On a nonlocal functional arising in the study of thin-film blistering

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Abstract

The energy of a Von Kármán circular plate is described by a nonlocal nonconvex one-dimensional functional depending on the thickness ε. Here we perform the asymptotic analysis via Γ-convergence as the parameter ε goes to zero.

1 Introduction

In this paper we propose a perturbed variational problem which arises in the study of thin-film blistering via energy methods for the Von Kármán model. The phenomenon of the blistering concerns the debonding of a compressed thin film from its substrate in some regions where the cohesion film/substrate is feeble or lacking. The damage then may evolve by means of the forces acting on the fracture interface. Two different problems arise in the study of thin-film blistering: the static problem connected to the appearance of the blister and the dynamical problem connected to the propagation of the damage. Here we focus our attention to the static problem of compressed plates described by the Von Kármán model. There is a copious mathematical literature on the topic, we quote among others [1],[3],[4],[5],[10],[12],[13], [14],[19]. Most of these papers concern the variational analysis of the singularly perturbed functional

$$F_\varepsilon(u) = \varepsilon^2 \int_\Omega |\nabla\nabla u|^2 d\mathbf{x} + \int_\Omega (1 - |\nabla u|^2)^2 d\mathbf{x}. \tag{1.1}$$

Our starting point is the Von Kármán model in the form of a unique functional equation in the unknown w that is a scalar function denoting the vertical displacement of the blister. In hypothesis of radial symmetry the one-dimensional energy functional we

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consider is

$$I_\varepsilon(w) \,=\, \varepsilon^2 \int_0^1 \Big(w_{rr} + \frac{w_r}{r} \Big)^2 \, r dr - \lambda^2 + \frac{1}{2} \int_0^1 r \, \Big(\int_r^1 \frac{1}{s} \, w_s^2 \, ds \, - 2\lambda \Big)^2 \, dr \qquad (1.2)$$

where λ is the compression parameter.
The morphology of the blister is the aim of the variational analysis. In particular the interest is focused on capturing the folding effects concentrated near the boundary that are experimentally observed. Although this purpose requires the treatment of a two-dimensional model, we propose a one-dimensional analysis to test the effect of the nonlocal term. Indeed the proposed functional (1.2) is slightly different from the one-dimensional analogous of (1.1) (see (1.4)) since the contribution of the in-plane displacements (missing in (1.1)) is here taken into account through the nonlocal term. As emphasized also by other authors (see for example [15]) the contribution of these displacements can produce a deep effect on the solutions of the problem.

The main result of this paper is Theorem 3.4 where we proved that the functional

$$J_\varepsilon(u) \,=\, \varepsilon^2 \int_0^1 \Big(u_r^2 + \frac{u^2}{r^2} \Big) \, r dr - \lambda^2 + \frac{1}{2} \int_0^1 r \, \Big(\int_r^1 \frac{1}{s} \, u^2 \, ds \, - 2\lambda \Big)^2 \, dr \qquad (1.3)$$

Γ-converges, with respect to the weak-$L^2_{\rm loc}$ topology, to

$$\Gamma(\text{w-}L^2_{\rm loc})\text{-}\lim_{\varepsilon\to 0} J_\varepsilon(u) = \frac{1}{2}\, \overline{G}(u) - \lambda^2$$

for every $u \in L^2_{\rm loc}(0,1)$. The functional $\overline{G}(u)$ is the lower semicontinuous envelope, with respect to the weak-$L^2_{\rm loc}$ topology, of the potential

$$G(u) = \int_0^1 r \left(\int_r^1 \frac{u^2}{s} \, ds - 2\lambda \right)^2 dr \, .$$

It is described by a minimum problem

$$\overline{G}(u) := \min\Big\{ \int_0^1 r \, (\mu((r,1]) - 2\lambda)^2 \, dr \, : \, \mu \geq (1/s) u^2 \, ds \Big\} \, ,$$

where the minimum is taken in the space $\mathcal{M}^+((0,1])$ of locally finite positive measures on $(0,1]$. The set of minimizers for $\overline{G}(u)$ is given only by $\{u \equiv 0\}$. It easily follows by testing the definition of $\overline{G}(u)$ with the Dirac measure $\mu = 2\lambda\delta_1$.

The asymptotic behavior of minimizing sequences of $G(u)$ highlights concentration phenomena at the boundary (see Remark 3.3); this suggests that branching and folding effects may arise in the boundary of the blister. Their characterization and shapes could be obtained through the two-dimensional variational analysis at different scales.

Under the same hypothesis of radial symmetry, the model described by the functional F_ε in (1.1) reduces to the following one-dimensional functional

$$K_\varepsilon(w) = \varepsilon^2 \int_0^1 \left(w_{rr}^2 + \frac{w_r^2}{r^2} \right) rdr + \int_0^1 \left(1 - w_r^2\right)^2 rdr \,. \tag{1.4}$$

In this case the Γ-limit is given by

$$\bar{K}_0(w) = \int_0^1 \left([w_r^2 - 1]^+\right)^2 rdr$$

which is the lower semicontinuous envelope of the two-well type-potential $K_0(w) = \int_0^1 \left(1 - w_r^2\right)^2 rdr$. Note that, now the set of minimizers for $\bar{K}_0(w)$ is $\{|w_r| \leq 1\}$ while, since $I_\varepsilon(w) = J_\varepsilon(w_r)$, we have that the set of minimizers for $I_\varepsilon(w)$ is given by $\{w_r \equiv 0\}$.

The results obtained studying the minimizers for the functionals (1.2) and (1.4) do not allow us to compare in an exhaustive way the two models suggested for the phenomenon of the blistering. It could be necessary further rescale the two functionals to be able to compare them.

In Section 2 we derive from the Von Kármán model the proposed one-dimensional functional (1.2). The asymptotic analysis with the computation of the Γ-limit is given in Section 3.

2 The model

Let Ω be a plate and h its thickness. We denote by E and ν two material parameters ($E > 0$ and $|\nu| < 1$) and assume that the plate is under a uniform initial compression σ_0. The Von Kármán equations in the unknowns F and w, Airy function and vertical displacement respectively, read

$$\begin{aligned} \frac{Eh^2}{12(1-\nu^2)}\Delta^2 w + \sigma_0 \Delta w &= F_{xx}w_{yy} + F_{yy}w_{xx} - 2F_{xy}w_{xy} \\ \Delta^2 F &= -E\left(w_{xx}w_{yy} - w_{xy}^2\right). \end{aligned}$$

The blister shape is described by the vertical displacement $w(x, y)$ while the plane displacements u_1, u_2 can be deduced by the following equations established by Landau & Lifschitz (see [16])

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= \frac{1}{E}\left(\frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial u_2}{\partial y} &= \frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \nu \frac{\partial^2 F}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} &= -\frac{2(1+\nu)}{E}\frac{\partial^2 F}{\partial x \partial y} - \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial w}{\partial x}\right). \end{aligned}$$

Setting

$$F = \frac{E}{2}\, f, \qquad \sigma_0 = \frac{E}{2}\, \lambda, \qquad \varepsilon^2 = \frac{h^2}{6(1-\nu^2)}$$

we get the normalized equations

$$\begin{aligned} \varepsilon^2 \Delta^2 w + \lambda \Delta w &= f_{xx} w_{yy} + f_{yy} w_{xx} - 2 f_{xy} w_{xy} = [f, w] \qquad \text{in } \Omega & (2.1) \\ \Delta^2 f &= -2(w_{xx} w_{yy} - w_{xy}^2) = -[w, w] \qquad \text{in } \Omega & (2.2) \end{aligned}$$

(see [8] for a comprehensive derivation of the Von Kármán model). The clamped conditions for w are imposed on the boundary of the blister region

$$w = w_n = 0, \qquad \text{on } \partial\Omega. \tag{2.3}$$

Let $f = G_0[w, w]$ be the solution of the equation (2.2) with respect to the boundary conditions

$$f = \Delta f = 0, \qquad \text{on } \partial\Omega\, .$$

Hence, we can consider the following energy functional depending only on the scalar function w

$$E_\varepsilon(w) \;=\; \varepsilon^2 \int_\Omega |\Delta w|^2 d\mathbf{x} - \lambda \int_\Omega |\nabla w|^2 d\mathbf{x} + \frac{1}{2} \int_\Omega |\Delta G_0[w, w]|^2 d\mathbf{x}. \tag{2.4}$$

(see also [6] for the treatment of the functional E_ε).

The mathematical analysis of the phenomenon has been carried out by several authors. We quote in particular the first paper by Gioia & Ortiz [12] where the blister shape is described neglecting the contribution of the in-plane displacements. In the same hypothesis compactness results and Γ-convergence results are proved in [1], [9], [10], [17], [18] for the referential functional (1.1).
Although the functional (2.4) depends only on function w, we want to underline that the contribution of the plane displacements is taken into account thought the nonlocal term. We want to investigate on this contribution looking, for sake of simplicity, at the one-dimensional problem.

2.1 The one-dimensional model

Let $\Omega \subset \mathbb{R}^2$ be an unit disk. We rewrite the equations (2.1), (2.2) as

$$\begin{cases} \varepsilon^2 \mathcal{D}^2 w + \lambda \mathcal{D} w = \frac{1}{r}\, (f_r w_r)_r \\ \\ \mathcal{D}^2 f = -\frac{1}{r}\, (w_r^2)_r \end{cases} \tag{2.5}$$

where

$$\mathcal{D} w = w_{rr} + \frac{w_r}{r} \;=\; \frac{1}{r}\, (w_r r)_r \; .$$

The boundary conditions in $r = 0$ follow from the radial symmetry. Indeed w and f are even functions; hence, the following conditions can be prescribed only in the origin

$$w'(0) = 0, \qquad \left(\frac{w'(r)}{r}\right)'_{r=0} = 0, \qquad (\mathcal{D}w)'_{r=0} = 0 \tag{2.6}$$

$$f'(0) = 0, \qquad \left(\frac{f'(r)}{r}\right)'_{r=0} = 0, \qquad (\mathcal{D}f)'_{r=0} = 0\,. \tag{2.7}$$

In $r = 1$ we assume

$$w(1) = 0, \qquad w'(1) = 0 \tag{2.8}$$

$$f(1) = 0, \qquad \mathcal{D}f|_{r=1} = f''(1) + f'(1) = 0\,. \tag{2.9}$$

By the second equation in (2.5), (2.7) and (2.9) we get the following nonlocal equation

$$\varepsilon^2\mathcal{D}^2 w + \lambda\mathcal{D}w = \frac{1}{r}\left[w_r\,\frac{1}{r}\int_0^r s\left(\int_s^1 \frac{1}{\xi}\,w_\xi^2\,d\xi\right)ds\right]_r \,.$$

Hence, the functional (2.4) can be rewritten as

$$I_\varepsilon(w) \;=\; \varepsilon^2\int_0^1\left(w_{rr} + \frac{w_r}{r}\right)^2 rdr - \lambda\int_0^1 r\,w_r^2\,dr + \frac{1}{2}\int_0^1\left(\int_r^1 \frac{1}{s}\,w_s^2\,ds\right)^2 rdr\,.$$

Since,

$$\int_0^1 r\,w_r^2\,dr \;=\; 2\int_0^1 s\left(\int_s^1 \frac{1}{\xi}\,w_\xi^2\,d\xi\right)ds$$

we get the functional $I_\varepsilon(w)$ as defined in (1.2)

$$I_\varepsilon(w) \;=\; \varepsilon^2\int_0^1\left(w_{rr} + \frac{w_r}{r}\right)^2 rdr - \lambda^2 + \frac{1}{2}\int_0^1\left(\int_r^1 \frac{1}{s}\,w_s^2\,ds \,-\, 2\lambda\right)^2 rdr\,.$$

Moreover, by (2.6), (2.8) and replacing w_r by u we get

$$J_\varepsilon(u) \;=\; \varepsilon^2\int_0^1\left(u_r^2 + \frac{u^2}{r^2}\right) rdr - \lambda^2 + \frac{1}{2}\int_0^1\left(\int_r^1 \frac{1}{s}\,u^2\,ds \,-\, 2\lambda\right)^2 rdr\,.$$

In Section 3 we carry out the asymptotic analysis as the parameter ε tends to zero and we prove the Γ-convergence result for the functional (1.3).

3 Γ-convergence result

In order to choose the topology with respect to compute the Γ-limit of the functional $J_\varepsilon(u)$ as defined in (1.3) we have to study the compactness of sequences with bounded energy.

Theorem 3.1 (Compactness) *Let $\{u_\varepsilon\}$ be a sequence such that $\sup_\varepsilon J_\varepsilon(u_\varepsilon) < +\infty$, then, up to subsequences, the sequence of measures $\{(1/s)\, u_\varepsilon^2\, ds\}$ converge weakly* in in the space $\mathcal{M}^+((0,1])$ of locally finite positive measures on $(0,1]$. In particular $\{u_\varepsilon\}$ converges weakly in $L^2_{\rm loc}(0,1)$.*

Proof. By assumption $\sup_\varepsilon \int_0^1 r \left(\int_r^1 (1/s)u_\varepsilon^2\, ds - 2\lambda\right)^2 dr < +\infty$; hence, we have that $\sup_\varepsilon \int_0^1 r \left(\int_r^1 (1/s)u_\varepsilon^2\, ds\right)^2 dr < +\infty$. By Hölder inequality and for any $r_0 \in (0,1]$, we have that

$$\begin{aligned}\left(\int_0^1 r\left(\int_r^1 \frac{u_\varepsilon^2}{s}\, ds\right)^2 dr\right)^{1/2} &\geq \int_0^1 \sqrt{r}\left(\int_r^1 \frac{u_\varepsilon^2}{s}\, ds\right) dr\\ &\geq \int_0^{r_0} \sqrt{r}\left(\int_{r_0}^1 \frac{u_\varepsilon^2}{s}\, ds\right) dr\\ &= \frac{2}{3} r_0^{3/2}\left(\int_{r_0}^1 \frac{u_\varepsilon^2}{s}\, ds\right), \qquad (3.1)\end{aligned}$$

which implies the weak convergence of $\{u_\varepsilon\}$ in $L^2_{\rm loc}(0,1)$. Moreover, by (3.1), we have that for every $I \subset\subset (0,1]$ there exists a constant depending on I, c_I, such that $\int_I (1/s)u_\varepsilon^2\, ds \leq c_I$ for every $\varepsilon > 0$; hence, also the sequence of measures $\{(1/s)\, u_\varepsilon^2\, ds\}$ converge weakly* in $\mathcal{M}^+((0,1])$. □

Reasoning as in [2] we now study the Γ-convergence of J_ε with respect to the weak-$L^2_{\rm loc}$ topology. To this end we first compute the lower semicontinuous envelope, $\overline{G}(u)$, of the functional

$$G(u) = \int_0^1 r\left(\int_r^1 \frac{u^2}{s}\, ds - 2\lambda\right)^2 dr\,.$$

Lemma 3.2 *The lower semicontinuous envelope of G with respect to the weak- $L^2_{\rm loc}$ topology is*

$$\overline{G}(u) := \min\left\{\int_0^1 r\,(\mu((r,1]) - 2\lambda)^2\, dr \,:\, \mu \geq (1/s)u^2\, ds\right\},$$

where the minimum is taken in $\mathcal{M}^+((0,1])$.

Proof. Let $\{u_N\}$ be a sequence weakly converging to u in $L^2_{\rm loc}$ such that

$$\lim_{N\to+\infty} G(u_N) < +\infty$$

and the sequence of positive measures $\{(1/s)u_N^2\,ds\}$ converges weakly* to μ in $\mathcal{M}^+((0,1])$. Then, by the lower semicontinuity of the L^2-norm and the convergence of the measures of almost all intervals, we have that

$$\frac{\mu(r-\delta,r+\delta)}{2\delta} = \lim_{N\to+\infty} \fint_{r-\delta}^{r+\delta} \frac{u_N^2}{s}\,ds \geq \fint_{r-\delta}^{r+\delta} \frac{u^2}{s}\,ds$$

for a.a. $\delta \in (0,r)$; by the Besicovitch Derivation Theorem, we conclude that for almost every r

$$\frac{d\mu}{ds}(r) \geq \frac{u^2(r)}{r}\,.$$

Since μ has at most countably many atoms, by the weak*-convergence in $\mathcal{M}^+((0,1])$, we have that

$$\lim_{N\to+\infty} \int_r^1 \frac{u_N^2}{s}\,ds = \mu((r,1]) \qquad \text{for a.e. } r \in (0,1)\,. \tag{3.2}$$

By assumption $\lim_{N\to+\infty} G(u_N) < +\infty$; hence, by Fatou's Lemma we have that

$$\lim_{N\to+\infty} G(u_N) \geq \int_0^1 r\,(\mu((r,1]) - 2\lambda)^2\,dr \geq \overline{G}(u)\,.$$

Note that the functional

$$\mu \mapsto \int_0^1 r\,(\mu((r,1]) - 2\lambda)^2\,dr$$

is weakly lower semicontinuous and coercive in $\mathcal{M}^+((0,1])$. Moreover, the set $\{\mu \geq (1/s)u^2\,ds\}$ is convex; hence, the minimum is attained.

We now check the limsup inequality for every $u \in L^2_{\mathrm{loc}}(0,1)$ such that $\overline{G}(u) < +\infty$. Let $\mu \in \mathcal{M}^+((0,1])$ be such that

$$\overline{G}(u) = \int_0^1 r\,(\mu((r,1]) - 2\lambda)^2\,dr\,.$$

For $0 < a < 1$ we define

$$u^a(r) = \begin{cases} u(r) & a \leq r \leq 1-a \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mu^a((r,1]) = \begin{cases} \mu((a,1]) & \text{if } r \leq a \\ \mu((r,1]) & \text{if } a < r \leq 1-a \\ 0 & \text{if } r > 1-a\,. \end{cases}$$

Moreover, we define

$$\mu_N^a((r,1]) = \begin{cases} \mu((a,1]) & \text{if } r \leq a \\ \mu((r,1]) & \text{if } a < r \leq 1-a \\ \mu((1-a,1]) & \text{if } 1-a < r \leq 1-a+\frac{1}{N} \\ 0 & \text{if } r > 1-a+\frac{1}{N} \end{cases}\,.$$

for $N \in \mathbb{N}$ and $N > 1$. Then, $u^a \in L^2(0,1)$, $u^a \to u$ in $L^2_{\rm loc}(0,1)$ as $a \to 0^+$, μ^a_N converges to the measure μ^a (this can be checked, e.g., by using Theorem 1.16 in [11]) and

$$\overline{G}(u^a) \le \int_0^1 r\,(\mu^a((r,1]) - 2\lambda)^2\,dr \le \overline{G}(u) + o(1) \tag{3.3}$$

as $a \to 0^+$. Indeed,

$$\begin{aligned}\int_0^1 r\,(\mu^a((r,1]) - 2\lambda)^2\,dr &= \int_a^1 r\,(\mu^a((r,1]) - 2\lambda)^2\,dr + o(1)\\ &= \int_a^1 r\,(\mu((r,1]) - 2\lambda)^2\,dr - \int_{1-a}^1 r\,\mu((r,1])^2\,dr\\ &\quad +4\lambda\int_{1-a}^1 r\,\mu((r,1])\,dr + o(1)\\ &\le \overline{G}(u) + o(1)\,,\end{aligned}$$

as $a \to 0^+$.

We denote by $I_N = \{0, \cdots, N-1\}$ and by

$$\begin{cases} \overline{u}_N(r) = \displaystyle -\!\!\!\!\!\!\int_{i/N}^{(i+1)/N} u^a\,ds\,, & i/N \le r \le (i+1)/N\,, \quad i \in I_N\\ v_N(r) = N\,\mu^a_N((i/N,(i+1)/N])\,, & i/N \le r \le (i+1)/N\,, \quad i \in I_N\,.\end{cases}$$

By definition of μ^a_N, we have that

$$\mu^a_N((i/N,(i+1)/N]) = \begin{cases} 0\,, & i = 0, \cdots, [Na]-1\\ \mu((a,([Na]+1)/N]) & i = [Na]\\ \mu((i/N,(i+1)/N])\,, & i = [Na]+1, \cdots, [N(1-a)]-1\\ \mu([N(1-a)]/N, 1-a])\,, & i = [N(1-a)]\\ \mu((1-a,1])\,, & i = [N(1-a)]+1\\ 0\,, & i = [N(1-a)]+2, \cdots, N-1\,.\end{cases} \tag{3.4}$$

Note that, in particular

$$\overline{u}_N(r) = 0\,, \qquad 0 \le r \le [Na]/N\,, \qquad ([N(1-a)]+1)/N \le r \le 1$$

and

$$v_N(r) = \begin{cases} 0\,, & 0 \le r \le [Na]/N\\ N\mu((1-a,1])\,, & ([N(1-a)]+1)/N \le r \le ([N(1-a)]+2)/N\\ 0\,, & ([N(1-a)]+2)/N \le r \le 1\,.\end{cases}$$

Hence, we define

$$u_N(r)=\begin{cases}0\,, & 0\le r\le [Na]/N\\ \overline{u}_N+\sqrt{r_N^i\,v_N-(\overline{u}_N)^2}\,, & i/N\le r\le i/N+1/2N,\\ & \quad i=[Na],\cdots,[N(1-a)]\\ c_N\Big(\overline{u}_N-\sqrt{r_N^i\,v_N-(\overline{u}_N)^2}\Big)\,, & i/N+1/2N\le r\le (i+1)/N,\\ & \quad i=[Na],\cdots,[N(1-a)]\\ \sqrt{N\,r\,\mu((1-a,1])}\,, & ([N(1-a)]+1)/N\le r\le ([N(1-a)]+2)/N\\ 0 & ([N(1-a)]+2)/N\le r\le 1\end{cases}\tag{3.5}$$

where

$$c_N(r)=\sqrt{\frac{2N\,r}{2N\,r-1}}\,,\quad [Na]/N+1/2N\le r\le ([N(1-a)]+1)/N$$

($[t]$ denotes the *integer part* of t). In particular we may extend the definition of c_N to the interval $(a/2,1]$ for N big enough. Finally, it remains to fix r_N^i for $i=[Na],\cdots,[N(1-a)]$ such that

$$v_N(r)\ge\frac{(\overline{u}_N(r))^2}{r_N^i}\,,\quad i/N\le r\le (i+1)/N\,,\quad i=[Na],\cdots,[N(1-a)]\,.$$

In fact, since $\mu_N^a\ge (1/s)\,(u^a)^2\,ds$, for every $i=[Na],\cdots,[N(1-a)]$, there exists $r_N^i\in[i/N,(i+1)/N]$ such that

$$\begin{aligned}\mu_N^a((i/N,(i+1)/N]) &\ge \int_{i/N}^{(i+1)/N}\frac{(u^a)^2}{s}\,ds=\frac{1}{r_N^i}\int_{i/N}^{(i+1)/N}(u^a)^2\,ds\\ &\ge \frac{1}{r_N^i}\frac{1}{N}\Big(\fint_{i/N}^{(i+1)/N}u^a\,ds\Big)^2=\frac{1}{r_N^i}\frac{1}{N}(\overline{u}_N)^2\,.\end{aligned}$$

By definition $\{u_N\}$ is bounded in L^2; hence, up to subsequences, it converges weakly in L^2. To identify the weak limit function with u^a it is sufficient to check that $\lim_{N\to+\infty}\int_b^d u_N=\int_b^d u^a$, for every $(b,d)\subseteq(0,1)$. In fact,

$$\begin{aligned}\int_{([N(1-a)]+1)/N}^{1}u_N\,dr &= \sqrt{N\mu((1-a,1])}\int_{([N(1-a)]+1)/N}^{([N(1-a)]+2)/N}\sqrt{r}\,dr\\ &\le \sqrt{\mu((1-a,1])}\sqrt{\frac{[N(1-a)]+2}{N^2}}\,;\end{aligned}\tag{3.6}$$

while, for $i=[Na],\cdots,[N(1-a)]$, we have that

$$\int_{i/N}^{(i+1)/N}u_N\,dr = \int_{i/N}^{i/N+1/2N}\overline{u}_N+\sqrt{r_N^i\,v_N-(\overline{u}_N)^2}\,dr$$

$$
\begin{aligned}
&\quad + \int_{i/N+1/2N}^{(i+1)/N} c_N(r) \left(\overline{u}_N - \sqrt{r_N^i\, v_N - (\overline{u}_N)^2} \right) dr \\
&= \int_{i/N}^{(i+1)/N} \overline{u}_N\, dr + \int_{i/N+1/2N}^{(i+1)/N} (c_N(r) - 1)\, \overline{u}_N\, dr \\
&\quad + \int_{i/N+1/2N}^{(i+1)/N} (1 - c_N(r)) \sqrt{r_N^i\, v_N - (\overline{u}_N)^2}\, dr \\
&= \int_{i/N}^{(i+1)/N} u^a\, dr + \int_{i/N+1/2N}^{(i+1)/N} (c_N(r) - 1)\, \overline{u}_N\, dr \\
&\quad + \int_{i/N+1/2N}^{(i+1)/N} (1 - c_N(r)) \sqrt{r_N^i\, v_N - (\overline{u}_N)^2}\, dr\,. \qquad (3.7)
\end{aligned}
$$

If we sum up on i, by Hölder inequality, we get that

$$
\begin{aligned}
&\Big| \sum_{i=[Na]}^{[N(1-a)]} \int_{i/N+1/2N}^{(i+1)/N} (c_N(r) - 1)\, \overline{u}_N\, dr \Big| \\
&\leq \Big(\int_0^1 (\overline{u}_N)^2\, dr \Big)^{1/2} \Big(\int_{a/2}^1 (c_N(r) - 1)^2\, dr \Big)^{1/2} \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big| \sum_{i=[Na]}^{[N(1-a)]} \int_{i/N+1/2N}^{(i+1)/N} (1 - c_N(r)) \sqrt{r_N^i\, v_N - (\overline{u}_N)^2}\, dr \Big| \\
&\leq \Big(\sum_{i=[Na]}^{[N(1-a)]} \int_{i/N+1/2N}^{(i+1)/N} r_N^i\, v_N + (\overline{u}_N)^2\, dr \Big)^{1/2} \Big(\int_{a/2}^1 (1 - c_N(r))^2\, dr \Big)^{1/2} \\
&\leq \Big(\mu((a, 1-a]) + \int_0^1 (\overline{u}_N)^2\, dr \Big)^{1/2} \Big(\int_{a/2}^1 (1 - c_N(r))^2\, dr \Big)^{1/2}. \qquad (3.9)
\end{aligned}
$$

Note that the sequence $\{\overline{u}_N\}$ is bounded in the L^2 since it converges to u^a strongly; moreover,

$$
\lim_{N \to +\infty} \int_{a/2}^1 (1 - c_N(r))^2\, dr = 0\,.
$$

Hence, by (3.6), (3.7), (3.8) and (3.9), we can easily conclude that $\lim_{N \to +\infty} \int_b^d u_N = \int_b^d u^a$, for every $(b, d) \subseteq (0, 1)$ and, therefore, the weak convergence of $\{u_N\}$ to u^a in $L^2(0, 1)$.

We now examine

$$
\begin{aligned}
G(u_N) &= \int_0^{([N(1-a)]+1)/N} r \Big(\int_r^1 \frac{u_N^2}{s}\, ds - 2\lambda \Big)^2 dr \\
&\quad + \int_{([N(1-a)]+1)/N}^1 r \Big(\int_r^1 \frac{u_N^2}{s}\, ds - 2\lambda \Big)^2 dr\,. \qquad (3.10)
\end{aligned}
$$

In particular, we have that

$$
\begin{aligned}
& \int_{([N(1-a)]+1)/N}^{1} r\Big(\int_r^1 \frac{u_N^2}{s}\,ds-2\lambda\Big)^2\,dr \\
= & \int_{([N(1-a)]+1)/N}^{1} r\Big(\int_r^{([N(1-a)]+2)/N} N\,\mu((1-a,1])\,ds-2\lambda\Big)^2\,dr \\
\leq & \; 2\int_{([N(1-a)]+1)/N}^{1} r\Big(\mu((1-a,1])^2+4\lambda^2\Big)\,dr\,. \qquad (3.11)
\end{aligned}
$$

Hence,

$$
\limsup_{N\to+\infty}\int_{([N(1-a)]+1)/N}^{1} r\Big(\int_r^1 \frac{u_N^2}{s}\,ds-2\lambda\Big)^2\,dr=o(1)\,, \qquad (3.12)
$$

as $a\to 0^+$. It remains to study the first term in (3.10). Note that,

$$
\int_0^{[Na]/N} r\Big(\int_r^1 \frac{u_N^2}{s}\,ds-2\lambda\Big)^2\,dr=\int_0^{[Na]/N} r\Big(\int_{[Na]/N}^1 \frac{u_N^2}{s}\,ds-2\lambda\Big)^2\,dr\,;
$$

hence, for $i=[Na],\cdots,[N(1-a)]$, we compute the following integral

$$
\begin{aligned}
& \int_{i/N}^{(i+1)/N} \frac{1}{s}\,u_N^2\,ds \\
= & \int_{i/N}^{i/N+1/2N} \frac{1}{s}\Big(r_N^i\,v_N+2\,\overline{u}_N\sqrt{r_N^i\,v_N-(\overline{u}_N)^2}\Big)\,ds \\
& +\int_{i/N+1/2N}^{(i+1)/N}\Big(\frac{2N\,s}{2N\,s-1}\Big)\frac{1}{s}\Big(r_N^i\,v_N-2\,\overline{u}_N\sqrt{r_N^i\,v_N-(\overline{u}_N)^2}\Big)\,ds \\
= & \; 2\int_{i/N}^{i/N+1/2N} \frac{1}{s}\,(r_N^i\,v_N)\,ds \\
= & \; \mu_N^a((i/N,(i+1)/N])\,\ln\Big(1+\frac{1}{2i}\Big)^{2N\,r_N^i}\,. \qquad (3.13)
\end{aligned}
$$

We recall that $r_N^i\in[i/N,(i+1)/N]$, $i=[Na],\cdots,[N(1-a)]$; hence,

$$
\lim_{N\to+\infty}\ln\Big(1+(1/2i)\Big)^{2N\,r_N^i}=1.
$$

Moreover,

$$
\int_{([N(1-a)]+1)/N}^{1}\frac{1}{s}\,u_N^2\,ds=\mu((1-a,1])\,;
$$

hence, by (3.13) and (3.4), we have that

$$
\lim_{N\to+\infty}\int_r^1\frac{1}{s}\,u_N^2\,ds=\mu^a((r,1])\quad \text{a.e.}\,r\in(0,1-a)\,.
$$

Since $\{u_N\}$ is bounded in L^2 we can apply the Lesbegue's Theorem in the first term of (3.10) and by (3.12), (3.3) we have

$$\limsup_{N\to+\infty} G(u_N) \leq \overline{G}(u) + o(1)\,, \quad \text{as} \quad a \to 0^+\,. \tag{3.14}$$

Since $u^a \to u$ in $L^2_{\rm loc}(0,1)$, as $a \to 0^+$, we can conclude that, passing to a further subsequence, $\{u_N\}$ converges weakly to u in $L^2_{\rm loc}(0,1)$ as $N \to +\infty$ and

$$\limsup_{N\to+\infty} G(u_N) \leq \overline{G}(u)$$

as desired. □

Remark 3.3 Note that the set of minimizers for $\overline{G}(u)$ is given only by $\{u \equiv 0\}$. It easily follows by testing the definition of $\overline{G}(u)$ with the Dirac measure $\mu = 2\lambda\delta_1$. In fact, by definition of Dirac measure, we have that $\mu(r,1] \equiv 2\lambda$ for every $r \in (0,1]$ while $\mu(0,r] = 0$ for every $r \in (0,1)$ and $\mu(0,1] = 2\lambda$. The constraint $\mu \geq (1/r)u^2 dr$ obliges u to be identically equal to zero. In this case a minimizing sequence for G, constructed as in (3.5), is given by

$$u_N(r) = \begin{cases} 0\,, & 0 \leq r \leq 1 - 1/N \\ \sqrt{2\lambda N r}\,, & r \in (1-1/N, 1]\,. \end{cases}$$

In fact, we may easily prove that $\{u_N\}$ converges weakly to $u = 0$ in $L^2(0,1)$ and

$$\lim_{N\to+\infty} \int_r^1 \frac{1}{s}\, u_N^2\, ds = \mu(r,1] \quad \text{a.e.}\, r \in (0,1]\,,$$

where $\mu = 2\lambda\delta_1$. Hence,

$$\lim_{N\to+\infty} G(u_N) = \int_0^1 r\,(\mu(r,1] - 2\lambda)^2\, dr = \overline{G}(0) = 0\,.$$

Moreover, if we consider the measures $\{\mu_N\}$ defined by

$$\mu_N(0,r] = \begin{cases} 0\,, & 0 < r \leq 1 - 1/N \\ 2\lambda\,, & 1 - 1/N < r \leq 1 \end{cases}\,,$$

we have, in particular, that

$$\mu_N(r,1] = \begin{cases} 2\lambda\,, & 0 \leq r \leq 1 - 1/N \\ 0\,, & 1 - 1/N < r \leq 1 \end{cases}\,,$$

and such measures satisfy the constraint $\mu_N \geq u_N^2/s\, ds$. Hence,

$$0 \leq \overline{G}(u_N) \leq \int_0^1 r\,(\mu_N(r,1] - 2\lambda)^2\, dr = 4\lambda^2 \int_{1-1/N}^1 r\ dr\,,$$

which implies that $\lim_{N\to+\infty} \overline{G}(u_N) = \overline{G}(0)$. The construction of the sequences $\{u_N\}$ and $\{\mu_N\}$ highlights the concentration phenomena at the boundary.

Theorem 3.4 (Γ-convergence result) *We have*

$$\Gamma(\text{w-}L^2_{\rm loc})\text{-}\lim_{\varepsilon\to 0} J_\varepsilon(u) = \frac{1}{2}\,\overline{G}(u) - \lambda^2$$

for every $u \in L^2_{\rm loc}(0,1)$.

Proof. Let $u \in L^2_{\rm loc}(0,1)$. By definition of $\overline{G}$ for every sequence $\{u_\varepsilon\}$ $L^2_{\rm loc}$-weakly converging to u we have

$$J_\varepsilon(u_\varepsilon) \geq \frac{1}{2}\,G(u_\varepsilon) - \lambda^2 \geq \frac{1}{2}\,\overline{G}(u_\varepsilon) - \lambda^2\,;$$

hence by the weak lower semicontinuity of $\overline{G}$ we get the liminf inequality.

Vice versa, let $u \in L^2_{\rm loc}(0,1)$ and let

$$u^a(r) = \begin{cases} u(r) & \text{if } a \leq r \leq 1-a \\ 0 & \text{otherwise} \end{cases}$$

with $0 < a < 1$; hence, $u^a \in L^2(0,1)$ and $u^a \to u$ in $L^2_{\rm loc}(0,1)$ as $a \to 0^+$. By Lemma 3.2, there exists a sequence $\{u_N\} \in L^2(\mathbb{R})$ weakly converging to u^a in $L^2(0,1)$ such that

$$\overline{G}(u^a) = \lim_{N\to+\infty} G(u_N)\,.$$

Let $\eta : \mathbb{R} \mapsto [0,+\infty)$ be a mollifier, we define $\eta_\varepsilon(r) = \frac{1}{\sqrt{\varepsilon}}\,\eta(\frac{r}{\sqrt{\varepsilon}})$ then $u_\varepsilon^N = u_N * \eta_\varepsilon \in C_c^\infty(\mathbb{R})$ and $u_\varepsilon^N \to u_N$ in $L^2(0,1)$ as $\varepsilon \to 0$ for every N. Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon^N) &= \lim_{\varepsilon\to 0}\Big(\varepsilon^2 \int_0^1 r\,(u_N * \eta'_\varepsilon)^2 + \frac{(u_N*\eta_\varepsilon)^2}{r}\, dr + \frac{1}{2}\,G(u_\varepsilon^N) - \lambda^2\Big) \\ &= \frac{1}{2}\lim_{\varepsilon\to 0} G(u_\varepsilon^N) - \lambda^2 = \frac{1}{2}\,G(u_N) - \lambda^2\end{aligned}$$

and, by the lower semicontinuity of the Γ-limsup and (3.3), we have that

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon\to 0} J_\varepsilon(u^a) &\leq \liminf_{N\to+\infty}\Big(\Gamma\text{-}\limsup_{\varepsilon\to 0} J_\varepsilon(u_N)\Big) \\ &\leq \frac{1}{2}\liminf_{N\to+\infty} G(u_N) - \lambda^2 = \frac{1}{2}\,\overline{G}(u^a) - \lambda^2 \leq \frac{1}{2}\,\overline{G}(u) - \lambda^2 + o(1)\end{aligned}$$

as $a \to 0^+$. We apply again the lower semicontinuity of the Γ-limsup to get, as $a \to 0^+$, that

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} J_\varepsilon(u) \leq \frac{1}{2}\,\overline{G}(u) - \lambda^2$$

which concludes the proof of the limsup inequality (see [7] Remark 1.29). □

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