THE FRACTIONAL MAKAI-HAYMAN INEQUALITY

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ABSTRACT. We prove that the first eigenvalue of the fractional Dirichlet-Laplacian of order s on a simply connected set of the plane can be bounded from below in terms of its inradius only. This is valid for $1/2 < s < 1$ and we show that this condition is sharp, i. e. for $0 < s \leq 1/2$ such a lower bound is not possible. The constant appearing in the estimate has the correct asymptotic behaviour with respect to s, as it permits to recover a classical result by Makai and Hayman in the limit $s \nearrow 1$. The paper is as self-contained as possible.

CONTENTS

1. INTRODUCTION

1.1. **Background.** For an open set $\Omega \subset \mathbb{R}^N$, we indicate by $W_0^{1,2}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,2}(\Omega)$. We then consider the following quantity

$$\lambda_1(\Omega) := \inf_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega |u|^2\,dx},$$

which coincides with the bottom of the spectrum of the Dirichlet-Laplacian on Ω. Observe that for a general open set, such a spectrum may not be discrete and the infimum value $\lambda_1(\Omega)$ may not be attained. Whenever a minimizer $u_1 \in W_0^{1,2}(\Omega)$ of the problem above exists, we call $\lambda_1(\Omega)$ the *first eigenvalue of the Dirichlet-Laplacian on* Ω.

2010 *Mathematics Subject Classification.* 47A75, 39B72, 35R11.

Key words and phrases. Poincaré inequality, fractional Laplacian, inradius, simply connected sets, Cheeger inequality.

By definition, such a quantity is different from zero if and only if Ω supports the Poincaré inequality

$$c\int_\Omega |u|^2\,dx \le \int_\Omega |\nabla u|^2\,dx, \qquad \text{for every } u\in C_0^\infty(\Omega).$$

It is well-known that this happens for example if Ω is bounded or with finite measure or even bounded in one direction only. However, in general it is quite complicate to give more general geometric conditions, assuring positivity of λ_1. In this paper, we will deal with the two-dimensional case $N = 2$.

In this case, there is a by now classical result which asserts that

$$\lambda_1(\Omega) \ge \frac{C}{r_\Omega^2}, \tag{1.1}$$

for every *simply connected* set $\Omega \subset \mathbb{R}^2$. Here $C > 0$ is a universal constant and the geometric quantity r_Ω is the *inradius* of Ω, i.e. the radius of a largest disk contained in Ω. More precisely, this is given by

$$r_\Omega := \sup\Big\{\rho>0\,:\,\exists\, x\in\Omega \text{ such that } B_\rho(x)\subset\Omega\Big\}.$$

Inequality (1.1) is in scale invariant form, by recalling that λ_1 scales like a length to the power -2, under dilations.

Such a result is originally due to Makai (see [25, equation (5)]). It permits in particular to prove that for a simply connected set in the plane, we have the following remarkable equivalence

$$\lambda_1(\Omega) > 0 \qquad \Longleftrightarrow \qquad r_\Omega < +\infty. \tag{1.2}$$

Indeed, if the inradius is finite, we immediately get from (1.1) that $\lambda_1(\Omega)$ must be positive. The converse implication is simpler and just based on the easy (though sharp) inequality

$$\lambda_1(\Omega) \le \frac{\lambda_1(B_1)}{r_\Omega^2}.$$

Here $B_1 \subset \mathbb{R}^2$ is any disk with radius 1 and the estimate simply follows from the monotonicity with respect to set inclusion of λ_1, together with its scaling properties.

The proof in [25] runs very similarly to that of the *Faber-Krahn inequality*, based on symmetrization techniques (see [20, Chapter 3]). It starts by rewriting the Dirichlet integral and the L^2 norm by using the Coarea Formula; then the key ingredient is a clever use of a particular quantitative isoperimetric inequality in $\mathbb{R}^2$ (a *Bonnesen–type inequality*), which permits to obtain a lower bound in terms of r_Ω only.

It should be noticed that Makai's result has been overlooked or neglected for some years and then rediscovered independently by Hayman, by means of a completely different proof, see [19, Theorem 1]. For this reason, we will call (1.1) the *Makai-Hayman inequality.*

It is interesting to remark that the result by Makai is quantitatively better than the one by Hayman: indeed, the former obtains (1.1) with $C = 1/4$, while the latter is only able to get the poorer constant $C = 1/900$ by his method of proof.

This could suggest that the attribution of this result to both authors is maybe too generous. On the contrary, we will show in this paper that, in despite of providing a poorer constant, the method of proof by Hayman is elementary, flexible and robust enough to be generalized to other situations, where Makai's and other approaches become too complicate or do not seem feasible.

In any case, we point out that the exact determination of the sharp constant in (1.1), i. e.

$$C_{MH} := \inf \Big\{ \lambda_1(\Omega)\, r_\Omega^2 \,:\, \Omega \subset \mathbb{R}^2 \text{ simply connected with } r_\Omega < +\infty \Big\},$$

is still a challenging open problem. The best result at present is that

$$0.6197 < C_{MH} < 2.13,$$

obtained by Bañuelos and Carroll (see [2, Corollary 1] for the lower bound and [2, Theorem 2] for the upper bound). The upper bound has then been slightly improved by Brown in [14], by using a refinement of the method by Bañuelos and Carroll.

The inequality (1.1) has also been obtained by Ancona in [1], by using yet another proof. His result comes with the constant $C = 1/16$, much better than Hayman's one, but still worse than that obtained by Makai. The proof by Ancona is quite elegant: it is based on the use of conformal mappings and the so-called *Koebe's one quarter Theorem* (see [22, Chapter 12]), which permits to obtain the following Hardy inequality for a simply connected set in the plane

$$\frac{1}{16}\int_\Omega \frac{|\varphi|^2}{\mathrm{dist}(x,\partial\Omega)^2}\,dx \le \int_\Omega |\nabla\varphi|^2\,dx, \qquad \text{for every } \varphi \in C_0^\infty(\Omega).$$

From this, inequality (1.1) is easily obtained (with $C = 1/16$), by observing that

$$r_\Omega = \sup_{x\in\Omega} \mathrm{dist}(x,\partial\Omega),$$

and then using the definition of $\lambda_1(\Omega)$. The conformality of the Dirichlet integral plays a central role in this proof. The result by Ancona is quite remarkable, as the Hardy inequality is proved without any regularity assumption on $\partial\Omega$. A generalization of this result can be found in [23].

1.2. **Goal of the paper and main results.** Our work is aimed at investigating the validity of a result analogous to (1.1) for fractional Sobolev spaces. In order to be more precise, we need some definitions at first. Let $0 < s < 1$ and let us recall the definition of Gagliardo-Slobodeckiĭ seminorm

$$[u]_{W^{s,2}(\mathbb{R}^N)} = \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\,s}}\,dx\,dy\right)^{\frac{1}{2}}.$$

Accordingly, we consider the fractional Sobolev space

$$W^{s,2}(\mathbb{R}^N) = \Big\{ u \in L^2(\mathbb{R}^N) \,:\, [u]_{W^{s,2}(\mathbb{R}^N)} < +\infty \Big\},$$

endowed with the norm

$$\|u\|_{W^{s,2}(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + [u]_{W^{s,2}(\mathbb{R}^N)}.$$

Finally, we consider the space $\widetilde{W}_0^{s,2}(\Omega)$, defined as the closure of $C_0^\infty(\Omega)$ in $W^{s,2}(\mathbb{R}^N)$. Observe that by definition the elements of $\widetilde{W}_0^{s,2}(\Omega)$ have to be considered on the whole $\mathbb{R}^N$ and they come with a natural nonlocal homogeneous Dirichlet condition "at infinity", i.e. they identically vanish on the complement $\mathbb{R}^N \setminus \Omega$.

We then consider the quantity

$$\lambda_1^s(\Omega) := \inf_{u\in\widetilde{W}_0^{s,2}(\Omega)\setminus\{0\}} \frac{[u]^2_{W^{s,2}(\mathbb{R}^N)}}{\|u\|^2_{L^2(\Omega)}}. \tag{1.3}$$

Again by definition, this quantity is non-zero if and only if the open set Ω supports the fractional Poincaré inequality

$$c \int_{\Omega} |u|^2\, dx \leq \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\,s}}\, dx\, dy, \qquad \text{for every } u\in C^\infty_0(\Omega).$$

This happens, for example, if Ω is an open bounded set (see [9, Lemma 2.4]). As in the local case, whenever the infimum in (1.3) is attained, this quantity will be called *first eigenvalue of the fractional Dirichlet-Laplacian of order s*. We recall that the latter is the linear operator denoted by the symbol $(-\Delta)^s$ and defined in weak form by

$$\langle (-\Delta)^s u, \varphi\rangle = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+2\,s}}\, dx\, dy, \qquad \text{for every } \varphi\in C^\infty_0(\Omega).$$

In this paper we want to inquire to which extent the Makai-Hayman inequality (1.1) still holds for λ_1^s defined above, still in the case of simply connected sets in the plane. Our main results assert that such an inequality is possible, provided s is "large enough". More precisely, we have the following

Theorem 1.1 (Fractional Makai-Hayman inequality)**.** *Let $1/2<s<1$ and let $\Omega\subset\mathbb{R}^2$ be an open simply connected set, with finite inradius r_Ω. There exists an explicit universal constant $\mathcal{C}_s>0$ such that*

$$\lambda_1^s(\Omega) \geq \frac{\mathcal{C}_s}{r_\Omega^{2\,s}}. \tag{1.4}$$

Moreover, $\mathcal{C}_s$ has the following asymptotic behaviours[1]

$$\mathcal{C}_s \sim \left(s-\frac{1}{2}\right), \quad \textit{for } s\searrow \frac{1}{2} \qquad \textit{and} \qquad \mathcal{C}_s\sim \frac{1}{1-s}, \quad \textit{for } s\nearrow 1.$$

Remark 1.2. We point out that the constant $\mathcal{C}_s$ appearing in the above estimate exhibits the sharp asymptotic dependence on s, as $s\nearrow 1$. Indeed, by recalling that for every open set $\Omega\subset\mathbb{R}^N$ we have (see [8, Lemma A.1])

$$\limsup_{s\nearrow 1} (1-s)\,\lambda_1^s(\Omega) \leq C_N\,\lambda_1(\Omega), \tag{1.5}$$

from Theorem 1.1 we can obtain the usual Makai-Hayman inequality for the Dirichlet-Laplacian, possibly with a worse constant. We recall that (1.5) is based on the fundamental asymptotic result by Bourgain, Brezis and Mironescu for the Gagliardo-Slobodeckiĭ seminorm, see [6]. We refer to [13] for some interesting refinements of such a result.

The previous result is complemented by the next one, asserting that for $0<s\leq 1/2$ a fractional Makai-Hayman inequality *is not* possible. In this way, we see that even for $s\searrow 1/2$ the asymptotic behaviour of $\mathcal{C}_s$ is optimal, in a sense.

[1]Throughout the paper, the writing

$$f(s)\sim g(s), \qquad \text{for } s\to s_0,$$

has to be intended in the following sense: there exists $C\in\mathbb{R}\setminus\{0\}$ such that

$$\lim_{s\to s_0} \frac{f(s)}{g(s)} = C.$$

Theorem 1.3 (Counter-example for $0 < s \le 1/2$). *There exists a sequence $\{Q_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ of open bounded simply connected sets such that*

$$0 < r_{Q_n} \le C, \qquad \textit{for every } n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \lambda_1^s(Q_n) = 0, \qquad \textit{for every } 0 < s \le \frac{1}{2}.$$

Remark 1.4. In [16, Theorem 1.1], a different counter-example for $0 < s < 1/2$ is given. Apart from the fact that in [16] the borderline case $s = 1/2$ is not considered, one could observe that strictly speaking the counter-example in [16] is not a simply connected set, since it is made of countably many connected components.

As it will be apparent to the experienced reader, our example will clearly display the role of fractional $s-$capacity in the failure of the Makai-Hayman inequality for $0 < s \le 1/2$ (see for example [26, Chapter 10, Section 4] for fractional capacities). Indeed, the range $0 < s \le 1/2$ is precisely the one for which *lines have zero fractional $s-$capacity*. This implies that, by removing a finite number of segments from an open set, the first eigenvalue λ_1^s remains unchaged, while this operation heavily affects the inradius. However, even if this is the ultimate reason for such a failure, our proof will be elementary and will not explicitly appeal to the properties of capacities.

We point out that, for practical reasons, our sequence $\{Q_n\}_{n\in\mathbb{N}}$ is given by a square with side length $2\,n$, from which a periodical array of segments is removed. If we scale this sequence by a factor $1/n$, we could produce another sequence contradicting the fractional Makai-Hayman, with the additional property of being equi-bounded.

Remark 1.5. Geometric estimates for eigenvalues of $(-\Delta)^s$ aroused great interest in the last years, also in the field of stochastic processes. Indeed, it is well-known that this operator is the infinitesimal generator of a symmetric $(2\,s)-$stable Lévy process. We recall that the nonlocal homogeneous Dirichlet boundary condition considered above (i. e. $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$) corresponds to a process where particles are "killed" upon reaching the complement of the set Ω. The Gagliardo-Slobodeckiĭ seminorm corresponds to the so-called *Dirichlet form* associated to this process. For more details, we refer for example to [5, Section 2] and the references therein.

In this context, we wish to mention the papers [3, 4] and [28], where some geometric estimates for λ_1^s are obtained, by exploiting this probabilistic approach. In particular, the paper [4] is very much related to ours, since in [4, Corollary 1] it is proved the lower bound

$$\lambda_1^s(\Omega) \ge \frac{C}{r_\Omega^{2\,s}},$$

in the restricted class of open *convex* subsets of the plane, with the sharp constant C. This result can be seen as the fractional counterpart of a well-known result for the Laplacian, which goes under the name of *Hersch-Protter inequality*, see [21, 30].

1.3. **Method of proof.** As already announced at the beginning, we will achieve the result of Theorem 1.1 by adapting to our setting Hayman's proof. It is then useful to recall the key ingredients of such a proof. These are essentially two:

1. a covering lemma, asserting that it is possible to cover an open subset $\Omega \subset \mathbb{R}^2$ with $r_\Omega < +\infty$ by means of *boundary disks*, whose radius is universally comparable to r_Ω and which do not overlap "too much" with each other. Here by *boundary disk* we simply mean a disk centered at the boundary $\partial\Omega$;

2. a Poincaré inequality for boundary disks in a simply connected set.

Point 1. is purely geometrical and thus it can still be used in the fractional setting.

On the contrary, the proof of point 2. is very much local. Indeed, an essential feature of the proof in [19] is the fact that

$$|\nabla u|^2 \geq \frac{1}{\varrho^2}\,|\partial_\theta u|^2, \tag{1.6}$$

where (ϱ,θ) denote the usual polar coordinates. Then one observes that a boundary circle always meets the complement of Ω, when the latter is simply connected. Thus, taken a function $u\in C^\infty_0(\Omega)$, the periodic function $\theta\mapsto u(\varrho,\theta)$ vanishes somewhere in $[0,2\,\pi]$. Consequently, it satisfies the following one-dimensional Poincaré inequality on the interval

$$\int_0^{2\,\pi} |u(\varrho,\theta)|^2\,d\theta \leq C\,\int_0^{2\,\pi} |\partial_\theta u(\varrho,\theta)|^2\,d\theta. \tag{1.7}$$

In a nutshell, this permits to prove point 2. by "foliating" the boundary disk with concentric boundary circles, using (1.7) on each of these circles, then integrating with respect to the radius of the circle and finally appealing to (1.6).

In the fractional case, the property (1.6) has no counterpart, because of the nonlocality of the Gagliardo-Slobodeckiĭ seminorm. Consequently, adapting this method to prove a fractional Poincaré inequality for boundary disks is a bit involved. We will achieve this through a lengthy though elementary method, which we believe to be of independent interest.

Remark 1.6 (Other proofs?)**.** We conclude the introduction, by observing that it does not seem easy to prove (1.4) by adapting Makai's proof, because of the lack of a genuine Coarea Formula for Gagliardo-Slobodeckiĭ seminorms. The proof by Ancona seems to be even more prohibitive to be adapted, because of the rigid machinery of conformal mappings on which is based. In passing, we mention that it would be interesting to know whether his Hardy inequality for simply connected sets in the plane could be extended to fractional Sobolev spaces. For completeness, we refer to [17] for some fractional Hardy inequalities under minimal regularity assumptions.

1.4. **Plan of the paper.**

In Section 2 we set the main notations and present some technical tools, needed throughout the paper. In particular, we recall Hayman's covering lemma from [19] and present a couple of technical results on fractional Sobolev spaces.

In Section 3 we prove a Poincaré inequality for boundary disks. This is the main ingredient for the proof of the fractional Makai-Hayman inequality.

Section 4 is then devoted to the proof of Theorem 1.1, while the construction of the counterexample of Theorem 1.3 is contained in Section 5.

Finally, in Section 6 we highlight some consequences of our main result. Among these, we record a Cheeger-type inequality, a comparison result for λ_1^s and λ_1 and the fractional analogue of the characterization (1.2).

The paper concludes with Appendix A, containing a one-dimensional fractional Poincaré inequality for periodic functions vanishing at one point (see Proposition A.2). This is the cornerstone on which the result in Section 3 is built.

Acknowledgments. The first author is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). Both authors have been financially supported by the Italian grant FFABR *Fondo Per il Finanziamento delle attività di base*. We thank an anonymous referee for his careful reading and suggestions.

Part of this work has been written during the conference "Variational methods and applications", held at the *Centro di Ricerca Matematica "Ennio De Giorgi"* in September 2021. The organizers and the hosting institution are gratefully acknowledged.

2. Preliminaries

2.1. **Notations.** Given $x_0 \in \mathbb{R}^N$ and $R > 0$, we will denote by $B_R(x_0)$ the $N-$dimensional open ball with radius R and center x_0. When the center coincides with the origin, we will simply write B_R. We indicate by ω_N the $N-$dimensional Lebesgue measure of B_1, so that by scaling

$$|B_R(x_0)| = \omega_N\, R^N.$$

If $E \subset \mathbb{R}^N$ is a measurable set with positive measure and $u \in L^1(E)$, we will use the notation

$$\overline{u}_E := \fint_E u(x)\, dx = \frac{1}{|E|} \int_E u(x)\, dx.$$

For $0 < s < 1$ and for a measurable set $E \subset \mathbb{R}^N$, we will indicate by

$$W^{s,2}(E) = \Big\{ u \in L^2(E) \,:\, [u]_{W^{s,2}(E)} < +\infty \Big\},$$

where

$$[u]_{W^{s,2}(E)} = \left(\iint_{E\times E} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\,s}}\, dx\, dy \right)^{\frac{1}{2}}.$$

This space will be endowed with the norm

$$\|u\|_{W^{s,2}(E)} = \|u\|_{L^2(E)} + [u]_{W^{s,2}(E)}.$$

We observe that the following *Leibniz–type rule* holds

$$[u\,v]_{W^{s,2}(E)} \le [u]_{W^{s,2}(E)}\, \|v\|_{L^\infty(E)} + [v]_{W^{s,2}(E)}\, \|u\|_{L^\infty(E)}, \quad \text{for every } u, v \in W^{s,2}(E) \cap L^\infty(E). \tag{2.1}$$

This will be useful somewhere in the paper.

Finally, by $W^{s,2}_{\rm loc}(\mathbb{R}^N)$ we mean the collection of functions which are in $W^{s,2}(B_R)$, for every $R > 0$.

2.2. **Technical tools.** In order to prove Theorem 1.1, we will need the following covering Lemma, whose proof can be found in [19, Lemma 2]. The result in [19] is stated for bounded sets and, accordingly, the relevant covering is made of a *finite* number of balls. However, a closer inspection of the proof in [19] easily shows that the same result still holds by removing the boundedness assumption. In this case, the covering could be made of countably infinitely many balls: this is still enough for our purposes. We omit the proof, since it is exactly the same as in [19].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set, with finite inradius r_Ω. Then there exist at most countably many distinct points $\{z_n\}_{n\in\mathbb{N}} \subset \partial\Omega$ such that the family of disks*

$$\mathfrak{B} = \big\{B_r(z_n)\big\}_{n\in\mathbb{N}}, \qquad \textit{with } r = r_\Omega\, \left(1+\sqrt{2}\right),$$

is a covering of Ω. Moreover, $\mathfrak{B}$ can be split in at most 36 subfamilies $\mathfrak{B}_1, \dots, \mathfrak{B}_{36}$ such that

$$B_r(z_n) \cap B_r(z_m) = \emptyset \qquad \textit{if } B_r(z_n), B_r(z_m) \in \mathfrak{B}_k,\ \textit{with } m \neq n,$$

for every $k = 1, \dots, 36$.

In the following technical result, we explicitly construct a continuous extension operator for fractional Sobolev spaces defined on a ball. The result is certainly well-known (see for example [18, Theorem 5.4]), but here we pay particular attention to the constant appearing in the continuity estimate (2.2) below: indeed, this can be taken to be independent of the differentiability index s.

Lemma 2.2. *Let $0<s<1$, there exists a linear extension operator*

$$\mathcal{E}:W^{s,2}(B_1(x_0))\to W^{s,2}_{\rm loc}(\mathbb{R}^N),$$

such that for every $u\in W^{s,2}(B_1(x_0))$ and every $R>1$ we have

$$\Big[\mathcal{E}[u]\Big]_{W^{s,2}(B_R(x_0))}\le 4\,R^{4\,N}\,[u]_{W^{s,2}(B_1(x_0))},\qquad \|\mathcal{E}[u]\|_{L^2(B_R(x_0))}\le 2\,R^{2\,N}\,\|u\|_{L^2(B_1(x_0))}. \tag{2.2}$$

Proof. Without loss of generality, we can suppose that x_0 coincides with the origin. Then, let us recall the definition of *inversion with respect to* $\mathbb{S}^{N-1}$: this is the bijection $\mathcal{K}:\mathbb{R}^N\setminus\{0\}\to\mathbb{R}^N\setminus\{0\}$, given by

$$\mathcal{K}(x)=\frac{x}{|x|^2},\qquad \text{for every } x\in\mathbb{R}^N\setminus\{0\}.$$

It is easily seen that if $x\in B_R\setminus B_1$, then $\mathcal{K}(x)\in B_1\setminus B_{1/R}$. Moreover, we have

$$\mathcal{K}^{-1}(x)=\mathcal{K}(x)\qquad\text{and}\qquad |\det(D\mathcal{K}(x))|=\frac{1}{|x|^{2\,N}},\qquad \text{for every } x\in\mathbb{R}^N\setminus\{0\}.$$

For every $u\in W^{s,2}(B_1)$, we define the extended function $\mathcal{E}[u]$ given by

$$\mathcal{E}[u](x)=\begin{cases} u(x), & \text{if } x\in B_1,\\ \\ u(\mathcal{K}(x)) & \text{if } x\in\mathbb{R}^N\setminus B_1.\end{cases} \tag{2.3}$$

It is easily seen that the operator $u\mapsto\mathcal{E}[u]$ is linear. In order to prove that $\mathcal{E}[u]\in W^{s,2}_{\rm loc}(\mathbb{R}^N)$, together with the claimed estimate (2.2), we take $R>1$ and we split the seminorm of $\mathcal{E}[u]$ as follows

$$\begin{split}\Big[\mathcal{E}[u]\Big]^2_{W^{s,2}(B_R)}&=[u]^2_{W^{s,2}(B_1)}\\ &+\iint_{(B_R\setminus B_1)\times(B_R\setminus B_1)}\frac{|u(\mathcal{K}(x))-u(\mathcal{K}(y))|^2}{|x-y|^{N+2\,s}}\,dx\,dy\\ &+2\iint_{B_1\times(B_R\setminus B_1)}\frac{|u(x)-u(\mathcal{K}(y))|^2}{|x-y|^{N+2\,s}}\,dx\,dy.\end{split}$$

By performing the change of variable $z=\mathcal{K}(x)$ in the second term on the right-hand side and the change of variable $w=\mathcal{K}(y)$ in the second and third terms, we get

$$\begin{split}\Big[\mathcal{E}[u]\Big]^2_{W^{s,2}(B_R)}&=[u]^2_{W^{s,2}(B_1)}\\ &+\iint_{(B_1\setminus B_{\frac{1}{R}})\times(B_1\setminus B_{\frac{1}{R}})}\frac{|u(z)-u(w)|^2}{|\mathcal{K}^{-1}(z)-\mathcal{K}^{-1}(w)|^{N+2\,s}}\,|\det D\mathcal{K}^{-1}(z)|\,|\det D\mathcal{K}^{-1}(w)|\,dz\,dw\\ &+2\int_{B_1\times(B_1\setminus B_{\frac{1}{R}})}\frac{|u(x)-u(w)|^2}{|x-\mathcal{K}^{-1}(w)|^{N+2\,s}}\,|\det D\mathcal{K}^{-1}(w)|\,dx\,dw.\end{split}$$

By using the expression for the Jacobian determinant, we then obtain

$$
\begin{aligned}
\Big[\mathcal{E}[u]\Big]^2_{W^{s,2}(B_R)} &\le [u]^2_{W^{s,2}(B_1)} \\
&+ R^{4\,N} \iint_{(B_1\setminus B_{\frac{1}{R}})\times(B_1\setminus B_{\frac{1}{R}})} \frac{|u(z)-u(w)|^2}{|\mathcal{K}^{-1}(z)-\mathcal{K}^{-1}(w)|^{N+2\,s}}\,dz\,dw \\
&+ 2\,R^{2\,N} \iint_{B_1\times(B_1\setminus B_{\frac{1}{R}})} \frac{|u(x)-u(w)|^2}{|x-\mathcal{K}^{-1}(w)|^{N+2\,s}}\,dx\,dw.
\end{aligned}
\tag{2.4}
$$

In order to estimate the last two integrals, it is sufficient to use that

$$
|\mathcal{K}^{-1}(z)-\mathcal{K}^{-1}(w)| = \left|\frac{1}{|z|^2}\,z - \frac{1}{|w|^2}\,w\right| \ge |z-w|, \qquad \text{for every } z,w\in B_1\setminus\{0\}, \tag{2.5}
$$

and

$$
|x-\mathcal{K}^{-1}(w)| = \left|x - \frac{1}{|w|^2}\,w\right| \ge |x-w|, \qquad \text{for every } x,w\in B_1\setminus\{0\}. \tag{2.6}
$$

Indeed, by taking the square, we see that (2.5) is equivalent to

$$
\left(\frac{1}{|z|^2} - |z|^2\right) + \left(\frac{1}{|w|^2} - |w|^2\right) \ge 2\,\left(\frac{1}{|z|^2\,|w|^2} - 1\right)\,\langle z,w\rangle.
$$

This in turn follows from Young's inequality

$$
2\,\langle z,w\rangle \le |z|^2 + |w|^2,
$$

once we multiply both sides by the positive quantity

$$
\left(\frac{1}{|z|^2\,|w|^2} - 1\right).
$$

As for inequality (2.6), by taking again the square we see that the latter is equivalent to

$$
\frac{1}{|w|^2} - |w|^2 \ge 2\,\left(\frac{1}{|w|^2} - 1\right)\,\langle x,w\rangle. \tag{2.7}
$$

This in turn follows again from Young's inequality: more precisely, by using that $|x|<1$, we have

$$
2\,\langle x,w\rangle \le |x|^2 + |w|^2 \le 1 + |w|^2,
$$

and if we now multiply both sides by the positive quantity (here we use that $|w|<1$)

$$
\left(\frac{1}{|w|^2} - 1\right),
$$

we get (2.7), with some simple algebraic manipulations.

By applying the estimates (2.5) and (2.6) in (2.4), we finally get

$$\begin{aligned}\Big[\mathcal{E}[u]\Big]^2_{W^{s,2}(B_R)} &\le [u]^2_{W^{s,2}(B_1)}\\ &+ R^{4\,N}\iint_{(B_1\setminus B_{\frac{1}{R}})\times(B_1\setminus B_{\frac{1}{R}})} \frac{|u(z)-u(w)|^2}{|z-w|^{N+2\,s}}\,dz\,dw\\ &+ 2\,R^{2\,N}\iint_{B_1\times(B_1\setminus B_{\frac{1}{R}})} \frac{|u(x)-u(w)|^2}{|x-w|^{N+2\,s}}\,dx\,dw\\ &\le \left(1+R^{4\,N}+2\,R^{2\,N}\right)\,[u]^2_{W^{s,2}(B_1)},\end{aligned}$$

which proves the first estimate in (2.2).

We are left with estimating the L^2 norm of $\mathcal{E}[u]$. This is simpler and can be done as follows

$$\begin{aligned}\int_{B_R}|\mathcal{E}[u](x)|^2\,dx &= \int_{B_1}|u(x)|^2\,dx+\int_{B_R\setminus B_1}|u(\mathcal{K}(x))|^2\,dx\\ &\le \int_{B_1}|u(x)|^2\,dx + R^{2\,N}\int_{B_1\setminus B_{\frac{1}{R}}}|u(z)|^2\,dz \le \left(1+R^{2\,N}\right)\int_{B_1}|u(x)|^2\,dx.\end{aligned}$$

This concludes the proof. □

Remark 2.3. Another important feature of the previous result is that, rather than the usual continuity estimate

$$\big\|\mathcal{E}[u]\big\|_{W^{s,2}(B_R)} \le C\,\|u\|_{W^{s,2}(B_1)},$$

for the extension operator, we obtained the more precise estimate (2.2). This will be useful in the next result.

Proposition 2.4. *Let $0<s<1$ and let $E\subseteq B_R(x_0)\subset\mathbb{R}^N$ be a measurable set, with positive measure. There exists a constant $\mathcal{M}=\mathcal{M}(N)>0$ such that for every $u\in W^{s,2}(B_R(x_0))$ we have*

$$\|u-\overline{u}_E\|^2_{L^2(B_R(x_0))} \le \mathcal{M}\,(1-s)\,\frac{R^N}{|E|}\,R^{2\,s}\,[u]^2_{W^{s,2}(B_R(x_0))}.$$

Proof. By a standard scaling argument, it is sufficient to prove the result for $R=1$ and $x_0=0$. For every $t>0$, we denote by $Q_t=(-t/2,t/2)^N$ the $N-$dimensional open cube centered at the origin, with side length t.

We consider the extension $\mathcal{E}[u]$ of u to the whole $\mathbb{R}^N$, as in (2.3). For ease of notation, we will simply write $\widetilde{u}:=\mathcal{E}[u]$. By using the triangle inequality and the fact that $B_1\subset Q_2$, we have

$$\begin{aligned}\|u-\overline{u}_E\|^2_{L^2(B_1)} &\le \|\widetilde{u}-\overline{u}_E\|^2_{L^2(Q_2)}\\ &\le 2\,\|\widetilde{u}-\overline{\widetilde{u}}_{Q_2}\|^2_{L^2(Q_2)} + 2\,\|\overline{\widetilde{u}}_{Q_2}-\overline{u}_E\|^2_{L^2(Q_2)}.\end{aligned}\tag{2.8}$$

By using Jensen's inequality and the fact that $|Q_2|=2^N$, we can estimate the second term as follows

$$\begin{aligned}\|\overline{\widetilde{u}}_{Q_2}-\overline{u}_E\|^2_{L^2(Q_2)} &= 2^N\,|\overline{\widetilde{u}}_{Q_2}-\overline{u}_E|^2\\ &= 2^N\left|\fint_E(\widetilde{u}(x)-\overline{\widetilde{u}}_{Q_2})\,dx\right|^2\\ &\le 2^N\fint_E|\widetilde{u}(x)-\overline{\widetilde{u}}_{Q_2}|^2\,dx \le \frac{2^N}{|E|}\,\|\widetilde{u}-\overline{\widetilde{u}}_{Q_2}\|^2_{L^2(Q_2)}.\end{aligned}$$

Thus from (2.8) we get

$$\|u-\overline{u}_E\|^2_{L^2(B_1)} \le 2\left(1+\frac{2^N}{|E|}\right)\|\widetilde{u}-\overline{\widetilde{u}}_{Q_2}\|^2_{L^2(Q_2)}.$$

We can now apply the following fractional Poincaré inequality[2] proved by Maz'ya and Shaposnikova (see [27, page 300])

$$\|\varphi-\overline{\varphi}_{Q_2}\|^2_{L^2(Q_2)} \le C_N\,(1-s)\,[\varphi]^2_{W^{s,2}(Q_2)}, \qquad \text{for every } \varphi\in W^{s,2}(Q_2). \tag{2.9}$$

Here C_N is an explicit dimensional constant. This yields

$$\begin{aligned}\|u-\overline{u}_E\|^2_{L^2(B_1)} &\le 2\left(1+\frac{2^N}{|E|}\right)C_N\,(1-s)\,[\widetilde{u}]^2_{W^{s,2}(Q_2)}\\ &\le 2\,\frac{\omega_N+2^N}{|E|}\,C_N\,(1-s)\,[\widetilde{u}]^2_{W^{s,2}(B_{\sqrt{2}})},\end{aligned}$$

where we used that $Q_2\subset B_{\sqrt{2}}$. It is now sufficient to apply Lemma 2.2 with $R=\sqrt{2}$, to get the claimed conclusion. □

3. An expedient Poincaré inequality

The following result is a nonlocal counterpart of [19, Lemma 1] in Hayman's paper. In the proof we pay due attention to the dependence of the constant on the fractional parameter s, as always.

Proposition 3.1 (Poincaré for boundary disks)**.** *Let $1/2<s<1$ and let $\Omega\subset\mathbb{R}^2$ be an open simply connected set, with $\partial\Omega\neq\emptyset$. There exists a constant $\mathcal{T}_s>0$ depending on s only, such that for every $r>0$ and every $x_0\in\partial\Omega$, we have*

$$\frac{\mathcal{T}_s}{r^{2\,s}}\int_{B_r(x_0)}|u(x)|^2\,dx \le \iint_{B_r(x_0)\times\mathbb{R}^2}\frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy, \qquad \textit{for every } u\in C^\infty_0(\Omega).$$

Moreover, $\mathcal{T}_s$ has the following asymptotic behaviours

$$\mathcal{T}_s\sim\left(s-\frac{1}{2}\right),\quad \textit{for } s\searrow\frac{1}{2}\qquad \textit{and}\qquad \mathcal{T}_s\sim\frac{1}{1-s},\quad \textit{for } s\nearrow 1.$$

Proof. Up to scaling and translating, we can assume without loss of generality that $r=1$ and that x_0 coincides with the origin.

We split the proof in three main steps: we first show that it is sufficient to prove the claimed estimate for the boundary ring $B_1\setminus B_{1/2}$. Then we prove such an estimate and at last we discuss the asymptotic behaviour of the constant obtained.

[2] We remark that the presence of the factor $(1-s)$ is important for our scopes. If one is not interested in keeping track of this factor, actually the proof of (2.9) would be much simpler, see for example [29, page 297].

Step 1: reduction to a ring. Let $u \in C_0^\infty(\Omega)$, we then estimate the L^2 norm on B_1 as follows

$$\begin{aligned}\int_{B_1} |u(x)|^2\,dx &= \int_{B_1\setminus B_{1/2}} |u(x)|^2\,dx + \int_{B_{1/2}} |u(x)|^2\,dx\\ &\le \int_{B_1\setminus B_{1/2}} |u(x)|^2\,dx + 2\int_{B_{1/2}} |u(x)-\overline{u}_{B_1\setminus B_{1/2}}|^2\,dx + 2\int_{B_{1/2}} |\overline{u}_{B_1\setminus B_{1/2}}|^2\,dx\\ &\le \int_{B_1\setminus B_{1/2}} |u(x)|^2\,dx + 2\int_{B_1} |u(x)-\overline{u}_{B_1\setminus B_{1/2}}|^2\,dx + 2\int_{B_{1/2}} |\overline{u}_{B_1\setminus B_{1/2}}|^2\,dx\\ &\le \int_{B_1\setminus B_{1/2}} |u(x)|^2\,dx + 2\int_{B_1} |u(x)-\overline{u}_{B_1\setminus B_{1/2}}|^2\,dx + \frac{2}{3}\int_{B_1\setminus B_{1/2}} |u(x)|^2\,dx,\end{aligned}$$

where we used the elementary inequality $(a+b)^2 \le 2\,a^2 + 2\,b^2$ and Jensen's inequality. If we now apply Proposition 2.4 with $R=1$ and $E = B_1\setminus B_{1/2}$, we get

$$\int_{B_1} |u(x)|^2\,dx \le \frac{5}{3}\int_{B_1\setminus B_{1/2}} |u(x)|^2\,dx + \frac{8}{3\,\pi}\,\mathcal{M}\,(1-s)\,[u]^2_{W^{s,2}(B_1)}.$$

Thus, in order to conclude, it is sufficient to prove that there exists a constant $C = C(s) > 0$ such that

$$\int_{B_1\setminus B_{1/2}} |u(x)|^2\,dx \le C \iint_{B_1\times\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy, \qquad \text{for every } u \in C_0^\infty(\Omega). \tag{3.1}$$

Step 2: estimate on the ring. We start with a topological observation. Since we are assuming that $0 \in \partial\Omega$ and that Ω is simply connected, we have the following crucial property

$$\partial B_\varrho \cap (\mathbb{R}^2\setminus\Omega) \neq \emptyset, \qquad \text{for every } \varrho > 0. \tag{3.2}$$

Indeed, if this were not true, we would have existence of a circle entirely contained in Ω and centered on the boundary of $\partial\Omega$. Such a circle could not be null-homotopic in Ω, thus contradicting our topological assumption.

In the rest of the proof, we will use polar coordinates (ϱ,θ) and we will make the slight abuse of notation of writing $u(\varrho,\theta)$. Then, in light of the property (3.2), for each $\varrho \in (1/2,1)$ there exists $\theta_\varrho \in [0,2\pi)$ such that $\theta \mapsto u(\varrho,\theta)$ must vanish at θ_ϱ. Hence, for every $1/2 < \varrho < 1$ we can apply Proposition A.2 to the function $\theta \mapsto u(\varrho,\theta)$ and get

$$\int_0^{2\,\pi} |u(\varrho,\theta)|^2\,d\theta \le \frac{1}{\mu_s}\int_0^{2\pi}\int_0^{2\pi} \frac{|u(\varrho,\theta)-u(\varrho,\varphi)|^2}{|\theta-\varphi|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,d\varphi.$$

The constant μ_s is the same as in Proposition A.2 and

$$|\theta-\varphi|_{\mathbb{S}^1} := \min_{k\in\mathbb{Z}} |\theta-\varphi+2\,k\,\pi|, \qquad \text{for every } \theta,\varphi \in \mathbb{R}.$$

If we now multiply both sides by ϱ, integrate over the interval $(1/2,1)$ and write the L^2 norm in polar coordinates, we get

$$\int_{B_1\setminus B_{1/2}} |u(x)|^2\,dx \le \frac{1}{\mu_s}\int_{\frac{1}{2}}^1\int_0^{2\pi}\int_0^{2\,\pi} \frac{|u(\varrho,\theta)-u(\varrho,\varphi)|^2}{\varrho^{1+2\,s}\,|\theta-\varphi|_{\mathbb{S}^1}^{1+2\,s}}\,\varrho\,d\varrho\,d\theta\,d\varphi. \tag{3.3}$$

Observe that we further used the fact that $\varrho \le 1$, to let the term $\varrho^{-1-2\,s}$ appear.

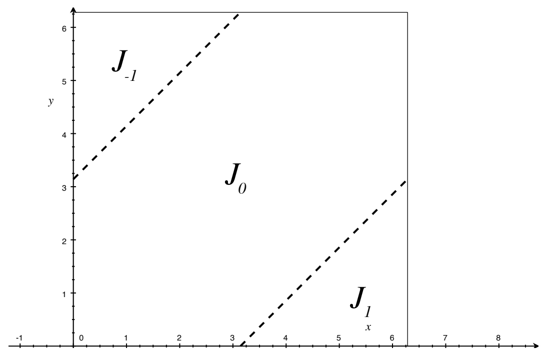

FIGURE 1. The partition of $[0,2\pi]\times[0,2\pi]$ needed to define the midpoint function.

In order to achieve (3.1), we need to show that the term on the right-hand side can be estimated by a two-dimensional Gagliardo-Slobodeckiĭ seminorm. To this aim, we follow an argument similar to that of [7, Lemma B.2]. At first, it is easily seen that

$$(\varrho\,|\theta-\varphi|_{\mathbb{S}^1})^{-1-2\,s}=(1+2\,s)\int_0^{+\infty}(t+\varrho\,|\theta-\varphi|_{\mathbb{S}^1})^{-2-2\,s}\,dt.$$

By inserting this in (3.3), we end up with

$$\begin{aligned}\int_{B_1\setminus B_{1/2}}&|u(x)|^2\,dx\\ &\le\frac{1+2\,s}{\mu_s}\int_0^{2\pi}\int_0^{2\pi}\int_{\frac12}^1\int_0^{+\infty}\frac{|u(\varrho,\theta)-u(\varrho,\varphi)|^2}{(t+\varrho\,|\theta-\varphi|_{\mathbb{S}^1})^{2+2\,s}}\,\varrho\,d\theta\,d\varphi\,d\varrho\,dt\\ &\le\frac{2\,(1+2\,s)}{\mu_s}\int_0^{2\pi}\int_0^{2\,\pi}\int_{\frac12}^1\int_0^{+\infty}\frac{|u(\varrho,\theta)-u(\varrho,\varphi)|^2}{(t+\varrho\,|\theta-\varphi|_{\mathbb{S}^1})^{2+2\,s}}\,\varrho\,(\varrho+t)\,d\theta\,d\varphi\,d\varrho\,dt,\end{aligned}\tag{3.4}$$

where we have used that $1/2\le\varrho+t$. We now split the set $[0,2\,\pi]\times[0,2\,\pi]=J_{-1}\cup J_0\cup J_1$, where

$$J_{-1}=\Big\{(\theta,\varphi)\,:\,\theta\in[0,\pi],\,\theta+\pi<\varphi\le2\,\pi\Big\},\qquad J_1=\Big\{(\theta,\varphi)\,:\,\theta\in[\pi,2\,\pi],\,0\le\varphi<\theta-\pi\Big\},$$

and

$$J_0=\Big\{(\theta,\varphi)\,:\,\theta\in[0,2\,\pi],\,\max\{0,\theta-\pi\}\le\varphi\le\min\{2\,\pi,\theta+\pi\}\Big\},$$

see Figure 1. Then, we define the *midpoint function* by

$$\overline{\theta\,\varphi}=\frac{\theta+\varphi}{2}+\ell\,\pi,\qquad\text{if }(\theta,\varphi)\in J_\ell,\ \text{with }\ell=-1,0,1.\tag{3.5}$$

Thanks to the triangle inequality, we estimate the numerator in the right-hand side of (3.4) as follows

$$|u(\varrho,\theta)-u(\varrho,\varphi)|^2\le2\left|u\left(\varrho,\theta\right)-u\left(\varrho+t,\overline{\theta\,\varphi}\right)\right|^2+2\left|u\left(\varrho,\varphi\right)-u\left(\varrho+t,\overline{\theta\,\varphi}\right)\right|^2.$$

As for the denominator, we observe that $|\theta-\overline{\theta\,\varphi}|_{\mathbb{S}^1}=|\varphi-\overline{\theta\,\varphi}|_{\mathbb{S}^1}$, thus we get

$$|\theta-\varphi|_{\mathbb{S}^1}=2\,|\theta-\overline{\theta\,\varphi}|_{\mathbb{S}^1}\ge2\,|e^{i\,\theta}-e^{i\,\overline{\theta\,\varphi}}|\quad\text{ and }\quad|\theta-\varphi|_{\mathbb{S}^1}=2\,|\varphi-\overline{\theta\,\varphi}|_{\mathbb{S}^1}\ge2\,|e^{i\,\varphi}-e^{i\,\overline{\theta\,\varphi}}|,$$

where the inequalities come from Lemma A.1. By using this fact, the identity $|e^{i\overline{\theta\,\varphi}}|=1$ and the triangle inequality again, we can estimate the denominator as

$$\begin{aligned}t+\varrho\,|\theta-\varphi|_{\mathbb{S}^1}&\geq t+\varrho\,|e^{i\theta}-e^{i\,\overline{\theta\,\varphi}}|\\&\geq\left|\varrho\,(e^{i\,\theta}-e^{i\,\overline{\theta\,\varphi}})-t\,e^{i\,\overline{\theta\,\varphi}}\right|=\left|\varrho\,e^{i\,\theta}-(\varrho+t)\,e^{i\,\overline{\theta\,\varphi}}\right|,\end{aligned}$$

and similarly

$$t+\varrho\,|\theta-\varphi|_{\mathbb{S}^1}\geq\left|\varrho\,e^{i\,\varphi}-(\varrho+t)\,e^{i\,\overline{\theta\,\varphi}}\right|.$$

These allow us to estimate the right-hand side in (3.4) in the following way:

$$\begin{aligned}&\int_{B_1\setminus B_{1/2}}|u(x)|^2\,dx\\&\qquad\leq\frac{4\,(1+2\,s)}{\mu_s}\int_0^{2\,\pi}\int_0^{2\,\pi}\int_{\frac{1}{2}}^1\int_0^{+\infty}\frac{\left|u\,(\varrho,\theta)-u\left(\varrho+t,\overline{\theta\,\varphi}\right)\right|^2}{\left|\varrho\,e^{i\,\theta}-(\varrho+t)\,e^{i\,\overline{\theta\,\varphi}}\right|^{2+2\,s}}\,\varrho\,(\varrho+t)\,d\theta\,d\varphi\,d\varrho\,dt\\&\qquad+\frac{4\,(1+2\,s)}{\mu_s}\int_0^{2\,\pi}\int_0^{2\,\pi}\int_{\frac{1}{2}}^1\int_0^{+\infty}\frac{\left|u\,(\varrho,\varphi)-u\left(\varrho+t,\overline{\theta\,\varphi}\right)\right|^2}{\left|\varrho\,e^{i\,\varphi}-(\varrho+t)\,e^{i\,\overline{\theta\,\varphi}}\right|^{2+2\,s}}\,\varrho\,(\varrho+t)\,d\theta\,d\varphi\,d\varrho\,dt\\&\qquad=\frac{8\,(1+2\,s)}{\mu_s}\int_0^{2\,\pi}\int_0^{2\,\pi}\int_{\frac{1}{2}}^1\int_0^{+\infty}\frac{\left|u\,(\varrho,\theta)-u\left(\varrho+t,\overline{\theta\,\varphi}\right)\right|^2}{\left|\varrho\,e^{i\,\theta}-(\varrho+t)\,e^{i\,\overline{\theta\,\varphi}}\right|^{2+2\,s}}\,\varrho\,(\varrho+t)\,d\theta\,d\varphi\,d\varrho\,dt.\end{aligned}$$

In the last identity we used that both multiple integrals coincide, by symmetry of the integrands. If we now make the change of variable $\tau=\varrho+t$ and use the decomposition $[0,2\,\pi]\times[0,2\,\pi]=J_{-1}\cup J_0\cup J_1$, we obtain

$$\begin{aligned}&\int_{B_1\setminus B_{1/2}}|u(x)|^2\,dx\\&\qquad\leq\frac{8\,(1+2\,s)}{\mu_s}\sum_{\ell=-1}^{1}\iint_{J_\ell}\int_{\frac{1}{2}}^1\int_{\varrho}^{+\infty}\frac{\left|u\,(\varrho,\theta)-u(\tau,\overline{\theta\,\varphi})\right|^2}{\left|\varrho\,e^{i\,\theta}-\tau\,e^{i\,\overline{\theta\,\varphi}}\right|^{2+2\,s}}\,\varrho\,\tau\,d\theta\,d\varphi\,d\varrho\,d\tau.\end{aligned}\tag{3.6}$$

If we now denote

$$\widetilde{J}_{-1}=\Big\{(\theta,\varphi)\,:\,\theta\in[0,\pi],\,\theta-\pi<\varphi\leq 0\Big\}\quad\text{and}\quad\widetilde{J}_1=\Big\{(\theta,\varphi)\,:\,\theta\in[\pi,2\,\pi],\,2\,\pi\leq\varphi<\theta+\pi\Big\},$$

use the definition of midpoint function (3.5) and make the change of variables

$$\begin{array}{ccc}J_{-1}&\to&\widetilde{J}_{-1}\\(\theta,\varphi)&\mapsto&(\theta,\varphi-2\,\pi)\end{array}\qquad\text{and}\qquad\begin{array}{ccc}J_1&\to&\widetilde{J}_1\\(\theta,\varphi)&\mapsto&(\theta,\varphi+2\,\pi),\end{array}$$

we obtain from (3.6)

$$\int_{B_1\setminus B_{1/2}}|u(x)|^2\,dx\leq\frac{8\,(1+2\,s)}{\mu_s}\iint_{\widetilde{J}_{-1}\cup J_0\cup\widetilde{J}_1}\int_{\frac{1}{2}}^1\int_{\varrho}^{+\infty}\frac{\left|u\,(\varrho,\theta)-u\left(\tau,\frac{\theta+\varphi}{2}\right)\right|^2}{\left|\varrho\,e^{i\,\theta}-\tau\,e^{i\,\frac{\theta+\varphi}{2}}\right|^{2+2s}}\,\varrho\,\tau\,d\theta\,d\varphi\,d\varrho\,d\tau.$$

For every $\theta \in [0, 2\pi]$, we now make the change of variable $\gamma = (\theta + \varphi)/2$, thus the above estimate becomes

$$\int_{B_1 \setminus B_{1/2}} |u(x)|^2 \, dx \le \frac{16\,(1+2\,s)}{\mu_s} \sum_{\ell=-1}^{1} \iint_{\widehat{J}_\ell} \int_{\frac{1}{2}}^{1} \int_{\varrho}^{+\infty} \frac{|u\,(\varrho,\theta) - u\,(\tau,\gamma)|^2}{|\varrho\, e^{i\,\theta} - \tau\, e^{i\,\gamma}|^{2+2s}}\, \varrho\, \tau\, d\theta\, d\gamma\, d\varrho\, d\tau, \tag{3.7}$$

where

$$\widehat{J}_{-1} = \left\{(\theta,\gamma)\,:\theta \in \left[0, \frac{\pi}{2}\right],\, \theta - \frac{\pi}{2} < \gamma \le 0\right\}, \quad \widehat{J}_1 = \left\{(\theta,\gamma)\,:\,\theta \in \left[\frac{3}{2}\,\pi, 2\,\pi\right],\, 2\,\pi \le \gamma < \theta + \frac{\pi}{2}\right\},$$

and

$$\widehat{J}_0 = \left\{(\theta,\gamma)\,:\,\theta \in [0, 2\,\pi],\, \max\left\{0,\, \theta - \frac{\pi}{2}\right\} \le \gamma \le \min\left\{2\pi,\, \theta + \frac{\pi}{2}\right\}\right\}.$$

If we now exploit the $2\,\pi-$periodicity of the integrand, we have

$$\begin{aligned} &\iint_{\widehat{J}_{-1}} \int_{\frac{1}{2}}^{1} \int_{\varrho}^{+\infty} \frac{|u\,(\varrho,\theta) - u\,(\tau,\gamma)|^2}{|\varrho\, e^{i\,\theta} - \tau\, e^{i\,\gamma}|^{2+2s}}\, \varrho\, \tau\, d\theta\, d\gamma\, d\varrho\, d\tau \\ &= \iint_{\widehat{J}_{-1}} \int_{\frac{1}{2}}^{1} \int_{\varrho}^{+\infty} \frac{|u\,(\varrho,\theta) - u\,(\tau,\gamma+2\,\pi)|^2}{\left|\varrho\, e^{i\,\theta} - \tau\, e^{i\,(\gamma+2\,\pi)}\right|^{2+2s}}\, \varrho\, \tau\, d\theta\, d\gamma\, d\varrho\, d\tau \\ &= \iint_{I_{-1}} \int_{\frac{1}{2}}^{1} \int_{\varrho}^{+\infty} \frac{|u\,(\varrho,\theta) - u\,(\tau,\varphi)|^2}{|\varrho\, e^{i\theta} - \tau\, e^{i\,\varphi}|^{2+2s}}\, \varrho\, \tau\, d\theta\, d\varphi\, d\varrho\, d\tau, \end{aligned} \tag{3.8}$$

where we set $\varphi = \gamma + 2\,\pi$ and

$$I_{-1} = \left\{(\theta,\varphi)\,:\theta \in \left[0, \frac{\pi}{2}\right],\, \theta + \frac{3}{2}\,\pi < \varphi \le 2\,\pi\right\}.$$

Similarly, we can obtain

$$\begin{aligned} &\iint_{\widehat{J}_1} \int_{\frac{1}{2}}^{1} \int_{\varrho}^{+\infty} \frac{|u\,(\varrho,\theta) - u\,(\tau,\gamma)|^2}{|\varrho\, e^{i\,\theta} - \tau\, e^{i\,\gamma}|^{2+2\,s}}\, \varrho\, \tau\, d\theta\, d\gamma\, d\varrho\, d\tau \\ &= \iint_{I_1} \int_{\frac{1}{2}}^{1} \int_{\varrho}^{+\infty} \frac{|u\,(\varrho,\theta) - u\,(\tau,\varphi)|^2}{|\varrho\, e^{i\,\theta} - \tau\, e^{i\,\varphi}|^{2+2s}}\, \varrho\, \tau\, d\theta\, d\varphi\, d\varrho\, d\tau, \end{aligned} \tag{3.9}$$

with the change of variable $\varphi = \gamma - 2\,\pi$ and

$$I_1 = \left\{(\theta,\varphi)\,:\,\theta \in \left[\frac{3}{2}\,\pi, 2\,\pi\right],\, 0 \le \varphi < \theta - \frac{3}{2}\,\pi\right\}.$$

By observing that $I_{-1} \cup \widehat{J}_0 \cup I_1 \subset [0, 2\,\pi] \times [0, 2\,\pi]$ and that the three sets I_{-1}, $\widehat{J}_0$ and I_1 are pairwise disjoint, from (3.7), (3.8) and (3.9) we finally obtain

$$\begin{aligned} \int_{B_1 \setminus B_{1/2}} |u(x)|^2 \, dx &\le \frac{16\,(1+2\,s)}{\mu_s} \iint_{[0,2\,\pi]\times[0,2\,\pi]} \int_{\frac{1}{2}}^{1} \int_{\varrho}^{+\infty} \frac{|u\,(\varrho,\theta) - u\,(\tau,\varphi)|^2}{|\varrho\, e^{i\,\theta} - \tau\, e^{i\,\varphi}|^{2+2\,s}}\, \varrho\, \tau\, d\theta\, d\varphi\, d\varrho\, d\tau \\ &\le \frac{16\,(1+2\,s)}{\mu_s} \iint_{B_1 \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^{2+2\,s}}\, dx\, dy. \end{aligned}$$

This concludes the proof of (3.1).

Step 3: asymptotics for the constant. From **Step 1** and **Step 2**, we obtained the Poincaré inequality claimed in the statement, with constant

$$\mathcal{T}_s = \left(\frac{80\,(1+2\,s)}{3\,\mu_s} + \frac{8}{3\,\pi}\,\mathcal{M}\,(1-s)\right)^{-1}.$$

By using the asymptotics for the constant μ_s (see Proposition A.2 below), we get the desired conclusion. □

Remark 3.2. The previous result can not hold for $0 < s \le 1/2$. Indeed, if the result were true for $0 < s \le 1/2$, this would permit to extend the fractional Makai-Hayman inequality to this range, as well (see the next section). However, this would contradict Theorem 1.3.

4. Proof of Theorem 1.1

Without loss of generality, we can consider $r_\Omega = 1$. We take $\mathfrak{B}$ and $\mathfrak{B}_1, \dots, \mathfrak{B}_{36}$ to be respectively the covering of Ω and the subclasses given by Lemma 2.1, made of ball with radius $r = 1 + \sqrt{2}$.

We take an index $k \in \{1, \dots, 36\}$, then we know that $\mathfrak{B}_k$ is composed of (possibly) countably many disjoint balls with radius r, centered on $\partial\Omega$. We indicate by $B^{j,k}$ each of these balls.

Then, for every $u \in C_0^\infty(\Omega) \setminus \{0\}$ we have

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy \ge \sum_{B^{j,k}\in\mathfrak{B}_k} \iint_{B^{j,k}\times\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy. \tag{4.1}$$

For each ball $B^{j,k}$, we can apply Proposition 3.1 so to obtain that

$$\sum_{B^{j,k}\in\mathfrak{B}_k} \iint_{B^{j,k}\times\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy \ge \frac{\mathcal{T}_s}{(1+\sqrt{2})^{2\,s}} \sum_{B^{j,k}\in\mathfrak{B}_k} \int_{B^{j,k}} |u(x)|^2\,dx.$$

We insert this estimate in (4.1) and then sum over $k = 1, \dots, 36$. We get

$$\begin{aligned} 36 \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy &\ge \sum_{k=1}^{36} \sum_{B^{j,k}\in\mathfrak{B}_k} \iint_{B^{j,k}\times\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy \\ &\ge \frac{\mathcal{T}_s}{(1+\sqrt{2})^{2\,s}} \sum_{k=1}^{36} \sum_{B^{j,k}\in\mathfrak{B}_k} \int_{B^{j,k}} |u(x)|^2\,dx \\ &\ge \frac{\mathcal{T}_s}{(1+\sqrt{2})^{2\,s}} \int_\Omega |u(x)|^2\,dx. \end{aligned}$$

In the last inequality we used that $\mathfrak{B}$ is a covering of Ω. By recalling the definition of $\lambda_1^s(\Omega)$, from the previous chain of inequalities we thus get the claimed estimate (1.1), with constant

$$\mathcal{C}_s := \frac{\mathcal{T}_s}{36\,(1+\sqrt{2})^{2\,s}}.$$

The asymptotic behaviour of $\mathcal{C}_s$ can now be inferred from that of $\mathcal{T}_s$, which in turn is contained in Proposition 3.1.

Remark 4.1. For suitable classes of open sets in $\mathbb{R}^N$ and every $0 < s < 1$, it is possible to give a Makai-Hayman–type lower bound on λ_1^s, by taking advantage of the nonlocality of the

Gagliardo-Slobodeckiĭ seminorm. More precisely, this is possible provided Ω satisfies the following mild regularity assumption: there exist[3] $\sigma > 1$ and $\alpha > 0$ such that

$$\frac{|B_{\sigma\, r_\Omega}(x) \setminus \Omega|}{|B_{\sigma\, r_\Omega}(x)|} \geq \alpha, \qquad \text{for every } x \in \Omega. \tag{4.2}$$

Indeed, in this case for every $u \in C_0^\infty(\Omega)$ we can simply estimate

$$\begin{aligned}\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\,s}}\, dx\, dy &\geq \int_\Omega \left(\int_{B_{\sigma\, r_\Omega}(x) \setminus \Omega} \frac{|u(x)|^2}{|x-y|^{N+2\,s}}\, dy \right) dx \\ &\geq \frac{1}{(\sigma\, r_\Omega)^{N+2\,s}} \int_\Omega |B_{\sigma\, r_\Omega}(x) \setminus \Omega|\, |u(x)|^2\, dx \\ &\geq \frac{\alpha\, \omega_N}{(\sigma\, r_\Omega)^{2\,s}} \int_\Omega |u(x)|^2\, dx,\end{aligned}$$

where in the last inequality we used the additional condition (4.2). By arbitrariness of u, we get

$$\lambda_1^s(\Omega) \geq \frac{\alpha\, \omega_N}{\sigma^{2\,s}}\, \frac{1}{r_\Omega^{2\,s}}.$$

One could observe that the additional condition (4.2) does not always hold for a simply connected set in the plane. Moreover, the constant obtained in this way is quite poor: first of all, it is not universal. It depends on the parameters α and σ and it deteriorates as $\sigma \searrow 1$, since in this case we must have $\alpha \searrow 0$. Secondly, it does not exhibit the correct asymptotic behaviour as s goes to 1.

5. Proof of Theorem 1.3

Let $0 < s \leq 1/2$ and $\{Q_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^2$ be the sequence of open squares $Q_k = (-k, k)^2$, with $k \in \mathbb{N} \setminus \{0, 1\}$. We introduce the one-dimensional set

$$\Sigma = \bigcup_{i \in \mathbb{Z}} \Sigma^{(i)}, \qquad \text{where } \Sigma^{(i)} := \bigcup_{i \in \mathbb{Z}} \left\{ (x_1, i) \in \mathbb{R}^2 \,:\, |x_1| \geq 1 \right\},$$

and then define, for every fixed $k \in \mathbb{N} \setminus \{0, 1\}$, the "cracked" square $\widetilde{Q}_k = Q_k \setminus \Sigma$ (see Figure 2).

First of all, we observe that

$$r_{\widetilde{Q}_k} = \frac{\sqrt{5}}{2}, \qquad \text{for every } k \geq 2.$$

Thus, if we can show that

$$\lim_{k \to \infty} \lambda_1^s(\widetilde{Q}_k) = 0, \tag{5.1}$$

we would automatically get the desired counter-example. We will obtain (5.1) by proving that

$$\lambda_1^s(\widetilde{Q}_k) = \lambda_1^s(Q_k), \qquad \text{for every } k \geq 2. \tag{5.2}$$

Indeed, if this were true, we would have

$$\lim_{k \to \infty} \lambda_1^s(\widetilde{Q}_k) = \lim_{k \to \infty} \lambda_1^s(Q_k) = \lim_{k \to \infty} k^{-2\,s}\, \lambda_1^s(Q_1) = 0,$$

by the scale properties of λ_s^1. This would prove (5.1), as claimed.

[3]It is not difficult to see that this property never holds for $\sigma = 1$.

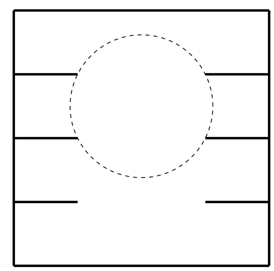

FIGURE 2. The set $\widetilde{Q}_k$ with $k=2$. In dashed line, a disk of maximal radius.

We are thus left with proving (5.2). We already know that

$$\lambda_1^s(\widetilde{Q}_k) \geq \lambda_1^s(Q_k),$$

thanks to the fact that λ_1^s is monotone with respect to set inclusion. In the remaining part of the proof, we focus our attention in proving the opposite inequality.

At this aim, for every $n \in \mathbb{N}\setminus\{0\}$ we introduce the neighborhoods

$$\Sigma^{(i)}_{k,n} = \left\{x \in \mathbb{R}^2\,:\, \mathrm{dist}(x, \Sigma^{(i)} \cap Q_k) \leq \frac{1}{n+1}\right\}, \qquad \text{for } i \in \{-(k-1), \dots, k-1\},$$

and consider a sequence of cut-off functions $\{\varphi^{(i)}_n\}_{n\in\mathbb{N}\setminus\{0\}} \subset C^\infty_0(\Sigma^{(i)}_{k,2\,n})$ such that

$$0 \leq \varphi^{(i)}_n \leq 1, \qquad \varphi^{(i)}_n \equiv 1 \text{ on } \Sigma^{(i)}_{k,4\,n}, \qquad |\nabla \varphi^{(i)}_n(x)| \leq C\,n,$$

for some constant $C>0$, independent of n. Observe that by construction we have

$$\mathrm{spt}(\varphi^{(i)}_n) \cap \mathrm{spt}(\varphi^{(j)}_n) = \emptyset, \qquad \text{for } i \neq j,$$

By using an interpolation inequality (see [10, Corollary 2.2]) and the properties of the cut-off functions, we can estimate the energy of each $\varphi^{(i)}_n$ as follows

$$\begin{aligned}[\varphi^{(i)}_n]^2_{W^{s,2}(\mathbb{R}^2)} &\leq C\left(\int_{\Sigma^{(i)}_{k,2\,n}} |\varphi^{(i)}_n|^2\,dx\right)^{1-s}\left(\int_{\Sigma^{(i)}_{k,2\,n}} |\nabla\varphi^{(i)}_n|^2\,dx\right)^{s}\\ &\leq C\,|\Sigma^{(i)}_{k,2\,n}|^{1-s}\,|\Sigma^{(i)}_{k,2\,n}|^{s}\,n^{2\,s} \leq C\,n^{2\,s-1},\end{aligned}$$

for a constant $C>0$ independent[4] of n. In particular, for every $i \in \{-(k-1),\dots,k-1\}$ we have

$$\lim_{n\to+\infty} [\varphi^{(i)}_n]^2_{W^{s,2}(\mathbb{R}^2)} = 0, \qquad \text{if } 0<s<\frac{1}{2}, \tag{5.3}$$

while

$$\sup_{n\geq 1}\,[\varphi^{(i)}_n]^2_{W^{s,2}(\mathbb{R}^2)} \leq C, \qquad \text{if } s=\frac{1}{2}. \tag{5.4}$$

[4]Observe that such a constant depends on k, through the length of the set $\Sigma^{(i)} \cap Q_k$. However this is not a problem, since in this part k is now fixed.

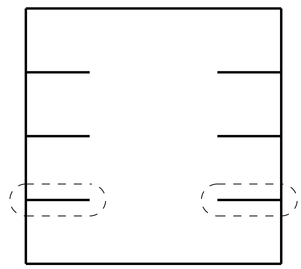

FIGURE 3. The dashed line encloses one of the set $\Sigma^{(i)}_{k,n}$.

From now on, for ease of notation, we denote

$$\Phi_{k,n} = \sum_{i=-(k-1)}^{k-1} \varphi_n^{(i)} \in C_0^\infty(Q_{2\,k}).$$

Due to the different behaviours (5.3) and (5.4), we need to consider the cases $0 < s < 1/2$ and $s = 1/2$ separately.

Case $0 < s < 1/2$. For every $u \in C_0^\infty(Q_k) \setminus \{0\}$, we simply take

$$u_n = (1 - \Phi_{k,n})\, u,$$

and observe that $u_n \in C_0^\infty(\widetilde{Q}_k)$ for every $n \in \mathbb{N} \setminus \{0\}$. Since each u_n is admissible for the problem (1.3), we get

$$\sqrt{\lambda_1^s(\widetilde{Q}_k)} \le \frac{[u_n]_{W^{s,2}(\mathbb{R}^2)}}{\|u_n\|_{L^2(\widetilde{Q}_k)}} \le \frac{[u]_{W^{s,2}(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)}\ [1 - \Phi_{k,n}]_{W^{s,2}(\mathbb{R}^2)}}{\|u\,(1 - \Phi_{k,n})\|_{L^2(\widetilde{Q}_k)}}, \tag{5.5}$$

where in the last inequality we have used the Leibniz–type rule (2.1) and the fact $|1 - \Phi_{k,n}| \le 1$. We now observe that

$$\lim_{n\to\infty} \|u\,(1 - \Phi_{k,n})\|_{L^2(\widetilde{Q}_k)} = \|u\|_{L^2(Q_k)},$$

which follows from a standard application of the Lebesgue Dominated Convergence Theorem, together with the properties of $\Phi_{k,n}$. Moreover, it holds

$$\lim_{n\to\infty} [1 - \Phi_{k,n}]_{W^{s,2}(\mathbb{R}^2)} = 0.$$

This simply follows by using the definition of $\Phi_{k,n}$, the triangle inequality and (5.3). By using these two limits in (5.5), we get

$$\sqrt{\lambda_1^s(\widetilde{Q}_k)} \le \lim_{n\to\infty} \frac{[u]_{W^{s,2}(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)}\ [1 - \Phi_{k,n}]_{W^{s,2}(\mathbb{R}^2)}}{\|u\,(1 - \Phi_{k,n})\|_{L^2(\widetilde{Q}_k)}} = \frac{[u]_{W^{s,2}(\mathbb{R}^2)}}{\|u\|_{L^2(Q_k)}}.$$

By arbitrariness of $u \in C_0^\infty(Q_k) \setminus \{0\}$, we get

$$\lambda_1^s(\widetilde{Q}_k) \le \lambda_1^s(Q_k).$$

and thus the desired conclusion (5.2).

Borderline case $s = 1/2$. This is more delicate, we can not use directly the sequence $\{\Phi_{k,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ to construct an approximation of $u \in C_0^\infty(Q_k)$. Indeed, by owing to (5.4), we can now guarantee that $\{\Phi_{k,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ only converges weakly to 0 in $W^{1/2,2}(\mathbb{R}^2)$ as n goes to ∞, up to a subsequence.

In order to "boost" such a sequence, we make a suitable application of *Mazur's Lemma* (see for example [24, Theorem 2.13]). More precisely, we define the sequence $\{F_{k,n}\}_{n\in\mathbb{N}\setminus\{0\}} \subset L^2(\mathbb{R}^2\times\mathbb{R}^2)$, given by

$$F_{k,n}(x,y) = \frac{\Phi_{k,n}(x) - \Phi_{k,n}(y)}{|x-y|^{1+\frac{1}{2}}}.$$

By construction, we have that

$$\|F_{k,n}\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)} = [\Phi_{k,n}]_{W^{\frac{1}{2},2}(\mathbb{R}^2)} \le C,$$

and $\{F_{k,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ converges weakly to 0 in $L^2(\mathbb{R}^2\times\mathbb{R}^2)$, up to a subsequence. Thanks to Mazur's Lemma, we can enforce this weak convergence to the strong one, by passing to a sequence of convex combinations. More precisely, we know that for every $n \in \mathbb{N}\setminus\{0\}$ there exists

$$\big\{t_\ell(n)\big\}_{\ell=1}^n \subset [0,1], \qquad \text{such that} \qquad \sum_{\ell=1}^n t_\ell(n) = 1,$$

and such that the new sequence made of convex combinations

$$\widetilde{F}_{k,n}(x,y) = \sum_{\ell=1}^n t_\ell(n)\, F_{k,\ell}(x,y),$$

strongly converges in $L^2(\mathbb{R}^2\times\mathbb{R}^2)$ to 0, as n goes to ∞. Observe that by construction we have

$$\begin{aligned}\|\widetilde{F}_{k,n}\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}^2)} &= \left\|\sum_{\ell=1}^n t_\ell(n)\, F_{k,\ell}\right\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}\\ &= \iint_{\mathbb{R}^2\times\mathbb{R}^2}\left|\sum_{\ell=1}^n t_\ell(n)\frac{\Phi_{k,\ell}(x)-\Phi_{k,\ell}(y)}{|x-y|^{1+\frac{1}{2}}}\right|^2 dx\,dy\\ &= \iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\left|\sum_{\ell=1}^n t_\ell(n)\,\Phi_{k,\ell}(x) - \sum_{\ell=1}^n t_\ell(n)\,\Phi_{k,\ell}(y)\right|^2}{|x-y|^3}\, dx\,dy.\end{aligned}$$

Thus, if we set

$$\widetilde{\Phi}_{k,n} = \sum_{\ell=1}^n t_\ell(n)\,\Phi_{k,\ell} \in C_0^\infty(Q_{2\,k}),$$

the previous observations give that

$$\lim_{n\to\infty}[\widetilde{\Phi}_{k,n}]^2_{W^{\frac{1}{2},2}(\mathbb{R}^2)} = \lim_{n\to\infty}\|\widetilde{F}_{k,n}\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}^2)} = 0. \tag{5.6}$$

Moreover, by using the fractional Poincaré inequality with $s = 1/2$ for the open bounded set $Q_{2\,k}$, we also have

$$\lim_{n\to\infty}\|\widetilde{\Phi}_{k,n}\|^2_{L^2(Q_{2\,k})} \le \frac{1}{\lambda_1^{\frac{1}{2}}(Q_{2\,k})}\lim_{n\to\infty}[\widetilde{\Phi}_{k,n}]^2_{W^{\frac{1}{2},2}(\mathbb{R}^2)} = 0. \tag{5.7}$$

We take as in the previous case $u \in C_0^\infty(Q_k) \setminus \{0\}$. In order to approximate u with functions compactly supported in $\widetilde{Q}_k$, we now define

$$\widetilde{u}_n = (1 - \widetilde{\Phi}_{k,n})\, u.$$

We observe that this function belongs to $C_0^\infty(\widetilde{Q}_k)$. Indeed, observe that

$$\Phi_{k,\ell}(x) = 1, \qquad \text{for every } x \in \Sigma^{(i)}_{k,4\,\ell},\ i \in \{-(k-1),\dots,k-1\} \text{ and } \ell \in \{1,\dots,n\},$$

thus in particular

$$\widetilde{\Phi}_{k,n}(x) = \sum_{\ell=1}^{n} t_\ell(n)\, \Phi_{k,\ell}(x) = \sum_{\ell=1}^{n} t_\ell(n) = 1, \qquad \text{for every } x \in \Sigma^{(i)}_{k,4\,n},\ i \in \{-(k-1),\dots,k-1\},$$

thanks to the fact that

$$\Sigma^{(i)}_{k,4\,n} \subset \Sigma^{(i)}_{k,4\,\ell}, \qquad \text{for } \ell \in \{1,\dots,n\}.$$

Clearly, we still have

$$|1 - \widetilde{\Phi}_{k,n}| \le 1 \qquad \text{and} \qquad \lim_{n\to\infty} \|\widetilde{u}_n\|_{L^2(\widetilde{Q}_k)} = \|u\|_{L^2(Q_k)}. \tag{5.8}$$

The second fact in (5.8) can be proved by observing that

$$\begin{aligned}
\left| \int_{\widetilde{Q}_k} |\widetilde{u}_n|^2\, dx - \int_{Q_k} |u|^2\, dx \right| &= \left| \int_{\widetilde{Q}_k} |u|^2 \left[|1 - \widetilde{\Phi}_{k,n}|^2 - 1 \right] dx \right| \\
&\le \int_{\widetilde{Q}_k} |u|^2 \left[1 - |1 - \widetilde{\Phi}_{k,n}|^2 \right] dx \\
&\le 2 \int_{\widetilde{Q}_k} |u|^2 \left[1 - |1 - \widetilde{\Phi}_{k,n}| \right] dx \le 2\, \|u\|^2_{L^\infty(Q_k)} \int_{\widetilde{Q}_k} |\widetilde{\Phi}_{k,n}|\, dx,
\end{aligned}$$

and then using (5.7).

We can now use $\widetilde{u}_n$ as a competitor for the variational problem defining $\lambda_1^s(\widetilde{Q}_k)$ and proceed exactly as in the case $0 < s < 1/2$, by using (5.6) and (5.8). This finally concludes the proof.

Remark 5.1. With the notation above, we obtain in particular that the *infinite complement comb* $\Theta := \mathbb{R}^2 \setminus \Sigma$ is an open simply connected set such that

$$r_\Theta = \frac{\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_1^s(\Theta) = 0, \quad \text{for } 0 < s \le \frac{1}{2}.$$

Indeed, by domain monotonicity and (5.1), we have

$$0 \le \lambda_1^s(\Theta) \le \lim_{k\to\infty} \lambda_1^s(\widetilde{Q}_k) = 0.$$

6. Some consequences

We highlight in this section some consequences of our main result, by starting with a fractional analogue of property (1.2) seen in the Introduction.

Corollary 6.1. *Let $\Omega \subset \mathbb{R}^2$ be an open simply connected set. Then we have:*

- *for $1/2 < s < 1$*

$$\lambda_1^s(\Omega) > 0 \qquad \Longleftrightarrow \qquad r_\Omega < +\infty;$$

- *for* $0 < s \le 1/2$
$$\lambda_1^s(\Omega) > 0 \qquad \Longrightarrow \qquad r_\Omega < +\infty,$$
but
$$r_\Omega < +\infty \qquad \not\Longrightarrow \qquad \lambda_1^s(\Omega) > 0.$$

Proof. Let $0 < s < 1$ and assume that $\lambda_1^s(\Omega) > 0$. Let $r > 0$ be such that there exists $x_0 \in \Omega$ with $B_r(x_0) \subset \Omega$. By using the monotonicity of λ_1^s with respect to set inclusion, we get

$$\lambda_1^s(\Omega) \le \lambda_1^s(B_r(x_0)) = \frac{\lambda_1^s(B_1)}{r^{2\,s}}.$$

The previous estimate gives

$$r < \left(\frac{\lambda_1^s(B_1)}{\lambda_1^s(\Omega)}\right)^{\frac{1}{2\,s}}.$$

By taking the supremum over admissible r, we get $r_\Omega < +\infty$ by definition of inradius.

For the converse implication in the case $s > 1/2$, it is sufficient to apply Theorem 1.1. Finally, by taking Θ as in Remark 5.1, we get an open set with finite inradius, but vanishing λ_1^s for $0 < s \le 1/2$. □

Our main results permit to compare two different Sobolev spaces, built up of functions "vanishing at the boundary". More precisely, let us denote by $\mathcal{D}_0^{s,2}(\Omega)$ the *completion* of $C_0^\infty(\Omega)$ with respect to the norm

$$u \mapsto [u]_{W^{s,2}(\mathbb{R}^N)}, \qquad \text{for every } u \in C_0^\infty(\Omega).$$

Observe that this is indeed a norm on $C_0^\infty(\Omega)$. We refer to [11] for more details on this space. We also recall that by $\widetilde{W}_0^{s,2}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{s,2}(\mathbb{R}^N)$.

We have the following

Corollary 6.2. *Let $1/2 < s < 1$ and let $\Omega \subset \mathbb{R}^2$ be an open simply connected set, with finite inradius. Then*

$$\mathcal{D}_0^{s,2}(\Omega) = \widetilde{W}_0^{s,2}(\Omega).$$

On the contrary, for $0 < s \le 1/2$ and Θ the infinite complement comb of Remark 5.1, the two spaces

$$\mathcal{D}_0^{s,2}(\Theta) \qquad \text{and} \qquad \widetilde{W}_0^{s,2}(\Theta),$$

can not be identified with each other.

Proof. For $1/2 < s < 1$ and an open simply connected set $\Omega \subset \mathbb{R}^2$, by Theorem 1.1 the two norms

$$[u]_{W^{s,2}(\mathbb{R}^2)} \qquad \text{and} \qquad \|u\|_{W^{s,2}(\mathbb{R}^2)},$$

are equivalent on $C_0^\infty(\Omega)$. This proves the first point.

As for the second statement, it is sufficient to observe that $\widetilde{W}_0^{s,2}(\Theta)$ is always continuously embedded in $L^2(\Theta)$, by its very definition. On the other hand, for $0 < s \le 1/2$ such an embedding does not hold for $\mathcal{D}_0^{s,2}(\Theta)$, since $\lambda_1^s(\Theta) = 0$ by Remark 5.1. □

We now show how Theorem 1.1 implies some fractional versions of the classical *Cheeger's inequality*, a fundamental result in Spectral Geometry. At this aim, for an open set $\Omega \subset \mathbb{R}^N$ we recall the definition of *Cheeger constant*

$$h_1(\Omega) = \inf\left\{\frac{P(E)}{|E|} \,:\, E \subset \Omega \text{ bounded and measurable with } |E| > 0\right\},$$

and $s-$*Cheeger constant* (for $0 < s < 1$)

$$h_s(\Omega) = \inf\left\{\frac{P_s(E)}{|E|}\,:\,E\subset\Omega \text{ bounded and measurable with } |E|>0\right\},$$

see [9] for some properties of this constant. Here P stands for the *perimeter* of a set in the sense of De Giorgi, while P_s is the $s-$perimeter of a set, defined by

$$P_s(E) = [1_E]_{W^{s,1}(\mathbb{R}^N)} = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|1_E(x)-1_E(y)|}{|x-y|^{N+s}}\,dx\,dy,$$

for any measurable set $E\subset\mathbb{R}^N$. Then we have the following

Corollary 6.3 (Fractional Cheeger inequality)**.** *Let $1/2<s<1$ and let $\Omega\subset\mathbb{R}^2$ be an open simply connected set, with finite inradius. Then we have*

$$\lambda_1^s(\Omega)\ge \mathcal{C}_s\left(\frac{h_1(\Omega)}{2}\right)^{2\,s},$$

and

$$\lambda_1^s(\Omega)\ge \mathcal{C}_s\left(\frac{\pi}{P_s(B_1)}\,h_s(\Omega)\right)^2.$$

where $\mathcal{C}_s$ is the same constant as in Theorem 1.1.

Proof. Let $r<r_\Omega$, by definition of inradius there exists a disk $B_r(x_0)\subset\Omega$. By using this disk as a competitor for the minimization problem defining $h_1(\Omega)$, we get

$$h_1(\Omega)\le\frac{2\,\pi\,r}{\pi\,r^2}=\frac{2}{r}.$$

By taking the supremum over admissible r, we get

$$h_1(\Omega)\le\frac{2}{r_\Omega}.$$

By raising to the power $2\,s$ and using Theorem 1.1, we get the first inequality. The second one can be obtained in exactly the same way. □

Finally, we have the following result, which permits to compare $\lambda_1^s(\Omega)$ and $\lambda_1(\Omega)$, for simply connected sets in the plane. We refer to [12, Theorem 6.1] and [15, Theorem 4.5] for a similar result in general dimension $N\ge 2$, under stronger regularity assumptions on the sets.

Corollary 6.4 (Comparison of eigenvalues)**.** *Let $1/2<s<1$ and let $\Omega\subset\mathbb{R}^2$ be an open simply connected set, with finite inradius. Then we have*

$$\alpha_s\left(\lambda_1(\Omega)\right)^s\le\lambda_1^s(\Omega)\le\beta_s\left(\lambda_1(\Omega)\right)^s, \tag{6.1}$$

where α_s,β_s are two positive constants depending on s only, such that

$$\alpha_s\sim\left(s-\frac{1}{2}\right),\quad \textit{for } s\searrow\frac{1}{2},\qquad \textit{and}\qquad \alpha_s\sim\frac{1}{1-s},\quad \textit{for } s\nearrow 1,$$

$$\beta_s\sim\frac{1}{1-s},\quad \textit{for } s\nearrow 1.$$

Proof. The upper bound follows directly from the general result of [12, Theorem 6.1], see equation (6.1) there. From this reference, we can also extract a value for the constant β_s, which is given by

$$\beta_s = \frac{4^{1-s}}{s\,(1-s)}\,\pi.$$

For the lower bound, the proof is similar to that of Corollary 6.3, it is sufficient to join the estimate

$$\lambda_1(\Omega) \le \frac{\lambda_1(B_1)}{r_\Omega^2},$$

with Theorem 1.1. This gives the claimed estimate, with constant

$$\alpha_s = \frac{\mathcal{C}_s}{(\lambda_1(B_1))^s},$$

and $\mathcal{C}_s$ is the same as in (1.4). □

Remark 6.5. The lower bound in estimate (6.1) degenerates as s approaches $1/2$. This behaviour is optimal: indeed, observe that for the set Θ of Remark 5.1 we have

$$\lambda_1(\Theta) > 0 \qquad \text{and} \qquad \lambda_1^s(\Theta) = 0, \quad \text{for } 0 < s \le \frac{1}{2}.$$

The first fact follows from the classical Makai-Hayman inequality (1.1), for example. Thus the lower bound can not hold for this range of values.

Appendix A. A one-dimensional Poincaré inequality

In what follows, we recall the definition of the following norm on the one-dimensional torus $\mathbb{S}^1 = \mathbb{R}/(2\,\pi\,\mathbb{Z})$

$$|\alpha|_{\mathbb{S}^1} := \min_{k\in\mathbb{Z}} |\alpha + 2\,k\,\pi|, \qquad \text{for every } \alpha \in \mathbb{R}.$$

We observe that in particular for $\alpha \in [0, 2\,\pi]$ this quantity is given by

$$|\alpha|_{\mathbb{S}^1} = \left\{ \begin{array}{cc} \alpha, & \text{if } 0 \le \alpha \le \pi, \\ 2\,\pi - \alpha, & \text{if } \pi < \alpha \le 2\,\pi. \end{array} \right. \tag{A.1}$$

Lemma A.1. *We have*

$$\frac{2}{\pi}\,|\theta-\varphi|_{\mathbb{S}^1} \le |e^{i\,\theta} - e^{i\,\varphi}| \le |\theta-\varphi|_{\mathbb{S}^1}, \qquad \textit{for every } \theta, \varphi \in \mathbb{R}.$$

Moreover, both inequalities are sharp.

Proof. We first observe that we can write

$$\begin{aligned} |e^{i\,\theta} - e^{i\,\varphi}| &= |e^{i\,\varphi}|\,|e^{i\,(\theta-\varphi)} - 1| \\ &= |e^{i\,(\theta-\varphi)} - 1| \\ &= \sqrt{(1-\cos(\theta-\varphi))^2 + \sin^2(\theta-\varphi)} = 2\,\left|\sin\left(\frac{\theta-\varphi}{2}\right)\right|, \end{aligned} \tag{A.2}$$

thanks to standard trigonometric formulas. In order to conclude the proof, it is sufficient to prove that

$$\frac{2}{\pi}\,|\alpha|_{\mathbb{S}^1} \le 2\,\left|\sin\left(\frac{\alpha}{2}\right)\right| \le |\alpha|_{\mathbb{S}^1}, \qquad \text{for every } \alpha \in \mathbb{R}. \tag{A.3}$$

It is easily seen that both functions

$$\alpha \mapsto |\alpha|_{\mathbb{S}^1} \qquad \text{and} \qquad \alpha \mapsto \left|\sin\left(\frac{\alpha}{2}\right)\right|,$$

are $2\,\pi-$periodic, thus is it sufficient to prove (A.3) for $\alpha \in [0, 2\,\pi]$. We thus seek for the maximum and the minimum on $[0, 2\,\pi]$ of the function

$$\alpha \mapsto 2\,\frac{|\sin(\alpha/2)|}{|\alpha|_{\mathbb{S}^1}},$$

extended by continuity to the whole interval. By keeping in mind (A.1), on $[0, 2\,\pi]$ this function can be rewritten as

$$\alpha \mapsto \begin{cases} 2\,\dfrac{\sin(\alpha/2)}{\alpha}, & \text{if } 0 \le \alpha \le \pi, \\[2ex] 2\,\dfrac{\sin(\alpha/2)}{2\,\pi - \alpha}, & \text{if } \pi \le \alpha \le 2\,\pi, \end{cases} = \begin{cases} \dfrac{\sin(\alpha/2)}{\alpha/2}, & \text{if } 0 \le \alpha \le \pi, \\[2ex] \dfrac{\sin(\pi - \alpha/2)}{\pi - \alpha/2}, & \text{if } \pi \le \alpha \le 2\,\pi. \end{cases}$$

By recalling that the *sinc function* $t \mapsto (\sin t)/t$ is monotone decreasing on the interval $[0, \pi/2]$, in light of the above discussion we now easily obtain

$$\frac{2}{\pi} \le 2\,\frac{|\sin(\alpha/2)|}{|\alpha|_{\mathbb{S}^1}} \le 1.$$

This gives (A.3), thus concluding the proof. □

The main result of this appendix is the following one-dimensional Poincaré inequality, for periodic functions vanishing at a point. The result is probably well-known, but as always we want to pay particular attention to the dependence of the constant on the parameter s. For $T > 0$, we define the one-dimensional torus $\mathbb{S}^1_T = \mathbb{R}/(T\,\mathbb{Z})$, endowed with the norm

$$|\theta - \varphi|_{\mathbb{S}^1_T} = \min_{k \in \mathbb{Z}} |\theta - \varphi + k\,T|, \qquad \text{for } \theta, \varphi \in \mathbb{R}.$$

Proposition A.2. *Let $1/2 < s < 1$ and $T > 0$. Let $\theta_0 \in [0, T]$, there exists a constant $\mu_s > 0$ depending on s only such that for every Lipschitz function $w : \mathbb{R} \to \mathbb{R}$ which is $T-$periodic and vanishing at θ_0, we have*

$$\mu_s \left(\frac{2\,\pi}{T}\right)^{2\,s} \int_0^T |w(\theta)|^2\,d\theta \le \iint_{[0,T]\times[0,T]} \frac{|w(\theta) - w(\varphi)|^2}{|\theta - \varphi|^{1+2\,s}_{\mathbb{S}^1_T}}\,d\theta\,d\varphi, \tag{A.4}$$

Moreover, the constant μ_s has the following asymptotic behaviours

$$\mu_s \sim \left(s - \frac{1}{2}\right), \quad \textit{for } s \searrow \frac{1}{2} \qquad \textit{and} \qquad \mu_s \sim \frac{1}{1 - s}, \quad \textit{for } s \nearrow 1.$$

Proof. Without loss of generality, we can assume that $\theta_0 = 0$ and $T = 2\,\pi$. Thus, in this case we have $|\cdot|_{\mathbb{S}^1_{2\pi}} = |\cdot|_{\mathbb{S}^1}$, with the notation of Lemma A.1.

Thanks to the periodicity of w, we can expand it in Fourier series, i.e. we can write

$$w(\theta) = \sum_{n \in \mathbb{Z}} \widehat{w}(n)\,e^{in\theta}, \qquad \text{where} \qquad \widehat{w}(n) = \frac{1}{2\,\pi} \int_0^{2\,\pi} w(\theta)\,e^{-i\,n\,\theta}\,d\theta.$$

The series is uniformly converging, thanks to the assumption on w. We will achieve the claimed result by joining the following two estimates

$$\iint_{[0,2\,\pi]\times[0,2\,\pi]} \frac{|w(\theta)-w(\varphi)|^2}{|\theta-\varphi|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,d\varphi \ge C_{1,s}\sum_{n\in\mathbb{Z}}|n|^{2\,s}\,|\widehat{w}(n)|^2, \tag{A.5}$$

and

$$\int_0^{2\,\pi}|w(\theta)|^2\,d\theta \le C_{2,s}\sum_{n\in\mathbb{Z}}|n|^{2\,s}\,|\widehat{w}(n)|^2, \tag{A.6}$$

that we prove separately. This would give (A.4), with constant $\mu_s = C_{1,s}/C_{2,s}$. In the last part of the proof, we will then prove that such a constant has the claimed asymptotics.

Proof of (A.5). We proceed similarly as in the proof of [18, Proposition 3.4], with suitable adaptations. The latter deals with $W^{s,2}$ functions on $\mathbb{R}$ and their Fourier transform.

First of all, we rewrite the Gagliardo-Slobodeckiĭ seminorm as follows: let us apply the change of variable $h=\varphi-\theta$, so to get

$$\begin{aligned}\int_0^{2\,\pi}\int_0^{2\,\pi}\frac{|w(\varphi)-w(\theta)|^2}{|\varphi-\theta|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,d\varphi &= \int_0^{2\,\pi}\int_{-\theta}^{2\,\pi-\theta}\frac{|w(\theta+h)-w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,dh\\ &= \int_0^{2\,\pi}\int_{-\theta}^{0}\frac{|w(\theta+h)-w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,dh\\ &+ \int_0^{2\,\pi}\int_{0}^{2\,\pi}\frac{|w(\theta+h)-w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,dh\\ &- \int_0^{2\,\pi}\int_{2\,\pi-\theta}^{2\,\pi}\frac{|w(\theta+h)-w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,dh.\end{aligned}$$

On the third integral, we can use that the integrand is $2\,\pi-$periodic, thus we get

$$\begin{aligned}\int_0^{2\,\pi}\int_{2\,\pi-\theta}^{2\,\pi}\frac{|w(\theta+h)-w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,dh &= \int_0^{2\,\pi}\int_{2\,\pi-\theta}^{2\,\pi}\frac{|w(\theta+h-2\,\pi)-w(\theta)|^2}{|h-2\,\pi|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,dh\\ &= \int_0^{2\,\pi}\int_{-\theta}^{0}\frac{|w(\theta+\eta)-w(\theta)|^2}{|\eta|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,d\eta.\end{aligned}$$

This finally permits to infer that

$$\int_0^{2\,\pi}\int_0^{2\,\pi}\frac{|w(\varphi)-w(\theta)|^2}{|\varphi-\theta|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,d\varphi = \int_0^{2\,\pi}\int_0^{2\,\pi}\frac{|w(\theta+h)-w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,dh.$$

By recalling (A.1), we can conclude that

$$\begin{aligned}\int_0^{2\,\pi}\int_0^{2\,\pi}\frac{|w(\theta+h)-w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2\,s}}\,d\theta\,dh &= \int_0^{\pi}\frac{1}{h^{1+2\,s}}\left(\int_0^{2\,\pi}|w(\theta+h)-w(\theta)|^2\,d\theta\right)dh\\ &+ \int_\pi^{2\,\pi}\frac{1}{(2\,\pi-h)^{1+2\,s}}\left(\int_0^{2\,\pi}|w(\theta+h)-w(\theta)|^2\,d\theta\right)dh.\end{aligned} \tag{A.7}$$

Now, for every h we denote by $w_h(\theta)$ the translation $w_h(\theta)=w(\theta+h)$. Thanks to the well-known properties of the Fourier coefficients, we have

$$\widehat{w}_h(n) = e^{i\,h\,n}\,\widehat{w}(n), \qquad \text{for every } n\in\mathbb{Z}. \tag{A.8}$$

By using Plancherel's identity in (A.7) and then applying (A.8), we finally obtain

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi}\frac{|w(\theta+h)-w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}}\,d\theta\,dh=\\ =2\pi\sum_{n\in\mathbb{Z}\setminus\{0\}}\left(\int_0^{\pi}\frac{|e^{i\,h\,n}-1|^2}{h^{1+2s}}\,dh+\int_{\pi}^{2\pi}\frac{|e^{i\,h\,n}-1|^2}{(2\pi-h)^{1+2s}}\,dh\right)|\widehat{w}(n)|^2.\end{aligned}\tag{A.9}$$

By recalling the identities (A.2), we have

$$|e^{i\,h\,n}-1|^2=2\,(1-\cos(h\,n)),$$

and applying the change of variable $\tau=h\,n$ with $n\in\mathbb{Z}\setminus\{0\}$, we can rewrite the first integral on the right-hand side of (A.9) as

$$\begin{aligned}\int_0^{\pi}\frac{|e^{i\,h\,n}-1|^2}{h^{1+2s}}\,dh&=2\int_0^{\pi}\frac{1-\cos(h\,n)}{h^{1+2s}}\,dh\\ &=2\int_0^{\pi\,n}\frac{1-\cos\tau}{\left(\dfrac{\tau}{n}\right)^{1+2s}}\,\frac{d\tau}{n}\geq 2\,|n|^{2s}\int_0^{\pi}\frac{1-\cos\tau}{\tau^{2s}}\,\frac{d\tau}{\tau}.\end{aligned}$$

For the second integral, it is sufficient to observe that by periodicity

$$\begin{aligned}\int_{\pi}^{2\pi}\frac{|e^{i\,h\,n}-1|^2}{(2\pi-h)^{1+2s}}\,dh&=2\int_{\pi}^{2\pi}\frac{1-\cos(h\,n)}{(2\pi-h)^{1+2s}}\,dh\\ &=2\int_{\pi}^{2\pi}\frac{1-\cos(2\pi\,n-h\,n)}{(2\pi-h)^{1+2s}}\,dh\\ &=2\int_0^{\pi}\frac{1-\cos(h\,n)}{h^{1+2s}}\,dh\geq 2\,|n|^{2s}\int_0^{\pi}\frac{1-\cos\tau}{\tau^{2s}}\,\frac{d\tau}{\tau}.\end{aligned}$$

Thus, from (A.9) we get in particular

$$\int_0^{2\pi}\int_0^{2\pi}\frac{|w(\theta+h)-w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}}\,d\theta\,dh\geq 8\pi\left(\int_0^{\pi}\frac{1-\cos\tau}{\tau^{2s}}\,\frac{d\tau}{\tau}\right)\sum_{n\in\mathbb{Z}}|n|^{2s}\,|\widehat{w}(n)|^2.$$

This finally proves (A.5), with constant

$$C_{1,s}=8\pi\int_0^{\pi}\frac{1-\cos\tau}{\tau^{2s}}\,\frac{d\tau}{\tau}.$$

Proof of (A.6): from Plancherel's identity, we know that

$$\frac{1}{2\pi}\int_0^{2\pi}|w(\theta)|^2\,d\theta=\sum_{n\in\mathbb{Z}}|\widehat{w}(n)|^2.\tag{A.10}$$

By using the Fourier expansion for w and the assumption $w(0)=w(2\pi)=0$, we can infer that

$$0=w(0)=\sum_{n\in\mathbb{Z}}\widehat{w}(n).$$

This in turn implies that

$$|\widehat{w}(0)|=\left|\sum_{n\in\mathbb{Z}\setminus\{0\}}\widehat{w}(n)\right|\leq\sum_{n\in\mathbb{Z}\setminus\{0\}}|\widehat{w}(n)|,$$

and so we can obtain

$$\sum_{n\in\mathbb{Z}} |\widehat{w}(n)|^2 = \sum_{n\in\mathbb{Z}\setminus\{0\}} |\widehat{w}(n)|^2 + |\widehat{w}(0)|^2 \le \sum_{n\in\mathbb{Z}\setminus\{0\}} |\widehat{w}(n)|^2 + \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} |\widehat{w}(n)|\right)^2. \tag{A.11}$$

We now estimate the last term in (A.11) by using Hölder's inequality

$$\sum_{n\in\mathbb{Z}} |a_n|\,|b_n| \le \left(\sum_{n\in\mathbb{Z}} |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} |b_n|^2\right)^{\frac{1}{2}},$$

with the choices $|a_n| = 1/|n|^s$ and $|b_n| = |\widehat{w}(n)|\,|n|^s$. This yields

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\widehat{w}(n)|^2 &\le \sum_{n\in\mathbb{Z}\setminus\{0\}} |\widehat{w}(n)|^2 + \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{|n|^{2\,s}}\right) \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{2\,s}\,|\widehat{w}(n)|^2\right)\\ &\le C\left(\sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{2s}\,|\widehat{w}(n)|^2\right),\end{aligned}$$

where we set

$$C = 1 + 2\sum_{n=1}^{\infty} \frac{1}{n^{2\,s}}.$$

Observe that this is a finite quantity, thanks to the crucial assumption $s > 1/2$. By using this estimate in (A.10), we then obtain the claimed inequality (A.6), with constant

$$C_{2,s} = 2\,\pi\left(1 + 2\sum_{n=1}^{\infty} \frac{1}{n^{2\,s}}\right).$$

Asymptotic behaviour of the constant. As we said, from the above discussion we get the inequality (A.4), with $\mu_s = C_{1,s}/C_{2,s}$. It is easily seen that

$$\lim_{s\to\left(\frac{1}{2}\right)^+} C_{1,s} = 8\,\pi \int_0^{\pi} \frac{1-\cos\tau}{\tau}\,\frac{d\tau}{\tau} < +\infty,$$

while

$$\lim_{s\to\left(\frac{1}{2}\right)^+} (2\,s-1)\,C_{2,s} = 2\,\pi \lim_{s\to\left(\frac{1}{2}\right)^+} (2\,s-1)\left(1 + 2\sum_{n=1}^{\infty}\frac{1}{n^{2\,s}}\right) = 4\,\pi,$$

by using the fact that the Riemann zeta function has a simple pole with residue 1 at $z = 1$ (see [22, Section 13.2.6]). This proves that μ_s has the claimed asymptotic behaviour, as s goes to 1/2.

As for the behaviour at $s \nearrow 1$, we observe that

$$\lim_{s\to 1^-} C_{2,s} = 2\,\pi\left(1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2}\right) = 2\,\pi\left(1 + \frac{\pi^2}{3}\right),$$

while

$$\begin{aligned}\lim_{s\to 1^-}(1-s)\,C_{1,s} &= 8\,\pi\,\lim_{s\to 1^-}(1-s)\int_0^\pi \frac{1-\cos\tau}{\tau^{2\,s}}\,\frac{d\tau}{\tau}\\ &= 4\,\pi\,\lim_{s\to 1^-}(1-s)\int_0^\pi \tau^{2-2\,s}\,\frac{d\tau}{\tau}\\ &\quad -4\,\pi\,\lim_{s\to 1^-}(1-s)\int_0^\pi \frac{\int_0^\tau(\tau-\ell)^2\,\sin\ell\,d\ell}{\tau^{2\,s}}\,\frac{d\tau}{\tau}=2\,\pi,\end{aligned}$$

where we used the third order Taylor expansion

$$f(\tau)=f(0)+f'(0)\,\tau+\frac{1}{2}\,f''(0)\,\tau^2+\frac{1}{2}\int_0^\tau f'''(\ell)\,(\tau-\ell)^2\,d\ell,$$

for the cosine function. This eventually leads to the conclusion of the proof. □

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