FRACTIONAL SEMILINEAR EIGENVALUES

G. FRANZINA

Abstract. We discuss a semilinear non-local eigenvalue problem under homogeneous Dirichlet boundary conditions on open sets supporting a compact embedding of a fractional Sobolev space into $L^q(\Omega)$, and we present some applications to an initial-boundary value problem for the fractional porous media equation.

Contents

1. Introduction

This manuscript concerns a semilinear eigenvalue problem for the fractional Laplace operator in N-dimensional euclidean space $\mathbb{R}^N$ with applications to a model for non-local filtration in a porous medium. We recall that, given $s \in (0,1)$, the s-Laplacian of a smooth function u is defined, up to a normalisation constant depending on N and s only, by the formula[1]

$$(-\Delta)^s u(x) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy\,. \tag{1.1}$$

By classical spectral theory in Hilbert spaces, it is known that the eigenvalue problem

$$(-\Delta)^s u = \lambda u$$

in an open bounded open set Ω in $\mathbb{R}^N$, with Dirichlet conditions $u = 0$ in the complement $\mathbb{R}^N \setminus \Omega$, has non trivial solutions for a discrete set of real numbers λ, which either is empty or consists of an unbounded non-decreasing sequence of *eigenvalues*. The corresponding *eigenfunctions* are the stationary points of the double integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy \tag{1.2}$$

subject to an $L^2(\Omega)$-constraint.

The variational problem under an $L^q(\Omega)$-constraint, with $q \neq 2$, leads one to a different non-local semilinear elliptic boundary value problem, formally

$$\begin{cases} (-\Delta)^s u = \lambda \|u\|_{L^q(\Omega)}^{2-q} |u|^{q-2} u & \text{in } \Omega,\\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.3}$$

Any fixed solution u of (1.3), if multiplied by a specific constant depending on u, solves the *fractional Lane-Emden Equation*

$$(-\Delta)^s u = |u|^{q-2} u \qquad \text{in } \Omega, \tag{1.4}$$

with $u = 0$ in $\mathbb{R}^N \setminus \Omega$.

The largest lower bound for the collection $\mathfrak{S}(\Omega; s, q)$ of all positive numbers λ for which (1.3) admits a non-trivial solution is called the *first q-semilinear s-eigenvalue*

$$\lambda_1(\Omega; s, q) = \inf_{\varphi\in C_0^\infty(\Omega)} \left\{ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}}\,dx\,dy :\ \int_\Omega |\varphi|^q\,dx = 1 \right\}. \tag{1.5}$$

In some cases, e.g. if Ω has finite N-dimensional volume, the embedding $\mathcal{D}_0^{s,2}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, which assures the infimum to be a minimum.

For $q \in (1, 2)$, in fact, a necessary and *sufficient* condition that the embedding be compact is that it be continuous (see [9, Theorem 1.3]). Moreover, we also have the following.

Theorem A. *Let $N \geq 2$, $s \in (0, 1)$, $q \in (1, 2)$, and let Ω be an open set in $\mathbb{R}^N$ with $\lambda_1(\Omega; s, q) > 0$. Up to a multiplicative constant, there exists a unique eigenfunction achieving the minimum in (1.5). The first eigenfunction has constant sign and the first eigenvalue is the unique one admitting eigenfunctions with this property.*

The conclusion of Theorem A implies the uniqueness of positive *least energy solutions* of (1.4), i.e., positive solutions of the fractional Lane-Emden Equation, under homogeneous Dirichlet boundary conditions, that minimise the energy functional

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{q}\int_\Omega |\varphi|^q\,dx\,. \tag{1.6}$$

[1] The right hand side in (1.1) is usually mutiplied by the quantity $\pi^{\frac{n}{2}-2s}\Gamma(\frac{n}{2}-s)/|\Gamma(-s)|$ which blows up both as $s \to 0^+$ and as $s \to 1^-$. The specific normalisation choice has no bearing for us and will be therefore omitted.

Thus, for $q \in (1, 2)$, to every open set Ω with $\lambda_1(\Omega, s, q) > 0$ we can associate the positive least energy solution $w_{\Omega,s,q}$, also called the *fractional Lane Emden density of* Ω (in fact, the definition can be given for arbitrary open sets in $\mathbb{R}^N$, see Section 5 for details).

In this manuscript, we also present some applications of the uniqueness properties that we discuss for the first eigenvalue $\lambda_1(\Omega; s, q)$ and the Lane Emden density $w_{\Omega;s,q}$, in case $q \in (1, 2)$ and $\lambda_1(\Omega; s, q) > 0$, to the initial-boundary value problem for the the *fractional porous media equation*

$$\begin{cases} \partial_t v + (-\Delta)^s(|v|^{m-1}v) = 0\,, & \text{in } \Omega \times (0,T)\,, \\ v = 0\,, & \text{on } (\mathbb{R}^N \setminus \Omega) \times (0,T)\,, \\ v = v_0\,, & \text{in } \Omega \times \{0\}\,, \end{cases} \tag{1.7}$$

with $m > 1$. Solutions of the latter are understood in a weak sense, for which we refer to Section 2. Weak solutions with non-negative initial data satisfy the following estimate.

Theorem B. *Let $N \geq 2$, $s \in (0, 1)$, $q \in (1, 2)$, and let Ω be an open set in $\mathbb{R}^N$ with $\lambda_1(\Omega; s, q) > 0$. Let $m = 1/(q - 1)$ and let v_0 be a bounded non-negative function, with $v_0^m \in \mathcal{D}_0^{s,2}(\Omega)$. Then,*

$$\|v\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C t^{\frac{1}{1-m}} \lambda_1(\Omega; s, q)^{-\gamma}\,, \tag{1.8}$$

for all weak solutions v of (1.7), where the constant C and the exponent γ depend on N, s, q, only.

The positive least energy solution encodes the long-time behaviour of solutions v of the fractional porous media equation.

Theorem C. *Let $N \geq 2$, $s \in (0, 1)$, $q \in (1, 2)$, let Ω be an open set in $\mathbb{R}^N$ with $\lambda_1(\Omega; s, q) > 0$, and let $m = 1/(q - 1)$. Then every non-negative weak solution v of (1.7) satisfies*

$$\lim_{t\to+\infty} \left| t^{\frac{1}{m-1}} v(x,t) - \left(\tfrac{1}{m-1}\right)^{\frac{m-1}{m}} w_{\Omega,s,q}(x)^{q-1} \right| = 0\,, \tag{1.9}$$

where $w_{\Omega,s,q}$ is the positive least energy solution of the fractional Lane-Emden equation (1.4) on Ω. The convergence in (1.9) is monotone non-decreasing in time and uniform in space.

Plan of the paper. In Section 2, after framing our problem in the appropriate function spaces, we introduce the fractional semilinear eigenvalue problem and the non-local Lane Emden density. The former is presented with more details in Section 4, whereas the latter is defined on arbitrary open sets, and used as a weight in a Hardy-type inequality, in Section 5. Section 3 is devoted to survey and prove a number of comparison principles for the elliptic and parabolic problems considered in this paper. Eventually, the results of Sections 3, Section 4, and Section 5 are used in Section 6 to prove our main results.

2. Framework and (pseudo) differential equations

Let $N \in \mathbb{N}$, with $N \geq 1$, let $s \in (0, 1)$, and let Ω be an open set in $\mathbb{R}^N$. The square root of

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}}\, dx\, dy \tag{2.1}$$

is a norm on the vector space $C_0^\infty(\Omega)$. The metric completion of this space is denoted by $\mathcal{D}_0^{s,2}(\Omega)$.

Remark 2.1 (Analogies and differences with other spaces). If Ω is bounded with Lipschitz boundaries, then for all $s \in (0,1)$ except the special case $s = \frac{1}{2}$, the space $\mathcal{D}^{s,2}_0(\Omega)$ coincides with the closure $H^s_0(\Omega)$ of $C^\infty_0(\Omega)$ in the Sobolev-Slobodeckij space $H^s(\Omega)$ of all $u \in L^2(\Omega)$ with

$$[u]_{H^s(\Omega)} := \int_\Omega \int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy < +\infty\,.$$

Indeed, if Ω is a bounded Lipschitz set then the Sobolev norm $\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)}$ is equivalent to

$$\|u\|_{L^2(\Omega)} + \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{\frac{1}{2}}$$

(see [6, Appendix B]), and the latter is equivalent to the norm in $\mathcal{D}^{s,2}_0(\Omega)$ because Lipschitz sets support a Poincaré-type inequality. On the contrary, if $\partial\Omega$ is not Lipschitz regular then the existence of functions $u \in H^s(\Omega)$ for which the integral

$$\int_\Omega \int_{\mathbb{R}^N\setminus\Omega} \frac{u(x)^2}{|x-y|^{N+2s}}\,dx\,dy$$

diverges cannot be ruled out.

For a general open set, it is not true that all the elements of $\mathcal{D}^{s,2}_0(\Omega)$ are functions; $\mathcal{D}^{s,2}_0(\Omega)$ is not even a distribution space, in general: see, e.g., [11, 14]. A restriction that clears off this difficulty is to consider open sets Ω supporting a Sobolev-type inequality, on which $\mathcal{D}^{s,2}_0(\Omega)$ is a function space; namely, assuming that the infimum in (1.5) is a positive number.

2.1. **Semilinear fractional spectrum.** Given $s \in (0,1)$, we denote by 2^*_s the fractional Sobolev conjugate exponent, defined by $2N/(N-2s)$, if $2s < N$, and $+\infty$, otherwise.

Definition 2.2 (Semilinear fractional eigenvalues)**.** Let $s \in (0,1)$, $q \in (1, 2^*_s)$, and let Ω be an open set in $\mathbb{R}^N$. We consider the constrained critical points of the double integral (2.1) along the submanifold

$$\left\{ u \in \mathcal{D}^{s,2}_0(\Omega) : \int_\Omega |u|^q\,dx = 1 \right\}\,. \tag{2.2}$$

We call *q-semilinear s-eigenvalues* the corresponding constrained critical values. Their collection is denoted by $\mathfrak{S}(\Omega; s, q)$ and is said to be the *q-semilinear s-spectrum* of Ω.

Clearly, the infimum in (1.5) is the largest lower bound for $\mathfrak{S}(\Omega; s, q)$, and it is its minimum whenever the variational problem (1.5) has a solution. The restriction $q < 2^*_s$ in Definition 2.2 is natural because for $q > 2^*_s$ loss of compactness occur regardless of the properties of Ω. In the borderline case $q = 2^*_s$, the infimum in (1.5) is independent of Ω and gives the best constant $\mathcal{S}(N,s)$ in Sobolev inequality

$$\mathcal{S}\cdot\left(\int_\Omega |v|^{2^*_s}\,dx\right)^{\frac{2}{2^*_s}} \le \iint_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}\,dx\,dy\,, \qquad \text{for all } v \in C^\infty_0(\mathbb{R}^N). \tag{2.3}$$

By Lagrange's multipliers rule, the q-semilinear s-eigenvalues are those $\lambda > 0$ for which

$$(-\Delta)^s u = \lambda \|u\|^{2-q}_{L^q(\Omega)} |u|^{q-2} u \tag{2.4}$$

has a non-trivial solution $u \in \mathcal{D}^{s,2}_0(\Omega)$ in the weak sense, viz.

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy = \lambda\|u\|^{2-q}_{L^q(\Omega)} \int_\Omega |u|^{q-2}u\varphi\,dx\,, \tag{2.5}$$

for all $\varphi \in \mathcal{D}_0^{s,2}(\Omega)$.

2.2. **Fractional Lane-Emden equation.** After a renormalisation, the equation (2.4) for Dirichlet q-semilinear s-eigenfunctions becomes the *fractional Lane Emden equation* (1.4). We will say a *weak supersolution* (resp., *subsolution*) of the latter any function $u \in \mathcal{D}_0^{s,2}(\Omega)$ such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \geq \int_\Omega |u|^{q-2}u\varphi\,dx \qquad (\textit{resp.}, \leq)\,, \tag{2.6}$$

for all $\varphi \in \mathcal{D}_0^{s,2}(\Omega)$. A function that is both a weak super- and a weak subsolution will be called a *weak solution*. Clearly, the weak solutions of (1.4) are the critical points of the free energy

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}}dx\,dy - \frac{1}{q}\int_\Omega |\varphi|^q\,dx\,, \qquad \varphi \in \mathcal{D}_0^{s,2}(\Omega). \tag{2.7}$$

Definition 2.3. (Fractional Lane-Emden densities) Let $N \geq 2$, $s \in (0,1)$, $q \in (1,2)$, and let Ω be an open set in $\mathbb{R}^N$ with $\lambda_1(\Omega; s, q) > 0$. We denote by $w_{\Omega;s,q}$ the unique solution of the variational problem

$$\min_{\varphi \in \mathcal{D}_0^{s,2}(\Omega)} \left\{ \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{q}\int_\Omega |\varphi|^q\,dx : \varphi \geq 0\,,\ \text{a.e. in } \Omega \right\}. \tag{2.8}$$

and we call it the (s,q)-Lane Emden Density of Ω.

Remark 2.4. The constrained minimisation problem (2.8) is equivalent to the minimisation over $\mathcal{D}_0^{s,2}(\Omega)$ of the free energy (2.7) (for, see Lemma 3.2 below). By [9, Theorem 1.3], the assumption $\lambda_1(\Omega; s, q) > 0$ assures, in fact, the compactness of the embedding $\mathcal{D}_0^{s,2}(\Omega) \hookrightarrow L^q(\Omega)$; then, minimisers exist by direct methods in calculus of variations, and clearly they are non-negative solutions of (2.5) for an appropriate λ; uniqueness then follows (see Proposition 4.3 and Proposition 4.4 below).

2.3. **Fractional porous media equation.** Let $T > 0$, $s \in (0,1)$, and let $m > 1$. We shall consider also an initial-boundary value problem for the parabolic equation

$$\partial_t v + (-\Delta)^s\big(|v|^{m-1}v\big) = 0\,, \qquad \text{in } \Omega \times (0,T). \tag{2.9}$$

We shall say *local weak supersolution* (resp., *subsolution*) any function $v \in C([0,T]; L^1(\Omega))$ with the property that, for every smooth open set $\Omega' \subset \Omega$ and for every $\varphi \in C^\infty(\Omega' \times (0,T))$, with support in $\Omega' \times [0,T]$ and $\varphi \geq 0$, the restriction of $|v|^{m-1}v$ belongs to $L^2(0,T; H^s(\mathbb{R}^N))$, and

$$-\int_{T_0}^{T_1}\int_\Omega v\partial_t\varphi\,dx\,dt + \int_{T_0}^{T_1}\iint_{\mathbb{R}^{2N}} \frac{(|v(x,t)|^{m-1}v(x,t) - |v(y,t)|^{m-1}v(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}\,dx\,dy\,dt$$
$$\geq \int_\Omega u(x,T_0)\varphi(x,T_0)\,dx - \int_\Omega u(x,T_1)\varphi(x,T_1)\,dx\,, \quad (\textit{resp.}, \leq 0),$$

for all $0 < T_0 \leq T_1 \leq T$. The function v is said to be a *weak solution* if it is both a weak supersolution and a weak subsolution.

Definition 2.5. Let $v_0 \in L^1(\Omega)$, with $|v_0|^{m-1}v_0 \in \mathcal{D}_0^{s,2}(\Omega)$, and let $g \in C([0,T]; L^1(\Omega))$, with $|g|^{m-1}g \in L^2(0,T; H^s(\mathbb{R}^N))$. A function $v \in C([0,T]; L^1(\Omega))$ is said to be a *weak supersolution* (resp., *subsolution*) *of initial-boundary value problem for* (2.9) *with Dirichlet conditions* $v = g$ *on* $\partial\Omega \times (0,T)$*, and initial datum* v_0, if it is a a local weak supersolution (*resp.*, subsolution), with

$$|v|^{m-1}v - |g|^{m-1}g \in L^2(0,T; \mathcal{D}_0^{s,2}(\Omega))\,,$$

and $v(\cdot, 0) = v_0$.

Remark 2.6. If Ω supports a continuous embedding $\mathcal{D}^{s,2}_0(\Omega) \hookrightarrow L^q(\Omega)$, where $q = 1 + 1/m$, then all weak sub- and supersolutions of (1.7) belong to $L^{2m}([0,T]; L^{m+1}(\Omega))$ with the estimate

$$\int_0^T \left(\int_\Omega |v(t)|^{m+1}\,dx\right)^{\frac{2m}{m+1}} dt \le \lambda_1(\Omega; s, q) \int_0^T \iint_{\mathbb{R}^{2N}} \frac{\big(|v|^{m-1}v)(x,t) - (|v|^{m-1}v)(y,t)\big)^2}{|x-y|^{N+2s}}\,dx\,dy\,dt\,.$$

Since $m > 1$, by Hölder inequality they all belong to $L^{m+1}([0,T]; L^{m+1}(\Omega))$. The weaker conclusion, that weak sub- and supersolutions belong to $L^2([0,T]; L^{m+1}(\Omega))$ hold true on every open set Ω, because of Gagliardo-Nirenberg-Sobolev inequality (see, e.g., [9, Lemma 2.3]).

Remark 2.7 (Uniqueness, Positivity). For the initial-boundary value problem (1.7), uniqueness of weak solutions is an immediate consequence of the comparison principle (see Proposition 3.4 below). By Lemma 3.7, weak supersolutions (*resp.*, subsolution), starting from initial data that are non-negative (*resp.*, non-positive) a.e. in Ω, remain such.

3. Fractional comparison principles

This section is devoted to state and prove the relevant comparison principles for the non-local elliptic and parabolic equations considered in this paper.

3.1. **Fractional elliptic strong maximum principle.** We will need the following fact.

Proposition 3.1. *Let $s \in (0,1)$, let $q \in (1,2)$, let Ω_1 and Ω_2 be bounded open sets and for $i \in \{1,2\}$ let w_i be the fractional Lane Emden density $w_{\Omega_i,s,q}$ on Ω_i (cfr. Definition 2.3). Then*

$$\Omega_1 \subset \Omega_2 \qquad \Longrightarrow \qquad w_1 \le w_2\,.$$

We split the proof of Proposition 3.1 in easier steps. First, the following Lemma is an immediate consequence of inequality $(|a| - |b|)^2 \le (a-b)^2$ that is strict if and only if $ab < 0$.

Lemma 3.2. *Let $s \in (0,1)$, let $\Omega \subset \mathbb{R}^N$ be an open set, and let $u \in \mathcal{D}^{s,2}_0(\Omega)$. Then*

$$\iint_{\mathbb{R}^{2N}} \frac{(|u(x)| - |u(y)|)^2}{|x-y|^{N+2s}}\,dx\,dy \le \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}\,dx\,dy \tag{3.1}$$

and the inequality is strict unless either $u \ge 0$ or $u \le 0$ a.e. in Ω.

Then we recall the following form of the minimum principle for weak supersolutions, observed by L. Brasco and E. Parini in [7]. We present the proof for convenience of the reader; notice that Ω is not required to be connected.

Proposition 3.3. *Let $s \in (0,1)$, let $u \in \mathcal{D}^{s,2}_0(\Omega)$ satisfy*

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \ge 0 \qquad \text{for all non-negative } \varphi \in C^\infty_0(\Omega)\,,$$

and assume that $u \ge 0$ a.e. in Ω. Then either $u = 0$ a.e. in Ω or $u > 0$ a.e. in Ω.

Proof. By [4, Theorem A.1], $u > 0$ in each connected component where it is not identically zero. Then we argue as in the proof of [7, Proposition 2.6] and we prove a contrapositive statement: if $u \equiv 0$ in a connected component Ω_0 of Ω, then by assumption for all $\varphi \in C^\infty_0(\Omega_0) \setminus \{0\}$, with $\varphi \ge 0$,

$$\int_{\Omega\setminus\Omega_0} \int_{\Omega_0} \frac{u(x)\varphi(y)}{|x-y|^{n+2s}}\,dx\,dy - \frac{1}{2} \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy \le 0\,,$$

which, by Fubini's theorem, implies $u = 0$ a.e. in $\Omega \setminus \Omega_0$, hence a.e. in Ω. □

Eventually, we can prove Proposition 3.1.

Proof of Proposition 3.1. Let us write $w_i = w_{\Omega_i,s,q}$ for $i = 1, 2$. The minimality property of w_1 implies

$$\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{(f(x)-f(y)^2}{|x-y|^{N+2s}}dx\,dy - \frac{1}{q}\int_{\Omega_1} f^q\,dx \geq \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{(w_1(x)-w_1(y)^2}{|x-y|^{N+2s}}dx\,dy - \frac{1}{q}\int_{\Omega_1} w_1^q\,dx\,,$$

for all $f \in \mathcal{D}_0^{s,2}(\Omega)$. The choice $f = \min\{w_1\,,w_2\}$ yields

$$\begin{aligned}
&\frac{1}{2}\Big\{J^{22}\big(\{w_2<w_1\},\{w_2<w_1\}\big) + J^{21}\big(\{w_2<w_1\},\{w_1\leq w_2\}\big) + J^{21}\big(\{w_2<w_1\},\Omega_2\setminus\Omega_1\big)\\
&\qquad + J^{12}\big(\{w_1\leq w_2\},\{w_2<w_1\}\big) + J^{12}\big(\Omega_2\setminus\Omega_1,\{w_2<w_1\}\big)\Big\} - \frac{1}{q}\int_{\{w_2<w_1\}} w_2^q\\
&\geq \frac{1}{2}\Big\{J^{11}\big(\{w_2<w_1\},\{w_2<w_1\}\big) + J^{11}\big(\{w_2<w_1\},\{w_1\leq w_2\}\big) + J^{11}\big(\{w_2<w_1\},\Omega_2\setminus\Omega_1\big)\\
&\qquad + J^{11}\big(\{w_1\leq w_2\},\{w_2<w_1\}\big) + J^{11}\big(\Omega_2\setminus\Omega_1,\{w_2<w_1\}\big)\Big\} - \frac{1}{q}\int_{\{w_2<w_1\}} w_1^q\,,
\end{aligned}$$

where we introduced the notation

$$J^{ij}\big(A,B\big) = \iint_{A\times B} \frac{(w_i(x)-w_j(y))^2}{|x-y|^{N+2s}}dx\,dy\,, \qquad i,j\in\{1,2\}\,,\ A,B\subset\Omega_2\,.$$

Subtracting from the previous inequality the quantity

$$\begin{aligned}
\underline{\mathcal{R}} := &\, J^{21}(\{w_2<w_1\},\{w_1\leq w_2\}) - J^{22}(\{w_2<w_1\},\{w_1\leq w_2\})\\
&+ J^{21}(\{w_2<w_1\},\Omega_2\setminus\Omega_1) - J^{22}(\{w_2<w_1\},\Omega_2\setminus\Omega_1)\\
&+ J^{12}(\{w_1\leq w_2\},\{w_2<w_1\}) - J^{22}(\{w_1\leq w_2\},\{w_2<w_1\})\\
&+ J^{12}(\Omega_2\setminus\Omega_1,\{w_2<w_1\}) - J^{22}(\Omega_2\setminus\Omega_1,\{w_2<w_1\})\,,
\end{aligned}$$

we get

$$\begin{aligned}
&\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{(w_2(x)-w_2(y))^2}{|x-y|^{N+2s}}dxdy - \frac{1}{q}\int_{\{w_2<w_1\}} w_2^q\,dx\\
&\qquad\geq \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{(g(x)-g(y))^2}{|x-y|^{N+2s}}dxdy - \frac{1}{q}\int_{\{w_2<w_1\}} w_1^q\,dx + \overline{\mathcal{R}} - \underline{\mathcal{R}}\,,
\end{aligned} \tag{3.2}$$

where $g = \max\{w_1\,,w_2\}$ and

$$\begin{aligned}
\overline{\mathcal{R}} := &\, J^{11}(\{w_2<w_1\},\{w_1\leq w_2\}) - J^{12}(\{w_2<w_1\},\{w_1\leq w_2\})\\
&+ J^{11}(\{w_2<w_1\},\Omega_2\setminus\Omega_1) - J^{12}(\{w_2<w_1\},\Omega_2\setminus\Omega_1)\\
&+ J^{11}(\{w_1\leq w_2\},\{w_2<w_1\}) - J^{21}(\{w_1\leq w_2\},\{w_2<w_1\})\\
&+ J^{11}(\Omega_2\setminus\Omega_1,\{w_2<w_1\}) - J^{21}(\Omega_2\setminus\Omega_1,\{w_2<w_1\})\,.
\end{aligned}$$

By direct inspection, we see that $\overline{\mathcal{R}} \geq \underline{\mathcal{R}}$. Thus, by adding to both sides in (3.2) the term

$$-\frac{1}{q}\int_{\{w_1\leq w_2\}} w_2^q$$

we arrive at

$$\frac{1}{2}\iint_{\mathbb{R}^{2N}}\frac{(w_2(x)-w_2(y))^2}{|x-y|^{N+2s}}dxdy-\frac{1}{q}\int_{\Omega_2}w_2^q\,dx\geq\frac{1}{2}\iint_{\mathbb{R}^{2N}}\frac{(g(x)-g(y))^2}{|x-y|^{N+2s}}dxdy-\frac{1}{q}\int_{\Omega_2}g(x)^q\,dx\,.$$

By uniqueness and minimality of w_2 we deduce that $w_2=g$, which is the desired conclusion. □

3.2. **Fractional parabolic comparison principle.** The following comparison principle is proved in [13, Theorem 7.2] for bounded domains. The proof presented here for non-negative local weak solutions works under the abstract compactness assumption for the fractional Sobolev embedding and does not make use of the s-harmonic extension theory.

Proposition 3.4 (Comparison)**.** *Let $T>0$, let $s\in(0,1)$, and let $m>1$. Let Ω be an open set for which the embedding of $\mathcal{D}_0^{s,2}(\Omega)$ in $L^{\frac{m+1}{m}}(\Omega)$ is compact. Let u (resp., v) be a bounded non-negative weak subsolution (resp., supersolution) of initial-boundary value problem (2.9) with homogeneous Dirichlet conditions on $\partial\Omega\times(0,T)$. Then,*

$$t\longmapsto\int_\Omega(u(t)-v(t))_+\,dx \tag{3.3}$$

is a non-increasing function on $[0,T]$, and

$$u_0\leq v_0\ \text{a.e. in}\ \Omega\qquad\text{implies}\qquad u\leq v\ \text{a.e. in}\ \Omega\times(0,T). \tag{3.4}$$

As a preliminary step towards the proof of Proposition 3.4, we first prove that regularising local weak solutions yields local quasi-strong solutions.

Lemma 3.5. *Let $T>0$, let $s\in(0,1)$, let $m>1$, let Ω be an open set for which the embedding of $\mathcal{D}_0^{s,2}(\Omega)$ in $L^{\frac{m+1}{m}}(\Omega)$ is compact, and let χ be a local weak supersolution (*resp., subsolution*). Fix $\eta\in C_0^\infty(\mathbb{R})$, with $0\leq\eta\leq1$ and $\int_{\mathbb{R}}\eta=1$, let $0<T_0<T_1<T$, and let $\varphi\in C^\infty(\Omega\times[0,T])$ be a non-negative function with support contained in $\Omega\times[0,T]$. Then, there exists a function $\omega\colon[0,\frac{1}{2}\min\{T_0/2\,,T-T_1\})\to\mathbb{R}$, with $\omega(\varepsilon)\to0$ as $\varepsilon\to0^+$, such that*

$$\int_{T_0}^{T_1}\int_\Omega\varphi\partial_t\chi_\varepsilon+\int_{T_0}^{T_1}\iint_{\mathbb{R}^{2N}}\frac{(|\chi|^{m-1}\chi)_\varepsilon(x,t)-(|\chi|^{m-1}\chi)_\varepsilon(y,t)}{|x-y|^{N+2s}}\times\\ \times\big(\varphi(x,t)-\varphi(y,t)\big)\,dx\,dy\,dt\geq\omega(\varepsilon),\quad(\text{resp.}, \leq), \tag{3.5}$$

where, for every $\varepsilon\in[0,\frac{1}{2}\min\{T_0/2\,,T-T_1\})$, z_ε is the Friedrichs regularisation of z in time, i.e.,

$$z_\varepsilon(x,t)=\frac{1}{\varepsilon}\int_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}}\eta\left(\frac{t-\tau}{\varepsilon}\right)z(x,\tau)\,d\tau\,, \tag{3.6}$$

for all $x\in\Omega$ and for all $t\in[T_0,T_1]$.

Proof. Let us assume that χ be a local weak supersolution. The proof for local weak subsolutions will be similar. We define

$$\triangle_0^\varepsilon=\Big\{(t,\tau)\in\mathbb{R}^2\,:\ T_0<t<T_0+\varepsilon\,,\ t-\frac{\varepsilon}{2}<\tau<t+\frac{\varepsilon}{2}\Big\}$$
$$\triangle_1^\varepsilon=\Big\{(t,\tau)\in\mathbb{R}^2\,:\ T_1-\varepsilon<t<T_1\,,\ t-\frac{\varepsilon}{2}<\tau<t+\frac{\varepsilon}{2}\Big\}$$

and set $\triangle_c^\varepsilon=\{(t,\tau)\in[T_0,T_1]\times\mathbb{R}:\ \ |t-\tau|<\varepsilon\}\setminus(\triangle_0^\varepsilon\cup\triangle_1^\varepsilon)$. By Fubini's theorem

$$\int_{T_0}^{T_1}\int_\Omega\varphi\,\partial_t\chi_\varepsilon\,dx\,dt=\iint_{\triangle_0^\varepsilon}\Phi_\varepsilon(t,\tau)\,dt\,d\tau+\iint_{\triangle_1^\varepsilon}\Phi_\varepsilon(t,\tau)\,dt\,d\tau+\iint_{\triangle_c^\varepsilon}\Phi_\varepsilon(t,\tau)\,dt\,d\tau\,, \tag{3.7a}$$

where

$$\Phi_\varepsilon(t,\tau)=\int_\Omega \varphi(x,t)\partial_t\left(\frac{1}{\varepsilon}\eta\left(\frac{t-\tau}{\varepsilon}\right)\chi(x,\tau)\right)dx\,. \tag{3.7b}$$

<u>Step 1</u>*(Integrals on $\triangle_i^\varepsilon$).* We claim that

$$\lim_{\varepsilon\to 0^+}\iint_{\triangle_i^\varepsilon}\Phi_\varepsilon(t,\tau)\,dt\,d\tau=-\int_\Omega \chi(x,T_0)\varphi(x,T_0)\,dx\,,\qquad \text{for } i=0,1. \tag{3.8}$$

Indeed, an integration by parts in (3.7b) yields

$$\begin{aligned}\iint_{\triangle_0^\varepsilon}\Phi_\varepsilon(t,\tau)\,dt\,d\tau&=\int_{T_0-\frac{\varepsilon}{2}}^{T_0+\frac{\varepsilon}{2}}\int_\Omega \varphi(x,\tau+\frac{\varepsilon}{2})\frac{1}{\varepsilon}\eta\left(\tfrac12\right)\chi(x,\tau)\,dx\,d\tau\\&\quad-\int_{T_0-\frac{\varepsilon}{2}}^{T_0+\frac{\varepsilon}{2}}\int_\Omega \varphi(x,T_0)\frac{1}{\varepsilon}\eta\left(\frac{T_0-\tau}{\varepsilon}\right)\chi(x,\tau)\,dx\,d\tau+\mathcal{R}_\varepsilon\end{aligned}$$

where

$$|\mathcal{R}_\varepsilon|\le\varepsilon\cdot\|\partial_t\varphi\|_{L^\infty(\Omega\times[0,T])}\|\eta\|_{L^\infty(\mathbb{R})}\fint_{T_0-\frac{\varepsilon}{2}}^{T_0+\frac{\varepsilon}{2}}\int_\Omega|\chi(x,\tau)|\,dx\,d\tau\,,$$

and the right hand side in the latter goes to zero, as $\varepsilon\to 0^+$, because $\chi\in C([0,T];L^1(\Omega))$. Also,

$$\begin{aligned}\left|\int_{T_0-\frac{\varepsilon}{2}}^{T_0+\frac{\varepsilon}{2}}\int_\Omega \varphi(x,\tau+\frac{\varepsilon}{2})\frac{1}{\varepsilon}\eta\left(\tfrac12\right)\chi(x,\tau)\,dx\,d\tau\right|&\le\eta\left(\tfrac12\right)\int_\Omega\int_{T_0-\frac{\varepsilon}{2}}^{T_0+\frac{\varepsilon}{2}}|\chi(x,\tau)|\fint_{\tau-\varepsilon}^{\tau+\varepsilon}|\varphi(x,t)|\,dt\,d\tau\,dx\\&\le C\int_{T_0-\frac{\varepsilon}{2}}^{T_0+\frac{\varepsilon}{2}}\int_\Omega|\chi\varphi|\,dx\,dt\to 0\,,\qquad \text{as } \varepsilon\to 0^+.\end{aligned}$$

Then, the claim for $i=0$ follows because

$$-\int_{T_0-\frac{\varepsilon}{2}}^{T_0+\frac{\varepsilon}{2}}\int_\Omega \varphi(x,T_0)\frac{1}{\varepsilon}\eta\left(\frac{T_0-\tau}{\varepsilon}\right)\chi(x,\tau)\,dx\,d\tau=-\int_\Omega \chi_\varepsilon(x,T_0)\varphi(x,T_0)\,dx$$

and $\chi_\varepsilon\to\chi$ as distributions. The proof for $i=1$ is similar. We shall use (3.8) later in this proof.

<u>Step 2</u>*(Integral on $\triangle_c^\varepsilon$).* We now deal with the third term in (3.7b). By Fubini's theorem and the fact that η is supported in $(-\frac12,\frac12)$, and using the assumption that χ is a weak supersolution, we arrive at

$$\begin{aligned}\iint_{\triangle_c^\varepsilon}\Phi_\varepsilon(t,\tau)\,dt\,d\tau&=-\int_{T_0^\varepsilon}^{T_1^\varepsilon}\int_\Omega\chi\partial_t\chi_\varepsilon\,dx\,dt\\&\ge-\int_{T_0^\varepsilon}^{T_1^\varepsilon}\iint_{\mathbb{R}^{2N}}\frac{|\chi|^{m-1}\chi(x,t)-|\chi|^{m-1}\chi(y,t)}{|x-y|^{N+2s}}\big(\varphi_\varepsilon(x,t)-\varphi_\varepsilon(y,t)\big)\,dx\,dy\,dt\\&\quad+\int_\Omega\chi(x,T_0)\varphi(x,T_0^\varepsilon)\,dx-\int_\Omega\chi(x,T_1)\varphi(x,T_1^\varepsilon)\,dx\,,\end{aligned} \tag{3.9}$$

where for all $\delta>0$ we have set $T_0^\delta=T_0+\delta/2$ and $T_1^\delta=T_1-\delta/2$. Now we set

$$\begin{aligned}\hat\triangle_0^\varepsilon&=\left\{(t,\tau):\ T_0<t<T_0+\varepsilon\,,\ T_0+\frac{\varepsilon}{2}<\tau<t+\frac{\varepsilon}{2}\right\}\,,\\\hat\triangle_1^\varepsilon&=\left\{(t,\tau):\ T_1-\varepsilon<t<T_1\,,\ t-\frac{\varepsilon}{2}<\tau<T_1-\frac{\varepsilon}{2}\right\}\,,\end{aligned}$$

and $\hat{\triangle}_c^\varepsilon = \triangle_c^\varepsilon \setminus (\hat{\triangle}_0^\varepsilon \cup \hat{\triangle}_1^\varepsilon)$. Then, for $i = 0, 1$ the 2-dimensional measure of $\hat{\triangle}_i^\varepsilon$ converges to 0 as $\varepsilon \to 0^+$. Since φ is smooth and $|\chi|^{m-1}\chi \in L^2(0, T\,;\mathcal{D}_0^{s,2}(\Omega))$, by Cauchy-Schwartz inequality it follows that

$$\lim_{\varepsilon\to 0^+} \iint_{\hat{\triangle}_i^\varepsilon} \iint_{\mathbb{R}^{2N}} \frac{|\chi|^{m-1}\chi(x,t) - |\chi|^{m-1}\chi(y,t)}{|x-y|^{N+2s}} \big(\varphi_\varepsilon(x,\tau) - \varphi_\varepsilon(y,\tau)\big)\, d\tau\, dx\, dy\, dt = 0\,. \tag{3.10}$$

Also, it is easily seen that

$$\lim_{\varepsilon\to 0^+} \int_\Omega \chi(x, T_i^\varepsilon)\varphi_\varepsilon(x, T_i^\varepsilon)\, dx = \int_\Omega \chi(x, T_i)\varphi(x, T_i)\, dx\,, \qquad \text{for } i = 0, 1. \tag{3.11}$$

Indeed, given τ we can pick $\varepsilon_0 > 0$ so small that $|\varphi(x,t) - \varphi(x, T_i)| \le \frac{\tau}{2}$ for all $x \in \Omega$ and for all $t \in [T_i, T_i + \varepsilon_0]$. Then, by Jensen's inequality, for every $\varepsilon \in (0, \varepsilon_0)$ we have

$$\int_\Omega |\chi(x, T_i^\varepsilon)\varphi_\varepsilon(x, T_i^\varepsilon) - \chi(x, T_i^\varepsilon)\varphi(x, T_i^\varepsilon)|\, dx \le \fint_{T_i^\varepsilon - \frac{\varepsilon}{2}}^{T_i^\varepsilon + \frac{\varepsilon}{2}} \eta\left(\frac{T_i^\varepsilon - t}{\varepsilon}\right) |\varphi(x,\tau) - \varphi(x, T_i^\varepsilon)|\, dx \le \tau\,.$$

Since $\chi \in C([0, T]\,; L^1(\Omega))$ and φ is smooth, the right hand side in

$$\begin{aligned}\int_\Omega |\chi(x, T_i^\varepsilon)\varphi(x, T_i^\varepsilon) - \chi(x, T_i)\varphi(x, T_i)|\, dx &\le \sup|\varphi| \int_\Omega |\chi(x, T_i^\varepsilon) - \chi(x, T_i)|\, dx \\ &\quad + \int_\Omega |\chi(x, T_i)||\varphi(x, T_i^\varepsilon) - \varphi(x, T_i)|\, dx\end{aligned}$$

converges to zero as $\varepsilon \to 0^+$. As $\tau > 0$ was arbitrary, we obtain (3.11) by triangle inequality.

Step 3. Recall that we are writing $T_0^\delta = T_0 + \frac{\delta}{2}$ and $T_1^\delta = T_1 - \frac{\delta}{2}$ for all $\delta > 0$, and fix $\tau > 0$. Then, (3.9), (3.10), and (3.11) imply

$$\begin{aligned}&\iint_{\triangle_c^\varepsilon} \Phi_\varepsilon(t,\tau)\, dt\, d\tau \\ &\quad \ge - \int_{T_0^{2\varepsilon}}^{T_1^{2\varepsilon}} \iint_{\mathbb{R}^{2N}} \frac{(|\chi|^{m-1}\chi)_\varepsilon(x,t) - (|\chi|^{m-1}\chi)_\varepsilon(y,t)}{|x-y|^{N+2s}} \big(\varphi_\varepsilon(x,t) - \varphi_\varepsilon(y,t)\big)\, dx\, dy\, dt \\ &\qquad + \int_\Omega \chi(x, T_0)\varphi(x, T_0)\, dx - \int_\Omega \chi(x, T_1)\varphi(x, T_1)\, dx - \tau\,,\end{aligned}$$

provided that ε is small enough. Recalling (3.7) and (3.8), it follows that

$$\int_{T_0}^{T_1} \int_\Omega \varphi\, \partial_t \chi_\varepsilon\, dx\, dt + \int_{T_0^{2\varepsilon}}^{T_1^{2\varepsilon}} \iint_{\mathbb{R}^{2N}} \frac{(|\chi|^{m-1}\chi)_\varepsilon(x,t) - (|\chi|^{m-1}\chi)_\varepsilon(y,t)}{|x-y|^{N+2s}} \big(\varphi(x,t) - \varphi(y,t)\big)\, dx\, dy\, dt \ge -\tau\,.$$

Hence and by arbitrariness of $\tau > 0$, to conclude it suffices that

$$\lim_{\varepsilon\to 0^+} \int_{T_0^{2\varepsilon}}^{T_1^{2\varepsilon}} g_\varepsilon(t)\, dt = \int_{T_0}^{T_1} g(t)\, dx \tag{3.12}$$

where for $\varepsilon \in [0, \varepsilon_0)$ we set

$$g_\varepsilon(t) = \iint_{\mathbb{R}^{2N}} \frac{(|\chi|^{m-1}\chi)_\varepsilon(x,t) - (|\chi|^{m-1}\chi)_\varepsilon(y,t)}{|x-y|^{N+2s}} \big(\varphi(x,t) - \varphi(y,t)\big)\, dx\, dy\,.$$

<u>*Step 4*</u>. *Identity* (3.12). By Jensen's inequality,

$$\big((|\chi|^{m-1}\chi)_\varepsilon(x,t)-(|\chi|^{m-1}\chi)_\varepsilon(y,t)\big)^2 \le \sup|\eta| \fint_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}} \eta\left(\frac{t-\tau}{\varepsilon}\right)\big(|\chi|^{m-1}\chi(x,\tau)-|\chi|^{m-1}\chi(y,\tau)\big)^2\, d\tau\,.$$

Thus, setting $h(t)=\Big[|\chi(\cdot,t)|^{m-1}\chi(\cdot,t)\Big]^2_{H^s(\mathbb{R}^N)}$

$$\int_{T_0}^{T_0^{2\varepsilon}} \Big[\big(|\chi|^{m-1}\chi\big)_\varepsilon(\cdot,t)\Big]^2_{H^s(\mathbb{R}^N)}\, dt \le \sup|\eta|\cdot \int_{T_0}^{T_0^{2\varepsilon}} h_\varepsilon(t)\, dt$$

and the right hand side in the latter converges to 0 as $\varepsilon\to 0^+$, because $h\in L^1(0,T)$. By Cauchy-Schwartz inequality, this entails that

$$\lim_{\varepsilon\to 0^+}\int_{T_0}^{T_0^{2\varepsilon}} g_\varepsilon(t)\, dt = 0\,.$$

The proof that

$$\lim_{\varepsilon\to 0^+}\int_{T_0}^{T_0^{2\varepsilon}} g_\varepsilon(t)\, dt = 0\,,$$

too, is analogous. As a consequence, we get (3.12) and this concludes the proof. ☐

Lemma 3.6. *Let* $0<T_0<T_1<T$, *let* $z\in L^2(0,T;H^s(\mathbb{R}^N))$, *and let* z_ε *be as in* (3.6). *Then*

$$\sup\left\{\|z_\varepsilon\|_{L^2(T_0,T_1;H^s(\mathbb{R}^N))}\,:\; 0<\varepsilon<\tfrac{1}{2}\max\{T_0\,,T-T_1\}\right\}<+\infty\,.$$

In addition, we have $z_\varepsilon\to z$ *strongly in* $L^2(T_0,T_1\,;H^s(\mathbb{R}^N))$.

Proof. By Jensen's inequality,

$$\|z_\varepsilon\|^2_{L^2(T_0,T_1;H^s(\mathbb{R}^N))}\le \|\eta\|_{L^\infty(\mathbb{R})}\|f_\varepsilon\|_{L^1(T_0,T_1)}$$

where f_ε is the regularisation of $f(t)=[z(\cdot,t)]^2_{H^s(\mathbb{R}^N)}$, and the first statement follows by the general fact that $f_\varepsilon\to f$ in $L^1(T_0,T_1)$. To prove the last one, we fix an infinitesimal sequence $(\varepsilon_j)_j$, we set

$$g_j(t)=\fint_{t-\frac{\varepsilon_j}{2}}^{t+\frac{\varepsilon_j}{2}} \eta\left(\frac{t-\tau}{\varepsilon_j}\right)[z(\cdot,t)-z(\cdot,\tau)]^2_{H^s(\mathbb{R}^N)}\, d\tau\,,$$

and we observe that

$$\int_{T_0}^{T_1}[z_{\varepsilon_j}(\cdot,t)-z(\cdot,t)]^2_{H^s(\mathbb{R}^N)}\, dt\le \|\eta\|_{L^\infty(\mathbb{R})}\int_{T_0}^{T_1} g_j(t)\, dt\,.$$

To prove that the right hand side converges to zero, we note that

$$g_j(t)\le \|\eta\|_{L^\infty(\mathbb{R})}\fint_{t-\frac{\varepsilon_j}{2}}^{t+\frac{\varepsilon_j}{2}}[z(\cdot,\tau)-z(\cdot,t)]^2_{H^s(\mathbb{R}^N)}\, d\tau$$

and the right hand side, by Lebesgue differentiation theorem, converges to zero a.e. in $[T_0,T_1]$, because $z\in L^2(T_0,T_1;H^s(\mathbb{R}^N))$. Hence,

$$g_j\to 0\,,\qquad \text{a.e. in } [T_0,T_1].$$

To prove that we can pass to the limit in the integral, we define $h_j(t) = 2\|\eta\|_{L^\infty(\mathbb{R})}\big(f_{\varepsilon_j}(t) + f(t)\big)$ and we note that $|g_j| \le h_j$, that $h_j \to 4\|\eta\|_{L^\infty(\mathbb{R})} f$ a.e. in $[T_0, T_1]$, and that

$$\lim_{j\to\infty} \int_{T_0}^{T_1} h_j(t)\,dt = 4\|\eta\|_{L^\infty(\mathbb{R}^N)} \int_{T_0}^{T_1} f(t)\,dt\,.$$

Therefore, the desired conclusion follows by the generalised dominated convergence theorem. □

The following lemma implies that solutions starting from non-negative data never change sign. It is a maximum principle for weak subsolutions of the equation.

Lemma 3.7. *Let $T > 0$, let $s \in (0,1)$, let $m > 1$, and let Ω be an open set for which the embedding of $\mathcal{D}^{s,2}_0(\Omega)$ in $L^{m+1}(\Omega)$ is compact. Let $v \in C([0,T]; L^1(\Omega))$ with $|v|^{m-1}v \in L^2(0,T;\mathcal{D}^{s,2}_0(\Omega))$, be a local weak supersolution of the fractional porous media equation (2.9). Then $v(\cdot,0) \ge 0$ a.e. in Ω implies $v(\cdot,t) \ge 0$ a.e. in Ω, for all $t \in [0,T]$.*

Proof. Let $0 < T_0 < T_1 < T$, and let $\tau > 0$. We claim that

$$\liminf_{\varepsilon\to 0^+} \int_{T_0}^{T_1} \Big\langle |v|^{m-1}v_-\,, \partial_t v\Big\rangle\,dt \ge -\frac{\tau}{m+1}\,. \tag{3.13}$$

Here and henceforth, $\langle\cdot\,,\cdot\rangle$ denotes the pairing between $\mathcal{D}^{s,2}_0(\Omega)$ and its dual. The claim implies

$$\int_{\{v(\cdot,T_1)<0\}} |v(x,T_1)|^{m+1}\,dx \le \int_{\{v(\cdot,T_0)<0\}} |v(x,T_0)|^{m+1}\,dx + \tau\,. \tag{3.14}$$

To prove the claim, we first note that

$$-\int_{T_0}^{T_1}\iint_{\mathbb{R}^{2N}} \frac{\big((|v|^{m-1}v)_\varepsilon(x,t) - (|v|^{m-1}v)_\varepsilon(y,t)\big)}{|x-y|^{N+2s}}\Big(\big((|v|^{m-1}v)_\varepsilon\big)_-(x,t) - \big((|v|^{m-1}v)_\varepsilon\big)_-(y,t)\Big)dx\,dy\,dt$$

is a non-negative quantity, which can be seen by direct inspection. In view of Lemma 3.5, we deduce

$$\liminf_{\varepsilon\to 0^+} \int_{T_0}^{T_1} \int_\Omega \big((|v|^{m-1}v)_\varepsilon\big)_- \partial_t v_\varepsilon\,dx\,dt \ge -\frac{1}{3}\cdot\frac{\tau}{m+1}\,. \tag{3.15}$$

By Lemma 3.6,

$$\sup_{\varepsilon>0} \int_{T_0}^{T_1} [(|v|^{m-1}v)_\varepsilon]^2_{H^s(\mathbb{R}^N)}\,dt < +\infty\,. \tag{3.16a}$$

and

$$\lim_{j\to\infty} \int_{T_0}^{T_1} [\varphi_j - \varphi]^2_{H^s(\mathbb{R}^N)}\,dt = 0\,, \tag{3.16b}$$

along an appropriate sequence $(\varepsilon_j)_j$, with $\varepsilon_j \to 0^+$ as $j \to +\infty$, where φ_j and φ denote the negative parts of $(|v|^{m-1}v)_{\varepsilon_j}$ and $|v|^{m-1}v$, respectively. In view of Lemma 3.5, condition (3.16a) implies that

(3.17) $\partial_t z_\varepsilon$ belongs to a bounded set in the dual of $L^2(T_0,T_1;\mathcal{D}^{s,2}_0(\Omega))$.

By (3.16b), (3.17), and by Cauchy-Schwartz inequality in $L^2([T_0,T_1];\mathcal{D}^{s,2}_0(\Omega))$, we have

$$\lim_{j\to\infty} \int_{T_0}^{T_1} \int_\Omega \Big[\big((|v|^{m-1}v)_{\varepsilon_j}\big)_- - |v|^{m-1}v_-\Big]\cdot\partial_t v_{\varepsilon_j}\,dx\,dt = 0\,. \tag{3.18}$$

By (3.17) and Banach-Alaouglu theorem, we also have

$$\liminf_{j\to\infty}\int_{T_0}^{T_1}\Big\langle |v|^{m-1}v_-\,,\,\partial_t v-\partial_t v_{\varepsilon_j}\Big\rangle\,dt\geq -\frac{1}{3}\frac{\tau}{m+1}\,. \tag{3.19}$$

The claim then follows by (3.15), (3.18), and (3.19).

Since $\tau>0$ was arbitrary, from (3.14) we infer

$$\int_{\{v(\cdot,T_1)<0\}}|v(x,T_1)|^{m+1}\,dx\leq\int_{\{v(\cdot,T_0)<0\}}|v(x,T_0)|^{m+1}\,dx\,. \tag{3.20}$$

We can repeat this argument for every $T_0\in(0,T)$ and for all $T_1\in(T_0,T)$. Thus,

$$t\longmapsto\int_{\{v(\cdot,t)<0\}}|v(x,t)|^{m+1}\,dx\qquad\text{is non-increasing on }(0,T). \tag{3.21}$$

Moreover,

$$\frac{1}{m+1}\int_0^T\int_\Omega\partial_t(v^{m+1})\,dx\,dt=\int_0^T\big\langle v^m\,,\partial_t v\big\rangle\,dt=-\int_0^T\iint_{\mathbb{R}^{2N}}\frac{\big(v^m(x,t)-v^m(y,t)\big)^2}{|x-y|^{N+2s}}\,dx\,dy\,dt\,.$$

As $v^m\in L^2(0,T\,;\mathcal{D}_0^{s,2}(\Omega))$, we have $v^{m+1}\in W^{1,2}(0,T\,;L^2(\Omega))$. Then $v^{m+1}\in C([0,T]\,;L^2(\Omega))$ and

$$\sup_{t\in(0,T)}\int_{\{v(\cdot,t)<0\}}|v(x,t)|^{m+1}\,dx<+\infty\,. \tag{3.22}$$

Then, from (3.21) and (3.22), for all $t\in(0,T)$ we infer

$$\int_{\{v(\cdot,t)<0\}}|v(x,t)|^{m+1}\,dx\leq\int_{\{v(\cdot,t)<0\}}|v(x,0)|^{m+1}\,dx=0\,,$$

where in the equality we used that $v(\cdot,0)\geq 0$ a.e. As a consequence, $v(\cdot,t)\geq 0$ a.e., as well. □

Remark 3.8. Some unessential changes in the proof of Lemma 3.7 provide the comparison principle with respect to constants: if $v_0\geq\delta>0$ and v is a local weak supersolution with $v(\cdot,0)=v_0$, then $v(\cdot,t)\geq\delta$ a.e. in Ω, for all $t\in[0,T]$.

Eventually, we prove the comparison principle.

Proof of Proposition 3.4. Let $0<T_0\leq T_1\leq T$ and let $\tau>0$ be fixed. Let us abbreviate $u_\tau=u+\tau$ and $v_\tau=v+\tau$. In view of Lemma 3.7 and Remark 3.8, the assumption that $u_0\geq 0$ and $v_0\geq 0$ implies that $u_\tau(\cdot,t),v_\tau(\cdot,t)\geq\tau$ a.e. in Ω, for all $t\in[0,T]$. Then, let $M=\max\{\|u\|_{L^\infty(\Omega)},\|v\|_{L^\infty(\Omega)}\}$ and set $M_\tau=M+\tau$. Thus, u_τ^m (*resp.*, v_τ^m) is a local weak subsolution (*resp.*, supersolution) of

$$\partial_t\varphi+c(x)(-\Delta)^s\varphi=0\,,\quad\text{in }\Omega\times(0,T),\qquad\text{where }m\cdot\tau^{1-\frac{1}{m}}\leq c(x)\leq m\cdot M_\tau^{1-\frac{1}{m}}.$$

By [8, Theorem 4.1], $u_\tau^{m-1}\partial_t u_\tau$ is continuous on $\Omega\times[T_0,T_1]$. The results of [1] imply that so is u_τ^{m-1}, by composition. As $u_\tau\geq\tau$, $\partial_t u_\tau$ is continuous on $\Omega\times[T_0,T_1]$. Then, u_τ is a strong subsolution, i.e., for every $\varphi\in C^\infty(\Omega\times[T_0,T_1])$, with support in $\Omega\times[T_0,T_1]$, we have

$$\int_{T_0}^{T_1}\int_\Omega\varphi\partial_t u_\tau\,dx\,dt+\int_{T_0}^{T_1}\iint_{\mathbb{R}^{2N}}\frac{u_\tau^m(x,t)-u_\tau^m(y,t)}{|x-y|^{N+2s}}\big(\varphi(x,t)-\varphi(y,t)\big)\,dx\,dy\,dt\leq 0\,,$$

and, similarly, the reverse inequality holds with u_τ replaced by v_τ. By difference, we then have

$$\int_{T_0}^{T_1}\int_\Omega\varphi\partial_t\chi_\tau\,dx\,dt+\int_{T_0}^{T_1}\iint_{\mathbb{R}^{2N}}\frac{\xi_\tau(x,t)-\xi_\tau(y,t)}{|x-y|^{N+2s}}\big(\varphi(x,t)-\varphi(y,t)\big)\,dx\,dy\,dt\leq 0\,,$$

where we set $\chi_\tau = u_\tau - v_\tau$, and $\xi_\tau = u_\tau^m - v_\tau^m$.

Let $(f_j)_j \subset C^1(\mathbb{R})$ be a pointwise increasing sequence such that, for all $j \in \mathbb{N}$, f_j is strictly increasing on $(0, 2^{-j})$ and $0 \le f_j \le 1$, $f_j(s) = 0$ if $s \le 0$, $f_j(s) = 1$ if $s \ge 2^{-j}$. Then,

$$\iint_{\mathbb{R}^{2N}} \frac{\xi_\tau(x,t) - \xi_\tau(y,t)}{|x-y|^{N+2s}} \big(f_j(\xi_\tau(x,t)) - f_j(\xi_\tau(y,t))\big)\, dx\, dy \ge 0\,,$$

because f_j is a non-decreasing function on $\mathbb{R}$. Thus, when we plug in $\varphi = f_j(\xi_\tau)$ we get

$$\int_{T_0}^{T_1} \int_\Omega f_j(\xi_\tau) \partial_t \chi_\tau \, dx\, dt \le 0\,.$$

By monotone convergence, we arrive at

$$\int_{T_0}^{T_1} \int_\Omega \operatorname{sgn}(\chi_\tau)_+ \partial_t \chi_\tau \, dx\, dt = \int_{T_0}^{T_1} \int_\Omega \operatorname{sgn}(\xi_\tau)_+ \partial_t \chi_\tau \, dx\, dt \le 0\,,$$

where in the equality we also used that $\operatorname{sgn}(\chi_\tau)_+ = \operatorname{sgn}(\xi_\tau)_+$. Since for a.e. $t \in [T_0, T_1]$ we have

$$\frac{d}{dt} \int_\Omega (u(x,t) - v(x,t))_+ \, dx = \frac{d}{dt} \int_\Omega (u_\tau(x,t) - v_\tau(x,t))_+ \, dx = \int_\Omega \operatorname{sgn}(\chi_\tau(x,t))_+ \partial_t \chi_\tau(x,t)\, dx\,,$$

it follows that

$$\int_\Omega (u(x,T_1) - v(x,T_1))_+ \, dx \le \int_\Omega (u(x,T_0) - v(x,T_0))_+ \, dx\,.$$

Recall that T_0, T_1 were arbitrary in $(0,T)$, with $T_0 < T_1$. By the material above, the function defined by (3.3) is non-increasing on the interval $(0,T)$, as desired. Since both u and v belong to $C([0,T]; L^1(\Omega))$, it is also bounded and

$$\sup_{t \in (0,T)} \int_\Omega (u(x,t) - v(x,t))_+ \, dx = \int_\Omega (u_0 - v_0)_+ \, dx\,,$$

which clearly implies (3.4). □

4. The fractional semilinear spectral problem

We provide L^∞ bounds for q-semilinear s-eigenfunctions u corresponding to $\lambda \in \mathfrak{S}(\Omega; s, q)$ in terms of the $L^q(\Omega)$-norm of u and of the eigenvalue λ.

Proposition 4.1. *Let $N \ge 2$, $s \in (0,1)$, $q \in (1, 2_s^*)$, and let Ω be an open set and assume the embedding $\mathcal{D}_0^{s,2}(\Omega) \hookrightarrow L^q(\Omega)$ to be compact. Let $\lambda \in \mathfrak{S}(\Omega; s, q)$ and let $u \in \mathcal{D}_0^{s,2}(\Omega)$ be a corresponding q-semilinear s-eigenfunction. Then*

$$\|u\|_{L^\infty(\Omega)} \le \mathcal{C} \lambda^{\frac{2_s^*}{2(2_s^* - q)}} \|u\|_{L^q(\Omega)}\,, \tag{4.1}$$

where the constant $\mathcal{C}$ depends on N, s, and q, only.

Proof. Fix $\beta > 1$. For all $a, b \in \mathbb{R}$, the inequality

$$\tfrac{4\beta^2}{(\beta+1)^2} (a^{\frac{\beta+1}{2}} - b^{\frac{\beta+1}{2}})^2 \le (a-b)(a^\beta - b^\beta)$$

follows by the fundamental theorem of calculus and Jensen's inequality, see e.g. [6, Lemma C.1]. Fixing $M > 0$, writing $a = \min\{u(x)\,, M\}$, $b = \min\{u(y)\,, M\}$, and integrating against the singular kernel we get

$$\begin{aligned}\frac{4\beta}{(\beta+1)^2}\iint_{\mathbb{R}^{2N}}&\frac{(\min\{u(x)\,,M\}^{\frac{\beta+1}{2}}-\min\{u(y)\,,M\}^{\frac{\beta+1}{2}})^2}{|x-y|^{N+2s}}\,dx\,dy\\&\le\iint_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))(\min\{u(x)\,,M\}^{\beta}-\min\{u(y)\,,M\}^{\beta})}{|x-y|^{N+2s}}\,dx\,dy\,.\end{aligned}\tag{4.2}$$

On the one hand, the choice $\varphi=\min\{u\,,M\}^\beta$ in (2.5) implies that the right hand side in (4.2) does not exceed

$$\lambda\|u\|_{L^q(\Omega)}^{2-q}\int_\Omega u(x)^{\beta+q-1}\,dx\,.$$

On the other hand, by the compactness of the embedding $\mathcal{D}_0^{s,2}(\Omega)\hookrightarrow L^q(\Omega)$ and by a density argument, Sobolev inequality (2.3) holds with $v=\min\{u\,,M\}^{\frac{\beta+1}{2}}$. Thus, the left hand side in (4.2) is at least

$$\mathcal{S}(N,s)\frac{4\beta}{(\beta+1)^2}\left(\int\min\{u\,,M\}^{\frac{\beta+1}{2}2^*}\right)^{\frac{2}{2^*}}\,.$$

Since $M>0$ was arbitrary, setting $\Lambda=\lambda\|u\|_{L^q(\Omega)}^{2-q}$, from (4.2) we then infer

$$\mathcal{S}(N,s)\left(\int u^{\frac{\beta+1}{2}2^*}\right)^{\frac{2}{2^*}}\le\Lambda\frac{(\beta+1)^2}{4\beta}\int_\Omega u(x)^{\beta+q-1}\,dx\,.\tag{4.3}$$

If $1<q<2$, (4.3) implies (4.1) (for a proof, see the second part of the proof of [5, Lemma C.1]). If instead $2\le q<2^*$, then by Hölder inequality

$$\int_\Omega u(x)^{\beta+q-1}\,dx\le\|u\|_{L^q(\Omega)}^{q-2}\left(\int_\Omega u^{\frac{\beta+1}{2}}q\,dx\right)^{\frac{2}{q}}\,,$$

whence if follows that

$$\mathcal{S}(N,s)\left(\int u^{\frac{\beta+1}{2}2^*}\right)^{\frac{2}{2^*}}\le\Lambda\frac{(\beta+1)^2}{4\beta}\|u\|_{L^q(\Omega)}^{q-2}\left(\int_\Omega u^{\frac{\beta+1}{2}}q\,dx\right)^{\frac{2}{q}}\,,$$

and from this we can infer the desired estimate (4.1) again (see, e.g., the iteration scheme in first part of the proof of [5, Lemma C.1].) □

4.1. **The first eigenvalue.** The following elementary Proposition contains a general property of the first semilinear fractional eigenvalue. We stress that in this subsection the open set Ω is not required to satisfy any specific assumption, except supporting a compact embedding.

Proposition 4.2. *Let $s\in(0,1)$, let $q\in(1,2_s^*)$, and let Ω be an open set for which the embedding of $\mathcal{D}_0^{s,2}(\Omega)$ into $L^q(\Omega)$ is compact. Then the infimum in (1.5) is a minimum. Moreover, any minimiser is either a strictly positive or a strictly negative function.*

Proof. The existence of a minimiser is an immediate consequence of the direct methods in the calculus of variations. The fact that it must have constant sign follows by Lemma 3.2. Then, the last statement follows by the strong minimum principle of Proposition 3.3. □

We prove some properties of $\lambda_1(\Omega;s,q)$ that hold conditionally, depending on the value of q.

Proposition 4.3. *Let $s\in(0,1)$, $q\in(1,2)$, and let Ω be an open set with $\lambda_1(\Omega;s,q)>0$. If $\lambda\in\mathfrak{S}(\Omega;s,q)$ and u is a corresponding eigenfunction, then $u\ge0$ a.e. in Ω implies $\lambda=\lambda_1(\Omega;s,q)$.*

Proof. By assumption, the embedding $\mathcal{D}_0^{s,2}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. Then, since $q \in (1,2)$, by Gagliardo-Nirenberg interpolation inequality (see [9, Lemma 2.3]) it is also compact. Thus, the assumptions of Proposition 4.2 are valid.

Let $v \in \mathcal{D}_0^{s,2}(\Omega)$ be a first eigenfunction and assume that $v > 0$ a.e. in Ω. Then, let $\lambda \in \mathfrak{S}(\Omega; s, q)$, let u be a corresponding eigenfunction, and assume that $u \geq 0$ a.e. in Ω, as well. This implies $u > 0$ a.e. in Ω by the strong minimum principle (Proposition 3.3). Being free to multiply by constants, we shall also assume both u and v to have unit norm in $L^q(\Omega)$.

Fix $\varepsilon > 0$ and write $u_\varepsilon = u + \varepsilon$. For every $x, y \in \mathbb{R}^N$, by [4, Proposition 4.2] with $p = 2$ we have

$$(u(x) - u(y)) \left(\frac{v(x)^q}{u_\varepsilon(x)^{q-1}} - \frac{v(y)^q}{u_\varepsilon(y)^{q-1}} \right) \leq |v(x) - v(y)|^q |u(x) - u(y)|^{2-q}.$$

Multiplying by the kernel $|x-y|^{N+2s} = |x-y|^{N\frac{q}{2}+sq+N(1-\frac{q}{2})+s(2-q)}$ and integrating yields

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))}{|x-y|^{N+2s}} \left(\frac{v(x)^q}{u_\varepsilon(x)^{q-1}} - \frac{v(y)^q}{u_\varepsilon(y)^{q-1}} \right) dxdy \leq \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^q}{|x-y|^{N\frac{q}{2}+sq}} \frac{|u(x) - u(y)|^{2-q}}{|x-y|^{N(1-\frac{q}{2})+s(2-q)}} dxdy.$$

By Hölder inequality with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$, the right hand side is bounded from above by

$$\lambda_1(\Omega; s, q)^{\frac{q}{2}} \lambda^{\frac{2-q}{2}},$$

because of the equations satisfied by u and v and of their normalisation in $L^q(\Omega)$. Since $\varphi = v^q/u_\varepsilon^{q-1}$ is an admissible test function in (2.5), we have

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))}{|x-y|^{N+2s}} \left(\frac{v(x)^q}{u_\varepsilon(x)^{q-1}} - \frac{v(y)^q}{u_\varepsilon(y)^{q-1}} \right) dxdy = \lambda \int_\Omega u(x)^{q-1} \frac{v(x)^q}{(u(x)+\varepsilon)^{q-1}}\, dx.$$

Therefore, for every $\varepsilon > 0$ we end up with inequality

$$\lambda \int_\Omega u(x)^{q-1} \frac{v(x)^q}{(u(x)+\varepsilon)^{q-1}}\, dx \leq \lambda_1(\Omega; s, q)^{\frac{q}{2}} \lambda^{\frac{2-q}{2}}. \tag{4.4}$$

Since $u > 0$ a.e. in Ω, applying Fatou's Lemma and dividing λ out we arrive at

$$1 = \int_\Omega v(x)^q\, dx \leq \left(\frac{\lambda_1(\Omega; s, q)}{\lambda} \right)^{\frac{q}{2}},$$

which gives $\lambda \leq \lambda_1(\Omega; s, q)$. The opposite inequality is obvious by definition of $\lambda_1(\Omega; s, q)$. □

Proposition 4.4. *Let $s \in (0,1)$, let $q \in (1,2)$, and let Ω be an open set in $\mathbb{R}^N$ with $\lambda_1(\Omega; s, q) > 0$. Then $\lambda_1(\Omega; s, q)$ is simple, i.e., all the corresponding eigenfunctions are mutually proportional.*

Proof. Let u and v be first eigenfunctions. With no loss of generality, assume that both u and v are non-negative functions. We may also assume both u and v to have unit norm in $L^q(\Omega)$. For all $t \in [0,1]$, consider the function $\xi_t : \Omega \to \mathbb{R}^2$ defined by $\xi_t(x) = (t^{\frac{1}{q}} u(x), (1-t)^{\frac{1}{q}} v(x))$. Let $\|\cdot\|_{\ell^q}$ denote the ℓ^q-norm in $\mathbb{R}^2$. Then, the convexity of $\tau \mapsto |\tau|^{\frac{2}{q}}$ implies

$$\|\xi_t(x) - \xi_t(y)\|_{\ell^q}^2 \leq t(u(x) - u(y))^2 + (1-t)(v(x) - v(y))^2, \qquad \text{for all } x, y. \tag{4.5a}$$

Also, for every $t \in [0,1]$, set $\sigma_t(x) = \|\xi_t(x)\|_{\ell^q}$, for $x \in \Omega$, and $\sigma_t(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. Then

$$(\sigma_t(x) - \sigma_t(y))^2 = \Big| \|\xi_t(x)\|_{\ell^q} - \|\xi_t(y)\|_{\ell^q} \Big|^2 \qquad \text{for all } x, y. \tag{4.5b}$$

Hence, by triangle inequality, $\sigma_t \in \mathcal{D}_0^{s,2}(\Omega)$ and we have

$$\iint_{\mathbb{R}^{2N}} \frac{(\sigma_t(x)-\sigma_t(y))^2}{|x-y|^{N+2s}} dx\, dy \le t \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dx\, dy + (1-t) \iint_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))^2}{|x-y|^{N+2s}} dx\, dy\,.$$

The normalisation in $L^q(\Omega)$ of u and of v implies that the right hand side in the latter equals $\lambda_1(\Omega; s, q)$. On the other hand, the left hand side is larger than or equal to $\lambda_1(\Omega; s, q)$, because

$$\int_\Omega |\sigma_t(x)|^q\, dx = \int_\Omega \|\xi_t(x)\|_{\ell^q}^q\, dx = t \int_\Omega u(x)^q\, dx + (1-t)\int_\Omega v(x)^q\, dx = 1\,,$$

and thus σ_t is admissible for the minimisation problem that defines $\lambda_1(\Omega; s, q)$. Therefore, for every $t \in [0, 1]$ the previous integral inequality is an equality. As a consequence, the pointwise identity

$$(\sigma_t(x) - \sigma_t(y))^2 = t(u(x) - u(y))^2 + (1-t)(v(x) - v(y))^2$$

holds for all $t \in [0, 1]$ and for a.e. $x, y \in \Omega$. In view of (4.5), the latter yields the equality case in triangle inequality

$$\Big| \|\xi_t(x)\|_{\ell^q} - \|\xi_t(y)\|_{\ell^q} \Big| \le \|\xi_t(x) - \xi_t(y)\|_{\ell^q}\,,$$

which occurs if and only if there exists $\alpha(x, y) \in \mathbb{R}$ with $\xi_t(x) = \alpha(x, y)\xi_t(y)$. Owing to the definition of ξ_t, it follows that $u(x) = \alpha(x, y)u(y)$ and $v(x) = \alpha(x, y)v(y)$. In conclusion, for a.e. x, y we have

$$\frac{u(x)}{v(x)} = \frac{u(y)}{v(y)}\,,$$

and this concludes the proof. □

4.2. **Higher eigenvalues.** Besides the first eigenvalue (1.5), higher eigenvalues also exist. In fact, it is straightforward to check that the squared norm (1.2) in $\mathcal{D}_0^{s,2}(\Omega)$ satisfies the Palais-Smale condition. Hence, in view of [16, Theorem 5.7], $\mathfrak{S}(\Omega; q, s)$ is an infinite set. More precisely, for all $n \in \mathbb{N}$ we denote by $\mathfrak{T}_n(\Omega; s, q)$ the collection of all subsets A of

$$\left\{ u \in \mathcal{D}_0^{s,2}(\Omega) : \int_\Omega |u|^q\, dx = 1 \right\} \tag{4.6}$$

that are symmetric and compact in $\mathcal{D}_0^{s,2}(\Omega)$ and satisfy the following property: for every $k < n$, there exist no odd and continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. Then, setting

$$\lambda_n(\Omega; s, q) = \inf_{A \in \mathfrak{T}_n(\Omega; s,q)} \max_{u \in A} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dx\, dy \tag{4.7}$$

one defines an unbounded non-decreasing sequence of q-semilinear s-eigenvalues.

Remark 4.5. In general, $\mathfrak{S}(\Omega; s, q)$ is closed in $\mathbb{R}$. Indeed, given a sequence $(\lambda_j)_j \subset \mathfrak{S}(\Omega; s, q)$ converging to a positive number λ, there is a corresponding sequence of q-semilinear s-eigenfunctions (obtained by renormalisation in L^2) which has constant L^2 norm and converging norm in $\mathcal{D}_0^{s,2}(\Omega)$. By uniform convexity, some subsequence is converging strongly to a limit u in $L^2(\Omega)$ and this implies that u is a q-semilinear s-eigenfunction corresponding to λ.

Little more is known about $\mathfrak{S}(\Omega; s, q) \setminus \{\lambda_1(\Omega; s, q)\}$. In particular, it is not known if (4.7) gives a complete descritpion of $\mathfrak{S}(\Omega; s, q)$, nor is it known if the latter is a discrete set.

5. The fractional semilinear Lane Emden density

In the present section, we introduce a Hardy-type inequality with a negative power of the Lane-Emden density acting as a singular weight. Since this inequality is of use on arbitrary open sets, we extend Definition 2.3.

Lemma 5.1. *Let $N \geq 2$, let $s \in (0,1)$, let $q \in (1,2)$, and let Ω be an open set in $\mathbb{R}^N$ with $\lambda_1(\Omega, s, q) > 0$. Then*

$$w_{\Omega;s,q}(x) = \lim_{r\to\infty} w_{\Omega\cap B_r;s,q}(x)\,, \qquad \text{for all } x \in \Omega. \tag{5.1}$$

where for every $r > 0$ we set $w_r(x) = w_{\Omega\cap B_r;s,q}(x)$ if $x \in B_r$ and $w_r(x) = 0$ otherwise.

By Proposition 3.1, the limit (5.1) always exists, for any open set Ω. Then, the following definition is well posed. Lemma 5.1 implies that it is consistent with Definition 2.3.

Definition 5.2. Let $N \geq 2$, let $s \in (0,1)$, let $q \in (1,2)$, and let Ω be an open set in $\mathbb{R}^N$. Then we call the function defined by (5.1) the (s,q)- Lane Emden density of Ω.

Proof of Lemma 5.1. As $r \to +\infty$, the (s,q)-Lane Emden density w_r on $\Omega \cap B_r$ converges to an appropriate function $\overline{w} \leq w_{\Omega,s,q}$. By minimality, for every given $\varphi \in C_0^\infty(\Omega)$ there exists $R_\varphi > 0$ such that for all $r \geq R_\varphi$ we have

$$\iint_{\mathbb{R}^{2N}} \frac{(w_r(x)-w_r(y))^2}{|x-y|^{N+2s}} dxdy - \int_{\Omega\cap B_r} w_r^q\,dx \leq \iint_{\mathbb{R}^{2N}} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}} dxdy - \int_{\Omega\cap B_r} \varphi^q\,dx\,. \tag{5.2}$$

Note that the equation for w_r is (2.5) with $\Omega \cap B_r$ in place of Ω, $u = w_r$, and $\lambda = \|w_r\|^{q-2}_{L^q(\Omega\cap B_r)}$. Testing with $\varphi = w_r$ the equation for w_r, we get

$$\iint_{\mathbb{R}^{2N}} \frac{(w_r(x)-w_r(y))^2}{|x-y|^{N+2s}} dx\,dy = \int_{\Omega\cap B_r} w_r(x)^q\,dx \leq \lambda(\Omega,s,q)^{\frac{q}{2}} \left[\iint_{\mathbb{R}^{2N}} \frac{(w_r(x)-w_r(y))^2}{|x-y|^{N+2s}} dx\,dy\right]^{\frac{q}{2}},$$

where in the second inequality we also used that $w_r = 0$ in $\Omega \setminus B_r$. Since $q < 2$, we deduce that w_r converges to $\overline{w}$ weakly in $\mathcal{D}_0^{s,2}(\Omega)$ and strongly in $L^q(\Omega)$. Thus, passing to the limit as $r \to \infty$ in (5.2) we obtain

$$\iint_{\mathbb{R}^{2N}} \frac{(\overline{w}(x)-\overline{w}(y))^2}{|x-y|^{N+2s}} dxdy - \int_{\Omega\cap B_r} \overline{w}(x)^q\,dx \leq \iint_{\mathbb{R}^{2N}} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}} dxdy - \int_{\Omega\cap B_r} \varphi(x)^q\,dx$$

for all $\varphi \in C_0^\infty(\Omega)$, which by uniqueness implies that $\overline{w} = w_{\Omega,s,q}$. □

The following Proposition contains the Hardy-type inequality.

Proposition 5.3. *Let $s \in (0,1)$, let $q \in (1,2)$, let Ω be an open set in $\mathbb{R}^N$. Then for all $u \in C_0^\infty(\Omega)$*

$$\int_\Omega \frac{u^2}{w_{\Omega,s,q}^{2-q}}\,dx \leq \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dx\,dy\,, \tag{5.3}$$

with the agreement that the integrand in the left hand side is 0 at all points where $w_{\Omega,s,q} = +\infty$.

Proof. We first prove (5.3) in the special case of a (s,q)-admissible open set. We write $w = w_{\Omega,s,q}$, we fix $u \in C_0^\infty(\Omega)$, and we take $\varepsilon > 0$. By Proposition 4.1, $w \in \mathcal{D}_0^{s,2}(\Omega) \cap L^\infty(\Omega)$. Hence, so does $(w+\varepsilon)^{-1}$, because $t \mapsto (t+\varepsilon)^{-1}$ is a Lipschitz function on $(0,\infty)$. Then, by [3, Lemma 2.4] we can plug $\varphi = u^2/(w+\varepsilon)$ into the equation for w and get

$$\int_\Omega w(x)^{q-1} \frac{u(x)^2}{w(x)+\varepsilon}\,dx = \iint_{\mathbb{R}^{2N}} \frac{(w(x)-w(y))}{|x-y|^{N+2s}} \left(\frac{u(x)^2}{w(x)+\varepsilon} - \frac{u(y)^2}{w(y)+\varepsilon}\right) dx\,dy\,,$$

for all $\varepsilon > 0$. In view of [4, Proposition 4.2], by Fatou's Lemma it follows that

$$\int_{\{w>0\}} \frac{u^2}{w^{2-q}}\,dx \le \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dx\,dy\,,$$

and the conclusion follows, once we recall that $w > 0$ a.e. in Ω.

In the general case, we take $R > 0$ such that the support of u is contained in Ω_r for all $r \ge R$. For all such radii r, denoting $\Omega_r = \Omega \cap B_r$ and $w_r = w_{\Omega_r,s,q}$, by the material above we have

$$\int_{\Omega_r} \frac{u^2}{w_r^{2-q}}\,dx \le \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dx\,dy\,.$$

In view of Definition 2.3, by Fatou's Lemma we get the conclusion passing to the limit as $r \to \infty$. □

We present a very general L^∞ bound for fractional Lane Emden densities on open sets.

Proposition 5.4. *Let $s \in (0,1)$, let $q \in (1,2)$, and let Ω be an open set in $\mathbb{R}^N$. If $\lambda_1(\Omega,s,2) > 0$ then $w_{\Omega,s,q} \in L^\infty(\Omega)$ and there exists a constant $\mathcal{C} > 1$ depending only on N, s, and q, with*

$$\|w_{\Omega,s,q}\|_{L^\infty(\Omega)}^{2-q}\lambda_1(\Omega,s,2) \le \mathcal{C}\,. \tag{5.4}$$

Conversely, if $w_{\Omega,s,q} \in L^\infty(\Omega)$ then $\lambda_1(\Omega,s,2) \ge \|w_{\Omega,s,q}\|_{L^\infty(\Omega)}^{q-2}$.

Proof. Let us write $w = w_{\Omega,s,q}$. The last statement is a direct consequence of Proposition 5.3. Then, we assume that $\lambda_1(\Omega,s,2) > 0$ and we prove (5.4). To do so, up to an approximation argument we may assume with no restriction Ω to be smooth and bounded; then, $w \in L^\infty(\Omega)$ by Proposition 4.1, and we are left to prove the estimate (5.4).

To this aim we follow the lines of the proof of a similar result [2, Theorem 9]. An unessential translation in $\mathbb{R}^N$ allows us to assume that $w(0) = \|w\|_{L^\infty(\Omega)}$, so as to simplify notations.

We take $R > 0$ and a cut-off function $\zeta \in C_0^\infty(\Omega)$ from $B_{\frac{R}{2}}$ to B_R, satisfying $|\nabla\zeta| \le 2R^{-1}$. By the localised Caccioppoli estimate [7, Proposition 3.5], with $F = w^{q-1}$, $p = 2$, $\beta = 1$, $\delta = 0$, $L = 1$, and $\Omega' = B_R$, we have

$$\iint_{B_R\times B_R} \frac{(w(x)\zeta(x)-w(y)\zeta(y))^2}{|x-y|^{N+2s}} dx\,dy \le C_1\Big(w(0)^q R^N + w(0)^2 R^{N-2s}\Big)\,, \tag{5.5}$$

with a constant $C_1 > 0$ depending only on N, s. Moreover, the mere fact that $w \in L^\infty(\Omega)$ implies

$$\iint_{B_R\times(\mathbb{R}^n\setminus B_R)} \frac{(w(x)\zeta(x)-w(y)\zeta(y))^2}{|x-y|^{n+2s}} dx\,dy \le C_2 w(0)^2 R^{n-2s}\,, \tag{5.6}$$

where $C_2 > 0$ depends only on N, s, again.

Then we observe that for a constant $C_3 > 0$, depending only on N and s, we have

$$w(0) \le C_3\left[\frac{1}{\delta}\left(\fint_{B_{2R}} w^2\,dx\right)^{\frac12} + R^{2s}\left(w(0)^{q-1} + \delta^2\int_{\mathbb{R}^N\setminus B_R}\frac{w(x)}{|x|^{N+2s}}dx\right)\right]. \tag{5.7}$$

To see that (5.7) holds, one may repeat the proof of [7, Theorem 3.8] in case $F = w^{q-1}$, with little modification to make the interpolating parameter δ appear. With the choices

$$R = \left(\frac{w(0)^{2-q}}{2C_3}\right)^{\frac{1}{2s}} \qquad \text{and} \qquad \delta = \frac{1}{2\sqrt{C_3}}\,, \tag{5.8}$$

from (5.7) we infer

$$\int w^2\zeta^2\,dx \geq R^N\delta^2\left(\frac{w(0)}{2C_3}-\delta^2 w(0)\right)^2=\frac{R^N}{64C_3^3}w(0)^2\,. \tag{5.9}$$

From (5.5), (5.6), and (5.9) we deduce that

$$\frac{\displaystyle\iint_{\mathbb{R}^{2N}}\frac{(w(x)\zeta(x)-w(y)\zeta(y))^2}{|x-y|^{N+2s}}dx\,dy}{\displaystyle\int_\Omega w^2\zeta^2\,dx}\leq C_4\frac{w(0)^qR^N+w(0)^2R^{N-2s}}{w(0)^2R^N}=C_4w(0)^{q-2}\,,$$

where in the last equality the value of R fixed in (5.8) entered. Since

$$\lambda_1(\Omega,s,2)=\inf_{u\in\mathcal{D}_0^{s,2}(\Omega)\setminus\{0\}}\frac{\displaystyle\iint_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}dx\,dy}{\displaystyle\int_\Omega u(x)^2\,dx}$$

and the function $u=w\zeta$ is admissible for the infimum, this ends the proof. □

Under the stronger assumption that $\lambda_1(\Omega,s,q)>0$, we have the following estimate.

Proposition 5.5. *Let $s\in(0,1)$, let $q\in(1,2)$, and let Ω be an open set in $\mathbb{R}^N$ with $\lambda_1(\Omega,s,q)>0$. Then $w_{\Omega,s,q}\in L^\infty(\Omega)$ with the estimate*

$$\|w_{\Omega,s,q}\|_{L^\infty(\Omega)}\leq\mathcal{C}\lambda_1(\Omega,s,q)^{-\gamma}\,,$$

where the constant $\mathcal{C}>0$ and the exponent $\gamma>0$ depend on N, s, and q, only.

Proof. We note that $w_{\Omega,s,q}$ is a the first q-semilinear s-eigenfunction with $L^q(\Omega)$-norm $\lambda_1(\Omega,s,q)^{\frac{1}{q-2}}$. Then, the estimate follows at once by Proposition 4.1. □

Remark 5.6. We notice that Proposition 5.5 can also be seen as a particular case of the general estimate (5.4) of Proposition 5.4. Indeed, the positivity of the greatest lower bound $\lambda_1(\Omega;s,2)$ for the spectrum of the fractional (linear) s-Laplacian is, by definition, equivalent to the continuity of the embedding of $\mathcal{D}_0^{s,2}(\Omega)$ into $L^2(\Omega)$: domains with this property are not necessarily bounded, nor are they required to have finite measure; also, an open set Ω may support a Sobolev-Poincaré inequality that makes $\lambda_1(\Omega;s,2)$ strictly positive even if $\lambda_1(\Omega,s,q)=0$ for all $q\in(1,2)$ (examples are provided by domains of the form $\omega\times(-M,M)$ with $M>0$ and ω bounded in $\mathbb{R}^{N-1}$.) Conversely, given any $q\in(1,2)$, the fact that $\lambda_1(\Omega,s,q)>0$ implies that $\lambda_1(\Omega;s,2)>0$, too (in fact, it implies that the embedding $\mathcal{D}_0^{s,2}(\Omega)$ into $L^2(\Omega)$ is compact, by interpolation: see, e.g., [9, Lemma 2.3]).

6. Proof of the main results

6.1. **Proof of Theorem A.** Because $q\in(1,2)$, the assumption $\lambda_1(\Omega;s,q)>0$ implies the compactness of the embedding $\mathcal{D}_0^{s,2}(\Omega)\hookrightarrow L^q(\Omega)$, by [9, Theorem 1.3]. Then, the existence of a first eigenfunction follows by Proposition 4.2. Uniqueness up to proportionality is proved in Proposition 4.4. Eventually, the last statement is stated and proved in Proposition 4.3. □

6.2. **Proof of Theorem B.** By Proposition 3.4 and by an approximation argument, we can assume that Ω is bounded, that $v \in L^\infty(0,T;L^\infty(\Omega))$, and prove (1.8) under these assumptions. We set $M = \|v\|_{L^\infty(\Omega)}$, and we fix $\varepsilon > 0$. By (1.5), we can pick an open set Ω_ε in $\mathbb{R}^N$, with $\overline{\Omega} \subset \Omega_\varepsilon$, and

$$\lambda_1(\Omega, s, q) - \varepsilon < \lambda_1(\Omega_\varepsilon; s, q) < \lambda_1(\Omega, s, q) \tag{6.1}$$

Also, let w_ε be the (s,q)-Lane Emden density of Ω_ε, let $\tau > 0$, and let

$$z_{\varepsilon,\tau}(x,t) = \left(\frac{q-1}{2-q}\right)^{\frac{q-1}{2-q}} (t+\tau)^{\frac{1-q}{2-q}} w_\varepsilon(x)\,,$$

for all $x \in \Omega$ and all $t \in [0,T]$. By Proposition 5.5, $\|w_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \le \mathcal{C} \cdot \lambda_1(\Omega_\varepsilon, s, q)^{-\gamma}$, where $\mathcal{C} > 0$ and $\gamma > 0$ depend on N, s, and q, only. We recall that bounded solutions of the fractional Lane-Emden equation (1.4) in Ω_ε belong to $w_\varepsilon \in C^s(\overline{\Omega}_\varepsilon)$ (see, e.g., [15, Theorem 1.1]). Then, there exists $c_\varepsilon > 0$ with $w_\varepsilon \ge c_\varepsilon$ in $\overline{\Omega}$. Therefore,

$$z_{\varepsilon,\tau}(x,0) \ge c_\varepsilon \left(\frac{q-1}{2-q}\right)^{\frac{q-1}{2-q}} \tau^{\frac{1-q}{2-q}}\,.$$

Thus, there exists $\tau_0 = \tau_0(\varepsilon) > 0$ such that for every $\tau \in (0,\tau_0)$ we have $z_{\varepsilon,\tau}(x,0) \ge v(x,0)$. The functions v and $z_{\varepsilon,\tau}$ are, respectively, a weak subsolution and a weak supersolution of (2.9) in $\Omega \times (0,T)$, under homogeneous Dirichlet conditions on $\partial\Omega \times (0,T)$. By Proposition 3.4, $v \le z_{\varepsilon,\tau}$ a.e. in $\Omega \times (0,T)$, which implies

$$v(x,t) \le \left(\frac{q-1}{2-q}\right)^{\frac{q-1}{2-q}} t^{\frac{1-q}{2-q}} \Big(\lambda_1(\Omega,s,q) - \varepsilon\Big)^{-\gamma(N,s,q)}, \qquad \text{a.e. in } \Omega \times (0,T),$$

where we also used (6.1). As $\varepsilon > 0$ was arbitrary and $m = \frac{1}{q-1}$, we get the desired conclusion. □

6.3. **Proof of Theorem C.** We follow verbatim the monotonicity method of Vasquez [17, §20]. Note that for every $\varepsilon > 0$ the function $v + \varepsilon$ (henceforth abbreviated to v^ε) is a local weak solution of (2.9) in $\Omega \times (0,+\infty)$ with $v^\varepsilon = \varepsilon$ at the spatial boundary. By Lemma 3.7 and Theorem B, we have $\varepsilon \le v^\varepsilon \le M + \varepsilon$, where the constant M depends on N, s, q, and Ω, only. Thus, $|v^\varepsilon|^m$ is a local weak solution of

$$\partial_t u + c(x)(-\Delta)^s u = 0\,, \quad \text{in } \Omega \times (0,+\infty), \qquad \text{where } m \cdot \varepsilon^{1-\frac{1}{m}} \le c(x) \le m \cdot (M+\varepsilon)^{1-\frac{1}{m}}.$$

By uniform parabolic regularity (see, e.g., [8]), we deduce that $\partial_t |v^\varepsilon|^m$ is continuous on $\Omega \times (0,+\infty)$. In view of the results of [1], by composition so is $(v^\varepsilon)^{m-1}$ and we deduce the continuity of $\partial_t v = \partial_t v^\varepsilon$ on $\Omega \times (0,+\infty)$. Then, given $\varphi \in C_0^\infty(\Omega \times (0,+\infty))$ we can integrate by parts in time in the weak equation for v and deduce that v is a strong solution of (2.9), i.e., for a.e. $t > 0$ we arrive at

$$\int_\Omega \varphi \partial_t v \, dx + \iint_{\mathbb{R}^{2N}} \frac{v^m(x,t) - v^m(y,t)}{|x-y|^{N+2s}} \big(\varphi(x,t) - \varphi(y,t)\big)\, dx\, dy = 0\,.$$

Then, the function defined on $\Omega \times \mathbb{R}$ by

$$\theta(x,\tau) = \exp\left(\frac{q-1}{2-q} \cdot \tau\right) v(x, e^\tau) \tag{6.2}$$

is such that

$$\int_\Omega \varphi(x,\tau)\partial_\tau \theta(x,\tau)\, dx + \iint_{\mathbb{R}^{2N}} \frac{\theta^m(x,\tau) - \theta^m(y,\tau)}{|x-y|^{N+2s}} \big(\varphi(x,\tau) - \varphi(y,\tau)\big)\, dx = \frac{1}{m-1} \int_\Omega \theta \varphi \, dx \tag{6.3}$$

for a.e. $\tau \in \mathbb{R}$, for all $\varphi \in C_0^\infty(\Omega \times \mathbb{R})$. Also, we claim that

$$0 \le \theta(x,\tau) \le C(N,s,q)\lambda_1(\Omega,s,q)^{-\gamma(N,s,q)}\,, \qquad \partial_\tau \theta(x,\tau) \ge 0\,, \tag{6.4}$$

a.e. in $\Omega \times \mathbb{R}$. Indeed, the first inequality is a consequence of (6.2) and Theorem B. Then, for $\tau \ge 1$ we consider the rescaled weak solution $v_\tau(x,t) = \tau v(x, \tau^{m-1} t)$. Since $v_\tau(x,0) \ge v(x,0)$ for all $\tau \ge 1$, by Proposition 3.4 we have

$$0 \le \lim_{\tau \to 1^+} \frac{v_\tau(x,t) - v(x,t)}{\tau - 1} = v(x,t) + (m-1)\, t\, \partial_t v(x,t)\,, \qquad \text{a.e. in } \Omega,$$

for all $t \in (0,T)$, whence it follows that the second inequality in (6.4) also holds.

In view of (6.4), when we plug in $\varphi = \theta_\varepsilon^m$ in (6.3), recalling that $m = 1/(q-1)$ we obtain

$$\iint_{\mathbb{R}^{2N}} \frac{\big(\theta^m(x,\tau) - \theta^m(y,\tau)\big)^2}{|x-y|^{N+2s}}\, dx \le \frac{1}{m-1} \int_\Omega \theta^{m+1}\, dx = \frac{1}{m-1} \int_\Omega (\theta^m)^q\, dx\,.$$

Since $\lambda_1(\Omega,s,q) > 0$, recalling (1.5) and reabsorbing a term we arrive at

$$\iint_{\mathbb{R}^{2N}} \frac{\big(\theta^m(x,\tau) - \theta^m(y,\tau)\big)^2}{|x-y|^{N+2s}}\, dx \le \left(\frac{q-1}{2-q}\right)^{\frac{4}{2-q}} \lambda_1(\Omega,s,q)^{\frac{2q}{2-q}}\,, \qquad \text{for a.e. } \tau \in \mathbb{R}. \tag{6.5}$$

By (6.4) and by monotone convergence theorem, there exists $\chi \in L^\infty(\Omega)$, with $\chi \ge 0$ a.e. in Ω, such that $\theta(\cdot,\tau) \to \chi$ as $\tau \to +\infty$. In view of the energy bound (6.5), we have $\theta^m(\cdot,\tau) \rightharpoonup \chi^m$ weakly in $\mathcal{D}_0^{s,2}(\Omega)$. Thus, given $\phi \in C_0^\infty(\Omega)$ and $\ell > 0$, we have

$$\begin{aligned} \lim_{\tau_0 \to +\infty} \int_{\tau_0}^{\tau_0+\ell} \iint_{\mathbb{R}^{2N}} & \frac{\theta^m(x,\tau) - \theta^m(y,\tau)}{|x-y|^{N+2s}} \big(\phi(x) - \phi(y)\big)\, dx\, dy\, d\tau \\ &= \ell \iint_{\mathbb{R}^{2N}} \frac{\chi^m(x,\tau) - \chi^m(y,\tau)}{|x-y|^{N+2s}} \big(\phi(x) - \phi(y)\big)\, dx\, dy\,. \end{aligned} \tag{6.6a}$$

Since $v \in C((0,+\infty); L^1(\Omega))$, recalling (6.2) we see that condition (6.4) and monotone convergence theorem imply $\theta(\cdot,\tau) \to \chi$, as $\tau \to +\infty$, strongly in $L^1(\Omega)$. Then, for $\phi \in C_0^\infty(\Omega)$ and $\ell > 0$,

$$\lim_{\tau_0 \to +\infty} \int_{\tau_0}^{\tau_0+\ell} \int_\Omega \theta(x,\tau)\phi(x)\, dx\, d\tau = \ell \cdot \int_\Omega \chi\phi\, dx\,, \tag{6.6b}$$

By (6.2) and by monotone convergence, for all $\phi \in C_0^\infty(\Omega)$ and $\ell > 0$ we also have

$$\lim_{\tau_0 \to +\infty} \int_\Omega \theta(x,\tau_0+\ell)\phi(x)\, dx = \lim_{\tau_0 \to +\infty} \int_\Omega \theta(x,\tau_0)\phi(x)\, dx = \int_\Omega \chi\phi\, dx\,. \tag{6.6c}$$

Choosing a test function of the form $\varphi(x,\tau) = \phi(x)$, with $\phi \in C_0^\infty(\Omega)$, in (6.3) and taking into account (6.6), after the passage of the limit we arrive at

$$\iint_{\mathbb{R}^{2N}} \frac{\big(\chi^m(x) - \chi^m(y)\big)\big(\phi(x) - \phi(y)\big)}{|x-y|^{N+2s}}\, dx\, dy = \frac{1}{m-1} \int_\Omega \chi\phi\, dx\,,$$

Recalling that $m = 1/(q-1)$, by Proposition 4.3 and Proposition 4.4 we deduce that the function $u = \chi^m$ coincides, up to a multiplicative factor, to the fractional Lane-Emden density $w_{\Omega,s,q}$ of Ω.

Owing to [1, Theorem 3.3] (see also [13, Theorem 9.2]), v satisfies local Hölder estimates in space-time with an absolute Hölder exponent $\gamma \in (0,1)$. Then we can repeat the last part of the proof of [17, Theorem 20.1], by observing that all the functions $v_\rho(x,t) = \rho^{\frac{1}{m-1}} v(x,\rho t)$ satisfy (1.8) and form, as a consequence, an equicontinuous family on every compact subset of $\Omega \times (0,+\infty)$, and deduce the last statement by Ascoli-Arzelá theorem. □

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