Hybrid multi-population traffic flow model: Optimal control for a mean-field limit

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***Abstract*— Heterogeneous and multi-lane traffic flow modeling is fundamental to understand the dynamics and control of complex traffic systems. In this article, we consider three populations of vehicles: two classes of human-driven vehicles (cars and trucks) and autonomous vehicles. The latter is distinguished by the presence of control in the acceleration. We model the multi-lane traffic by hybrid systems because of its hybrid nature: the continuous dynamics on each lane and the discrete events due to lane-changing maneuvers and study the optimal control problem associated with the hybrid systems. In particular, we investigate controlled hybrid systems from both a microscopic and macroscopic point of view. Furthermore, using Γ-convergence, we prove the well-posedness of an optimal control problem for a mean-field limit of a finite-dimensional hybrid system modeling heterogeneous multi-lane traffic with controlled autonomous vehicles.**

***Index Terms*— Heterogeneous traffic, car-following models, autonomous vehicles, trucks, multi-lane traffic, hybrid system, mean-field limit, optimal control, Γ-convergence**

I. Introduction

A delicate problem in traffic flow modeling is how to represent multi-lane traffic. The difficulty lies in its hybrid nature since it presents continuous dynamics on each lane and discrete events for lane-change. Moreover an accurate description requires taking into account the multiplicity of vehicle types which constitute the traffic. A step in this direction has been done in [12] where the authors introduced mean-field equations coupled with ODEs to capture the mixed nature of this type of traffic in presence of human-driven vehicles and a limited number of autonomous vehicles. The system was formally derived from a microscopic model based on a combined dynamics of Bando ([2]) and Follow-the-Leader (FtL) ([26], [25]) type together with some lane changing condition rules for each vehicle inspired by [19]. More recently this model has been generalized to the case of two populations of human-driven vehicles (cars and trucks) and autonomous vehicles ([6]).

In the present work, we aim to develop and study qualitative properties of models for traffic which are motivated by the idea of considering, simultaneously, two important aspects: lane-change maneuvers and heterogeneous composition of the flow. The former is one of the most common maneuvers, source of interaction and risk ([16]) among vehicles on motorways. Currently, multi-lane traffic is modeled either by two-dimensional models ([15], [31]), in which lane changing rules are not explicitly prescribed, or by treating lanes as discrete entities ([17], [28]). The latter aspect, instead, is becoming more and more important with the increasingly interest in automated-driven vehicles and their effects within the vehicular traffic flow ([18]). Experiments [8], [29] and mathematical models [9], [24] have shown that a small number of controlled vehicles can stabilize traffic flow damping unstable phenomena.

We consider an optimal control problem for a hybrid system ([7], [14]) which models the traffic on multi-lane with cars, trucks and autonomous vehicles, extending in this way the analysis already started in [13]. Specifically, the control is introduced in the acceleration of autonomous vehicles with the idea that they can influence the general dynamics of the other two population. The optimal control problem consists in the minimization of a functional which depends on each location, velocity, control and switching times. It can represent the distance from a target desirable traffic profile or fuel consumption or CO2 emissions ([4], [30], [33]).

In general the optimal control of hybrid systems has attracted enormous attention in recent years, we can mention for instance [3], [5], [11], [22], [27] and [32]. The paper is organized as follows. In Section II we present a microcopic model for heterogeneous traffic involving cars, trucks and autonomous vehicles on L lanes, specify the conditions in terms of acceleration and probability measure under which a vehicle can perform a lane change and introduce our model assumption on ”cool-down” time. In Section III we introduce the notion of controlled finite-dimensional hybrid system and the associated optimal control problem. The mean-field limit is described in Section IV together with the correspondent optimal control problem given by the Γ-limit of the finite dimensional case. Section V contains the extension of the hybrid system to the case of M + 1 vehicle populations and finally the last section summarizes the contributions of this work, the applicative relevance of a model with cars and trucks and future issues that need to be explored.

II. Preliminary

In this section, we first introduce the convolutional form of a first order car-following model, Bando-Follow-the-Leader, for heterogeneous traffic containing cars, trucks and autonomous vehicles. Then we state the lane-changing rules for multi-lane traffic in both finite-dimensional and infinite-dimensional cases. We point out that the abundance of parameters considered comes from the necessity to make the model sensitive to vehicle dimensions and to the interactions of vehicles of different types. In the end, we emphasize our lane-changing model assumption, the ”cool-down” time.

A. Car-following models for heterogeneous traffic

For convenience, we introduce now the notation used throughout the paper. Let $T > 0$ be fixed, $\mathcal{I}$ be the set of indices for vehicles on the open stretch road. Assume that the vehicles can move on L lanes and $\mathcal{K}$ is the set of lane labels $\{1, \dots, L\}$. Denote with $\mathcal{I}_P$,$\mathcal{I}_Q$, $\mathcal{I}_S$ the set of indices for cars, autonomous vehicles and trucks respectively, and with $\mathcal{I}_P^k$, $\mathcal{I}_Q^k$, $\mathcal{I}_S^k$ the correspondent set of indices on lane $k \in \mathcal{K}$. Let $(x_i(t), v_i(t))$ be the position-velocity for the vehicle i at time $t \in [0, T]$, and $P_k(t), Q_k(t), S_k(t)$ be the number of cars, trucks and autonomous vehicles at time $t \in [0, T]$

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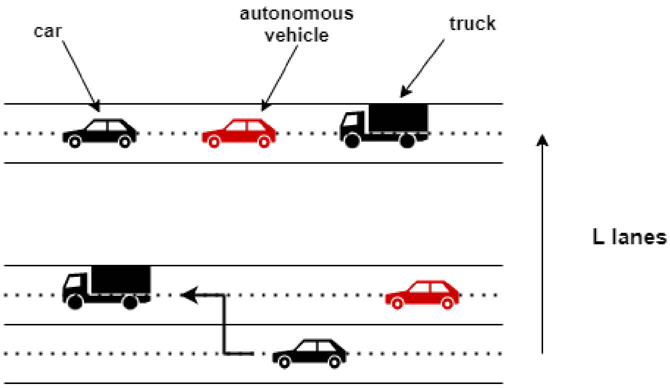

Fig. 1. A schematic view of our setting: we consider autonomous vehicles (red) cars and trucks (black) on L lanes. A vehicle can perform lane-change with a probability given by (7) which depends on the Incentive and Safety conditions (5)-(6).

on lane $k \in \mathcal{K}$, respectively. Moreover we consider $\mathcal{P}$ space of probability measures and $\mathcal{M}^+$ space of positive Borel measures.

The first order car-following model, Bando-Follow-the-Leader (Bando-FtL), which was first introduced in [29], assumes that the acceleration of the vehicle $i \in \mathcal{I}$ depends on its headway h_i, the difference between its current velocity v_i and its leading vehicle's velocity v_{L_i} (L_i is the label of the leading vehicle of vehicle i), and the difference between its current velocity and its optimal velocity V_i. In this paper, we consider the convolutional form of the Bando-FtL model which allows a vehicle to modify its acceleration according the positions and velocities of its nearby front vehicles instead of only its leading vehicle's. Consider the atomic measures supported on absolutely continuous trajectories $t \in [0,T] \to (x_i(t), v_i(t)) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$:

$$\mu_P^k(t) = \frac{1}{P_k(t)} \sum_{i \in \mathcal{I}_P^k(t)} \delta_{(x_i(t), v_i(t))}, \tag{1}$$

$$\mu_Q^k(t) = \frac{1}{Q_k(t)} \sum_{i \in \mathcal{I}_Q^k(t)} \delta_{(x_i(t), v_i(t))}, \tag{2}$$

$$\mu_S^k(t) = \frac{1}{S_k(t)} \sum_{i \in \mathcal{I}_S^k(t)} \delta_{(x_i(t), v_i(t))}. \tag{3}$$

Having in mind the four different combinations Car-Car, Car-Truck, Truck-Car, Truck-Truck, we define the following convolution kernels

$$H_1^{cf} : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R} \quad \text{with } cf \in \{cc, tc, ct, tt\}$$

$$(x, v) \mapsto \alpha_{cf} h_{cf}(x)(V_{cf}(-x) - v)$$

where α_{cf} are positive parameters denoting the speed of response, V_{cf} is the optimal velocity function, and $h_{cf} : \mathbb{R} \mapsto \mathbb{R}_{\geq 0}$ is a smooth function with compact support $[-\varepsilon_{cf}, 0]$ measuring the interaction of two vehicles depending on their distance and types, where $\varepsilon_{cf} > 0$. Then, for instance, for a car $i \in \mathcal{I}_P^k$ on lane $k \in \mathcal{K}$, the Bando-term of the Bando-FtL model can be written as

$$\left(H_1^{cc} *_1 (\mu_P^k + \mu_Q^k) + H_1^{tc} *_1 \mu_S^k\right)(x_i, v_i), i \in \mathcal{I}_P^k.$$

Similarly, we introduce the following convolution kernels for the Follow-the-Leader term of the Bando-FtL term,

$$H_2^{cf} : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R} \quad \text{with } cf \in \{cc, tc, ct, tt\}$$

$$(x, v) \mapsto \beta_{cf} h_{cf}(x) \frac{-v}{x^2}$$

where β_{cf} is positive. The vehicles' dynamic is described by the following system

$$\dot{x}_i = v_i \quad i \in \mathcal{I}$$

$$\dot{v}_i = \begin{cases} \left(H_1^{cc} *_1 (\mu_P^k + \mu_Q^k) + H_1^{tc} *_1 \mu_S^k\right)(x_i, v_i) \\ + \left(H_2^{cc} * (\mu_P^k + \mu_Q^k) + H_2^{tc} * \mu_S^k\right)(x_i, v_i) & i \in \mathcal{I}_P^k \\ \\ \left(H_1^{cc} *_1 (\mu_P^k + \mu_Q^k) + H_1^{tc} *_1 \mu_S^k\right)(x_i, v_i) \\ + \left(H_2^{cc} * (\mu_P^k + \mu_Q^k) + H_2^{tc} * \mu_S^k\right)(x_i, v_i) + u_i & i \in \mathcal{I}_Q^k \\ \\ \left(H_1^{ct} *_1 (\mu_P^k + \mu_Q^k) + H_1^{tt} *_1 \mu_S^k\right)(x_i, v_i) \\ + \left(H_2^{ct} * (\mu_P^k + \mu_Q^k) + H_2^{tt} * \mu_S^k\right)(x_i, v_i) & i \in \mathcal{I}_S^k \end{cases} \tag{4}$$

B. Lane-changing rules for multi-lane traffic

Now we will introduce the lane-changing rules used in this article for the heterogeneus multi-lane traffic. We consider both the finite-dimensional case when we have finitely many cars, trucks and autonomous vehicles and the infinite-dimensional case when we still consider finitely many autonomous vehicles but infinitely many cars and trucks.

The finite-dimensional case

We consider P cars, Q autonomous vehicles and S trucks on the road. Let a_i^k be the acceleration of vehicle $i \in \mathcal{I}$ on lane $k \in \mathcal{K}$ and $\bar{a}_i^{k'}$ the expected acceleration of vehicle i on its target lane $k' \in \{k-1, k+1\} \cap \mathcal{K}$. Let $i_F^{k'}$ be the index of the following vehicle of vehicle i on its target lane k' if vehicle i performs lane-changing from lane k to lane k'. Let $\Delta^{cf} > 0$, $cf \in \{cc, tc, ct, tt\}$.

The "incentive" and "safety" conditions for lane-changing are defined as follows:

$$\text{Incentive: } \bar{a}_i^{k'} \geq \begin{cases} a_i^k + \Delta^{cc} \text{ if } i, i_F^{k'} \in \mathcal{I}_P \cup \mathcal{I}_Q, \\ a_i^k + \Delta^{tc} \text{ if } i \in \mathcal{I}_P \cup \mathcal{I}_Q, i_F^{k'} \in \mathcal{I}_S, \\ a_i^k + \Delta^{ct} \text{ if } i \in \mathcal{I}_S, i_F^{k'} \in \mathcal{I}_P \cup \mathcal{I}_Q, \\ a_i^k + \Delta^{tt} \text{ if } i, i_F^{k'} \in \mathcal{I}_S; \end{cases} \tag{5}$$

$$\text{Safety: } \bar{a}_i^{k'} \geq \begin{cases} -\Delta^c \text{ and } \bar{a}_{i_F^{k'}}^{k'} \geq -\Delta^c \text{ if } i, i_F^{k'} \in \mathcal{I}_P \cup \mathcal{I}_Q, \\ -\Delta^c \text{ and } \bar{a}_{i_F^{k'}}^{k'} \geq -\Delta^t \text{ if } i \in \mathcal{I}_P \cup \mathcal{I}_Q, \in \mathcal{I}_S, \\ -\Delta^t \text{ and } \bar{a}_{i_F^{k'}}^{k'} \geq -\Delta^c \text{ if } i \in \mathcal{I}_S, i_F^{k'} \in \mathcal{I}_P \cup \mathcal{I}_Q, \\ -\Delta^t \text{ and } \bar{a}_{i_F^{k'}}^{k'} \geq -\Delta^t \text{ if } i, i_F^{k'} \in \mathcal{I}_S. \end{cases} \tag{6}$$

Let $\Delta = \min\{\Delta^{cc}, \Delta^{ct}, \Delta^{tc}, \Delta^{tt}, \Delta^c, \Delta^t\}$. It can be proved that there exists $M \in \mathbb{R}_{\geq 0}$, such that for every $t \in [0,T]$ and $i \in \mathcal{I}$, $|a_i(t)| < M$, i.e. the acceleration is bounded. Define three probability functions as follows

$$p_j : (\mathbb{R}_{\geq 0})^5 \to [0,1]; \tag{7}$$
$$p_j(b_1, b_2, b_3, b_4, b_5) = \frac{1}{C_j}\left(1 - e^{-\gamma_j b_1 b_2 b_3 b_4 b_5}\right),$$

where $\gamma_j > 0$, C_j are renormalization constants given by

$$C_j = \max_{[0,2M-\Delta]^5} \left(1 - e^{-\gamma_j b_1 b_2 b_3 b_4 b_5}\right) = 1 - e^{-\gamma_j (2M-\Delta)^5},$$

and $j = 1, 2, 3$. The third probability function p_3 will be used in the infinite-dimensional case.

A car or an autonomous vehicle $i \in \mathcal{I}_P \cup \mathcal{I}_Q$ will perform lane-change from lane $k \in \mathcal{K}$ to lane $k' \in \{k-1, k+1\} \cap \mathcal{K}$ under both the incentive and safety conditions with a probability given by

$$p_1\Big([\bar{a}_i^{k'} - a_i^k - \Delta^{cc}]_+, [\bar{a}_i^{k'} - a_i^k - \Delta^{tc}]_+, [\bar{a}_i^{k'} + \Delta^c]_+, [\bar{a}_{i_F^{k'}}^{k'} + \Delta^c]_+, [\bar{a}_{i_F^{k'}}^{k'} + \Delta^t]_+\Big).$$

The probability of a truck $i \in \mathcal{I}_S$ performing lane-change is

$$p_2\Big([\bar{a}_i^{k'} - a_i^k - \Delta^{ct}]_+, [\bar{a}_i^{k'} - a_i^k - \Delta^{tt}]_+, [\bar{a}_i^{k'} + \Delta^t]_+, [\bar{a}_{i_F^{k'}}^{k'} + \Delta^c]_+, [\bar{a}_{i_F^{k'}}^{k'} + \Delta^t]_+\Big).$$

The infinite-dimensional case

Now we still consider Q autonomous vehicles but infinitely many cars and trucks. In this case we investigate the lane-changing behavior of all vehicles by looking into the following average acceleration of cars A_P^k, and the average acceleration of trucks A_S^k, on lane $k \in \mathcal{K}$:

$$A_P^k = H_1^{cc} *_1 (\mu_P^k + \mu_Q^k) + H_1^{tc} *_1 \mu_S^k + H_2^{cc} * (\mu_P^k + \mu_Q^k) + H_2^{tc} * \mu_S^k,$$
$$A_S^k = H_1^{ct} *_1 (\mu_P^k + \mu_Q^k) + H_1^{tt} *_1 \mu_S^k + H_2^{ct} * (\mu_P^k + \mu_Q^k) + H_2^{tt} * \mu_S^k.$$

The probability of cars changing from lane $k \in \mathcal{K}$ to lane $k' \in \{k-1, k+1\} \cap \mathcal{K}$ lane is

$$p_1\Big([A_P^{k'} - A_P^k - \Delta^{cc}]_+, [A_S^{k'} - A_P^k - \Delta^{tc}]_+, [A_P^{k'} + \Delta^c]_+, [A_S^{k'} + \Delta^t]_+, [A_S^{k'} + \Delta^t]_+\Big),$$

the probability of trucks performing lane-change from lane $k \in \mathcal{K}$ to lane $k' \in \{k-1, k+1\} \cap \mathcal{K}$ is

$$p_2\Big([A_P^{k'} - A_S^k - \Delta^{ct}]_+, [A_S^{k'} - A_S^k - \Delta^{tt}]_+, [A_P^{k'} + \Delta^c]_+, [A_S^{k'} + \Delta^t]_+, [A_S^{k'} + \Delta^t]_+\Big),$$

and finally the probability of an autonomous vehicle $i \in \mathcal{I}_Q$ performing lane-change from lane $k \in \mathcal{K}$ to lane $k' \in \{k-1, k+1\} \cap \mathcal{K}$ is

$$p_3\Big([\bar{a}_i^{k'} - A_P^k - \Delta^{cc}]_+, [\bar{a}_i^{k'} - A_S^k - \Delta^{tc}]_+, [\bar{a}_i^{k'} + \Delta^c]_+, [A_S^{k'} + \Delta^t]_+, [A_P^{k'} + \Delta^t]_+\Big),$$

where the probability functions p_j, $j = 1, 2, 3$ are defined in eq. (7).

C. Cool-down time

Now we introduce the model assumption, "cool-down" time, which is critical to describe the frequencies of the vehicles' lane-changing behavior and to prove the well-posedness of our heterogeneus multi-lane traffic model.

By empirical observations, the lane-changing frequency of vehicles on the highway is low. For instance, a study analyzing a two dimensional dataset recorded on a German highway shows that only 15% of the vehicles performed lane-change while traveling the recorded road segment. For this reason, the chance of two vehicles performing lane-change at exactly the same time is even lower. Therefore it is reasonable to assume that there are not two vehicles changing lane at the same time. To achieve this, we associate each vehicle $i \in \mathcal{I}$ a timer τ_i and assume that the initial timers for two different vehicles are different. We also introduce the "cool-down" time $\bar{\tau} = \frac{T}{N_\tau}$, where $N_\tau \in \mathbb{N}_{\geq 0}$ is large and assume that vehicle $i \in \mathcal{I}$ checks the lane-changing conditions only when its timer reaches the cool-down time, τ_1. In addition, we reset the vehicle's timer to 0 once its timer reaches the cool-down time $\bar{\tau}$. Specifically, for each vehicle $i \in \mathcal{I}$, its timer τ_i satisfies the following

$$\dot{\tau}_i(t) = 1,\ \tau_i(0) = \tau_{i,0},\ t \in [0, \bar{\tau})$$

where $\tau_{i_1,0} \neq \tau_{i_2,0}$ if $i_1 \neq i_2 \in \mathcal{I}$. Note that one can also model large lane-changing frequencies by choosing small cool-down time $\bar{\tau}$.

In the case of finitely many vehicles, the presence of the cool-down time, $\bar{\tau}$ allows us to consider a small time interval $[0, t_1]$ when there is no vehicle changing lane. Similarly, in the case of infinitely many cars and trucks but finitely many autonomous vehicles, due to the definition of the cool-down time, there is a small time interval $[0, t_2]$ when there is no autonomous vehicle changing lane. In particular, $t_1 = \min_{i \in \mathcal{I}}\{\bar{\tau} - \tau_{i,0}\}$ and $t_2 = \min_{i \in \mathcal{I}_Q}\{\bar{\tau} - \tau_{i,0}\}$.

III. The optimal control problem on a finite-dimensional hybrid system

In this subsection, we again consider the multi-lane and multi-population traffic with P cars, S trucks and Q autonomous vehicles. The continuous dynamics of the finitely many vehicles without lane-change and the discrete events generated by the vehicles' lane-changing behaviors lead us to consider a finite-dimensional hybrid system Σ_1. We distinguish the autonomous vehicles from the others by adding control terms to their accelerations.

Let $X = \mathbb{R} \times \mathbb{R}_{\geq 0} \times [0, \bar{\tau})$ and $\mathcal{L} = \Big\{\ell = (\ell_i)_{i \in \mathcal{I}} \in \mathcal{K}^{P+Q+S}\Big\}$ be the set of symbols that represent all possible lane labels of all vehicles including cars, trucks and autonomous vehicles. Additionally, before giving the definition of the finite-dimensional hybrid system Σ_1, we define the following two sets: the set A_ℓ containing the position-velocity-timer vectors of all vehicles among which there are at least two vehicles occupying the same lane and position at certain time and the set $LC(\Sigma_1)$ representing the lane-changing mechanism of the finitely many vehicles:

$$A_\ell = \big\{(x_i, v_i, \tau_i)_{i \in \mathcal{I}} \in X \colon \exists t \in [0, T], i_1, i_2 \in \mathcal{I}, \quad s.t., x_{i_1}(t) = x_{i_2}(t) \wedge \ell_{i_1}(t) = \ell_{i_2}(t), \text{ with } \ell_{i_1}, \ell_{i_2} \in \mathcal{K}\big\}, \tag{8}$$

$$LC(\Sigma_1) = \Big\{(\ell, (x_i, v_i, \tau_i), \ell', (x_i', v_i', \tau_i'))_{i \in \mathcal{I}} \in (\mathcal{L} \times X)^2 \colon \exists i_0 \in \mathcal{I}, \exists t_0 \in [0, \bar{\tau}), s.t., j \neq i_0, (\ell_j(t_0), x_j(t_0), v_j(t_0), \tau_j(t_0)) = (\ell_j'(t_0), x_j'(t_0), v_j'(t_0), \tau_j'(t_0)) \wedge (x_{i_0}(t_0), v_{i_0}(t_0)) = (x_{i_0}'(t_0), v_{i_0}'(t_0)), \ell_{i_0}'(t_0) = \ell_{i_0}(t_0) \pm 1, \tau_{i_0}'(t_0) = 0\Big\}. \tag{9}$$

Now we are ready to define the controlled finite-dimensional hybrid system.

Definition III.1 (The controlled finite-dimensional hybrid system)**.** A finite dimensional hybrid system is a 6-tuple $\Sigma_1 = (\mathcal{L}, \mathcal{M}, U, \mathcal{U}, g, SW)$ where:

(1) $\mathcal{L} = \Big\{\ell = (\ell_i)_{i \in \mathcal{I}} \in \mathcal{K}^{P+Q+S}\Big\}$ is a finite set of symbols representing all possible lane labels of all vehicles including cars, autonomous vehicles and trucks. Here we call $\ell \in \mathcal{L}$ a location of the hybrid system Σ_1;

(2) $\mathcal{M} = \{\mathcal{M}_\ell\}_{\ell \in \mathcal{L}}$, where $\mathcal{M}_\ell = (X \setminus A_\ell)^{P+Q+S}$ is the space of position-velocity-timer vectors of all vehicles, with A_ℓ defined as in (8);

(3) $U = \{U_\ell\}_{\ell \in \mathcal{L}}$ represents the control space, $U_\ell = I^Q$, where $I \subset [-U_{\max}, U_{\max}]$ is compact with $U_{\max} > 0$;

(4) $\mathcal{U} = \{\mathcal{U}_\ell\}_{\ell \in \mathcal{L}}$ is such that $\mathcal{U}_\ell = \{u : [0,T] \subset \mathbb{R}_0^+ \to U_\ell$ measurable$\}$ which represents the set of admissible controls at location ℓ;

(5) $g = \{g_\ell\}_{\ell \in \mathcal{L}}$ with $g_\ell \colon \mathcal{M}_\ell \times \mathcal{U}_\ell \mapsto \mathbb{R}^{3(P+Q+S)}$, is such that for every $(x_i, v_i, \tau_i, u_i) \in \mathcal{M}_{\ell_i} \times \mathcal{U}_{\ell_i}$, it holds $g_{(\ell_i)}(x_i, v_i, \tau_i, u_i) = (v_i, a_i, 1)$, where $a_i = \dot{v}_i$ is defined as in systems (4);

(6) SW is a subset of $LC(\Sigma_1)$, where $LC(\Sigma_1)$ is the set of states for which a lane-changing can occur, that is (9).

For the well-posedness of the above finite-dimensional hybrid system see [6].

Now we are ready to introduce an optimal control problem associated to the finite-dimensional hybrid system of definition III.1 on the time interval $[0, t_1)$ which can be extended to the whole time interval $[0, T]$. For more details, we refer again to [6].

Definition III.2 (Optimal control problem associated with a finite-dimensional hybrid system)**.** Find $u^* \in L^1([0,t_1); I)^Q$, such that

$$F_{P,S}(u^*) = \min_{u \in L^1([0,t_1),I)^Q} F_{P,S}(u). \tag{10}$$

where the functional $F_{P,S}$ is given by

$$\sum_{k \in \mathcal{K}} \int_0^{t_1} \left\{ L_k(x^k(t), v^k(t), \mu_P^k(t), \mu_S^k(t)) + \frac{1}{Q_k(t)} \sum_{j=1}^{Q_k(t)} |u_j^k(t)| \right\} \mathrm{d}t \tag{11}$$

where $(x^k, v^k, \mu_P^k, \mu_S^k) \in (\mathbb{R} \times \mathbb{R}_{\geq 0})^{Q_k(t)} \times \mathcal{P}(\mathbb{R} \times \mathbb{R}_{\geq 0})^2$ are solutions to the finite dimensional hybrid system (III.1) on the time interval $[0, t_1)$.

Notice that the cost functional defined in (11) contains two pieces of different nature: the former is a classic Lagrangian involving position and velocity of autonomous vehicles and the density measure of cars and trucks; the latter is a weighted L^1-norm of the vector of controls. For the existence of the optimal control in the case of the finite-dimensional hybrid system, we refer to [12].

IV. The optimal control problem on a infinite-dimensional hybrid systems

In the following we introduce a definition of controlled infinite-dimensional hybrid system. The rigorous proof of the passage from the finite to the infinite system is given in [6]. Here again the system describes the dynamics of multi-lane traffic with cars, trucks an autonomous vehicles. The difference is that the evolution for the first two type of vehicle is given in term of their density, while the number of autonomous vehicles is still finite. The lane changing behavior of the autonomous vehicles generates discrete event of the infinite dimensional hybrid system. Let (x_i, v_i) be the position-velocity vector of the autonomous vehicle $i \in \mathcal{I}_Q$ and $\nu_c^k, \nu_t^k \in \mathcal{M}^+(\mathbb{R} \times \mathbb{R}_{\geq 0})$ the density distribution of cars and trucks on the k-th lane with μ_Q^k the empirical measure for autonomous vehicles defined in (2). The continuous dynamics is given by

$$\begin{aligned}
\dot{x}_i &= v_i \quad i \in \mathcal{I}_Q \\
\dot{v}_i &= \left(H_1^{cc} *_1 (\nu_c^k + \mu_Q^k) + H_1^{tc} *_1 \nu_t^k \right)(x_i, v_i) \\
&\quad + \left(H_2^{cc} * (\nu_c^k + \nu_Q^k) + H_2^{tc} * \nu_t^k \right)(x_i, v_i) + u_i \quad i \in \mathcal{I}_Q^k
\end{aligned}$$

$$\begin{aligned}
&\partial_t \nu_c^k + v \partial_x \nu_c^k + \partial_v \Big[\big(H_1^{cc} *_1 (\nu_c^k + \nu_Q^k) + H_1^{ct} *_1 \nu_t^k \\
&+ H_2^{cc} * (\nu_c^k + \nu_Q^k) + H_2^{ct} * \nu_t^k \big) \nu_c^k \Big] = G_1(\nu_c^k, \nu_t^k, \nu_c^{k'}, \nu_t^{k'}) \\
&\partial_t \nu_t^k + v \partial_x \nu_t^k + \partial_v \Big[\big(H_1^{ct} *_1 (\nu_c^k + \nu_Q^k) + H_1^{tt} *_1 \nu_t^k \\
&\quad + H_2^{ct} * (\nu_c^k + \nu_Q^k) + H_2^{tt} * \nu_t^k \big) \nu_t^k \Big] = G_2(\nu_c^k, \nu_t^k, \nu_c^{k'}, \nu_t^{k'}),
\end{aligned} \tag{12}$$

where $k \in \mathcal{K}$, the source terms G_1 and G_2 are generated by the lane-changing behavior of the cars and trucks. To define the infinite-dimensional controlled hybrid system we consider $X = \mathbb{R} \times \mathbb{R}_{\geq 0} \times [0, \bar{\tau})$ and $\tilde{\mathcal{L}} = \left\{ \ell = (\ell_i)_{i \in \mathcal{I}_Q} \in \mathcal{K}^Q \right\}$ the set of symbols representing all the possible lane labels for autonomous vehicles. Moreover we introduce $\tilde{A}_\ell$ set of triples position-velocity-timer and $LC(\Sigma_2)$ representing the lane-changing mechanism of the finitely many autonomous vehicles:

$$\begin{aligned}
\tilde{A}_\ell = \{ &(x_i, v_i, \tau_i)_{i \in \mathcal{I}_Q} \in X \colon \exists t \in [0,T], i_1, i_2 \in \mathcal{I}_Q, \\
&s.t., x_{i_1}(t) = x_{i_2}(t) \wedge \ell_{i_1}(t) = \ell_{i_2}(t), \text{ with } \ell_{i_1}, \ell_{i_2} \in \mathcal{K} \},
\end{aligned} \tag{13}$$

$$\begin{aligned}
&LC(\Sigma_2) = \Big\{ \big(\ell, (x_i, v_i, \tau_i), \ell', (x_i', v_i', \tau_i') \big)_{i \in \mathcal{I}_Q} \in (\tilde{\mathcal{L}} \times X)^2 \colon \\
&\exists\, i_0 \in \mathcal{I}_Q, \exists\, t_0 \in [0, \bar{\tau}), s.t., j \neq i_0, (\ell_j(t_0), x_j(t_0), v_j(t_0), \tau_j(t_0)) \\
&= (\ell_j'(t_0), x_j'(t_0), v_j'(t_0), \tau_j'(t_0)) \wedge (x_{i_0}(t_0), v_{i_0}(t_0)) \\
&= (x_{i_0}'(t_0), v_{i_0}'(t_0)), \ell_{i_0}'(t_0) = \ell_{i_0}(t_0) \pm 1, \tau_{i_0}'(t_0) = 0 \Big\}.
\end{aligned} \tag{14}$$

Definition IV.1 (The controlled infinite-dimensional hybrid system)**.** A infinite dimensional hybrid system is a 6-tuple $\Sigma_2 = (\mathcal{L}, \mathcal{M}, U, \mathcal{U}, g, SW)$ where:

(1) $\tilde{\mathcal{L}} = \left\{ \ell = (\ell_i)_{i \in \mathcal{I}_Q} \in \mathcal{K}^Q \right\}$ is a finite set of symbols that represent all possible lane labels of the autonomous vehicles;

(2) $\mathcal{M} = \{\mathcal{M}_\ell\}_{\ell \in \tilde{\mathcal{L}}}$, where $\mathcal{M}_\ell = (X \setminus \tilde{A}_\ell)^Q \times (\mathcal{M}^+(\mathbb{R} \times \mathbb{R}_{\geq 0}))^{2L}$, with $\tilde{A}_\ell$ given by (13);

(3) $U = \{U_\ell\}_{\ell \in \tilde{\mathcal{L}}}$ represents the control space;

(4) $\mathcal{U} = \{\mathcal{U}_\ell\}_{\ell \in \tilde{\mathcal{L}}}$ is such that $\mathcal{U}_\ell = \{u : \mathrm{Dom}(u) \subset \mathbb{R}_0^+ \to U_\ell$ measurable$\}$ which represents the set of admissible controls at location ℓ;

(5) $g = \{g_\ell\}_{\ell \in \tilde{\mathcal{L}}}$, $g_\ell \colon \mathcal{M}_\ell \times \mathcal{U}_\ell \mapsto \mathbb{R}^{3Q}$, such that for every $(x_i, v_i, \tau_i, \nu_c, \nu_t, u_i) \in \mathcal{M}_{\ell_i} \times \mathcal{U}_{\ell_i}$, $g_{(\ell_i)}(x_i, v_i, \tau_i, \nu_c, \nu_t, u_i) = (v_i, a_i, 1)$, where $a_i = \dot{v}_i$, $i \in \mathcal{I}_Q$ as defined in systems (12);

(6) S is a subset of $LC(\Sigma_2)$, where $LC(\Sigma_2)$ is the set of states for which a lane-changing can occur, that is (14).

The well-posedness of the infinite-dimensional hybrid system is again established in [6]. For the next results we consider the product space

$$\mathcal{X} \colon = \left\{ (x, v, \mu, \nu) \in (\mathbb{R} \times \mathbb{R}_{\geq 0})^M \times \mathcal{M}^+(\mathbb{R} \times \mathbb{R}_{\geq 0})^2 \right\}$$

and endow it with the following metric: for any $(x_1, v_1, \mu_1, \nu_1), (x_2, v_2, \mu_2, \nu_2) \in \mathcal{X}$,

$$\begin{aligned}
&\|(x_1, v_1, \mu_1, \nu_1) - (x_2, v_2, \mu_2, \nu_2)\|_{\mathcal{X}} = \\
&= \sum_{j=1}^{M} (|x_{1,j} - x_{2,j}| + |v_{1,j} - v_{2,j}|) + \mathcal{W}_1^{1,1}(\mu_1, \mu_2) + \mathcal{W}_1^{1,1}(\nu_1, \nu_2),
\end{aligned} \tag{15}$$

where $M \in \mathbb{N}_0$, $\mathcal{W}_1^{1,1}$ is the generalization of the standard Wasserstein distance (see [1]) introduced in [23].

Definition IV.2 (Optimal control problem associated with an infinite-dimensional hybrid system)**.** Find $u^* \in L^1([0,t_2); I)^Q$, such that

$$F(u^*) = \min_{u \in L^1([0,t_2),I)^Q} F(u). \tag{16}$$

and the functional F is given by

$$\sum_{k \in J} \int_0^{t_2} \left\{ L_k(x^k(t), v^k(t), \nu_c^k(t), \nu_t^k(t)) + \frac{1}{Q_k(t)} \sum_{j=1}^{Q_k(t)} |u_j^k(t)| \right\} \mathrm{d}t \tag{17}$$

where $(x^k, v^k, \nu_c^k, \nu_t^k) \in (\mathbb{R} \times \mathbb{R}_{\geq 0})^{Q_k(t)} \times \mathcal{M}^+(\mathbb{R} \times \mathbb{R}_{\geq 0})^2$ are solutions to system (12) and each function $L_k : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is continuous with respect to the metric defined in (15).

Definition IV.3. Let X be a separable metric space and consider the sequence of functionals $F_N : X \mapsto (-\infty, \infty]$, $N \in \mathbb{N}$. Then F_N Γ- converges to $F : X \mapsto (-\infty, \infty]$ if the following conditions are satisfied:

1. (**Lim inf inequality**) For every $u \in X$ and every sequence $u_N \to u$,
$$F(u) \leq \liminf_{N \to \infty} F_N(u_N);$$
2. (**Lim sup inequality**) For every $u \in X$, there exists a sequence $u_N \to u$, s.t.
$$F(u) \geq \limsup_{N \to \infty} F_N(u_N).$$

Theorem IV.1. *For every $k \in J$, the sequence of functionals $(F^k_{P,S})_{P,S \in \mathbb{N}^+}$ on $L^1([0,t_2), \mathcal{U})^Q$ as defined in* (11) *Γ-converges to the functional F^k in* (17)*.*

Proof. We start showing that the Lim inf inequality is satisfied. Let $(u_N)_{N=1}^\infty$ be a sequence in $L^1([0,t_2))^Q$ such that $u_N \rightharpoonup u \in L^1([0,t_2))^Q$. Furthermore, we require that as $N \to \infty$, the number of human-driven vehicles for each type goes to infinity (i.e. $P, S \to \infty$). Then for each $k \in J$,

$$F^k(u) \leq \liminf_{P \to \infty,\, S \to \infty} F^k_{P,S}(u_N); \tag{18}$$

For each $N \in \mathbb{N}_{\geq 1}$, there exists a unique solution $(x_N^k, v_N^k, \mu_{N,P}^k, \mu_{N,S}^k) \in (\mathbb{R} \times \mathbb{R}_{\geq 0})^{Q_k} \times \mathcal{M}^+(\mathbb{R} \times \mathbb{R}_{\geq 0})$, $k \in J$, to the finite-dimensional hybrid system.

The key property needed to prove (18) is

$$\lim_{N \to \infty} (x_N^k, v_N^k, \mu_{N,P}^k, \mu_{N,S}^k) = (x^k(t), v^k(t), \nu_c^k(t), \nu_t^k(t))$$

where the limit is in the norm defined in (15) and it holds by construction. This indeed implies that

$$\lim_{N \to \infty} \int_0^{t_0} L_k(x_N^k(t), v_N^k(t), \mu_{N,P}^k(t), \mu_{N,S}^k(t)) dt = \int_0^{t_0} L_k(x^k(t), v^k(t), \nu_c^k(t), \nu_t^k(t)) dt$$

which together the lower-continuity of the L^1-norm gives (18). The Lim sup inequality follows easily by choosing u_N constantly equal to u. □

Corollary IV.1.1. *The optimal control problem in Definition (IV.2) has solutions.*

Proof. For each $N \in \mathbb{N}_{\geq 1}$, there is an optimal control $u_{*,N}$ for the finite dimensional system (see [10] for more details). Note that since the sequence of controls $(u_{*,N})$ is bounded in $L^1([0,t_2), I)^Q$ which is compact in the week topology, there exists a subsequence (for simplicity, we still use the same notaion), such that $u_{*,N} \rightharpoonup u^* \in L^1([0,t_2), I)^Q$. By Theorem 7.8 in [21], u^* minimizes F. □

V. Generalization to multi-population models

In this section we derive a general formulation for both the finite dimensional and the infinite dimensional hybrid system in the case of $M+1$ populations of vehicles. Indeed the theory built in the preliminary papers of Gong, Piccoli and Visconti ([12], [13]) can be extended to more complex situations just by using the same technique.

Consider A_k the number of autonomous vehicles on the lane $k \in \{1, \ldots, L\}$, with $\mathcal{I}_0^k$ the correspondent sets of indices. On the other side let $V_{n,k}$ with $n = 1, \ldots, M$ be the number of human-driven vehicles on lane k with indices in $\mathcal{I}_n^k$.

The dynamic of this multi-population frame is given by the following system of first order ODE representing the Bando-FtL model in convolutional form:

$$\dot{x}_i = v_i \quad i \in \mathcal{I}$$
$$\dot{v}_i = \begin{cases} \Big(\sum_{m=0}^{M} H_1^{mn} *_1 \mu_m^k + \sum_{m=0}^{M} H_2^{mn} * \mu_m^k \Big)(x_i, v_i) + u_i & i \in \mathcal{I}_0^k, \\ \Big(\sum_{m=0}^{M} H_1^{mn} *_1 \mu_m^k + \sum_{m=0}^{M} H_2^{mn} * \mu_m^k \Big)(x_i, v_i) & i \in \mathcal{I}_n^k. \end{cases} \tag{19}$$

Here μ_n^k, with $n \in \{0, \ldots, M\}$, represents again the atomic measures supported on absolutely continuous trajectories $t \in [0,T] \to (x_i(t), v_i(t)) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$ for each type of vehicle in the lane k, explicitly we have:

$$\mu_0^k(t) = \frac{1}{A_k(t)} \sum_{i \in \mathcal{I}_0^k(t)} \delta_{(x_i(t), v_i(t))},$$
$$\mu_1^k(t) = \frac{1}{V_{1,k}(t)} \sum_{i \in \mathcal{I}_1^k(t)} \delta_{(x_i(t), v_i(t))},$$
$$\vdots$$
$$\mu_M^k(t) = \frac{1}{V_{M,k}(t)} \sum_{i \in \mathcal{I}_M^k(t)} \delta_{(x_i(t), v_i(t))},$$

The convolutional kernels in (19) are defined in such a way that they can represents all the possible combinations of vehicles depending on their order, indeed

$$H_q^{mn} : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R} \quad \text{with } q \in \{1,2\}, \quad m, n \in \{0, \ldots, M\}$$

and the structure of these maps is the same described for the two populations case in II-A. The control appears only in the acceleration of autonomous vehicles and aims to influence the dynamic of the other populations. Observe that in the case of $M = 2$, i.e. with autonomous vehicles and two population of human-driven vehicles (cars and trucks), if we assume that the convolutional kernel for the first two types is the same, then we find again the dynamic (4)

We point out that the lane changing conditions (Incentive and Safety), the probability for a vehicle of performing lane change and the definition of controlled hybrid system can be formulated in analogous way up to heavier notation. It can be also proved rigorously that the mean field limit (i.e, assuming that the number of vehiches for each class, apart AV, goes to $+\infty$) for (19) is given by

$$\dot{x}_i = v_i \quad i \in \mathcal{I}_0$$
$$\dot{v}_i = \Big(\sum_{m=0}^{M} H_1^{m0} *_1 \nu_m^k + \sum_{m=0}^{M} H_2^{m0} * \nu_m^k \Big)(x_i, v_i) + u_i \qquad i \in \mathcal{I}_0^k$$

$$\partial_t \nu_n^k + v\partial_x \nu_n^k + \partial_v \Big[\Big(\sum_{m=0}^{M} H_1^{mn} *_1 \nu_m^k + \sum_{m=0}^{M} H_2^{mn} * \nu_m^k\Big)\nu_n^k\Big] \tag{20}$$
$$= G_n(\nu_n^k, \nu_m^k, \nu_n^{k'}, \nu_m^{k'}) \qquad n \in \{1, \dots, M\}$$

with $\nu_n^k \in \mathcal{M}^+(\mathbb{R} \times \mathbb{R}_{\geq 0})$ representing the density of vehicles of the class n. Observe that in the new source term the input ν_m^k is the density of vehicles of the class $m \neq n$ entering the contiguous lane, therefore it stands for $M-1$ densities (same for $\nu_m^{k'}$).

Similarly to what seen in the previous sections, we can associate an optimal control problem both to the finite and infinite dimensional hybrid system and study the well posedness . This extension of the model to multiple populations allows for a fine analysis of traffic including for example small, medium and large vehicles, motorcycles, bicycles and so on.

VI. Conclusion and future work

In this paper we have analyzed the existence of an optimal control for a minimization problem associated to the dynamic of a multi-population traffic model on multi-lane both on a microscopic and macroscopic scale (through the mean-field limit). This is a preliminary step towards a deep understanding of the mechanisms regulating this type of traffic and how to improve it. A question still open is, for instance, the identification of appropriate ranges for the parameters in the convolution kernels and in the Incentive (5) and Safety (6) conditions. For this purpose it is relevant to perform a comparison with empirical data. Also the choice of the probability (7) can be improved by including parameters which take into account the position along the stretch of road, the proximity of intersections, weather conditions and the time slots. Consequently it will be possible to implement optimal control and develop numerical methods for predictions. We emphasize that the control of traffic in the presence of large vehicles is of vital importance also for safety purposes, in fact studies and data show that their presence often increases the number and extent of road accidents ([20]).

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