Discrete regularity for elliptic equations on graphs

Sun-ra J.N. Mosconi *

May 28, 2001

Abstract

The paper deals with the regularity properties of solutions to elliptic difference equation on a graph G under suitable hypotheses on the geometry of the latter, namely a doubling condition and a Poincarè inequality involving discrete derivatives. The difference equation we deal with is of the kind $\sum_{x\sim y} a(x,y)(u(y)-u(x))+c(x)u(x)=f(x)$ where $x\sim y$ means that the vertices x and y of G are connected each other in the graph structure. The main hypotheses on the coefficients of the equation is that a is symmetric in its arguments and positive. A Harnack inequality for positive solutions is proved, thus giving an Hölder estimate on the oscillation of a solution in term of suitable L^p norms of the coefficients.

1 Introduction

This paper is devoted to the regularity properties of solution to certain difference equation on graphs. By a graph we will mean an at most countable set of vertices G, some of which are connected in a non-oriented way. More precisely, we are given a symmetric (around the diagonal) subset E of $G\times G$, called the set of the edges: in this way $(x,y)\in E$ means that the vertices x and y are connected, or adjacent each other, and in this case we write $x\sim y$. This structure gives a natural, integer-valued metric on the graph: given to vertices x and y the distance between them is given by least number of steps needed to go from one to the other passing through adjacent vertices, see (2).

The difference equations we deal with are of the kind

$$\sum_{y\sim x} a(x,y)\frac{(u(y)-u(x))}{h^2}+\sum_{y\sim x} b(x,y)\frac{(u(y)-u(x))}{h}+c(x)u(x)=f(x), \quad (1)$$

for some parameter $h>0$ and functions $a,b\in\mathbb{R}^E$, $c,f\in\mathbb{R}^G$. This kind of equation and the regularity property of its solutions arise in many questions regarding approximation problems, spectral graph theory and decay rate of Markov processes, see, to cite only few in the vast literature on these subjects,

*Scuola Normale Superiore, Piazza dei Cavalieri 7, 56100 Pisa, Italy. email: mosconi@cibs.sns.it

[9, 8, 7], [2], [9, 3, 4] respectively. By a regularity result we mean an estimate on the modulus of continuity (with respect to the metric structure described above) of the solutions of (1) depending only on some "structural" constants given by the coefficients of the equation. This kind of result is definitely not possible unless we specify somehow the coefficients in (1). The first condition we will impose to the equation is to be of elliptic type, i.e., the coefficient $a(x, y)$ in (1) has to be positive. However, example 2.2 shows that this is not jet sufficient to obtain reasonable regularity of the solutions. One natural way in which we can furthermore specify the equation is to ask the latter to be in a certain sense "variational", i.e. that the coefficient $a(x, y)$ is a symmetric function of its arguments. Regularity results in this direction have been obtained by [6] and [10] in the case that the only non-zero coefficient is the "second order" one a, under some geometrical assumption on the graph, namely a doubling condition and a Poincaré inequality. Examples (see [11]) suggest that these condition are in a certain sense necessary, and this is known to be true in the Riemaniann setting (see [13]). However a direct (in particular, non-parabolic) proof of this necessity result has not yet been found in this context.

The structure of the paper is as follows: in section 2 we set the problem and state the hypotheses assumed in the sequel. In section 3 we prove some lemmas helping us in performing difference calculus on the solutions, obtaining as a first result a bound on the BMO-norm of the logarithm of a solution. In section 4 we will prove a Caccioppoli inequality for the solutions, giving an integral maximum principle for the latter. In section 5, although it is by now a standard technique, for sake of completeness we deduce from the maximum principle the desired regularity result. We remark that the main difference between the approach adopted, for example, in [6] and ours is in introducing the parameter h in equation (1), in order to obtain suitable L^p bound condition on the lower order term of the equation; see Remark 2.1 or [11] for details.

This paper is taken out from author's degree thesis written under the supervision of professor M.Giaquinta and P.Tilli.

2 Setting of the problem and notations

By a graph we will mean a couple (G, E) of vertices and edges such that G (the vertex set) is a countable set and E (the edge set) is a symmetric subset of $G \times G$ disjoint from the diagonal, namely

$$(x, y) \in E \Rightarrow x \neq y \text{ and } (y, x) \in E.$$

We will often write $x \sim y$ for $(x, y) \in E$ and G for the couple (G, E), at least when this causes no confusion. For $A \subset G$, $\#A$ will be the number of elements of A if it is finite and $+\infty$ otherwise. Let us define a path $\gamma_{x,y}$ between two vertices x and y of a graph (G, E) as any finite set of vertices $\gamma_{x,y} = \{x_0, \dots, x_N\}$ such that $x_0 = x$, $x_N = y$ and $x_i \sim x_{i+1}$ for every $i = 0, \dots, N-1$. We say that (G, E) is connected if for every $x, y \in G$, there exists a path between x and y. *Henceforth every graph* (G, E) *will be connected and locally finite, i.e.* $\#\{y :$

$x \sim y\} < \infty \ \forall x \in G$.
We equip the graph (G, E) with a metric space structure on its vertex set given by the distance function

$$d(x, y) := \min_{\gamma_{x,y}} \{\#\gamma_{x,y}\} - 1. \tag{2}$$

We write $B_x(r)$ meaning the open ball with centre $x \in G$ and radius r with respect to this distance. Furthermore we let

$$KB_x(r) = B_x(Kr),$$

for every $K \geq 0$. Given any $f \in \mathbb{R}^G$, $x \in G$ and $r > 0$, we define

$$\mathrm{Osc}(f, x, r) = \max_{B_x(r)} f - \min_{B_x(r)} f.$$

A subgraph (G', E') of (G, E) is a graph (G', E') whose vertex set is a subset of G and such that if $(x, y) \in E'$, then $(x, y) \in E$. When we consider a subset G' of G without specifying any subgraph structure, we will always consider the graph (G', E'), given by

$$E' = \{(x, y) \in G' \times G' \subset G \times G : (x, y) \in E\}.$$

Every ball $B = B_x(r)$ of G has in this way a natural subgraph structure, and given a subgraph G' of G we write $B \subset G'$ if every vertex of B is in G'. The boundary of a subgraph Ω of G is defined as one of the following

$$\partial\Omega := \{x \in \Omega : \exists y \sim x, y \notin \Omega\} \subset G,$$

$$\partial_E\Omega := \{(x, y) \in E : x \in \Omega, y \notin \Omega\} \subset E,$$

and its interior as

$$\mathring{\Omega} = \{x \in \Omega : y \sim x \Rightarrow y \in \Omega\}.$$

We introduce now a useful parameter, which we will call *edge length*, associated to a graph. The edge length will be a positive real number h whose use is to distinguish between functions and derivatives, or else, between subsets and boundaries of subsets. Usually this number will be proportional to the inverse of the diameter of the graph. Let us fix a graph (G, E), an edge length $h \in \mathbb{R}^+$ and a *dimension* $\delta > 0$. With these settings we will consider the measures

$$\begin{aligned} \mu_\delta(A) := |A|_\delta = h^\delta \#A, \qquad & \text{for } A \subset G, \\ \tilde{\mu}_\delta(B) = h^{\delta-1} \#B, \qquad & \text{for } B \subset E, \end{aligned} \tag{3}$$

usually omitting the subscribed δ when the dimension is fixed. Given a function $f \in \mathbb{R}^G$, for any $\Omega \subset G$ and $p > 0$, the $L^p(\Omega)$ of f with respect to the measure μ will be denoted by $\|f\|_{L^p(\Omega)}$, or simply $\|f\|_p$ when $\Omega = G$.

We call a function $\mathbb{F} \in \mathbb{R}^E$ a *vector field* if $\mathbb{F}(x, y) = -\mathbb{F}(y, x)$, and given any field $\mathbb{F}$ we define its divergence (with respect to edge length h) as the function $\operatorname{div}_h \mathbb{F} \in \mathbb{R}^G$ defined by

$$G \ni x \to \operatorname{div}_h \mathbb{F}(x) := \frac{1}{h} \sum_{y \sim x} \mathbb{F}(x, y) \in \mathbb{R}.$$

We will frequently omit the subscribed h when the edge length is fixed. A particular case of vector field is the *derivative* of a function: given any function $f \in \mathbb{R}^G$ its derivative $D_h f$ is defined as the vector field

$$E \ni (x, y) \to D_h f(x, y) := \frac{f(y) - f(x)}{h} \in \mathbb{R},$$

and we will use the same convention about the subscribed h as for the divergence. One can easily prove the following statement (see [11]):

Proposition 2.1. *Suppose $\Omega \subset G$ is finite, and $\mathbb{F} \in \mathbb{R}^E$ is a vector field. Then the following Gauss-Green formula holds:*

$$\int_\Omega \operatorname{div} \mathbb{F} \, d\mu = \int_{\partial_E \Omega} \mathbb{F} \, d\tilde{\mu}.$$

We define a scalar product between two functions $\mathbb{F}, \mathbb{G} \in \mathbb{R}^E$ as the function $\mathbb{F} \cdot \mathbb{G} \in \mathbb{R}^G$ defined by

$$\mathbb{F} \cdot \mathbb{G}(x) := \frac{1}{2} \sum_{y \sim x} \mathbb{F}(x, y) \mathbb{G}(x, y),$$

and thus we denote by $|\mathbb{F}| \in \mathbb{R}^G$ the function pointwise obtained by taking the norm associated to this scalar product. When the dot is not present the product $\mathbb{F}\mathbb{G}$ between $\mathbb{F}$ and $\mathbb{G}$ in $\mathbb{R}^E$ is simply taken pointwise. Finally we define a map between functions on the graph and (symmetric) functions on the edges given by

$$\mathbb{R}^G \ni u \to u^* \in \mathbb{R}^E, \qquad u^*(x, y) = \frac{u(x) + u(y)}{2}. \tag{4}$$

Note that, with this setting, we have $Du^2(x, y) = 2u^* Du$. We deal with the following question:
Given $\Omega \subset G$, study the regularity properties of solutions $u \in \mathbb{R}^\Omega$ of

$$\operatorname{div}(\mathbf{a} Du)(x) + \mathbf{b} \cdot Du(x) + cu(x) = f(x), \qquad \forall x \in \mathring{\Omega}, \tag{5}$$

for some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^E$, $c, f \in \mathbb{R}^G$.
Since we are interested in local properties, we can suppose Ω finite. By a regularity result, we mean an estimate on the oscillation of the solutions, thus appealing to the metric structure given in (2). We observe that since Ω is finite, every function $u \in \mathbb{R}^\Omega$ is trivially Lipschitz continuous with a sufficiently high Lipschitz constant. What we are looking for is an estimate that involves only some

constant depending on the structure of the subgraph Ω and on the coefficients of equation (5). Such a result allows us to perform approximation processes taking care only of some "intrinsic" constants, thus bringing the regularity estimates to the limit. Let us formalise this regularity result.

Definition 2.1. Consider a sequence of finite graphs $\{G_n\}$ with edge length $h_n \to 0$. Given $f_n \in \mathbb{R}^{G_n}$, $n \in \mathbb{N}$, let, for $(x_n) \in \prod_n G_n$ and $r > 0$,

$$\mathrm{Osc}((f_n),(x_n),r) = \limsup_n \mathrm{Osc}(f_n, x_n, \frac{r}{h_n}).$$

We say that an equibounded sequence (f_n) is α-Hölder continuous if

$$\mathrm{Osc}((f_n),(x_n),r)) \leq Cr^{\alpha},$$

for some $C > 0$ and every $r > 0$.

Suppose we are given the following sequences of graph, edge length, coefficients and equations:

$$\begin{aligned} &\Omega_n \subset G_n, \quad h_n \to 0, \quad \mathbf{a}_n, \mathbf{b}_n \in \mathbb{R}^{E_n}, \quad c_n, f_n \in \mathbb{R}^{G_n} \\ &\mathrm{div}_{h_n}(\mathbf{a}_n D_{h_n} u_n) + \mathbf{b}_n \cdot D_{h_n} u_n + c_n u_n = f_n \qquad in\ \overset{\circ}{\Omega}. \end{aligned} \tag{6}$$

We will find in this paper condition on the geometries of Ω_n and on some L^p-norms of the coefficients, under which (u_n) is α-Holder continuous.

Remark 2.1. Let us consider the relationships between the L^p-norm of the coefficients and the seemingly "free" parameters h and δ. Given a sequence of equations as in (6), suppose for the moment that edge length in each equation is 1. In all concrete cases it holds

$$\operatorname*{diam}_n(\Omega_n) \to +\infty \quad and \quad \lim_n \frac{diam_n(\Omega_n)}{h_n} = k,$$

for some $k > 0$, and thus it is natural to suppose $h_n = diam(\Omega_n)$. Suppose furthermore we have some "natural" choice of dimension $\delta > 0$ for the whole sequence of subgraphs Ω_n. By scaling the equations with respect to edge length h_n, (6) becomes

$$\mathrm{div}_{h_n}(\mathbf{a}_n D_{h_n} u_n) + \mathbf{b}'_n \cdot D_{h_n} u_n + c'_n u_n = f'_n, \tag{7}$$

where

$$\mathbf{b}'_n = \frac{\mathbf{b}_n}{h_n}, \qquad c'_n = \frac{c_n}{h_n^2}, \qquad f'_n = \frac{f_n}{h_n^2}.$$

In order to get regularity results for (u_n), we will need, among other hypotheses, the following bounds:

$$\begin{aligned} \Lambda \geq \mathbf{a}_n \geq \lambda > 0, &\qquad \sup_n \|\mathbf{b}'_n\|_q < +\infty \\ \sup_n \|c'_n\|_{q/2} < +\infty, &\qquad \sup_n \|f'_n\|_{q/2} < +\infty, \end{aligned}$$

for some $q > \delta$. We remark that this implies that the coefficients in (6) go to zero at least at certain rate. Indeed, recalling (3), for the zero-order term we have

$$|f_n(x)|\, h_n^{2(\delta/q-1)} \leq (|f_n'(x)|^{q/2}\, h_n^{\delta})^{2/q} \leq \|f_n'\|_{q/2} \leq C < +\infty,$$

and thus $f_n \to 0$, whatever sequence $h_n \to 0$ we choose. Obviously analogous statements hold for the other terms (except, of course, for $\mathbf{a}_n$). Indeed, it easy to construct counterexamples to regularity when some of the lower order coefficients does not go to zero rapidly enough at some point $x \in G$ (see [11]).

We are now going to list the hypotheses we assume in order to obtain Hölder continuity of the solutions of (9).

$\boxed{\textit{1) Symmetry}}$

In order to obtain the regularity result, we are going to apply the De Giorgi-Nash-Moser method (see [5, 12]) to solutions of equation (5), and thus we will need a variational formulation of this equation. This can be done applying proposition 2.1 but, in order to satisfy its hypothesis, $\mathbf{a}Du$ has to be a vector field: we accomplish this condition by requiring $\mathbf{a}$ to be a symmetric function of the edges:

$$\mathbf{a}(x,y) = \mathbf{a}(y,x).$$

This condition characterizes equation (5) as a "variational equation". Counterexamples shows that this is the only natural hypothesis to assume in this abstract setting: indeed we recall a classical example.

Example 2.2. Let us consider the graph $\mathbb{Z}$ with its Cayley graph structure (see [11] for the definition of Cayley graph) given by the generators $S := \{1, -1\}$, i.e.

$$E := \{(x,y) \in \mathbb{Z}^2 : x - y \in S\}.$$

Consider furthermore the sequence of subgraphs $G_n = [-n, +n] \cap \mathbb{Z}$ with the natural subgraph structure. Let p and q be two positive real numbers such that $p > q$, $p + q = 1$ and consider the Dirichlet problems

$$\mathrm{div}_{h_n}(\mathbf{a}_n D u_n) = 0 \quad \text{in } \overset{\circ}{G}_n, \quad u_n(-n) = 0, \quad u_n(n) = 1,$$

where

$$\begin{aligned} \mathbf{a}_n(x, x+1) &= \begin{cases} q & if\ -n \leq x < 0, \\ p & if\ n \geq x > 0, \\ 1/2 & if\ x = 0, \end{cases} \\ \mathbf{a}_n(x, x-1) &= \begin{cases} p & if\ -n \leq x < 0, \\ q & if\ n \geq x > 0, \\ 1/2 & if\ x = 0. \end{cases} \end{aligned}$$

Note that since the equation involves only the second order term, it is invariant by edge length, and hence we can assume the natural hypothesis $h_n = 1/2n$. Let us now explicitly compute the solutions: one can easily verify that, letting $\alpha = p/q$

$$u_n(x) = \begin{cases} c_n \sum_{j=0}^{|x|-1} \alpha^j & if -n < x \leq 0, \\ u_n(0) + c_n \alpha^n \sum_{j=0}^{x-1} \alpha^{-j} & if 0 < x < n, \end{cases}$$

with

$$c_n = \frac{\alpha - 1}{2(\alpha^n - 1)},$$

is the (unique) solution of our Dirichlet problem. Noting that

$$\mathrm{Osc}(u_n, 0, 1) = \frac{\alpha^n}{\alpha^n - 1} \frac{\alpha - 1}{2} \to \frac{\alpha - 1}{2},$$

one gets, for $(x_n) \equiv (0) \in \prod G_n$

$$\mathrm{Osc}(u_n, (x_n), r) = \limsup_n \mathrm{Osc}(u_n, 0, 2nr) \geq \limsup_n \mathrm{Osc}(u_n, 0, 1) = \frac{\alpha - 1}{2},$$

and thus $\{u_n\}$ is equibounded but not Hölder continuous. Note that if $p = q = 1/2$ our variational hypothesis holds, and indeed in this case $\alpha - 1 = 0$.

2) Ellipticity

We furthermore suppose the equation being *strictly elliptic*, i.e.

$$\mathbf{a}(x, y) = \mathbf{a}(y, x) \geq \lambda > 0.$$

By virtue of Remark (2.1), we can strengthen the ellipticity hypothesis, supposing the equation being of *positive type*, i.e.

$$\mathbf{a}(x, y) + h\mathbf{b}(x, y) > 0, \qquad \forall y \sim x, \tag{8}$$

holds for any $x \in \mathring{\Omega}$, since edge length can be made arbitrarily small. Note that by virtue of Proposition 2.1 equation (5) is equivalent to the following weak variational formulation:

$$\int_G \mathbf{A}(x, u, Du) \cdot Dv - B(x, u, Du) v \, d\mu = 0, \quad \forall v \quad s.t. \ \operatorname{supp} v \subset \mathring{\Omega}, \tag{9}$$

where

$$\begin{aligned} \mathbf{A}(x, u, Du)(z, y) &= \delta_x(z) \mathbf{a}(x, y) Du(x, y), \\ B(x, u, Du) &= \mathbf{b} \cdot Du(x) + c(x) u(x) - f(x). \end{aligned}$$

Letting, for $k > 0$ fixed,

$$\overline{u} = |u| + k \ , \quad \overline{a} = \|\mathbf{b}\|^2 + |c| + k^{-1} |f| \ , \tag{10}$$

it is easy to check that the following estimates hold for every $\varepsilon > 0$:

$$\begin{cases} Du \cdot \mathbf{A}(x,u,Du) \geq |Du|^2 \ , \\ |\mathbf{A}(x,u,Du)(x,y)| \leq \Lambda \, |Du| \qquad\qquad (\Lambda \geq 1), \\ |\overline{u}B(x,u,Du)| \leq \epsilon \, |Du|^2 + \frac{1}{\epsilon} \overline{a}\overline{u}^2 . \end{cases} \tag{11}$$

$\boxed{\textit{3) Doubling}}$

We will suppose that G has the doubling volume property:

$$\exists \ C_1 \quad \text{s.t.} \quad \forall B_x(n) \quad \# B_x(2n) \leq C_1 \# B_x(n). \tag{12}$$

Note, by the way, that this implies that every vertex has at most m neighbouring vertices for some $m < \infty$ fixed. Henceforth we will suppose that

$$\delta := \frac{\log C_1}{\log 2} > 2,$$

although it is not strictly necessary, δ being therefore the dimension of G. We will frequently use the following well known lemma, which holds in any metric space equipped with a doubling measure (see for example [1]).

Lemma 2.3. *The graph G has doubling property if and only if there exists constants δ, C such that*

$$\frac{|B_x(r)|}{|B_y(R)|} \geq C \left(\frac{r}{R} \right)^{\delta} \quad \forall x, y \in G \ , \quad R \geq r \ , \quad x \in B_y(R).$$

Note that taking in the previous lemma

$$r = 1, \qquad y = x, \qquad R = n,$$

we obtain

$$|B(n)| \leq C(hn)^{\delta}. \tag{13}$$

$\boxed{\textit{4) Poincaré inequality}}$

In addition to (12), we will suppose another strong geometrical assumption, i.e. that G supports a Poincaré inequality such as

$$\exists C_2 \ \text{ s.t. } \ \forall B \quad \int_B |f(x) - f_B|^2 \, d\mu \leq C_2 h^2 n^2 \int_{2B} |Df|^2 \, d\mu, \tag{14}$$

where f_B stands for the mean of f on B. We will use a result proved in [6]:

Theorem 2.4. *(Sobolev inequality) Let G satisfies hypotheses* (12) *and* (14)*. For every $f : G \to \mathbb{R}$ such that* $\operatorname{supp} f \subset B_{x_0}(n-1)$*, we have*

$$\left(\fint_{B_{x_0}(n)} |f|^{2^*} \, d\mu \right)^{2/2^*} \leq C_2' \left(h^2 n^2 \fint_{B_{x_0}(n)} |Df|^2 \, d\mu + \fint_{B_{x_0}(n)} f^2 d\mu \right), \quad (15)$$

where $2^* = 2\delta/2 - \delta$*.*

Using estimate (13), formula (15) of Theorem 2.4 becomes

$$\left(\int_{B_{x_0}(n)} |f|^{2^*} \, d\mu \right)^{2/2^*} \leq C_2' \left(\int_{B_{x_0}(n)} |Df|^2 \, d\mu + \frac{1}{h^2 n^2} \int_{B_{x_0}(n)} f^2 d\mu \right), \quad (16)$$

and we will use the latter in the sequel.

3 Logarithm's BMO-norm

In this section we will study the BMO-norm of $\log(u)$, where u is a solution of 9 in $\overset{\circ}{\Omega} \subset \Omega \subset G$. First of all, let us introduce the BMO-norm of a function:

Definition 3.1. Let $f \in L^1(G)$; the BMO-norm (Bounded Mean Oscillation) of f in $\Omega \subset G$ is defined as:

$$\|f\|_{BMO(\Omega)} = \sup_{B \subset \Omega} \left\{ \frac{1}{|B|} \int_B |f(x) - f_B| \, d\mu \right\}.$$

We will need a generalization of the classical Jhon-Niremberg Lemma, proved in [6] .

Lemma 3.1. *There exists two constants α and β depending only on* (12)*, such that for every $f \in \mathbb{R}^G$ it holds*

$$\fint_B \exp(\alpha f(x)/\|f\|_{BMO(4B)}) \, d\mu \cdot \fint_B \exp(-\alpha f(x)/\|f\|_{BMO(4B)}) \, d\mu \leq \beta^2.$$

We remark that:
a structural constant *will mean a constant depending only on the geometry of G (i.e.* (12) *and* (14)*) and on the coefficients of equation* (9)*.*

We begin with a first regularity result, which is definitely not sufficient to prove Hölder regularity.

Lemma 3.2. *Let $u > 0$ be a solution of* (9)*, $x \in \overset{\circ}{\Omega}$ and $k > 0$. There exists a structural constant $\theta > 0$, (thus independent of k), such that for every $y \sim x$ we have $\overline{u}(x) \geq \theta \overline{u}(y)$ (recall* (10) *for the definition of $\overline{u}$).*

Proof By hypothesis (8) we can choose $\varepsilon_0 > 0$ such that $\mathbf{a} + h\mathbf{b} \geq \varepsilon_0$. Recalling (5) and splitting c as $c = c^+ - c^-$, ($c^+ \geq 0$, $c^- \geq 0$) we have that u pointwise satisfies

$$\operatorname{div}(\mathbf{a}Du) + \mathbf{b}Du - (c^- + 1 + \frac{f^-}{u})u = -(c^+ + 1 + \frac{f^+}{u})u.$$

Defining $c' = c^- + 1 + f^- \geq 1$, we then obtain

$$\operatorname{div}(\mathbf{a}D\overline{u}) + \mathbf{b}D\overline{u} - c'\overline{u} \leq 0,$$

or, more explicitly

$$\begin{aligned} 0 \quad \geq \quad & -\overline{u}(x)\left(c'(x) + \frac{1}{h^2}\sum_{y\sim x}\mathbf{a}(x,y) + \frac{1}{h}\sum_{y\sim x}\mathbf{b}(x,y)\right) + \\ & + \frac{1}{h^2}\sum_{y\sim x}(\mathbf{a}(x,y) + h\mathbf{b}(x,y))\overline{u}(y). \end{aligned}$$

Hence, using the definition of ε_0 and multiplying both sides by h^2, we obtain

$$\overline{u}(x)\left(h^2c'(x) + \sum_{y\sim x}\mathbf{a}(x,y) + h\mathbf{b}(x,y)\right) \geq \varepsilon_0 u(y),$$

for every $y \sim x$, and the lemma follows taking the maximum value attained by the term multiplying $u(x)$. □

Remark 3.1. Note that, by virtue of Remark 2.1, in an hypothetic approximation process the constant $\theta(|\mathbf{b}|, |c|, h)$ has a good asymptotic behaviour:

$$\lim_{h\to 0^+}\theta(|\mathbf{b}|, |c|, h) = \frac{\lambda}{m\Lambda}\ .$$

Recalling (4), we now use the previous lemma to perform approximate differential calculus on positive solutions of (9). Note that none of the last two points in the following proposition holds for generic functions. In particular, the constants c_2 and c_4 degenerate for $\theta \to 0$.

Proposition 3.3. *(Chain rule)*
Let $u : \Omega \to \mathbb{R}$ be a positive function and θ as in Lemma 3.2.

1. *For every x, $y \in \Omega$*

$$|D(\log(u))(x,y)| \leq \left|(\frac{1}{u})^*(x,y)Du(x,y)\right|.$$

2. *If u is a positive solution of (9) in Ω , for every $p > 0$ there exist constants $c_1(p)$, $c_2(p,\theta)$ such that $\forall k > 0$, x, $y \in \overset{\circ}{\Omega}$*

$$c_1\left|(p\overline{u}^{p-1})^*(x,y)D\overline{u}(x,y)\right| \geq |D(\overline{u}^p)(x,y)| \geq c_2\left|(p\overline{u}^{p-1})^*(x,y)D\overline{u}(x,y)\right|.$$

3. *If u is a positive solution of* (9) *in* $\overset{\circ}{\Omega}$, *for every $p < 0$ there exist constants $c_3(p)$, $c_4(p,\theta)$ such that* $\forall k > 0$, x, $y \in \overset{\circ}{\Omega}$

$$c_3 \left|(p\overline{u}^{p-1})^*(x,y)D\overline{u}(x,y)\right| \geq |D(\overline{u}^p)(x,y)| \geq c_4 \left|(p\overline{u}^{p-1})^*(x,y)D\overline{u}(x,y)\right|.$$

Proof The estimates being trivial if $u(x) = u(y)$, we can suppose $u(x) < u(y)$ without loss in generality.
1) Letting $t = u(y)/u(x)$, $t > 1$, we simply have :

$$\frac{1}{2}\left(\frac{1}{u(y)} + \frac{1}{u(x)}\right)(u(y) - u(x)) = \frac{1}{2}\left(\frac{1}{t} + 1\right)(t-1) \geq \log(t).$$

2) Letting $s = \overline{u}(x)/\overline{u}(y) < 1$ we want to find c_1, c_2 such that

$$c_1 \geq \frac{2}{p}\frac{1 - s^p}{(1 + s^{p-1})(1-s)} \geq c_2 \quad \forall s \in [\theta, 1).$$

But the function we want to bound is positive and continuous in $[\theta, 1]$, and so it achieves finite maximum and positive minimum.
3) The proof is similar to the previous one and will be omitted. $\square$

Remark 3.2. Explicitly computing the constants c_1, c_2, c_3, c_4 of the previous Proposition one gets:

$$c_1(|p|) = c_3(|p|) = \begin{cases} 2 & \text{if } |p| \geq 1, \\ \dfrac{2}{|p|} & \text{if } |p| \leq 1, \end{cases}$$

and

$$c_2(|p|) = c_4(|p|) = \begin{cases} \dfrac{\theta}{|p|} & \text{if } |p| \geq 1, \\ \theta & \text{if } |p| \leq 1. \end{cases}$$

where θ is given in Lemma 3.2.

Henceforth we will denote by u a *positive* solution of (9) in $\overset{\circ}{\Omega} \subset G$ and by B a ball such that $4B \subset \overset{\circ}{\Omega}$.

Lemma 3.4. *Let $\eta \geq 0$ be such that* $\operatorname{supp}\eta \subset B$. *There exists a structural constant C such that, for every $k > 0$, it holds:*

$$\int_B |\eta^* D\log(\overline{u})|^2 \, d\mu \leq C \int_B \left(\overline{a}\eta^2 + |D\eta|^2\right) d\mu. \tag{17}$$

Proof Using $v = \eta^2 1/\overline{u}$ as test function in (9) we have

$$-\int_B (\eta^2)^* D(\frac{1}{\overline{u}})\mathbf{a} \cdot D\overline{u} d\mu = 2\int_B (\frac{1}{\overline{u}})^* \eta^* D\eta \mathbf{a} \cdot D\overline{u} d\mu - \int_B \eta^2 \frac{1}{\overline{u}} B(x, \overline{u}, D\overline{u}) d\mu.$$

Applying Young's inequality and using (11) we obtain the estimate

$$
\begin{aligned}
-\int_B (\eta^2)^* D(\frac{1}{\overline{u}}) \cdot D\overline{u}\, d\mu \quad &\le \quad \Lambda(\delta+4\epsilon)\int_B \left| (\frac{1}{\overline{u}})^* (\eta^2)^* D\overline{u} \right|^2 d\mu + \\
&\quad + \int_B \left(\frac{1}{\epsilon}\overline{a}\eta^2 + \frac{\Lambda}{\delta}\left|D\eta\right|^2 \right) d\mu,
\end{aligned}
$$

where we used the inequality $D(\overline{u}^{-1})(x,y)D\overline{u}(x,y) \le 0$. We now use Proposition 3.3, point 1, for the left hand term whereas for the first term on the right we choose ϵ and δ such that $2\Lambda(\delta+4\epsilon) \le c_4(-1)$ and apply Proposition 3.3, point 3; we therefore obtain

$$
\Lambda(\delta+4\epsilon)\int_B \left|\eta^* D\log(\overline{u})\right|^2 d\mu \le \int_B \left(\frac{1}{\epsilon}\overline{a}\eta^2 + \frac{\Lambda}{\delta}\left|D\eta\right|^2 \right) d\mu,
$$

and this gives the claim with a constant $C \simeq \Lambda^2/\theta^2$. □

Corollary 3.5. *There exists a structural constant C such that for every $k > 0$ it holds:*

$$
\fint_B \left|\log(\overline{u}) - \log(\overline{u})_B\right| d\mu \le C.
$$

Proof Let $B = B_{x_0}(r)$ and let us define the test function $\eta_{r,s}$ in (17) as

$$
\eta_{r,s}(y) = \begin{cases} 1 & \text{if } y \in B_x(r), \\ 1 - \frac{d(y,B)}{s} & \text{if } y \in B_x(r+s) \setminus B_x(r), \\ 0 & \text{if } y \notin B_x(r+s), \end{cases} \tag{18}
$$

and observe that $\left|D\eta_{r,s}\right|^2 \le m/4s^2h^2$; substituting in (17) η with $\eta_{r,r}$ we have

$$
\begin{aligned}
\int_B \left|D\log(\overline{u})\right|^2(x)\, d\mu \quad &\le \quad C\int_{2B}\left(\overline{a}\eta_{r,r}^2 + \left|D\eta_{r,r}\right|^2\right) d\mu \\
&\le \quad C\left(\|\overline{a}\|_{\delta/2}\left(\int_{2B}\eta_{r,r}^{2^*} d\mu\right)^{2/2^*} + \int_{2B}\left|D\eta_{r,r}\right|^2 d\mu \right) \\
&\le \quad C(\|\overline{a}\|_{\delta/2}+1)\left(\int_{2B}\frac{\eta_{r,r}^2}{h^2r^2}d\mu + \int_{2B}\left|D\eta_{r,r}\right|^2 d\mu\right) \\
&\le \quad Cm(\|\overline{a}\|_{\delta/2}+1)\frac{|B|}{h^2r^2},
\end{aligned}
$$

where we have used Remark 16. We now obtain the claim applying Hölder and Poincaré inequalities:

$$
\begin{aligned}
\fint_B \left|\log(\overline{u}) - \log(\overline{u})_B\right| d\mu \quad &\le \quad \frac{1}{|B|^{1/2}}\left(Cr^2h^2\int_B \left|D\log(\overline{u})\right|^2 d\mu\right)^{1/2} \\
&\le \quad Cm^{1/2}(\|\overline{a}\|_{\delta/2}+1)^{1/2}.
\end{aligned}
$$

□

Theorem 3.6. *(Weak Harnack Inequality). There exist two structural constants $\tilde{p} > 0$, $\tilde{C}$, such that for every $k > 0$ it holds:*

$$\left(\fint_B \overline{u}^{\tilde{p}} \, d\mu \right)^{1/\tilde{p}} \leq \tilde{C} \left(\fint_B \overline{u}^{-\tilde{p}} \, d\mu \right)^{-1/\tilde{p}}.$$

Proof Applying Lemma 3.1 to $\log(\overline{u})$, we have

$$\fint_B \overline{u}^{\alpha / \| \log(\overline{u}) \|_{BMO(4B)}} \, d\mu \fint_B \overline{u}^{-\alpha / \| \log(\overline{u}) \|_{BMO(4B)}} \, d\mu \leq \beta^2.$$

Note that, by the previous corollary, $\| \log(\overline{u}) \|_{BMO(4B)} \leq C$; hence, letting $\tilde{p} = \alpha / C$ and $\tilde{q} = \alpha / \| \log(\overline{u}) \|_{BMO(4B)}$ $(\tilde{p} \leq \tilde{q})$ and applying Hölder inequality we obtain

$$\left(\fint_B \overline{u}^{\tilde{p}} d\mu \right)^{1/\tilde{p}} \leq \left(\fint_B \overline{u}^{\tilde{q}} d\mu \right)^{1/\tilde{q}} \leq \beta^2 \left(\fint_B \overline{u}^{-\tilde{q}} d\mu \right)^{-1/\tilde{q}} \leq \beta^2 \left(\fint_B \overline{u}^{-\tilde{p}} d\mu \right)^{-1/\tilde{p}}.$$

□

4 The local maximum principles

In this section we will prove a Caccioppoli inequality, in order to obtain an integral maximum principle for positive solutions of the elliptic equation. Although infinite iteration is clearly not possible in a discrete space such as a graph, due to this very discrete structure finite iterations suffice to obtain a bound for the sup-norm in term of (sufficiently strong) L^p norms, at least (as in Lemma 4.2) in a controlled way. Using this estimate, Caccioppoli inequality helps us in performing iterative controls between stronger and stronger L^p norms of the solution until Lemma 4.2 applies, thus giving the desired maximum principle.

Throughout this section u will be a *positive* solution of equation (9) in $\Omega \subset G$.

Lemma 4.1. *(Caccioppoli inequality)*
Let $\beta \neq -1$ and $\eta \geq 0$ s.t. $\operatorname{supp} \eta \subset B \subset \mathring{\Omega}$. Then for every $k > 0$ we have:

$$\int_B \left| \eta^* D \overline{u}^{\beta} \right|^2 \, d\mu \leq C(|\beta|) \int_B (\overline{a} \eta^2 + |D\eta|^2) \left| (\overline{u}^{\beta})^* \right|^2 \, d\mu. \tag{19}$$

Proof The proof runs similarly to the one of Lemma 3.4. Consider as test function in (9)

$$v = \eta^2 \overline{u}^p, \qquad p = 2\beta - 1.$$

A straightforward application of Young inequality gives, for $\theta > 0$

$$\mathbb{F} \cdot (\mathbb{G}(fg)^*)(x) \leq \theta \left| \mathbb{F} f^* \right|^2 (x) + \frac{1}{\theta} \left| \mathbb{G} g^* \right|^2 (x),$$

for every couple of vector fields $\mathbb{F}$, $\mathbb{G}$ and nonnegative functions f, g. Letting then $\mathbb{F} = \eta^* D\overline{u}$, $\mathbb{G} = D\eta$, $f = \overline{u}^{(p-1)/2}$, $g = \overline{u}^{(p+1)/2}$ and substituting in (9), we obtain

$$\begin{aligned}
\pm \int_B (\eta^2)^* D(\overline{u}^p) \cdot D\overline{u}\, d\mu \quad &\leq \quad \Lambda\Big(\int_B \theta \left|(\overline{u}^{(p-1)/2})^* \eta^* D\overline{u}\right|^2 d\mu + \\
&\qquad + \frac{1}{\theta} \int_B \left|(\overline{u}^{(p+1)/2})^* D\eta\right|^2 d\mu \Big) + \\
&\qquad + \int_B \eta^2 \overline{u}^{p-1} \left|\overline{u} B(x,u,Du)\right| d\mu,
\end{aligned}$$

with the sign depending on whether p is positive or negative (note that if $p < 0$, $D(\overline{u}^p) \cdot D\overline{u} \leq 0$).
Using the estimates (11), the Chain Rule, and noting that for every $f \geq 0$, $f \leq 2\left|(f^{1/2})^*\right|^2$, we find out, for any $\varepsilon > 0$ and $\theta > 0$

$$\begin{aligned}
{c_{2,4}}^2(|p|) \int_B \left|\eta^* (\overline{u}^{(p-1)/2})^* D\overline{u}\right|^2 d\mu \quad &\leq \quad \pm \int_B (\eta^2)^* D(\overline{u}^p) \cdot D\overline{u}\, d\mu \\
&\leq \quad (\Lambda\theta + 16\epsilon) \int_B \left|\eta^* (\overline{u}^{\beta-1})^* D\overline{u}\right|^2 d\mu + \\
&\quad + \int_B \left(\frac{\Lambda}{\theta} |D\eta|^2 + \frac{2}{\epsilon} \overline{a} \eta^2 \right) \left|(\overline{u}^\beta)^*\right|^2 d\mu,
\end{aligned}$$

(recalling that $\beta = (p+1)/2$); choosing ϵ and θ properly we then have

$$\int_B \left|\eta^* (\overline{u}^{\beta-1})^* D\overline{u}\right|^2 d\mu \leq C \int_B \left(|D\eta|^2 + \overline{a}\eta^2 \right) \left|(\overline{u}^\beta)^*\right|^2 d\mu,$$

where $C = K(\Lambda/c(\pm|p|))^2$. Applying once more the Chain Rule we finally obtain the claim. $\square$

Remark 4.1. Using Remark 3.2 we estimate the constant of the previous lemma as:

$$C(|\beta|) = \begin{cases} \tilde{C}\,|\beta|^2 & \text{if } |\beta| > 1, \\ \tilde{C}\,|\beta|^{-2} & \text{if } |\beta| < 1. \end{cases}$$

What we need now is a bound for the maximum in a ball of a given function f in term of its L^p mean. The fact that we have a discrete doubling measure actually allows such an estimate, at least for "sufficiently high" p's.

Lemma 4.2. *For every $r > 0$ there exists a constant $\tilde{q}(r) > 0$ such that for every $f \geq 0$, $B = B_x(r) \subset G$, it holds:*

$$\left(⨍_B f^{\tilde{q}}\, d\mu \right)^{1/\tilde{q}} \geq C \max_B f;$$

$$\left(⨍_B f^{-\tilde{q}}\, d\mu \right)^{-1/\tilde{q}} \leq C \min_B f.$$

Moreover $\tilde{q}(r)$ has a logarithmic behaviour in r.

Proof We can suppose $f > 0$, since the second inequality reduces to $0 \leq 0$; hence, being $\min f^{-1} = (\max f)^{-1}$, it is enough to prove the first one. Let $i = i(r)$ be such that $2^{i-1} \leq r \leq 2^i$: we have then

$$|B_x(r)| \leq \left|B_x(2^i)\right| \leq C_1 \left|B_x(2^{i-1})\right| \leq \cdots \leq C_1^i \left|B_x(1)\right| \leq C_1^i m h^\delta,$$

(where δ is given in Lemma 2.3). Hence, for $\tilde{q}(r) \geq i(r)$,

$$\left(\fint_B f^{\tilde{q}}\, d\mu\right)^{1/\tilde{q}} \geq h^{\delta/\tilde{q}} \max_B f \, |B|^{-1/\tilde{q}} \geq \max_B f C_1^{-1} m^{1/i}.$$

The last statement follows from the inequality $i \leq \log(2r)$. □

Let $\delta > 2$ be the dimension of G and $q' > \delta$. Letting

$$\frac{1}{q} = \frac{1}{2}\left(\frac{1}{q'} + \frac{1}{\delta}\right).$$

we have then $\delta < q < q'$. We therefore define

$$\mu := \delta\left(\frac{2}{q} - \frac{2}{q'}\right) = 1 - \frac{\delta}{q'}, \tag{20}$$

and we will furthermore let

$$k = k(x,r) = (hr)^\mu \|f\|_{L^{q'}(B_x(r))}. \tag{21}$$

Remark 4.2. Note that, with the choice (21) of k, $\|\overline{a}\|_{L^{q/2}(B_x(r))}$ depends only on $\mathbf{b}$, Λ and c, and not on f; indeed (recalling definition (10)) we can bound the integral term involving f as follows:

$$\frac{1}{k(x,r)}\|f\|_{L^{q/2}(B_x(r))} \leq \frac{1}{k(x,r)}\|f\|_{L^{q'/2}(B_x(r))} \left|B_x(r)\right|^{2/q-2/q'} \leq C,$$

where we have used (20) and the polynomial growth of the volume $|B_x(r)| \leq C(hr)^\delta$. Therefore, with this choice of k, the constants in Theorem 3.6 depend only on the quantities Λ, $\|\mathbf{b}\|_q$ and $\|c\|_{q/2}$.

Theorem 4.3. *(Local maximum principle)*
For every $\tilde{p} > 0$ there exist two constants C and τ such that, if $B_x(\tau r) \subset \mathring{\Omega}$ and u is a positive solution to (9) in $\mathring{\Omega}$, then

$$\max_{B_x(r)} u \leq C\left(\fint_{B_x(\tau r)} u^{\tilde{p}}\, d\mu\right)^{1/\tilde{p}} + Ck(x,r),$$

holds, where k is given by (21).

Proof. Let $v = \overline{u}^{\beta} = (u + k(x,r))^{\beta}$, $B := B_x(R)$ and $\eta \geq 0$ with $\operatorname{supp} \eta \subset B$. Using Sobolev and Young inequality, we have

$$\begin{aligned}
\left(\int_B |\eta v|^{2^*} \, d\mu\right)^{2/2^*} &\leq C(\int_B |D(\eta v)|^2 \, d\mu + \int_B \frac{|\eta v|^2}{h^2 R^2} d\mu) \\
&\leq 2mC \int_B (|\eta^* Dv|^2 + |v^* D\eta)|^2 + \frac{|\eta v|^2}{h^2 R^2}) d\mu.
\end{aligned}$$

We can now use Caccioppoli inequality to estimate the first term on the right, obtaining

$$\left(\int_B |\eta v|^{2^*} \, d\mu\right)^{2/2^*} \leq C(|\beta|) \int_B \left[(\overline{a}\eta^2 + |D\eta|^2) \, |v^*|^2 + \frac{|\eta v|^2}{h^2 R^2} \right] \, d\mu.$$

Using Hölder and Interpolation inequality we estimate the first term on the right, getting, for $2 < 2q/(q-2) < 2^*$, $\sigma = -q^*/q$ and $\varepsilon > 0$

$$\begin{aligned}
\int_B \overline{a}(\eta v)^2 d\mu &\leq \left(\int_B \overline{a}^{\frac{q}{2}} d\mu\right)^{\frac{2}{q}} \left(\int_B (\eta v)^{\frac{2q}{(q-2)}} d\mu\right)^{\frac{(q-2)}{q}} \\
&\leq \left(\int_B \overline{a}^{\frac{q}{2}} d\mu\right)^{\frac{2}{q}} \left(\epsilon \left(\int_B (\eta v)^{2^*} d\mu\right)^{\frac{1}{2^*}} + \frac{1}{\epsilon^{\sigma}} \left(\int_B (\eta v)^2 d\mu\right)^{\frac{1}{2}}\right)^2,
\end{aligned}$$

and hence we finally have

$$\begin{aligned}
\|\eta v\|_{L^{2^*}(B)} &\leq C(|\beta|)\epsilon^{-2\sigma} \|\overline{a}\|_{q/2} \int_B (\eta^2 + \frac{\eta^2}{h^2 R^2} + |D\eta|^2) \, |v^*|^2 \, d\mu + \\
&\quad + C(|\beta|)\epsilon^2 \|\overline{a}\|_{q/2} \left(\int_B |\eta v|^{2^*} \, d\mu\right)^{2/2^*}.
\end{aligned}$$

Note that, by Remark 4.2, $\|\overline{a}\|_{q/2}$ depends only on $\mathbf{b}$, Λ and c. We can, with an appropriate choice of ϵ, bring the last term on the right of the previous formula to the left, obtaining

$$|\eta v|^2_{L^{2^*}} \leq C(|\beta|)^{\sigma+1} \|\overline{a}\|^{\sigma+1}_{q/2} \int_B (\eta^2 + \frac{\eta^2}{h^2 R^2} + |D\eta|^2) \, |v^*|^2 \, d\mu.$$

Letting $2^* = 2\chi$, $\chi > 1$ and $2\beta = p$, let us use as test function the one given in (18) with $R = r + s$, $r \geq s$. Note that with this choice $\eta = \eta_{r,s}$ we have

$$\eta^2 + \frac{\eta^2}{h^2 R^2} + |D\eta|^2 \leq \frac{C}{h^2 s^2},$$

at least for sufficiently small h. Applying this estimate we therefore obtain

$$\|\overline{u}\|_{L^{p\chi}(B(r))} \leq \left(\frac{C(p)}{h^2 s^2}\right)^{1/p} \|\overline{u}\|_{L^p(B(r+s))}, \tag{22}$$

and hence

$$\left(\fint_{B(r)} \overline{u}^{p\chi}\, d\mu\right)^{1/p\chi} \le \left(\frac{C(p)}{h^2 s^2}\right)^{1/p} \frac{|B(r+s)|^{1/p}}{|B(r)|^{1/p\chi}} \left(\fint_{B(r+s)} \overline{u}^{p}\, d\mu\right)^{1/p}.$$

We now use Lemma 2.3 to estimate the term $\frac{|B(r+s)|^{1/p}}{|B(r)|^{1/p\chi}}$:

$$\frac{|B(r+s)|^{1/p\chi}}{|B(r)|^{1/p\chi}} |B(r+s)|^{1/p-1/p\chi} \le C_1^2(1+\frac{s}{r})^{\delta/p\chi} |B(r+s)|^{2/p\delta}.$$

But, again from Lemma 2.3, we have $|B(r+s)|^{2/p\delta} \le C_1^2(h(r+s))^{2/p}$, therefore (22) becomes (recalling $r \ge s$):

$$\left(\fint_{B(r)} \overline{u}^{p\chi}\, d\mu\right)^{1/p\chi} \le C(p)(1+\frac{s}{r})^{\delta/p\chi}(\frac{r}{s})^{2/p} \left(\fint_{B(r+s)} \overline{u}^{p}\, d\mu\right)^{1/p}. \tag{23}$$

We can now begin the iteration process.
Without loss in generality, we can suppose $r = 2^n$, $n \ge 1$. Being $\tilde{p}$ fixed, we want to iterate (23) N times in such a way that Lemma 4.2 holds: because of the logarithmic behaviour mentioned in Lemma 4.2, N should therefore be such that $\chi^N \tilde{p} \ge n+1$. Let us choose

$$\chi^N \tilde{p} = 2n, \qquad N = \log(2n/\tilde{p})/\log(\chi), \tag{24}$$

and define

$$B_i = B(r_i), \quad r = r_1 \le r_2 \cdots \le r_N, \quad r_j \ge s_j = r_{j+1} - r_j, \quad j \ge 1.$$

Then, by Lemma 4.2, we obtain by an induction argument

$$C \max_{B(r)} \overline{u} \le \prod_{j=1}^{N-1} \left(C(\tilde{p}\chi^{N-j})\right)^{\frac{\delta}{\tilde{p}\chi^{N-j}}} (\frac{r_j}{s_j})^{\frac{2}{\tilde{p}\chi^{N-j}}} \left(\fint_{B_N} \overline{u}^{\tilde{p}} d\mu\right)^{1/\tilde{p}}, \tag{25}$$

again by $r_j \ge s_j$ and iterating (23) with $p = \tilde{p}\chi^j$. To estimate the product on the right define, for $N > j \ge 1$,

$$r_j = 2^n + 2^{j-1}\,; \quad s_j = 2^{j-1}\,; \quad \frac{r_j}{s_j} \le 4 \cdot 2^{n-j},$$

a straightforward computation shows that, being by Remark 4.1 $C(\tilde{p}\chi^{N-j}) \approx K\chi^{\alpha(N-j)}$, the infinite product in (25) is uniformly bounded by a structural constant C. Let us now estimate the constant $\tau = r_N/r_0$. Choosing N as in (24), we have

$$\tau = 1 + 2^{N-n-1} \le C \exp\left(\frac{\log(cn/\tilde{p})}{\log(\chi)} - n\right) \le 1 + C\tilde{p}^{-C},$$

for some $C \geq 1$. We can conclude then that τ can be chosen depending only on $\tilde{p}$, and with this choice we have

$$\max_{B_x(r)} u \leq \max_{B_x(r)} \overline{u} \leq C \left(\fint_{B_x(\tau r)} u^{\tilde{p}} \, d\mu \right)^{1/\tilde{p}} + Ck(x,r),$$

which concludes the proof. □

In exactly the same way, but using Caccioppoli inequality with $\beta < 0$, one can prove the previous estimate for $\overline{u}^{-1}$ obtaining the following theorem.

Theorem 4.4. *For every $\tilde{p} > 0$ there exist two constants C and τ such that, if $B_x(\tau r) \subset \overset{\circ}{\Omega}$ and $\overline{u}$ is a positive solution to* (9) *in $\overset{\circ}{\Omega}$, then*

$$\min_{B_x(r)} \overline{u} \geq C \left(\fint_{B_x(\tau r)} \overline{u}^{-\tilde{p}} \, d\mu \right)^{-1/\tilde{p}},$$

where k is given by (21).

5 Discrete regularity

In this section we will prove a Harnack inequality for solutions of elliptic equation (9); in a standard way this inequality will provide us with a Hölder regularity result. Throughout this section we will use the notation

$$\mathbb{L}(u) = \operatorname{div}(\mathbf{a}Du) + \mathbf{b}Du + cu.$$

Theorem 5.1. *(Harnack Inequality)*
Let u be a positive solution in $\overset{\circ}{\Omega} \subset G$ of (9). *There exist a $\tau = \tau(G, \Omega, \mathbb{L}) > 0$ such that if $B = B_x(\tau r) \subset \overset{\circ}{\Omega}$ it holds*

$$\max_B u \leq C(\min_B u + (hr)^{\mu} \|f\|_{q'/2}), \tag{26}$$

for $C = C(G, \Omega, \mathbb{L})$ and μ given by (20).

Dim It suffice to use Theorems 4.3 and 4.4 and then use Corollary 3.6, obtaining

$$\begin{aligned} \max_{B_x(r)} u + k(x,r) \quad &\leq \quad C \left(\fint_{B_x(\tau r)} \overline{u}^{\tilde{p}} \, d\mu \right)^{1/\tilde{p}} \leq C \left(\fint_{B_x(\tau r)} \overline{u}^{-\tilde{p}} \, d\mu \right)^{-1/\tilde{p}} \leq \\ &\leq \quad C \left(\min_{B_x(\tau r)} (u + k(x,r)) \right), \end{aligned}$$

(recall (21) for the definition of k). The dependence mentioned in the statement follows from Remark 4.2. □

Theorem 5.2. *Let u be a solution in $\mathring{\Omega} \subset G$ of equation* (9). *There exist $\tau = \tau(G, \Omega, \mathbb{L}) > 0$ and $\alpha = \alpha(G, \Omega, \mathbb{L}) > 0$ such that for every $x \in \mathring{\Omega}$ and $\rho < \operatorname{dist}(x, \partial\Omega)/\tau$ it holds*

$$\operatorname{Osc}(u, x, \rho) \leq C \left(\max_{B_x(\tau \operatorname{dist}(x,\partial\Omega))} |u| + \sigma(hR)^\mu \right) \cdot \left(\frac{\rho}{\operatorname{dist}(x, \partial\Omega)} \right)^\alpha,$$

for some $C = C(G, \Omega, \mathbb{L})$, $\sigma = \sigma(f)$ and μ given in (20).

Dim Let the constant C in the previous theorem satisfy $C > \max\{3, (1 + 2^{-\mu})/(1 - 2^{-\mu})\}$, with μ given in (20). Being x fixed, for every r such that $\tau r \leq \operatorname{dist}(x, \partial\Omega)$ consider the positive functions $V_r = \max_{B(2r)} u - u + 1$ and $v_r = u - \min_{B(2r)} u + 1$ for which it holds

$$\mathbb{L}(V_r) = c(\max_{B(2r)} u + 1) - f, \qquad \mathbb{L}(v_r) = f + c(1 - \min_{B(2r)} u).$$

Letting then

$$\overline{\sigma} = \max\{|\max_{B(2r)} u + 1|, |1 - \min_{B(2r)} u|\} \|c\|_{q'/2} + \|f\|_{q'/2},$$

we can apply (26) to these function, obtaining

$$\begin{aligned}
\max_{B(2r)} u - \min_{B(r)} u &\leq C(\max_{B(2r)} u - \max_{B(r)} u + \overline{\sigma}(rh)^\mu), \\
\max_{B(r)} u - \min_{B(2r)} u &\leq C(\max_{B(r)} u - \min_{B(2r)} u + \overline{\sigma}(rh)^\mu),
\end{aligned}$$

which, summing up and manipulating, gives

$$\operatorname{Osc}(u, x, r) \leq \gamma \operatorname{Osc}(u, x, 2r) + \overline{\sigma}\gamma(rh)^\mu,$$

where $\gamma = (C - 1)/(C + 1) < 1$. Iterating this last inequality gives

$$\operatorname{Osc}(u, x, \frac{r}{2^n}) \leq \gamma^n \operatorname{Osc}(u, x, r) + \overline{\sigma}\gamma(rh)^\mu \frac{(2^\mu\gamma)^n}{2^{\mu n}(2^\mu\gamma - 1)},$$

(since $2^\mu\gamma > 1$ for the choice of the constant C at the beginning of the proof). Letting now $\rho = r/2^n$ and $r = \operatorname{dist}(x, \partial\Omega)/\tau$ easily gives the claim. $\square$

References

[1] L. Ambrosio and P. Tilli. *Selected Topics in Analysis in Metric Spaces.* Appunti. Scuola Normale Superiore, Pisa, 2000.

[2] F. R. K. Chung. *Spectral Graph Theory*, volume 92 of *Regional Conference Series in Mathematics.* American Mathematical Society, Providence, 1997.

[3] T. Coulhon and A. Grigoryan. Random walks on graphs with regular volume growth. *G.A.F.A.*, 8:656–701, 1998.

[4] T. Coulhon and L. Saloff-Coste. Isopérimétrie pour les groupes et les variétés. *Rev. Math. Iberoam.*, 11(3):687–726, 1993.

[5] E. De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat.*, 3(3):25–43, 1957.

[6] T. Delmotte. Inégalité de Harnack elliptique sur le graphes. *Colloq. Math.*, 72(1):19–37, 1997.

[7] H. J. Kuo and N. S. Trudinger. Discrete methods for fully nonlinear elliptic equation. *SIAM J. Numer. Anal.*, 29:123–135, 1992.

[8] H. J. Kuo and N. S. Trudinger. Positive difference operators on general meshes. *Duke Math. J.*, 83:415–433, 1996.

[9] G. F. Lawler. Inequality for difference operators coming from random walks with symmetric, spatially inhomogeneous, increments. *Proc. London Math. Soc.*, 63(3):552–568, 1991.

[10] A. B. Merkov. Second-order elliptic equation on graphs. *Math. USSR-Sb*, 5:493–509, 1986.

[11] S. J. N. Mosconi. *Equazioni Ellittiche su Grafi.* Degree Thesis. Universitá degli studi di Pisa, 1999. Downloadable at http://cvgmt/papers/.

[12] J. Moser. On Harnack's Theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14:577–591, 1961.

[13] L. Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. *Duke math. J.*, I.M.R.N(2):27–38, 1992.