)\setminus\overline{T_\varepsilon(\alpha)}\quad\text{and}\quad Y_{\bar\varepsilon}:=S_\varepsilon\cup\mathcal{G}_{\psi\llcorner T^{\bar\varepsilon}_\varepsilon(\alpha)}\cup\mathcal{G}_{-\psi\llcorner T^{\bar\varepsilon}_\varepsilon(\alpha)}\,.$$

Since S_ε is a disc-type surface and ψ is analytic in $T^{\bar\varepsilon}_\varepsilon(\alpha)$ it turns out that $Y_{\bar\varepsilon}$ is also a disc-type surface satisfying $\partial Y_{\bar\varepsilon}=\Gamma_{\bar\varepsilon}$. Therefore using that $S_{\bar\varepsilon}$ and S_ε are solutions to the Plateau problems corresponding to $\Gamma_{\bar\varepsilon}$ and Γ_ε respectively, we have

$$\begin{aligned}\mathcal{H}^2(S_{\bar\varepsilon})\le\mathcal{H}^2(Y_{\bar\varepsilon})&=2\mathcal{H}^2(\mathcal{G}_{\psi\llcorner T^{\bar\varepsilon}_\varepsilon(\alpha)})+\mathcal{H}^2(S_\varepsilon)\\&\le2\mathcal{H}^2(\mathcal{G}_{\psi\llcorner T_{\bar\varepsilon}(\alpha)})+2\int_{\alpha_\varepsilon\cup L_\varepsilon}\psi\llcorner T_{\bar\varepsilon}(\alpha)\,d\mathcal{H}^1\\&=2\mathcal{H}^2(\mathcal{G}_{\psi\llcorner T_{\bar\varepsilon}(\alpha)})+2\int_{\alpha_\varepsilon}\psi\llcorner T_{\bar\varepsilon}(\alpha)\,d\mathcal{H}^1+2\int_{L_\varepsilon}\psi\llcorner T_{\bar\varepsilon}(\alpha)\,d\mathcal{H}^1\,.\end{aligned}$$

Passing to the limit as $\varepsilon\to0^+$, by (5.13) and the fact that $\mathcal{H}^1(L_\varepsilon)\to0$, we obtain

$$\mathcal{H}^2(S_{\bar\varepsilon})\le2\mathcal{H}^2(\mathcal{G}_{\psi\llcorner T_{\bar\varepsilon}(\alpha)})+2\int_\alpha\psi\llcorner T_{\bar\varepsilon}(\alpha)\,d\mathcal{H}^1,$$

which yields

$$\mathcal{A}(\psi^{\bar\varepsilon}_{p,q};U^{\bar\varepsilon}_{p,q})=\mathcal{H}^2(S^+_{\bar\varepsilon})\le\mathcal{H}^2(\mathcal{G}_{\psi\llcorner T_{\bar\varepsilon}(\alpha)})+\int_\alpha\psi\llcorner T_{\bar\varepsilon}(\alpha)\,d\mathcal{H}^1=\mathcal{A}(\psi;T_{\bar\varepsilon}(\alpha))+\int_\alpha\psi\llcorner T_{\bar\varepsilon}(\alpha)\,d\mathcal{H}^1,$$

and (5.12) is proved.

We now define $E^\alpha:=(E(\sigma)\cup T_{\bar\varepsilon}(\alpha))\setminus U^{\bar\varepsilon}_{p,q}$ and

$$\psi^\alpha:=\begin{cases}0&\text{in }E^\alpha\\ \psi^{\bar\varepsilon}_{p,q}&\text{in }U^{\bar\varepsilon}_{p,q}\\ \psi&\text{otherwise}\,.\end{cases}$$

By (5.12) and using that $U^{\bar\varepsilon}_{p,q}\cup E^\alpha=E(\sigma)\cup T_{\bar\varepsilon}(\alpha)$ we derive

$$\begin{aligned}\mathcal{A}(\psi^\alpha;\Omega)-|E^\alpha|&=\mathcal{A}(\psi^{\bar\varepsilon}_{p,q};U^{\bar\varepsilon}_{p,q})+\mathcal{A}(\psi;\Omega\setminus(U^{\bar\varepsilon}_{p,q}\cup E^\alpha))\\&=\mathcal{A}(\psi^{\bar\varepsilon}_{p,q};U^{\bar\varepsilon}_{p,q})+\mathcal{A}(\psi;\Omega\setminus(T_{\bar\varepsilon}(\alpha)\cup E(\sigma)))\\&\le\mathcal{A}(\psi;T_{\bar\varepsilon}(\alpha))+\int_\alpha\psi\llcorner T_{\bar\varepsilon}(\alpha)\,d\mathcal{H}^1+\mathcal{A}(\psi;\Omega\setminus T_{\bar\varepsilon}(\alpha))-|E(\sigma)|\\&=\mathcal{A}(\psi;\Omega)-|E(\sigma)|\,.\end{aligned}\tag{5.14}$$

It remains to construct $\sigma^\alpha\in\Sigma_{\rm conv}$. Without loss of generality we may assume

$$\sigma_1([0,1]),\dots,\sigma_h([0,1])\subset F\quad\text{and}\quad\sigma_{h+1}([0,1]),\dots,\sigma_n([0,1])\not\subset F$$

for some $h \leq n$, notice that if $h = n$ the second family of curves is empty. Then we define $\sigma^\alpha := (\sigma_1^\alpha, \dots, \sigma_h^\alpha, \sigma_{h+1}, \dots, \sigma_n) \in \mathrm{Lip}([0,1]; \overline{\Omega})^n$ where if $h > 1$

$$\sigma_i^\alpha([0,1]) = \begin{cases} \overline{q_i p_{i+1}} & \text{for} \quad i = 1, \dots, h-1 \\ \partial(F \cup T_{\bar\varepsilon}(\alpha) \setminus U_{p,q}^{\bar\varepsilon}) \setminus \left((\cup_{i=1}^h \partial_i^0 \Omega) \cup (\cup_{i=1}^{h-1} \overline{q_i p_{i+1}}) \right) & \text{for} \quad i = h, \end{cases}$$

where $\overline{q_i p_{i+1}}$ is the segment joining q_i to p_{i+1}; if instead $h = 1$ we simply set

$$\sigma_1^\alpha([0,1]) = \partial(F \cup T_{\bar\varepsilon}(\alpha) \setminus U_{p,q}^{\bar\varepsilon}) \setminus \partial_1^0 \Omega.$$

Clearly the pair $(\sigma^\alpha, \psi^\alpha)$ belongs to $\mathcal{W}_{\mathrm{conv}}$, and by (5.14) it satisfies

$$\mathcal{F}(\sigma^\alpha, \psi^\alpha) = \mathcal{F}(\sigma, \psi)\,.$$

Moreover $\partial E(\sigma^\alpha) \cap \partial\Omega = \partial E(\sigma) \cap \partial\Omega$ and ψ^α is continuous and null on α', where

$$\alpha' := \beta_{p,q}^{\bar\varepsilon} \subset \Omega \cap \partial E(\sigma^\alpha)\,. \tag{5.15}$$

Summarizing, we have replaced the curve α with α', ensuring that the new function ψ^α is now continuous and null on α'.

Step 2: Iterative case. In this step we construct a minimizer $(\widehat\sigma, \widehat\psi) \in \mathcal{W}_{\mathrm{conv}}$ that satisfies the thesis by iterating step one at most a countable number of times.
We first consider $F = F_1$ and apply step 1 for each $\alpha_{1,j}$ with $j \geq 1$. More precisely we define the pair $(\sigma_{1,j}, \psi_{1,j}) \in \mathcal{W}_{\mathrm{conv}}$ as follows:

- if $j = 1$ we set
$$(\sigma_{1,1}, \psi_{1,1}) := (\sigma^{\alpha_{1,1}}, \psi^{\alpha_{1,1}})\,,$$
where $(\sigma^{\alpha_{1,1}}, \psi^{\alpha_{1,1}}) \in \mathcal{W}_{\mathrm{conv}}$ is a minimizer constructed as in step 1 with $\alpha = \alpha_{1,1}$;
- if $j > 1$ we set
$$(\sigma_{1,j}, \psi_{1,j}) := (\sigma_{1,j-1}^{\alpha_{1,j}}, \psi_{1,j-1}^{\alpha_{1,j}})\,,$$
where $(\sigma_{1,j-1}^{\alpha_{1,j}}, \psi_{1,j-1}^{\alpha_{1,j}}) \in \mathcal{W}_{\mathrm{conv}}$ is a minimizer constructed as in step 1 with $(\sigma, \psi) = (\sigma_{1,j-1}, \psi_{1,j-1})$ and $\alpha = \alpha_{1,j}$.

Since $\mathcal{F}(\sigma_{1,j}, \psi_{1,j}) = \mathcal{F}(\sigma, \psi)$ for all $j \geq 1$, by Lemma 3.4 it follows that $(\sigma_{1,j}, \psi_{1,j})$ converges to $(\sigma_1, \psi_1) \in \mathcal{W}_{\mathrm{conv}}$ in the sense of Definition 3.2. Moreover by construction we have that for every $j \geq 1$ the pair $(\sigma_{1,j}, \psi_{1,j})$ satisfies

$$\partial E(\sigma_{1,j}) \cap \partial\Omega = \partial E(\sigma) \cap \partial\Omega\,,$$

and $\psi_{1,j}$ is continuous and null on $\cup_{h=1}^j \alpha'_{1,h} \subset \Omega \cap \partial E(\sigma_{1,j}) \cap \partial F_1$, where $\alpha'_{1,h}$ are defined as in (5.15). As a consequence (σ_1, ψ_1) satisfies

$$\partial E(\sigma_1) \cap \partial\Omega = \partial E(\sigma) \cap \partial\Omega\,,$$

and ψ_1 is continuous and null on $\cup_{j=1}^\infty \alpha'_{1,j} \subset \Omega \cap \partial E(\sigma_1) \cap \partial F_1$. Moreover

$$\Omega \cap \partial E(\sigma_1) = (\cup_{j=1}^\infty \alpha'_{1,j}) \cup (\cup_{i=2}^k \cup_{j=1}^\infty \alpha_{i,j})\,,$$

Now repeating the argument above for the pair (σ_1, ψ_1) and $i = 2$ we obtain a new minimizer $(\sigma_2, \psi_2) \in \mathcal{W}_{\mathrm{conv}}$ satisfying

$$\partial E(\sigma_2) \cap \partial\Omega = \partial E(\sigma) \cap \partial\Omega\,,$$

ψ_2 is continuous and null on $\cup_{j=1}^{\infty}(\alpha'_{1,j}\cup\alpha'_{2,j})\subset\Omega\cap\partial E(\sigma_1)\cap(\partial F_1\cup\partial F_2)$ and

$$\Omega\cap\partial E(\sigma_2)=(\cup_{i=1}^{2}\cup_{j=1}^{\infty}\alpha'_{i,j})\cup(\cup_{i=3}^{k}\cup_{j=1}^{\infty}\alpha_{i,j})\,.$$

Iterating this process a finite number of times we finally get a minimizer $(\widehat{\sigma},\widehat{\psi})\in\mathcal{W}_{\rm conv}$ with the required properties. □

We are finally in the position to conclude the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ be any minimizer as in Theorem 4.1. By Lemma 5.3 we know that (σ,ψ) satisfies properties 1, 2 and

$$\psi=\varphi\quad\text{on}\quad\partial^D\Omega\setminus\partial E(\sigma)\,.$$

Moreover by Lemma 5.11 there is a minimizer $(\widehat{\sigma},\widehat{\psi})\in\mathcal{W}_{\rm conv}$ such that

$$\partial E(\widehat{\sigma})\cap\partial\Omega=\partial E(\sigma)\cap\partial\Omega\,,\tag{5.16}$$

and $\widehat{\psi}$ is continuous and null on $\Omega\cap\partial E(\widehat{\sigma})$.

It remains to show that if $\partial_i^D\Omega$ is not straight for some $i=1,\dots,n$, then

$$\partial E(\sigma)\cap\partial_i^D\Omega=\partial E(\widehat{\sigma})\cap\partial_i^D\Omega=\emptyset\,.$$

If instead $\partial_i^D\Omega$ is straight for some $i=1,\dots,n$ we prove that property 4 holds. Eventually we show that there is a minimizer that satisfies property 5. This will be achieved in a number of steps.

Step 1: Assuming that there is $i\in\{1,\dots,n\}$ such that $\partial_i^D\Omega$ is not straight, we show that $\partial_i^D\Omega\cap E(\widehat{\sigma})=\emptyset$. To prove this we proceed by analysing three different cases.

Case A: Suppose, to the contrary, that there is a non-straight[8] arc $\widehat{ab}$ (with endpoints $a\neq b$) in $\partial_i^D\Omega\cap\partial E(\widehat{\sigma})$. Thus in particular $\widehat{ab}\subset\cup_{j=1}^{n}\widehat{\sigma}_j([0,1])$. We may assume without loss of generality that $\widehat{ab}\subset\widehat{\sigma}_1([0,1])$.
Then we consider the curves

$$\Gamma:=\Gamma^+\cup\Gamma^-\,,\quad\Gamma^+:=\mathcal{G}_{\varphi\llcorner\widehat{ab}}\cup l^+\,,\quad\Gamma^-:=\mathcal{G}_{-\varphi\llcorner\widehat{ab}}\cup l^-\,,\tag{5.17}$$

where

$$l^+:=(\{a\}\times[0,\varphi(a)])\cup(\{b\}\times[0,\varphi(b)])\,,\quad l^-:=(\{a\}\times[-\varphi(a),0])\cup(\{b\}\times[-\varphi(b),0])\,.$$

In this way Γ satisfies the assumptions of Lemma 5.4 and hence a solution S to the Plateau problem spanning Γ is a disc-type surface such that:

i. $\beta_{a,b}:=S\cap(\mathbb{R}^2\times\{0\})$ is a simple curve of class C^∞ joining a and b;

ii. S is symmetric with respect to $\mathbb{R}^2\times\{0\}$;

iii. the surface $S^+:=S\cap\{x_3\geq0\}$ is the graph of a function $\psi_{a,b}\in W^{1,1}(U_{a,b})\cap C^0(\overline{U}_{a,b})$, where $U_{a,b}\subset E(\widehat{\sigma}_1)$ is the open region enclosed between $\widehat{ab}$ and $\beta_{a,b}$;

iv. the curve $\beta_{a,b}$ is contained in the closed convex hull of Γ and $E(\widehat{\sigma}_1)\setminus U_{a,b}$ is convex.

[8]Namely, $\widehat{ab}$ is not contained in a line.

The inclusion $U_{a,b} \subset E(\widehat{\sigma}_1)$ follows since $\widehat{ab} \subset \widehat{\sigma}_1([0,1])$, $E(\widehat{\sigma}_1)$ is convex, and S is contained in the convex envelope of Γ. Furthermore by the minimality of S one has

$$\mathcal{A}(\psi_{a,b}; U_{a,b}) = \mathcal{H}^2(S^+) < \int_{\widehat{ab}} \varphi \, d\mathcal{H}^1 = \int_{\widehat{ab}} |\widehat{\psi} - \varphi| \, d\mathcal{H}^1 \,. \tag{5.18}$$

Here the strict inequality follows since the vertical wall spanning Γ given by $\{(x', x_3) \colon x' \in \widehat{ab},\ x_3 \in [-\varphi(x'), \varphi(x')]\}$ is a disc-type surface but, since $\widehat{ab}$ is not a segment, cannot be a solution to the Plateau problem. We now consider the pair $(\widetilde{\sigma}, \widetilde{\psi}) \in \mathcal{W}_{\rm conv}$ given by

$$\widetilde{\sigma} := (\widetilde{\sigma}_1, \widehat{\sigma}_2, \dots, \widehat{\sigma}_n) \,, \qquad \widetilde{\psi} := \begin{cases} 0 & \text{in } \widetilde{E} \,, \\ \psi_{a,b} & \text{in } U_{a,b} \,, \\ \widehat{\psi} & \text{otherwise} \,, \end{cases} \tag{5.19}$$

where $\widetilde{\sigma}_1$ is such that $\widetilde{\sigma}_1([0,1]) = (\widehat{\sigma}_1([0,1]) \setminus \widehat{ab}) \cup \beta_{a,b}$ and $\widetilde{E} := E(\widehat{\sigma}) \setminus U_{a,b} = E(\widetilde{\sigma})$. Then noticing that $\widehat{\psi} = 0$ in $U_{a,b}$, $E(\widehat{\sigma}) = E(\widetilde{\sigma}) \cup U_{a,b}$, and recalling (5.18), we get

$$\begin{aligned}
\mathcal{F}(\widetilde{\sigma}, \widetilde{\psi}) &= \mathcal{A}(\widetilde{\psi}; \Omega) - |E(\widetilde{\sigma})| + \int_{\partial\Omega} |\widetilde{\psi} - \varphi| \, d\mathcal{H}^1 \\
&= \mathcal{A}(\widehat{\psi}; \Omega \setminus U_{a,b}) + \mathcal{A}(\psi_{a,b}; U_{a,b}) - |E(\widetilde{\sigma})| + \int_{\partial\Omega} |\widetilde{\psi} - \varphi| \, d\mathcal{H}^1 \\
&= \mathcal{A}(\widehat{\psi}; \Omega) + \mathcal{A}(\psi_{a,b}; U_{a,b}) - |E(\widehat{\sigma})| + \int_{\partial\Omega} |\widehat{\psi} - \varphi| \, d\mathcal{H}^1 \\
&< \mathcal{A}(\widehat{\psi}; \Omega) - |E(\widehat{\sigma})| + \int_{\partial\Omega} |\widetilde{\psi} - \varphi| \, d\mathcal{H}^1 + \int_{\widehat{ab}} |\widehat{\psi} - \varphi| \, d\mathcal{H}^1 \\
&= \mathcal{A}(\widehat{\psi}; \Omega) - |E(\widehat{\sigma})| + \int_{\partial\Omega} |\widehat{\psi} - \varphi| \, d\mathcal{H}^1 = \mathcal{F}(\widehat{\sigma}, \widehat{\psi}) \,,
\end{aligned}$$

where the penultimate equality follows from the fact that $\widetilde{\psi}$ is continuous and equal to φ on $\widehat{ab}$ while the traces of $\widetilde{\psi}$ and $\widehat{\psi}$ coincide on $\partial\Omega \setminus \widehat{ab}$. This contradicts the minimality of $(\widehat{\sigma}, \widehat{\psi})$.

Case B: Suppose by contradiction that the set $\partial_i^D\Omega \cap \partial E(\widehat{\sigma})$ contains an isolated point c or has a straight segment $\overline{cc'}$ as isolated connected component. Then there are two arcs $\widehat{ab} \subset \partial_i^D\Omega$ and $\widehat{a'b'} \subset \partial E(\widehat{\sigma})$ with either $a \neq a'$ or $b \neq b'$ (and with endpoints $a \neq b$ and $a' \neq b'$) such that $\overline{aa'} \cap \overline{bb'} = \emptyset$ and $\widehat{ab} \cap \widehat{a'b'} = \{c\}$ (respectively $\widehat{ab} \cap \widehat{a'b'} = \overline{cc'}$). Notice also that, since $\partial_i^D\Omega$ is not straight, the segment $\overline{cc'}$ does not coincide with $\partial_i^D\Omega$ and hence the arc $\widehat{ab}$ can be chosen so that it properly contains the segment $\overline{cc'}$. We consider the curves

$$\Gamma := \Gamma^+ \cup \Gamma^- \,, \quad \Gamma^+ := \mathcal{G}_{\varphi \llcorner \widehat{ab}} \cup \mathcal{G}_{\widehat{\psi} \llcorner \overline{aa'}} \cup \mathcal{G}_{\widehat{\psi} \llcorner \overline{bb'}} \,, \quad \Gamma^- := \mathcal{G}_{-\varphi \llcorner \widehat{ab}} \cup \mathcal{G}_{-\widehat{\psi} \llcorner \overline{aa'}} \cup \mathcal{G}_{-\widehat{\psi} \llcorner \overline{bb'}} \,. \tag{5.20}$$

Notice that $\Gamma^\pm$ connect a' to b'. By applying again Lemma 5.4 to the nonplanar curve Γ and arguing as in case A we obtain the contradiction also in this case.

Case C: More generally, assume by contradiction that both the sets $\partial_i^D\Omega \cap \partial E(\widehat{\sigma})$ and $\partial_i^D\Omega \setminus \partial E(\widehat{\sigma})$ are nonempty. Then we can find a not flat arc $\widehat{ab} \subset \partial_i^D\Omega$ such that the following holds[9]: there are

[9] This is a consequence of the fact that $\widehat{ab} \setminus \partial E(\widehat{\sigma})$ is relatively open in $\widehat{ab}$, so it is an at most countable union of disjoint relatively open arcs.

pairs of points $\{c_j, d_j\}_{j\in\mathbb{N}} \subset \partial_i^D\Omega \cap \partial E(\widehat{\sigma})$ such that the arcs $\widehat{ad_0}$, $\widehat{c_0b}$, and $\{\widehat{c_jd_j}\}_{j=1}^{\infty}$ are mutually disjoint and

$$\widehat{ab} \setminus \partial E(\widehat{\sigma}) = \widehat{ad_0} \cup (\cup_{j=1}^{\infty}\widehat{c_jd_j}) \cup \widehat{c_0b}\,.$$

Without loss of generality, we might assume that all the points $c_j, d_j \in \widehat{\sigma}_1([0,1])$. For all $j \geq 1$ we denote by V_j the region enclosed by $\widehat{c_jd_j}$ and $\partial E(\widehat{\sigma})$[10]. We now argue as in case B and choose $a', b' \in \widehat{\sigma}_1([0,1])$. Additionally, let $V_0 = V_0^a \cup V_0^b$, with V_0^a (respectively V_0^b) be the region enclosed between $\partial E(\widehat{\sigma})$ and $\overline{aa'}\cup\widehat{ad_0}$ ($\partial E(\widehat{\sigma})$ and $\overline{bb'}\cup\widehat{c_0b}$, respectively). We finally define Γ correspondingly, as in (5.20). Again by Lemma 5.4 the solution S to the Plateau problem corresponding to Γ satisfies properties i.-iv. with a' and b' in place of a and b respectively. Moreover by the minimality of S for every $N \geq 1$ there holds[11]

$$\mathcal{A}(\psi_{a',b'}; U_{a',b'}) = \mathcal{H}^2(S^+) \leq \int_{\widehat{ab}} \varphi\, d\mathcal{H}^1 - \int_{\widehat{ad_0}\cup\widehat{c_0b}} \varphi\, d\mathcal{H}^1 - \sum_{j=1}^{N}\int_{\widehat{c_jd_j}} \varphi\, d\mathcal{H}^1 + \sum_{j=0}^{N} \mathcal{A}(\psi, V_j)\,. \quad (5.21)$$

In particular by taking the limit as $N \to \infty$ in (5.21) we get

$$\mathcal{A}(\psi_{a',b'}; U_{a',b'}) = \mathcal{H}^2(S^+) \leq \int_{\widehat{ab}\setminus\partial E(\widehat{\sigma})} \varphi\, d\mathcal{H}^1 + \mathcal{A}(\widehat{\psi}, \cup_{j=0}^{\infty} V_j)\,. \quad (5.22)$$

Let $(\widetilde{\sigma}, \widetilde{\psi}) \in \mathcal{W}_{\rm conv}$ be defined as in (5.19), then observing that $\widehat{\psi} = 0$ in $U_{a',b'} \setminus (\cup_{j=0}^{\infty} V_j)$, $E(\widehat{\sigma}) = E(\widetilde{\sigma}) \cup (U_{a',b'} \setminus \cup_{j=0}^{\infty} V_j)$ and using (5.22) we deduce

$$\begin{aligned}
\mathcal{F}(\widetilde{\sigma}, \widetilde{\psi}) &= \mathcal{A}(\widehat{\psi}; \Omega \setminus U_{a',b'}) + \mathcal{A}(\psi_{a',b'}; U_{a',b'}) - |E(\widetilde{\sigma})| + \int_{\partial\Omega} |\widetilde{\psi} - \varphi|\, d\mathcal{H}^1 \\
&= \mathcal{A}(\widehat{\psi}; \Omega \setminus (\cup_{j=0}^{\infty} V_j)) + \mathcal{A}(\psi_{a',b'}; U_{a',b'}) - |E(\widehat{\sigma})| + \int_{\partial\Omega} |\widetilde{\psi} - \varphi|\, d\mathcal{H}^1 \\
&\leq \mathcal{A}(\widehat{\psi}; \Omega \setminus (\cup_{j=0}^{\infty} V_j)) - |E(\widehat{\sigma})| + \int_{\partial\Omega} |\widetilde{\psi} - \varphi|\, d\mathcal{H}^1 + \int_{\widehat{ab}\cap\partial E(\widehat{\sigma})} \varphi\, d\mathcal{H}^1 + \mathcal{A}(\widehat{\psi}; \cup_{j=0}^{\infty} V_j) \\
&= \mathcal{A}(\widehat{\psi}; \Omega) - |E(\widehat{\sigma})| + \int_{\partial\Omega} |\widehat{\psi} - \varphi|\, d\mathcal{H}^1 = \mathcal{F}(\widehat{\sigma}, \widehat{\psi})\,,
\end{aligned}$$

which in turn implies

$$\mathcal{F}(\widetilde{\sigma}, \widetilde{\psi}) \leq \mathcal{F}(\widehat{\sigma}, \widehat{\psi})\,. \quad (5.23)$$

To conclude we need to show that the inequality in (5.23) is strict. To this aim we choose $c \in \{c_j\}_{j=1}^{\infty}$. Consider the curves Γ_1 and Γ_2 defined as follows

$$\begin{aligned}
\Gamma_1 &:= \Gamma_1^+ \cup \Gamma_1^-\,, \quad \Gamma_1^+ := \mathcal{G}_{\varphi\llcorner\widehat{ac}} \cup \mathcal{G}_{\widehat{\psi}\llcorner\overline{aa'}} \cup l^+\,, \quad \Gamma_1^- := \mathcal{G}_{-\varphi\llcorner\widehat{ac}} \cup \mathcal{G}_{-\widehat{\psi}\llcorner\overline{aa'}} \cup l^-\,, \\
\Gamma_2 &:= \Gamma_2^+ \cup \Gamma_2^-\,, \quad \Gamma_2^+ := \mathcal{G}_{\varphi\llcorner\widehat{cb}} \cup \mathcal{G}_{\widehat{\psi}\llcorner\overline{bb'}} \cup l^+\,, \quad \Gamma_2^- := \mathcal{G}_{-\varphi\llcorner\widehat{cb}} \cup \mathcal{G}_{-\widehat{\psi}\llcorner\overline{bb'}} \cup l^-\,,
\end{aligned}$$

where

$$l^+ := (\{c\} \times [0, \varphi(c)])\,, \quad l^- := (\{c\} \times [-\varphi(c), 0])\,.$$

Let S_1 and S_2 be the solutions to the Plateau problem corresponding to Γ_1 and Γ_2 respectively, so that properties i.-iv. are satisfied with c in place of b' and a' respectively. By the minimality of S we have

$$\mathcal{A}(\psi_{a',b'}, U_{a',b'}) < \mathcal{A}(\psi_{a',c}, U_{a',c}) + \mathcal{A}(\psi_{c,b'}, U_{c,b'})\,. \quad (5.24)$$

[10]These regions are simply connected since $c_j, d_j \in \widehat{\sigma}_1([0,1])$.

[11]The right-hand side is the area of the surface given by the (positive) subgraph of φ on $\widehat{ab} \setminus \cup_{j=1}^{N}\widehat{c_jd_j}$ and the graph of $\widehat{\psi}$ on the region $\cup_{j=0}^{N} V_j$, which is of disc-type. To see this we use that the trace of $\widehat{\psi}$ on the subarcs of $\partial E(\widehat{\sigma})$ between the points c_j and d_j is zero (and between a' and d_0, and d_0 and b').

On the other hand by arguing as above[12] we conclude

$$\mathcal{A}(\psi_{a',c}, U_{a',c}) \leq \int_{\widehat{ac}\cup\partial E(\widehat{\sigma})} \varphi \, d\mathcal{H}^1 + \mathcal{A}(\widehat{\psi}, \cup_{j\in I_1} V_j \cup V_0^a)\,, \tag{5.25}$$

and

$$\mathcal{A}(\psi_{c,b'}, U_{c,b'}) \leq \int_{\widehat{cb}\cup\partial E(\widehat{\sigma})} \varphi \, d\mathcal{H}^1 + \mathcal{A}(\widehat{\psi}, \cup_{j\in I_2} V_i \cup V_0^b)\,, \tag{5.26}$$

where $I_1 := \{j\colon \widehat{c_j d_j} \subset \widehat{ac}\}$ and $I_2 := \{j\colon \widehat{c_j d_j} \subset \widehat{cb}\}$. Gathering together (5.24)-(5.26) we derive

$$\mathcal{A}(\psi_{a',b'}, U_{a',b'}) < \int_{\widehat{ab}\cup\partial E(\widehat{\sigma})} \varphi \, d\mathcal{H}^1 + \mathcal{A}(\widehat{\psi}, \cup_{j=0}^{\infty} V_j)\,,$$

which in turn implies

$$\mathcal{F}(\widetilde{\sigma}, \widetilde{\psi}) < \mathcal{F}(\widehat{\sigma}, \widehat{\psi})\,,$$

and thus the contradiction.

Step 2: Assuming there is $i \in \{1, \dots, n\}$ such that $\partial_i^D\Omega$ is a straight segment, and we show that either $\partial E(\widehat{\sigma}) \cap \partial_i^D\Omega = \emptyset$ or $\partial E(\widehat{\sigma}) \cap \partial_i^D\Omega = \partial_i^D\Omega$.
Suppose by contradiction that $\partial E(\widehat{\sigma}) \cap \partial_i^D\Omega \neq \emptyset$ and also $\partial_i^D\Omega \setminus \partial E(\widehat{\sigma}) \neq \emptyset$. Without loss of generality we can restrict to the case $\partial E(\widehat{\sigma}) \cap \partial_i^D\Omega = \partial F \cap \partial_i^D\Omega$ with F any connected component of $E(\widehat{\sigma})$. Since F is convex and $\partial_i^D\Omega$ is a segment $\partial F \cap \partial_i^D\Omega$ has to be connected, i.e., it is either a single point a or a segment $\overline{aa'} \neq \partial_i^D\Omega$.
In both cases we then consider a (small enough) ball B centred at a such that $B \cap E(\widehat{\sigma}) = B \cap F$ (in the second case we also require that the radius of B is smaller than $\overline{aa'}$).

If $\partial F \cap \partial_i^D\Omega = \{a\}$ we let $\{p, q\} := \partial B \cap \partial F$ and $\{b, c\} := \partial B \cap \partial_i^D\Omega$ (with b, p and c, q lying on the same side with respect to a). Then we define the curves

$$\Gamma := \Gamma^+ \cup \Gamma^-\,, \quad \Gamma^+ := \mathcal{G}_{\varphi\llcorner\overline{bc}} \cup \mathcal{G}_{\psi\llcorner\widehat{bp}} \cup \mathcal{G}_{\psi\llcorner\widehat{cq}}\,, \quad \Gamma^- := \mathcal{G}_{-\varphi\llcorner\overline{bc}} \cup \mathcal{G}_{-\psi\llcorner\widehat{bp}} \cup \mathcal{G}_{-\psi\llcorner\widehat{cq}}\,,$$

where $\widehat{bp}$, $\widehat{cq}$ denote the arcs in ∂B joining b to p and c to q respectively.

If $\partial F \cap \partial_i^D\Omega = \overline{aa'}$ we let $\{p, q\} := \partial B \cap \partial F$ and $\{b, c\} := \partial B \cap \partial_i^D\Omega$ where we identify q and c. Then we consider the curves

$$\Gamma := \Gamma^+ \cup \Gamma^-\,, \quad \Gamma^+ := \mathcal{G}_{\varphi\llcorner\overline{bc}} \cup \mathcal{G}_{\psi\llcorner\widehat{bp}} \cup (\{c\}\times[0, \varphi(c)])\,, \quad \Gamma^- := \mathcal{G}_{-\varphi\llcorner\overline{bc}} \cup \mathcal{G}_{-\psi\llcorner\widehat{bp}} \cup (\{c\}\times[-\varphi(c), 0])\,.$$

By applying again Lemma 5.4 to Γ and arguing as above we get the contradiction.

Step 3: We show that there is a minimizer $(\widetilde{\sigma}, \widetilde{\psi})$ that satisfies property 5.
We first notice that $\widehat{\psi}$ is continuous and null on $\partial E(\widehat{\sigma}) \setminus \partial^D\Omega$. Moreover by steps 1 and 2 it follows that $\partial E(\widehat{\sigma}) \cap \Omega$ is the union of a finite number of pairwise disjoint Lipschitz curves each of them joining each p_i for $i = 1, \dots, n$ to each of the q_j for some $j = 1, \dots, n$. To conclude it is enough to replace each curve, without increasing the energy, with a smooth one having the same endpoints. More precisely, let γ be any of such curves. Reasoning as in the proof of Lemma 5.11 step 1, we can replace $(\widehat{\sigma}, \widehat{\psi})$ with a new minimizer $(\sigma^\gamma, \psi^\gamma) \in \mathcal{W}_{\rm conv}$ such that $\partial E(\sigma^\gamma) \cap \partial\Omega = \partial E(\sigma) \cap \partial\Omega$ and $\psi^\gamma = 0$ on γ', where $\gamma' \subset \partial E(\sigma^\gamma) \cap \Omega$ is a suitable smooth curve that replaces γ and has the same endpoints of γ. In particular ψ^γ is continuous and null on $\partial E(\sigma^\gamma) \setminus \partial^D\Omega$. Eventually iterating this procedure for each curve in $\partial E(\widehat{\sigma}) \setminus \partial\Omega$ we can construct a new minimizer $(\widetilde{\sigma}, \widetilde{\psi})$ with the required properties. □

[12] With the arc $\widehat{ac}$ ($\widehat{cb}$, respectively) in place of $\widehat{ab}$.

5.1 The example of the catenoid containing a segment

Consider the setting depicted in Figure 4. Here $\Omega = R_{2l} = (0, 2l) \times (-1, 1)$, $n = 1$, $\partial^D\Omega = (\{0, 2l\} \times (-1, 1)) \cup ((0, 2l) \times \{-1\})$ and $\partial^0\Omega = (0, 2l) \times \{1\}$, $p = (0, 1)$, $q = (2l, 1)$. The map φ is $\varphi(w_1, w_2) = \sqrt{1 - w_2^2}$, and thus vanishes on $[0, 2l] \times \{-1\}$; for this reason this case is not covered by our analysis. However we can find a solution as in Theorem 1.1 also in this case, by an approximation procedure.

Precisely, for $\varepsilon > 0$ and consider an approximating sequence $(\varphi)_\varepsilon$ of continuous Dirichlet data, with $\mathcal{G}_{\varphi_\varepsilon}$ Lipschitz, which tends to φ uniformly and satisfy $\varphi_\varepsilon = 0$ on $\partial^0\Omega$, $\varphi_\varepsilon > 0$ on $\partial^D\Omega$. Let $(\sigma_\varepsilon, \psi_\varepsilon)$ be a solution as in Theorem 1.1 corresponding to the boundary datum φ_ε; as $\mathcal{F}(\sigma_\varepsilon, \psi_\varepsilon)$ is equibounded[13], arguing as in the proof of Lemma 3.4, we can see that, up to a subsequence, $((\sigma_\varepsilon, \psi_\varepsilon))$ tends to some $(\sigma, \psi) \in \mathcal{W}_{\text{conv}}$, which minimizes the functional $\mathcal{F}$ with Dirichlet condition φ. In this case however we cannot guarantee that σ does not touch $\partial^D\Omega$, even if this is not a straight segment. This is essentially due to the presence of the portion $[0, 2l] \times \{-1\}$ of $\partial\Omega$ where φ is zero, which does not allow to apply the arguments used in the proof of Theorem 5.1.

In particular, it can be seen that if l is large enough, the solution (σ, ψ) splits and becomes degenerate, being $\psi \equiv 0$ and the functional $\mathcal{F}$ pays only the area of two vertical half discs of radius 1. Under a certain threshold instead the solution satisfies the regularity properties stated in Theorem 5.1, and in particular $\psi = \varphi$ on $\partial^D\Omega$, and σ is the graph of a smooth convex function passing through p and q. We refer to [6] for details and comprehensive proofs of these facts.

6 Comparison with the parametric Plateau problem: The case $n = 1, 2$

In this section we compare the solutions provided by Theorems 3.1 and 5.1 with the solutions to the classical Plateau problem in parametric form. Specifically, motivated by the example of the catenoid, we will restrict our analysis to the classical disc-type and annulus-type Plateau problem. These configurations correspond to the cases $n = 1$ and $n = 2$ respectively, i.e., the Dirichlet boundary $\partial^D\Omega$ is either an open arc or the union of two open arcs of $\partial\Omega$ with disjoint closure. Due to the highly involved geometric arguments, we do not discuss the case $n > 2$, which requires further investigation.

Thus, in this section we assume $n = 1, 2$. We first discuss the case $n = 1$ which is a consequence of Lemma 5.4, and then the case $n = 2$.

6.1 The case $n = 1$

Let $n = 1$. Let $p_1, q_1 \in \partial\Omega$, $\partial^D\Omega = \partial_1^D\Omega$, φ be as in Section 2.3 and consider the space curve $\gamma_1 := \mathcal{G}_{\varphi \llcorner \partial_1^D\Omega}$ joining p_1 to q_1. We define the curve

$$\Gamma := \gamma_1 \cup \text{Sym}(\gamma_1),$$

where $\text{Sym}(\gamma_1) := \mathcal{G}_{-\varphi \llcorner \partial_1^D\Omega}$, and consider the classical Plateau problem in parametric form spanning Γ. More precisely we look for a solution to

$$m_1(\Gamma) := \inf_{\Phi \in \mathcal{P}_1(\Gamma)} \int_{B_1} |\partial_{w_1}\Phi \wedge \partial_{w_2}\Phi| dw, \tag{6.1}$$

[13] We can indeed always bound it from above by $|\Omega| + \int_{\partial_D\Omega} |\varphi_\varepsilon| d\mathcal{H}^1$.

where

$$\mathcal{P}_1(\Gamma) := \Big\{ \Phi \in H^1(B_1;\mathbb{R}^3) \cap C^0(\overline{B}_1;\mathbb{R}^3) \text{ such that } \Phi \llcorner \partial B_1 : \partial B_1 \to \Gamma \\ \text{is a weakly monotonic parametrization of } \Gamma \Big\}. \tag{6.2}$$

Then the following holds:

Theorem 6.1 (**The disc-type Plateau problem** ($n = 1$)). *Let $\Phi \in \mathcal{P}_1(\Gamma)$ be a solution to* (6.1) *and let*

$$S^+ := \Phi(\overline{B}_1) \cap \{x_3 \geq 0\} \qquad \text{and} \qquad S^- := \Phi(\overline{B}_1) \cap \{x_3 \leq 0\}.$$

Then there exists a minimizer $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1-5 of Theorem 5.1 and such that

$$S^{\pm} = \mathcal{G}_{\pm\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}. \tag{6.3}$$

Conversely let $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1-5 of Theorem 5.1. Then the disc-type surface

$$S := \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$$

is a solution to the classical Plateau problem associated to Γ, i.e., there is $\Phi \in \mathcal{P}_1(\Gamma)$ solution to (6.1) *such that $\Phi(\overline{B_1}) = S$.*

6.2 The case $n = 2$

Let $n = 2$. Let Ω, $p_1, q_1, p_2, q_2 \in \partial\Omega$, $\partial^D\Omega$, $\partial_1^D\Omega$, $\partial_2^D\Omega$, φ be as in Section 2.3 and consider the space curve $\gamma_i := \mathcal{G}_{\varphi \llcorner \partial_i^D\Omega}$ joining p_i to q_i for $i = 1, 2$. We define the curves

$$\Gamma_1 := \gamma_1 \cup \mathrm{Sym}(\gamma_1), \qquad \Gamma_2 := \gamma_2 \cup \mathrm{Sym}(\gamma_2),$$

where $\mathrm{Sym}(\gamma_i) := \mathcal{G}_{-\varphi \llcorner \partial_i^D\Omega}$ for $i = 1, 2$. We consider the classical Plateau problem in parametric form spanning the curve

$$\Gamma := \Gamma_1 \cup \Gamma_2.$$

Precisely we set $\Sigma_{\rm ann} \subset \mathbb{R}^2$ to be an open annulus enclosed between two concentric circles C_1 and C_2, and we look for a solution to

$$m_2(\Gamma) := \inf_{\Phi \in \mathcal{P}_2(\Gamma)} \int_{\Sigma_{\rm ann}} |\partial_{w_1}\Phi \wedge \partial_{w_2}\Phi| dw, \tag{6.4}$$

where

$$\mathcal{P}_2(\Gamma) := \Big\{ \Phi \in H^1(\Sigma_{\rm ann};\mathbb{R}^3) \cap C^0(\overline{\Sigma}_{\rm ann};\mathbb{R}^3) \text{ such that } \Phi(\partial\Sigma_{\rm ann}) = \Gamma \text{ and } \Phi \llcorner C_j : C_j \to \Gamma_j \\ \text{is a weakly monotonic parametrization of } \Gamma_j \text{ for } j = 1, 2 \Big\}.$$

Here the crucial assumption that we require is that the curves Γ_j have the orientation inherited by the orientation[14] of the graph of φ on $\partial_j^D\Omega$.
Due to the specific geometry of Γ we can appeal to Theorem 6.4 below (which is a consequence of [18, Theorem 1 and Theorem 5]) to deduce the existence of a minimizer. This might not be true

[14] Once we fix an orientation of $\partial\Omega$, the orientation of the graph $\mathcal{G}_\varphi$ of φ is inherited, since $\mathcal{G}_\varphi$ is standardly defined as the push-forward of the current of integration on $\partial_D\Omega$ by the map $x \mapsto (x, \varphi(x))$.

for a more general Γ. To this purpose for $j = 1, 2$ we consider the minimization problem defined in (6.1) for the curve Γ_j, namely

$$m_1(\Gamma_j) = \inf_{\Phi \in \mathcal{P}_1(\Gamma_j)} \int_{B_1} |\partial_{w_1}\Phi \wedge \partial_{w_2}\Phi| dw, \tag{6.5}$$

with $\mathcal{P}_1(\Gamma_j)$ defined as in (6.2).

Remark 6.2. By standard arguments one sees that $m_2(\Gamma) \leq m_1(\Gamma_1) + m_1(\Gamma_2)$. Indeed, two disc-type surfaces can be joined by a very thin tube (with arbitrarily small area) in order to change the topology of the two discs into an annulus-type surface.

Definition 6.3. *Let $\Phi \in \mathcal{P}_2(\Gamma)$ be a solution to (6.4). We say that Φ is a $\mathcal{MY}$ solution to (6.4) if Φ is harmonic, conformal, and it is an embedding. In particular, in such a case, $m_2(\Gamma) = \mathcal{H}^2(\Phi(\overline{\Sigma}_{\rm ann}))$.*

Theorem 6.4 (Meeks and Yau). *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$. Then there exists a $\mathcal{MY}$ solution $\Phi \in \mathcal{P}_2(\Gamma)$ to (6.4). Furthermore, every minimizer of (6.4) is a $\mathcal{MY}$ solution.*

Proof. See [18]. □

This result allows to prove the following:

Theorem 6.5 (The annulus-type Plateau problem ($n = 2$)). *The following hold:*

(i) Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$. Let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4) and let

$$S := \Phi(\overline{\Sigma}_{\rm ann}), \qquad S^+ := S \cap \{x_3 \geq 0\}, \qquad S^- := S \cap \{x_3 \leq 0\}.$$

Then there exists a minimizer $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1-5 of Theorem 5.1 and such that

$$S^{\pm} = \mathcal{G}_{\pm\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}. \tag{6.6}$$

(ii) Suppose $m_2(\Gamma) = m_1(\Gamma_1) + m_1(\Gamma_2)$. For $j = 1, 2$ let $\Phi_j \in \mathcal{P}_1(\Gamma_j)$ be a solution to (6.5) and let $S_j := \Phi_j(\overline{B}_1)$. Let also

$$S^+ := (S_1 \cup S_2) \cap \{x_3 \geq 0\} \qquad \text{and} \qquad S^- := (S_1 \cup S_2) \cap \{x_3 \leq 0\}.$$

Then $S_1 \cap S_2 = \emptyset$ and there exists a minimizer $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1-5 of Theorem 5.1 and such that (6.6) holds.

(iii) Conversely, let $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1-5 of Theorem 5.1. Then the surface

$$S := \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$$

is either an annulus-type surface or the union of two disjoint disc-type surfaces, and is a solution to the classical Plateau problem associated to Γ. More precisely, either there is a $\mathcal{MY}$ solution $\Phi \in \mathcal{P}_2(\Gamma)$ to (6.4) with $S = \Phi(\overline{\Sigma}_{\rm ann})$, or there are $\Phi_j \in \mathcal{P}_1(\Gamma_j)$ solutions to (6.5) for $j = 1, 2$, such that $S = \Phi_1(\overline{B}_1) \cup \Phi_2(\overline{B}_1)$ and $\Phi_1(\overline{B}_1) \cap \Phi_2(\overline{B}_1) = \emptyset$.

6.3 Toward the proofs of Theorems 6.1 and 6.5: preliminary lemmas

In order to prove Theorems 6.1 and 6.5, we collect some technical lemmas.

Lemma 6.6. *Let $n = 2$, and $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1-5 of Theorem 5.1.*

(a) Suppose that $\overline{\Omega \setminus E(\sigma)}$ is simply connected. Then there exists an injective map $\Phi \in W^{1,1}(\Sigma_{\rm ann}; \mathbb{R}^3) \cap C^0(\overline{\Sigma_{\rm ann}}; \mathbb{R}^3)$ such that

$$\Phi(\overline{\Sigma}_{\rm ann}) = \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})},$$

and $\Phi \llcorner C_j \colon C_j \to \Gamma_j$ is a weakly monotonic parametrization of Γ_j for $j = 1, 2$.

(b) Suppose that $\Omega \setminus E(\sigma)$ consists of two connected components, whose closures F_1 and F_2 are disjoint, with $F_j \supseteq \partial^D_j \Omega$ for $j = 1, 2$. Then there exist two injective maps $\Phi_1, \Phi_2 \in W^{1,1}(B_1; \mathbb{R}^3) \cap C^0(\overline{B_1}; \mathbb{R}^3)$ such that

$$\Phi_j(\overline{B_1}) = \mathcal{G}_{\psi \llcorner F_j} \cup \mathcal{G}_{-\psi \llcorner F_j}, \qquad j = 1, 2,$$

and $\Phi_j \llcorner \partial B_1 \colon \partial B_1 \to \Gamma_j$ is a weakly monotonic parametrization of Γ_j for $j = 1, 2$.

Proof. (a). Since $\overline{\Omega \setminus E(\sigma)}$ is simply connected, the maps

$$\widetilde{\Psi}^{\pm} \in W^{1,1}(\Omega \setminus E(\sigma); \mathbb{R}^3) \cap C^0(\overline{\Omega \setminus E(\sigma)}; \mathbb{R}^3), \qquad \widetilde{\Psi}^{\pm}(p) := (p, \pm\psi(p)), \tag{6.7}$$

are disc-type parametrizations of $\mathcal{G}_{\pm\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$.

Now, using a homeomorphism of class H^1 between $\overline{\Omega \setminus E(\sigma)}$ and a disc, we can parametrize[15] $\overline{\Omega \setminus E(\sigma)}$ with a half-annulus, obtained as the region enclosed between two concentric half-circles with endpoints A_1, A_2, A_3, A_4 (in the order) on the same diameter, and the two segments $\overline{A_1 A_2}$ and $\overline{A_3 A_4}$. Then we construct a parametrization Ψ^+ of $\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ from the half-annulus, such that $\Psi^+(A_1) = (q_1, 0)$, $\Psi^+(A_2) = (p_2, 0)$, $\Psi^+(A_3) = (q_2, 0)$, $\Psi^+(A_4) = (p_1, 0)$, and sending weakly monotonically the two half-circles into γ_1 and γ_2, and the two segments into σ_1 and σ_2, respectively. Similarly, we construct a parametrization Ψ^- of $\mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ from another copy of a half-annulus, just by setting $\Psi^- := \mathrm{Sym}(\Psi^+)$ (the symmetric of Ψ^+ with respect to the plane containing Ω). Eventually, glueing the two half-annuli along the two segments, we obtain a parametrization Φ of $\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ defined on $\overline{\Sigma}_{\rm ann}$. By the continuity of ψ on $\partial^D \Omega$ we have that Φ parametrizes Γ_i on C_i, $i = 1, 2$.

(b). It is sufficient to argue as in case (a), by replacing $\Omega \setminus E(\sigma)$ in turn with F_1 and F_2 and $\Sigma_{\rm ann}$ with B_1 to find Φ_1 and Φ_2, respectively. □

Lemma 6.7. *Let $n = 2$, and $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1-5 of Theorem 5.1.*

(a) Suppose that $\overline{\Omega \setminus E(\sigma)}$ is simply connected and

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}) \le m_2(\Gamma). \tag{6.8}$$

Let Φ be the parametrization given by Lemma 6.6 (a). Then there exists a reparametrization of the annulus $\Sigma_{\rm ann}$ such that, using it to reparametrize Φ, the corresponding map (still denoted by Φ) belongs to $\mathcal{P}_2(\Gamma)$ and solves (6.4).

[15] For instance, we can consider a (flat) disc-type Plateau solution spanning $\partial(\Omega \setminus E(\sigma))$. Then we can employ a Lipschitz homeomorphism between the disc and the half-annulus.

(b) Suppose that $\Omega \setminus E(\sigma)$ consists of two connected components whose F_1 and F_2 are disjoint, and $F_j \supset \partial_j^D \Omega$ for $j = 1, 2$, and

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner F_j} \cup \mathcal{G}_{-\psi \llcorner F_j}) \leq m_1(\Gamma_j), \qquad j = 1, 2.$$

Let Φ_1, Φ_2 be the maps given by Lemma 6.6 (b). Then, for $j = 1, 2$, there is a reparametrization of Φ_j belonging to $\mathcal{P}_1(\Gamma_j)$ and solving (6.5).

Proof. (a). Fix a point $\widetilde{p} \in \Omega \setminus E(\sigma)$ and set $\widetilde{\Psi}_k^+ := \widetilde{\Psi}^+ \llcorner H_k$, where $\widetilde{\Psi}$ is defined in (6.7) and H_k is the connected component of

$$\widetilde{H}_k := \{p \in \overline{\Omega \setminus E(\sigma)} : \operatorname{dist}(p, \partial(\overline{\Omega \setminus E(\sigma)})) \geq 1/k\}$$

containing $\widetilde{p}$.

For $k \in \mathbb{N}$ large enough H_k is simply connected with rectifiable boundary. In particular $\widetilde{\Psi}_k^+$ parametrizes a disc-type surface, and using the regularity of ψ in $\Omega \setminus E(\sigma)$, it follows that $\widetilde{\Psi}_k^+$ is Lipschitz continuous. Furthermore, $\widetilde{\Psi}_k^+ \llcorner \partial H_k$ parametrizes a Jordan curve, and these curves converge, in the sense of Fréchet (see [10, Theorem 4, Section 4.3]) as $k \to +\infty$, to the curve having image $\widetilde{\Psi}^+(\partial(\Omega \setminus E(\sigma)))) =: \lambda$. Notice that

$$\lambda = \sigma_1 \cup \sigma_2 \cup \gamma_1 \cup \gamma_2. \tag{6.9}$$

Call λ_k the image of the curve given by $\widetilde{\Psi}_k^+ \llcorner \partial H_k$. Let $\mathcal{P}_1(\lambda_k)$, $\mathcal{P}_1(\lambda)$, $m_1(\lambda_k)$, $m_1(\lambda)$ be defined as in (6.2) and (6.1) with λ_k and λ in place of Γ respectively. Up to reparametrizing $\overline{B}_1$ (see footnote 15), $\widetilde{\Psi}_k^+$ belongs to $\mathcal{P}_1(\lambda_k)$, therefore

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner H_k}) = \int_{H_k} |\partial_{w_1} \widetilde{\Psi}_k^+ \wedge \partial_{w_2} \widetilde{\Psi}_k^+| dw \geq m_1(\lambda_k) \qquad \forall k \geq 1.$$

We claim that equality holds in the previous expression, namely

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner H_k}) = m_1(\lambda_k) \qquad \forall k \geq 1. \tag{6.10}$$

Indeed, assume by contradiction that $\mathcal{H}^2(\mathcal{G}_{\psi \llcorner H_{k_0}}) > m_1(\lambda_{k_0})$ for some $k_0 \geq 1$, and pick $\delta > 0$ with

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner H_{k_0}}) \geq \delta + m_1(\lambda_{k_0}). \tag{6.11}$$

Take $\Phi_{k_0} \in \mathcal{P}_1(\lambda_{k_0})$ a solution to $m_1(\lambda_{k_0})$. For $k > k_0$, as $H_{k_0} \subset H_k$, by a glueing argument[16], we can find $\Phi_k \in \mathcal{P}_1(\lambda_k)$ such that $\Phi_k(\overline{B_1}) = \Phi_{k_0}(\overline{B_1}) \cup \mathcal{G}_{\psi \llcorner (H_k \setminus H_{k_0})}$. Thus by (6.11) we have

$$\begin{aligned} \mathcal{H}^2(\mathcal{G}_{\psi \llcorner H_k}) \geq & \delta + m_1(\lambda_{k_0}) + \mathcal{H}^2(\mathcal{G}_{\psi \llcorner (H_k \setminus H_{k_0})}) \\ = & \delta + \mathcal{H}^2(\Phi_{k_0}(\overline{B}_1)) + \mathcal{H}^2(\mathcal{G}_{\psi \llcorner (H_k \setminus H_{k_0})}) \geq \delta + m_1(\lambda_k) \qquad \forall k > k_0. \end{aligned}$$

Letting $k \to +\infty$, since $\lambda_k \to \lambda$ in the sense of Fréchet, we have $m_1(\lambda_k) \to m_1(\lambda)$ [10, Theorem 4, Section 4.3]. In particular, from the previous inequality we infer

$$\mathcal{F}(\sigma, \psi) = \mathcal{H}^2(\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}) \geq \delta + m_1(\lambda).$$

[16] This is done, for instance, by glueing an external annulus to a disc, and using Φ_{k_0} from the disc, and a reparametrization of $\mathcal{G}_{\psi \llcorner (H_k \setminus H_{k_0})}$ from the annulus.

Hence we conclude

$$\mathcal{H}^2(\mathcal{G}_{\psi\llcorner(\overline{\Omega\setminus E(\sigma)})}\cup\mathcal{G}_{-\psi\llcorner(\overline{\Omega\setminus E(\sigma)})})\geq 2\delta+2m_1(\lambda)\geq 2\delta+m_2(\Gamma),$$

which contradicts (6.8). In the last inequality we have used that $2m_1(\lambda)\geq m_2(\Gamma)$; this follows from the fact that a disc-type parametrization of a minimizer for $m_1(\lambda)$ can be reparametrized on a half-annulus (as in the proof of Lemma 6.6), and glued with another reparametrization of it on the other half-annulus, so to obtain a parametrization of an annulus-type surface spanning Γ which is admissible for (6.4). Hence claim (6.10) follows.

Now, since ψ is Lipschitz continuous on $\overline{H}_k$, for all $k\in\mathbb{N}$ there exists a parametrization $\Psi_k\in H^1(B_1;\mathbb{R}^3)\cap C^0(\overline{B_1};\mathbb{R}^3)$ with $\Psi_k(\partial B_1)=\lambda_k$ monotonically which solves the classical disc-type Plateau problem spanning λ_k and such that

$$\Psi_k(B_1)=\mathcal{G}_{\psi\llcorner H_k}.$$

Letting $k\to+\infty$ and using that the Dirichlet energy of Ψ_k equals the area of $\mathcal{G}_{\psi\llcorner H_k}$, we conclude that (Ψ_k) tends to a map $\Psi\in H^1(B_1;\mathbb{R}^3)\cap C^0(\overline{B_1};\mathbb{R}^3)$ with $\Psi(\partial B_1)=\lambda$ weakly monotonically, and that is a solution of the classical disc-type Plateau problem with

$$\Psi(\overline{B}_1)=\mathcal{G}_{\psi\llcorner(\overline{\Omega\setminus E(\sigma)})}.$$

Arguing as in the proof of Lemma 6.6 we finally get a parametrization $\Phi:\overline{\Sigma}_{\rm ann}\to\mathbb{R}^3$ which belongs to $\mathcal{P}_2(\Gamma)$ and parametrizes $\mathcal{G}_{\psi\llcorner(\overline{\Omega\setminus E(\sigma)})}\cup\mathcal{G}_{-\psi\llcorner(\overline{\Omega\setminus E(\sigma)})}$. This concludes the proof of (a).

(b). It is sufficient to argue as in case (a), by replacing $\Omega\setminus E(\sigma)$ in turn with F_1 and F_2 and $\Sigma_{\rm ann}$ with B_1 to find Φ_1 and Φ_2, respectively. □

We can now start the proof of Theorems 6.1 and 6.5.

6.4 Proof of Theorem 6.1

Proof of Theorem 6.1. Let $\Phi\in\mathcal{P}_1(\Gamma)$ be a solution to (6.1). The curve Γ satisfies the assumptions of Lemma 5.4, hence the minimal disc-type surface $S:=\Phi(\overline{B_1})$ satisfies the following properties:

- $\beta_{p_1,q_1}:=S\cap(\mathbb{R}^2\times\{0\})\subset\overline{\Omega}$ is a simple curve of class C^∞ joining p_1 and q_1 and such that $\beta_{p_1,q_1}\cap\partial\Omega=\{p_1,q_1\}$;
- S is symmetric with respect to $\mathbb{R}^2\times\{0\}$;
- the surface $S^+=S\cap\{x_3\geq 0\}$ is the graph of a function $\widetilde{\psi}\in W^{1,1}(U_{p_1,q_1})\cap C^0(\overline{U}_{p_1,q_1})$, where $U_{p_1,q_1}\subset\Omega$ is the open region enclosed between $\partial_1^D\Omega$ and β_{p_1,q_1}. Moreover $\widetilde{\psi}$ is analytic in U_{p_1,q_1};
- the curve β_{p_1,q_1} is contained in the closed convex hull of Γ, and $\Omega\setminus U_{p_1,q_1}$ is convex.

Let $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ be given by

$$\sigma:=\sigma_1\qquad\text{and}\qquad\psi:=\begin{cases}0 & \text{in }\Omega\setminus U_{p_1,q_1}\\ \widetilde{\psi} & \text{in }U_{p_1,q_1},\end{cases}$$

where $\sigma_1([0,1])=\beta_{p_1,q_1}$. Clearly (6.3) holds. Moreover $\mathcal{H}^2(S)=2\mathcal{F}(\sigma,\psi)=m_1(\Gamma)$. It remains to show that this is a minimizer of $\mathcal{F}$. Let $(\sigma',\psi')\in\mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ that satisfies

properties 1-5 of Theorem 5.1 and consider the disc-type surface with boundary Γ given by $S' := \mathcal{G}_{\psi' \llcorner (\overline{\Omega \setminus E(\sigma')})} \cup \mathcal{G}_{-\psi' \llcorner (\overline{\Omega \setminus E(\sigma')})}$. Since (σ, ψ) is admissible for $\mathcal{F}$, we deduce

$$\mathcal{H}^2(S') = 2\mathcal{F}(\sigma', \psi') \leq m_1(\Gamma).$$

Then we are in the hypotheses of Lemma 6.7 and so there is a parametrization $\Phi' \in \mathcal{P}_1(\Gamma)$ with $\Phi'(\overline{B}_1) = S'$. By minimality of (σ', ψ') and of S we have

$$\mathcal{H}^2(S) \leq \mathcal{H}^2(S') = 2\mathcal{F}(\sigma', \psi') \leq 2\mathcal{F}(\sigma, \psi) = \mathcal{H}^2(S). \tag{6.12}$$

Hence (σ, ψ) is a minimizer of $\mathcal{F}$ in $\mathcal{W}$ and Φ' is a solution to (6.1).

Conversely, let $(\sigma, \psi) \in \mathcal{W}_{\text{conv}}$ be a solution that satisfies properties 1-5 of Theorem 5.1. Let $\widetilde{\Phi}$ be a solution to (6.1); then we can find $(\widetilde{\sigma}, \widetilde{\psi}) \in \mathcal{W}$ whose doubled graph $\widetilde{S} = \mathcal{G}_{\widetilde{\psi} \llcorner (\overline{\Omega \setminus E(\widetilde{\sigma})})} \cup \mathcal{G}_{-\widetilde{\psi} \llcorner (\overline{\Omega \setminus E(\widetilde{\sigma})})}$ satisfies

$$\mathcal{H}^2(S) = 2\mathcal{F}(\sigma, \psi) \leq 2\mathcal{F}(\widetilde{\sigma}, \widetilde{\psi}) = \mathcal{H}^2(\widetilde{S}) = m_1(\Gamma).$$

Arguing as before we find a map $\Phi \in \mathcal{P}_1(\Gamma)$ parametrizing S. We conclude that Φ is a solution to (6.1), and the theorem is proved. □

6.5 Proof of Theorem 6.5

The proof of Theorem 6.5 is much more involved, so we divide it in a number of steps. We start with a result (which can be seen as the counterpart of Lemma 5.4 for the Plateau problem defined in (6.4)) that will be crucial to prove (i). In what follows we denote by $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$ the orthogonal projection.

Theorem 6.8. *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4). Then the minimal surface $\Phi(\overline{\Sigma}_{\text{ann}})$ satisfies the following properties:*

(1) *The set $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is simply connected in $\overline{\Omega}$; $\Omega \cap \partial\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ consists of two disjoint embedded curves β_1 and β_2 of class C^∞ joining q_1 to p_2, and q_2 to p_1, respectively. Moreover, the closed region E_i enclosed between $\partial_i^0\Omega$ and β_i, $i = 1, 2$, is convex;*

(2) *$\Phi(\overline{\Sigma}_{\text{ann}})$ is symmetric with respect to the plane $\mathbb{R}^2 \times \{0\}$;*

(3) *$\Phi(\overline{\Sigma}_{\text{ann}}) \cap (\mathbb{R}^2 \times \{0\}) = \beta_1 \cup \beta_2$;*

(4) *$S^+ := \Phi(\overline{\Sigma}_{\text{ann}}) \cap \{x_3 \geq 0\}$ is Cartesian. Precisely, it is the graph of a function $\widetilde{\psi} \in W^{1,1}(\text{int}(\pi(\Phi(\overline{\Sigma}_{\text{ann}})))) \cap C^0(\pi(\Phi(\overline{\Sigma}_{\text{ann}})))$.*

The proof of Theorem 6.8 is a consequence of Lemmas 6.9, 6.10, 6.11, 6.13, 6.14, and 6.15 below.

Lemma 6.9. *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4). Then $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is a simply connected region in $\overline{\Omega}$ containing $\partial_1^D\Omega \cup \partial_2^D\Omega$.*

Proof. We recall that $\Phi : \overline{\Sigma}_{\text{ann}} \to \mathbb{R}^3$ is an embedding. The fact that $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is a subset of $\overline{\Omega}$ and contains $\partial_1^D\Omega \cup \partial_2^D\Omega$ follows from the fact that the interior of $\Phi(\overline{\Sigma}_{\text{ann}})$ is contained in the convex hull of Γ. So it remains to show that $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is simply connected.
Suppose by contradiction that $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is not simply connected. Let H be a hole of it, namely a region in Ω surrounded by a loop contained in $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ and such that $H \cap \pi(\Phi(\overline{\Sigma}_{\text{ann}})) = \emptyset$; choose a point $P \in H$. We will search for a contradiction by exploiting that Σ_{ann} is an annulus and using that the map Φ is analytic and harmonic.

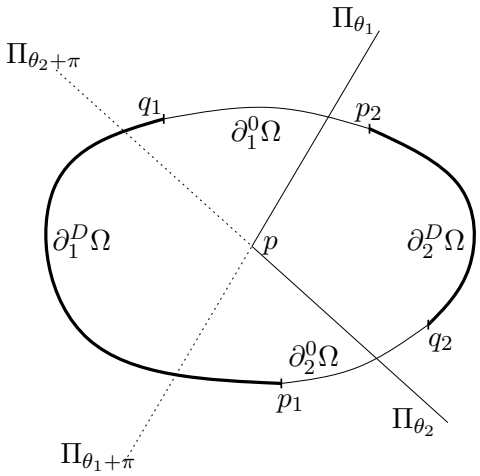

Figure 5: The horizontal section of two planes Π_{θ_1} and Π_{θ_2} intersecting $\partial_1^0\Omega$ and $\partial_2^0\Omega$, respectively.

Let θ be the angular coordinate of a cylindrical coordinate system (ρ, θ, z) in $\mathbb{R}^3$ centred at P and with z-axis the vertical line $\pi^{-1}(P)$. For $\theta \in [0, 2\pi)$ we consider the half-plane orthogonal to $\mathbb{R}^2 \times \{0\}$ defined by

$$\Pi_\theta := \{(\rho, \theta, z) \colon \rho > 0, z \in \mathbb{R}\}.$$

Now we fix two values θ_1 and θ_2 so that Π_{θ_1} and Π_{θ_2} intersect (the interior of) $\partial_1^0\Omega$ and $\partial_2^0\Omega$ respectively. The half-planes[17] $\Pi_{\theta_1+\pi}$ and $\Pi_{\theta_2+\pi}$ might intersect $\partial^D\Omega$ (see Figure 5). However, since the points p_1, q_1, p_2, q_2, are in clockwise order on $\partial\Omega$, and Ω is convex, it is not difficult to conclude the following assertion:

The half-planes $\Pi_{\theta_1+\pi}$ and $\Pi_{\theta_2+\pi}$ cannot intersect the two components $\partial_1^D\Omega$ and $\partial_2^D\Omega$ of $\partial^D\Omega$ at the same time.

In other words: If, for instance, $\Pi_{\theta_1+\pi}$ intersects $\partial_1^D\Omega$, then $\Pi_{\theta_2+\pi}$ does not intersect $\partial_2^D\Omega$. Let us prove the assertion in the form of the last statement, being the other cases similar. This is trivial, since, if Π_{θ_1} intersects $\partial_1^0\Omega$ and $\Pi_{\theta_1+\pi}$ intersects $\partial_1^D\Omega$ (as in Figure 5), we have that $\Pi_{\widehat{\theta}}$ intersects $\partial_1^D\Omega \cup \partial_1^0\Omega$ for all $\widehat{\theta} \in [\theta_1, \theta_1 + \pi]$. As either θ_2 or $\theta_2 + \pi$ belongs to $[\theta_1, \theta_1 + \pi]$, we have that $\Pi_{\theta_2} \cup \Pi_{\theta_2+\pi}$ intersects $\partial_1^D\Omega \cup \partial_1^0\Omega$. Since by hypothesis Π_{θ_2} intersects $\partial_2^0\Omega$, it follows that $\Pi_{\theta_2+\pi}$ does not intersect $\partial_2^D\Omega$, and the statement follows.

Moreover, since Π_{θ_1} intersects $\partial_1^0\Omega$ and Π_{θ_2} intersects $\partial_2^0\Omega$, it is straightforward that:

If $\Pi_{\theta_1+\pi}$ intersects $\partial_1^0\Omega$ then also $\Pi_{\theta_2+\pi}$ intersects $\partial_1^0\Omega$.

We are now ready to conclude the proof of the lemma. We have to discuss the following cases:

(1) $\Pi_{\theta_1+\pi}$ intersects $\partial^0\Omega$;

(2) $\Pi_{\theta_1+\pi}$ intersects $\partial_1^D\Omega$;

(3) $\Pi_{\theta_1+\pi}$ intersects $\partial_2^D\Omega$.

By hypothesis on P, for all $\theta \in [0, 2\pi)$ the intersection between $\Phi(\overline{\Sigma}_{\text{ann}})$ and Π_θ consists of a family of smooth simple curves, either closed or with endpoints on Γ. Correspondingly, $\Phi^{-1}(\Phi(\overline{\Sigma}_{\text{ann}}) \cap \Pi_\theta)$ is a family of closed curves in $\overline{\Sigma}_{\text{ann}}$, possibly with endpoints on $C_1 \cup C_2$.

[17] The angles are considered (mod 2π).

In particular, since $\Pi_{\theta_1} \cap \partial_1^0\Omega \neq \emptyset$, the set[18] $\Phi^{-1}(\Phi(\overline{\Sigma}_{\rm ann}) \cap \Pi_{\theta_1})$ is a family of closed curves in $\Sigma_{\rm ann}$.

In case (1) also $\Phi^{-1}(\Phi(\overline{\Sigma}_{\rm ann}) \cap \Pi_{\theta_1+\pi})$ consists of closed curves in $\Sigma_{\rm ann}$. Take two loops α and α' in $\Phi^{-1}(\Phi(\overline{\Sigma}_{\rm ann}) \cap \Pi_{\theta_1})$ and in $\Phi^{-1}(\Phi(\overline{\Sigma}_{\rm ann}) \cap \Pi_{\theta_1+\pi})$ respectively. Let d_1 be the signed distance function from the plane $\overline{\Pi}_{\theta_1} \cup \Pi_{\theta_1+\pi}$, positive on $\partial_2^D\Omega$. Since $d_1 \circ \Phi$ changes its sign when one crosses transversally α and α', we easily see that both α and α' cannot be homotopically trivial in $\Sigma_{\rm ann}$ (by harmonicity of $d_1 \circ \Phi$, if for instance α is homotopically trivial in $\Sigma_{\rm ann}$, $d_1 \circ \Phi = 0$ in the region enclosed by α, i.e. the image of Φ is locally flat, contradicting the analyticity of Φ). Hence, since Φ is an embedding, they run exactly one time around C_1; as a consequence, they must be homotopically equivalent to each other in $\Sigma_{\rm ann}$. On the other hand, they do not intersect each other (Φ is an embedding), so they bound an annulus-type region in $\Sigma_{\rm ann}$, and by harmonicity $d_1 \circ \Phi$ is constantly null in this region. This would imply again that the image by Φ of this annulus is contained in $\overline{\Pi}_{\theta_1} \cup \Pi_{\theta_1+\pi}$, a contradiction.

In case (2), by recalling our assertion, we deduce that $\Pi_{\theta_2+\pi}$ might intersect either $\partial^0\Omega$ or $\partial_1^D\Omega$. Further we can exclude that $\Pi_{\theta_2+\pi}$ intersects $\partial^0\Omega$ (otherwise, we repeat the argument for case (1) switching the role of θ_1 and θ_2). Therefore the only remaining possibility is that $\Pi_{\theta_2+\pi}$ intersects $\partial_1^D\Omega$ (see Figure 5). Let d_2 be the signed distance function from $\Pi_{\theta_2} \cup \Pi_{\theta_2+\pi}$ positive on $\partial_2^D\Omega$. In particular, $d_i \circ \Phi$, $i = 1, 2$, is positive on the circle C_2 of $\overline{\Sigma}_{\rm ann}$. By hypothesis on d_i, $i = 1, 2$, we see that d_1 is positive on Π_{θ_2}, and d_2 is positive on Π_{θ_1}.

As in case (1), let $\alpha \in \Phi^{-1}(\Phi(\overline{\Sigma}_{\rm ann}) \cap \Pi_{\theta_1})$ and $\beta \in \Phi^{-1}(\Phi(\overline{\Sigma}_{\rm ann}) \cap \Pi_{\theta_2})$ be two loops. We know that α and β are closed in $\Sigma_{\rm ann}$. Again, we conclude that α and β are homotopically equivalent in $\Sigma_{\rm ann}$, and both run one time around C_2. Assume without loss of generality that α encloses β, which in turn encloses C_2. Since $d_2 \circ \Phi$ is positive on both α and C_2, $d_2 \circ \Phi$ must be positive in the region enclosed between them, contradicting the fact that it vanishes on β.

If instead we are in case (3) we can argue analogously to case (2) and get a contradiction. In all cases (1), (2), and (3), we reach a contradiction which derives by assuming that $\pi(\Phi(\overline{\Sigma}_{\rm ann}))$ is not simply connected. The proof is achieved. □

We next proceed to characterize the geometry of $\Omega \cap \partial\pi(\Phi(\overline{\Sigma}_{\rm ann}))$.

Lemma 6.10. *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4). Then $\Omega \cap \partial\pi(\Phi(\overline{\Sigma}_{\rm ann}))$ consists of two disjoint Lipschitz embedded curves β_1 and β_2 joining q_1 to p_2, and q_2 to p_1, respectively. Moreover, the closed regions E_i enclosed between $\partial_1^0\Omega$ and β_i are convex for $i = 1, 2$.*

Proof. By Lemma 6.9, $\pi(\Phi(\overline{\Sigma}_{\rm ann}))$ is simply connected in $\overline{\Omega}$, and contains $\partial^D\Omega$. Therefore $\overline{\Omega} \setminus \pi(\Phi(\overline{\Sigma}_{\rm ann}))$ consists of two simply connected components, one containing $\partial_1^0\Omega$ and the other containing $\partial_2^0\Omega$. Let E_1 and E_2 be the closures of these two components[19], so that in particular the boundary of E_i is a simple Jordan curve of the form $\beta_i \cup \partial_i^0\Omega$ for some embedded curve $\beta_i \subset \overline{\Omega}$ joining the endpoints of $\partial_i^0\Omega$. We will prove that E_i is convex for $i = 1, 2$. This will also imply that β_i are Lipschitz.

Take $i = 1$, and assume by contradiction that E_1 is not convex. Thus we can find a line l in $\mathbb{R}^2$ and three different points A_1, A_2, A_3 on l, with $A_2 \in \overline{A_1A_3}$, so that A_2 is contained in $\Omega \setminus E_1$, and A_1 and A_3 belong to the interior of E_1.

Consider the region $\pi(\Phi(\overline{\Sigma}_{\rm ann})) \setminus l$, which consists in several (open) connected components. There is one of these connected components, say U, which does not intersect $\partial^D\Omega$ and whose boundary contains A_2. In addition, $\overline{U} \cap \partial^D\Omega = \emptyset$. Indeed, ∂U is the union of a segment L (containing A_2)

[18] Since $\Pi_{\theta_1} \cap \partial^D\Omega = \emptyset$ these curves must be closed in $\Sigma_{\rm ann}$.

[19] The sets E_1 and E_2 have nonempty interior, since $\Phi(\Sigma_{\rm ann})$ is contained in the interior of the convex hull of $\Phi(\partial\Sigma_{\rm ann})$, hence contained in the cylinder $\Omega \times \mathbb{R}$.

and a curve γ (contained in $\beta_1 \subseteq \partial(\pi(\Phi(\overline{\Sigma}_{\rm ann})))$ joining its endpoints. Hence, $\overline{U} \setminus U = \gamma \cup L$, and L cannot intersect $\partial^D\Omega$ by the hypothesis on A_1, A_2, and A_3.

Let $\Pi_l \subset \mathbb{R}^3$ be the plane containing l and orthogonal to the plane containing Ω; As usual, $\Pi_l \cap \Phi(\overline{\Sigma}_{\rm ann})$ is a family of closed curves, possibly with endpoints on $\Gamma \cap \Pi_l$. Now, pick a point P on $\partial U \setminus L$, and let Q be a point on $\Phi(\overline{\Sigma}_{\rm ann})$ so that $\pi(Q) = P$. Let $d_l : \mathbb{R}^3 \to \mathbb{R}$ be the signed distance from Π_l, with $d_l(Q) = d_l(P) > 0$. We claim that, if D is the connected component of $\{w \in \overline{\Sigma}_{\rm ann} : d_l \circ \Phi(w) > 0\}$ containing the point $\Phi^{-1}(Q)$, then $D \cap \partial\Sigma_{\rm ann} = \emptyset$. This would contradict the harmonicity of $d_l \circ \Phi$, since $d_l \circ \Phi$ would be zero on D, but $d_l(Q) > 0$.

Assume by contradiction that the converse holds. Then there is an arc $\alpha : [0,1] \to D \cup \partial\Sigma_{\rm ann}$ joining $\Phi^{-1}(Q)$ to $\partial\Sigma_{\rm ann}$. The image of the map $\pi \circ \Phi \circ \alpha$ is an arc in $\overline{\Omega}$ joining P to $\partial^D\Omega$ and such that $d_l \geq 0$ on it. Clearly this arc is a subset of $\pi(\Phi(\overline{\Sigma}_{\rm ann}))$. Since $\pi \circ \Phi \circ \alpha(0) = P$, it follows that the image of $\pi \circ \Phi \circ \alpha$ is contained in $\overline{U}$. Now $\overline{U}$ does not intersect $\partial^D\Omega$, contradicting that $\pi \circ \Phi \circ \alpha(1) \in \partial^D\Omega$. This concludes the proof. □

In the next step we show that there exists a set $E \subset \mathbb{R}^3$ of finite perimeter such that

$$\partial E = \partial^* E = \Phi(\Sigma_{\rm ann}) \cup \overline{\Delta}_1 \cup \overline{\Delta}_2,$$

where

$$\Delta_i := \{P = (P', P_3) \in \mathbb{R}^3 : P' = (P_1, P_2) \in \partial_i^D\Omega,\ P_3 \in (-\varphi(P'), \varphi(P'))\}, \qquad i = 1, 2. \tag{6.13}$$

In particular $\overline{\Delta}_1 \cup \overline{\Delta}_2 \subset (\partial\Omega) \times \mathbb{R}$ and $(\Omega \times \mathbb{R}) \cap \partial E = \Phi(\Sigma_{\rm ann})$.

We first fix some notation. We let $[\![E]\!] \in \mathcal{D}_3(\mathbb{R}^3)$ be the 3-current given by integration over E with $E \subset \mathbb{R}^3$ being a set of finite perimeter. To every $\mathcal{MY}$ solution $\Phi \in \mathcal{P}_2(\Gamma)$ to (6.4) we associate the push-forward 2-current $\Phi_\sharp[\![\Sigma_{\rm ann}]\!] \in \mathcal{D}_2(\mathbb{R}^3)$ given by integration over the (suitably oriented) surface $\Phi(\Sigma_{\rm ann})$ [17, Section 7.4.2]. Finally if $\mathcal{T} \in \mathcal{D}_k(U)$ with $U \subset \mathbb{R}^3$ open and $k = 2, 3$, we denote by $|\mathcal{T}|$ the mass of $\mathcal{T}$ in U [see [11, p. 358]].

Lemma 6.11 (**Region enclosed by $\Phi(\Sigma_{\rm ann})$**). *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to* (6.4)*. Then there is a closed finite perimeter set $E \subset \overline{\Omega} \times \mathbb{R}$ such that $\partial E = \Phi(\Sigma_{\rm ann})$ in $\Omega \times \mathbb{R}$.*

Proof. As $\Phi_\sharp[\![\Sigma_{\rm ann}]\!]$ is a boundaryless integral 2-current in $\Omega \times \mathbb{R}$, there exists (see, e.g., [17, Theorem 7.9.1]) an integral 3-current $\mathcal{E} \in \mathcal{D}_3(\Omega \times \mathbb{R})$ with $\partial\mathcal{E} = \Phi_\sharp[\![\Sigma_{\rm ann}]\!]$, and we might also assume that the support of $\mathcal{E}$ is compact in $\overline{\Omega} \times \mathbb{R}$. We claim that, up to switching the orientation of $\Phi_\sharp[\![\Sigma_{\rm ann}]\!]$, $\mathcal{E}$ has multiplicity in $\{0, 1\}$, and hence is the integration $[\![E]\!]$ over a bounded measurable set E. This is a finite perimeter set if we show that the integration over $(\Omega \times \mathbb{R}) \cap \partial^* E$ coincides with $\Phi_\sharp[\![\Sigma_{\rm ann}]\!]$.

By Federer decomposition theorem [11, Section 4.2.25, p. 420] (see also [11, Section 4.5.9] and [17, Theorem 7.5.5]) there is a sequence $(E_k)_{k\in\mathbb{N}}$ of finite perimeter sets such that

$$\mathcal{E} = \sum_{k=1}^{+\infty} \sigma_k [\![E_k]\!], \qquad \sigma_k \in \{-1, 1\}, \tag{6.14}$$

moreover

$$|\mathcal{E}| = \sum_{k=1}^{+\infty} |E_k| \quad \text{and} \quad |\partial\mathcal{E}| = \mathcal{H}^2(\Phi(\Sigma_{\rm ann})) = \sum_{k=1}^{+\infty} \mathcal{H}^2(\partial^* E_k). \tag{6.15}$$

We start by observing that

$$\partial^* E_k \subseteq \Phi(\Sigma_{\rm ann}) \qquad \forall k \in \mathbb{N}. \tag{6.16}$$

Indeed, fixing $k \in \mathbb{N}$, by the second equation in (6.15), we have that $\partial^* E_k$ is contained in the support of $\partial\mathcal{E}$, which in turn is $\Phi(\Sigma_{\rm ann})$. As a consequence, if $P = (P_1, P_2, P_3) \in (\Omega \times \mathbb{R}) \cap \overline{\partial^* E_k}$, then $P \in \Phi(\Sigma_{\rm ann})$. Around P we can find suitable coordinates and a cube $U = (P_1 - \varepsilon, P_1 + \varepsilon) \times (P_2 - \varepsilon, P_2 + \varepsilon) \times (P_3 - \varepsilon, P_3 + \varepsilon)$ such that $\Phi(\Sigma_{\rm ann}) \cap U$ is the graph $\mathcal{G}_h$ of a smooth function $h : (P_1 - \varepsilon, P_1 + \varepsilon) \times (P_2 - \varepsilon, P_2 + \varepsilon) \to (P_3 - \varepsilon, P_3 + \varepsilon)$. Moreover, $\Phi_\sharp[\![\Sigma_{\rm ann}]\!] = [\![\mathcal{G}_h]\!]$ in U. We conclude[20] that $\mathcal{E} \llcorner U = [\![SG_h \cap U]\!] + m[\![U]\!]$, with SG_h the subgraph of h, and $m \in \mathbb{Z}$.
We claim that

$$\forall k \quad \text{either} \quad E_k \cap U = SG_h \cap U \quad \text{or} \quad E_k \cap U = U \setminus SG_h.$$

Indeed, assume for instance that $|E_k \cap SG_h \cap U| > 0$ and $|(SG_h \setminus E_k) \cap U| > 0$; by the constancy lemma it follows that $\partial[\![E_k]\!]$ is nonzero in the simply connected open set SG_h, contradicting (6.16).

As a consequence of the preceding claim, we have that $U \cap \partial^* E_k = U \cap \Phi(\Sigma_{\rm ann})$. Since this argument holds for any choice of $P \in (\Omega \times \mathbb{R}) \cap \overline{\partial^* E_k}$, we have proved that $(\Omega \times \mathbb{R}) \cap \overline{\partial^* E_k}$ is relatively open (and relatively closed at the same time) in $\Phi(\Sigma_{\rm ann})$, which in turn being a connected open set, implies

$$\Phi(\overline{\Sigma}_{\rm ann}) = \overline{\partial^* E_k} \qquad \forall k \in \mathbb{N}.$$

Denote by $\mathcal{I}^\pm := \{k \in \mathbb{N} : \sigma_k = \pm 1\}$, where σ_k appears in (6.14). Going back to the local behaviour around $P \in \Phi(\Sigma_{\rm ann})$, if U is a neighbourhood as above, we see that for all $k \in \mathcal{I}^+$ either $E_k \cap U = SG_h$ or $E_k = U \setminus SG_h$ (namely, all the E_k's coincide in U), since otherwise, there will be cancellations in the series $\sum_{k \in \mathcal{I}^+} \partial[\![E_k]\!]$, in contradiction with the second formula in (6.15). Assume without loss of generality that for all $k \in \mathcal{I}^+$ we have $E_k \cap U = SG_h$; thus, arguing as before, for all $k \in \mathcal{I}^-$ we must have $E_k \cap U = U \setminus SG_h$.

We obtain that $\mathcal{E} \llcorner U = m[\![SG_h]\!] - n[\![U \setminus SG_h]\!]$ for some nonnegative integers n, m. Since $(\partial\mathcal{E}) \llcorner U = (m+n)[\![\mathcal{G}_h]\!]$ and also $(\partial\mathcal{E}) \llcorner U = \Phi_\sharp[\![\Sigma_{\rm ann}]\!] = [\![\mathcal{G}_h]\!]$ in U, we conclude $m + n = 1$. Hence either $m = 1$ and $n = 0$, or $m = 0$ and $n = 1$. On the other hand, we know that $\mathcal{E} \llcorner U = \sum_{k \in \mathcal{I}^+} [\![E_k \cap U]\!] - \sum_{k \in \mathcal{I}^-} [\![E_k \cap U]\!]$, from which it follows that $\mathcal{I}^+$ has cardinality m and $\mathcal{I}^-$ has cardinality n. Namely, one of the sets $\mathcal{I}^\pm$ is empty, and the other contains only one index.

We conclude that the sum in (6.14) involves only one index, that is, there is only one compact set E in $\overline{\Omega} \times \mathbb{R}$ such that (up to switching the orientation)

$$\mathcal{E} = [\![E]\!].$$

This concludes the proof. □

Remark 6.12. From the fact that $(\overline{\Omega} \times \mathbb{R}) \cap \partial E = \Phi(\overline{\Sigma}_{\rm ann}) \cup \overline{\Delta}_1 \cup \overline{\Delta}_2$, we easily see that $\pi(E) = \pi(\Phi(\overline{\Sigma}_{\rm ann}))$ which, by Lemma 6.9, is simply connected.

We denote by $\mathrm{sym}_{\rm st}(E)$ the set (symmetric with respect to the horizontal plane $\mathbb{R}^2 \times \{0\}$) obtained applying to E the Steiner symmetrization with respect to $\mathbb{R}^2 \times \{0\}$.
Clearly $\mathrm{sym}_{\rm st}(E) \cap (\partial_i^D \Omega \times \mathbb{R}) = \overline{\Delta_i}$ with Δ_i defined as in (6.13). We define the surfaces

$$S := \partial(\mathrm{sym}_{\rm st}(E)) \setminus (\Delta_1 \cup \Delta_2), \quad S^+ := S \cap \{x_3 \geq 0\}, \quad S^- := S \cap \{x_3 \leq 0\}. \tag{6.17}$$

Since $P(\mathrm{sym}_{\rm st}(E)) \leq P(E)$ (here $P(\cdot)$ is the perimeter [2]) we have $\mathcal{H}^2(S) \leq \mathcal{H}^2(\Phi(\overline{\Sigma}_{\rm ann}))$.

[20] This is a consequence of the constancy lemma and the fact that $\partial\mathcal{E} - \Phi_\sharp[\![\Sigma_{\rm ann}]\!] = 0$ in U.

Lemma 6.13 (**Graphicality of** $\partial(\mathrm{sym}_{\mathrm{st}}(E))$)**.** *Suppose* $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ *and let* $\Phi \in \mathcal{P}_2(\Gamma)$ *be a* $\mathcal{MY}$ *solution to* (6.4)*. Let* E *be the finite perimeter set given by Lemma 6.11 and* $S^{\pm}$ *be as in* (6.17)*. Then there is* $\widetilde{\psi} \in BV(\mathrm{int}(\pi(E)) \cap C^0(\pi(E))$ *such that* $S^{\pm} = \mathcal{G}_{\pm\widetilde{\psi}}$*. In particular* $S^{\pm} \cap (\mathbb{R}^2 \times \{0\}) = \overline{\Omega \cap \partial(\pi(E))}$*.*

The proof of Lemma 6.13 essentially follows from the fact that $\Phi(\overline{\Sigma}_{\mathrm{ann}})$ is a minimal surface in $\overline{\Omega} \times \mathbb{R}$.

Proof. Since E has finite perimeter, there exists a function $\widetilde{\psi} \in BV(\pi(E))$ such that $S^{\pm} = \mathcal{G}_{\pm\widetilde{\psi}}$ [9]. So, we only need to show that $\widetilde{\psi}$ is continuous. Take a point P' in the interior of $\pi(E)$; if $P' = \pi(\Phi(w))$ for some w, then $w \in \Sigma_{\mathrm{ann}}$, since $\pi(\Phi(C_i)) \subset \partial\Omega$ for $i = 1, 2$. If at none of the points of $\pi^{-1}(P') \cap \Phi(\overline{\Sigma}_{\mathrm{ann}})$ the tangent plane to $\Phi(\overline{\Sigma}_{\mathrm{ann}})$ is vertical, then $\widetilde{\psi}$ is C^∞ in a neighbourhood of P', since it is the linear combination of smooth functions (see the discussion after formula (6.21) below, where details are given). Therefore we only have to check continuity of $\widetilde{\psi}$ at those points P' for which there is $P \in \pi^{-1}(P') \cap \Phi(\overline{\Sigma}_{\mathrm{ann}})$ such that $\Phi(\overline{\Sigma}_{\mathrm{ann}})$ has a vertical tangent plane Π at P.

Consider a system of Cartesian coordinates centred at P, with the (x, y)-plane coinciding with Π, the x-axis coinciding with the line $\pi^{-1}(P')$, and let $z = z(x, y)$ (defined at least in a neighbourhood of 0) be the analytic function whose graph coincides with $\Phi(\overline{\Sigma}_{\mathrm{ann}})$. This map, restricted to the x-axis, is analytic and it vanishes at $x = 0$; hence it is either constantly zero or it has a discrete set of zeroes (in the neighbourhood where it exists). We now exclude the former case: If $z(\cdot, 0)$ is constantly zero, it means that around P there is a vertical open segment included in $\pi^{-1}(P')$, which is contained in $\Phi(\overline{\Sigma}_{\mathrm{ann}})$. Let Q be an extremal point of this segment, and let Π_Q be the tangent plane to $\Phi(\overline{\Sigma}_{\mathrm{ann}})$ at Q. This plane must contain as tangent vector the above segment, hence Π_Q is vertical and contains $\pi^{-1}(P')$. Choosing again a suitable Cartesian coordinate system centred at Q we can express locally the surface $\Phi(\overline{\Sigma}_{\mathrm{ann}})$ as the graph of an analytic function defined in a neighbourhood of Q in Π_Q, and so the restriction of this map to $\pi^{-1}(P')$ is analytic in a neighbourhood of Q, hence it must be constantly zero since it is zero in a left (or right) neighbourhood of Q. What we found is that we can properly extend the segment $\overline{PQ}$ on the Q side to a segment $\overline{PR}$ contained in $\Phi(\overline{\Sigma}_{\mathrm{ann}})$. By iterating this argument we conclude that the whole line $\pi^{-1}(P')$ is contained in $\Phi(\overline{\Sigma}_{\mathrm{ann}})$, which is impossible since $\Phi(\overline{\Sigma}_{\mathrm{ann}})$ is bounded.

Hence the zeroes of the function $z(\cdot, 0)$ are isolated, so the next assertion follows:

Assertion A: Let $P \in \pi^{-1}(P') \cap \Phi(\overline{\Sigma}_{\mathrm{ann}})$*. Then in a neighbourhood of* P *the only intersection between* $\Phi(\overline{\Sigma}_{\mathrm{ann}})$ *and* $\pi^{-1}(P')$ *is* P *itself.*

We can now conclude the proof of the continuity of the function $\widetilde{\psi}$. Let P' be in the interior of $\pi(E)$, and write $\pi^{-1}(P') \cap \Phi(\overline{\Sigma}_{\mathrm{ann}}) = \{Q_1, Q_2, \dots, Q_m\} \subset \Omega \times \mathbb{R}$. It follows that

$$2\widetilde{\psi}(P') = \mathcal{H}^1(\pi^{-1}(P') \cap E) = \sum_{j=1}^{m} \sigma_j (Q_j)_3, \tag{6.18}$$

where $(Q_j)_3$ is the vertical coordinate of Q_j and $\sigma_j \in \{-1, 0, 1\}$ is defined as

$$\sigma_j = \begin{cases} -1 & \text{if } \overline{Q_{j-1}Q_j} \subset \overline{\mathbb{R}^3 \setminus E} \text{ and } \overline{Q_jQ_{j+1}} \subset E, \\ 1 & \text{if } \overline{Q_{j-1}Q_j} \subset E \text{ and } \overline{Q_jQ_{j+1}} \subset \overline{\mathbb{R}^3 \setminus E}, \\ 0 & \text{otherwise,} \end{cases} \qquad j = 1, \dots, m. \tag{6.19}$$

Let $P'_k \in \mathrm{int}(\pi(E))$ be such that the sequence (P'_k) converges to P', and write $\pi^{-1}(P'_k) \cap \Phi(\overline{\Sigma}_{\mathrm{ann}}) = \{Q_1^k, Q_2^k, \dots, Q_{m_k}^k\} \subset \Omega \times \mathbb{R}$. With a similar notation as above, we have

$$2\widetilde{\psi}(P'_k) = \mathcal{H}^1(\pi^{-1}(P'_k) \cap E) = \sum_{j=1}^{m_k} \sigma_j^k (Q_j^k)_3. \tag{6.20}$$

Now, if P' is such that at every point Q_j the tangent plane to $\Phi(\overline{\Sigma}_{\rm ann})$ is not vertical, then $\Phi(\overline{\Sigma}_{\rm ann})$ is a smooth Cartesian surface in a neighbourhood of Q_j, and so it is clear that, for k large enough,

$$m = m_k, \qquad Q_j^k \to Q_j, \qquad \sigma_j^k \to \sigma_j \qquad \text{for all } j = 1, \dots, m, \tag{6.21}$$

and the continuity of (6.18) follows. Therefore it remains to check continuity in the case that the tangent plane to some Q_j is vertical.

Let $\widetilde{Q}$ be one of these points, with associated sign $\widetilde{\sigma}$. By assertion A there is $\delta > 0$ so that $\widetilde{Q}$ is the unique intersection between $\pi^{-1}(P')$ and $\Phi(\overline{\Sigma}_{\rm ann})$ with vertical coordinate in $[\widetilde{Q}_3 - \delta, \widetilde{Q}_3 + \delta]$. This means that the segments $\pi^{-1}(P') \cap \{\widetilde{Q}_3 - \delta < x_3 < \widetilde{Q}_3\}$ and $\pi^{-1}(P') \cap \{\widetilde{Q}_3 < x_3 < \widetilde{Q}_3 + \delta\}$ are either subsets of $\mathrm{int}(E)$ or subsets of $\mathbb{R}^3 \setminus E$. In particular, there is a neighbourhood $U \subset \Omega$ of P' such that the discs $U \times \{x_3 = \widetilde{Q}_3 - \delta\}$ and $U \times \{x_3 = \widetilde{Q}_3 + \delta\}$ are subsets of $\mathrm{int}(E)$ or of $\mathbb{R}^3 \setminus E$. Suppose without loss of generality that both these discs are inside $\mathbb{R}^3 \setminus E$ (the other cases being similar), so that $\widetilde{\sigma} = 0$. We infer that, for k large enough so that $P'_k \in U$, there is a finite subfamily $\{Q_j^k : j \in J\}$ of $\{Q_1^k, Q_2^k, \dots, Q_{m_k}^k\}$ contained in $\{\widetilde{Q}_3 < x_3 < \widetilde{Q}_3 + \delta\}$ and which satisfies the following: The sum in (6.20) restricted to such subfamily reads as:

$$\sum_{j \in J} \sigma_j^k (Q_j^k)_3 = (Q_{j_l}^k)_3 - (Q_{j_{l-1}}^k)_3 + \cdots + (Q_{j_2}^k)_3 - (Q_{j_1}^k)_3,$$

where $J = \{j_1, j_2, \dots, j_l\}$ and $(Q_{j_l}^k)_3 > (Q_{j_{l-1}}^k)_3 > \cdots > (Q_{j_2}^k)_3 > (Q_{j_1}^k)_3$ (in the case that $j_l = 1$ necessarily $\sigma_{j_1}^k = 0$ and the sum is zero). We have to show that this sum tends to $\widetilde{\sigma}\widetilde{Q}_3 = 0$ as $k \to +\infty$, which is true, since each Q_j^k tends to $\widetilde{Q}$. Repeating this argument for each point $\widetilde{Q}$ appearing in (6.18) with a vertical tangent plane to $\Phi(\overline{\Sigma}_{\rm ann})$, we conclude the proof of continuity of $\widetilde{\psi}$ in the interior of $\pi(E)$.

Let now $P' \in \partial(\pi(E))$. If $P' \in \partial(\pi(E)) \cap \Omega$ then every point in $\pi^{-1}(P') \cap \Phi(\overline{\Sigma}_{\rm ann})$ has vertical tangent plane and we can argue as in the previous case. It remains to show continuity of $\widetilde{\psi}$ on $\partial\pi(E) \cap \partial\Omega$. In this case we exploit the fact that the interior of $\Phi(\overline{\Sigma}_{\rm ann})$ is contained in $\Omega \times \mathbb{R}$. We sketch the proof without details since it is very similar to the previous arguments. Let $P' \in \partial_1^D\Omega$, thus $\pi^{-1}(P') \cap \Gamma_1$ consists of two points Q_1 and Q_2. Let (P'_k) be a sequence of points in $\overline{\pi(E)}$ converging to P. For $P'_k \in \partial_1^D\Omega$ it follows $\pi^{-1}(P'_k) \cap \Gamma_1 = \{Q_1^k, Q_2^k\}$ and the continuity of $\widetilde{\psi}$ follows from the continuity of φ on $\partial_1^D\Omega$, whereas if P'_k is in the interior of $\pi(E)$ there holds $\pi^{-1}(P'_k) \cap \Gamma_1 = \{Q_1^k, Q_2^k, \dots, Q_{m_k}^k\}$. Using the continuity of Φ up to C_1, it is easily seen that all such points must converge, as $k \to +\infty$, either to Q_1 or to Q_2. Hence we can repeat an argument similar to the one used before. □

Lemma 6.14. *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to* (6.4). *Let E be the finite perimeter set given in Lemma 6.11 and let S be defined as in* (6.17). *Then there is an injective map $\widetilde{\Phi} \in H^1(\Sigma_{\rm ann}; \mathbb{R}^3) \cap C^0(\overline{\Sigma}_{\rm ann}; \mathbb{R}^3)$ which maps $\partial\Sigma_{\rm ann}$ weakly monotonically to Γ and such that $\widetilde{\Phi}(\overline{\Sigma}_{\rm ann}) = S$, and also*

$$\mathcal{H}^2(S) = \int_{\Sigma_{\rm ann}} |\partial_{w_1}\widetilde{\Phi} \wedge \partial_{w_2}\widetilde{\Phi}| dw = \int_{\Sigma_{\rm ann}} |\partial_{w_1}\Phi \wedge \partial_{w_2}\Phi| dw = m_2(\Gamma). \tag{6.22}$$

In particular, $\widetilde{\Phi}$ is a solution of (6.4).

Proof. By Lemma 6.13 there is $\widetilde{\psi} \in BV(\mathrm{int}(\pi(E)) \cap C^0(\pi(E))$ such that $S^{\pm} = \mathcal{G}_{\pm\widetilde{\psi}}$. As a consequence, for $p \in \partial^D\Omega$ we have $\widetilde{\psi}(p) = \varphi(p)$ and for $p \in \partial(\pi(E)) \cap \Omega$ we have $\widetilde{\psi}(p) = 0$.

By Lemma 6.9 $\pi(E)$ is simply connected, and so the maps $\widetilde{\Psi}^{\pm} : \pi(E) \to \mathbb{R}^3$ given by $\widetilde{\Psi}^{\pm}(p) := (p, \pm\widetilde{\psi}(p))$ are disc-type parametrizations of $S^{\pm}$. Moreover S^+ and S^- glue to each other along $\partial(\mathrm{sym}_{\mathrm{st}}(E)) \cap (\mathbb{R}^2 \times \{0\}) = \beta_1 \cup \beta_2$, where β_1 and β_2 are the curves given by Lemma 6.10 .

Let $(\sigma, \psi) \in \mathcal{W}_{\mathrm{conv}}$ be a minimizer of $\mathcal{F}$ which satisfies properties 1-5 of Theorem 5.1. Setting $\widetilde{\sigma} := (\beta_1, \beta_2)$ and by extending $\widetilde{\psi}$ to zero in $\overline{\Omega} \setminus \pi(E)$, without relabelling it, by minimality we get

$$2\mathcal{F}(\sigma, \psi) \leq 2\mathcal{F}(\widetilde{\sigma}, \widetilde{\psi}) = \mathcal{H}^2(S),$$

whence, by Remark 6.12

$$2\mathcal{F}(\sigma, \psi) \leq \mathcal{H}^2(S) \leq \mathcal{H}^2(\Phi(\overline{\Sigma}_{\mathrm{ann}})) = \int_{\Sigma_{\mathrm{ann}}} |\partial_{w_1}\Phi \wedge \partial_{w_2}\Phi| dw = m_2(\Gamma). \tag{6.23}$$

We are in the hypotheses of Lemma 6.7, and therefore there exists a map $\widetilde{\Phi} \in P_2(\Gamma)$ parametrizing $\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ which is a minimizer of (6.4). In particular, $2\mathcal{F}(\sigma, \psi) = m_2(\Gamma)$, and all the inequalities in (6.23) are equalities. We deduce also that $(\widetilde{\sigma}, \widetilde{\psi})$ is a minimizer of $\mathcal{F}$ in $\mathcal{W}_{\mathrm{conv}}$, so that by Theorem 5.1 $\widetilde{\psi}$ is analytic in $\mathrm{int}(\pi(E))$. As a consequence it belongs to $W^{1,1}(\pi(E); \mathbb{R}^3)$.

We now conclude the proof of the lemma by invoking again Lemma 6.7. □

Lemma 6.15. *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4). Let E be the finite perimeter set given in Lemma 6.11 and let S be defined as in (6.17). Then $\Phi(\overline{\Sigma}_{\mathrm{ann}}) = S$ and in particular $E = \mathrm{sym}_{\mathrm{st}}(E)$.*

Proof. By Lemma 6.14 we have that $\mathcal{H}^2(S) = m_2(\Gamma)$ from which it follows that $P(\mathrm{sym}_{\mathrm{st}}(E)) = P(E)$. Then we can apply [9, Theorem 1.1] to deduce the existence of two functions $f, g : \pi(E) \to \mathbb{R}$ of bounded variation, such that $\partial^* E = \mathcal{G}_f \cup \mathcal{G}_g$ (up to $\mathcal{H}^2$-negligible sets). We will show that $f = \widetilde{\psi}$ and $g = -\widetilde{\psi}$. To this aim, thanks again to [9, Theorem 1.1], we know that for a.e. $p \in \pi(E)$, the two unit normal vectors $\nu^f = (\nu_1^f, \nu_2^f, \nu_3^f)$ and $\nu_g = (\nu_1^g, \nu_2^g, \nu_3^g)$ to $\mathcal{G}_f$ and $\mathcal{G}_g$ at the points $(p, f(p))$ and $(p, g(p))$, respectively, satisfy

$$(\nu_1^f, \nu_2^f, \nu_3^f) = (\nu_1^g, \nu_2^g, -\nu_3^g). \tag{6.24}$$

To conclude the proof it is then sufficient to show that $f = -g$ a.e. on $\pi(E)$: indeed this would readily imply $E = \mathrm{sym}_{\mathrm{st}}(E)$ and hence $f = \widetilde{\psi}$.

Let $p \in \mathrm{int}(\pi(E))$; if

$$\pi^{-1}(p) \cap S = \{P_1, P_2, \ldots, P_k\}, \tag{6.25}$$

then for a.e. $p \in \mathrm{int}(\pi(E))$ it is $k \leq 2$. We now show that, for all $p \in \mathrm{int}(\pi(E))$, if $k > 1$, none of the points $\{P_1, P_2, \ldots, P_k\}$ has vertical tangent plane. Assume by contradiction that P_1 has vertical tangent plane Π_1. In this case $\Pi_1 \cap S$ consists, in a neighbourhood U of P_1, of at least 2 curves crossing transversally at P_1. These curves, by assertion A in the proof of Lemma 6.13, intersect $\pi^{-1}(p)$ only at P_1. Moreover, in a neighbourhood V of P_2, with $U \cap V = \varnothing$, $\Pi_1 \cap S$ consists of (at least) one (or more) curve passing through P_2. This curve is locally Cartesian if $\pi^{-1}(p)$ crosses S transversally in P_2, otherwise it can be locally the union of two curves ending at P_2, with vertical tangent plane, which lie on the same side of Π_1 with respect to $\pi^{-1}(p)$. In both cases, we deduce that there is a point $q \in \Pi_1 \cap (\Omega \times \{0\})$ for which $\pi^{-1}(q)$ intersects transversally S in at least three points. As a consequence, for all q' in a neighbourhood of q in Ω, the line $\pi^{-1}(q')$ intersects S at more than two points, which is a contradiction. We have proved the following:

Assertion: for all $p \in int(E)$ the line $\pi^{-1}(p)$ either intersects S transversally at two points P_1, P_2, or it intersects S at only one point P_1.

We now see that the latter case cannot happen. Indeed, first one checks that in this case the intersection cannot be transversal[21], and that $\pi^{-1}(p)$ must be tangent to S at P_1. Let Π_1 be the vertical tangent plane to S at P_1. Let $\Pi_1^\perp$ be the vertical plane orthogonal to Π_1 passing through P_1. In a neighbourhood of P_1, the unique curve in $S \cap \Pi_1^\perp$ must be the union of two curves joining at P_1, and these curves must belong to the same half-plane of $\Pi_1^\perp$ with boundary $\pi^{-1}(p)$. As a consequence, if $p' \in \Omega \cap \Pi_1^\perp$ is in that half-plane, then $\pi^{-1}(p')$ consists of at least two points; if p' lies in the opposite half-plane, then $\pi^{-1}(p')$ is empty. This means that necessarily $p \in \partial\pi(E)$. Namely, the previous assertion can be strengthened to:

For all $p \in int(E)$ the line $\pi^{-1}(p)$ intersects S transversally at exactly two points P_1, P_2.

The consequence of this is that f and g belong to $W^{1,1}(\mathrm{int}(\pi(E)))$ and are also smooth in $\mathrm{int}(\pi(E))$. Indeed, let $p \in \mathrm{int}(\pi(E))$, so $f(p) \neq g(p)$, and

$$\pi^{-1}(p) \cap S = \{(p, f(p)), (p, g(p))\}. \tag{6.26}$$

Since S is locally the graph of smooth functions around $(p, f(p))$ and $(p, g(p))$, these functions coincide with f and g, respectively. We can now conclude the proof of the lemma: let us choose a simple curve $\alpha : [0,1] \to \pi(E)$ with $\alpha(0) \in \partial^D\Omega$ and $\alpha(1) = p$ such that (6.24) holds for $\mathcal{H}^1$ a.e. $p \in \alpha([0,1])$. Since $f \circ \alpha$ and $g \circ \alpha$ are differentiable in $[0,1]$, condition (6.24) uniquely determines the tangent planes to $\mathcal{G}_f$ and $\mathcal{G}_g$, and hence it implies that the derivatives of $f \circ \alpha$ and $g \circ \alpha$ satisfy

$$(f \circ \alpha)'(t) + (g \circ \alpha)'(t) = 0, \qquad \text{for a.e. } t \in [0,1]. \tag{6.27}$$

By continuity of f and g one infers $f \circ \alpha + g \circ \alpha = c$ a.e. on $[0,1]$ (actually everywhere since $f + g$ is continuous), for some constant $c \in \mathbb{R}$. To show that $c = 0$ it is sufficient to observe that $f \circ \alpha(0) = \varphi(\alpha(0))$ and $g \circ \alpha(0) = -\varphi(\alpha(0))$. Hence $f(p) = -g(p)$, and the thesis of Lemma 6.15 is achieved. □

We are now in a position to conclude the proof of Theorem 6.8.

Proof of Theorem 6.8. Property (1) follows by Lemma 6.9 and Lemma 6.10. Properties (2)–(4) follow by Lemma 6.13 and Lemma 6.15. To see that β_i are C^∞ it is sufficient to observe that, in view of the Cartesianity of S^+ and S^-, their union coincides with the set $S \cap \{x_3 = 0\}$ which, by standard arguments, is the image of the zero-set of Φ_3, which is smooth. □

Theorem 6.16. *There holds*

$$2 \min_{(s,\zeta)\in\mathcal{W}_{\mathrm{conv}}} \mathcal{F}(s,\zeta) = m_2(\Gamma). \tag{6.28}$$

Proof. Step 1: $2\min_{(s,\zeta)\in\mathcal{W}_{\mathrm{conv}}} \mathcal{F}(s,\zeta) \leq m_2(\Gamma)$.

Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$. Let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4) and let $S := \Phi(\overline{\Sigma}_{\mathrm{ann}})$. By Theorem 6.8 the following properties hold:

- $S \cap (\mathbb{R}^2 \times \{0\}) = \beta_1 \cup \beta_2$ with β_1 and β_2 disjoint embedded curves of class C^∞ joining q_1 to p_2 and q_2 to p_1, respectively;
- S is symmetric with respect to $\mathbb{R}^2 \times \{0\}$;
- for $i = 1, 2$ the closed region E_i enclosed between $\partial_i^0\Omega$ and β_i is convex;

[21] This is a consequence of the fact that the line $\pi^{-1}(p)$ must lie outside the set E, with the only exception of the point P_1. Indeed, otherwise, there must be some other point in $\pi^{-1}(p) \cap S$, E being compact in $\mathbb{R}^3$.

- $S^+ = S \cap \{x_3 \geq 0\}$ is the graph of $\widetilde{\psi} \in W^{1,1}(U) \cap C^0(\overline{U})$, where $U = \Omega \setminus (E_1 \cup E_2)$ is the open region enclosed between $\partial^D\Omega$ and $\beta_1 \cup \beta_2$.

Let $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be given by

$$\sigma := (\sigma_1, \sigma_2) \quad \text{and} \quad \psi := \begin{cases} 0 & \text{in } \Omega \setminus U, \\ \widetilde{\psi} & \text{in } U, \end{cases}$$

where $\sigma_i([0,1]) = \beta_i$ for $i = 1, 2$. Then clearly $S^+ = \mathcal{G}_{\psi \llcorner (\Omega \setminus E(\sigma))}$ and

$$\min_{(s,\zeta) \in \mathcal{W}_{\rm conv}} \mathcal{F}(s,\zeta) \leq \mathcal{F}(\sigma,\psi) = \mathcal{H}^2(S^+) = \frac{1}{2} m_2(\Gamma).$$

Suppose now $m_2(\Gamma) = m_1(\Gamma_1) + m_1(\Gamma_2)$. Let $\Phi_j \in \mathcal{P}_1(\Gamma_j)$ be a solution to (6.1) and $S_j := \Phi_j(\overline{B}_1)$ $(j = 1, 2)$. For $j = 1, 2$, let D_j be the closed convex hull of Γ_j: clearly $D_1 \cap D_2 = \emptyset$. By Lemma 5.4 each S_j satisfies the following properties:

- $S_j \cap (\mathbb{R}^2 \times \{0\}) = \beta_j \subset D_j$ is a simple smooth curve joining p_j to q_j;
- S_j is symmetric with respect to $\mathbb{R}^2 \times \{0\}$;
- $S_j^+ := S \cap \{x_3 \geq 0\}$ is the graph of a function $\widetilde{\psi}_j \in W^{1,1}(U_j) \cap C^0(\overline{U}_j)$, where $U_j \subset D_j$ is the open region enclosed between $\partial^D\Omega_j$ and β_j;
- β_j is contained in D_j and $F_j \setminus U_j$ is convex.

Let $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be given by

$$\sigma := (\sigma_1, \sigma_2) \quad \text{and} \quad \psi := \begin{cases} 0 & \text{in } \Omega \setminus \{U_1 \cup U_2\}, \\ \widetilde{\psi}_j & \text{in } U_j \text{ for } j = 1, 2, \end{cases}$$

where $\sigma_1([0,1]) := \overline{p_1 q_2}$ and $\sigma_2([0,1]) := \beta_2 \cup \overline{q_2 p_1} \cup \beta_1$. Then $S^+ := S_1^+ \cup S_2^+ = \mathcal{G}_{\psi \llcorner (\Omega \setminus E(\sigma))}$ and

$$\min_{(s,\zeta) \in \mathcal{W}_{\rm conv}} \mathcal{F}(s,\zeta) \leq \mathcal{F}(\sigma,\psi) = \mathcal{H}^2(S^+) = \frac{1}{2}(m_1(\Gamma_1) + m_1(\Gamma_2)) = \frac{1}{2} m_2(\Gamma),$$

and the proof of step 1 is concluded.

Step 2: $2 \min_{(s,\zeta) \in \mathcal{W}_{\rm conv}} \mathcal{F}(s,\zeta) \geq m_2(\Gamma)$.

Let $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be a minimizer satisfying properties 1-5 of Theorem 5.1.
If $E(\sigma_1) \cup E(\sigma_2) = \emptyset$, by Step 1 we can apply Lemma 6.7 and find an injective parametrization $\Phi \in \mathcal{P}_2(\Gamma)$ such that $\Phi_i(\partial\Sigma_{\rm ann}) = \Gamma$ monotonically $\Phi(\overline{\Sigma}_{\rm ann}) = \mathcal{G}_\psi \cup \mathcal{G}_{-\psi}$, and

$$2\mathcal{F}(\sigma,\psi) = \int_{\Sigma_{\rm ann}} |\partial_{w_1}\Phi \wedge \partial_{w_2}\Phi| dw \geq m_2(\Gamma).$$

If instead $E(\sigma_1) \cup E(\sigma_2) \neq \emptyset$, similarly we find injective parametrizations $\Phi_1 \in \mathcal{P}_1(\Gamma_1)$ and $\Phi_2 \in \mathcal{P}_1(\Gamma_2)$ such that $\Phi_j(\partial B_1) = \Gamma_j$ monotonically for $j = 1, 2$, $\Phi_1(\overline{B}_1) \cup \Phi_2(\overline{B}_1) = \mathcal{G}_\psi \cup \mathcal{G}_{-\psi}$, and

$$2\mathcal{F}(\sigma,\psi) = \int_{B_1} |\partial_{w_1}\Phi_1 \wedge \partial_{w_2}\Phi_1| dw + \int_{B_1} |\partial_{w_1}\Phi_2 \wedge \partial_{w_2}\Phi_2| dw \geq m_1(\Gamma_1) + m_1(\Gamma_2) \geq m_2(\Gamma).$$

This concludes the proof. □

Now the proof of Theorem 6.5 is easily achieved.

Proof of Theorem 6.5. (i). Let $\Phi \in \mathcal{P}_2(\Gamma)$, S, S^+, S^- be as in the statement. By arguing as in the proof of Theorem 6.16 we can find $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ such that $S^{\pm} = \mathcal{G}_{\pm\psi \llcorner (\Omega \setminus E(\sigma))}$. Then by Theorem 6.16 we have

$$\mathcal{F}(\sigma, \psi) = \frac{1}{2} m_2(\Gamma) = \min_{(s,\zeta) \in \mathcal{W}_{\rm conv}} \mathcal{F}(s, \zeta) \tag{6.29}$$

Hence (σ, ψ) is a minimizer for $\mathcal{F}$ in $\mathcal{W}$; moreover by the properties of S it also satisfies properties 1-5 of Theorem 5.1.

(ii). Let $\Phi_j \in \mathcal{P}_1(\Gamma_j)$, S_j for $j = 1, 2$, S^+, S^- be as in the statement. Again arguing as in the proof of Theorem 6.16, we can find $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ such that $S^{\pm} = \mathcal{G}_{\pm\psi \llcorner (\Omega \setminus E(\sigma))}$ and (6.29) holds, so that (σ, ψ) is a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1-5 of Theorem 5.1.

(iii). Let $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1-5 of Theorem 5.1. Let also

$$S := \mathcal{G}_{\psi \llcorner (\Omega \setminus E(\sigma))} \cup \mathcal{G}_{-\psi \llcorner (\Omega \setminus E(\sigma))}.$$

Suppose $E(\sigma_1) \cap E(\sigma_2) = \emptyset$. Then there is $\Phi \in \mathcal{P}_2(\Gamma)$ which is a $\mathcal{MY}$ solution to (6.4) such that $\Phi(\overline{\Sigma}_{\rm ann}) = S$: indeed, to see this, it is sufficient to apply Lemma 6.7, since by Theorem 6.16 we have

$$2\mathcal{F}(\sigma, \psi) = m_2(\Gamma). \tag{6.30}$$

Suppose now $E(\sigma_1) \cap E(\sigma_2) \neq \emptyset$; then with a similar argument we can construct $\Phi_j \in \mathcal{P}_1(\Gamma_j)$ for $j = 1, 2$ solutions to (6.1) such that $\Phi_1(\overline{B}_1) \cup \Phi_2(\overline{B}_1) = S$. The proof is achieved. □

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