A non-parametric Plateau problem with partial free boundary

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Abstract

We consider a Plateau problem in codimension 1 in the non-parametric setting, where a Dirichlet boundary datum is assigned only on part of the boundary $\partial\Omega$ of a bounded convex domain $\Omega \subset \mathbb{R}^2$. Where the Dirichlet datum is not prescribed, we allow a free contact with the horizontal plane. We show existence of a solution, and prove regularity for the corresponding area-minimizing surface. We compare these solutions with the classical minimal surfaces of Meeks and Yau, and show that they are equivalent when the Dirichlet boundary datum is assigned on at most 2 disjoint arcs of $\partial\Omega$.

Key words: Plateau problem, area functional, minimal surfaces, relaxation, Cartesian currents.

AMS (MOS) 2020 Subject Classification: 49J45, 49Q05, 49Q15, 28A75.

1 Introduction

The Plateau problem is a classical problem in the Calculus of Variations modelling configurations of soap films obtained by immersing a wire frame into soapy water. Roughly speaking, it consists in seeking for an area minimizing surface over all surfaces with prescribed boundary a given closed Jordan curve in space. Over the years several approaches and variants were proposed, each corresponding to a specific choice of the class of admissible surfaces. In the following we list just few of them and we refer for example to [31] and references therein for a list of the main approaches available in the literature. One of the first result is due to Weierstrass and Riemann who studied a non-parametric Plateau problem in $\mathbb{R}^3$ obtained by minimizing the area over all cartesian surfaces; this gave rise to theory of minimal surfaces. Successively Douglas and Radó developed independently [23, 38] the classical parametric approach for disk type solutions. This method was later generalized by Jost [33] to study the Plateau problem for surfaces with higher genus (see also the paper [36] by Meeks and Yau). A more general approach which accounts for a large class of surfaces was instead proposed by Federer and Fleming [25], based on integral currents. Another remarkable work is due to Reifenberg [40] which adopts completely different techniques involving the concept of Čech homology. Relevant is also Almgren's contribution with three different approaches, one of these using the notion of varifolds [2]. Among all possible variants one might consider a partial free boundary version of the Plateau problem where the boundary datum is partially fixed and

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partially free to move within a given surfaces. This type of problem has been exhaustively studied (see for instance [22]) in the parametric framework but never investigated, to our best knowledge, with the non-parametric approach. To this aim, in the present paper we will analyse existence and regularity of solutions of a non-parametric partial free boundary Plateau problem. More precisely, we look for an area-minimizing surface which can be written as a graph over a bounded open convex set $\Omega \subset \mathbb{R}^2$, and spanning a Jordan curve $\Gamma_\sigma = \gamma \cup \sigma \subset \mathbb{R}^2 \times [0, +\infty)$ that is partially fixed. Namely, γ is fixed (Dirichlet condition) and is given by a family $\{\gamma_i\}_{i=1}^n \subset \partial\Omega \times [0, +\infty)$ of $n \in \mathbb{N}$ curves each joining distinct pairs of points $\{(p_i, q_i)\}_{i=1}^n$ of $\partial\Omega$. Whereas σ, which represents the *free boundary*, is an unknown and consists of (the image of) n curves $\sigma_1, \dots, \sigma_n$ sitting in $\overline{\Omega}$, and joining the endpoints of γ in order that $\gamma \cup \sigma$ forms a Jordan curve Γ_σ in $\mathbb{R}^3$. We assume that each γ_i is Cartesian, i.e., it can be expressed as the graph of a given nonnegative function φ defined on a corresponding portion of $\partial\Omega$. This allows to restrict ourselves to the Cartesian setting, and to assume that the competitors for the Plateau problem are expressed by graphs of functions ψ defined on a suitable subdomain of Ω depending on σ; see Figure 1 when $n = 3$. A peculiarity of our problem is the presence of a free boundary.

The purpose of this paper is twofold. We start addressing the question of existence and regularity of solutions. Our first main result (Theorems 1.1, 3.1 and 5.1) asserts that there are always solutions (which can be degenerate, in the sense that they may consist of more than one connected component, see the example of the catenoid below) and that, under suitable hypotheses on the boundary datum, there is at least one regular solution continuous up to the boundary. Next we compare our solutions with solutions to a parametric Plateau problem when $n = 1, 2$. Roughly speaking, our second main result (Theorems 1.2, 6.1 and 6.4) shows that any regular solution to our minimization problem is a minimal embedding in the sense of Meeks and Yau [36], and vice-versa.

Existence and regularity of solutions: We describe here our main results with few details, referring to Section 2 for the precise description of the mathematical framework. We fix $n \in \mathbb{N}$ and $2n$ distinct points $p_1, q_1, p_2, q_2, \dots, p_n, q_n \in \partial\Omega$ in clockwise order, and set $q_{n+1} := p_1$. The relatively open arc of $\partial\Omega$ between the points p_i and q_i is noted by $\partial_i^D\Omega$, and the relatively open arc between q_i and p_{i+1} by $\partial_i^0\Omega$. We fix a nonnegative continuous function $\varphi \colon \partial\Omega \to [0, +\infty)$ positive on $\partial^D\Omega := \cup_{i=1}^n \partial_i^D\Omega$ and vanishing on $\{p_i, q_i\}_{i=1}^n \cup \partial^0\Omega$, where¡ $\partial^0\Omega := \cup_{i=1}^n \partial_i^0\Omega$. For every $i = 1, \dots, n$ we denote by γ_i the graph of φ over $\overline{\partial_i^D\Omega}$ and we consider curves $\sigma_i \colon [0, 1] \to \overline{\Omega}$ with the following properties:

(i) σ_i is injective, $\sigma_i(0) = q_i$ and $\sigma_i(1) = p_{i+1}$, for all $i = 1, \dots, n$;

(ii) $\mathrm{int}(E(\sigma_i)) \cap \mathrm{int}(E(\sigma_j)) = \emptyset$ for $i, j = 1, \dots, n$, $i \neq j$, where int denotes the interior part.

Note carefully that σ_i and σ_j are allowed to partially overlap.

We suppose the graph of φ over $\partial^D\Omega$ to be a Lipschitz curve in $\mathbb{R}^3$ (see Figure 1). Finally we set

$$E(\sigma) := \bigcup_{i=1}^n E(\sigma_i), \tag{1.1}$$

and define the two classes

$$\Sigma := \Big\{\sigma = (\sigma_1, \dots, \sigma_n) \in (\mathrm{Lip}([0,1]; \overline{\Omega}))^n \text{ satisfies (i)-(ii)}\Big\}, \tag{1.2}$$

$$\mathcal{X}_\varphi := \{(\sigma, \psi) \in \Sigma \times W^{1,1}(\Omega) : \psi = 0 \text{ a.e. in } E(\sigma) \text{ and } \psi = \varphi \text{ on } \partial^D\Omega\}. \tag{1.3}$$

If $(\sigma, \psi) \in \mathcal{X}_\varphi$, then the graph of ψ over $\Omega \setminus E(\sigma)$ is a surface spanning the curve Γ_σ. We look for a pair (σ, ψ) minimizing the area of such surfaces, that is, we want to find a solution to the

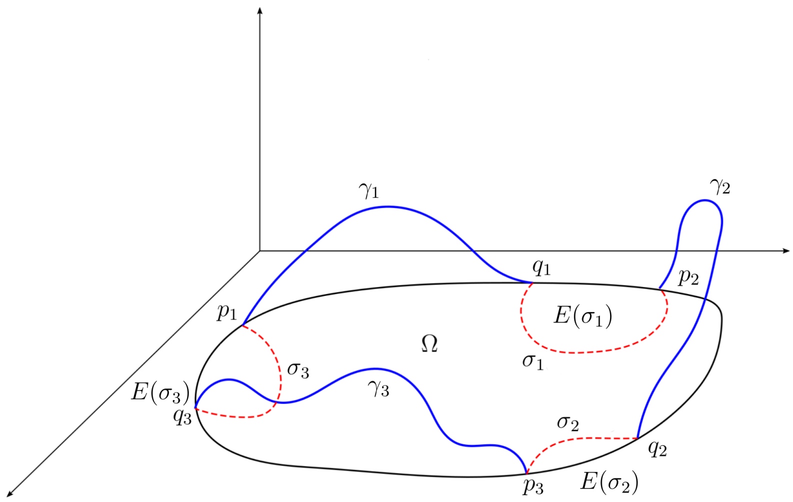

Figure 1: An example of the setting (in 3D), when $n = 3$. On the boundary of the convex set Ω fix the points p_i, q_i; the arc of $\partial\Omega$ joining p_i to q_i is $\partial_i^D\Omega$, while the arc joining q_i to p_{i+1} is $\partial_i^0\Omega$ ($p_4 := p_1$). On $\partial^D\Omega$ the Dirichlet boundary datum φ is imposed, whose graph has been depicted. The dotted arcs are the free planar curves σ_i joining q_i and p_{i+1}.

minimum problem

$$\inf_{(\sigma,\psi)\in\mathcal{X}_\varphi} \int_{\Omega\setminus E(\sigma)} \sqrt{1+|\nabla\psi|^2}\,dx. \tag{1.4}$$

We then prove the following result, accounting for existence and regularity of solutions to (1.4).

Theorem 1.1. *Let Ω be strictly convex. Then there exists a solution $(\sigma,\psi) \in \mathcal{X}_\varphi$ to (1.4) such that ψ is continuous on $\overline{\Omega}$, analytic in $\Omega \setminus E(\sigma)$, and $\Omega \cap \partial E(\sigma)$ consists of a family of mutually disjoint analytic curves (joining p_i and q_j in some order). Moreover each connected component of $E(\sigma)$ is convex.*

We emphasize that convexity of Ω is necessary (even for the classical non-parametric Plateau problem with no free boundary, existence of regular solutions is not guaranteed if Ω is not convex). The proof of existence relies on direct methods; however, since the class $\mathcal{X}_\varphi$ is not closed under weak* convergence in BV, they cannot be applied directly to (1.4) but rather to a suitable weak formulation. For this reason we replace $\mathcal{X}_\varphi$ in (1.3) with a larger class $\mathcal{W}$ of admissible pairs, and relax accordingly the functional in (1.4). We set

$$\mathcal{W} := \Big\{(\sigma,\psi) \in \Sigma \times BV(\Omega) : \psi = 0 \text{ a.e. in } E(\sigma)\Big\}. \tag{1.5}$$

The weak formulation consists in looking for solutions to the problem

$$\inf_{(\sigma,\psi)\in\mathcal{W}} \mathcal{F}(\sigma,\psi), \tag{1.6}$$

where $\mathcal{F}$ is the functional defined by

$$\begin{aligned}\mathcal{F}(\sigma,\psi) :&= \int_\Omega \sqrt{1+|\nabla\psi|^2}\,dx + |D^s\psi|(\Omega) - |E(\sigma)| + \int_{\partial\Omega} |\psi-\varphi|\,d\mathcal{H}^1 \\ &= \int_{\Omega\setminus E(\sigma)} \sqrt{1+|\nabla\psi|^2}\,dx + |D^s\psi|(\Omega) + \int_{\partial\Omega} |\psi-\varphi|\,d\mathcal{H}^1,\end{aligned} \tag{1.7}$$

with $D^s\psi$ the singular part of the measure $D\psi$ and $|E(\sigma)|$ the Lebesgue measure of $E(\sigma)$. Observe that $\mathcal{F}(\sigma,\psi)$ equals the integral in (1.4) when $\psi \in W^{1,1}(\Omega)$ attains the boundary value φ. The existence of solutions to (1.6) is shown in two steps. In the first step we prove existence of minimizers of $\mathcal{F}$ in a smaller class $\mathcal{W}_{\rm conv} \subset \mathcal{W}$ of admissible pairs (σ,ψ), where compactness is easier and allows to make use of the direct method. The class $\mathcal{W}_{\rm conv}$ accounts only for specific geometries of the free boundary σ, namely, each set $E(\sigma_i)$ is required to be convex (see (2.6) for its precise definition). In the second step we show, by means of a convexification procedure, that every minimizer $(\sigma,\psi) \in \mathcal{W}_{\rm conv}$ is actually a solution to (1.6). Eventually we prove that there exists at least a minimizer $(\sigma,\psi) \in \mathcal{W}_{\rm conv}$ which satisfies certain regularity properties, and in particular is a solution to (1.4). The fact that, for minimizers, all connected components of $E(\sigma)$ are convex, is somehow a consequence of the maximum principle, i.e., every minimal surface is contained in the convex hull of its boundary. The existence and regularity of a solution to (1.6) are contained in Theorems 3.1 and 5.1 respectively, which in turn imply Theorem 1.1. We stress that Theorems 3.1 and 5.1 are actually stated in the more general case of a convex planar domain Ω. However, if Ω is convex but not strictly convex it may happen that a solution to (1.6) is "less regular", in the sense that ψ may not achieve the boundary condition (as in the next example), thus failing to be a solution to (1.4).

The example of the catenoid: Our prototypical example is given by (half of) the catenoid. Consider a cylinder in $\mathbb{R}^3$ with basis a circle of radius r and height ℓ. Choose Cartesian coordinates for which the x_1x_2-plane contains the cylinder axis, and restrict attention to the half-space $\{x_3 \geq 0\}$ as in Figure 2, where $\Omega = R_\ell := (0,\ell) \times (-r,r)$ and $n = 2$. Write

$$\partial\Omega = \overline{\partial_1^D\Omega} \cup \partial_1^0\Omega \cup \overline{\partial_2^D\Omega} \cup \partial_2^0\Omega,$$

where $\partial_1^D\Omega = \{0\} \times (-r,r)$, $\partial_1^0\Omega = (0,\ell) \times \{r\}$, $\partial_2^D\Omega = \{\ell\} \times (-r,r)$ and $\partial_2^0\Omega = (0,\ell) \times \{-r\}$. On the Dirichlet boundary $\partial^D\Omega = \partial_1^D\Omega \cup \partial_2^D\Omega$ we prescribe the continuous function φ whose graph consists of the two half-circles γ_1 and γ_2. The endpoints of γ_1 and γ_2 live on the free boundary plane (the horizontal plane) and are $p_1 = (0,-r)$, $q_1 = (0,r)$, and $p_2 = (\ell,r)$, $q_2 = (\ell,-r)$, respectively. The free boundary σ consists of two curves σ_1 and σ_2 with endpoints q_1, p_2, and q_2, p_1, respectively, constrained to stay in $\overline{\Omega}$. The concatenation of $\gamma = \gamma_1 \cup \gamma_2$ and σ forms a Jordan curve

$$\Gamma_\sigma = \gamma_1 \cup \sigma_1 \cup \gamma_2 \cup \sigma_2 \subset \mathbb{R}^3. \tag{1.8}$$

Therefore we look for an area-minimizer among all Cartesian surfaces S with boundary Γ_σ *keeping σ free*, i.e. we look for a solution to (1.4) for this specific geometry. In this case a minimizing sequence $(\sigma_k,\psi_k) \subset \mathcal{W}$ of the weak formulation (1.7) tends (in the sense of Definition 4.3) to a minimizer $(\sigma,\psi) \in \mathcal{W}_{\rm conv}$ which allows for two different possibilities. If ℓ is small, σ_1 and σ_2 remain disjoint and $(\sigma,\psi) \in \mathcal{X}_\varphi$. In particular the area-minimizing surface S (given by the graph of ψ over $\Omega \setminus E(\sigma)$) is the classical (half) catenoid (namely the intersection between the catenoid and the half-space $\{x_3 \geq 0\}$). If instead ℓ is large, the two curves σ_1 and σ_2 merge, the region $\Omega \setminus E(\sigma)$ collapses (i.e., it reduces to the two segments $\partial_1^D\Omega \cup \partial_2^D\Omega$) and $\psi = 0$ and therefore $(\sigma,\psi) \notin \mathcal{X}_\varphi$. In particular the surface S is the union of two vertical (half) disks. We emphasize that this example is classical and, due to the rotational symmetry of the curve Γ, it can be reduced to a 1-dimensional problem (see [16,30]).

Let us now quickly describe the second part of the paper.

Comparison with embedded minimal surfaces: We recall that γ_i is the graph of the map φ on $\overline{\partial_i^D\Omega}$. We consider $\mathrm{sym}(\gamma_i)$, namely the graph of $-\varphi$ on $\overline{\partial_i^D\Omega}$, which is symmetric to γ_i with respect to the plane containing Ω. Setting $\Gamma_i := \gamma_i \cup \mathrm{sym}(\gamma_i)$, this turns out to be a simple Jordan curve in $\mathbb{R}^3$, for all $i = 1,\ldots,n$. Hence we can consider the classical Plateau problem for

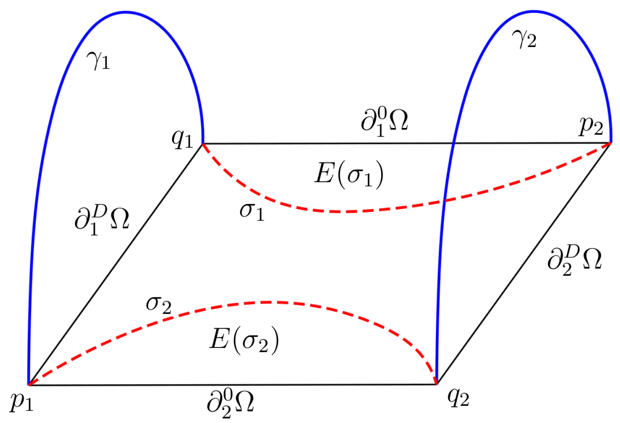

Figure 2: The setting for the catenoid: for ℓ large enough (the basis of the rectangle) the dotted curves σ_1 and σ_2 merge and the (generalized) graph of ψ reduces to two vertical half-circles on $\partial^D\Omega = \partial_1^D\Omega \cup \partial_2^D\Omega$. In this case $\partial^D\Omega \subset \partial E(\sigma_1) \cup \partial E(\sigma_2)$.

the curve $\Gamma := \cup_{i=1}^n \Gamma_i$. In the case $n = 1$ a solution is an area minimizing disk-type surface S spanning $\Gamma = \Gamma_1$. Whereas in the case $n = 2$ a solution is either an annulus-type surface spanning $\Gamma = \Gamma_1 \cup \Gamma_2$ or the union of two disjoint disks spanning Γ_1 and Γ_2, respectively. Then the following result holds true:

Theorem 1.2. *Let Ω be strictly convex. For $n \in \{1, 2\}$ let $(\sigma, \psi) \in \mathcal{X}_\varphi$ be a minimizer as in Theorem 1.1. Let S^+ be the graph of ψ over $\Omega \setminus E(\sigma)$ and let S^- be the symmetric of S^+ with respect to the plane containing Ω. Then the set $S = S^+ \cup S^-$ is a solution to the classical Plateau problem associated to $\Gamma = \cup_{i=1}^n \Gamma_i$. Vice-versa every solution S to the classical Plateau problem associated to $\Gamma = \cup_{i=1}^n \Gamma_i$ is symmetric with respect to the plane containing Ω. Moreover $S^+ := S \cap \{x_3 \geq 0\}$ is the graph of ψ over $\Omega \setminus E(\sigma)$ for some $(\sigma, \psi) \in \mathcal{X}_\varphi$, a minimizer as in Theorem 1.1.*

The above theorem is rigorously stated in Theorems 6.1 ($n = 1$) and 6.4 ($n = 2$) in the more general case of Ω convex. In particular, if Ω is convex, we prove that there is a correspondence between a regular solution to the weak formulation (1.6) and a solution to the classical Plateau problem (as in the example of the catenoid). A relevant consequence of this equivalence is that when the boundary closed curve Γ is symmetric with respect to the plane containing Ω, and its upper part is Cartesian, then the same property holds for the corresponding Meeks and Yau solution.

The proof of Theorem 1.2 for $n = 1$ is not difficult, whereas for $n = 2$ it is considerably more complicated, and requires several lemmas: we strongly use the convexity of the domain Ω, which implies that the cylinder $\Omega \times \mathbb{R}$, whose boundary contains Γ, is convex, and so the existence results of Meeks and Yau [36] (see also Theorem 6.3) are applicable.

The main steps of the proof are the following: if S is a Meeks-Yau annulus-type minimal surface, we perform a Steiner symmetrization of the 3-dimensional finite perimeter set in $\Omega \times \mathbb{R}$ enclosed by S to obtain a set (symmetric with respect to the plane containing Ω) whose boundary is an annulus-type minimal surface $\widetilde{S}$ spanning Γ which is symmetric and such that $\widetilde{S}^+ := \widetilde{S} \cap \{x_3 \geq 0\}$ is Cartesian. In turn, using standard results on the case of equality for the perimeter of a set and its symmetrization, we show that the original surface S was already symmetric with respect to the plane containing Ω, so S^+ was already Cartesian, and the conclusion of the proof for $n = 2$ is achieved. Note that the aim of Theorem 1.2 is not to provide new examples of minimal surfaces; rather, it enlights (among other things) some interesting qualitative properties of the Meeks-Yau solutions. Due to the highly nontrivial arguments, we have restricted our analysis to the cases

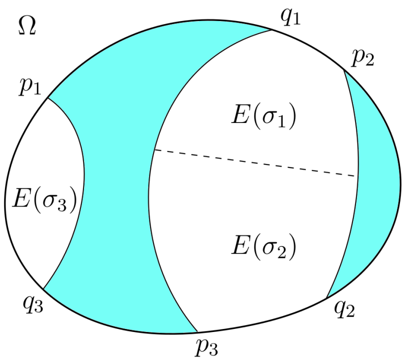

Figure 3: A possible configuration of the sets $E(\sigma_i)$ in the case $n = 3$. On the (clockwise oriented) arcs $\widehat{p_1q_1} = \partial_1^D\Omega$, $\widehat{p_2q_2} = \partial_2^D\Omega$, and $\widehat{p_3q_3} = \partial_3^D\Omega$ the function φ is prescribed and positive. On $\partial^0\Omega = \widehat{q_1p_2} \cup \widehat{q_2p_3} \cup \widehat{q_3p_1}$ and on $E(\sigma) = E(\sigma_1) \cup E(\sigma_2) \cup E(\sigma_3)$ we prescribe $\psi = 0$. The curves σ_i joining q_i to p_{i+1} (with the corresponding set $E(\sigma_i)$) are indicated. On the dotted segment σ_1 and σ_2 overlaps with opposite orientations. On the dark region $\Omega \setminus E(\sigma)$, ψ is not necessarily null.

$n \in \{1, 2\}$, since a generalization to the case $n > 2$ probably requires heavy modifications. Indeed, some lemmas needed to prove Theorem 6.4 employ crucially the fact that $\partial^0\Omega$ consists of just two connected components. For this reason we leave the case $n > 2$ for future investigations.

Some motivation: The setting of our problem models a cluster of soap films which are constrained to wet a given system of wires γ emanating from a given free boundary plane (representing a table, or a water surface, on which the soap films can freely moves). Our results show that if the system of wires describes the graphs of functions on $\partial\Omega$ as above, then the (Meeks and Yau) solutions of the "parametric" Plateau problem are in fact Cartesian, and coincide with the solutions obtained by the non-parametric approach. This result can be viewed as a generalization of the well-known theorem of Radó stating that any minimal disk spanning a Jordan curve in $\mathbb{R}^3$ whose projection on a plane is a bijection with a convex Jordan curve is the graph of a function defined on the plane [39].

However, the scope of this article goes beyond this generalization, and the solutions we look for are strongly related with the vertical parts of Cartesian currents arising in the analysis of the relaxation of the non parametric area functional in dimension 2 and codimension 2. We further comment on this in Section 7 where we go more into details.

Structure of the paper: The paper is organized as follows. In Section 2 we introduce the setting of the problem in detail. In Section 3 we show how to reduce the minimum problem from the wider class $\mathcal{W}$ to the class $\mathcal{W}_{\rm conv}$ (Theorem 3.1). Next, in Section 4 we prove the existence of minimizers in $\mathcal{W}_{\rm conv}$. As a consequence, we gain the existence of minimizers in class $\mathcal{W}$ (Corollary 4.2). In Section 5 we study the regularity of minimizers. Specifically, we state and prove Theorem 5.1 which, together with Theorem 3.1, generalize Theorem 1.1. Theorem 1.1 follows from Theorem 4.1, Corollary 4.2, and Theorem 5.1. Eventually, in Section 6 we compare our solutions with the classical minimal surfaces spanning Γ. Here, as anticipated, we restrict the analysis to $n = 1, 2$, the case $n = 2$ essentially giving rise to either a catenoid-type minimal surface, or two disk-type surfaces spanning Γ_1 and Γ_2. The main theorems here are Theorems 6.1 and 6.4. In Section 7 we briefly point out our motivations for the present study and some open problems. The paper concludes with an appendix containing some rather classical results on convex sets and Hausdorff distance, needed in Section 5.

2 Preliminaries

2.1 Area of the graph of a BV function

Let $U \subset \mathbb{R}^2$ be a bounded open set. For any $\psi \in BV(U)$ we denote by $D\psi$ its distributional gradient, so that

$$D\psi = \nabla\psi\mathcal{L}^2 + D^s\psi,$$

where $\nabla\psi$ is the approximate gradient of ψ and $D^s\psi$ denotes the singular part of $D\psi$. We recall that the L^1-relaxed area functional reads as [29]

$$\mathcal{A}(\psi; U) := \int_U \sqrt{1 + |\nabla\psi|^2}\, dx + |D^s\psi|(U). \tag{2.1}$$

In what follows we denote by $\partial^* A$ the reduced boundary of a set of finite perimeter $A \subset \mathbb{R}^3$ (see [4]). For any $\psi \in BV(U)$ we denote by $R_\psi \subset U$ the set of regular points of ψ, namely the set of points $x \in U$ which are Lebesgue points for ψ, $\psi(x)$ coincides with the Lebesgue value of ψ at x, and ψ is approximately differentiable at x. We define the subgraph SG_ψ of ψ as

$$SG_\psi := \{(x, y) \in R_\psi \times \mathbb{R} \colon y < \psi(x)\},$$

which is a finite perimeter set in $U \times \mathbb{R}$. Its reduced boundary in $U \times \mathbb{R}$ is the generalised graph $\mathcal{G}_\psi := \{(x, \psi(x)) \colon x \in R_\psi\}$ of ψ, which turns out to be 2-rectifiable. If $[\![SG_\psi]\!] \in \mathcal{D}_3(\mathbb{R}^3)$ denotes the integral current given by integration over SG_ψ and $\partial[\![SG_\psi]\!] \in \mathcal{D}_2(\mathbb{R}^3)$ is its boundary in the sense of currents, then

$$[\![\mathcal{G}_\psi]\!] = \partial[\![SG_\psi]\!] \llcorner (U \times \mathbb{R}),$$

with $[\![\mathcal{G}_\psi]\!]$ the integer multiplicity 2-current given by integration over $\mathcal{G}_\psi$ (suitably oriented; see [27] for more details).

2.2 Setting of the problem

We fix $\Omega \subset \mathbb{R}^2$ to be an open bounded convex set (strict convexity is not required) which will be our reference domain. Given two points $p, q \in \partial\Omega$ in clockwise order, $\widehat{pq}$ stands for the relatively open arc on $\partial\Omega$ joining p and q.

Let $n \in \mathbb{N}$, $n \geq 1$, and let $\{p_i\}_{i=1}^n$ be distinct points on $\partial\Omega$ chosen in clockwise order; we set $p_{n+1} := p_1$. For all $i = 1, \ldots, n$ let q_i be a point in $\widehat{p_i p_{i+1}} \subset \partial\Omega$. We set

$$\partial_i^D\Omega := \widehat{p_i q_i}, \qquad \partial_i^0\Omega := \widehat{q_i p_{i+1}} \qquad \text{for } i = 1, \ldots, n, \tag{2.2}$$

and

$$\partial^D\Omega := \bigcup_{i=1}^n \partial_i^D\Omega, \qquad \partial^0\Omega := \bigcup_{i=1}^n \partial_i^0\Omega. \tag{2.3}$$

Since $\partial_i^D\Omega$ and $\partial_i^0\Omega$ are relatively open in $\partial\Omega$, so are $\partial^D\Omega$ and $\partial^0\Omega$. It follows that $\partial\Omega$ is the disjoint union

$$\partial\Omega = \cup_{i=1}^n \{p_i, q_i\} \cup \partial^D\Omega \cup \partial^0\Omega.$$

We fix a continuous function $\varphi : \partial\Omega \to [0, +\infty)$ such that

$$\varphi = 0 \text{ on } \partial^0\Omega \qquad \text{and} \qquad \varphi > 0 \text{ on } \partial^D\Omega, \tag{2.4}$$

see Figures 2, 1. We will make a further regularity assumption on φ: we require that the graph $\mathcal{G}_{\varphi \llcorner \partial_i^D\Omega} = \{(x, \varphi(x)) : x \in \partial_i^D\Omega\}$ of φ on $\partial_i^D\Omega$ is a Lipschitz curve in $\mathbb{R}^3$, for all $i = 1, \ldots, n$.

Remark 2.1. The hypothesis $\varphi > 0$ on $\partial^D\Omega$ excludes from our analysis the example in Figure 6 of the Introduction. We will further comment on this later on (see Section 5.1); the presence of pieces of $\partial^D\Omega$ where $\varphi = 0$ brings to some additional technical difficulties that we prefer to avoid here. However, the setting in Figure 6 can be achieved by an approximation argument. Namely, one considers a suitable regularization φ_ε of φ on $\partial^D\Omega$ such that $\varphi_\varepsilon > 0$, and then letting $\varepsilon \to 0$ one obtains a solution to the problem with Dirichlet datum φ.

Remark 2.2. By definition (1.2) any $\sigma \in \Sigma$ satisfies the injectivity property in (i) which guarantees that the sets $E(\sigma_i)$ are simply connected (but not necessarily connected). Assumption (ii) means essentially that the curves σ_i cannot cross transversally each other, but might overlap. Notice that $\mathrm{int}(E(\sigma_i))$ might be empty, the case $\partial_i^0\Omega = \sigma_i([0,1])$ being not excluded.

In what follows we will study existence and regularity of solutions to problem (1.6). A first step in this direction is to show in Section 3 that

$$\inf_{(s,\zeta)\in\mathcal{W}} \mathcal{F}(s,\zeta) = \inf_{(s,\zeta)\in\mathcal{W}_{\mathrm{conv}}} \mathcal{F}(s,\zeta)\,, \tag{2.5}$$

where $\mathcal{F}$ is the functional in (1.7) and

$$\begin{aligned} \mathcal{W}_{\mathrm{conv}} &:= \Big\{(\sigma,\psi) \in \Sigma_{\mathrm{conv}} \times BV(\Omega) :\ \psi = 0 \text{ a.e. in } E(\sigma)\Big\}, \\ \Sigma_{\mathrm{conv}} &:= \Big\{\sigma = (\sigma_1,\dots,\sigma_n) \in \Sigma : E(\sigma_i) \text{ is convex for all } i = 1,\dots,n\Big\}. \end{aligned} \tag{2.6}$$

Notice that, by definition

$$\Sigma_{\mathrm{conv}} \subset \Sigma \quad \text{and} \quad \mathcal{W}_{\mathrm{conv}} \subset \mathcal{W}. \tag{2.7}$$

Moreover, we already know that the sets $\mathrm{int}(E(\sigma_i))$ might be empty, since from assumption (i) in (1.2) we cannot exclude that σ_i overlaps $\partial_i^0\Omega$: Recalling that Ω is convex, by (ii) and the convexity of each $E(\sigma_i)$, this can happen, only if $\widehat{q_i p_{i+1}}$ is a straight segment[1]. Afterwards, in Section 4, we prove the existence of $(\sigma,\psi) \in \mathcal{W}_{\mathrm{conv}}$ which is a solution to (1.6) by showing that there exists a minimizer to

$$\mathcal{F}(\sigma,\psi) = \inf_{(s,\zeta)\in\mathcal{W}_{\mathrm{conv}}} \mathcal{F}(s,\zeta)\,. \tag{2.8}$$

Eventually in Section 5 we prove existence of solutions to (2.8) which belong to $\mathcal{X}_\varphi$.

Remark 2.3. Exploiting the characterization of the boundaries of convex sets given in Corollary 8.3 in the Appendix 8, we see that conditions (i),(ii) and the convexity of $E(\sigma_i)$ for the curves in Σ_{conv} imply the following:

(P) Let $\sigma \in \Sigma_{\mathrm{conv}}$; then for all $i = 1,\dots,n$ there are an injective (non-relabelled) reparametrization of σ_i in $[0,1]$, and a nondecreasing function $\theta_i \colon [0,1] \to \mathbb{R}$ with $\theta_i(1) - \theta_i(0) \le 2\pi$, such that, setting $\gamma_i(t) := (\cos(\theta_i(t))\,,\, \sin(\theta_i(t)))$ for all $t \in [0,1]$, we have

$$\sigma_i(t) = q_i + \ell(\sigma_i)\int_0^t \gamma_i(s)\,ds \qquad \forall t \in [0,1],$$

where $\ell(\sigma_i)$ denotes the length of σ_i.

[1] We will show that for a minimizer, $\sigma_i([0,1])$ cannot intersect $\partial^D\Omega$ unless $\partial^D\Omega$ is locally a segment (Theorem 5.1).

3 Reduction from $\mathcal{W}$ to $\mathcal{W}_{\rm conv}$

The main result of this section is contained in Theorem 3.1 where we prove the equivalence given in (2.5). The reason being that in minimizing the functional $\mathcal{F}$ on $\mathcal{W}$ one issue is that the class Σ in (1.2) is not closed under uniform convergence, since a uniform limit of elements in Σ needs not be formed by injective curves. However, we can always modify a minimizing sequence of curves to curves in $\Sigma_{\rm conv}$, since the modification can be done decreasing the energy.

The fact that the infimum of $\mathcal{F}$ over $\mathcal{W}$ coincides with that over $\mathcal{W}_{\rm conv}$ is due to the following geometric property: whenever a set $E(\sigma_i)$ is not convex, we can always convexify it reducing the energy. The procedure of convexification is described in Lemmas 3.3, 3.4, and 3.5. Again, the convexification of $E(\sigma_i)$ is still contained in $\overline{\Omega}$ thanks to the convexity of Ω.

Theorem 3.1 (**Reduction from $\mathcal{W}$ to $\mathcal{W}_{\rm conv}$**). *For every $(s,\zeta)\in\mathcal{W}$ there exists $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ such that every connected component of $E(\sigma)$ is convex, and*

$$\mathcal{F}(\sigma,\psi)\le\mathcal{F}(s,\zeta). \tag{3.1}$$

In particular (2.5) *holds true. Further, if the connected components of $E(\zeta)$ are not convex, then the strict inequality holds in* (3.1).

Remark 3.2. Since the σ_i's may overlap, the convexity of each $E(\sigma_i)$ does not imply in general that every connected component of $E(\sigma)=\cup_{i=1}^n E(\sigma_i)$ is convex.

For the reader convenience we split the proof of Theorem 3.1 into a sequence of intermediate results: Lemmas 3.3, 3.4, 3.5, and the conclusion. First we need to introduce some notation.

Let $(\sigma,\psi)\in\mathcal{W}$. We fix an extension $\widehat{\varphi}\in W^{1,1}(B)$ of φ on an open ball $B\supset\overline{\Omega}$, where we recall φ is the boundary datum in (2.4). Extending ψ in $B\setminus\overline{\Omega}$ as $\widehat{\varphi}$, and still denoting by ψ such an extension, we can rewrite $\mathcal{F}(\sigma,\psi)$ as

$$\mathcal{F}(\sigma,\psi)=\mathcal{A}(\psi;B)-|E(\sigma)|-\mathcal{A}(\psi;B\setminus\overline{\Omega}). \tag{3.2}$$

Lemma 3.3 (**Trace estimate**). *Let $u\in BV(\mathbb{R}\times(0,+\infty))$ be a nonnegative function with compact support in an open ball $B_r\subset\mathbb{R}^2$. Then*

$$\int_{(\mathbb{R}\times\{0\})\cap B_r}u(s)\,d\mathcal{H}^1(s)\le\mathcal{A}(u;B_r\cap(\mathbb{R}\times(0,+\infty)))-|E_{B_r}|, \tag{3.3}$$

where

$$E_{B_r}:=\{x\in B_r\cap(\mathbb{R}\times(0,+\infty)):u(x)=0\}.$$

Moreover, inequality (3.3) *is always strict, unless $u=0$ a.e. on $\mathbb{R}\times(0,+\infty)$.*

Notice that the function u is defined only on the half-plane $\mathbb{R}\times(0,+\infty)$, and in (3.3) the symbol $u(s)$ denotes its trace on the line $\mathbb{R}\times\{0\}$ (which is integrable).

Proof. We denote by $x=(x_1,x_2)\in\mathbb{R}^2$ the coordinates in $\mathbb{R}^2$. Set $H^+:=\mathbb{R}\times(0,+\infty)$, $Z:=(B_r\cap H^+)\times\mathbb{R}\subset\mathbb{R}^3$. Let

$$L_u:=\{(x,y)\in Z\colon x\in R_u,\ y\in(-u(x),u(x))\}\subset\mathbb{R}^3,$$

where R_u is the set of regular points of u. We have, recalling the notation in Section 2.1,

$$\begin{aligned}2\mathcal{A}(u;B_r\cap H^+)&=\mathcal{A}(u;B_r\cap H^+)+\mathcal{A}(-u;B_r\cap H^+)\\&=\mathcal{H}^2(\partial^*(Z\cap SG_u))+\mathcal{H}^2(\partial^*(Z\cap SG_{-u}))\\&=\mathcal{H}^2(Z\cap\partial^*L_u)+2|E_{B_r}|.\end{aligned} \tag{3.4}$$

Write $B_r \cap (\mathbb{R} \times \{0\}) = (a, b) \times \{0\}$. Then a slicing argument of the current $[\![\mathcal{G}_u]\!]$ yields

$$\begin{aligned} \mathcal{H}^2(Z \cap \partial^* L_u) &\geq \int_a^b \mathcal{H}^1(Z \cap \{x_1 = t\} \cap \partial^* L_u) dt \\ &= \int_a^b \mathcal{H}^1(Z \cap \{x_1 = t\} \cap (\mathrm{spt}([\![\mathcal{G}_u]\!] - [\![\mathcal{G}_{-u}]\!]))) dt \\ &\geq \int_a^b 2u(t,0) dt = 2 \int_{(\mathbb{R}\times\{0\})\cap B_r} u(s)\, d\mathcal{H}^1(s)\,, \end{aligned} \tag{3.5}$$

where the last inequality follows from the following fact: If we denote by $[\![\mathcal{G}_u]\!]_t$ the slice of the current $[\![\mathcal{G}_u]\!]$ on $\{x_1 = t\}$, then

$$\partial [\![\mathcal{G}_u]\!]_t = \delta_{(t,0,u(t,0))} - \delta_{(t,s_t,0)} \qquad \text{for a.e. } t \in (a, b),$$

where $s_t \geq 0$ is such that $(t, s_t) = B_r \cap (\{t\} \times \mathbb{R}^+)$, and in writing $\delta_{(t,s_t,0)}$ we are using that u has compact support in B_r. This can be seen, for instance, by approximating[2] u with a sequence of smooth functions. Therefore

$$\partial([\![\mathcal{G}_u]\!]_t - [\![\mathcal{G}_{-u}]\!]_t) = \delta_{(t,0,u(t,0))} - \delta_{(t,0,-u(t,0))} \qquad \text{for a.e. } t \in (a, b).$$

This justifies the last inequality in (3.5) and, using (3.4), the proof is achieved. Notice that, from the last formula, it follows that the last inequality in (3.5) is strict if $[\![\mathcal{G}_u]\!]_t - [\![\mathcal{G}_{-u}]\!]_t$ is not the straight segment connecting $(t, 0, u(t, 0))$ and $(t, 0, -u(t, 0))$ on a set of positive $\mathcal{H}^1$-measure. This implies that inequality in (3.3) is an equality if and only if $u = 0$ a.e. on H^+. □

We now turn to two technical lemmas needed to prove Theorem 3.1. We introduce a class of sets whose boundaries are regular enough to support the trace of a BV function. Precisely we say that an open subset of $\mathbb{R}^2$ is *piecewise Lipschitz* if it can be written as the union of a finite family of (not necessarily disjoint) Lipschitz open sets. Using that, for a Lipschitz set $E \subset \mathbb{R}^2$, the symmetric difference $(\partial^* E)\Delta\partial E$ has null $\mathcal{H}^1$ measure, one can see[3] that the same property holds also for a piecewise Lipschitz set. In particular, by (2.1) if $V \subset\subset U$ is a piecewise Lipschitz subset of a bounded open set $U \subset \mathbb{R}^2$, then

$$\mathcal{A}(\psi, \overline{V}) = \mathcal{A}(\psi, V) + \int_{\partial V} |\psi^+ - \psi^-| d\mathcal{H}^1, \tag{3.6}$$

where ψ^+ (respectively ψ^-) denotes the trace of $\psi \llcorner V$ (respectively $\psi \llcorner (U \setminus \overline{V})$) on ∂V.

Lemma 3.4 (Reduction of energy, I). *For $N \geq 1$ let $F_1, \dots, F_N$ be nonempty connected subsets of $\overline{\Omega}$, each F_i being the closure of a piecewise Lipschitz set, with $F_i \cap F_j = \emptyset$ for $i, j \in \{1, \dots, N\}$, $i \neq j$. Let $\psi \in BV(B)$ satisfy*

$$\psi = 0 \quad a.e.\ in \quad G := \bigcup_{i=1}^N F_i \quad and \quad \psi = \widehat{\varphi} \quad a.e.\ in \quad B \setminus \Omega\,. \tag{3.7}$$

[2] With respect to the strict convergence of $BV(B_r \cap (\mathbb{R} \times \{0\}))$, which guarantees the approximation also of the trace of u on $\partial\big(B_r \cap (\mathbb{R} \times \{0\})\big)$.

[3] The conclusion $\mathcal{H}^1((\partial^* V)\Delta\partial V) = 0$ for a piecewise Lipschitz set $V = \cup_{i=1}^m A_i$, with A_i Lipschitz open sets, can be proven by induction on m, using also the following fact: If B_1 and B_2 are open sets with $\mathcal{H}^1((\partial^* B_i)\Delta\partial B_i) = 0$ for $i = 1, 2$, then $B = B_1 \cup B_2$ satisfies $\mathcal{H}^1((\partial^* B)\Delta\partial B) = 0$. This follows by the identity $\partial(B_1 \cup B_2) = ((\partial B_1) \setminus \overline{B_2}) \cup ((\partial B_2) \setminus \overline{B_1}) \cup ((\partial B_1) \cap \partial B_2)$, which shows that $\partial(B_1 \cup B_2)$ is a $\mathcal{H}^1$-measurable subset of $\partial B_1 \cup \partial B_2$.

Then, for any $i \in \{1, \dots, N\}$,

$$\mathcal{A}(\psi_i^\star; B) - |G_i^\star| - \mathcal{A}(\psi_i^\star; B \setminus \overline{\Omega}) \le \mathcal{A}(\psi; B) - |G| - \mathcal{A}(\psi; B \setminus \overline{\Omega}), \tag{3.8}$$

where

$$G_i^\star := \bigcup_{j \neq i} F_j \cup \mathrm{conv}(F_i) \quad \textit{and} \quad \psi_i^\star := \begin{cases} 0 & \text{in } \mathrm{conv}(F_i) \\ \psi & \text{otherwise.} \end{cases} \tag{3.9}$$

Further, inequality in (3.8) *is strict unless* $\psi = \psi_i^\star$ *a.e..*

Proof. Fix $i \in \{1, \dots, N\}$. By the convexity of Ω, we have $\psi = \psi_i^\star$ in $B \setminus \overline{\Omega}$, hence it suffices to show that

$$\mathcal{A}(\psi_i^\star; B) - |G_i^\star| \le \mathcal{A}(\psi; B) - |G|.$$

We start by observing that we may assume F_i to be simply connected. Indeed, if not, we can replace it with the set obtained by filling the holes of F_i, and by setting ψ equal to zero in the holes[4]. This procedure reduces the energy since F_i is piecewise Lipschitz, and any hole H of it has the property that the external trace of $\psi \llcorner (B \setminus H)$ on ∂H vanishes.

We have that $(\partial \mathrm{conv}(F_i)) \setminus \partial F_i$ is a countable union of segments. We will next modify ψ by iterating at most countably many operations, setting $\psi = 0$ in the region between each of these segments and ∂F_i.

Step 1: Base case. Let l be one of such segments, and U be the open region enclosed between ∂F_i and l. We define $\psi' \in BV(\Omega)$ as

$$\psi' := \begin{cases} 0 & \text{in } U \\ \psi & \text{otherwise}. \end{cases}$$

We claim that

$$\mathcal{A}(\psi'; B) - |G'| \le \mathcal{A}(\psi; B) - |G|, \tag{3.10}$$

with strict inequality unless $\psi' = \psi$ a.e., where $G' := G \cup \overline{U}$. To prove the claim we introduce the sets

$$H := \mathrm{int}(F_i \cup U), \qquad V := U \cap (\cup_{j \neq i} F_j).$$

Note that H is a piecewise Lipschitz set. By construction

$$|G'| = |H| + |\cup_{j \neq i} F_j| - |V|,$$

and (3.10) will follow if we show that

$$\mathcal{A}(\psi'; B) - |H| \le \mathcal{A}(\psi; B) - |\cup_j F_j| + |\cup_{j \neq i} F_j| - |V| = \mathcal{A}(\psi; B) - |F_i \cup V|,$$

with strict inequality unless $\psi' = \psi$ a.e. in Ω. Since $|H| = |F_i \cup V| + |U \setminus V|$, this can also be written as

$$\mathcal{A}(\psi'; B) \le \mathcal{A}(\psi; B) + |U \setminus V|.$$

In turn $\mathcal{A}(\psi'; B) = \mathcal{A}(\psi'; \overline{U}) + \mathcal{A}(\psi'; B \setminus \overline{U})$ (and similarly for ψ), so we have reduced ourselves with proving

$$\mathcal{A}(\psi'; \overline{U}) \le \mathcal{A}(\psi; \overline{U}) + |U \setminus V|. \tag{3.11}$$

[4] If H is a hole of F_i and it happens that $F_j \subset H$ for some $j \neq i$, we redefine F_i as the union of it with H, and set $F_j = \emptyset$. This procedure does not invalidate the following argument.

In view of the definition of ψ' which is zero in U, we have[5] $\mathcal{A}(\psi';\overline{U}) = \int_l |\psi^+| d\mathcal{H}^1 + |U|$ (ψ^+ denoting the trace of $\psi \llcorner (B \setminus U)$ on the segment l) implying that (3.11) is equivalent to

$$\int_l |\psi^+| d\mathcal{H}^1 \leq \mathcal{A}(\psi;\overline{U}) - |V|\,.$$

Finally, if ψ_U denotes the trace of $\psi \llcorner U$ on l, we write $\mathcal{A}(\psi;\overline{U}) = \mathcal{A}(\psi;\overline{U}\setminus l) + \int_l |\psi^+ - \psi_U| d\mathcal{H}^1$, and the expression above is equivalent to

$$\int_l |\psi^+| d\mathcal{H}^1 \leq \int_l |\psi^+ - \psi_U| d\mathcal{H}^1 + \mathcal{A}(\psi;\overline{U}\setminus l) - |V|\,. \tag{3.12}$$

We now prove (3.12). Fix a Cartesian coordinate system (x_1, x_2) so that l belongs to the x_1-axis and U belongs to the half-plane $\{x_2 > 0\}$. Let u be an extension of ψ in $\mathbb{R}\times(0,+\infty)$ which vanishes outside U. Lemma 3.3, applied to u with the ball $B_r = B$, implies

$$\int_l |\psi_U| d\mathcal{H}^1 = \int_{\{x_2=0\}\cap B} u\, d\mathcal{H}^1 \leq \mathcal{A}(u; B\cap(\mathbb{R}\times(0,+\infty))) - |E_B| \leq \mathcal{A}(\psi;\overline{U}\setminus l) - |V|.$$

Here the last inequality follows by recalling that ψ (and thus u) vanishes on V. From this and the inequality $\int_l |\psi^+| d\mathcal{H}^1 \leq \int_l |\psi^+ - \psi_U| d\mathcal{H}^1 + \int_l |\psi_U| d\mathcal{H}^1$ the proof of (3.12) is achieved, so that (3.10) follows. Notice that in applying Lemma 3.3 the inequality holds strict when ψ' does not coincide with ψ a.e..

Step 2: Iterative case. We set $\partial(\mathrm{conv}(F_i)) \setminus \partial F_i = \cup_{j=1}^{\infty} l_j$ with l_j mutually disjoint segments. For every $h \geq 1$ we define the pair (ψ_h, G_h) as follows:

- if $h = 1$
$$\psi_1 := \begin{cases} 0 & \text{in } U_1 \\ \psi & \text{otherwise,} \end{cases} \qquad \text{and} \quad G_1 := G \cup \overline{U}_1\,,$$
where U_1 is the open region enclosed between ∂F_i and l_1. We also define $H_1 := \mathrm{int}(\overline{F_i \cup U_1})$;
- if $h \geq 2$
$$\psi_h := \begin{cases} 0 & \text{in } U_h \\ \psi_{h-1} & \text{otherwise,} \end{cases} \qquad \text{and} \quad G_h := G_{h-1} \cup \overline{U}_h\,,$$
where U_h is the open region enclosed between ∂H_{h-1} and l_h and $H_h := \mathrm{int}(\overline{H_{h-1} \cup U_h})$.

By construction each H_h is simply connected and piecewise Lipschitz, $H_h \subset H_{h+1}$, $G_h \subset G_{h+1} \subset \overline{\Omega}$ for every $h \geq 1$, and moreover

$$\lim_{h\to+\infty} |H_h| = |\mathrm{conv}(F_i)|\,, \qquad \lim_{h\to+\infty} |G_h| = |G_i^\star|\,, \tag{3.13}$$

where $G_i^\star := \cup_{h=1}^{\infty} G_h = \cup_{j\neq i} F_j \cup \mathrm{conv}(F_i)$. For any $h \geq 2$ we apply step 1, and after h iterations we get

$$\mathcal{A}(\psi_h; B) - |G_h| \leq \mathcal{A}(\psi_{h-1}; B) - |G_{h-1}| \leq \cdots \leq \mathcal{A}(\psi_1; B) - |G_1| \leq \mathcal{A}(\psi; B) - |G|\,. \tag{3.14}$$

In particular,

$$|D\psi_h|(B) \leq \mathcal{A}(\psi_h; B) \leq \mathcal{A}(\psi; B) + |G_h \setminus G| \leq \mathcal{A}(\psi; B) + |\Omega \setminus G|\,,$$

[5] We use the precise integral formula (3.6) thanks to the boundary regularity of U, where we have $\partial U \setminus l \subset \partial F_i$.

for all $h \geq 1$, and then we easily see that, up to a subsequence, $\psi_h \overset{*}{\rightharpoonup} \psi_i^\star$ in $BV(B)$, where ψ_i^* is defined as in (3.9). Now the lower semicontinuity of $\mathcal{A}(\cdot; B)$ yields

$$\liminf_{h\to+\infty} \mathcal{A}(\psi_h, B) \geq \mathcal{A}(\psi_i^\star; B)\,. \tag{3.15}$$

Finally, gathering together (3.13)-(3.15) we infer

$$\mathcal{A}(\psi_i^\star; B) - |G_i^\star| \leq \liminf_{h\to+\infty} \mathcal{A}(\psi_h; B) - \lim_{h\to+\infty} |G_h| \leq \mathcal{A}(\psi; B) - |G|\,.$$

Again we have strict inequality unless $\psi_h = \psi_{h-1}$ for all h a.e. in Ω. This concludes the proof. □

Lemma 3.5 (**Reduction of energy, II**). *Let $N \geq 1$, $F_1, \ldots, F_N, G$ and ψ be as in Lemma 3.4. Then there exist $\tilde{n} \in \{1, \ldots, N\}$ and mutually disjoint closed convex sets $\widetilde{F}_1, \ldots, \widetilde{F}_{\tilde{n}} \subset \overline{\Omega}$ with nonempty interior such that*

$$G \subset \bigcup_{i=1}^{\tilde{n}} \widetilde{F}_i =: G^\star\,, \tag{3.16}$$

and

$$\mathcal{A}(\psi^\star; B) - |G^\star| - \mathcal{A}(\psi^\star; B \setminus \overline{\Omega}) \leq \mathcal{A}(\psi; B) - |G| - \mathcal{A}(\psi; B \setminus \overline{\Omega})\,, \tag{3.17}$$

where

$$\psi^\star := \begin{cases} 0 & \textit{in } G^\star \\ \psi & \textit{otherwise}\,. \end{cases} \tag{3.18}$$

Finally, inequality in (3.17) *is strict unless $\psi = \psi^\star$ a.e..*

Proof. Base case: If $N = 1$ we set $\widetilde{F}_1 := \operatorname{conv}(F_1) = G^\star$ and the thesis follows by Lemma 3.4. Suppose $N > 1$. We take the sets

$$\operatorname{conv}(F_1)\,,\ F_2\,,\ \ldots\,,\ F_N \quad \text{and} \quad G_1^\star := \bigcup_{i=2}^{N} F_i \cup \operatorname{conv}(F_1)\,, \tag{3.19}$$

and let

$$\psi_1^\star := \begin{cases} 0 & \text{in } G_1^\star \\ \psi & \text{otherwise}\,. \end{cases}$$

Then by Lemma 3.4,

$$\mathcal{A}(\psi_1^\star; B) - |G_1^\star| - \mathcal{A}(\psi_1^\star; B \setminus \overline{\Omega}) \leq \mathcal{A}(\psi; B) - |G| - \mathcal{A}(\psi; B \setminus \overline{\Omega})\,, \tag{3.20}$$

with strict inequality unless $\psi_1^\star = \psi$ a.e..

Iterative case: Let m, k, h be natural numbers such that $1 \leq k \leq m \leq N$, $1 < h \leq 2N - 1$, and let $F_{1,h}, \ldots, F_{m,h}$ be closed subsets of $\overline{\Omega}$ with nonempty interior that satisfy the following property:

(1) $F_{1,h}, \ldots, F_{k,h}$ are convex;

(2) $F_{i,h} \cap F_{j,h} = \emptyset$ for all $i, j \neq k$, $i \neq j$, $i, j = 1, \ldots, m$.

Notice that for $h = 2$ and $m = N$ the sets

$$F_{1,2} := \operatorname{conv}(F_1)\,,\ F_{2,2} := F_2\,,\ \ldots\,,\ F_{N,2} := F_N\,,$$

satisfy (1), (2) with $k = 1$ by the base case (so the iterative step can be applied to these sets).

We then set $I_{k,h} := \{1 \leq i \leq m,\ i \neq k\colon F_{i,h} \cap F_{k,h} \neq \emptyset\}$. If $I_{k,h} = \emptyset$ and $k = m$ we are done, otherwise we construct a new family of sets using the following algorithm, distinguishing the two cases (a) and (b):

(a) if $I_{k,h} = \emptyset$ and $k < m$ we define the sets

$$F_{i,h+1} := \begin{cases} F_{i,h} & \text{for } i \neq k+1 \\ \mathrm{conv}(F_{k+1,h}) & \text{for } i = k+1\,, \end{cases} \qquad \text{for } i = 1, \dots, m,$$

and $G^\star_{h+1} := \cup_{i=1}^m F_{i,h+1}$;

(b) if $I_{k,h} \neq \emptyset$, up to relabelling the indices, we may assume that

$$I_{k,h} = \{k_{h,1} \le i \le k_{h,2}\} \setminus \{k\},$$

for some $k_{h,1} \neq k_{h,2}$ with $1 \le k_{h,1} \le k \le k_{h,2} \le m$, so that

$$\{1, \dots, m\} \setminus \{k\} \setminus I_{k,h} = \{1 \le i \le k_{h,1} - 1\} \cup \{k_{h,2} + 1 \le i \le m\}.$$

Note that if $k_{h,1} = 1$ then $\{1 \le i \le k_{h,1} - 1\} = \emptyset$, and similarly if $k_{h,2} = m$ then $\{k_{h,2} + 1 \le i \le m\} = \emptyset$. Then we set

$$F_{i,h+1} := \begin{cases} F_{i,h} & \text{for } i = 1, \dots, k_{h,1} - 1 \\ \mathrm{conv}(F_{k,h} \cup (\cup_{j \in I_{k,h}} F_{j,h})) & \text{for } i = k_{h,1} \\ F_{i+k_{h,2}-k_{h,1},h} & \text{for } i = k_{h,1} + 1, \dots, m - k_{h,2} + k_{h,1}\,, \end{cases}$$

and $G^\star_{h+1} := \cup_{i=1}^{m-k_{h,2}+k_{h,1}} F_{i,h+1}$.

In both cases (a) and (b) a direct check shows that the produced sets satisfy properties (1) and (2) with m, $k+1$, $h+1$ and $m - k_{h,2} + k_{h,1}$, $k_{h,1}$, $h+1$ respectively.

In both cases we define also the function

$$\psi^\star_{h+1} := \begin{cases} 0 & \text{in } G^\star_{h+1} \\ \psi^\star_h & \text{otherwise}\,. \end{cases}$$

Then, by induction, for all $1 < h \le 2N - 1$ we use Lemma 3.4, and in view of (3.20) we infer

$$\begin{aligned} \mathcal{A}(\psi^\star_{h+1}; B) - |G^\star_{h+1}| - \mathcal{A}(\psi^\star_{h+1}; B \setminus \overline{\Omega}) \le& \mathcal{A}(\psi^\star_h; B) - |G^\star_h| - \mathcal{A}(\psi^\star_h; B \setminus \overline{\Omega}) \\ \le& \mathcal{A}(\psi; B) - |G| - \mathcal{A}(\psi; B \setminus \overline{\Omega})\,, \end{aligned}$$

with strict inequality unless $\psi^\star_{h+1} = \psi^\star_h$ for all h a.e. in Ω.

Conclusion. If $N = 1$ it is sufficient to apply the base case. If instead $N > 1$ after a finite number $h^\star \le 2N - 1$ of iterations we obtain a collections of mutually disjoint and closed convex sets with nonempty interiors $F_1 := F_{1,h^\star}, \dots, F_{\tilde{n}} := F_{\tilde{n},h^\star}$ with $1 \le \tilde{n} \le N$ such that

$$G \subset \cup_{i=1}^{\tilde{n}} F_i =: G^\star\,,$$

and

$$\mathcal{A}(\psi^\star; B) - |G^\star| - \mathcal{A}(\psi^\star; B \setminus \overline{\Omega}) \le \mathcal{A}(\psi; B) - |G| - \mathcal{A}(\psi; B \setminus \overline{\Omega})\,,$$

with

$$\psi^\star := \psi^\star_{h^\star} = \begin{cases} 0 & \text{in } G^\star \\ \psi & \text{otherwise}\,, \end{cases}$$

with strict inequality unless $\psi^\star = \psi$ a.e.. □

Proof of Theorem 3.1. We start by observing that (2.5) readily follows from (3.1). Indeed, this implies

$$\inf_{(\sigma,\psi)\in\mathcal{W}_{\rm conv}} \mathcal{F}(\sigma,\psi) \le \inf_{(\sigma,\psi)\in\mathcal{W}} \mathcal{F}(\sigma,\psi)\,.$$

Whereas from (2.7) it follows

$$\inf_{(\sigma,\psi)\in\mathcal{W}} \mathcal{F}(\sigma,\psi) \le \inf_{(\sigma,\psi)\in\mathcal{W}_{\rm conv}} \mathcal{F}(\sigma,\psi).$$

Thus, we only need to show (3.1). Take a pair $(\bar\sigma,\bar\psi)\in\mathcal{W}$; we suitably modify $(\bar\sigma,\bar\psi)$ into a new pair $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ such that every connected component of $E(\sigma)$ is convex and

$$\mathcal{F}(\sigma,\psi)\le\mathcal{F}(\bar\sigma,\bar\psi),$$

and this will conclude the proof. Once again we notice that strict inequality holds unless $\psi=\bar\psi$ a.e..

Let $E(\bar\sigma_1),\dots,E(\bar\sigma_n)$ be the closed sets with mutually disjoint interiors corresponding to $\bar\sigma$ (as in (ii) before (1.2)) and let $G:=\cup_{i=1}^n E(\bar\sigma_i)$. Let $F_1,\dots,F_N$ be the (closure of the) connected components of G, $N\le n$, which are piecewise Lipschitz. By Lemma 3.5 there exist $1\le\tilde n\le N$ and $\widetilde F_1,\dots,\widetilde F_{\tilde n}\subset\overline{\Omega}$ mutually disjoint closed and convex with nonempty interior satisfying (3.16), (3.17) and (3.18). Therefore, by construction, for every $i=1,\dots,n$, q_i and p_{i+1} belong to $\widetilde F_j$ for a unique $j\in\{1,\dots,\tilde n\}$. For every $j=1,\dots,\tilde n$ we denote by

$$q_{j_1},p_{j_1+1},\dots,q_{j_{n_j}},p_{j_{n_j}+1},$$

the ones that belong to $\widetilde F_j$. Then we conclude by taking $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ with $\sigma:=(\sigma_1,\dots,\sigma_n)$ and

$$\sigma_{j_k}([0,1])=\begin{cases}\overline{q_{j_k}p_{j_k+1}} & \text{for}\quad k=1,\dots,n_j-1\\ \partial\widetilde F_j\setminus\left(\cup_{h=1}^{n_j}\partial^0_{j_h}\Omega\right)\cup\left(\cup_{h=1}^{n_j-1}\overline{q_{j_h}p_{j_h+1}}\right) & \text{for}\quad k=n_j,\end{cases}$$

for every $j=1,\dots,\tilde n$ and $\psi:=\psi^\star$. □

4 Existence of minimizers of $\mathcal{F}$ in $\mathcal{W}_{\rm conv}$

The main result of this section reads as follows.

Theorem 4.1 (Existence of a minimizer of $\mathcal{F}$ in $\mathcal{W}_{\rm conv}$). *Let $\mathcal{F}$ and $\mathcal{W}_{\rm conv}$ be as in* (1.7) *and* (2.6) *respectively. Then there is $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ such that*

$$\mathcal{F}(\sigma,\psi)=\min_{(s,\zeta)\in\mathcal{W}_{\rm conv}}\mathcal{F}(s,\zeta). \tag{4.1}$$

Moreover, every minimizer (σ,ψ) of $\mathcal{F}$ in $\mathcal{W}_{\rm conv}$ is such that every connected component of $E(\sigma)$ is convex.

As a direct consequence of Theorem 3.1 and Theorem (4.1)we have:

Corollary 4.2. *Let $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ be a minimizer as in Theorem 4.1. Then (σ,ψ) is also a minimizer of $\mathcal{F}$ in the class $\mathcal{W}$. Moreover, every minimizer (σ,ψ) of $\mathcal{F}$ in $\mathcal{W}$ is such that every connected component of $E(\sigma)$ is convex.*

We prove Theorem 4.1 using the direct method. To this aim we need to introduce a notion of convergence in $\mathcal{W}_{\rm conv}$.

Definition 4.3 (**Convergence in $\mathcal{W}_{\rm conv}$**). *We say that the sequence $((\sigma)_k, \psi_k)_k \subset \mathcal{W}_{\rm conv}$, with $(\sigma)_k = ((\sigma_1)_k, \dots, (\sigma_n)_k)$, converges to $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ if:*

(a) *$((\sigma_i)_k)_\sharp[\![0,1]\!]$ converges to $(\sigma_i)_\sharp[\![0,1]\!]$ in the sense of currents in $\mathcal{D}_1(\mathbb{R}^2)$, for all $i = 1, \dots, n$;*

(b) *$(\psi_k)_k$ converges to ψ weakly* in $BV(\Omega)$, i.e., $\psi_k \to \psi$ in $L^1(\Omega)$ and $D\psi_k \rightharpoonup D\psi$ weakly* in Ω as measures as $k \to +\infty$.*

In Definition 4.3 $(\sigma_i)_\sharp[\![0,1]\!]$ denotes the push-forward by σ_i of the 1-current given by integration on the segment $[0,1]$, standardly oriented (see [34] for details).

In the next lemma we show a compactness property of $\mathcal{W}_{\rm conv}$. In particular given $(\sigma)_k \subset \Sigma_{\rm conv}$ with equibounded energies, for all $i = 1, \dots, n$, up to subsequences, a (not-relabelled) reparametrization of $(\sigma_i)_k$ converges uniformly to some $\widehat{\sigma}_i$, and there is a parametrization σ_i of the support of $(\widehat{\sigma}_i)_\sharp[\![0,1]\!]$ such that $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma_{\rm conv}$. This, together with a uniform bound on the lengths of $(\sigma_i)_k$, implies the convergence of the push-forwards as currents. Notice that $(\sigma_i)_\sharp[\![0,1]\!]$ is invariant under reparametrization of σ_i.

Lemma 4.4 (**Compactness of $\mathcal{W}_{\rm conv}$**). *Let $\big((\sigma)_k, \psi_k\big)_k \subset \mathcal{W}_{\rm conv}$ be a sequence with $\sup_k \mathcal{F}((\sigma)_k, \psi_k) < +\infty$. Then $\big((\sigma)_k, \psi_k\big)_k$ admits a subsequence converging to an element of $\mathcal{W}_{\rm conv}$.*

Proof. We divide the proof in two steps.

Step 1: Compactness of $(\sigma)_k$. For simplicity we use the notation $\sigma_{ik} = (\sigma_i)_k$ for every $k \in \mathbb{N}$ and $i \in \{1, \dots, n\}$. By condition (P) in Remark 2.3, for every $k \in \mathbb{N}$ and $i \in \{1, \dots, n\}$ there exists a non-decreasing function $\theta_{ik} \colon [0,1] \to \mathbb{R}$, $\theta_{ik}(1) - \theta_{ik}(0) \leq 2\pi$, such that, for a reparametrization $\widehat{\sigma}_{ik}$ of σ_{ik},

$$\widehat{\sigma}_{ik}(t) = q_i + \ell(\sigma_{ik}) \int_0^t \gamma_{ik}(s) ds, \quad \gamma_{ik}(t) := (\cos\theta_{ik}(t)\,, \sin\theta_{ik}(t)) \quad \forall t \in [0,1],$$

and with $\widehat{\sigma}_{ik}(1) = p_{i+1}$. We observe that

$$\ell(\sigma_{ik}) = \int_0^1 |\sigma'_{ik}(t)| dt \leq \mathcal{H}^1(\partial\Omega), \tag{4.2}$$

since the orthogonal projection $\Pi_{ki} \colon \partial\Omega \setminus \partial_i^0\Omega \to E(\sigma_{ik})$ is a contraction and $\mathcal{H}^1(\partial\Omega \setminus \partial_i^0\Omega) \leq \mathcal{H}^1(\partial\Omega)$. Hence, up to a (not relabelled) subsequence, $\ell(\sigma_{ik}) \to m_i \in \mathbb{R}^+$ as $k \to +\infty$. The number m_i is positive since, for all k and i, we have $\ell(\sigma_{ik}) \geq |q_i - p_{i+1}| > 0$. Moreover

$$\int_0^1 |\theta'_{ik}(t)| dt = \int_0^1 \theta'_{ik}(t) dt \leq 2\pi;$$

hence, up to a not relabelled subsequence, $\theta_{ki} \overset{*}{\rightharpoonup} \theta_i$ in $BV(0,1)$ and θ_i is non-decreasing with $\theta_i(1) - \theta_i(0) \leq 2\pi$. Furthermore $\gamma_{ik} \overset{*}{\rightharpoonup} \gamma_i$ in $BV((0,1);\mathbb{R}^2)$ with $\gamma_i(t) = (\cos(\theta_i(t))\,, \sin(\theta_i(t)))$. Thus, arguing as in (8.2) and using (4.2), we get $\widehat{\sigma}_{ik} \to \widehat{\sigma}_i$ in $W^{1,1}([0,1];\mathbb{R}^2)$, where

$$\widehat{\sigma}_i(t) := q_i + m_i \int_0^t \gamma_i(s) ds = q_i + \ell(\sigma_i) \int_0^t \gamma_i(s) ds. \tag{4.3}$$

Thus $\lim_{k\to+\infty} \widehat{\sigma}_{ik} = \widehat{\sigma}_i$ uniformly, hence we also conclude that $\widehat{\sigma}_i$ takes values in $\overline{\Omega}$. Since by (H3)

$$d_H(E(\sigma_{ik}), E(\sigma_{ih})) = d_H(\partial E(\sigma_{ik}), \partial E(\sigma_{ih})) \leq \|\sigma_{ik} - \sigma_{ih}\|_{L^\infty}$$

for all $h,k>0$, the uniform convergence of $(\widehat{\sigma}_{ik})$ implies that $(E(\sigma_{ik}))_k$ is a Cauchy sequence with respect to the Hausdorff distance. Hence, by (H2) there is $K_i \in \mathcal{K}$ such that $d_H(E(\sigma_{ik}), K_i) \to 0$, and K_i is also convex by (H5).

We now show that $\widehat{\sigma}_i$ is injective, unless a pathological case that might happen only if $\partial_i^0\Omega$ is a straight segment[6]. Notice that, if $\partial_i^0\Omega$ is not straight, K_i must have nonempty interior, since it contains the region enclosed between $\overline{q_i p_{i+1}}$ and $\partial_i^0\Omega$.

First observe that $\widehat{\sigma}_i([0,1]) \subseteq \partial K_i$. Assume by contradiction that $\widehat{\sigma}_i(t_1) = \widehat{\sigma}_i(t_2)$ for some $t_1, t_2 \in [0,1]$, $t_1 < t_2$. Since K_i is convex, the curve $\widehat{\sigma}_i \llcorner [t_1,t_2]$ is closed and its image is contained in ∂K_i. If $\widehat{\sigma}_i \llcorner [t_1,t_2]$ is constant and equals to $\widehat{\sigma}_i(t_1)$ we get a contradiction with (4.3) and the fact that $|\gamma_i| = 1$ a.e. in $[t_1,t_2]$. Hence there is a point $t_3 \in (t_1,t_2)$ such that $\widehat{\sigma}_i(t_3) \neq \widehat{\sigma}_i(t_1)$. Let ℓ_{13}^k and ℓ_{23}^k denote the half-lines in $\mathbb{R}^2$ with endpoint $\widehat{\sigma}_{ik}(t_3)$ and passing through $\widehat{\sigma}_{ik}(t_1)$ and $\widehat{\sigma}_{ik}(t_2)$, respectively. Since $E(\sigma_{ik})$ is convex, we infer that $\widehat{\sigma}_{ik}([0,t_1]) \cup \widehat{\sigma}_{ik}([t_2,1])$ is contained in the closed angular sector of $\mathbb{R}^2$ enclosed between ℓ_{13}^k and ℓ_{23}^k. Since $(\widehat{\sigma}_{ik})$ converges uniformly to $\widehat{\sigma}_i$, we have $\widehat{\sigma}_{ik}(t_j) \to \widehat{\sigma}_i(t_j)$ for $j = 1,2,3$, and $\widehat{\sigma}_i(t_3) \neq \widehat{\sigma}_i(t_1) = \widehat{\sigma}_i(t_2)$, so we easily conclude that $\widehat{\sigma}_{ik}([0,t_1]) \cup \widehat{\sigma}_{ik}([t_2,1])$ must be contained in the line passing through $\widehat{\sigma}_i(t_1) = \widehat{\sigma}_i(t_2)$ and $\widehat{\sigma}_i(t_3)$. As a consequence also K_i, being convex, is a segment contained in such a line, and has empty interior. Hence this leads to a contradiction if $\partial_i^0\Omega$ is not a straight segment. In this case we set $\sigma_i := \widehat{\sigma}_i$.

If instead $\partial_i^0\Omega$ is a straight segment, it might happen that the image of $\widehat{\sigma}_i$ is contained in a line, which must be the one passing through q_i and p_{i+1}. Since uniform convergence of $(\widehat{\sigma}_{ik})$ and the fact that $\ell(\sigma_{ik}) \to \ell(\widehat{\sigma}_i)$ imply that $(\widehat{\sigma}_{ik})_\sharp [\![0,1]\!] = (\sigma_{ik})_\sharp [\![0,1]\!] \to (\widehat{\sigma}_i)_\sharp [\![0,1]\!]$ as currents, and since $\partial(\sigma_{ik})_\sharp [\![0,1]\!] = \delta_{p_{i+1}} - \delta_{q_i}$ for all k, also $\partial(\widehat{\sigma}_i)_\sharp [\![0,1]\!] = \delta_{p_{i+1}} - \delta_{q_i}$. We conclude that $(\widehat{\sigma}_i)_\sharp [\![0,1]\!]$ is the integration over the segment $\overline{q_i p_{i+1}}$, and hence there is a Lipschitz injective curve σ_i which parametrizes $\overline{q_i p_{i+1}}$ such that

$$(\sigma_i)_\sharp [\![0,1]\!] = (\widehat{\sigma}_i)_\sharp [\![0,1]\!], \quad \text{and} \quad (\sigma_{ik})_\sharp [\![0,1]\!] \to (\sigma_i)_\sharp [\![0,1]\!].$$

We next show that $E(\sigma_i)$ is convex for any $i \in \{1,\dots,n\}$. If σ_i parametrizes the segment $\overline{q_i p_{i+1}}$ then $E(\sigma_i)$ is that segment, and there is nothing to prove. Assume then that $\sigma_i([0,1]) \neq \overline{q_i p_{i+1}}$. As shown above, the uniform limit σ_i of $(\widehat{\sigma}_{ik})$ is injective. We will show that $K_i = E(\sigma_i)$. Indeed, the uniform convergence of $(\widehat{\sigma}_{ik})$ yields

$$\lim_{k\to+\infty} d_H(\partial E(\sigma_{ik}), \partial E(\sigma_i)) = 0.$$

From (H3) we get

$$\begin{aligned} &d_H(\partial K_i, \partial E(\sigma_i)) \le d_H(\partial E(\sigma_{ik}), \partial K_i) + d_H(\partial E(\sigma_{ik}), \partial E(\sigma_i)) \\ =&d_H(E(\sigma_{ik}), K_i) + d_H(\partial E(\sigma_{ik}), \partial E(\sigma_i)) \to 0 \quad \text{as} \quad k \to +\infty. \end{aligned}$$

Thus $\partial K_i = \partial E(\sigma_i)$, so $K_i = E(\sigma_i)$ and the convexity is shown. This implies $\sigma \in \Sigma_{\mathrm{conv}}$, and since $(\sigma_{ik})_\sharp [\![0,1]\!] \to (\sigma_i)_\sharp [\![0,1]\!]$ as currents, the compactness of $(\sigma)_k$ is achieved.

Step 2: Compactness of (ψ_k). Setting $F_k = \cup_{i=1}^n E(\sigma_{ik})$ we have

$$|D\psi_k|(\Omega) \le \mathcal{A}(\psi_k;\Omega) \le \mathcal{F}((\sigma)_k, \psi_k) + |F_k| \le C < +\infty \qquad \forall k > 0,$$

where we used that $|F_k| \le |\Omega|$. Therefore, up to a subsequence, $\psi_k \overset{*}{\rightharpoonup} \psi$ in $BV(\Omega)$ and almost everywhere in Ω as $k \to +\infty$. To conclude it remains to show that $\psi = 0$ in $E(\sigma) = \cup_i E(\sigma_i)$. If for some $i \in \{1,\dots,n\}$ it happens that $\partial_i^0\Omega$ is straight and σ_i is the straight segment $\overline{q_i p_{i+1}}$, then

[6] This case corresponds to $E(\sigma_{ik})$ a possibly curvilinear triangle with vertices p_i, q_{i+1} and a third point $r_k \in \Omega$ converging to a point $r \in \partial\Omega$ which is on the same line as p_i, q_{i+1}, but outside the segment $\overline{p_i q_{i+1}}$.

$E(\sigma_i)$ has empty interior, and so there is nothing to prove. Otherwise, for the other indeces, by $\lim_{k\to+\infty} d_H(E(\sigma_{ik}), E(\sigma_i)) = 0$, property (H6) yields

$$\text{if} \quad x \in \text{int}(E(\sigma_i)) \quad \text{then} \quad x \in E(\sigma_{ik}) \quad \text{for } k \text{ sufficiently large,}$$

and hence, since $\lim_{k\to+\infty} \psi_k = \psi$ a.e. in Ω, we infer $\psi = 0$ a.e. in $E(\sigma)$. □

Remark 4.5. The previous proof shows a slightly stronger result: under the assumption of Lemma 4.4, for every $i = 1, \ldots, n$, we can find σ_i with $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma_{\text{conv}}$, $\widehat{\sigma}_i \in \text{Lip}([0,1]; \overline{\Omega})$, and reparametrizations $\widehat{\sigma}_{ik}$ of σ_{ik} such that

$$(\widehat{\sigma}_i)_\sharp[\![0,1]\!] = (\sigma_i)_\sharp[\![0,1]\!],$$

$$\widehat{\sigma}_{ik} \to \widehat{\sigma}_i \quad \text{uniformly on} \quad [0,1].$$

Moreover $(\sigma_{ik})_\sharp[\![0,1]\!]$ converges to $(\sigma_i)_\sharp[\![0,1]\!]$ in the sense of currents in $\mathcal{D}_1(\mathbb{R}^2)$. Finally $E(\sigma_{ik}) = E(\widehat{\sigma}_{ik})$ converges to $E(\widehat{\sigma}_i) = E(\sigma_i)$ in $(\mathcal{K}, d_H)$, and $\widehat{\sigma}_i = \sigma_i$ unless $\partial_i^0\Omega$ is a straight segment. In the latter case it might happen that $\widehat{\sigma}_i$ is not injective, but this happens only if $\widehat{\sigma}_i([0,1])$ is a segment, σ_i is a parametrization of $\overline{q_i p_{i+1}}$, and $E(\sigma_i) = \overline{q_i p_{i+1}}$.

Remark 4.6. We have also shown that if $(\widehat{\sigma}_{ik})$ converges uniformly to $\sigma_i \in \Sigma_{\text{conv}}$ for some $i = 1, \ldots, n$ then

$$\lim_{k\to+\infty} d_H(E(\sigma_{ik}), E(\sigma_i)) = 0.$$

Lemma 4.7 (Lower semicontinuity of $\mathcal{F}$ in $\mathcal{W}_{\text{conv}}$). *Let $\big((\sigma)_k, \psi_k\big)_k \subset \mathcal{W}_{\text{conv}}$ be a sequence converging to $(\sigma, \psi) \in \mathcal{W}_{\text{conv}}$. Then*

$$\mathcal{F}(\sigma, \psi) \le \liminf_{k\to+\infty} \mathcal{F}((\sigma)_k, \psi_k).$$

Proof. By a standard argument [29], the functional

$$\psi \in BV(\Omega) \mapsto \mathcal{A}(\psi; \Omega) + \int_{\partial\Omega} |\psi - \varphi| d\mathcal{H}^1$$

is $L^1(\Omega)$-lower semicontinuous. We now show that the map $\sigma \in \Sigma_{\text{conv}} \mapsto |E(\sigma)|$ is continuous. Let $(\sigma)_k \subset \Sigma_{\text{conv}}$, $\sigma \in \Sigma_{\text{conv}}$, and suppose that $((\sigma_i)_k)_\sharp[\![0,1]\!]$ converges to $(\sigma_i)_\sharp[\![0,1]\!]$ in $\mathcal{D}_1(\mathbb{R}^2)$ for all $i = 1, \ldots, n$ as $k \to +\infty$. Set $F_k := \cup_{i=1}^n E((\sigma_i)_k)$ and recall that $E(\sigma) = \cup_{i=1}^n E(\sigma_i)$. Thanks to Remark 4.5, we can always assume that there are reparametrizations $\widehat{\sigma}_{ik}$ of σ_{ik} such that $\widehat{\sigma}_{ik}$ converges uniformly to $\widehat{\sigma}_i$ with $(\widehat{\sigma}_i)_\sharp[\![0,1]\!] = (\sigma_i)_\sharp[\![0,1]\!]$. Let us suppose first that $\widehat{\sigma}_i$ is injective for all $i = 1, \ldots, n$, and so $\widehat{\sigma}_i = \sigma_i$. By Remark 4.6 $\lim_{h\to+\infty} d_H(E((\sigma_i)_k), E(\sigma_i)) = 0$ for all $i = 1, \ldots, n$ and therefore $d_H(F_k, E(\sigma)) =: \varepsilon_k \to 0^+$.

By invoking (H7) we have $E(\sigma) \subset (F_k)^+_{\varepsilon_k}$. Moreover, since $d_H((F_k)^+_{\varepsilon_k}, E(\sigma)) \le 2\varepsilon_k$, we get $(F_k)^+_{\varepsilon_k} \subseteq (E(\sigma))^+_{2\varepsilon_k}$, and so

$$|E(\sigma)| \le |(F_k)^+_{\varepsilon_k}| \le |(E(\sigma))^+_{2\varepsilon_k}|.$$

This implies

$$\limsup_{k\to+\infty} |F_k| \le \limsup_{k\to+\infty} |(F_k)^+_{\varepsilon_k}| \le |E(\sigma)|.$$

The converse inequality is a consequence of Fatou's Lemma and (H6), indeed

$$|E(\sigma)| \le \int_\Omega \liminf_{k\to+\infty} \chi_{F_k}(x)\, dx \le \liminf_{k\to+\infty} \int_\Omega \chi_{F_k}(x)\, dx = \liminf_{k\to+\infty} |F_k|.$$

If instead $\widehat{\sigma}_i$ is not injective for some i, we have $\widehat{\sigma}_i \in \text{Lip}([0,1]; \overline{\Omega})$ with $(\widehat{\sigma}_i)_\sharp[\![0,1]\!] = (\sigma_i)_\sharp[\![0,1]\!]$, and we are in the case that $E(\widehat{\sigma}_i)$ has empty interior (see Remark 4.5). Thus $E(\sigma_{ik}) = E(\widehat{\sigma}_{ik})$ converges to a segment $K_i \supsetneq E(\sigma_i)$ in the Hausdorff distance. Since $|K_i| = 0$, the thesis of the lemma follows along the same argument above replacing the symbol $E(\sigma_i)$ by K_i. □

Proof of Theorem 4.1. By Lemma 4.4 and Lemma 4.7 we can apply the direct method and conclude that there exists $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ such that (4.1) holds. Moreover, since $\mathcal{W}_{\rm conv} \subset \mathcal{W}$ by Theorem (3.1) we can choose (σ, ψ) such that every connected component of $E(\sigma)$ is convex.

□

5 Regularity of minimizers

In this section we investigate regularity properties of minimizers of $\mathcal{F}$. We recall that our boundary datum φ satisfies the conditions in (2.4), and $\widehat{\varphi} \in W^{1,1}(B)$ denotes a fixed extension of φ in the open ball $B \supset \overline{\Omega}$. The main result here reads as follows.

Theorem 5.1 (**Structure of minimizers**). *Every minimizer $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ of $\mathcal{F}$ in $\mathcal{W}$, namely*

$$\mathcal{F}(\sigma, \psi) = \min_{(s,\zeta)\in\mathcal{W}} \mathcal{F}(s, \zeta)\,,$$

satisfies the following properties:

1. *Each connected component of $E(\sigma)$ is convex;*

2. *ψ is positive and real analytic in $\Omega \setminus E(\sigma)$;*

3. *If $\partial_i^D\Omega$ is not a segment for some $i = 1, \dots, n$, then $\partial E(\sigma) \cap \partial_i^D\Omega = \emptyset$, ψ is continuous up to $\partial_i^D\Omega$, and $\psi = \varphi$ on $\partial_i^D\Omega$;*

4. *If $\partial_i^D\Omega$ is a segment for some $i = 1, \dots, n$, then either $\partial E(\sigma) \cap \partial_i^D\Omega = \emptyset$ or $\partial E(\sigma) \cap \partial_i^D\Omega = \partial_i^D\Omega$. In the first case ψ is continuous up to $\partial_i^D\Omega$ and $\psi = \varphi$ on $\partial_i^D\Omega$.*

Moreover, there is a minimizer $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ such that

5. *$\Omega \cap \partial E(\sigma)$ consists of a finite number of disjoint analytic curves, and ψ is continuous and null on $\partial E(\sigma) \setminus \partial^D\Omega$.*

Remark 5.2. If $\partial_i^D\Omega$ is a straight segment for some $i = 1, \dots, n$, nothing ensures that $\partial E(\sigma) \cap \partial_i^D\Omega = \emptyset$. However, if this intersection is nonempty, then necessarily $\partial_i^D\Omega \subset \partial E(\sigma)$. The prototypical example is given by the classical catenoid, as explained in the Introduction (see Figure 2) where, if the basis of the rectangle $\Omega = R_\ell$ is large enough, a solution ψ is identically zero, and $\partial^D\Omega \subset \partial E(\sigma)$. This also explains why in point 5. of Theorem 5.1 we write $\partial E(\sigma) \setminus \partial^D\Omega$.

A consequence of Theorem 5.1 is that a regular solution ψ belongs to $W^{1,1}(\Omega)$ and, if Ω is strictly convex, it also attains the boundary values. In particular Theorem 5.1 implies Theorem 1.1.

For the reader convenience we divide the proof in a number of steps.

Lemma 5.3. *Every minimizer $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ of $\mathcal{F}$ in $\mathcal{W}$ satisfies* 1., 2. *and $\psi = \varphi$ on $\partial^D\Omega \setminus \partial E(\sigma)$.*

Proof. Item 1. follows by Theorem 3.1. By [29, Theorem 14.13] we also have that ψ is real analytic in $\Omega \setminus E(\sigma)$. Together with the strong maximum principle [29, Theorem C.4], this implies that, in $\Omega \setminus E(\sigma)$, either $\psi > 0$ or $\psi \equiv 0$. On the other hand, since Ω is convex we can apply [29, Theorem 15.9] and get that ψ is continuous up to $\partial^D\Omega \setminus \partial E(\sigma)$; in particular

$$\psi = \varphi > 0 \quad \text{on } \partial^D\Omega \setminus \partial E(\sigma)\,, \tag{5.1}$$

which in turn implies $\psi > 0$ in $\Omega \setminus E(\sigma)$. □

Lemma 5.4. *Let $\Gamma \subset \mathbb{R}^3$ be a rectifiable, simple, closed and non-planar curve satisfying the following properties:*

(1) $\Gamma \subset \partial(F \times \mathbb{R})$ *for some closed bounded convex set $F \subset \mathbb{R}^2$ with nonempty interior;*

(2) Γ *is symmetric with respect to the horizontal plane $\mathbb{R}^2 \times \{0\}$;*

(3) *There are a nonempty relatively open arc $\widehat{pq} \subset \partial F$ with endpoints p and q, and $f \in C^0(\widehat{pq} \cup \{p,q\}; [0,+\infty))$ such that f is positive in $\widehat{pq}$ and*

$$\Gamma \cap \{x_3 \geq 0\} = \mathcal{G}_f \cup (\{p\} \times [0, f(p)]) \cup (\{q\} \times [0, f(q)]). \tag{5.2}$$

Let S be a solution to the classical Plateau problem for Γ, i.e., a disk-type surface minimizing area among all disk-type surfaces spanning Γ. Then:

(1′) $\beta_{p,q} := S \cap (\mathbb{R}^2 \times \{0\}) \subset F$ *is a simple analytic curve joining p and q with $\beta_{p,q} \cap \partial F = \{p,q\}$;*

(2′) S *is symmetric with respect to $\mathbb{R}^2 \times \{0\}$;*

(3′) *The surface $S^+ := S \cap \{x_3 \geq 0\}$ is the graph of a function $\widetilde{\psi} \in W^{1,1}(U_{p,q}) \cap C^0(\overline{U}_{p,q} \setminus \{p,q\})$, where $U_{p,q} \subset \mathrm{int}(F)$ is the open region enclosed between $\widehat{pq}$ and $\beta_{p,q}$. Moreover $\widetilde{\psi}$ is analytic in $U_{p,q}$, and if $f(p) = 0$ (resp. $f(q) = 0$) then ψ is also continuous at p (resp. at q);*

(4′) *The curve $\beta_{p,q}$ is contained in the closed convex hull of Γ, and $F \setminus U_{p,q}$ is convex.*

Remark 5.5. If the function f in (3) is such that $f(p) = f(q) = 0$ then (5.2) becomes $\Gamma \cap \{x_3 \geq 0\} = \mathcal{G}_f$. For later convenience we prove Lemma 5.4 under the more general assumption (3).

Proof. Even though several arguments are standard, we give the proof for completeness.

Step 1: $\beta_{p,q}$ is a simple analytic curve joining p and q.
Let $B_1 \subset \mathbb{R}^2$ be the open unit disk centred at the origin. Let $\Phi = (\Phi_1, \Phi_2, \Phi_3) \colon \overline{B}_1 \to S \subset \mathbb{R}^3$ be a parametrization of S with $\Phi(\partial B_1) = \Gamma$, that is harmonic, conformal, and therefore analytic in B_1, continuous up to ∂B_1. Further, by (1), Φ is an embedding (see [36] and also [22, page 343]).
By assumption (5.2) we have $\{w \in \partial B_1 \colon \Phi_3(w) = 0\} = \{\Phi^{-1}(p,0), \Phi^{-1}(q,0)\}$, so that Φ_3 changes sign only twice on ∂B_1. By applying Rado's lemma (see e.g. [22, Lemma 2, page 295]) to the harmonic function Φ_3 we deduce that $\nabla \Phi_3 \neq 0$ in B_1 and in particular $\{w \in B_1 \colon \Phi_3(w) > 0\}$ and $\{w \in B_1 \colon \Phi_3(w) < 0\}$ are connected, and $\{w \in B_1 \colon \Phi_3(w) = 0\}$ is a simple smooth curve in B_1 joining $\Phi^{-1}(p,0)$ and $\Phi^{-1}(q,0)$. By the injectivity of Φ we have that $S \cap (\mathbb{R}^2 \times \{0\}) = \Phi(\{w \in B_1 \colon \Phi_3(w) = 0\})$ is a simple analytic curve joining p and q.

Step 2: S is symmetric with respect to the horizontal plane $\mathbb{R}^2 \times \{0\}$.
By step 1 the sets $\{w \in \overline{B}_1 \colon \Phi_3(w) \geq 0\}$ and $\{w \in \overline{B}_1 \colon \Phi_3(w) \leq 0\}$ are simply connected and the two surfaces

$$S^+ := \Phi(\{w \in \overline{B}_1 \colon \Phi_3(w) \geq 0\}), \quad S^- := \Phi(\{w \in \overline{B}_1 \colon \Phi_3(w) \leq 0\})$$

have the topology of the disk. We assume without loss of generality that $\mathcal{H}^2(S^+) \leq \mathcal{H}^2(S^-)$. Let

$$\mathrm{Sym}(S^+) := \{(x', x_3) \colon (x', -x_3) \in S^+\}, \quad \widetilde{S} := S^+ \cup \mathrm{Sym}(S^+).$$

Then $\widetilde{S}$ is a symmetric surface of disk-type with $\partial \widetilde{S} = \Gamma$ and

$$\mathcal{H}^2(\widetilde{S}) = 2\mathcal{H}^2(S^+) \leq \mathcal{H}^2(S^+) + \mathcal{H}^2(S^-) = \mathcal{H}^2(S).$$

In particular $\widetilde{S}$ is a symmetric solution to the Plateau problem for Γ. Further $S = \widetilde{S}$ on a relatively open subset of S; hence, since they are real analytic surfaces, they must coincide, $S = \widetilde{S}$.

Step 3: S^+ is the graph of a function $\widetilde{\psi} \in W^{1,1}(U_{p,q}) \cap C^0(\overline{U}_{p,q} \setminus \{p,q\})$.
To show this it is enough to check the validity of the following

Claim: Every vertical plane Π is tangent to $\mathrm{int}(S)$ at most at one point.

We prove the claim arguing by contradiction as in [8, page 97], that is we assume there is a vertical plane Π tangent to $\mathrm{int}(S)$ at x' and x'' with $x' \neq x''$. We define the linear map $d_\nu(x) := (x - x') \cdot \nu$ with ν a unit normal to Π, so that clearly $\Pi = \{x \in \mathbb{R}^3 \colon d_\nu(x) = 0\}$. Since F is convex, $\Pi \cap (\partial F \times \{0\})$ contains at most two points. By properties (1)-(3) each of these points is either the projection on the horizontal plane of one or two points of $\Pi \cap \Gamma$, or the projection on the horizontal plane of one of the vertical segments $\{p\} \times [0, f(p)]$ and $\{q\} \times [0, f(q)]$. Hence $\Pi \cap \Gamma$ contains either: (a) at most two points and a segment, (b) two segments, (c) four points. Without loss of generality we restrict our analysis to the last case (the others are simpler to treat), namely we assume that there are four (clockwise ordered) points $w_1, \dots, w_4 \in \partial B_1$ such that $\Pi \cap \Gamma = \{\Phi(w_1), \dots, \Phi(w_4)\}$, that is $d_\nu \circ \Phi(w_i) = 0$ for $i = 1, \dots, 4$. We may also assume $d_\nu \circ \Phi > 0$ on $\widehat{w_1 w_2} \cup \widehat{w_3 w_4}$ and $d_\nu \circ \Phi < 0$ on $\widehat{w_2 w_3} \cup \widehat{w_4 w_1}$. Here $\widehat{w_i w_j}$ denotes the relatively open arc in ∂B_1 joining w_i and w_j for $i, j \in \{1, \dots, 4\}$. Notice that the function $d_\nu \circ \Phi \colon \overline{B}_1 \to \mathbb{R}$ is harmonic in B_1, continuous up to ∂B_1 and vanishes at $w_1, \dots, w_4$; hence, by classical arguments [37, Section 437] we see that the set $\{w \in B_1 \colon d_\nu \circ \Phi = 0\}$, in a neighbourhood of $w' := \Phi^{-1}(x')$ (respectively $w'' := \Phi^{-1}(x'')$), is the union of a number $m \geq 2$ of analytic curves crossing at w' (respectively w''). Thus near w' and w'' the set $\{w \in B_1 \colon d_\nu \circ \Phi(w) > 0\}$ is the union of at least two disjoint open regions $A_{1,1}$, $A_{1,2}$ and $A_{2,1}$, $A_{2,2}$ respectively such that $\overline{A}_{1,1} \cap \overline{A}_{1,2} = \{w'\}$, $\overline{A}_{2,1} \cap \overline{A}_{2,2} = \{w''\}$. Moreover each $A_{i,j}$ belongs either to the connected component of $\{w \in B_1 \colon d_\nu \circ \Phi(w) > 0\}$ containing $\widehat{w_1 w_2}$, or to the one containing $\widehat{w_3 w_4}$. Up to relabelling the indices we have two possibilities.

Case 1: $A_{1,1}$ and $A_{1,2}$ belong to the same connected component containing $\widehat{w_1 w_2}$. Then we can find two simple curves α_1, α_2 contained in $A_{1,1}$ and $A_{1,2}$ respectively, that connect w' to a point in $\widehat{w_1 w_2}$ and such that the region enclosed by the curve $\alpha_1 \cup \alpha_2$ intersects $\{w \in B_1 \colon d_\nu \circ \Phi(w) < 0\}$. Since $d_\nu \circ \Phi > 0$ on $\alpha_1 \cup \alpha_2$ by the maximum principle we have a contradiction.

Case 2: $A_{1,1}$ and $A_{2,1}$ belong to the connected component containing $\widehat{w_1 w_2}$ while $A_{1,2}$ and $A_{2,2}$ belong to the connected component containing $\widehat{w_3 w_4}$. Then we can find four simple curves $\alpha_{i,j}$ (with $i, j = 1, 2$) contained respectively in $A_{i,j}$, such that $\alpha_{1,1}$ (respectively $\alpha_{2,1}$) connects w' (respectively w'') to a point in $\widehat{w_1 w_2}$ and $\alpha_{1,2}$ (respectively $\alpha_{2,2}$) connects w' (respectively w'') to $\widehat{w_3 w_4}$. Then the region enclosed by the curve $\cup_{i,j} \alpha_{i,j}$ intersects $\{w \in B_1 \colon d_\nu \circ \Phi(w) < 0\}$, while $d_\nu \circ \Phi > 0$ on $\cup_{i,j} \alpha_{i,j}$, which again by the maximum principle gives a contradiction.

Thus the claim follows. Now, by step 2, the claim readily implies that $\mathrm{int}(S^+)$ has no points with vertical tangent plane and hence $\mathrm{int}(S^+)$ is the graph of a function $\widetilde{\psi}$ defined on $U_{p,q}$. Since $\widetilde{\psi}$ must minimize (locally) the area functional, it is also real analytic in $U_{p,q}$. Moreover, the claim also implies that $\widetilde{\psi}$ must vanish on $\beta_{p,q}$ and that it must attain the boundary values on $\widehat{pq}$. If f vanishes on p or q, then also the continuity of $\widetilde{\psi}$ at these points is achieved.

Step 4: The curve $\beta_{p,q}$ is contained in the closed convex hull of Γ, and the set $F \setminus U_{p,q}$ is convex.
Let $\pi(\Gamma) \subset \partial F$ be the projection of Γ onto the plane $\mathbb{R}^2 \times \{0\}$. By [22, Theorem 3, pag. 343] the relative interior of S is strictly contained in the convex hull of Γ, thus in particular the curve $\beta_{p,q}$ (respectively $\beta_{p,q} \setminus \{p,q\}$) is contained (respectively strictly contained) in the same half-plane (with respect to the line $\overline{pq}$) that contains $\pi(\Gamma)$.

Now, assume by contradiction that $F \setminus U_{p,q}$ is not convex. Then there are $p', q' \in \beta_{p,q}$ with the following properties:

- The open region U' enclosed by $\beta_{p,q}$ and the segment $\overline{p'q'}$ is nonempty and contained in $U_{p,q}$;
- the points p and q and the set U' lie on the same side with respect to the line containing $\overline{p'q'}$.

Let then $d_W : \mathbb{R}^3 \to \mathbb{R}$ be an affine function that vanishes on the vertical plane containing $\overline{p'q'}$ and is positive in the half-space W^+ containing p, q and U'. We now observe that $\Gamma \cap W^+$ is the union of two connected subcurves $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$, containing p and q respectively. As a consequence $\Phi^{-1}(\widehat{\Gamma}_1) = \widehat{w_1 w_2}$ and $\Phi^{-1}(\widehat{\Gamma}_2) = \widehat{w_3 w_4}$ for some $w_1, w_2, w_3, w_4 \in \partial B_1$ (clockwise oriented).
On the other hand since $d_W > 0$ on U' we can find $t' \in \partial U' \setminus \overline{p'q'}$ such that $d_W \circ \Phi(\Phi^{-1}(t')) = d_W(t') > 0$ with $\Phi^{-1}(t') \in B_1$. Once again by the harmonicity of $d_W \circ \Phi : \overline{B}_1 \to \mathbb{R}$ we deduce the existence of a curve $\alpha \subset \{w \in B_1 : d_W \circ \Phi(w) > 0\}$ joining $\Phi^{-1}(t')$ either to $\widehat{w_1 w_2}$ or $\widehat{w_3 w_4}$. Hence $\Phi(\alpha) \subset \Phi(B_1)$ is a curve joining t' either to $\widehat{\Gamma}_1$ or $\widehat{\Gamma}_2$, say $\widehat{\Gamma}_1$. This implies that the projection $\pi(\Phi(\alpha))$ of $\Phi(\alpha)$ onto the horizontal plane $\mathbb{R}^2 \times \{0\}$ is a curve contained in $U_{p,q}$ that connects t' to $\pi(\widehat{\Gamma}_1)$. So in particular, the curve $\pi(\Phi(\alpha))$ cannot be included in the half-space W^+. But this contradicts the fact that $\alpha \subset \{w \in B_1 : d_W \circ \Phi(w) > 0\}$ (this is because the values of d_W at a point x and $\pi(x)$ are the same). □

We need also the following technical results on the distance function d_F from a convex set F. Recall the definition of E^+_ε given in (H7) in the Appendix, for $\varepsilon > 0$ and $E \subset \mathbb{R}^2$.

Lemma 5.6. *Let $F \subset \mathbb{R}^2$ be bounded, closed and convex. Then $\Delta\,\mathrm{d}_F \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus F) \cap L^1(B \setminus F)$ for every ball B with $F \subset\subset B$.*

Proof. By [18, Theorem 3.6.7 pag. 75] it follows that $\mathrm{d}_F \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2 \setminus F)$, hence $\nabla^2 \mathrm{d}_F \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus F; \mathbb{R}^{2\times 2})$. Therefore we only have to check that $\Delta\,\mathrm{d}_F \in L^1(B \setminus F)$.
Let $\eta > 0$ be fixed sufficiently small. Select $(f_k)_{k\in\mathbb{N}} \subset C^1_c(\mathbb{R}^2; \mathbb{R}^2)$ such that $f_k \to \nabla\,\mathrm{d}_F$ in $W^{1,1}(B \setminus F^+_{\eta/2})$ as $k \to +\infty$. By the divergence theorem we have

$$\int_{B\setminus F^+_\eta} \mathrm{div} f_k\, dx = \int_{\partial B \cup \partial(F^+_\eta)} f_k \cdot \nu_\eta\, d\mathcal{H}^1, \tag{5.3}$$

with ν_η the outer unit normal to $\partial B \cup \partial(F^+_\eta)$. By taking the limit as $k \to \infty$ we get

$$\lim_{k\to+\infty} \int_{B\setminus F^+_\eta} \mathrm{div} f_k\, dx = \int_{B\setminus F^+_\eta} \Delta\,\mathrm{d}_F\, dx\,, \tag{5.4}$$

and

$$\lim_{k\to+\infty} \int_{\partial B \cup \partial(F^+_\eta)} f_k \cdot \nu_\eta\, d\mathcal{H}^1 = \int_{\partial B \cup \partial(F^+_\eta)} \nabla\,\mathrm{d}_F \cdot \nu_\eta\, d\mathcal{H}^1\,, \tag{5.5}$$

where (5.5) follows by using that $\partial(F^+_\eta)$ is of class $C^{1,1}$ and hence $f_k \llcorner (\partial B \cup \partial(F^+_\eta)) \to \nabla\,\mathrm{d}_F \llcorner (\partial B \cup \partial(F^+_\eta))$ in $L^1(\partial B \cup \partial(F^+_\eta))$. Since d_F is convex we have $\Delta\,\mathrm{d}_F \geq 0$ a.e. in $\mathbb{R}^2 \setminus F$, moreover $|\nabla\,\mathrm{d}_F| = 1$ in $\mathbb{R}^2 \setminus F$; then gathering together (5.3), (5.4), (5.5) we have

$$\int_{B\setminus F^+_\eta} |\Delta\,\mathrm{d}_F|\, dx = \int_{B\setminus F^+_\eta} \Delta\,\mathrm{d}_F\, dx = \int_{\partial B \cup \partial(F^+_\eta)} \nabla\,\mathrm{d}_F \cdot \nu_\eta\, d\mathcal{H}^1 \leq \mathcal{H}^1(\partial B \cup \partial(F^+_\eta)) \leq C,$$

with $C > 0$ independent of η. By the arbitrariness of $\eta > 0$, the thesis follows. □

Corollary 5.7. *Let $U \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. Let $F \subset \mathbb{R}^2$ be closed and convex such that $U \cap F = \emptyset$ and let $\psi \in W^{1,1}(U) \cap L^\infty(U) \cap C^0(U)$. Then the following formula holds:*

$$-\int_U \psi \Delta\,\mathrm{d}_F\, dx = \int_U \nabla\psi \cdot \nabla\,\mathrm{d}_F\, dx - \int_{\partial U} \psi\, \gamma\, d\mathcal{H}^1,$$

where γ denotes the normal trace of $\nabla\,\mathrm{d}_F$ on ∂U.

Proof. We have $|\nabla \mathrm{d}_F| = 1$ in $\mathbb{R}^2 \setminus F$, moreover since $U \cap F = \emptyset$, by Lemma 5.6 we deduce also $\Delta \mathrm{d}_F \in L^1(U)$. Therefore the thesis readily follows by applying [5, Theorem 1.9]. □

Remark 5.8. The normal trace γ of $\nabla \mathrm{d}_F$ on ∂F equals 1 $\mathcal{H}^1$-a.e. on ∂F. Indeed, from Corollary 5.7 we have that for all $\varphi \in C^1_c(\mathbb{R}^2;\mathbb{R}^2)$ it holds

$$\begin{aligned} -\int_{\mathbb{R}^2\setminus F^+_\eta} \varphi \Delta \mathrm{d}_F \,dx &= \int_{\mathbb{R}^2\setminus F^+_\eta} \nabla\varphi\cdot\nabla \mathrm{d}_F\,dx - \int_{\partial(F^+_\eta)} \varphi\,\gamma\,d\mathcal{H}^1 \\ &= \int_{\mathbb{R}^2\setminus F^+_\eta} \nabla\varphi\cdot\nabla \mathrm{d}_F\,dx - \int_{\partial(F^+_\eta)} \varphi\,d\mathcal{H}^1, \end{aligned}$$

where we have used that $\partial(F^+_\eta)$ being a level set of d_F, it results $\nabla \mathrm{d}_F = \nu_\eta$ on it. Letting $\eta \to 0$ and using that $\Delta \mathrm{d}_F \in L^1(B \setminus F)$ for all balls B, we infer

$$-\int_{\mathbb{R}^2\setminus F} \varphi \Delta \mathrm{d}_F \,dx = \int_{\mathbb{R}^2\setminus F} \nabla\varphi\cdot\nabla \mathrm{d}_F\,dx - \int_{\partial F} \varphi\,d\mathcal{H}^1.$$

By the arbitrariness of φ and again by Corollary 5.7, the claim follows.

Lemma 5.9. *Let $F \subset \overline{\Omega}$ be closed and convex with nonempty interior, and let $\delta > 0$. Let $\psi \in W^{1,1}((F^+_\delta \setminus F) \cap \Omega) \cap L^\infty((F^+_\delta \setminus F) \cap \Omega) \cap C^0((F^+_\delta \setminus F) \cap \Omega)$. Then*

$$\lim_{\varepsilon\to 0^+} \int_{\Omega\cap\partial(F^+_\varepsilon)} \psi\,d\mathcal{H}^1 = \int_{\Omega\cap\partial F} \psi\,d\mathcal{H}^1\,. \tag{5.6}$$

Proof. Let $\varepsilon \in (0,\delta)$ and $T_\varepsilon := (F^+_\varepsilon \setminus F) \cap \Omega$. Since $T_\varepsilon \cap F = \emptyset$, by Corollary 5.7 we get

$$-\int_{T_\varepsilon} \psi \Delta \mathrm{d}_F\,dx = \int_{T_\varepsilon} \nabla\psi\cdot\nabla \mathrm{d}_F\,dx - \int_{\partial T_\varepsilon} \psi\,\gamma\,d\mathcal{H}^1\,, \tag{5.7}$$

which by Remark 5.8 becomes

$$\begin{aligned} -\int_{T_\varepsilon} \psi \Delta \mathrm{d}_F\,dx = &\int_{T_\varepsilon} \nabla\psi\cdot\nabla \mathrm{d}_F\,dx \\ &+ \int_{\Omega\cap\partial F} \psi\,d\mathcal{H}^1 - \int_{\Omega\cap\partial(F^+_\varepsilon)} \psi\,d\mathcal{H}^1 - \int_{((F^+_\varepsilon)\setminus F)\cap\partial\Omega} \psi\,\gamma\,d\mathcal{H}^1\,. \end{aligned} \tag{5.8}$$

Now

$$\lim_{\varepsilon\to 0^+} \Big|\int_{T_\varepsilon} \nabla\psi\cdot\nabla \mathrm{d}_F\,dx\Big| \le \lim_{\varepsilon\to 0^+} \int_{T_\varepsilon} |\nabla\psi|\,dx = 0\,, \tag{5.9}$$

and

$$\lim_{\varepsilon\to 0^+} \Big|\int_{(F^+_\varepsilon\setminus F)\cap\partial\Omega} \psi\,\gamma\,d\mathcal{H}^1\Big| \le \lim_{\varepsilon\to 0^+} \int_{(F^+_\varepsilon\setminus F)\cap\partial\Omega} \psi\,d\mathcal{H}^1 = 0\,. \tag{5.10}$$

Moreover, since $\Delta \mathrm{d}_F \in L^1(T_\varepsilon)$ by Lemma 5.6, we deduce also

$$\lim_{\varepsilon\to 0^+} \Big|\int_{T_\varepsilon} -\psi\,\Delta \mathrm{d}_F\,dx\Big| \le \|\psi\|_{L^\infty} \lim_{\varepsilon\to 0^+} \int_{T_\varepsilon} |\Delta \mathrm{d}_F|\,dx = 0\,. \tag{5.11}$$

Finally gathering together (5.8)-(5.11) we infer (5.6). □

Remark 5.10. Let F, δ and ψ be as in Lemma 5.9. Let α be any connected component of $\Omega\cap\partial F$, and for every $0<\varepsilon<\delta$ let α_ε be the corresponding component of $\Omega\cap\partial(F^+_\varepsilon)$; namely, if π_F is the orthogonal projection onto the convex closed set F, setting

$$\widehat{\alpha}_\varepsilon := \{x\in\partial(F^+_\varepsilon) : \pi_F(x)\in\alpha\},$$

then one has $\alpha_\varepsilon := \widehat{\alpha}_\varepsilon\cap\Omega$. Arguing as in Lemma 5.9, we can show that

$$\lim_{\varepsilon\to 0^+}\int_{\alpha_\varepsilon}\psi\, d\mathcal{H}^1 = \int_\alpha \psi\, d\mathcal{H}^1 .$$

Lemma 5.11. *Let $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ in $\mathcal{W}$ as in Theorem 3.1. Then there is a minimizer $(\widehat{\sigma},\widehat{\psi})\in\mathcal{W}_{\rm conv}$ of $\mathcal{F}$ in $\mathcal{W}$ with the following properties:*

1. *$(\partial E(\widehat{\sigma}))\cap\partial\Omega = (\partial E(\sigma))\cap\partial\Omega$;*
2. *$\widehat{\psi}$ is continuous and null on $\Omega\cap\partial E(\widehat{\sigma})$.*

The second condition means essentially that $\widehat{\psi}$ vanishes on $\Omega\cap\partial E(\widehat{\sigma})$ when considering its trace from the side of $\Omega\setminus E(\widehat{\sigma})$.

Proof. We know by Lemma 5.3 that (σ,ψ), $\sigma=(\sigma_1,\dots,\sigma_n)$, satisfies the following properties:

- Each connected component of $E(\sigma)$ is convex;
- ψ is positive and real analytic in $\Omega\setminus E(\sigma)$;
- $\psi=\varphi$ on $\partial^D\Omega\setminus\partial E(\sigma)$.

In what follows we are going to modify (σ,ψ) near each arc of $\partial E(\sigma)$ using an iterative argument in order to get a new minimizer $(\widehat{\sigma},\widehat{\psi})\in\mathcal{W}_{\rm conv}$ that satisfies conditions 1 and 2. To this aim we denote by $F_1,\dots,F_k$ with $1\le k\le n$ the closure of the connected components of $E(\sigma)$ and set $\delta_0:=\min_{i\neq j}\operatorname{dist}(F_i,F_j)>0$. Moreover by the first property we deduce that $\Omega\cap\partial E(\sigma)$ is the union of an at most countable family of pairwise disjoint arcs with endpoints in $\partial\Omega$, i.e., $\Omega\cap\partial E(\sigma)=\bigcup_{i=1}^k\bigcup_{j=1}^\infty\alpha_{i,j}$, where $\alpha_{i,j}$ is a connected component of $\Omega\cap\partial F_i$ for $i\in\{1,\dots,k\}$, $j\ge 1$[7].

Step 1: Base case. Let α be one of the connected components of $\Omega\cap\partial F$, with $F:=F_i$ for some $i\in\{1,\dots,k\}$. In this step we construct a new minimizer $(\sigma^\alpha,\psi^\alpha)\in\mathcal{W}_{\rm conv}$ such that $(\partial E(\sigma^\alpha))\cap\partial\Omega=(\partial E(\sigma))\cap\partial\Omega$ and ψ^α is continuous and null on α', where $\alpha'\subset\Omega\cap\partial E(\sigma^\alpha)$ is a suitable curve that replaces α and has the same endpoints as α. For $\varepsilon\in(0,\delta_0/2)$ we define the stripe

$$\widehat{T}_\varepsilon(\alpha) := \{x\in\Omega\setminus F\colon \operatorname{dist}(x,\alpha)<\varepsilon\}\subset F^+_\varepsilon\setminus F\,,$$

and consider the planar curve α_ε in $\overline{\Omega}$ defined as in Remark 5.10. Let $T_\varepsilon(\alpha)$ be the connected component of $\widehat{T}_\varepsilon(\alpha)$ whose boundary contains α_ε. Let L_ε be defined as

$$L_\varepsilon := (\partial T_\varepsilon(\alpha))\cap\partial\Omega,$$

so that in particular $\partial T_\varepsilon(\alpha)=\alpha\cup\alpha_\varepsilon\cup L_\varepsilon$. Let $p,q\in\partial\Omega$ be the endpoints of α (and then also the endpoints of $\alpha_\varepsilon\cup L_\varepsilon$, which are independent of ε). We define the curves

$$\Gamma_\varepsilon := \Gamma^+_\varepsilon\cup\Gamma^-_\varepsilon\,,\quad \Gamma^+_\varepsilon := \mathcal{G}_{\psi\llcorner\alpha_\varepsilon}\cup\mathcal{G}_{\varphi\llcorner L_\varepsilon}\cup l^+\,,\quad \Gamma^-_\varepsilon := \mathcal{G}_{-\psi\llcorner\alpha_\varepsilon}\cup\mathcal{G}_{-\varphi\llcorner L_\varepsilon}\cup l^-\,,$$

[7]Notice that at this stage we do not have any information about the geometry of the set $(\partial E(\sigma))\cap\partial\Omega$, and $\Omega\cap\partial F_i$ could a priori be the union of countably many connected components.

where

$$l^+ := (\{p\} \times [0, \varphi(p)]) \cup (\{q\} \times [0, \varphi(q)])\,, \quad l^- := (\{p\} \times [-\varphi(p), 0]) \cup (\{q\} \times [-\varphi(q), 0])\,.$$

Observing that $L_\varepsilon \subset \partial^D\Omega \setminus \partial E(\sigma)$ and recalling that $\psi = \varphi$ on $\partial^D\Omega \setminus \partial E(\sigma)$ we deduce that Γ_ε is a closed non-planar curve in $\mathbb{R}^3$ that satisfies assumptions (1)-(3) of Lemma 5.4. Therefore, a solution S_ε to the classical Plateau problem corresponding to Γ_ε is a disk-type surface such that:

1. $\beta^\varepsilon_{p,q} := S_\varepsilon \cap (\mathbb{R}^2 \times \{0\})$ is a simple analytic curve joining p and q;
2. S_ε is symmetric with respect to the horizontal plane;
3. the surface $S^+_\varepsilon := S_\varepsilon \cap \{x_3 \geq 0\}$ is the graph of a function $\psi^\varepsilon_{p,q} \in W^{1,1}(U^\varepsilon_{p,q}) \cap C^0(\overline{U}^\varepsilon_{p,q} \setminus \{p, q\})$, where $U^\varepsilon_{p,q} \subset F \cup T_\varepsilon(\alpha)$ is the open region enclosed between $\alpha_\varepsilon \cup L_\varepsilon$ and $\beta^\varepsilon_{p,q}$;
4. the curve $\beta^\varepsilon_{p,q}$ is contained in the closed convex hull of Γ_ε and $(F \cup T_\varepsilon(\alpha)) \setminus U^\varepsilon_{p,q}$ is convex.

We would like to compare the area of S^+_ε with the area of the generalized graph of ψ on $\overline{T_\varepsilon(\alpha)}$. This is not immediate since, due to the fact that ψ is just BV, we cannot, a priori, conclude that its generalized graph is of disk-type[8]. Hence we proceed as follows. We fix $\bar\varepsilon \in (0, \delta_0/2)$; we claim that

$$\mathcal{A}(\psi^{\bar\varepsilon}_{p,q}; U^{\bar\varepsilon}_{p,q}) \leq \mathcal{A}(\psi; T_{\bar\varepsilon}(\alpha)) + \int_\alpha \psi \llcorner T_{\bar\varepsilon}(\alpha)\, d\mathcal{H}^1\,. \tag{5.12}$$

Since ψ is analytic in $T_{\bar\varepsilon}(\alpha) \subset \Omega \setminus E(\sigma)$, by Lemma 5.9 and Remark 5.10 it follows that

$$\lim_{\varepsilon \to 0^+} \int_{\alpha_\varepsilon} \psi \llcorner T_{\bar\varepsilon}(\alpha)\, d\mathcal{H}^1 = \int_\alpha \psi \llcorner T_{\bar\varepsilon}(\alpha)\, d\mathcal{H}^1\,. \tag{5.13}$$

We take

$$T^{\bar\varepsilon}_\varepsilon(\alpha) := T_{\bar\varepsilon}(\alpha) \setminus \overline{T_\varepsilon(\alpha)} \quad \text{and} \quad Y_{\bar\varepsilon} := S_\varepsilon \cup \mathcal{G}_{\psi \llcorner T^{\bar\varepsilon}_\varepsilon(\alpha)} \cup \mathcal{G}_{-\psi \llcorner T^{\bar\varepsilon}_\varepsilon(\alpha)}\,.$$

Since S_ε is a disk-type surface and ψ is analytic in $T^{\bar\varepsilon}_\varepsilon(\alpha)$ it turns out that $Y_{\bar\varepsilon}$ is also a disk-type surface satisfying $\partial Y_{\bar\varepsilon} = \Gamma_{\bar\varepsilon}$. Therefore using that $S_{\bar\varepsilon}$ and S_ε are solutions to the Plateau problems corresponding to $\Gamma_{\bar\varepsilon}$ and Γ_ε respectively, we have

$$\begin{aligned}
\mathcal{H}^2(S_{\bar\varepsilon}) \leq \mathcal{H}^2(Y_{\bar\varepsilon}) &= 2\mathcal{H}^2(\mathcal{G}_{\psi \llcorner T^{\bar\varepsilon}_\varepsilon(\alpha)}) + \mathcal{H}^2(S_\varepsilon) \\
&\leq 2\mathcal{H}^2(\mathcal{G}_{\psi \llcorner T_{\bar\varepsilon}(\alpha)}) + 2\int_{\alpha_\varepsilon \cup L_\varepsilon} \psi \llcorner T_{\bar\varepsilon}(\alpha)\, d\mathcal{H}^1 \\
&= 2\mathcal{H}^2(\mathcal{G}_{\psi \llcorner T_{\bar\varepsilon}(\alpha)}) + 2\int_{\alpha_\varepsilon} \psi \llcorner T_{\bar\varepsilon}(\alpha)\, d\mathcal{H}^1 + 2\int_{L_\varepsilon} \psi \llcorner T_{\bar\varepsilon}(\alpha)\, d\mathcal{H}^1\,.
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0^+$, by (5.13) and the fact that $\mathcal{H}^1(L_\varepsilon) \to 0$, we obtain

$$\mathcal{H}^2(S_{\bar\varepsilon}) \leq 2\mathcal{H}^2(\mathcal{G}_{\psi \llcorner T_{\bar\varepsilon}(\alpha)}) + 2\int_\alpha \psi \llcorner T_{\bar\varepsilon}(\alpha)\, d\mathcal{H}^1,$$

which yields

$$\mathcal{A}(\psi^{\bar\varepsilon}_{p,q}; U^{\bar\varepsilon}_{p,q}) = \mathcal{H}^2(S^+_{\bar\varepsilon}) \leq \mathcal{H}^2(\mathcal{G}_{\psi \llcorner T_{\bar\varepsilon}(\alpha)}) + \int_\alpha \psi \llcorner T_{\bar\varepsilon}(\alpha)\, d\mathcal{H}^1 = \mathcal{A}(\psi; T_{\bar\varepsilon}(\alpha)) + \int_\alpha \psi \llcorner T_{\bar\varepsilon}(\alpha)\, d\mathcal{H}^1,$$

and (5.12) is proved.

[8] This is due to the jump of ψ on ∂F which is, in general, not regular enough.

We now define $E^\alpha := (E(\sigma) \cup T_{\bar\varepsilon}(\alpha)) \setminus U^{\bar\varepsilon}_{p,q}$ and

$$\psi^\alpha := \begin{cases} 0 & \text{in } E^\alpha \\ \psi^{\bar\varepsilon}_{p,q} & \text{in } U^{\bar\varepsilon}_{p,q} \\ \psi & \text{otherwise}. \end{cases}$$

By (5.12) and using that $U^{\bar\varepsilon}_{p,q} \cup E^\alpha = E(\sigma) \cup T_{\bar\varepsilon}(\alpha)$ we derive

$$\begin{aligned} \mathcal{A}(\psi^\alpha;\Omega) - |E^\alpha| &= \mathcal{A}(\psi^{\bar\varepsilon}_{p,q}; U^{\bar\varepsilon}_{p,q}) + \mathcal{A}(\psi; \Omega \setminus (U^{\bar\varepsilon}_{p,q} \cup E^\alpha)) \\ &= \mathcal{A}(\psi^{\bar\varepsilon}_{p,q}; U^{\bar\varepsilon}_{p,q}) + \mathcal{A}(\psi; \Omega \setminus (T_{\bar\varepsilon}(\alpha) \cup E(\sigma))) \\ &\le \mathcal{A}(\psi; T_{\bar\varepsilon}(\alpha)) + \int_\alpha \psi \llcorner T_{\bar\varepsilon}(\alpha)\, d\mathcal{H}^1 + \mathcal{A}(\psi; \Omega \setminus T_{\bar\varepsilon}(\alpha)) - |E(\sigma)| \\ &= \mathcal{A}(\psi;\Omega) - |E(\sigma)|. \end{aligned} \tag{5.14}$$

It remains to construct $\sigma^\alpha \in \Sigma_{\mathrm{conv}}$. Without loss of generality we may assume

$$\sigma_1([0,1]), \dots, \sigma_h([0,1]) \subset F \quad \text{and} \quad \sigma_{h+1}([0,1]), \dots, \sigma_n([0,1]) \not\subset F$$

for some $h \le n$; notice that if $h = n$ the second family of curves is empty. Then we define $\sigma^\alpha := (\sigma^\alpha_1, \dots, \sigma^\alpha_h, \sigma_{h+1}, \dots, \sigma_n) \in \mathrm{Lip}([0,1]; \overline{\Omega})^n$ as follows: if $h > 1$

$$\sigma^\alpha_i([0,1]) = \begin{cases} \overline{q_i p_{i+1}} & \text{for} \quad i = 1, \dots, h-1 \\ \partial(F \cup T_{\bar\varepsilon}(\alpha) \setminus U^{\bar\varepsilon}_{p,q}) \setminus \Big((\cup_{i=1}^h \partial^0_i \Omega) \cup (\cup_{i=1}^{h-1} \overline{q_i p_{i+1}}) \Big) & \text{for} \quad i = h, \end{cases}$$

where $\overline{q_i p_{i+1}}$ is the segment joining q_i to p_{i+1}; if instead $h = 1$ we simply set

$$\sigma^\alpha_1([0,1]) = \partial(F \cup T_{\bar\varepsilon}(\alpha) \setminus U^{\bar\varepsilon}_{p,q}) \setminus \partial^0_1 \Omega.$$

Clearly the pair $(\sigma^\alpha, \psi^\alpha)$ belongs to $\mathcal{W}_{\mathrm{conv}}$, and by (5.14) it satisfies

$$\mathcal{F}(\sigma^\alpha, \psi^\alpha) = \mathcal{F}(\sigma, \psi).$$

Moreover $(\partial E(\sigma^\alpha)) \cap \partial\Omega = (\partial E(\sigma)) \cap \partial\Omega$ and ψ^α is continuous and null on α', where

$$\alpha' := \beta^{\bar\varepsilon}_{p,q} \subset \Omega \cap \partial E(\sigma^\alpha). \tag{5.15}$$

Summarizing, we have replaced the curve α with α', ensuring that the new function ψ^α is now continuous and null on α'.

Step 2: Iterative case. In this step we construct a minimizer $(\widehat{\sigma}, \widehat{\psi}) \in \mathcal{W}_{\mathrm{conv}}$ of $\mathcal{F}$ in $\mathcal{W}$ that satisfies the thesis by iterating step one at most a countable number of times.
We first consider $F = F_1$ and apply step 1 for each $\alpha_{1,j}$ with $j \ge 1$. More precisely we define the pair $(\sigma_{1,j}, \psi_{1,j}) \in \mathcal{W}_{\mathrm{conv}}$ as follows:

- if $j = 1$ we set
$$(\sigma_{1,1}, \psi_{1,1}) := (\sigma^{\alpha_{1,1}}, \psi^{\alpha_{1,1}}),$$
where $(\sigma^{\alpha_{1,1}}, \psi^{\alpha_{1,1}}) \in \mathcal{W}_{\mathrm{conv}}$ is a minimizer constructed as in step 1 with $\alpha = \alpha_{1,1}$;
- if $j > 1$ we set
$$(\sigma_{1,j}, \psi_{1,j}) := (\sigma^{\alpha_{1,j}}_{1,j-1}, \psi^{\alpha_{1,j}}_{1,j-1}),$$
where $(\sigma^{\alpha_{1,j}}_{1,j-1}, \psi^{\alpha_{1,j}}_{1,j-1}) \in \mathcal{W}_{\mathrm{conv}}$ is a minimizer constructed as in step 1 with $(\sigma, \psi) = (\sigma_{1,j-1}, \psi_{1,j-1})$ and $\alpha = \alpha_{1,j}$.

Since $\mathcal{F}(\sigma_{1,j},\psi_{1,j}) = \mathcal{F}(\sigma,\psi)$ for all $j \geq 1$, by Lemma 4.4 it follows that $(\sigma_{1,j},\psi_{1,j})$ converges to $(\sigma_1,\psi_1) \in \mathcal{W}_{\rm conv}$ in the sense of Definition 4.3. Moreover by construction we have that for every $j \geq 1$ the pair $(\sigma_{1,j},\psi_{1,j})$ satisfies

$$(\partial E(\sigma_{1,j})) \cap \partial\Omega = (\partial E(\sigma)) \cap \partial\Omega\,,$$

and $\psi_{1,j}$ is continuous and null on $\cup_{h=1}^{j}\alpha'_{1,h} \subset \Omega \cap (\partial E(\sigma_{1,j})) \cap \partial F_1$, where $\alpha'_{1,h}$ are defined as in (5.15). As a consequence (σ_1,ψ_1) satisfies

$$(\partial E(\sigma_1)) \cap \partial\Omega = (\partial E(\sigma)) \cap \partial\Omega\,,$$

and ψ_1 is continuous and null on $\cup_{j=1}^{\infty}\alpha'_{1,j} \subset \Omega \cap (\partial E(\sigma_1)) \cap \partial F_1$. Moreover

$$\Omega \cap \partial E(\sigma_1) = (\cup_{j=1}^{\infty}\alpha'_{1,j}) \bigcup (\cup_{i=2}^{k} \cup_{j=1}^{\infty} \alpha_{i,j})\,,$$

Now repeating the argument above for the pair (σ_1,ψ_1) and $i = 2$ we obtain a new minimizer $(\sigma_2,\psi_2) \in \mathcal{W}_{\rm conv}$ of $\mathcal{F}$ in $\mathcal{W}$ satisfying

$$(\partial E(\sigma_2)) \cap \partial\Omega = (\partial E(\sigma)) \cap \partial\Omega\,,$$

with ψ_2 continuous and null on $\cup_{j=1}^{\infty}(\alpha'_{1,j} \cup \alpha'_{2,j}) \subset \Omega \cap (\partial E(\sigma_1)) \cap ((\partial F_1) \cup \partial F_2)$ and

$$\Omega \cap (\partial E(\sigma_2)) = (\cup_{i=1}^{2} \cup_{j=1}^{\infty} \alpha'_{i,j}) \cup (\cup_{i=3}^{k} \cup_{j=1}^{\infty} \alpha_{i,j})\,.$$

Iterating this process a finite number of times we finally get a minimizer $(\widehat{\sigma},\widehat{\psi}) \in \mathcal{W}_{\rm conv}$ of $\mathcal{F}$ in $\mathcal{W}$ with the required properties. □

We are finally in the position to conclude the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $(\sigma,\psi) \in \mathcal{W}_{\rm conv}$ be any minimizer of $\mathcal{F}$ in $\mathcal{W}$ as in Theorem 3.1. By Lemma 5.3 we know that (σ,ψ) satisfies properties 1., 2. and the boundary datum is attained, namely

$$\psi = \varphi \quad \text{on} \quad \partial^D\Omega \setminus \partial E(\sigma)\,.$$

Moreover by Lemma 5.11 there is a minimizer $(\widehat{\sigma},\widehat{\psi}) \in \mathcal{W}_{\rm conv}$ such that

$$\partial E(\widehat{\sigma}) \cap \partial\Omega = \partial E(\sigma) \cap \partial\Omega\,, \tag{5.16}$$

and $\widehat{\psi}$ is continuous and null on $\Omega \cap \partial E(\widehat{\sigma})$.

It remains to show that if $\partial_i^D\Omega$ is not straight for some $i = 1,\ldots,n$, then

$$\partial E(\sigma) \cap \partial_i^D\Omega = \partial E(\widehat{\sigma}) \cap \partial_i^D\Omega = \emptyset\,,$$

and if instead $\partial_i^D\Omega$ is straight for some $i = 1,\ldots,n$, then property 4. holds. Eventually we show that there is a minimizer that satisfies property 5.. This will be achieved in a number of steps.

Step 1: Assuming that there is $i \in \{1,\ldots,n\}$ such that $\partial_i^D\Omega$ is not straight, we show that $E(\widehat{\sigma}) \cap \partial_i^D\Omega = \emptyset$. To prove this we proceed by analysing three different cases.

Case A: Suppose, to the contrary, that there is a non-straight[9] arc $\widehat{ab}$ (with endpoints $a \neq b$) in $\partial_i^D\Omega \cap \partial E(\widehat{\sigma})$ (Case A in Figure 4). Thus in particular $\widehat{ab} \subset \cup_{j=1}^{n}\widehat{\sigma}_j([0,1])$. We may assume without loss of generality that $\widehat{ab} \subset \widehat{\sigma}_1([0,1])$. Then we consider the curves

$$\Gamma := \Gamma^+ \cup \Gamma^-\,, \quad \Gamma^+ := \mathcal{G}_{\varphi \llcorner \widehat{ab}} \cup l^+\,, \quad \Gamma^- := \mathcal{G}_{-\varphi \llcorner \widehat{ab}} \cup l^-\,, \tag{5.17}$$

[9]Namely, $\widehat{ab}$ is not contained in a line.

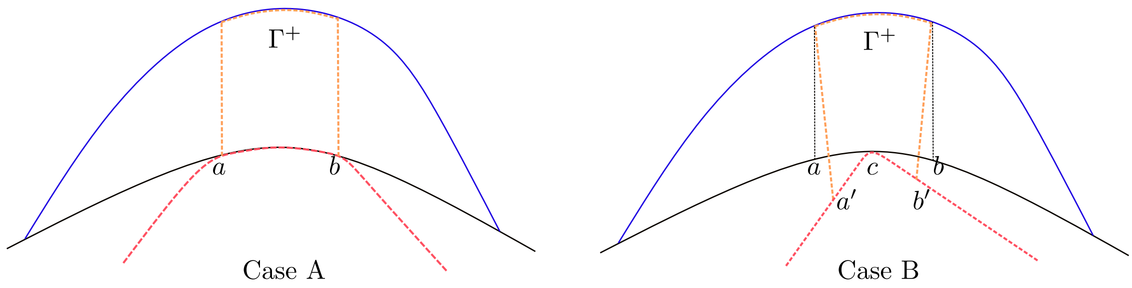

Figure 4: Case A. $\partial_i^D\Omega \cap \partial E(\hat\sigma) = \widehat{ab}$. The orange dotted curve represents Γ^+ in (5.17). Case B. $\partial_i^D\Omega \cap \partial E(\hat\sigma) = \{c\}$. The orange dotted curve represents the curve Γ^+ in (5.20).

where

$$l^+ := (\{a\} \times [0, \varphi(a)]) \cup (\{b\} \times [0, \varphi(b)]) , \quad l^- := (\{a\} \times [-\varphi(a), 0]) \cup (\{b\} \times [-\varphi(b), 0]) .$$

In this way Γ satisfies the assumptions of Lemma 5.4 and hence a solution S to the Plateau problem spanning Γ is a disk-type surface such that:

i. $\beta_{a,b} := S \cap (\mathbb{R}^2 \times \{0\})$ is a simple analytic curve joining a and b;

ii. S is symmetric with respect to $\mathbb{R}^2 \times \{0\}$;

iii. the surface $S^+ := S \cap \{x_3 \geq 0\}$ is the graph of a function $\psi_{a,b} \in W^{1,1}(U_{a,b}) \cap C^0(\overline{U}_{a,b} \setminus \{a, b\})$, where $U_{a,b} \subset E(\widehat\sigma_1)$ is the open region enclosed between $\widehat{ab}$ and $\beta_{a,b}$;

iv. the curve $\beta_{a,b}$ is contained in the closed convex hull of Γ and $E(\widehat\sigma_1) \setminus U_{a,b}$ is convex.

The inclusion $U_{a,b} \subset E(\widehat\sigma_1)$ follows since $\widehat{ab} \subset \widehat\sigma_1([0, 1])$, $E(\widehat\sigma_1)$ is convex, and S is contained in the convex envelope of Γ. Furthermore by the minimality of S one has

$$\mathcal{A}(\psi_{a,b}; U_{a,b}) = \mathcal{H}^2(S^+) < \int_{\widehat{ab}} \varphi \, d\mathcal{H}^1 = \int_{\widehat{ab}} |\widehat\psi - \varphi| \, d\mathcal{H}^1 . \tag{5.18}$$

Here the strict inequality follows since the vertical wall spanning Γ given by $\{(x', x_3) \colon x' \in \widehat{ab},\ x_3 \in [-\varphi(x'), \varphi(x')]\}$ is a disk-type surface but, since $\widehat{ab}$ is not a segment, it cannot be a solution to the Plateau problem. We now consider the pair $(\widetilde\sigma, \widetilde\psi) \in \mathcal{W}_{\mathrm{conv}}$ given by

$$\widetilde\sigma := (\widetilde\sigma_1, \widehat\sigma_2, \ldots, \widehat\sigma_n) , \qquad \widetilde\psi := \begin{cases} 0 & \text{in } \widetilde{E} , \\ \psi_{a,b} & \text{in } U_{a,b} , \\ \widehat\psi & \text{otherwise} , \end{cases} \tag{5.19}$$

where $\widetilde\sigma_1$ is such that $\widetilde\sigma_1([0, 1]) = (\widehat\sigma_1([0, 1]) \setminus \widehat{ab}) \cup \beta_{a,b}$ and $\widetilde{E} := E(\widehat\sigma) \setminus U_{a,b} = E(\widetilde\sigma)$. Then noticing that $\widehat\psi = 0$ in $U_{a,b}$, $E(\widehat\sigma) = E(\widetilde\sigma) \cup U_{a,b}$, and recalling (5.18), we get

$$\begin{aligned} \mathcal{F}(\widetilde\sigma, \widetilde\psi) &= \mathcal{A}(\widetilde\psi; \Omega) - |E(\widetilde\sigma)| + \int_{\partial\Omega} |\widetilde\psi - \varphi| \, d\mathcal{H}^1 \\ &= \mathcal{A}(\widehat\psi; \Omega \setminus U_{a,b}) + \mathcal{A}(\psi_{a,b}; U_{a,b}) - |E(\widetilde\sigma)| + \int_{\partial\Omega} |\widetilde\psi - \varphi| \, d\mathcal{H}^1 \\ &= \mathcal{A}(\widehat\psi; \Omega) + \mathcal{A}(\psi_{a,b}; U_{a,b}) - |E(\widehat\sigma)| + \int_{\partial\Omega} |\widehat\psi - \varphi| \, d\mathcal{H}^1 \\ &< \mathcal{A}(\widehat\psi; \Omega) - |E(\widehat\sigma)| + \int_{\partial\Omega} |\widetilde\psi - \varphi| \, d\mathcal{H}^1 + \int_{\widehat{ab}} |\widehat\psi - \varphi| \, d\mathcal{H}^1 \\ &= \mathcal{A}(\widehat\psi; \Omega) - |E(\widehat\sigma)| + \int_{\partial\Omega} |\widehat\psi - \varphi| \, d\mathcal{H}^1 = \mathcal{F}(\widehat\sigma, \widehat\psi) , \end{aligned}$$

where the penultimate equality follows from the fact that $\widetilde{\psi}$ is continuous and equal to φ on $\widehat{ab}$ while the traces of $\widetilde{\psi}$ and $\widehat{\psi}$ coincide on $\partial\Omega \setminus \widehat{ab}$. This contradicts the minimality of $(\widehat{\sigma}, \widehat{\psi})$.

Case B: Suppose by contradiction that the set $\partial_i^D\Omega \cap \partial E(\widehat{\sigma})$ contains an isolated point c or has a straight segment $\overline{cc'}$ as isolated connected component (Case B in Figure 4). Then there are two arcs $\widehat{ab} \subset \partial_i^D\Omega$ and $\widehat{a'b'} \subset \partial E(\widehat{\sigma})$ with either $a \neq a'$ or $b \neq b'$ (and with endpoints $a \neq b$ and $a' \neq b'$) such that $\overline{aa'} \cap \overline{bb'} = \emptyset$ and $\widehat{ab} \cap \widehat{a'b'} = \{c\}$ (respectively $\widehat{ab} \cap \widehat{a'b'} = \overline{cc'}$). Notice also that, since $\partial_i^D\Omega$ is not straight, the segment $\overline{cc'}$ does not coincide with $\partial_i^D\Omega$ and hence the arc $\widehat{ab}$ can be chosen so that it properly contains the segment $\overline{cc'}$. We consider the curves

$$\Gamma := \Gamma^+ \cup \Gamma^-\,, \quad \Gamma^+ := \mathcal{G}_{\varphi \llcorner \widehat{ab}} \cup \mathcal{G}_{\widehat{\psi} \llcorner \overline{aa'}} \cup \mathcal{G}_{\widehat{\psi} \llcorner \overline{bb'}}\,, \quad \Gamma^- := \mathcal{G}_{-\varphi \llcorner \widehat{ab}} \cup \mathcal{G}_{-\widehat{\psi} \llcorner \overline{aa'}} \cup \mathcal{G}_{-\widehat{\psi} \llcorner \overline{bb'}}\,. \tag{5.20}$$

Notice that $\Gamma^\pm$ connect a' to b'. By applying again Lemma 5.4 to the nonplanar curve Γ and arguing as in case A we obtain the contradiction also in this case.

Case C: More generally, assume by contradiction that both the sets $\partial_i^D\Omega \cap \partial E(\widehat{\sigma})$ and $\partial_i^D\Omega \setminus \partial E(\widehat{\sigma})$ are nonempty. Then we can find a not flat arc $\widehat{ab} \subset \partial_i^D\Omega$ such that the following holds[10]: there are pairs of points $\{c_j, d_j\}_{j\in\mathbb{N}} \subset \partial_i^D\Omega \cap \partial E(\widehat{\sigma})$ such that the arcs $\widehat{ad_0}$, $\widehat{c_0b}$, and $\{\widehat{c_jd_j}\}_{j=1}^\infty$ are mutually disjoint and

$$\widehat{ab} \setminus \partial E(\widehat{\sigma}) = \widehat{ad_0} \cup (\cup_{j=1}^\infty \widehat{c_jd_j}) \cup \widehat{c_0b}\,.$$

Without loss of generality, we might assume that all the points $c_j, d_j \in \widehat{\sigma}_1([0,1])$. For all $j \geq 1$ we denote by V_j the region enclosed by $\widehat{c_jd_j}$ and $\partial E(\widehat{\sigma})$[11]. We now argue as in case B and choose $a', b' \in \widehat{\sigma}_1([0,1])$. Additionally, let $V_0 = V_0^a \cup V_0^b$, with V_0^a (respectively V_0^b) be the region enclosed between $\partial E(\widehat{\sigma})$ and $\overline{aa'} \cup \widehat{ad_0}$ ($\partial E(\widehat{\sigma})$ and $\overline{bb'} \cup \widehat{c_0b}$, respectively). We finally define Γ correspondingly, as in (5.20). Again by Lemma 5.4 the solution S to the Plateau problem corresponding to Γ satisfies properties i.-iv., with a' and b' in place of a and b respectively. Moreover by the minimality of S for every $N \geq 1$ there holds[12]

$$\mathcal{A}(\psi_{a',b'}; U_{a',b'}) = \mathcal{H}^2(S^+) \leq \int_{\widehat{ab}} \varphi\, d\mathcal{H}^1 - \int_{\widehat{ad_0} \cup \widehat{c_0b}} \varphi\, d\mathcal{H}^1 - \sum_{j=1}^N \int_{\widehat{c_jd_j}} \varphi\, d\mathcal{H}^1 + \sum_{j=0}^N \mathcal{A}(\psi; V_j)\,. \tag{5.21}$$

In particular by taking the limit as $N \to +\infty$ in (5.21) we get

$$\mathcal{A}(\psi_{a',b'}; U_{a',b'}) = \mathcal{H}^2(S^+) \leq \int_{\widehat{ab} \setminus \partial E(\widehat{\sigma})} \varphi\, d\mathcal{H}^1 + \mathcal{A}(\widehat{\psi}; \cup_{j=0}^\infty V_j)\,. \tag{5.22}$$

Let $(\widetilde{\sigma}, \widetilde{\psi}) \in \mathcal{W}_{\rm conv}$ be defined as in (5.19), then observing that $\widehat{\psi} = 0$ in $U_{a',b'} \setminus (\cup_{j=0}^\infty V_j)$, $E(\widehat{\sigma}) =$

[10] This is a consequence of the fact that $\widehat{ab} \setminus \partial E(\widehat{\sigma})$ is relatively open in $\widehat{ab}$, so it is an at most countable union of disjoint relatively open arcs.

[11] These regions are simply connected since $c_j, d_j \in \widehat{\sigma}_1([0,1])$.

[12] The right-hand side is the area of the surface given by the (positive) subgraph of φ on $\widehat{ab} \setminus \cup_{j=1}^N \widehat{c_jd_j}$ and the graph of $\widehat{\psi}$ on the region $\cup_{j=0}^N V_j$, which is of disc-type. To see this we use that the trace of $\widehat{\psi}$ on the subarcs of $\partial E(\widehat{\sigma})$ between the points c_j and d_j is zero (and between a' and d_0, and d_0 and b').

$E(\widetilde{\sigma}) \cup (U_{a',b'} \setminus \cup_{j=0}^{\infty} V_j)$ and using (5.22) we deduce

$$
\begin{aligned}
\mathcal{F}(\widetilde{\sigma}, \widetilde{\psi}) &= \mathcal{A}(\widehat{\psi}; \Omega \setminus U_{a',b'}) + \mathcal{A}(\psi_{a',b'}; U_{a',b'}) - |E(\widetilde{\sigma})| + \int_{\partial\Omega} |\widetilde{\psi} - \varphi| \, d\mathcal{H}^1 \\
&= \mathcal{A}(\widehat{\psi}; \Omega \setminus (\cup_{j=0}^{\infty} V_j)) + \mathcal{A}(\psi_{a',b'}; U_{a',b'}) - |E(\widehat{\sigma})| + \int_{\partial\Omega} |\widetilde{\psi} - \varphi| \, d\mathcal{H}^1 \\
&\leq \mathcal{A}(\widehat{\psi}; \Omega \setminus (\cup_{j=0}^{\infty} V_j)) - |E(\widehat{\sigma})| + \int_{\partial\Omega} |\widetilde{\psi} - \varphi| \, d\mathcal{H}^1 + \int_{\widehat{ab} \cap \partial E(\widehat{\sigma})} \varphi \, d\mathcal{H}^1 + \mathcal{A}(\widehat{\psi}; \cup_{j=0}^{\infty} V_j) \\
&= \mathcal{A}(\widehat{\psi}; \Omega) - |E(\widehat{\sigma})| + \int_{\partial\Omega} |\widehat{\psi} - \varphi| \, d\mathcal{H}^1 = \mathcal{F}(\widehat{\sigma}, \widehat{\psi}) \,,
\end{aligned}
$$

which in turn implies

$$
\mathcal{F}(\widetilde{\sigma}, \widetilde{\psi}) \leq \mathcal{F}(\widehat{\sigma}, \widehat{\psi}) \,. \tag{5.23}
$$

To conclude we need to show that the inequality in (5.23) is strict. To this aim we choose $c \in \{c_j\}_{j=1}^{\infty}$. Consider the curves Γ_1 and Γ_2 defined as follows

$$
\begin{aligned}
\Gamma_1 := \Gamma_1^+ \cup \Gamma_1^- \,, \quad \Gamma_1^+ := \mathcal{G}_{\varphi \llcorner \widehat{ac}} \cup \mathcal{G}_{\widehat{\psi} \llcorner \overline{aa'}} \cup l^+ \,, \quad \Gamma_1^- := \mathcal{G}_{-\varphi \llcorner \widehat{ac}} \cup \mathcal{G}_{-\widehat{\psi} \llcorner \overline{aa'}} \cup l^- \,, \\
\Gamma_2 := \Gamma_2^+ \cup \Gamma_2^- \,, \quad \Gamma_2^+ := \mathcal{G}_{\varphi \llcorner \widehat{cb}} \cup \mathcal{G}_{\widehat{\psi} \llcorner \overline{bb'}} \cup l^+ \,, \quad \Gamma_2^- := \mathcal{G}_{-\varphi \llcorner \widehat{cb}} \cup \mathcal{G}_{-\widehat{\psi} \llcorner \overline{bb'}} \cup l^- \,,
\end{aligned}
$$

where

$$
l^+ := (\{c\} \times [0, \varphi(c)]) \,, \quad l^- := (\{c\} \times [-\varphi(c), 0]) \,.
$$

Let S_1 and S_2 be the solutions to the Plateau problem corresponding to Γ_1 and Γ_2 respectively, so that properties i.-iv. are satisfied with c in place of b' and a' respectively. By the minimality of S we have

$$
\mathcal{A}(\psi_{a',b'}; U_{a',b'}) < \mathcal{A}(\psi_{a',c}; U_{a',c}) + \mathcal{A}(\psi_{c,b'}; U_{c,b'}) \,. \tag{5.24}
$$

On the other hand by arguing as above[13] we conclude

$$
\mathcal{A}(\psi_{a',c}; U_{a',c}) \leq \int_{\widehat{ac} \cup \partial E(\widehat{\sigma})} \varphi \, d\mathcal{H}^1 + \mathcal{A}(\widehat{\psi}; \cup_{j \in I_1} V_j \cup V_0^a) \,, \tag{5.25}
$$

and

$$
\mathcal{A}(\psi_{c,b'}; U_{c,b'}) \leq \int_{\widehat{cb} \cup \partial E(\widehat{\sigma})} \varphi \, d\mathcal{H}^1 + \mathcal{A}(\widehat{\psi}; \cup_{j \in I_2} V_i \cup V_0^b) \,, \tag{5.26}
$$

where $I_1 := \{j \colon \widehat{c_j d_j} \subset \widehat{ac}\}$ and $I_2 := \{j \colon \widehat{c_j d_j} \subset \widehat{cb}\}$. Gathering together (5.24)-(5.26) we derive

$$
\mathcal{A}(\psi_{a',b'}; U_{a',b'}) < \int_{\widehat{ab} \cup \partial E(\widehat{\sigma})} \varphi \, d\mathcal{H}^1 + \mathcal{A}(\widehat{\psi}; \cup_{j=0}^{\infty} V_j) \,,
$$

which in turn implies

$$
\mathcal{F}(\widetilde{\sigma}, \widetilde{\psi}) < \mathcal{F}(\widehat{\sigma}, \widehat{\psi}) \,,
$$

and thus the contradiction.

Step 2: Assuming there is $i \in \{1, \ldots, n\}$ such that $\partial_i^D \Omega$ is a straight segment, we show that either $(\partial E(\widehat{\sigma})) \cap \partial_i^D \Omega = \emptyset$ or $(\partial E(\widehat{\sigma})) \cap \partial_i^D \Omega = \partial_i^D \Omega$.
Suppose by contradiction that $(\partial E(\widehat{\sigma})) \cap \partial_i^D \Omega \neq \emptyset$ and also $\partial_i^D \Omega \setminus \partial E(\widehat{\sigma}) \neq \emptyset$. Without loss of generality we can restrict to the case $(\partial E(\widehat{\sigma})) \cap \partial_i^D \Omega = (\partial F) \cap \partial_i^D \Omega$ with F any connected component of $E(\widehat{\sigma})$. Since F is convex and $\partial_i^D \Omega$ is a segment $(\partial F) \cap \partial_i^D \Omega$ has to be connected,

[13] With the arc $\widehat{ac}$ ($\widehat{cb}$, respectively) in place of $\widehat{ab}$.

i.e., it is either a single point a or a segment $\overline{aa'} \neq \partial_i^D\Omega$. In both cases we then consider a (small enough) ball B centred at a such that $B \cap E(\widehat{\sigma}) = B \cap F$ (in the second case we also require that the radius of B is smaller than $\overline{aa'}$).

If $(\partial F) \cap \partial_i^D\Omega = \{a\}$ we let $\{p, q\} := (\partial B) \cap \partial F$ and $\{b, c\} := (\partial B) \cap \partial_i^D\Omega$ (with b, p and c, q lying on the same side with respect to a). Then we define the curves

$$\Gamma := \Gamma^+ \cup \Gamma^- , \quad \Gamma^+ := \mathcal{G}_{\varphi \llcorner \overline{bc}} \cup \mathcal{G}_{\psi \llcorner \widehat{bp}} \cup \mathcal{G}_{\psi \llcorner \widehat{cq}} , \quad \Gamma^- := \mathcal{G}_{-\varphi \llcorner \overline{bc}} \cup \mathcal{G}_{-\psi \llcorner \widehat{bp}} \cup \mathcal{G}_{-\psi \llcorner \widehat{cq}} ,$$

where $\widehat{bp}$, $\widehat{cq}$ denote the arcs in ∂B joining b to p and c to q respectively.

If $(\partial F) \cap \partial_i^D\Omega = \overline{aa'}$ we let $\{p, q\} := (\partial B) \cap \partial F$ and $\{b, c\} := (\partial B) \cap \partial_i^D\Omega$ where we identify q and c. Then we consider the curves

$$\Gamma := \Gamma^+ \cup \Gamma^- , \quad \Gamma^+ := \mathcal{G}_{\varphi \llcorner \overline{bc}} \cup \mathcal{G}_{\psi \llcorner \widehat{bp}} \cup (\{c\} \times [0, \varphi(c)]) , \quad \Gamma^- := \mathcal{G}_{-\varphi \llcorner \overline{bc}} \cup \mathcal{G}_{-\psi \llcorner \widehat{bp}} \cup (\{c\} \times [-\varphi(c), 0]) .$$

By applying again Lemma 5.4 to Γ and arguing as above we get the contradiction.

Step 3: We show that there is a minimizer $(\widetilde{\sigma}, \widetilde{\psi})$ that satisfies property 5.. We first notice that $\widehat{\psi}$ is continuous and null on $\partial E(\widehat{\sigma}) \setminus \partial^D\Omega$. Moreover by steps 1 and 2 it follows that $\Omega \cap \partial E(\widehat{\sigma})$ is the union of a finite number of pairwise disjoint Lipschitz curves each of them joining each p_i for $i = 1, \ldots, n$ to each of the q_j for some $j = 1, \ldots, n$. To conclude it is enough to replace each curve, without increasing the energy, with an analytic one having the same endpoints. More precisely, let γ be any of such curves. Reasoning as in the proof of Lemma 5.11 step 1, we can replace $(\widehat{\sigma}, \widehat{\psi})$ with a new minimizer $(\sigma^\gamma, \psi^\gamma) \in \mathcal{W}_{\rm conv}$ such that $(\partial E(\sigma^\gamma)) \cap \partial\Omega = (\partial E(\sigma)) \cap \partial\Omega$ and $\psi^\gamma = 0$ on γ', where $\gamma' \subset (\partial E(\sigma^\gamma)) \cap \Omega$ is a suitable analytic curve that replaces γ and has the same endpoints of γ. In particular ψ^γ is continuous and null on $\partial E(\sigma^\gamma) \setminus \partial^D R_{2\ell}$. Eventually iterating this procedure for each curve in $\partial E(\widehat{\sigma}) \setminus \partial\Omega$ we can construct a new minimizer $(\widetilde{\sigma}, \widetilde{\psi})$ with the required properties. □

5.1 The example of the catenoid containing a segment

Consider the setting of Figure 6. Recall that $\Omega = R_{2\ell} = (0, 2\ell) \times (-1, 1)$, $n = 1$, $\partial^D\Omega = (\{0, 2\ell\} \times (-1, 1)) \cup ((0, 2\ell) \times \{-1\})$ and $\partial^0\Omega = (0, 2\ell) \times \{1\}$, $p = (0, 1)$, $q = (2\ell, 1)$. The map φ given in (7.3) is $\varphi(z_1, z_2) = \sqrt{1 - z_2^2}$ on $\partial^D\Omega$, and thus vanishes on $[0, 2l] \times \{-1\}$; for this reason this case is not covered by our analysis. However we can find a solution as in Theorem 1.1 also in this case, by an approximation procedure. Precisely, for $\varepsilon > 0$ consider an approximating sequence (φ_ε) of continuous Dirichlet data, with $\mathcal{G}_{\varphi_\varepsilon}$ Lipschitz, which tends to φ uniformly and satisfies $\varphi_\varepsilon = 0$ on $\partial^0\Omega$, $\varphi_\varepsilon > 0$ on $\partial^D\Omega$. Let $(\sigma_\varepsilon, \psi_\varepsilon)$ be a solution as in Theorem 4.1 corresponding to the boundary datum φ_ε; since $\mathcal{F}(\sigma_\varepsilon, \psi_\varepsilon)$ is equibounded[14], arguing as in the proof of Lemma 4.4, we can see that, up to a subsequence, $((\sigma_\varepsilon, \psi_\varepsilon))$ tends to some $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$, which minimizes the functional $\mathcal{F}$ with Dirichlet condition φ. In this case however we cannot guarantee that σ does not touch $\partial^D\Omega$, even if this is not a straight segment. This is essentially due to the presence of the portion $[0, 2\ell] \times \{-1\}$ of $\partial\Omega$ where φ is zero, which does not allow to apply the arguments used in the proof of Theorem 5.1.

In particular, it can be seen that if ℓ is large enough, the solution (σ, ψ) splits and becomes degenerate, being $\psi \equiv 0$ and the value of $\mathcal{F}$ is just the area of two vertical half-disks of radius 1. For ℓ under a certain threshold, instead, the solution satisfies the regularity properties stated in Theorem 5.1, and in particular $\psi = \varphi$ on $\partial^D\Omega$, and σ is the graph of a smooth convex function passing through p and q. We refer to [10] for details and comprehensive proofs of these facts; we also notice that in this special case further regularity of solutions can be obtained.

[14]We can bound it from above by $|\Omega| + \int_{\partial_D\Omega} |\varphi_\varepsilon| d\mathcal{H}^1$.

6 Comparison with the parametric Plateau problem: The case $n = 1, 2$

In this section we compare the solutions of Theorems 3.1 and 5.1 with the solutions to the classical Plateau problem in parametric form. Specifically, motivated by the example of the catenoid, we restrict our analysis to the classical disk-type and annulus-type Plateau problem. These configurations correspond to the cases $n = 1$ and $n = 2$ respectively, i.e., the Dirichlet boundary $\partial^D\Omega$ is either an open arc or the union of two open arcs of $\partial\Omega$ with disjoint closure. Due to the highly involved geometric arguments, we do not discuss the case $n > 2$, which requires further investigation. Thus, in this section we assume $n = 1, 2$. We first discuss the case $n = 1$, which is a consequence of Lemma 5.4, and next the case $n = 2$.

6.1 The case $n = 1$

Let $n = 1$. Let $p_1, q_1 \in \partial\Omega$, $\partial^D\Omega = \partial_1^D\Omega$, φ be as in Section 2.2 and consider the space curve $\gamma_1 := \mathcal{G}_{\varphi \llcorner \partial_1^D\Omega}$ joining p_1 to q_1. We define the curve

$$\Gamma := \gamma_1 \cup \mathrm{Sym}(\gamma_1),$$

where $\mathrm{Sym}(\gamma_1) := \mathcal{G}_{-\varphi \llcorner \partial_1^D\Omega}$, and consider the classical Plateau problem in parametric form spanning Γ. More precisely we look for a solution to

$$m_1(\Gamma) := \inf_{\Phi \in \mathcal{P}_1(\Gamma)} \int_{B_1} |\partial_{w_1}\Phi \wedge \partial_{w_2}\Phi| dw, \tag{6.1}$$

where

$$\mathcal{P}_1(\Gamma) := \Big\{ \Phi \in H^1(B_1; \mathbb{R}^3) \cap C^0(\overline{B}_1; \mathbb{R}^3) \text{ such that } \Phi \llcorner \partial B_1 : \partial B_1 \to \Gamma \\ \text{is a weakly monotonic parametrization of } \Gamma \Big\}. \tag{6.2}$$

By classical arguments, every solution to (6.1) is a harmonic and conformal parametrization of an area-minimizing surface spanning Γ.

Theorem 6.1 (The disk-type Plateau problem ($n = 1$)). *Assume Γ is not planar, let $\Phi \in \mathcal{P}_1(\Gamma)$ be a solution to* (6.1) *and let*

$$S^+ := \Phi(\overline{B}_1) \cap \{x_3 \geq 0\}, \qquad S^- := \Phi(\overline{B}_1) \cap \{x_3 \leq 0\}.$$

Then there exists a minimizer $(\sigma, \psi) \in \mathcal{W}_{\mathrm{conv}}$ of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1.-5. *of Theorem 5.1 and such that*

$$S^{\pm} = \mathcal{G}_{\pm\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}. \tag{6.3}$$

Conversely let $(\sigma, \psi) \in \mathcal{W}_{\mathrm{conv}}$ be a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1.-5. *of Theorem 5.1. Then the disk-type surface*

$$S := \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$$

is a solution to the classical Plateau problem associated to Γ, i.e., there is a harmonic and conformal map $\Phi \in \mathcal{P}_1(\Gamma)$ solving (6.1) *and such that $\Phi(\overline{B_1}) = S$.*

We have assumed Γ is not planar, otherwise the classical solution is flat, and any solution to Theorem 5.1 satisfies $(\partial E(\sigma)) \cap \partial^D\Omega = \partial^D\Omega$.

6.2 The case $n = 2$

Let $n = 2$. Let Ω, $p_1, q_1, p_2, q_2 \in \partial\Omega$, $\partial^D\Omega$, $\partial_1^D\Omega$, $\partial_2^D\Omega$, φ be as in Section 2.2 and consider the space curve $\gamma_i := \mathcal{G}_{\varphi\llcorner\partial_i^D\Omega}$ joining p_i to q_i for $i = 1, 2$. We define the curves

$$\Gamma_1 := \gamma_1 \cup \mathrm{Sym}(\gamma_1), \qquad \Gamma_2 := \gamma_2 \cup \mathrm{Sym}(\gamma_2),$$

where $\mathrm{Sym}(\gamma_i) := \mathcal{G}_{-\varphi\llcorner\partial_i^D\Omega}$ for $i = 1, 2$. We consider the classical Plateau problem in parametric form spanning the curve

$$\Gamma := \Gamma_1 \cup \Gamma_2.$$

Precisely we set $\Sigma_{\mathrm{ann}} \subset \mathbb{R}^2$ to be an open annulus enclosed between two concentric circles $C_1 := \partial B_1(0)$ and $C_2 := \partial B_2(0)$, and we look for a solution to

$$m_2(\Gamma) := \inf_{\Phi\in\mathcal{P}_2(\Gamma)} \int_{\Sigma_{\mathrm{ann}}} |\partial_{w_1}\Phi \wedge \partial_{w_2}\Phi| dw, \tag{6.4}$$

where

$$\mathcal{P}_2(\Gamma) := \Big\{\Phi \in H^1(\Sigma_{\mathrm{ann}}; \mathbb{R}^3) \cap C^0(\overline{\Sigma}_{\mathrm{ann}}; \mathbb{R}^3) \text{ such that } \Phi(\partial\Sigma_{\mathrm{ann}}) = \Gamma \text{ and } \Phi\llcorner C_j : C_j \to \Gamma_j$$
$$\text{is a weakly monotonic parametrization of } \Gamma_j \text{ for } j = 1, 2\Big\}.$$

Here the crucial assumption that we require is that the curves Γ_j have the orientation inherited by the orientation[15] of the graph of φ on $\partial_j^D\Omega$.
Due to the specific geometry of Γ we can appeal to Theorem 6.3 below (which is a consequence of [36, Theorem 1 and Theorem 5]) to deduce the existence of a minimizer. This might not be true for a more general Γ. To this purpose for $j = 1, 2$ we consider the minimization problem defined in (6.1) for the curve Γ_j, namely

$$m_1(\Gamma_j) = \inf_{\Phi\in\mathcal{P}_1(\Gamma_j)} \int_{B_1} |\partial_{w_1}\Phi \wedge \partial_{w_2}\Phi| dw, \tag{6.5}$$

with $\mathcal{P}_1(\Gamma_j)$ defined as in (6.2).

By standard arguments one sees that $m_2(\Gamma) \le m_1(\Gamma_1) + m_1(\Gamma_2)$. Indeed, two disk-type surfaces can be joined by a thin tube (with arbitrarily small area) in order to change the topology of the two disks into an annulus-type surface.

Definition 6.2 (**$\mathcal{MY}$ solution**). *Let $\Phi \in \mathcal{P}_2(\Gamma)$ be a solution to* (6.4). *We say that Φ is a $\mathcal{MY}$ solution to* (6.4) *if Φ is harmonic, conformal, and it is an embedding. In particular, in such a case, $m_2(\Gamma) = \mathcal{H}^2(\Phi(\overline{\Sigma}_{\mathrm{ann}}))$.*

Theorem 6.3 (**Meeks and Yau**). *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$. Then there exists a $\mathcal{MY}$ solution $\Phi \in \mathcal{P}_2(\Gamma)$ to* (6.4). *Furthermore, every minimizer of* (6.4) *is a $\mathcal{MY}$ solution.*

Proof. See [36]. □

This result allows us to prove the following:

Theorem 6.4 (**The annulus-type Plateau problem** ($n = 2$)). *The following holds:*

[15] Once we fix an orientation of $\partial\Omega$, the orientation of the graph $\mathcal{G}_\varphi$ of φ is inherited, since $\mathcal{G}_\varphi$ is standardly defined as the push-forward of the current of integration on $\partial_D\Omega$ by the map $x \mapsto (x, \varphi(x))$.

(i) *Suppose* $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$. *Let* $\Phi \in \mathcal{P}_2(\Gamma)$ *be a* $\mathcal{MY}$ *solution to* (6.4) *and let*

$$S := \Phi(\overline{\Sigma}_{\rm ann}), \qquad S^+ := S \cap \{x_3 \geq 0\}, \qquad S^- := S \cap \{x_3 \leq 0\}.$$

Then there exists a minimizer $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ *of* $\mathcal{F}$ *in* $\mathcal{W}$ *satisfying properties* 1.-5. *of Theorem 5.1 and such that*

$$S^{\pm} = \mathcal{G}_{\pm\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}. \tag{6.6}$$

(ii) *Suppose* $m_2(\Gamma) = m_1(\Gamma_1) + m_1(\Gamma_2)$, *and assume that both* Γ_1 *and* Γ_2 *are not planar. For* $j = 1, 2$ *let* $\Phi_j \in \mathcal{P}_1(\Gamma_j)$ *be a solution to* (6.5) *and let* $S_j := \Phi_j(\overline{B}_1)$. *Let also*

$$S^+ := (S_1 \cup S_2) \cap \{x_3 \geq 0\} \qquad \text{and} \qquad S^- := (S_1 \cup S_2) \cap \{x_3 \leq 0\}.$$

Then $S_1 \cap S_2 = \emptyset$ *and there exists a minimizer* $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ *of* $\mathcal{F}$ *in* $\mathcal{W}$ *satisfying properties* 1.-5. *of Theorem 5.1 and such that* (6.6) *holds.*

(iii) *Conversely, let* $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ *be a minimizer of* $\mathcal{F}$ *in* $\mathcal{W}$ *satisfying properties* 1.-5. *of Theorem 5.1. Then the surface*

$$S := \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$$

is either an annulus-type surface or the union of two disjoint disk-type surfaces, and is a solution to the classical Plateau problem associated to Γ. *More precisely, either there is a* $\mathcal{MY}$ *solution* $\Phi \in \mathcal{P}_2(\Gamma)$ *to* (6.4) *with* $S = \Phi(\overline{\Sigma}_{\rm ann})$, *or there are* $\Phi_j \in \mathcal{P}_1(\Gamma_j)$ *solutions to* (6.5) *for* $j = 1, 2$, *such that* $S = \Phi_1(\overline{B}_1) \cup \Phi_2(\overline{B}_1)$ *and* $\Phi_1(\overline{B}_1) \cap \Phi_2(\overline{B}_1) = \emptyset$.

6.3 Toward the proof of Theorems 6.1 and 6.4: preliminary lemmas

In order to prove Theorems 6.1 and 6.4, we collect some technical lemmas.

Lemma 6.5 (Graphicality of minimizers for $n = 2$**).** *Let* $n = 2$, *and* $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ *be a minimizer of* $\mathcal{F}$ *in* $\mathcal{W}$ *satisfying properties* 1.-5. *of Theorem 5.1.*

(a) *Suppose that* $\overline{\Omega \setminus E(\sigma)}$ *is connected. Then there exists an injective map* $\Phi \in W^{1,1}(\Sigma_{\rm ann}; \mathbb{R}^3) \cap C^0(\overline{\Sigma}_{\rm ann}; \mathbb{R}^3)$ *such that*

$$\Phi(\overline{\Sigma}_{\rm ann}) = \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})},$$

and $\Phi \llcorner C_j \colon C_j \to \Gamma_j$ *is a weakly monotonic parametrization of* Γ_j *for* $j = 1, 2$.

(b) *Suppose that* $\Omega \setminus E(\sigma)$ *consists of two connected components, whose closures* F_1 *and* F_2 *are disjoint, with* $F_j \supseteq \partial_j^D \Omega$ *for* $j = 1, 2$. *Then there exist two injective maps* $\Phi_1, \Phi_2 \in W^{1,1}(B_1; \mathbb{R}^3) \cap C^0(\overline{B_1}; \mathbb{R}^3)$ *such that*

$$\Phi_j(\overline{B_1}) = \mathcal{G}_{\psi \llcorner F_j} \cup \mathcal{G}_{-\psi \llcorner F_j}, \qquad j = 1, 2,$$

and $\Phi_j \llcorner \partial B_1 \colon \partial B_1 \to \Gamma_j$ *is a weakly monotonic parametrization of* Γ_j *for* $j = 1, 2$.

Supposing that $\Omega \setminus E(\sigma)$ has two connected components as in (b), it readily follows that Γ_1 and Γ_2 cannot be planar (otherwise the solution will be flat on $\partial_j^D \Omega$ and $F_j = \emptyset$ for $j = 1, 2$).

Proof. (a). Since $\overline{\Omega \setminus E(\sigma)}$ is simply connected[16], the maps

$$\widetilde{\Psi}^{\pm} \in W^{1,1}(\Omega \setminus E(\sigma);\mathbb{R}^3) \cap C^0(\overline{\Omega \setminus E(\sigma)};\mathbb{R}^3), \qquad \widetilde{\Psi}^{\pm}(p) := (p, \pm\psi(p)), \tag{6.7}$$

are disk-type parametrizations of $\mathcal{G}_{\pm\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$, thanks to properties 1.-5. of Theorem 5.1.

Now, using a homeomorphism of class H^1 between $\overline{\Omega \setminus E(\sigma)}$ and a disk, we can parametrize[17] $\overline{\Omega \setminus E(\sigma)}$ with a half-annulus, obtained as the region enclosed between two concentric half-circles with endpoints A_1, A_2, A_3, A_4 (in the order) on the same diameter, and the two segments $\overline{A_1A_2}$ and $\overline{A_3A_4}$. Then we construct a parametrization Ψ^+ of $\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ as in (6.7) from the half-annulus, such that $\Psi^+(A_1) = (q_1, 0)$, $\Psi^+(A_2) = (p_2, 0)$, $\Psi^+(A_3) = (q_2, 0)$, $\Psi^+(A_4) = (p_1, 0)$, and mapping weakly monotonically the two half-circles into γ_1 and γ_2, and the two segments into $\sigma_1([0,1])$ and $\sigma_2([0,1])$, respectively. Similarly, we construct a parametrization Ψ^- of $\mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ from another copy of a half-annulus, just setting $\Psi^- := \mathrm{Sym}(\Psi^+)$, the symmetric of Ψ^+ with respect to the plane containing Ω.
Eventually, glueing the two half-annuli along the two segments, we obtain a parametrization Φ of $\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ defined on $\overline{\Sigma}_{\mathrm{ann}}$. By the continuity of ψ on $\partial^D \Omega$ we have that Φ parametrizes Γ_i on C_i, $i = 1, 2$.

(b). It is sufficient to argue as in case (a), by replacing $\Omega \setminus E(\sigma)$ in turn with F_1 and F_2 and Σ_{ann} with B_1 to find Φ_1 and Φ_2, respectively. $\square$

Lemma 6.6. *Let $n = 2$, and $(\sigma, \psi) \in \mathcal{W}_{\mathrm{conv}}$ be a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties* 1.-5. *of Theorem 5.1.*

(a) Suppose that $\overline{\Omega \setminus E(\sigma)}$ is connected and

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}) \leq m_2(\Gamma). \tag{6.8}$$

Let Φ be the parametrization given by Lemma 6.5 (a). Then there exists a reparametrization of the annulus Σ_{ann} such that, using it to reparametrize Φ, the corresponding map (still denoted by Φ) belongs to $\mathcal{P}_2(\Gamma)$ and solves (6.4).

(b) Suppose that $\Omega \setminus E(\sigma)$ consists of two connected components whose closures F_1 and F_2 are disjoint, $F_j \supseteq \partial^D_j \Omega$ for $j = 1, 2$, and

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner F_j} \cup \mathcal{G}_{-\psi \llcorner F_j}) \leq m_1(\Gamma_j), \qquad j = 1, 2.$$

Let Φ_1, Φ_2 be the maps given by Lemma 6.5 (b). Then, for $j = 1, 2$, there is a reparametrization of Φ_j belonging to $\mathcal{P}_1(\Gamma_j)$ and solving (6.5).

Proof. (a). Fix a point $\widetilde{p} \in \Omega \setminus E(\sigma)$ and set $\widetilde{\Psi}^+_k := \widetilde{\Psi}^+ \llcorner H_k$, where $\widetilde{\Psi}$ is defined in (6.7) and, for $k \in \mathbb{N}$ sufficiently large, H_k is the connected component of

$$\widetilde{H}_k := \{p \in \overline{\Omega \setminus E(\sigma)} : \mathrm{dist}(p, \partial(\overline{\Omega \setminus E(\sigma)})) \geq 1/k\}$$

containing $\widetilde{p}$. For $k \in \mathbb{N}$ large enough H_k is simply connected with rectifiable boundary, thanks to the simply-connectedness of $\Omega \setminus E(\sigma)$. In particular $\widetilde{\Psi}^+_k$ parametrizes a disk-type surface, and using the regularity of ψ in $\Omega \setminus E(\sigma)$, it follows that $\widetilde{\Psi}^+_k$ is Lipschitz continuous. Furthermore, $\widetilde{\Psi}^+_k \llcorner \partial H_k$ parametrizes a Jordan curve, and these curves, suitably parametrized, converge

[16] This is the region enclosed by $\partial^D \Omega \cup \sigma_1([0,1]) \cup \sigma_2([0,1])$.

[17] For instance, we can consider a (flat) disk-type Plateau solution spanning $\partial(\Omega \setminus E(\sigma))$. Then we can employ a Lipschitz homeomorphism between the disk and the half-annulus.

in the sense of Fréchet (see [22, Theorem 4, Section 4.3]) as $k \to +\infty$, to the curve having image $\widetilde{\Psi}^+(\partial(\Omega \setminus E(\sigma))) =: \lambda$. Notice that

$$\lambda = \sigma_1([0,1]) \cup \sigma_2([0,1]) \cup \gamma_1 \cup \gamma_2. \tag{6.9}$$

Call λ_k the image of the curve given by $\widetilde{\Psi}_k^+ \llcorner \partial H_k$. Let $\mathcal{P}_1(\lambda_k)$, $\mathcal{P}_1(\lambda)$, $m_1(\lambda_k)$, $m_1(\lambda)$ be defined as in (6.2) and (6.1) with λ_k and λ in place of Γ respectively. Up to reparametrizing $\overline{B}_1$ (see footnote 15), $\widetilde{\Psi}_k^+$ belongs to $\mathcal{P}_1(\lambda_k)$, therefore

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner H_k}) = \int_{H_k} |\partial_{w_1} \widetilde{\Psi}_k^+ \wedge \partial_{w_2} \widetilde{\Psi}_k^+| dw \geq m_1(\lambda_k) \qquad \forall k \geq 1.$$

We claim that equality holds in the previous expression, namely

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner H_k}) = m_1(\lambda_k) \qquad \forall k \geq 1. \tag{6.10}$$

Indeed, assume by contradiction that $\mathcal{H}^2(\mathcal{G}_{\psi \llcorner H_{k_0}}) > m_1(\lambda_{k_0})$ for some $k_0 \geq 1$, and pick $\delta > 0$ with

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner H_{k_0}}) \geq \delta + m_1(\lambda_{k_0}). \tag{6.11}$$

Take $\Phi_{k_0} \in \mathcal{P}_1(\lambda_{k_0})$ a solution to $m_1(\lambda_{k_0})$. For $k > k_0$, as $H_{k_0} \subset H_k$, by a glueing argument[18], we can find $\Phi_k \in \mathcal{P}_1(\lambda_k)$ such that $\Phi_k(\overline{B_1}) = \Phi_{k_0}(\overline{B_1}) \cup \mathcal{G}_{\psi \llcorner (H_k \setminus H_{k_0})}$. Thus by (6.11) we have

$$\begin{aligned}\mathcal{H}^2(\mathcal{G}_{\psi \llcorner H_k}) \geq& \delta + m_1(\lambda_{k_0}) + \mathcal{H}^2(\mathcal{G}_{\psi \llcorner (H_k \setminus H_{k_0})}) \\ =& \delta + \mathcal{H}^2(\Phi_{k_0}(\overline{B}_1)) + \mathcal{H}^2(\mathcal{G}_{\psi \llcorner (H_k \setminus H_{k_0})}) \geq \delta + m_1(\lambda_k) \qquad \forall k > k_0.\end{aligned}$$

Letting $k \to +\infty$, since $\lambda_k \to \lambda$ in the sense of Fréchet, we have $m_1(\lambda_k) \to m_1(\lambda)$ [22, Theorem 4, Section 4.3]. In particular, from the previous inequality we infer

$$\mathcal{F}(\sigma, \psi) = \mathcal{H}^2(\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}) \geq \delta + m_1(\lambda).$$

Hence we conclude

$$\mathcal{H}^2(\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}) \geq 2\delta + 2m_1(\lambda) \geq 2\delta + m_2(\Gamma),$$

which contradicts (6.8). In the last inequality we have used that $2m_1(\lambda) \geq m_2(\Gamma)$; this follows from the fact that a disk-type parametrization of a minimizer for $m_1(\lambda)$ can be reparametrized on a half-annulus (as in the proof of Lemma 6.5), and glued with another reparametrization of it on the other half-annulus, so to obtain a parametrization of an annulus-type surface spanning Γ which is admissible for (6.4). Hence claim (6.10) follows.

Now, since ψ is Lipschitz continuous on $\overline{H}_k$, for all $k \in \mathbb{N}$ sufficently large there exists a map $\Psi_k \in H^1(B_1; \mathbb{R}^3) \cap C^0(\overline{B_1}; \mathbb{R}^3)$ with $\Psi_k(\partial B_1) = \lambda_k$ monotonically which solves the classical disk-type Plateau problem spanning λ_k and such that

$$\Psi_k(B_1) = \mathcal{G}_{\psi \llcorner H_k}.$$

Letting $k \to +\infty$ and using that the Dirichlet energy of Ψ_k equals the area of $\mathcal{G}_{\psi \llcorner H_k}$, we conclude that (Ψ_k) tends to a map $\Psi \in H^1(B_1; \mathbb{R}^3) \cap C^0(\overline{B_1}; \mathbb{R}^3)$ with $\Psi(\partial B_1) = \lambda$ weakly monotonically, and that is a solution of the classical disk-type Plateau problem with

$$\Psi(\overline{B}_1) = \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}.$$

[18] This is done, for instance, by glueing an external annulus to a disk, and using Φ_{k_0} from the disk, and a reparametrization of $\mathcal{G}_{\psi \llcorner (H_k \setminus H_{k_0})}$ from the annulus.

Arguing as in the proof of Lemma 6.5 we finally get a map $\Phi : \overline{\Sigma}_{\rm ann} \to \mathbb{R}^3$ which belongs to $\mathcal{P}_2(\Gamma)$ and parametrizes $\mathcal{G}_{\psi\llcorner(\overline{\Omega\setminus E(\sigma)})} \cup \mathcal{G}_{-\psi\llcorner(\overline{\Omega\setminus E(\sigma)})}$. This concludes the proof of (a).

(b). It is sufficient to argue as in case (a), by replacing $\Omega \setminus E(\sigma)$ in turn with F_1 and F_2 and $\Sigma_{\rm ann}$ with B_1 to find Φ_1 and Φ_2, respectively. □

Using the arguments above to show conditions (b) of Lemma 6.5 and Lemma 6.6, we deduce the following:

Corollary 6.7. *Let $n = 1$, assume that Γ is not planar, and let $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties* 1.-5. *of Theorem 5.1. Then there exists an injective map $\Phi \in W^{1,1}(B_1;\mathbb{R}^3) \cap C^0(\overline{B_1};\mathbb{R}^3)$ such that*

$$\Phi(\overline{B_1}) = \mathcal{G}_{\psi\llcorner(\overline{\Omega\setminus E(\sigma)})} \cup \mathcal{G}_{-\psi\llcorner(\overline{\Omega\setminus E(\sigma)})},$$

and $\Phi\llcorner\partial B_1 \colon \partial B_1 \to \Gamma$ is a weakly monotonic parametrization of Γ. Moreover, if $\mathcal{H}^2(\mathcal{G}_{\psi\llcorner(\overline{\Omega\setminus E(\sigma)})} \cup \mathcal{G}_{-\psi\llcorner(\overline{\Omega\setminus E(\sigma)})}) \leq m_1(\Gamma)$ then there is a reparametrization of Φ belonging to $\mathcal{P}_1(\Gamma)$ and solving (6.5).

Now we can start the proof of Theorems 6.1 and 6.4.

6.4 Proof of Theorem 6.1

Proof of Theorem 6.1. Let $\Phi \in \mathcal{P}_1(\Gamma)$ be a solution to (6.1). The curve Γ satisfies the assumptions of Lemma 5.4 (notice in this case we have $f(p_1) = f(q_1) = 0$), hence the minimal disk-type surface $S := \Phi(\overline{B_1})$ satisfies the following properties:

- $\beta_{p_1,q_1} := S \cap (\mathbb{R}^2 \times \{0\}) \subset \overline{\Omega}$ is a simple analytic curve joining p_1 and q_1 and such that $\beta_{p_1,q_1} \cap \partial\Omega = \{p_1, q_1\}$;
- S is symmetric with respect to $\mathbb{R}^2 \times \{0\}$;
- the surface $S^+ = S \cap \{x_3 \geq 0\}$ is the graph of a function $\widetilde{\psi} \in W^{1,1}(U_{p_1,q_1}) \cap C^0(\overline{U}_{p_1,q_1})$, where $U_{p_1,q_1} \subset \Omega$ is the open region enclosed between $\partial_1^D\Omega$ and β_{p_1,q_1}. Moreover $\widetilde{\psi}$ is analytic in U_{p_1,q_1};
- the curve β_{p_1,q_1} is contained in the closed convex hull of Γ, and $\Omega \setminus U_{p_1,q_1}$ is convex.

Let $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be given by

$$\sigma := \sigma_1 \qquad \text{and} \qquad \psi := \begin{cases} 0 & \text{in } \Omega \setminus U_{p_1,q_1} \\ \widetilde{\psi} & \text{in } U_{p_1,q_1}, \end{cases}$$

where $\sigma_1([0,1]) = \beta_{p_1,q_1}$. Clearly (6.3) holds, and $\mathcal{H}^2(S) = 2\mathcal{F}(\sigma,\psi) = m_1(\Gamma)$. It remains to show that (σ,ψ) is a minimizer of $\mathcal{F}$. Let $(\sigma',\psi') \in \mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ that satisfies properties 1.-5. of Theorem 5.1 and consider the disk-type surface with boundary Γ given by $S' := \mathcal{G}_{\psi'\llcorner(\overline{\Omega\setminus E(\sigma')})} \cup \mathcal{G}_{-\psi'\llcorner(\overline{\Omega\setminus E(\sigma')})}$. Since (σ,ψ) is admissible for $\mathcal{F}$, we deduce

$$\mathcal{H}^2(S') = 2\mathcal{F}(\sigma',\psi') \leq m_1(\Gamma).$$

Thus we are in the hypotheses of Corollary 6.7 and so there is a map $\Phi' \in \mathcal{P}_1(\Gamma)$ with $\Phi'(\overline{B}_1) = S'$. By minimality of (σ',ψ') and of S we have

$$\mathcal{H}^2(S) \leq \mathcal{H}^2(S') = 2\mathcal{F}(\sigma',\psi') \leq 2\mathcal{F}(\sigma,\psi) = \mathcal{H}^2(S). \tag{6.12}$$

Hence (σ, ψ) is a minimizer of $\mathcal{F}$ in $\mathcal{W}$ and Φ' is a solution to (6.1).

Conversely, let $(\sigma, \psi) \in \mathcal{W}_{\text{conv}}$ be a solution that satisfies properties 1.-5. of Theorem 5.1 and let $S := \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$. Let $\widetilde{\Phi}$ be a solution to (6.1); then we can find $(\widetilde{\sigma}, \widetilde{\psi}) \in \mathcal{W}$ whose doubled graph $\widetilde{S} = \mathcal{G}_{\widetilde{\psi} \llcorner (\overline{\Omega \setminus E(\widetilde{\sigma})})} \cup \mathcal{G}_{-\widetilde{\psi} \llcorner (\overline{\Omega \setminus E(\widetilde{\sigma})})}$ satisfies

$$\mathcal{H}^2(S) = 2\mathcal{F}(\sigma, \psi) \leq 2\mathcal{F}(\widetilde{\sigma}, \widetilde{\psi}) = \mathcal{H}^2(\widetilde{S}) = m_1(\Gamma).$$

Arguing as before we find a map $\Phi \in \mathcal{P}_1(\Gamma)$ parametrizing S. We conclude that Φ is a solution to (6.1), and the theorem is proved. □

6.5 Proof of Theorem 6.4

The proof of Theorem 6.4 is much more involved, so we divide it in a number of steps. We start with a result (which can be seen as the counterpart of Lemma 5.4 for the Plateau problem defined in (6.4)) that will be crucial to prove (i). In what follows we denote by $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$ the orthogonal projection.

Theorem 6.8. *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to* (6.4). *Then the minimal surface $\Phi(\overline{\Sigma}_{\text{ann}})$ satisfies the following properties:*

(1) *The set $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is simply connected in $\overline{\Omega}$; $\Omega \cap \partial\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ consists of two disjoint embedded analytic curves β_1 and β_2 joining q_1 to p_2, and q_2 to p_1, respectively. Moreover, for $i = 1, 2$, the closed region E_i enclosed between $\partial_i^0\Omega$ and β_i is convex;*

(2) $\Phi(\overline{\Sigma}_{\text{ann}})$ *is symmetric with respect to the plane $\mathbb{R}^2 \times \{0\}$;*

(3) $\Phi(\overline{\Sigma}_{\text{ann}}) \cap (\mathbb{R}^2 \times \{0\}) = \beta_1 \cup \beta_2$;

(4) $S^+ := \Phi(\overline{\Sigma}_{\text{ann}}) \cap \{x_3 \geq 0\}$ *is Cartesian. Precisely, it is the graph of a function* $\widetilde{\psi} \in W^{1,1}(\text{int}(\pi(\Phi(\overline{\Sigma}_{\text{ann}})))) \cap C^0(\pi(\Phi(\overline{\Sigma}_{\text{ann}})))$.

The proof of Theorem 6.8 is a consequence of Lemmas 6.9, 6.10, 6.11, 6.13 , 6.14, and 6.15 below.

Lemma 6.9 (Simply connectedness). *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to* (6.4). *Then $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is a simply connected region in $\overline{\Omega}$ and $\pi(\Phi(\overline{\Sigma}_{\text{ann}})) \cap \partial\Omega = \partial_1^D\Omega \cup \partial_2^D\Omega$.*

Proof. We recall that $\Phi : \overline{\Sigma}_{\text{ann}} \to \mathbb{R}^3$ is an embedding. The fact that $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is a subset of $\overline{\Omega}$ and $\pi(\Phi(\overline{\Sigma}_{\text{ann}})) \cap \partial\Omega = \partial_1^D\Omega \cup \partial_2^D\Omega$ follows from the fact that the interior of $\Phi(\overline{\Sigma}_{\text{ann}})$ is contained in the convex hull of Γ. So it remains to show that $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is simply connected.
Suppose by contradiction that $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is not simply connected. Let H be a hole of it, namely a region in Ω surrounded by a loop contained in $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ and such that $H \cap \pi(\Phi(\overline{\Sigma}_{\text{ann}})) = \emptyset$; choose a point $P \in H$. We will look for a contradiction by exploiting that Σ_{ann} is an annulus and using that the map Φ is analytic and harmonic.

Let θ be the angular coordinate of a cylindrical coordinate system (ρ, θ, z) in $\mathbb{R}^3$ centred at P and with z-axis the vertical line $\pi^{-1}(P)$. For $\theta \in [0, 2\pi)$ we consider the half-plane orthogonal to $\mathbb{R}^2 \times \{0\}$ defined by

$$\Pi_\theta := \{(\rho, \theta, z) \colon \rho > 0, z \in \mathbb{R}\}.$$

Now we fix two values θ_1 and θ_2 so that Π_{θ_1} and Π_{θ_2} intersect (the interior of) $\partial_1^0\Omega$ and $\partial_2^0\Omega$ respectively. The half-planes[19] $\Pi_{\theta_1+\pi}$ and $\Pi_{\theta_2+\pi}$ might intersect $\partial^D\Omega$ (see Figure 5). However,

[19]The angles are considered (mod 2π).

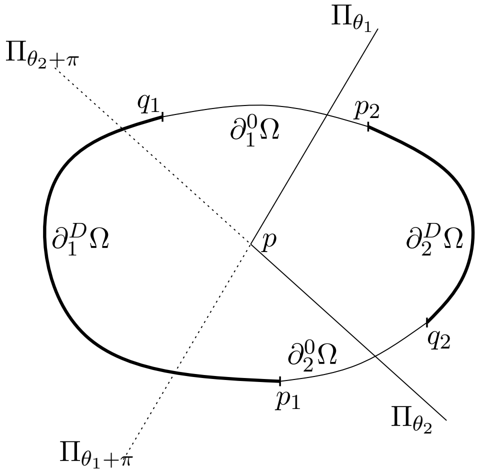

Figure 5: The horizontal section of two half-planes Π_{θ_1} and Π_{θ_2} intersecting $\partial_1^0\Omega$ and $\partial_2^0\Omega$, respectively.

since the points p_1, q_1, p_2, q_2, are in clockwise order on $\partial\Omega$, and Ω is convex, it is not difficult to conclude the following assertion:

The half-planes $\Pi_{\theta_1+\pi}$ and $\Pi_{\theta_2+\pi}$ cannot intersect the two components $\partial_1^D\Omega$ and $\partial_2^D\Omega$ of $\partial^D\Omega$ at the same time.

In other words: If, for instance, $\Pi_{\theta_1+\pi}$ intersects $\partial_1^D\Omega$, then $\Pi_{\theta_2+\pi}$ does not intersect $\partial_2^D\Omega$. Let us prove the assertion in the form of the last statement, being the other cases similar. This is trivial since, if Π_{θ_1} intersects $\partial_1^0\Omega$ and $\Pi_{\theta_1+\pi}$ intersects $\partial_1^D\Omega$ (as in Figure 5), we have that Π_θ intersects $\partial_1^D\Omega \cup \partial_1^0\Omega$ for all $\theta \in [\theta_1, \theta_1+\pi]$. As either θ_2 or $\theta_2+\pi$ belongs to $[\theta_1, \theta_1+\pi]$, we have that $\Pi_{\theta_2} \cup \Pi_{\theta_2+\pi}$ intersects $\partial_1^D\Omega \cup \partial_1^0\Omega$. Since by hypothesis Π_{θ_2} intersects $\partial_2^0\Omega$, it follows that $\Pi_{\theta_2+\pi}$ does not intersect $\partial_2^D\Omega$, and the statement follows.

Moreover, since Π_{θ_1} intersects $\partial_1^0\Omega$ and Π_{θ_2} intersects $\partial_2^0\Omega$, it is straightforward that:

If $\Pi_{\theta_1+\pi}$ intersects $\partial_1^0\Omega$ then also $\Pi_{\theta_2+\pi}$ intersects $\partial_1^0\Omega$.

We are now ready to conclude the proof of the lemma. We have to discuss the following cases:

(1) $\Pi_{\theta_1+\pi}$ intersects $\partial^0\Omega$;

(2) $\Pi_{\theta_1+\pi}$ intersects $\partial_1^D\Omega$;

(3) $\Pi_{\theta_1+\pi}$ intersects $\partial_2^D\Omega$.

By hypothesis on P, for all $\theta \in [0, 2\pi)$ the intersection between $\Phi(\overline{\Sigma}_{\text{ann}})$ and Π_θ consists of a family of smooth simple curves, either closed or with endpoints on Γ. Correspondingly, $\Phi^{-1}(\Phi(\overline{\Sigma}_{\text{ann}}) \cap \Pi_\theta)$ is a family of closed curves in $\overline{\Sigma}_{\text{ann}}$, possibly with endpoints on $C_1 \cup C_2$. In particular, since $\Pi_{\theta_1} \cap \partial_1^0\Omega \neq \emptyset$, the set[20] $\Phi^{-1}(\Phi(\overline{\Sigma}_{\text{ann}}) \cap \Pi_{\theta_1})$ is a family of closed curves in Σ_{ann}.

In case (1) also $\Phi^{-1}(\Phi(\overline{\Sigma}_{\text{ann}}) \cap \Pi_{\theta_1+\pi})$ consists of closed curves in Σ_{ann}. Take two loops α and α' in $\Phi^{-1}(\Phi(\overline{\Sigma}_{\text{ann}}) \cap \Pi_{\theta_1})$ and in $\Phi^{-1}(\Phi(\overline{\Sigma}_{\text{ann}}) \cap \Pi_{\theta_1+\pi})$ respectively. Let d_1 be the signed distance function from the plane $\overline{\Pi}_{\theta_1} \cup \Pi_{\theta_1+\pi}$, positive on $\partial_2^D\Omega$. Since $d_1 \circ \Phi$ changes its sign when one crosses transversally α and α', we easily see that both α and α' cannot be homotopically trivial in Σ_{ann} (by harmoniticy of $d_1 \circ \Phi$, if for instance α is homotopically trivial in Σ_{ann}, by the maximum principle $d_1 \circ \Phi = 0$ in the region enclosed by α, i.e. the image of Φ is locally flat, contradicting

[20]Since $\Pi_{\theta_1} \cap \partial^D\Omega = \emptyset$ these curves must be closed in Σ_{ann}.

the analyticity of Φ). Hence, since Φ is an embedding, they run exactly one time around C_1; as a consequence, they must be homotopically equivalent to each other in Σ_{ann}. On the other hand, they do not intersect each other (Φ is an embedding), so they bound an annulus-type region in Σ_{ann}, and by harmonicity $d_1 \circ \Phi$ is constantly null in this region. This would imply again that the image by Φ of this annulus is contained in $\overline{\Pi_{\theta_1}} \cup \Pi_{\theta_1+\pi}$, a contradiction.

In case (2), from our assertion, we deduce that $\Pi_{\theta_2+\pi}$ might intersect either $\partial^0\Omega$ or $\partial_1^D\Omega$. Further we can exclude that $\Pi_{\theta_2+\pi}$ intersects $\partial^0\Omega$ (otherwise, we repeat the argument for case (1) switching the role of θ_1 and θ_2). Therefore the only remaining possibility is that $\Pi_{\theta_2+\pi}$ intersects $\partial_1^D\Omega$ (see Figure 5). Let d_2 be the signed distance function from the plane $\overline{\Pi_{\theta_2}} \cup \Pi_{\theta_2+\pi}$ positive on $\partial_2^D\Omega$. In particular, $d_i \circ \Phi$, $i = 1, 2$, is positive on the circle C_2 of $\overline{\Sigma}_{\text{ann}}$. By hypothesis on d_i, $i = 1, 2$, we see that d_1 is positive on Π_{θ_2}, and d_2 is positive on Π_{θ_1}.

As in case (1), let $\alpha \subseteq \Phi^{-1}(\Phi(\overline{\Sigma}_{\text{ann}}) \cap \Pi_{\theta_1})$ and $\beta \subseteq \Phi^{-1}(\Phi(\overline{\Sigma}_{\text{ann}}) \cap \Pi_{\theta_2})$ be two loops. We know that α and β are closed in Σ_{ann}. Again, we conclude that α and β are homotopically equivalent in Σ_{ann}, and both run one time around C_1. Assume without loss of generality that β encloses α, which in turn encloses C_1. Since $d_2 \circ \Phi$ is positive on both α and C_2, $d_2 \circ \Phi$ must be positive in the region enclosed between them, contradicting the fact that it vanishes on β.

If instead we are in case (3) we can argue as in case (2) and get a contradiction. In all cases (1), (2), and (3), we reach a contradiction which derives by assuming that $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is not simply connected. The proof is achieved. □

We next proceed to characterize the geometry of $\Omega \cap \partial\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$.

Lemma 6.10 (Trace on the horizontal plane). *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4). Then $\Omega \cap \partial\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ consists of two disjoint Lipschitz embedded curves β_1 and β_2 joining q_1 to p_2, and q_2 to p_1, respectively. Moreover, the closed regions E_i enclosed between $\partial_1^0\Omega$ and β_i are convex for $i = 1, 2$.*

Proof. By Lemma 6.9, $\pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ is simply connected in $\overline{\Omega}$, and $\pi(\Phi(\overline{\Sigma}_{\text{ann}})) \cap \partial\Omega = \partial^D\Omega$. Therefore $\overline{\Omega} \setminus \pi(\Phi(\overline{\Sigma}_{\text{ann}}))$ consists of two simply connected components, one containing $\partial_1^0\Omega$ and the other containing $\partial_2^0\Omega$. Let E_1 and E_2 be the closures of these two components[21], so that in particular the boundary of E_i is a simple Jordan curve of the form $\beta_i \cup \partial_i^0\Omega$ for some embedded curve $\beta_i \subset \overline{\Omega}$ joining the endpoints of $\partial_i^0\Omega$. We will prove that E_i is convex for $i = 1, 2$. This will also imply that β_i are Lipschitz.

Take $i = 1$, and assume by contradiction that E_1 is not convex. Thus we can find a line l in $\mathbb{R}^2$ and three different points A_1, A_2, A_3 on l, with $A_2 \in \overline{A_1A_3}$, so that A_2 is contained in $\Omega \setminus E_1$, and A_1 and A_3 belong to the interior of E_1.

Consider the region $\pi(\Phi(\overline{\Sigma}_{\text{ann}})) \setminus l$, which consists in several (open) connected components. There is one of these connected components, say U, which does not intersect $\partial^D\Omega$ and whose boundary contains A_2. In addition, $\overline{U} \cap \partial^D\Omega = \emptyset$. Indeed, ∂U is the union of a segment L (containing A_2) and a curve γ (contained in $\beta_1 \subseteq \partial(\pi(\Phi(\overline{\Sigma}_{\text{ann}})))$ joining its endpoints. Hence, $\overline{U} \setminus U = \gamma \cup L$, and L cannot intersect $\partial^D\Omega$ by the hypothesis on A_1, A_2, and A_3.

Let $\Pi_l \subset \mathbb{R}^3$ be the plane containing l and orthogonal to the plane containing Ω; As usual, $\Pi_l \cap \Phi(\overline{\Sigma}_{\text{ann}})$ is a family of closed curves, possibly with endpoints on $\Gamma \cap \Pi_l$. Now, pick a point P on $\partial U \setminus L$, and let Q be a point on $\Phi(\overline{\Sigma}_{\text{ann}})$ so that $\pi(Q) = P$. Let $d_l : \mathbb{R}^3 \to \mathbb{R}$ be the signed distance from Π_l, with $d_l(Q) = d_l(P) > 0$. We claim that, if D is the connected component of $\{w \in \overline{\Sigma}_{\text{ann}} : d_l \circ \Phi(w) > 0\}$ containing the point $\Phi^{-1}(Q)$, then $D \cap \partial\Sigma_{\text{ann}} = \emptyset$. This would contradict the harmonicity of $d_l \circ \Phi$, since $d_l \circ \Phi$ would be zero on D, but $d_l(Q) > 0$, in contrast with the maximum principle.

[21] The sets E_1 and E_2 have nonempty interior, since $\Phi(\Sigma_{\text{ann}})$ is contained in the interior of the convex hull of $\Phi(\partial\Sigma_{\text{ann}})$, hence contained in the cylinder $\Omega \times \mathbb{R}$.

Assume by contradiction that the converse holds. Then there is an arc $\alpha : [0,1] \to D \cup \partial\Sigma_{\rm ann}$ joining $\Phi^{-1}(Q)$ to $\partial\Sigma_{\rm ann}$. The image of the map $\pi \circ \Phi \circ \alpha$ is an arc in $\overline{\Omega}$ joining P to $\partial^D\Omega$ and such that $d_l \geq 0$ on it. Clearly this arc is a subset of $\pi(\Phi(\overline{\Sigma}_{\rm ann}))$. Since $\pi \circ \Phi \circ \alpha(0) = P$, it follows that the image of $\pi \circ \Phi \circ \alpha$ is contained in $\overline{U}$. Now $\overline{U}$ does not intersect $\partial^D\Omega$, contradicting that $\pi \circ \Phi \circ \alpha(1) \in \partial^D\Omega$. This concludes the proof. $\square$

In the next step we show that there exists a set $E \subset \mathbb{R}^3$ of finite perimeter such that

$$\partial E = \partial^* E = \Phi(\Sigma_{\rm ann}) \cup \overline{\Delta}_1 \cup \overline{\Delta}_2,$$

where ∂^* denotes the reduced boundary, and

$$\Delta_i := \{P = (P', P_3) \in \mathbb{R}^3 : P' = (P_1, P_2) \in \partial_i^D\Omega,\ P_3 \in (-\varphi(P'), \varphi(P'))\}, \qquad i = 1, 2. \tag{6.13}$$

In particular $\overline{\Delta}_1 \cup \overline{\Delta}_2 \subset (\partial\Omega) \times \mathbb{R}$ and $(\Omega \times \mathbb{R}) \cap \partial E = \Phi(\Sigma_{\rm ann})$.

We first fix some notation. We let $[\![E]\!] \in \mathcal{D}_3(\mathbb{R}^3)$ be the 3-current given by integration over E with $E \subset \mathbb{R}^3$ a set of finite perimeter. To every $\mathcal{MY}$ solution $\Phi \in \mathcal{P}_2(\Gamma)$ to (6.4) we associate the push-forward 2-current $\Phi_\sharp[\![\Sigma_{\rm ann}]\!] \in \mathcal{D}_2(\mathbb{R}^3)$ given by integration over the (suitably oriented) surface $\Phi(\Sigma_{\rm ann})$ [34, Section 7.4.2]. Finally, if $\mathcal{T} \in \mathcal{D}_k(U)$ with $U \subset \mathbb{R}^3$ open and $k = 2, 3$, we denote by $|\mathcal{T}|$ the mass of $\mathcal{T}$ in U [24, p. 358].

Lemma 6.11 (Region enclosed by $\Phi(\Sigma_{\rm ann})$). *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4). Then there is a closed finite perimeter set $E \subset \overline{\Omega} \times \mathbb{R}$ such that $\partial E \cap (\Omega \times \mathbb{R}) = \Phi(\Sigma_{\rm ann})$.*

Proof. As $\Phi_\sharp[\![\Sigma_{\rm ann}]\!]$ is a boundaryless integral 2-current in $\Omega \times \mathbb{R}$, there exists (see, e.g., [34, Theorem 7.9.1]) an integral 3-current $\mathcal{E} \in \mathcal{D}_3(\Omega \times \mathbb{R})$ with $\partial\mathcal{E} = \Phi_\sharp[\![\Sigma_{\rm ann}]\!]$, and we might also assume that the support of $\mathcal{E}$ is compact in $\overline{\Omega} \times \mathbb{R}$. We claim that, up to switching the orientation of $\Phi_\sharp[\![\Sigma_{\rm ann}]\!]$, $\mathcal{E}$ has multiplicity in $\{0, 1\}$, and hence is the integration $[\![E]\!]$ over a bounded measurable set E. Since $\partial\mathcal{E} = \Phi_\sharp[\![\Sigma_{\rm ann}]\!]$, this will be a finite perimeter set, and $[\![(\Omega \times \mathbb{R}) \cap \partial^* E]\!] = \Phi_\sharp[\![\Sigma_{\rm ann}]\!]$.

By Federer decomposition theorem [24, Section 4.2.25, p. 420] (see also [24, Section 4.5.9] and [34, Theorem 7.5.5]) there is a sequence $(E_k)_{k\in\mathbb{N}}$ of finite perimeter subsets of $\Omega \times \mathbb{R}$ such that

$$\mathcal{E} = \sum_{k=1}^{+\infty} \sigma_k [\![E_k]\!], \qquad \sigma_k \in \{-1, 1\}, \tag{6.14}$$

and

$$|\mathcal{E}| = \sum_{k=1}^{+\infty} |E_k| \quad \text{and} \quad |\partial\mathcal{E}| = \mathcal{H}^2(\Phi(\Sigma_{\rm ann})) = \sum_{k=1}^{+\infty} \mathcal{H}^2(\partial^* E_k). \tag{6.15}$$

We start by observing that

$$\partial^* E_k \subseteq \Phi(\Sigma_{\rm ann}) \qquad \forall k \in \mathbb{N}. \tag{6.16}$$

Indeed, fixing $k \in \mathbb{N}$, by the second equation in (6.15), we have that $\partial^* E_k$ is contained in the support of $\partial\mathcal{E}$, which in turn is $\Phi(\Sigma_{\rm ann})$. As a consequence, if $P = (P_1, P_2, P_3) \in (\Omega \times \mathbb{R}) \cap \overline{\partial^* E_k}$, then $P \in \Phi(\Sigma_{\rm ann})$. Around P we can find suitable coordinates and a cube $U = (P_1 - \varepsilon, P_1 + \varepsilon) \times (P_2 - \varepsilon, P_2 + \varepsilon) \times (P_3 - \varepsilon, P_3 + \varepsilon)$ such that $\Phi(\Sigma_{\rm ann}) \cap U$ is the graph $\mathcal{G}_h$ of a smooth function $h : (P_1 - \varepsilon, P_1 + \varepsilon) \times (P_2 - \varepsilon, P_2 + \varepsilon) \to (P_3 - \varepsilon, P_3 + \varepsilon)$. Moreover, $\Phi_\sharp[\![\Sigma_{\rm ann}]\!] = [\![\mathcal{G}_h]\!]$ in U.

We claim that

$$\forall k \quad \text{either} \quad E_k \cap U = U \cap SG_h \quad \text{or} \quad E_k \cap U = U \setminus SG_h.$$

Indeed, assume for instance that $|E_k \cap U \cap SG_h| > 0$ and $|(SG_h \setminus E_k) \cap U| > 0$; by the constancy lemma [34] it follows that $\partial[\![E_k]\!]$ is nonzero in the simply connected open set SG_h, contradicting (6.16). As a consequence of the preceding claim, we have $U \cap \partial^* E_k = U \cap \Phi(\Sigma_{\rm ann})$. Since this argument holds for any choice of $P \in (\Omega \times \mathbb{R}) \cap \overline{\partial^* E_k}$, we have proved that $(\Omega \times \mathbb{R}) \cap \overline{\partial^* E_k}$ is relatively open (and relatively closed at the same time) in $\Phi(\Sigma_{\rm ann})$, which in turn being a connected open set, implies

$$\Phi(\overline{\Sigma}_{\rm ann}) = \overline{\partial^* E_k} \qquad \forall k \in \mathbb{N}.$$

Denote by $\mathcal{I}^\pm := \{k \in \mathbb{N} : \sigma_k = \pm 1\}$, with σ_k as in (6.14). Going back to the local behaviour around $P \in \Phi(\Sigma_{\rm ann})$, if U is a neighbourhood as above, we see that for all $k \in \mathcal{I}^+$ either $E_k \cap U = SG_h$ or $E_k = U \setminus SG_h$ (namely, all the E_k's coincide in U), since otherwise, there will be cancellations in the series $\sum_{k \in \mathcal{I}^+} \partial[\![E_k]\!]$, in contradiction with the second formula in (6.15). Assume without loss of generality that for all $k \in \mathcal{I}^+$ we have $E_k \cap U = SG_h$; thus, arguing as before, for all $k \in \mathcal{I}^-$ we must have $E_k \cap U = U \setminus SG_h$.

We obtain that $\mathcal{E} \llcorner U = m[\![SG_h]\!] - n[\![U \setminus SG_h]\!]$ for some nonnegative integers n, m. Since $(\partial \mathcal{E}) \llcorner U = (m+n)[\![\mathcal{G}_h]\!]$ and also $(\partial \mathcal{E}) \llcorner U = \Phi_\sharp[\![\Sigma_{\rm ann}]\!] = [\![\mathcal{G}_h]\!]$ in U, we conclude $m + n = 1$. Hence either $m = 1$ and $n = 0$, or $m = 0$ and $n = 1$. On the other hand, we know that $\mathcal{E} \llcorner U = \sum_{k \in \mathcal{I}^+} [\![E_k \cap U]\!] - \sum_{k \in \mathcal{I}^-} [\![E_k \cap U]\!]$, from which it follows that $\mathcal{I}^+$ has cardinality m and $\mathcal{I}^-$ has cardinality n. Namely, one of the sets $\mathcal{I}^\pm$ is empty, and the other contains one index only.

We conclude that the sum in (6.14) involves one index only, that is, there is only one compact set E in $\overline{\Omega} \times \mathbb{R}$ such that (up to switching the orientation)

$$\mathcal{E} = [\![E]\!].$$

This concludes the proof. □

For later convenience, from now on we denote by E the closure of a precise representative of the set found in Lemma 6.11.

Remark 6.12. From the fact that $(\overline{\Omega} \times \mathbb{R}) \cap \partial E = \Phi(\overline{\Sigma}_{\rm ann}) \cup \overline{\Delta}_1 \cup \overline{\Delta}_2$, we easily see that $\pi(E) = \pi(\Phi(\overline{\Sigma}_{\rm ann}))$ which, by Lemma 6.9, is simply connected.

We denote by $\mathrm{sym}_{\rm st}(E)$ the set (symmetric with respect to the horizontal plane $\mathbb{R}^2 \times \{0\}$) obtained applying to E the Steiner symmetrization with respect to $\mathbb{R}^2 \times \{0\}$.
Clearly $\mathrm{sym}_{\rm st}(E) \cap (\partial_i^D \Omega \times \mathbb{R}) = \overline{\Delta_i}$ with Δ_i defined as in (6.13). We define

$$S := \partial(\mathrm{sym}_{\rm st}(E)) \setminus (\Delta_1 \cup \Delta_2), \quad S^+ := S \cap \{x_3 \geq 0\}, \quad S^- := S \cap \{x_3 \leq 0\}. \qquad (6.17)$$

Since $P(\mathrm{sym}_{\rm st}(E)) \leq P(E)$ (here $P(\cdot)$ is the perimeter in $\mathbb{R}^3$ [4]) we have $\mathcal{H}^2(S) \leq \mathcal{H}^2(\Phi(\overline{\Sigma}_{\rm ann}))$.

Lemma 6.13 (Graphicality of $\partial(\mathrm{sym}_{\rm st}(E))$ and continuity up to the boundary). *Suppose that $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4). Let E be the finite perimeter set given by Lemma 6.11 and $S^\pm$ be as in (6.17). Then there is $\widetilde{\psi} \in BV(\mathrm{int}(\pi(E))) \cap C^0(\pi(E))$ such that $S^\pm = \mathcal{G}_{\pm\widetilde{\psi}}$. In particular $S^\pm \cap (\mathbb{R}^2 \times \{0\}) = \overline{\Omega \cap \partial(\pi(E))}$.*

Proof. Since E has finite perimeter, there exists a function $\widetilde{\psi} \in BV(\mathrm{int}(\pi(E)))$ such that $S^\pm = \mathcal{G}_{\pm\widetilde{\psi}}$ [20]. So, we only need to show that $\widetilde{\psi}$ is continuous (note that $\pi(E)$ is a closed set). Take a point P' in the interior of $\pi(E)$; if $P' = \pi(\Phi(w))$ for some w, then $w \in \Sigma_{\rm ann}$, since $\pi(\Phi(C_i)) \subset \partial\Omega$ for $i = 1, 2$ (recall C_1 and C_2 form the boundary of $\Sigma_{\rm ann}$). If at none of the points of $\pi^{-1}(P') \cap \Phi(\overline{\Sigma}_{\rm ann})$ the tangent plane to $\Phi(\overline{\Sigma}_{\rm ann})$ is vertical, then $\widetilde{\psi}$ is C^∞ in a neighbourhood of P', since it is the linear combination of smooth functions (see the discussion after formula (6.21) below, where details are given). Therefore we only have to check continuity of $\widetilde{\psi}$ at those points P' for which there is $P \in \pi^{-1}(P') \cap \Phi(\overline{\Sigma}_{\rm ann})$ such that $\Phi(\overline{\Sigma}_{\rm ann})$ has a vertical tangent plane Π at P.

Consider a system of Cartesian coordinates centred at P, with the (x, y)-plane coinciding with Π, the x-axis coinciding with the line $\pi^{-1}(P')$, and let $z = z(x, y)$ (defined at least in a neighbourhood of 0) be the analytic function whose graph coincides with $\Phi(\overline{\Sigma}_{\rm ann})$. This map, restricted to the x-axis, is analytic and vanishes at $x = 0$; hence it is either identically zero or it has a discrete set of zeroes (in the neighbourhood where it exists). We now exclude the former case: If $z(\cdot, 0)$ is identically zero, it means that around P there is a vertical open segment included in $\pi^{-1}(P')$, which is contained in $\Phi(\overline{\Sigma}_{\rm ann})$. Let Q be an extremal point of this segment, and let Π_Q be the tangent plane to $\Phi(\overline{\Sigma}_{\rm ann})$ at Q. This plane must contain as tangent vector the above segment, hence Π_Q is vertical and contains $\pi^{-1}(P')$. Choosing again a suitable Cartesian coordinate system centred at Q we can express locally the surface $\Phi(\overline{\Sigma}_{\rm ann})$ as the graph of an analytic function defined in a neighbourhood of Q in Π_Q, and so the restriction of this map to $\pi^{-1}(P')$ is analytic in a neighbourhood of Q, hence it must be identically zero since it is zero in a left (or right) neighbourhood of Q. What we found is that we can properly extend the segment $\overline{PQ}$ on the Q side to a segment $\overline{PR}$ contained in $\Phi(\overline{\Sigma}_{\rm ann})$. This proves that $\Phi(\overline{\Sigma}_{\rm ann}) \cap \pi^{-1}(P')$ is relatively open in $\pi^{-1}(P')$. Since it is also relatively closed, it coincides with the whole line $\pi^{-1}(P')$, which is impossible since $\Phi(\overline{\Sigma}_{\rm ann})$ is bounded.

Hence the zeroes of the function $z(\cdot, 0)$ are isolated, so we have shown:

Assertion A: Let $P \in \pi^{-1}(P') \cap \Phi(\overline{\Sigma}_{\rm ann})$. Then in a neighbourhood of P the only intersection between $\Phi(\overline{\Sigma}_{\rm ann})$ and $\pi^{-1}(P')$ is P itself.

Now, we can conclude the proof of the continuity of the function $\widetilde{\psi}$. Write $\pi^{-1}(P') \cap \Phi(\overline{\Sigma}_{\rm ann}) = \{Q_1, Q_2, \dots, Q_m\} \subset \Omega \times \mathbb{R}$. It follows that

$$2\widetilde{\psi}(P') = \mathcal{H}^1(\pi^{-1}(P') \cap E) = \sum_{j=1}^{m} \sigma_j (Q_j)_3, \tag{6.18}$$

where $(Q_j)_3$ is the vertical coordinate of Q_j and

$$\sigma_j = \begin{cases} -1 & \text{if } \overline{Q_{j-1}Q_j} \subset \overline{\mathbb{R}^3 \setminus E} \text{ and } \overline{Q_jQ_{j+1}} \subset E, \\ 1 & \text{if } \overline{Q_{j-1}Q_j} \subset E \text{ and } \overline{Q_jQ_{j+1}} \subset \overline{\mathbb{R}^3 \setminus E}, \\ 0 & \text{otherwise,} \end{cases} \qquad j = 1, \dots, m. \tag{6.19}$$

Let $(P'_k) \subset \mathrm{int}(\pi(E))$ be a sequence converging to P', and write $\pi^{-1}(P'_k) \cap \Phi(\overline{\Sigma}_{\rm ann}) = \{Q^k_1, Q^k_2, \dots, Q^k_{m_k}\} \subset \Omega \times \mathbb{R}$. With a similar notation as above, we have

$$2\widetilde{\psi}(P'_k) = \mathcal{H}^1(\pi^{-1}(P'_k) \cap E) = \sum_{j=1}^{m_k} \sigma^k_j (Q^k_j)_3. \tag{6.20}$$

Now, if at every point Q_j the tangent plane to $\Phi(\overline{\Sigma}_{\rm ann})$ is not vertical, then $\Phi(\overline{\Sigma}_{\rm ann})$ is a smooth Cartesian surface in a neighbourhood of Q_j, and so it is clear that, for k large enough,

$$m = m_k, \qquad Q^k_j \to Q_j, \qquad \sigma^k_j \to \sigma_j \qquad \text{for all } j = 1, \dots, m, \tag{6.21}$$

and the continuity of $\widetilde{\psi}$ at P' follows. Therefore it remains to check continuity in the case that the tangent plane to some Q_j is vertical.

Let $\widetilde{Q}$ be one of these points, with associated sign $\widetilde{\sigma} \in \{0, 1\}$. By assertion A there is $\delta > 0$ so that $\widetilde{Q}$ is the unique intersection between $\pi^{-1}(P')$ and $\Phi(\overline{\Sigma}_{\rm ann})$ with vertical coordinate in $[\widetilde{Q}_3 - \delta, \widetilde{Q}_3 + \delta]$.

This means that the segments $\pi^{-1}(P')\cap\{\widetilde{Q}_3-\delta<x_3<\widetilde{Q}_3\}$ and $\pi^{-1}(P')\cap\{\widetilde{Q}_3<x_3<\widetilde{Q}_3+\delta\}$ are contained in either $\mathrm{int}(E)$ or $\mathbb{R}^3\setminus E$. In particular, there is a neighbourhood $U\subset\Omega$ of P' such that $U\times\{x_3=\widetilde{Q}_3-\delta\}$ and $U\times\{x_3=\widetilde{Q}_3+\delta\}$ are subsets of $\mathrm{int}(E)$ or of $\mathbb{R}^3\setminus E$. Suppose without loss of generality that both are inside $\mathbb{R}^3\setminus E$ (the other cases being similar), so that $\widetilde{\sigma}=0$. We infer that, for k large enough so that $P'_k\in U$, there is a finite subfamily $\{Q^k_j : j\in J\}$ of $\{Q^k_1,Q^k_2,\dots,Q^k_{m_k}\}$ contained in $\{\widetilde{Q}_3<x_3<\widetilde{Q}_3+\delta\}$ and which satisfies the following: The sum in (6.20) restricted to such subfamily reads as:

$$\sum_{j\in J}\sigma^k_j(Q^k_j)_3=(Q^k_{j_l})_3-(Q^k_{j_{l-1}})_3+\cdots+(Q^k_{j_2})_3-(Q^k_{j_1})_3,$$

where $J=\{j_1,j_2,\dots,j_l : j_1<j_2<\cdots<j_l\}$ and $(Q^k_{j_l})_3>(Q^k_{j_{l-1}})_3>\cdots>(Q^k_{j_2})_3>(Q^k_{j_1})_3$ (if $j_l=1$ necessarily $\sigma^k_{j_1}=0$ and the sum is zero). We have to show that this sum tends to $\widetilde{\sigma}\widetilde{Q}_3=0$ as $k\to+\infty$, which is true, since each Q^k_j tends to $\widetilde{Q}$. Repeating this argument for each point $\widetilde{Q}$ with a vertical tangent plane to $\Phi(\overline{\Sigma}_{\rm ann})$, the proof of continuity of $\widetilde{\psi}$ in the interior of $\pi(E)$ follows.

Now, let $P'\in\partial(\pi(E))$. If $P'\in\Omega\cap\partial(\pi(E))$ then every point in $\pi^{-1}(P')\cap\Phi(\overline{\Sigma}_{\rm ann})$ has vertical tangent plane and we can argue as in the previous case. It remains to show continuity of $\widetilde{\psi}$ on $\partial\pi(E)\cap\partial\Omega$. In this case we exploit the fact that the interior of $\Phi(\overline{\Sigma}_{\rm ann})$ is contained in $\Omega\times\mathbb{R}$. We sketch the proof without details since it is very similar to the previous argument. Let $P'\in\partial^D_1\Omega$, thus $\pi^{-1}(P')\cap\Gamma_1$ consists of two distinct points Q_1 and Q_2. Let (P'_k) be a sequence of points in $\pi(E)$ converging to P. For $P'_k\in\partial^D_1\Omega$ it follows $\pi^{-1}(P'_k)\cap\Gamma_1=\{Q^k_1,Q^k_2\}$ and the continuity of $\widetilde{\psi}$ follows from the continuity of φ on $\partial^D_1\Omega$, whereas if P'_k is in the interior of $\pi(E)$ there holds $\pi^{-1}(P'_k)\cap\Gamma_1=\{Q^k_1,Q^k_2,\dots,Q^k_{m_k}\}$. Using the continuity of Φ up to C_1, it is easily seen that all such points must converge, as $k\to+\infty$, either to Q_1 or to Q_2. Hence we can repeat an argument similar to the one used before. ☐

Lemma 6.14. *Suppose $m_2(\Gamma)<m_1(\Gamma_1)+m_1(\Gamma_2)$ and let $\Phi\in\mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to* (6.4). *Let E be the finite perimeter set given in Lemma 6.11 and let S be defined as in* (6.17). *Then there is an injective map $\widetilde{\Phi}\in H^1(\Sigma_{\rm ann};\mathbb{R}^3)\cap C^0(\overline{\Sigma}_{\rm ann};\mathbb{R}^3)$ which maps $\partial\Sigma_{\rm ann}$ weakly monotonically to Γ and such that $\widetilde{\Phi}(\overline{\Sigma}_{\rm ann})=S$, and furthermore*

$$\mathcal{H}^2(S)=\int_{\Sigma_{\rm ann}}|\partial_{w_1}\widetilde{\Phi}\wedge\partial_{w_2}\widetilde{\Phi}|dw=\int_{\Sigma_{\rm ann}}|\partial_{w_1}\Phi\wedge\partial_{w_2}\Phi|dw=m_2(\Gamma).\tag{6.22}$$

In particular, $\widetilde{\Phi}$ is a solution of (6.4).

Proof. By Lemma 6.13 there is $\widetilde{\psi}\in BV(\mathrm{int}(\pi(E)))\cap C^0(\pi(E))$ such that $S^\pm=\mathcal{G}_{\pm\widetilde{\psi}}$. As a consequence, for $p\in\partial^D\Omega$ we have $\widetilde{\psi}(p)=\varphi(p)$ and for $p\in\partial(\pi(E))\cap\Omega$ we have $\widetilde{\psi}(p)=0$.
By Lemma 6.9 $\pi(E)$ is simply connected, and so the maps $\widetilde{\Psi}^\pm:\pi(E)\to\mathbb{R}^3$ given by $\widetilde{\Psi}^\pm(p):=(p,\pm\widetilde{\psi}(p))$ are disk-type parametrizations of $S^\pm$. Moreover S^+ and S^- glue to each other along $\partial(\mathrm{sym}_{\rm st}(E))\cap(\mathbb{R}^2\times\{0\})=\beta_1\cup\beta_2$, where β_1 and β_2 are the curves given by Lemma 6.10 .

Let $(\sigma,\psi)\in\mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ which satisfies properties 1.-5. of Theorem 5.1. Setting $\widetilde{\sigma}:=(\beta_1,\beta_2)$ and extending $\widetilde{\psi}$ to zero in $\overline{\Omega}\setminus\pi(E)$ (still calling $\widetilde{\psi}$ such an extension), by minimality we get

$$2\mathcal{F}(\sigma,\psi)\le2\mathcal{F}(\widetilde{\sigma},\widetilde{\psi})=\mathcal{H}^2(S),$$

whence

$$2\mathcal{F}(\sigma,\psi)\le\mathcal{H}^2(S)\le\mathcal{H}^2(\Phi(\overline{\Sigma}_{\rm ann}))=\int_{\Sigma_{\rm ann}}|\partial_{w_1}\Phi\wedge\partial_{w_2}\Phi|dw=m_2(\Gamma).\tag{6.23}$$

We are in the hypotheses of Lemma 6.6 (a), therefore there exists a map $\widehat{\Phi} \in P_2(\Gamma)$ parametrizing $\mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ which is a minimizer of (6.4). In particular, $2\mathcal{F}(\sigma, \psi) = m_2(\Gamma)$, and all inequalities in (6.23) are equalities. We deduce also that $(\widetilde{\sigma}, \widetilde{\psi})$ is a minimizer of $\mathcal{F}$ in $\mathcal{W}_{\rm conv}$, so that by Theorem 5.1 $\widetilde{\psi}$ is analytic in $\mathrm{int}(\pi(E))$. As a consequence it belongs to $W^{1,1}(\mathrm{int}(\pi(E)); \mathbb{R}^3)$. Applying Lemma 6.5 (a) and Lemma 6.6 (a), we get the existence of $\widetilde{\Phi} \in P_2(\Gamma)$ as in the statement, and we have concluded. □

Lemma 6.15. *Suppose $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$ and let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4). Let E be the finite perimeter set given in Lemma 6.11 and let S be defined as in (6.17). Then $\Phi(\overline{\Sigma}_{\rm ann}) = S$ and in particular*

$$E = \mathrm{sym}_{\rm st}(E).$$

Proof. By Lemma 6.14 we have $\mathcal{H}^2(S) = m_2(\Gamma)$ from which it follows $P(\mathrm{sym}_{\rm st}(E)) = P(E)$. Then we can apply [20, Theorem 1.1] to deduce the existence of two functions $f, g : \pi(E) \to \mathbb{R}$ of bounded variation, such that $\partial^* E = \mathcal{G}_f \cup \mathcal{G}_g$ (up to $\mathcal{H}^2$-negligible sets). We will show that $f = \widetilde{\psi}$ and $g = -\widetilde{\psi}$. To this aim, thanks again to [20, Theorem 1.1], we know that for a.e. $p \in \pi(E)$, the two unit (external to E) normal vectors $\nu^f = (\nu_1^f, \nu_2^f, \nu_3^f)$ and $\nu_g = (\nu_1^g, \nu_2^g, \nu_3^g)$ to $\mathcal{G}_f$ and $\mathcal{G}_g$ at the points $(p, f(p))$ and $(p, g(p))$, respectively, satisfy

$$(\nu_1^f, \nu_2^f, \nu_3^f) = (\nu_1^g, \nu_2^g, -\nu_3^g). \tag{6.24}$$

To conclude the proof it is then sufficient to show that $f = -g$ a.e. on $\pi(E)$: indeed this would readily imply $E = \mathrm{sym}_{\rm st}(E)$ and hence $f = \widetilde{\psi}$. Let $p \in \mathrm{int}(\pi(E))$; if

$$\pi^{-1}(p) \cap S = \{P_1, P_2, \ldots, P_k\}, \tag{6.25}$$

then for a.e. $p \in \mathrm{int}(\pi(E))$ it is $k \leq 2$. Now we show that, for all $p \in \mathrm{int}(\pi(E))$, if $k > 1$, none of the points $\{P_1, P_2, \ldots, P_k\}$ has vertical tangent plane. Assume by contradiction that P_1 has vertical tangent plane Π_1. In this case $\Pi_1 \cap S$ consists, in a neighbourhood U of P_1, of at least 2 curves crossing transversally (see [37, Section 373]) at P_1. These curves, by assertion A in the proof of Lemma 6.13 , intersect $\pi^{-1}(p)$ only at P_1. Moreover, in a neighbourhood V of P_2, with $U \cap V = \emptyset$, $\Pi_1 \cap S$ consists of (at least) one curve passing through P_2. This curve is locally Cartesian if $\pi^{-1}(p)$ crosses S transversally in P_2, otherwise it is locally the union of two curves ending at P_2, with vertical tangent plane, which lie on the same side of Π_1 with respect to $\pi^{-1}(p)$. In both cases, we deduce that there is a point $q \in \Pi_1 \cap (\Omega \times \{0\})$ for which $\pi^{-1}(q)$ intersects transversally S in at least three points. As a consequence, for all q' in a neighbourhood of q in Ω, the line $\pi^{-1}(q')$ intersects S at more than two points, which is a contradiction. We have proved the following:

Assertion: for all $p \in int(\pi(E))$ the line $\pi^{-1}(p)$ intersects S either transversally at two points P_1, P_2, or at only one point P_1.

Now we see that the latter case cannot happen. Indeed, first one checks that in this case the intersection cannot be transversal[22], and that $\pi^{-1}(p)$ must be tangent to S at P_1. Let Π_1 be the vertical tangent plane to S at P_1. Let $\Pi_1^\perp$ be the vertical plane orthogonal to Π_1 passing through P_1. In a neighbourhood of P_1, the unique curve in $S \cap \Pi_1^\perp$ must be the union of two curves joining at P_1, and these curves must belong to the same half-plane of $\Pi_1^\perp$ with boundary $\pi^{-1}(p)$. As a consequence, if $p' \in \Omega \cap \Pi_1^\perp$ is in that half-plane, then $\pi^{-1}(p')$ consists of at least two points; if p' lies in the opposite half-plane, then $\pi^{-1}(p')$ is empty. This means that necessarily $p \in \partial\pi(E)$. Namely, the previous assertion can be strengthened to:

[22] This is a consequence of the fact that the line $\pi^{-1}(p)$ must lie outside the set E, with the only exception of the point P_1. Indeed, otherwise, there must be some other point in $\pi^{-1}(p) \cap S$, E being compact in $\mathbb{R}^3$.

For all $p \in int(E)$ the line $\pi^{-1}(p)$ intersects S transversally at exactly two points P_1, P_2.

The consequence of this is that f and g belong to $W^{1,1}(\text{int}(\pi(E)))$ and are also smooth in $\text{int}(\pi(E))$. Indeed, let $p \in \text{int}(\pi(E))$, so $f(p) \neq g(p)$, and

$$\pi^{-1}(p) \cap S = \{(p, f(p)), (p, g(p))\}. \tag{6.26}$$

Since S is locally the graph of smooth functions around $(p, f(p))$ and $(p, g(p))$, these functions coincide with f and g, respectively. We can now conclude the proof of the lemma: let us choose a simple curve $\alpha : [0,1] \to \pi(E)$ with $\alpha(0) \in \partial^D\Omega$ and $\alpha(1) = p$ such that (6.24) holds for $\mathcal{H}^1$ a.e. $p \in \alpha([0,1])$. Since $f \circ \alpha$ and $g \circ \alpha$ are differentiable in $[0,1]$, condition (6.24) uniquely determines the tangent planes to $\mathcal{G}_f$ and $\mathcal{G}_g$, and hence it implies that the derivatives of $f \circ \alpha$ and $g \circ \alpha$ satisfy

$$(f \circ \alpha)'(t) + (g \circ \alpha)'(t) = 0, \qquad \text{for a.e. } t \in [0,1]. \tag{6.27}$$

By continuity of f and g one infers $f \circ \alpha + g \circ \alpha = c$ a.e. on $[0,1]$ (actually everywhere since $f + g$ is continuous), for some constant $c \in \mathbb{R}$. To show that $c = 0$ it is sufficient to observe that $f \circ \alpha(0) = \varphi(\alpha(0))$ and $g \circ \alpha(0) = -\varphi(\alpha(0))$. Hence $f(p) = -g(p)$, and the thesis of Lemma 6.15 is achieved. □

We are now in a position to conclude the proof of Theorem 6.8.

Proof of Theorem 6.8. Property (1) follows by Lemma 6.9 and Lemma 6.10. Properties (2)–(4) follow by Lemma 6.13 and Lemma 6.15. To see that β_i are C^∞ it is sufficient to observe that, since S^+ and S^- are Cartesian surfaces, their intersection coincides with the set $S \cap \{x_3 = 0\}$ which, by standard arguments, is the image of the zero-set of Φ_3, which is smooth. □

Theorem 6.16. *Assume $n = 2$ and Γ_j not planar for $j = 1, 2$. Then*

$$2 \min_{(s,\zeta)\in\mathcal{W}_{\text{conv}}} \mathcal{F}(s,\zeta) = m_2(\Gamma). \tag{6.28}$$

Proof. Step 1: $2\min_{(s,\zeta)\in\mathcal{W}_{\text{conv}}} \mathcal{F}(s,\zeta) \leq m_2(\Gamma)$.

Suppose first $m_2(\Gamma) < m_1(\Gamma_1) + m_1(\Gamma_2)$. Let $\Phi \in \mathcal{P}_2(\Gamma)$ be a $\mathcal{MY}$ solution to (6.4) and let $S := \Phi(\overline{\Sigma}_{\text{ann}})$. By Theorem 6.8 the following properties hold:

- $S \cap (\mathbb{R}^2 \times \{0\}) = \beta_1 \cup \beta_2$ with β_1 and β_2 disjoint embedded analytic curves joining q_1 to p_2 and q_2 to p_1, respectively;
- S is symmetric with respect to $\mathbb{R}^2 \times \{0\}$;
- for $i = 1, 2$ the closed region E_i enclosed between $\partial_i^0\Omega$ and β_i is convex;
- $S^+ = S \cap \{x_3 \geq 0\}$ is the graph of $\widetilde{\psi} \in W^{1,1}(U) \cap C^0(\overline{U})$, where $U = \Omega \setminus (E_1 \cup E_2)$ is the open region enclosed between $\partial^D\Omega$ and $\beta_1 \cup \beta_2$.

Let $(\sigma, \psi) \in \mathcal{W}_{\text{conv}}$ be given by

$$\sigma := (\sigma_1, \sigma_2) \quad \text{and} \quad \psi := \begin{cases} 0 & \text{in } \Omega \setminus U, \\ \widetilde{\psi} & \text{in } U, \end{cases}$$

where $\sigma_i([0,1]) = \beta_i$ for $i = 1, 2$. Then clearly $S^+ = \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ and

$$\min_{(s,\zeta)\in\mathcal{W}_{\text{conv}}} \mathcal{F}(s,\zeta) \leq \mathcal{F}(\sigma,\psi) = \mathcal{H}^2(S^+) = \frac{1}{2} m_2(\Gamma).$$

Now, suppose $m_2(\Gamma) = m_1(\Gamma_1) + m_1(\Gamma_2)$. For $j = 1, 2$, let $\Phi_j \in \mathcal{P}_1(\Gamma_j)$ be a solution to (6.1) and $S_j := \Phi_j(\overline{B}_1)$. Let D_j be the closed convex hull of Γ_j: clearly $D_1 \cap D_2 = \emptyset$. By Lemma 5.4 (with $F = \overline{\Omega}$) each S_j satisfies the following properties:

- $S_j \cap (\mathbb{R}^2 \times \{0\}) = \beta_j \subset D_j$ is a simple analytic curve joining p_j to q_j;
- S_j is symmetric with respect to $\mathbb{R}^2 \times \{0\}$;
- $S_j^+ := S \cap \{x_3 \geq 0\}$ is the graph of a function $\widetilde{\psi}_j \in W^{1,1}(U_j) \cap C^0(\overline{U}_j)$, where $U_j \subset D_j$ is the open region enclosed between $\partial_j^D \Omega$ and β_j;
- $\Omega \setminus U_j$ is convex.

Let $(\sigma, \psi) \in \mathcal{W}_{\mathrm{conv}}$ be given by

$$\sigma := (\sigma_1, \sigma_2) \quad \text{and} \quad \psi := \begin{cases} 0 & \text{in } \Omega \setminus (U_1 \cup U_2), \\ \widetilde{\psi}_j & \text{in } U_j \text{ for } j = 1, 2, \end{cases}$$

where $\sigma_1([0,1]) := \overline{p_1 q_2}$ and $\sigma_2([0,1]) := \beta_2 \cup \overline{q_2 p_1} \cup \beta_1$. Then $S^+ := S_1^+ \cup S_2^+ = \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ and

$$\min_{(s,\zeta) \in \mathcal{W}_{\mathrm{conv}}} \mathcal{F}(s, \zeta) \leq \mathcal{F}(\sigma, \psi) = \mathcal{H}^2(S^+) = \frac{1}{2}(m_1(\Gamma_1) + m_1(\Gamma_2)) = \frac{1}{2} m_2(\Gamma),$$

and the proof of step 1 is concluded.

Step 2: $2 \min_{(s,\zeta) \in \mathcal{W}_{\mathrm{conv}}} \mathcal{F}(s, \zeta) \geq m_2(\Gamma)$.

Let $(\sigma, \psi) \in \mathcal{W}_{\mathrm{conv}}$ be a minimizer satisfying properties 1.-5. of Theorem 5.1. If $E(\sigma_1) \cup E(\sigma_2) = \emptyset$, by Step 1 we can apply Lemma 6.6 and find an injective parametrization $\Phi \in \mathcal{P}_2(\Gamma)$ such that $\Phi_i(\partial \Sigma_{\mathrm{ann}}) = \Gamma$ weakly monotonically, $\Phi(\overline{\Sigma}_{\mathrm{ann}}) = \mathcal{G}_\psi \cup \mathcal{G}_{-\psi}$, and

$$2\mathcal{F}(\sigma, \psi) = \int_{\Sigma_{\mathrm{ann}}} |\partial_{w_1} \Phi \wedge \partial_{w_2} \Phi| dw \geq m_2(\Gamma).$$

If instead $E(\sigma_1) \cup E(\sigma_2) \neq \emptyset$, similarly we find injective parametrizations $\Phi_1 \in \mathcal{P}_1(\Gamma_1)$ and $\Phi_2 \in \mathcal{P}_1(\Gamma_2)$ such that $\Phi_j(\partial B_1) = \Gamma_j$ weakly monotonically for $j = 1, 2$, $\Phi_1(\overline{B}_1) \cup \Phi_2(\overline{B}_1) = \mathcal{G}_\psi \cup \mathcal{G}_{-\psi}$, and

$$2\mathcal{F}(\sigma, \psi) = \int_{B_1} |\partial_{w_1} \Phi_1 \wedge \partial_{w_2} \Phi_1| dw + \int_{B_1} |\partial_{w_1} \Phi_2 \wedge \partial_{w_2} \Phi_2| dw \geq m_1(\Gamma_1) + m_1(\Gamma_2) \geq m_2(\Gamma).$$

This concludes the proof. □

Now the proof of Theorem 6.4 is easily achieved.

Proof of Theorem 6.4. (i). Let $\Phi \in \mathcal{P}_2(\Gamma)$, S, S^+, S^- be as in the statement. By arguing as in the proof of Theorem 6.16 we can find $(\sigma, \psi) \in \mathcal{W}_{\mathrm{conv}}$ such that $S^\pm = \mathcal{G}_{\pm\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$. Then by Theorem 6.16 we have

$$\mathcal{F}(\sigma, \psi) = \frac{1}{2} m_2(\Gamma) = \min_{(s,\zeta) \in \mathcal{W}_{\mathrm{conv}}} \mathcal{F}(s, \zeta). \tag{6.29}$$

Hence (σ, ψ) is a minimizer for $\mathcal{F}$ in $\mathcal{W}$; moreover by the properties of S it also satisfies properties 1.-5. of Theorem 5.1.

(ii). Let $\Phi_j \in \mathcal{P}_1(\Gamma_j)$, S_j for $j = 1, 2$, S^+, S^- be as in the statement. Again arguing as in the proof of Theorem 6.16, we can find $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ such that $S^\pm = \mathcal{G}_{\pm\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}$ and (6.29) holds, so that (σ, ψ) is a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1.-5. of Theorem 5.1.

(iii). Let $(\sigma, \psi) \in \mathcal{W}_{\rm conv}$ be a minimizer of $\mathcal{F}$ in $\mathcal{W}$ satisfying properties 1.-5. of Theorem 5.1. Let also

$$S := \mathcal{G}_{\psi \llcorner (\overline{\Omega \setminus E(\sigma)})} \cup \mathcal{G}_{-\psi \llcorner (\overline{\Omega \setminus E(\sigma)})}.$$

Suppose first $E(\sigma_1) \cap E(\sigma_2) = \emptyset$. Then there is $\Phi \in \mathcal{P}_2(\Gamma)$ which is a $\mathcal{MY}$ solution to (6.4) such that $\Phi(\overline{\Sigma}_{\rm ann}) = S$: indeed, to see this, it is sufficient to apply Lemma 6.6, since by Theorem 6.16 we have

$$2\mathcal{F}(\sigma, \psi) = m_2(\Gamma). \tag{6.30}$$

Now, suppose $E(\sigma_1) \cap E(\sigma_2) \neq \emptyset$; then with a similar argument we can construct $\Phi_j \in \mathcal{P}_1(\Gamma_j)$ for $j = 1, 2$ solutions to (6.1) such that $\Phi_1(\overline{B}_1) \cup \Phi_2(\overline{B}_1) = S$. The proof is achieved. □

7 Final remarks and open problems

In this section we describe some motivations of the present study, possible applications and related problems. Furthermore, we briefly comment on the hypotheses of our setting and on possible extensions and generalizations of our results.

Connection with the Plateau problem in high codimension: The main motivation of our study is related to the classical non-parametric Plateau problem in codimension greater than 1. Specifically, our setting is suited for the description of the singular part of the L^1-relaxation $\mathcal{A}(\cdot, U)$ of the Cartesian 2-codimensional area functional

$$\int_U \sqrt{1 + |\nabla u_1|^2 + |\nabla u_2|^2 + (\det \nabla u)^2}\, dx, \qquad u = (u_1, u_2) \in C^1(U; \mathbb{R}^2), \tag{7.1}$$

computed on nonsmooth maps. The functional $\mathcal{A}(\cdot, U)$ computed out of $C^1(U, \mathbb{R}^2)$ is mostly unknown [1,28], up to a few exceptions, see [1,7,8,14,41]. One of the remarkable exceptions is given by the vortex map $u_V : B_\ell(0) \setminus \{0\} \subset \mathbb{R}^2 \to \mathbb{R}^2$, $u_V(x) := \frac{x}{|x|}$: in this case it can be proved [9–11] that

$$\mathcal{A}(u_V, B_\ell(0)) = \int_{B_\ell(0)} \sqrt{1 + |\nabla u_V|^2}\, dx + \inf \mathcal{F}(\sigma, \psi), \tag{7.2}$$

where $\mathcal{F}(\sigma, \psi)$ is as in (1.7) with $\Omega = R_{2\ell} = (0, 2\ell) \times (-1, 1)$ and the Dirichlet datum $\varphi : \partial R_{2\ell} \to [0, +\infty)$ is given by

$$\varphi(z_1, z_2) := \begin{cases} \sqrt{1 - z_2^2} & \text{on } \partial^D R_{2\ell}, \\ 0 & \text{on } \partial^0 R_{2\ell}, \end{cases} \tag{7.3}$$

with $\partial^D R_{2\ell} = (\{0\} \times (-1, 1)) \cup ([0, 2\ell] \times \{-1\}) \cup (\{2\ell\} \times (-1, 1))$ and $\partial^0 R_{2\ell} = (0, 2\ell) \times \{1\}$. Here the infimum is taken over all pairs $(\sigma, \psi) \in \Sigma \times BV(R_{2\ell})$ with σ a unique curve in $\overline{R}_{2\ell}$ joining $(0, 1)$ to $(2\ell, 1)$ and $\psi = 0$ a.e. in $E(\sigma)$. This setting is similar to the catenoid case, with the notable difference that the Dirichlet boundary is here extended to include the basis $(0, 2\ell) \times \{-1\}$ and the free curve is just one simple curve σ (see Figure 6).

To construct a recovery sequence for (7.2), it is crucial to analyse the existence and regularity of minimizers of $\mathcal{F}$. In particular, it is necessary to show that there is at least one sufficiently regular minimizer (σ, ψ). The shape of the curve σ and the graph of ψ are related to the vertical part of a Cartesian 2-current in $B_\ell(0) \times \mathbb{R}^2 \subset \mathbb{R}^4$ which arises as a limit of (the graphs of) a recovery sequence $(v_k) \subset C^1(B_\ell(0); \mathbb{R}^2)$ for $\mathcal{A}(u_V; B_\ell(0))$.

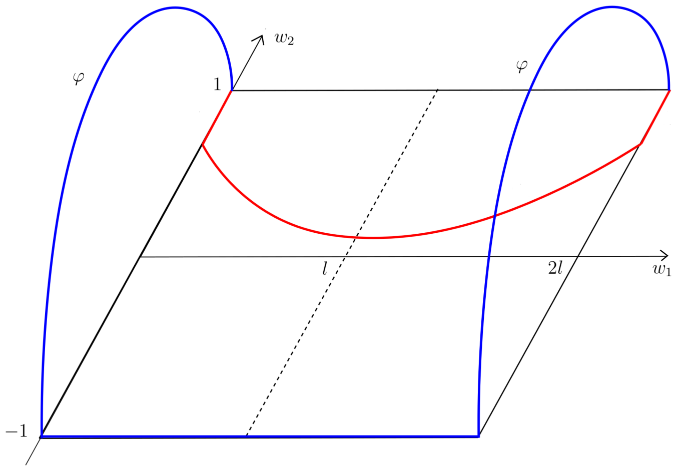

Figure 6: The domain $R_{2\ell}$ (example of the vortex map u_V). The graph of φ on $\partial^D R_{2\ell}$ is emphasized (in particular $\varphi = 0$ on the lower horizontal side), together with an admissible curve σ, which in this specific case partially overlaps the Dirichlet boundary. In this example $n = 1$.

According to what happens for the catenoid, also in this case we have a dichotomy for the behaviour of minimizers (σ, ψ). When $\ell > 0$ is small, the solution (σ, ψ) consists of a curve σ joining p and q having relative interior contained in $R_{2\ell}$, and so that $E(\sigma)$ is convex; at the same time the graph of ψ on $R_{2\ell} \setminus E(\sigma)$ is a sort of half-catenoid, so that if we double it considering also its symmetric with respect to the plane containing $R_{2\ell}$, it becomes a sort of catenoid spanning two unit circles and constrained to contain the segment $(0, 2\ell) \times \{-1\}$. When instead ℓ is larger than a certain threshold, the solution reduces to two circles spanning the two unit parallel circles. Notice however that in the setting of (7.3) on a part of the Dirichlet boundary we have $\varphi = 0$. This leads to a number of additional difficulties which must be treated separately with approximation techniques (we refer to [10] for the details).

Another relevant case in which the relaxation is known, is for the so-called *triple junction function* $u_T : B_l(0) \subset \mathbb{R}^2 \to \mathbb{R}^2$, a map taking only three values, vertices of an equilateral triangle of unit side-length (see [14, 41]). Also in this case, in order to compute the singular contribution of $\mathcal{A}(u_T; B_l(0))$, a Cartesian Plateau problem with a partial free boundary must be solved. Following our approach, it is possible to reduce this problem to our setting. In general[23], given $\Omega \subset \mathbb{R}^2$ and $u \in BV(\Omega; \mathbb{R}^2)$, the singular contribution of the relaxed area functional $\mathcal{A}(u; \Omega)$ coincides with the mass of vertical parts in the optimal Cartesian current T_u with underlying map u that arises as limit of the graphs G_{v_k} of a recovery sequence $v_k : \Omega \to \mathbb{R}^2$. Generally, a few can be said on the structure and properties of these vertical parts. However, for optimal Cartesian currents T_u as above, they enjoy minimality properties under suitable constraints. In the aforementioned known cases (a suitable projection[24] of) these vertical parts is exactly the area minimizing solution of Cartesian Plateau type problem with partial free boundary.

We emphasize that understanding the features of vertical parts of optimal Cartesian currents for the relaxed graph area is crucial in order to detect the behaviour of the area functional. In more

[23]We restrict the discussion to the 2 dimensional case (and codimension 2), although this is valid for any dimension and codimension.

[24]These currents live in $\Omega \times \mathbb{R}^2$, but stands above 1-dimensional subsets of Ω, so that, with suitable thecniques, they can be identified with integral currents of codimension 1 (we refer to [9–11, 14, 41] for more detail).

general setting and for general maps $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ only partial results are currently available, and specifically, without a finer analysis of the singularities of these Cartesian currents only upper and lower bounds can be obtained (see e.g. [15, 42], where some estimates are given for nonsmooth $\mathbb{S}^1$-valued maps).

Hypotheses: We assume that Ω is convex. Convexity is crucial to ensure existence of solutions even in the classical non-parametric setting with no free boundary. Indeed, there are examples in which Ω is not convex, and a minimizer of the area functional does not attain the Dirichlet boundary datum.

We also point out that we assume injectivity of the free-boundary curves σ_i (see hypothesis (i) in the Introduction). This assumption is crucial in order to define the sets $E(\sigma_i)$ and then to solve the problem in a non-parametric form. However, if one allows σ_i to have self-intersections, one can look for a disk spanning the curve Γ_σ in (1.8) which is not a Jordan curve anymore. In this case we have to face a singular Plateau problem such as the one recently studied in [21] using results of [35]. Notice that in this case the curves σ_i will be also planar and some additional hint to face this problem can be found in [19].

Further developments: In the present analysis we have assumed that the free boundary curves are included in a plane. Of course, one may ask for domains Ω which are subset of a manifold (not necessarily a plane), leading to additional difficulties, since the symmetrization with respect to the plane is strongly used in our arguments.

Furthermore, the correspondence between the Meeks and Yau solutions is obtained in the special cases $n = 1, 2$, although we believe that it holds also for $n \geq 3$. However, due to technical difficulties which renders the setting much more involved, we leave this generalization to future developments.

A further interesting question is the following. Suppose that $\partial\Omega$ is smooth; then one may ask whether each free boundary $\partial E(\sigma_j)$ is smooth up to $\partial\Omega$, and moreover if there is some special kind of contact angle condition at $\partial\Omega$, due to minimality. This question should need further investigation in the future.

The problem considered in this paper seems not directly related to the partial wetting phenomenon, an interesting behaviour of soap films pointed out in [3], see also [17] and [12], [13], where the soap film (typically in a non Cartesian context) does not attain the boundary condition, leaving unwetted a part of the wire Γ. However, when the boundary datum φ is allowed to vanish (a case not covered by the results of the present paper), as in the case of the "catenoid" constrained to contain the segment $[0, 2\ell] \times \{-1\}$ mentioned in Section 5.1, it may happen that the singular solution consisting of the two half-disks does not wet that segment.

We conclude this section with a couple of additional examples which are open problems and we consider to be interesting, relating the problem (and suitable variants) studied in this paper with the relaxation of the area functional (7.1) in dimension 2 and codimension 2.

Let $\widehat{u} : B_\ell(0) \subset \mathbb{R}^2 \to \mathbb{S}^1$ be the map defined in polar coordinates

$$\widehat{u}(\rho, \theta) = e^{2i\theta},$$

i.e., the vortex map of degree 2. Our conjecture is that the relaxed area functional $\mathcal{A}(\widehat{u}; B_\ell(0))$ is given by

$$\int_{B_\ell(0)} \sqrt{1 + |\nabla \widehat{u}|^2}\, dx + \inf\{\mathcal{F}_1(\sigma, \psi_1) + \mathcal{F}_2(\widehat{\sigma}, \psi_2)\}, \tag{7.4}$$

where both $\mathcal{F}_i$, $i = 1, 2$, are as the functional in (1.7), but applied to different domains and variables. Specifically, $\mathcal{F}_1$ is applied to $\Omega = R_{2\ell}$, and φ, $\sigma = (\sigma_1)$ and $\psi_1 = \psi$ are exactly as in the case of u_V (see (7.2) and (7.3)). Instead, for $\mathcal{F}_2$, $\Omega = R_{2\ell}$ while $\widehat{\sigma} = (\widehat{\sigma}_1, \widehat{\sigma}_2) = (\sigma_1, \widehat{\sigma}_2)$, and φ are as in the example of the catenoid in the introduction. Notice that minimizations of $\mathcal{F}_1$ and $\mathcal{F}_2$ are not independent each other, since $\sigma_1 = \widehat{\sigma}_1$.

Another nontrivial example is given by a map $u \in BV(B_\ell(0); \mathbb{R}^2)$ which we assume to jump on three radii of $B_\ell(0)$ (not necessarily at 120^o-degrees angles). Depending on the trace values of u on these radii, we consider the Plateau problem with partial free boundary as described below: we take as domain Ω a triangle and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are three curves in Ω connecting the three pairs of vertices. Let φ be a boundary datum on $\partial\Omega$ that is null on the three vertices of Ω, and denote by $H(\sigma_i)$ the region enclosed between σ_i and the side l_i of Ω with the same vertices. We conjecture that the singular contribution in $\mathcal{A}(u; B_\ell(0))$ is related to the infimum of the quantity

$$|\Omega \setminus (\cup_{i=1}^3 H(\sigma_i))| + \sum_{i=1}^{3} \Big(\int_\Omega \sqrt{1 + |\nabla \psi_i|^2}\, dx + |D^s \psi_i|(\Omega) - |\Omega \setminus H(\sigma_i)| + \int_{l_i} |\psi_i - \varphi|\, d\mathcal{H}^1 \Big).$$

8 Appendix

We recall here some classical facts about convex sets and Hausdorff distance.

If $A, B \subset \mathbb{R}^2$ are nonempty, the symbol $d_H(A, B)$ stands for the Hausdorff distance between A and B, that is

$$d_H(A, B) := \max \left\{ \sup_{a \in A} \mathrm{d}_B(a)\, ,\, \sup_{b \in B} \mathrm{d}_A(b) \right\},$$

where $\mathrm{d}_F(\cdot)$ is the distance from the nonempty set $F \subseteq \mathbb{R}^2$. If we restrict d_H to the class of closed sets, then d_H defines a metric. Moreover:

(H1) $\mathrm{d}_A(x) \leq \mathrm{d}_B(x) + d_H(A, B)$ for every $x \in \mathbb{R}^2$;

(H2) If $\mathcal{K} := \{K \subset \mathbb{R}^2 \text{ nonempty and compact}\}$ then $(\mathcal{K}, d_H)$ is a complete metric space;

(H3) If $A, B \in \mathcal{K}$ are convex then $d_H(A, B) = d_H(\partial A, \partial B)$;

(H4) If $A \in \mathcal{K}$ is convex, then there exists a sequence $(A_n)_n \subset \mathcal{K}$ of convex sets with boundary of class C^∞ such that $d_H(A_n, A) \to 0$ as $n \to +\infty$;

(H5) Let $(A_n)_n$ be a sequence of nonempty closed convex sets in $\mathbb{R}^2$, $A \subset \mathbb{R}^2$ is nonempty and $d_H(A_n, A) \to 0$ as $n \to +\infty$. Then A is convex as well;

(H6) Let $A_n, A \in \mathcal{K}$ be convex such that $d_H(A_n, A) \to 0$ and let $x \in \mathrm{int}(A)$; then $x \in A_n$ for all $n \in \mathbb{N}$ sufficiently large;

(H7) Let A and B be nonempty closed subsets of $\mathbb{R}^2$ with $d_H(A, B) = \varepsilon$. Then $A \subset B_\varepsilon^+$ and $B \subset A_\varepsilon^+$ where, for all nonempty $E \subset \mathbb{R}^2$, we have set $E_\varepsilon^+ := \{x \in \mathbb{R}^2 \colon \mathrm{d}_E(x) \leq \varepsilon\}$.

Remark 8.1. Property (H1) is straightforward, while (H2) is well-known. Also property (H3) is easily obtained (see, e.g. [43]). Concerning property (H4) we refer to, e.g., [6, Corollary 2]. To see (H5), from (H1) we have that $\mathrm{d}_{A_n} \to \mathrm{d}_A$ pointwise, and therefore since d_{A_n} is convex, also d_A is convex, which implies A convex[25]. Let us now prove (H6) by contradiction; assume that there exists a subsequence (n_k) such that $\mathrm{d}_{A_{n_k}}(x) > 0$ for all $k \in \mathbb{N}$; then $x \in \mathbb{R}^2 \setminus A_{n_k}$, $\mathrm{d}_{A_{n_k}}(x) = \mathrm{d}_{\partial A_{n_k}}(x)$, and using (H1) twice,

$$\begin{aligned} \mathrm{d}_{\partial A}(x) &\leq \mathrm{d}_{\partial A_{n_k}}(x) + d_H(\partial A_{n_k}, \partial A) = \mathrm{d}_{A_{n_k}}(x) + d_H(A_{n_k}, A) \\ &\leq \mathrm{d}_A(x) + 2 d_H(A, A_{n_k}) = 2 d_H(A, A_{n_k}) \to 0, \end{aligned}$$

[25] Since A is closed, it coincides with the sublevel $\{x : d(x, A) \leq 0\}$, which is convex.

the first equality following from (H3). This implies $x \in \partial A$, a contradiction. Finally property (H7) is immediate. Indeed if $a \in A$ then

$$\mathrm{d}_B(a) \le \sup_{x \in A} \mathrm{d}_B(x) \le d_H(A,B) = \varepsilon\,,$$

hence $a \in B^+_\varepsilon$ and so $A \subset B^+_\varepsilon$. In a similar way we get $B \subset A^+_\varepsilon$.

We begin with the following standard result that will be useful later:

Lemma 8.2. *Let $K \in \mathcal{K}$ be convex with nonempty interior. Then there exists a 1-periodic map $\widehat{\sigma} \in \mathrm{Lip}(\mathbb{R};\mathbb{R}^2)$, injective on $[0,1)$, such that $\widehat{\sigma}([0,1]) = \partial K$ and*

$$\widehat{\sigma}(t) = \widehat{\sigma}(0) + \ell(\widehat{\sigma}) \int_0^t \widehat{\gamma}(s)\,ds, \quad \widehat{\gamma}(t) = (\cos(\widehat{\theta}(t))\,,\,\sin(\widehat{\theta}(t))) \quad \textit{for all } t \in [0,1],$$

with $\widehat{\theta}\colon \mathbb{R} \to \mathbb{R}$ a non-decreasing function satisfying $\widehat{\theta}(t+1) - \widehat{\theta}(t) = 2\pi$ for all $t \in \mathbb{R}$, and $\ell(\widehat{\sigma}) := \int_0^1 |\widehat{\sigma}'(s)|ds$ the length of the curve.

Notice that $\widehat{\sigma}$ is differentiable a.e. in $\mathbb{R}$ and $\widehat{\sigma}'(t) = \ell(\widehat{\sigma})\widehat{\gamma}(t)$, so that the speed modulus of the curve $|\hat{\sigma}'(t)| = \ell(\widehat{\sigma})$ is constant.

Proof. We start by approximating K by convex sets with C^∞ boundary. By (H4) for all $n \in \mathbb{N}$ there is a convex compact set $K_n \subset \mathbb{R}^2$ with boundary of class C^∞ and such that $d_H(K_n, K) \to 0$ as $n \to +\infty$. For any $n \in \mathbb{N}$ we let $\widehat{\sigma}_n \in C^\infty(\mathbb{R};\mathbb{R}^2)$ be a 1-periodic function injectively parametrizing ∂K_n on $[0,1)$; therefore $\widehat{\sigma}_n([0,1]) = \partial K_n$, and

$$\widehat{\sigma}_n(t) = \widehat{\sigma}_n(0) + \ell(\widehat{\sigma}_n) \int_0^t \widehat{\gamma}_n(s)\,ds, \quad \widehat{\gamma}_n(t) = (\cos(\widehat{\theta}_n(t))\,,\,\sin(\widehat{\theta}_n(t))) \quad \forall t \in [0,1],$$

where $\widehat{\theta}_n \in C^\infty(\mathbb{R})$ is a non-decreasing function with $\widehat{\theta}_n(t+1) - \widehat{\theta}_n(t) = 2\pi$, for all $t \in \mathbb{R}$. In view of (H2), by construction we can find $x_0 \in K$, $R > r > 0$ such that $B_r(x_0) \subset K_n \subset B_R(x_0)$ for all $n \in \mathbb{N}$, and therefore $\mathcal{H}^1(\partial B_r(x_0)) \le \ell(\widehat{\sigma}_n) = \mathcal{H}^1(\partial K_n) \le \mathcal{H}^1(\partial B_R(x_0))$; where the last inequality follows since $\partial K_n = \pi_{K_n}(\partial B_R(x_0))$ and from the fact that, since K_n is convex, the projection π_{K_n} on K_n does not increase the lengths, thus, up to subsequence, $\ell(\widehat{\sigma}_n) \to \widehat{m} \in (0,+\infty)$ as $n \to +\infty$. Moreover, up to subsequence, we might assume $\widehat{\sigma}_n(0) \to p \in \partial K$. On the other hand, observing that

$$\int_t^{t+1} |\widehat{\theta}_n'(s)|ds = \int_t^{t+1} \widehat{\theta}_n'(s)ds = 2\pi, \text{ for all } t \in \mathbb{R},$$

we have that, again up to subsequence, $\widehat{\theta}_n \overset{*}{\rightharpoonup} \widehat{\theta} \in BV_{\mathrm{loc}}(\mathbb{R})$ and pointwise (by Helly selection principle), with $\widehat{\theta}$ a non-decreasing function with $\widehat{\theta}(t+1) - \widehat{\theta}(t) = 2\pi$ for all $t \in \mathbb{R}$. We also have $\widehat{\gamma}_n \overset{*}{\rightharpoonup} \widehat{\gamma}$ in $BV_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2)$ where $\widehat{\gamma}(t) = (\cos(\widehat{\theta}(t))\,,\,\sin(\widehat{\theta}(t)))$.
We let $\widehat{\sigma} \in \mathrm{Lip}(\mathbb{R};\mathbb{R}^2)$ be the 1-periodic curve, injective on $[0,1)$, defined as

$$\widehat{\sigma}(t) := p + \widehat{m} \int_0^t \widehat{\gamma}(s)\,ds \quad \forall t \in \mathbb{R}. \tag{8.1}$$

Note that $\widehat{m} = \ell(\widehat{\sigma})$. Then clearly $\widehat{\sigma}_n \to \widehat{\sigma}$ in $W^{1,1}([0,1];\mathbb{R}^2)$, since

$$\begin{aligned} \|\widehat{\sigma}_n' - \widehat{\sigma}'\|_{L^1([0,1];\mathbb{R}^2)} &= \int_0^1 |\ell(\widehat{\sigma}_n)\widehat{\gamma}_n(t) - \ell(\widehat{\sigma})\widehat{\gamma}(t)|dt \\ &\le |\ell(\widehat{\sigma}_n) - \ell(\widehat{\sigma})| + \ell(\widehat{\sigma}) \int_0^1 |\widehat{\gamma}_n(t) - \widehat{\gamma}(t)|dt \to 0. \end{aligned} \tag{8.2}$$

By the continuous embedding $W^{1,1}([0,1];\mathbb{R}^2) \subset C^0([0,1];\mathbb{R}^2)$ (and by 1-periodicity, on $\mathbb{R}$) we also get $\widehat{\sigma}_n \to \widehat{\sigma}$ uniformly on $[0,1]$. This, together with property (H3) gives

$$d_H(\partial K, \widehat{\sigma}([0,1])) \leq d_H(\partial K, \partial K_n) + d_H(\widehat{\sigma}_n([0,1]), \widehat{\sigma}([0,1])) \to 0,$$

which in turn implies $\widehat{\sigma}([0,1]) = \partial K$. The injectivity of $\widehat{\sigma}$ on $[0,1)$ follows from expression (8.1), the fact that $\widehat{m} > 0$ and that K is convex with nonempty interior.

□

Corollary 8.3. *Let $K \in \mathcal{K}$ be convex with nonempty interior. Let q, p be two distinct points on ∂K, and let $\widehat{pq} \subset \partial K$ be the relatively open arc with endpoints q and p clockwise oriented. Then there exists an injective curve $\sigma \in \mathrm{Lip}([0,1];\mathbb{R}^2)$ such that $\sigma((0,1)) = \widehat{pq}$, $\sigma(0) = q$, $\sigma(1) = p$, and*

$$\sigma(t) = q + \ell(\sigma) \int_0^t \gamma(s)\, ds, \quad \gamma(t) = (\cos(\theta(t)), \sin(\theta(t))) \quad \text{for all } t \in [0,1],$$

with θ a non-decreasing function satisfying $\theta(1) - \theta(0) \leq 2\pi$.

Proof. Lemma 8.2 provides $\widehat{\sigma} \in \mathrm{Lip}([0,1];\mathbb{R}^2)$ parametrizing ∂K. Then there are two values $t_1, t_2 \in [0,1]$, $t_1 < t_2$, with $q = \widehat{\sigma}(t_1)$ and $p = \widehat{\sigma}(t_2)$ so that the existence of σ follows by reparametrizing the interval $[t_1, t_2]$, and all the properties follows from the corresponding properties of $\widehat{\sigma}$. □

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