ELECTROMAGNETIC HYPOGENE CO-SEISMIC SOURCES

GIOVANNI FRANZINA

Contents

1. Introduction

The task of predicting time, location, and energy, of a seismic event in advance enough to deliver warnings is a tough task to undertake. Deterministic predictions are out-of-reach. Some estimation of earthquake probabilities is however possible. Long-term chances that an earthquake of magnitude larger than a threshold will (or will not) take place in a given area within a given time period are sometimes provided by research groups. For example, magnitude and location of the 2004 Parkfield earthquake were correctly predicted [7], without however a precise prediction of its time of occurrence; the same section of the San Andreas fault is still the object of long-term probabilistic prediction [22]. Besides the specific example, in general this kind of prediction usually regards a rather large time window, and short-term precursors are very difficult to detect.

Instead, the advance notice that a shock is going to hit an area, within several to tens of seconds, does not only involve inferential statistics, but also physics. Is called *early warning* and is sometimes made possibile by the difference in propagation velocity between primary waves and secondary waves: for instance, seismic wave propagation in elastic homogeneous isotropic media is described by a vector-valued displacement function u obeying the PDEs

$$\varrho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u}\,,$$

where λ, μ are Lamé's coefficients [25]. Recalling the vector calculus identity

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{\nabla}^2 \boldsymbol{u}\,,$$

where $\boldsymbol{\nabla}^2$ is the componentwise Laplace operator, we deduce that $\boldsymbol{\nabla} \cdot u$ and $\boldsymbol{\nabla} \times u$ solve wave equations with propagation speed respectively given by

$$\sqrt{\frac{\lambda + 2\mu}{\varrho}} \quad \text{and} \quad \sqrt{\frac{\mu}{\varrho}}\,,$$

whose ratio is $\sqrt{2}$ at least. The earlier arrival of "P-waves", oscillating parallel to propagation, can be used to warn in advance of the imminent effect of the "S-waves", travelling slightly more slowly along the same direction but shaking in the orthogonal directions, and being therefore likely to carry a larger energy.

It makes sense to wonder if other "imminent" seismic precursors exist: In particular, if it makes sense to seek electromagnetic seismicity-related signals that might provide advance notice of an imminent earthquake: the possibility of links between subsurface electric currents and earthquake physics have been investigated for occasionally in the literature. Yet, little is known, nonetheless.

Here, we present an elementary magneto-quasistatic model aiming to study hypogene co-seismic source reconstruction starting from subsurface measurements of magnetic signals. There is no pretence of originality, nor is this paper supplemented by data or any specific material.

1.1. **Magnetic anomalies of possible coseismic nature.** Telluric electric currents, flowing throughout Earth's crust, can be measured; in particular, it is sometimes conjectured that those acting as sources for signals with frequency ranging between 10^{-3} and 10^3 Hz may admit some relation to seismology [21]. Possible meaningful causal explanations are indicated in friction and piezoelectric effects within rocks, due to the relative movement of fault blocks.

Measurements in experimental seismo-electromagnetic research were carried out to figure out about evidences of aperiodic changes in electromagnetic fields, with controversial results; in fact, claims of magnetic anomalies in the low-frequency band are sometimes asserted. The first instances in literature of papers supporting with some data these hypothesises seem to be those concerning two seismic events, one in Spitak, in Armenia, and another one in Loma Prieta, in California in 1989; an anomalous electromagnetic emission in the ULF range was measured in both cases; the instruments were believed to reveal a transient signal for hours before and after the earthquake [14]. As a matter of fact, other scientists [26] refuse to acknowledge significance to the findings about Loma Prieta earthquake, suggesting that they would be due to a sensor malfunction. Also, a recent surge in research in geophysics in this topic is related to the DEMETER mission; in this case, the magnetic measurements are taken in orbit rather than at the surface; some authors related their findings to earthquakes in Sichuan [27] and Haiti [6].

Thenceforth, a number of experts started investigating the matter, with the object of understanding if the simultaneous occurrence of seismic activity in the crust and of electromagnetic anomalies in ULF bands does take place, and if this happens by chance, with no cause-effect relation, or if instead the two phaenomena are linked by a causal relationship [10, 13, 16, 18]. For instance, in [18] the authors conjecture a source-generating mechanism based on micro-crack propagation. Their considerations are based on a simple dimensional analysis. According to findings based on geometric deep sounding, the macroscopic crustal *dielectric permittivity* ε is small relative to the average *conductivity* σ of rocks: precisely, the ratio ε/σ is estimated to range between 10^{-7} s and 10^{-5} s. Since changes in geomagnetic fields, geoelectric potentials, and electrokinetic potential on the water-solid contact are not expected to cause fast ULF variation, a possible stress-induced mechanism with this time-scale is the opening of cracks with lengthscale between 10^{-4} and 10^{-1} m at the seismic velocity of 10^3 m/s. In this picture, the consequent EM noise would dissipate within the region interested by the phaenomenon, producing ULF emission under a cut-off at 1 Hz.

1.2. **General electrodynamic models in seismology.** At the occurrence of a seismic event, and in correspondence with its preparatory phase, the scalar parameters (azimuth, dip and depth) describing the (affine) fault plane, the local crustal strain, and the width of the portion of plane interested by yield stress, cope with ground motion, friction and crack opening. The induced movement of electrically charged particles generates an electromagnetic signal. Subsurface charge motions take place along the field lines for a vector field $\boldsymbol{v}$. By charge consevation, $\boldsymbol{J} = \rho^{-1}\boldsymbol{v}$ compensates the rate of change in time of the electric charge distribution in Earth's crust denoted by $\rho(\cdot, t)$ at time t, so that

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0\,. \tag{1.1}$$

By causality, $\boldsymbol{J}$ must be a non-stationary function; this yields a time-varying magnetic field. Indeed, (1.1) and Gauß's law $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$ for the electric displacement field $\boldsymbol{D}$ imply (up to harmonic fields)

$$-\boldsymbol{D} + \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}\,. \tag{1.2a}$$

Also, in view of Faraday's induction law, a non-stationary magnetic induction field $\boldsymbol{B}$ induces a non-conservative electric field $\boldsymbol{E}$ satisfying

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{E} = 0\,. \tag{1.2b}$$

In addition to (1.2a) and (1.2b), Maxwell's equations include Gauss's law

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho\,, \tag{1.2c}$$

and the constraint of absence of magnetic sources:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0\,. \tag{1.2d}$$

1.3. **Constitutive properties of the propagation medium.** The medium filling Earth's crust is described by the constitutive relations

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}\,, \quad \boldsymbol{B} = \mu \boldsymbol{H}\,, \quad \boldsymbol{J} = \boldsymbol{J}_0 + \sigma \boldsymbol{E}\,. \tag{1.3}$$

where ε is the *electric permittivity* $\varepsilon_0 = 8.8541878128(13) \times 10^{-12}$ F·m^{-1} and μ is the *magnetic permeability* $\mu_0 = 4\pi \times 10^{-7}$ H·m^{-1} in vacuum. The last equation in (1.3) includes a vector field $\boldsymbol{J}_0$, interpreted as the *source*, concentrated in region several to tens of kilometers deep and generating the EM signal, and the induced volume *eddy currents*, that depend on the crust stratification according to a linear Ohm-type law: the *electric conductivity* σ is a known piecewise scalar function [21], but anisotropic media can also be considered (in that case σ is tensor-valued). Inserting the constitutive laws (1.3) in the complete set of Maxwell's equations (1.2), we arrive at the system of equations

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{H} - \left(\sigma \boldsymbol{E} + \varepsilon \frac{\partial \boldsymbol{E}}{\partial t}\right) &= \boldsymbol{J}_0 \\ \boldsymbol{\nabla} \times \boldsymbol{E} + \mu \frac{\partial \boldsymbol{H}}{\partial t} &= 0\,, \end{aligned} \tag{1.4a}$$

subject to the differential constraint

$$\boldsymbol{\nabla} \cdot (\mu \boldsymbol{H}) = 0\,. \tag{1.4b}$$

Initial conditions, in this model, must be imposed both on the electric and on the magnetic field

$$\boldsymbol{E}(0) = \boldsymbol{E}_{0\varepsilon} \tag{1.5a}$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_{0\varepsilon} \tag{1.5b}$$

with $\boldsymbol{E}_{0\varepsilon}$ and $\boldsymbol{H}_{0\varepsilon}$ being given, and $\boldsymbol{\nabla} \cdot (\mu \boldsymbol{H}_{0\varepsilon}) = 0$.

1.4. **Grounds for magneto-quasistatic models.** In time-harmonic regime, the collection of all bulk terms between round brackets in the first equation of the system (1.4a) is given by multiplication of the electric field (in frequency domain) by the complex tensor

$$\sigma + i\omega\varepsilon\,.$$

Assuming smallness of the complex modulus $|\omega|$ would be consistent with findings of magnetic anomalies in ULF band. Also, the smallness of the time-scale $\frac{\varepsilon}{\sigma}$ (see Subsection 1.1) makes ε negligible, relative to σ. Thus, the interest in *ULF magnetic anomalies* in signals due to *hypogene* sources suggests one to consider the magneto-quasistatic model

$$\boldsymbol{\nabla} \times \boldsymbol{H} - \sigma \boldsymbol{E} = \boldsymbol{J}_0 \tag{1.6a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \mu \frac{\partial \boldsymbol{H}}{\partial t} = 0\,, \tag{1.6b}$$

with the constraint

$$\boldsymbol{\nabla} \cdot (\mu\, \boldsymbol{H}) = 0\,. \tag{1.6c}$$

In this case, we may provide an initial condition for the sole magnetic field, requiring that

$$\boldsymbol{H}(0) = \boldsymbol{H}_0\,, \tag{1.7}$$

where the vector field $\boldsymbol{H}_0$ is given and satisfies the compatibility condition $\boldsymbol{\nabla} \cdot (\mu \boldsymbol{H}_0) = 0$. Some comments on the singular limit as $\varepsilon \to 0^+$, in passing from (1.4) to (1.6), are made in Section 4. The loss of an initial condition may induce a boundary layer problem in time.

We point out that the limit problem (1.6) is or parabolic type. Indeed, multiplying (1.6a) by σ^{-1} and using the result to cancel the electric field from (1.6b) we arrive at the equation

$$\mu \frac{\partial \boldsymbol{H}}{\partial t} + \boldsymbol{\nabla} \times \left(\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{H}\right) = \boldsymbol{\nabla} \times (\sigma^{-1} \boldsymbol{J}_0)\,. \tag{1.8}$$

This equation involves the differential operator $\boldsymbol{H} \mapsto \boldsymbol{\nabla} \times \left(\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{H}\right)$, that is of elliptic type under natural assumptions on the coefficients (see Section 2.4).

1.5. **Boundary conditions.** Let $\Omega \subset \mathbb{R}^3$ be an open region with smooth boundaries, filled with a medium having permittivity ε, magnetic permeability μ, and electric conductivity σ, with appropriate assumptions on μ and σ being in force (we postpone the details to Section 2.4). In order to match the degrees of freedom in components of $\boldsymbol{J}_0, \boldsymbol{E}, \boldsymbol{H}$, both (1.4) and (1.6) list a number of equations that lack two scalar conditions. A convenient choice is to impose the boundary condition

$$\boldsymbol{H} \times \boldsymbol{n} = 0\,, \qquad \text{on } \partial\Omega \tag{1.9}$$

That, supplemented with these tangential boundary conditions, the magneto-quasistatic model (1.6) be well-posed is assured by suitable assumptions, that are discussed in Section 3.

The choice of limiting ourselves to consider homogeneous tangential boundary data is a mathematical artifice that causes no real restriction in Sections 3, 4, 5.

1.6. **Parabolic inverse source problems.** Notwithstanding, it may be interesting to consider solutions attaining inhomogeneous data at the boundary. Those data may model measurements, at least in some subregion of the boundary surface; in this spirit, we point out that, under some circumstances, partial measurements are enough to recover the constitutive properties of the medium [5]. But the relevant inverse problem in this context is different, because it points to recover the source appearing in the equations (1.6), that we couple with (1.9), from the knowledge of suitable data $\boldsymbol{\theta}$.

In time-harmonic regime [4] there exist *non-radiating sources*, i.e., non-trivial right hand sides $\boldsymbol{J}_0$ in (1.6) that are consistent with the homogeneous conditions (1.6). Thus, even a complete knowledge of the boundary data $\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{\theta}$ would not be sufficient to determine $\boldsymbol{J}_0$. Due to this ill-posedness, it is essential to subject the inverse source problem to some *a priori* assumptions on geometric and analytic structure of the source. Both in hyperbolic and in parabolic setting [1, 4], tangential boundary measurements uniquely dictate the source if the source is a priori known to be concentrated along a surface. Uniqueness holds for dipole sources, too. In Section 6, we survey this results in the time-dependent model.

Of course source reconstruction from the knowledge of tangential boundary measurements is not the only inverse source problem that can be considered, and there are a number of variants of the same idea. Another inverse problem that we introduce in Section 6 is that of determining the source in (1.6), under homogeneous boundary conditions (1.9), by assuming the complete knowledge of the magnetic field at the endpoints of a time interval [24]. Yet another example is provided by inverse source reconstruction for (1.6) with (1.9) from boundary normal measurements: the method of [23] for (1.4) can be adapted verbatim to the parabolic setting.

2. Mathematical framework

Unless otherwise specified, here and henceforth the spaces of L^2 scalar-valued, vector-valued, and tensor-valued functions will be denoted by $L^2(\Omega)$, $L^2(\Omega;\mathbb{R}^3)$, and $L^2(\Omega;\mathbb{R}^{3\times 3})$, respectively. Also, we shall denote throughout the paper by $(\cdot\,,\cdot)_{L^2}$ and by $\|\cdot\|_{L^2}$ the scalar product and the norm in all these spaces. Occasionally, we may opt for notation $\|\cdot\|_{L^2(f)}$ when referring to the weighted L^2-norm with the function f as a density.

2.1. **Energy space.** We recall that

$$H^1(\mathrm{curl}\,,\Omega) = \{\boldsymbol{\psi} \in L^2(\Omega;\mathbb{R}^3) :\ \boldsymbol{\nabla} \times \boldsymbol{\psi} \in L^2(\Omega;\mathbb{R}^3)\}$$

is a Hilbert space with the scalar product defined for all $\boldsymbol{\varphi}, \boldsymbol{\psi}$ by $(\boldsymbol{\varphi}\,, \boldsymbol{\psi})_{L^2} + (\boldsymbol{\nabla} \times \boldsymbol{\varphi}\,, \boldsymbol{\nabla} \times \boldsymbol{\psi})_{L^2}$. For all smooth surfaces Σ, in particular for all smooth portions of $\partial\Omega$, we set

$$H^{-\frac{1}{2}}(\mathrm{div}_\tau;\Sigma) = \left\{\boldsymbol{\lambda} \in H^{-\frac{1}{2}}(\Sigma\,;\mathbb{R}^3) \colon \boldsymbol{\lambda} \cdot \boldsymbol{n} = 0\,,\ \mathrm{div}_\tau \boldsymbol{\lambda} = 0\right\}. \tag{2.1}$$

We recall that the Gauss Green-type formula

$$(\boldsymbol{\varphi}\,,\boldsymbol{\nabla}\times\boldsymbol{\psi})_{L^2}-(\boldsymbol{\psi}\,,\boldsymbol{\nabla}\times\boldsymbol{\varphi})_{L^2}=\int_{\partial\Omega}\boldsymbol{\varphi}\cdot(\boldsymbol{n}\times\boldsymbol{\psi})\,dS\,, \tag{2.2}$$

holds for all $\boldsymbol{\varphi},\boldsymbol{\psi}\in C^1(\overline{\Omega};\mathbb{R}^3)$. As a consequence, the tangential trace $\boldsymbol{\psi}\longmapsto\boldsymbol{n}\times\boldsymbol{\psi}$ from $C^1(\overline{\Omega};\mathbb{R}^3)$ to $C(\partial\Omega;\mathbb{R}^3)$ extends to a bounded linear operator from $H^1(\mathrm{curl},\Omega)$ to the dual space $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)$ of $H^{\frac{1}{2}}(\partial\Omega\,;\mathbb{R}^3)$ (see, e.g, [11]), whose kernel is denoted by $H^1_0(\mathrm{curl}\,,\Omega)$.

We shall occasionally abbreviate $H^1(\mathrm{curl}\,,\Omega)$ to $\mathscr{H}^1$, for ease of notation. We also set

$$\mathscr{H}^1_0=\{\boldsymbol{\psi}\in H^1_0(\mathrm{curl}\,,\Omega)\,:\,\boldsymbol{\nabla}\cdot(\mu\boldsymbol{\psi})=0\}\,, \tag{2.3}$$

which defines a closed vector subspace of $\mathscr{H}^1$. In view of assumptions made in Section 1.5 the open set Ω supports the so-called Gaffney inequality [9] (a Friedrichs-Poincaré type functional inequality)

$$\int_\Omega|\boldsymbol{\psi}|^2\,d\boldsymbol{x}\le C\int_\Omega|\boldsymbol{\nabla}\times\boldsymbol{\psi}|^2\,d\boldsymbol{x}\,,\qquad\text{for all }\boldsymbol{\psi}\in\mathscr{H}^1_0\,,$$

and $\mathscr{H}^1_0$ is a subspace of $H^1(\Omega;\mathbb{R}^3)$, and with the equivalent norm $\boldsymbol{\psi}\mapsto\|\boldsymbol{\nabla}\times\boldsymbol{\psi}\|_{L^2}$, and $\mathscr{H}^1_0$ is contained in $L^2(\Omega;\mathbb{R}^3)$ with a compact embedding (see, e.g, [15, §2]). By induction, we also define

$$\mathscr{H}^n_0=\{\boldsymbol{\psi}\in\mathscr{H}^{n-1}_0\;:\;\boldsymbol{\nabla}\times\boldsymbol{\psi}\in\mathscr{H}^{n-1}_0\;,\;\boldsymbol{\nabla}\cdot\boldsymbol{\psi}=0\}$$

for $n\in\{1,2,3\}$.

2.2. **Time-dependent spaces.** We recall that, given $p\ge1$ and a Hilbert space $\mathcal{Z}$, a function ϕ belongs to $L^p(0,T;\mathcal{Z})$ if we have

$$\int_0^T\|\phi(t)\|^p_{\mathcal{Z}}\,dt<+\infty$$

and, in that case, the p-th root of left hand side is denoted by $\|\phi\|_{L^p(0,T;\mathcal{Z})}$. We also recall that this defines a complete norm on $L^p(0,T;\mathcal{Z})$. The same conclusion holds in the borderline case $p=\infty$, provided that the p-th root of the integral is replaced by the essential sup norm.

If $\partial_t\phi$ belongs to $L^p(0,T;\mathcal{Z})$ then so does ϕ, and in that case we write $\phi\in W^{1,p}(0,T;\mathcal{Z})$. Assume that $\mathcal{Z}\subset L^2(\Omega;\mathbb{R}^3)\subset\mathcal{Z}^*$, where $\mathcal{Z}^*$ is the dual of $\mathcal{Z}$, with continuous inclusions having dense images. Then, we recall Lions-Magenes Lemma (see [12, §5]): if $\phi\in L^2(0,T;\mathcal{Z})$ and $\partial_t\phi$ belongs to the dual space $L^2(0,T;\mathcal{Z}^*)$, then $\phi\in C([0,T];L^2(\Omega;\mathbb{R}^3))$ and the function $t\mapsto\|\phi(t)\|^2_{L^2}$ is absolutely continuous, with $\frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2_{L^2}=\langle\phi(t)\,,\partial_t\phi(t)\rangle$ for a.e. $0\le t\le T$, where $\langle\cdot\,,\cdot\rangle$ denotes the duality pairing. The same conclusions hold also if $\phi\in W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))$, because of one-dimensional Sobolev embedding.

2.3. **Regularity of the boundaries.** For all points $\boldsymbol{x}\in\mathbb{R}^3$ and for all $r>0$, we denote by $B_r(\boldsymbol{x})$ the ball of radius r centred at $\boldsymbol{x}$. An open set Ω in $\mathbb{R}^3$ is said to satisfy a *uniform two-sided ball condition* if there exists a positive $r>0$ with the property that, for every boundary point $\boldsymbol{\xi}\in\partial\Omega$, there exist a ball $B_r(\boldsymbol{x})$ contained in Ω and a ball $B_r(\boldsymbol{y})$ contained in its complement, such that $\boldsymbol{\xi}$ belongs both to the closure of $B_r(\boldsymbol{x})$ and to that of $B_r(\boldsymbol{y})$. Throughout this paper, we shall make the following assumptions:

(2.4a) Ω is a bounded open set in $\mathbb{R}^3$.

(2.4b) Ω is either convex or it satisfies a uniform two-sided ball condition.

(2.4c) The closure $\overline{\Omega}$ of Ω has the same boundary as Ω.

Incidentally, we point out that, under the assumption (2.4c), condition (2.4b) is equivalent to a uniform bound on the $C^{1,1}$ constants of the functions describing locally Ω as a subgraph.

2.4. **Assumptions on the coefficients.** We assume that σ be a bounded measurable function with values in the set of (3×3)-symmetric matrices with real coefficients such that

$$\sigma_0 |\boldsymbol{\xi}|^2 \le \sigma(x)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \le \sigma_0^{-1} |\boldsymbol{\xi}|^2 \,, \qquad \text{for a.e. } x \in \Omega, \text{ and for all } \boldsymbol{\xi} \in \mathbb{R}^3, \tag{2.5a}$$

for an appropriate constant $\sigma_0 > 1$. Here, for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3$ we are denoting by $\boldsymbol{\xi} \cdot \boldsymbol{\eta}$ the standard scalar product in $\mathbb{R}^3$. By μ we shall denote a fixed positive smooth scalar function, satisfying

$$\sigma_0 \le \inf\{\mu(x)\,, |\nabla \mu(x)\} \le \sup\{\mu(x)\,, |\nabla \mu(x)\} \le \sigma_0^{-1} \,, \tag{2.5b}$$

for all $x \in \Omega$.

2.5. **Total basis and magnetic eigenvalues.** Assuming (2.4) and (2.5b), the eigenvalue-type boundary value problem

$$\begin{cases} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\psi} = \lambda \boldsymbol{\psi}\,, & \text{in } \Omega, \\ \boldsymbol{\nabla} \cdot (\mu \boldsymbol{\psi}) = 0\,, & \text{in } \Omega, \\ \boldsymbol{n} \times \boldsymbol{\psi} = 0\,, & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

admits non-trivial (weak) solutions for a discrete set of real numbers λ, called *eigenvalues*. If $\lambda \ge 0$ is an eigenvalue, any non-trivial (weak) solution of (2.6) is called an *eigenfield*. The eigenvalues form an unbounded non-decreasing sequence, that is completely described by the variational principle

$$\lambda_m = \min_{\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_m \in \mathscr{H}_0^1} \max \left\{ \frac{\displaystyle\int_\Omega |\nabla \times \boldsymbol{\psi}(\boldsymbol{x})|^2 \, d\boldsymbol{x}}{\displaystyle\int_\Omega |\boldsymbol{\psi}(\boldsymbol{x})|^2 \, d\boldsymbol{x}} \;:\; \boldsymbol{\psi} \in \operatorname{Span}\{\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_m\} \setminus \{0\} \right\}. \tag{2.7}$$

If Ω is simply connected with a connected boundary $\partial\Omega$ then $\lambda_1 > 0$. In general, $\lambda_j = 0$ for all positive integers j smaller than the number of degrees of freedom behind conditions

$$\boldsymbol{\nabla} \times \boldsymbol{\psi} = 0\,,\ \boldsymbol{\nabla} \cdot (\mu \boldsymbol{\psi}) = 0\,,\ \boldsymbol{n} \times \boldsymbol{\psi}\big|_{\partial\Omega} = 0\,,$$

which is however always finite (it is the second Betti number of Ω as a Euclidean manifold).

3. Well-posedness for the forward problem

3.1. **Parabolic estimates.** In this section we provide solutions of the quasi-static Maxwell equations, understood in the following sense.

Definition 3.1. Given $\boldsymbol{J}_0 \in L^2(0,T; L^2(\Omega;\mathbb{R}^3))$ and $\boldsymbol{H}_0 \in L^2(\Omega;\mathbb{R}^3)$, with $\boldsymbol{\nabla} \cdot (\mu \boldsymbol{H}_0) = 0$ in Ω, we say that $\boldsymbol{H} \in L^2(0,T;\mathscr{H}_0^1)$, with $\partial_t \boldsymbol{H} \in L^2(0,T;(\mathscr{H}_0^1)^*)$ and $\boldsymbol{H}(0) = \boldsymbol{H}_0$, is a weak solution of (1.6) with the boundary conditions (1.9) if we have

$$-\int_{T_0}^{T_1} \left(\mu \boldsymbol{H} \,, \frac{\partial \boldsymbol{\phi}}{\partial t} \right)_{L^2} d\tau + \int_{T_0}^{T_1} \left(\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{H} \,, \boldsymbol{\nabla} \times \boldsymbol{\phi} \right)_{L^2} d\tau = \int_{T_0}^{T_1} \left(\sigma^{-1} \boldsymbol{J}_0 \,, \boldsymbol{\nabla} \times \boldsymbol{\phi} \right)_{L^2} d\tau \,,$$

for all $\boldsymbol{\phi} \in C^\infty(\Omega \times [0,T])$ with support contained in $\Omega \times [0,T]$, for all $0 < T_0 < T_1 < T$.

Note that, under the assumptions made in Definition 3.1, the initial condition on weak solutions makes sense because of Lions- Magenes Lemma. If certain better regularity criteria are met, the weak solutions are solutions in the following stronger sense.

Definition 3.2. Given $\boldsymbol{J}_0 \in L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ and $\boldsymbol{H}_0 \in \mathscr{H}^1_0$, with $\boldsymbol{\nabla}\cdot(\mu\boldsymbol{H}_0) = 0$ in Ω, we say that $\boldsymbol{H} \in L^2(0,T;\mathscr{H}^1_0)$, with $\partial_t\boldsymbol{H} \in L^2(0,T;L^2(\Omega;\mathbb{R}^3))$, is a strong solution of (1.6), with boundary conditions (1.9), if

$$\left(\mu\frac{\partial \boldsymbol{H}}{\partial t}(\tau)\,,\boldsymbol{\psi}\right)_{L^2} + \left(\sigma^{-1}\boldsymbol{\nabla}\times\boldsymbol{H}(\tau)\,,\boldsymbol{\nabla}\times\boldsymbol{\psi}\right)_{L^2} = \left(\sigma^{-1}\boldsymbol{J}_0(\tau)\,,\boldsymbol{\nabla}\times\boldsymbol{\psi}\right)_{L^2}\,, \tag{3.1}$$

for all $\boldsymbol{\psi} \in \mathscr{H}^1_0$ and for a.e. $0 \le \tau \le T$.

To construct solutions, we follow a specific Galerkin-type method. To do so, for all $m \in \mathbb{N}$, we denote by $\boldsymbol{\pi}_m$ the projection from $\mathscr{H}^1_0$ onto the vector space $\mathscr{H}^1_{0m}$ generated by the eigenfields associated with the eigenvalues $\lambda_1,\dots,\lambda_m$ introduced in (2.7). Given $\boldsymbol{H}_0 \in \mathscr{H}^1_0$, the standard results for linear systems of ordinary differential equations imply the existence of a (unique) $\boldsymbol{H}_m \in C^1([0,T];\mathscr{H}^1_{0m})$ for which

$$\begin{cases} \boldsymbol{\nabla}\times\boldsymbol{H}_m - \sigma\boldsymbol{E}_m = \boldsymbol{\pi}_m\boldsymbol{J}_0 \\ \boldsymbol{\nabla}\times\boldsymbol{E}_m + \mu\dfrac{\partial\boldsymbol{H}_m}{\partial t} = 0 \end{cases} \tag{3.2}$$

for an appropriate $\boldsymbol{E}_m \in C^1([0,T];\mathscr{H}^1_{0m})$, under initial conditions $\boldsymbol{H}_m(0) = \boldsymbol{\pi}_m\boldsymbol{H}_0$. Multiplying the first equation in (3.2) by $\boldsymbol{E}_m$ and the second one by $\boldsymbol{H}_m$, we arrive at the identity

$$\frac{1}{2}\frac{d}{dt}\left(\mu\boldsymbol{H}_m\,,\boldsymbol{H}_m\right)_{L^2} + \left(\sigma\boldsymbol{E}_m\,,\boldsymbol{E}_m\right)_{L^2} = -\left(\boldsymbol{\pi}_m\boldsymbol{J}_0\,,\boldsymbol{E}_m\right)_{L^2}\,. \tag{3.3}$$

Using Cauchy Schwartz inequality and the first equation in (3.2) again, from (3.3) we deduce

$$\frac{d}{dt}\left(\mu\boldsymbol{H}_m\,,\boldsymbol{H}_m\right)_{L^2} + \left(\sigma^{-1}\boldsymbol{\nabla}\times\boldsymbol{H}_m\,,\boldsymbol{\nabla}\times\boldsymbol{H}_m\right)_{L^2} \le \left(\sigma^{-1}\boldsymbol{\pi}_m\boldsymbol{J}_0\,,\boldsymbol{\pi}_m\boldsymbol{J}_0\right)_{L^2}\,.$$

Integrating in time, we get

$$\sup_{\tau\in[0,T]}\|\sqrt{\mu}\boldsymbol{H}_m\|^2_{L^2} + \int_0^T\|\boldsymbol{\nabla}\times\boldsymbol{H}_m\|^2_{L^2}\,d\tau \le C_1(\Lambda)\left(\|\boldsymbol{H}_0\|^2_{L^2} + \int_0^T\|\boldsymbol{J}_0\|^2_{L^2}\,d\tau\right)\,. \tag{3.4}$$

Also, recalling that (2.6) with $\lambda = \lambda_i$ holds for an appropriate $\boldsymbol{\psi}_i \in \mathscr{H}^1_{0m}$, we have

$$\begin{aligned} \|\boldsymbol{\nabla}\times\boldsymbol{H}_m(0)\|^2_{L^2} &= \sum_{i,j=1}^m\left(\boldsymbol{H}_m(0)\,,\boldsymbol{\psi}_i\right)_{L^2}\left(\boldsymbol{H}_m(0)\,,\boldsymbol{\psi}_j\right)_{L^2}\left(\boldsymbol{\nabla}\times\boldsymbol{\psi}_i\,,\boldsymbol{\nabla}\times\boldsymbol{\psi}_j\right)_{L^2} \\ &= \sum_{i=1}^m\lambda_i|\left(\boldsymbol{H}_0\,,\boldsymbol{\psi}_i\right)_{L^2}|^2 = \sum_{\substack{i\le m\\ \lambda_i>0}}\left|\left(\boldsymbol{\nabla}\times\boldsymbol{H}_0\,,\frac{\boldsymbol{\nabla}\times\boldsymbol{\psi}_i}{\sqrt{\lambda_i}}\right)_{L^2}\right| \le \|\boldsymbol{\nabla}\times\boldsymbol{H}_0\|^2_{L^2} \end{aligned} \tag{3.5}$$

where in the last passage we also used Bessel's inequality. We differentiate the first equation in (3.2) and we multiply the result by $\boldsymbol{E}_m$, then we multiply the second equation in (3.2) by $\partial_t\boldsymbol{H}_m$. Doing so, we arrive at

$$\left(\mu\frac{\partial\boldsymbol{H}_m}{\partial t}\,,\frac{\partial\boldsymbol{H}_m}{\partial t}\right)_{L^2} + \tfrac{1}{2}\tfrac{d}{dt}\left(\sigma\boldsymbol{E}_m\,,\boldsymbol{E}_m\right)_{L^2} \le \left(\sigma\boldsymbol{E}_m\,,\boldsymbol{E}_m\right)^{\frac{1}{2}}_{L^2}\left(\sigma^{-1}\boldsymbol{\pi}_m\frac{\partial\boldsymbol{J}_0}{\partial t}\,,\boldsymbol{\pi}_m\frac{\partial\boldsymbol{J}_0}{\partial t}\right)^{\frac{1}{2}}_{L^2}$$

where we also used Cauchy-Schwartz inequality. By a Grönwall-type argument, we infer that

$$\left(\sigma\boldsymbol{E}_m\,,\boldsymbol{E}_m\right)_{L^2} \le 2\left(\sigma\boldsymbol{E}_m(0)\,,\boldsymbol{E}_m(0)\right)_{L^2} + T\int_0^T\left(\sigma^{-1}\boldsymbol{\pi}_m\frac{\partial\boldsymbol{J}_0}{\partial t}\,,\boldsymbol{\pi}_m\frac{\partial\boldsymbol{J}_0}{\partial t}\right)_{L^2}d\tau\,.$$

Using the first equation in (3.2) with $t=0$ and recalling (3.5), after an integation in time from the last two inequalities we deduce

$$\int_0^T \left\|\sqrt{\mu}\frac{\partial \boldsymbol{H}_m}{\partial t}\right\|_{L^2}^2 d\tau + \sup_{\tau\in[0,T]} \|\nabla\times \boldsymbol{H}_m\|_{L^2}^2 \le C_2(\Lambda)\left(\|\nabla\times\boldsymbol{H}_0\|_{L^2}^2 + \int_0^T \left\|\frac{\partial \boldsymbol{J}_0}{\partial t}\right\|_{L^2}^2 d\tau\right). \tag{3.6}$$

If $\boldsymbol{\nabla}\times(\sigma^{-1}\boldsymbol{\nabla}\times\boldsymbol{H}_0)\in L^2(\Omega)$, the system (3.2) at initial time gives

$$\mu\partial_t \boldsymbol{H}_m(0) = -\boldsymbol{\nabla}\times(\sigma^{-1}\boldsymbol{\nabla}\times\boldsymbol{H}_0) + \boldsymbol{\nabla}\times(\sigma^{-1}\boldsymbol{J}_0(0))\,.$$

By (3.2) we also have

$$\left(\mu\frac{\partial^2}{\partial t^2}\boldsymbol{H}_m\,,\boldsymbol{\psi}\right)_{L^2} + \left(\sigma^{-1}\boldsymbol{\nabla}\times\frac{\partial \boldsymbol{H}_m}{\partial t}\,,\boldsymbol{\nabla}\times\boldsymbol{\psi}\right)_{L^2} = \left(\sigma^{-1}\boldsymbol{\pi}_m\frac{\partial \boldsymbol{J}_0}{\partial t}\,,\boldsymbol{\nabla}\times\boldsymbol{\psi}\right)_{L^2}, \qquad \text{for all } \boldsymbol{\psi}\in\mathscr{H}^1_0\,.$$

Then, for $\boldsymbol{\nabla}\times(\sigma^{-1}\boldsymbol{\nabla}\times\boldsymbol{H}_0)\in L^2(\Omega)$, choosing $\boldsymbol{\psi}=\dfrac{\partial \boldsymbol{H}_m}{\partial t}$ and integrating in time we arrive at

$$\begin{aligned}&\sup_{\tau\in[0,T]}\left\|\frac{\partial \boldsymbol{H}_m}{\partial t}(\tau)\right\|_{L^2}^2 + \int_0^T\left\|\frac{\partial(\nabla\times\boldsymbol{H}_m)}{\partial t}\right\|_{L^2}^2 d\tau\\ &\qquad\le C_3(\Lambda)\left(\|\sigma^{-1}\boldsymbol{\nabla}\times\boldsymbol{H}_0\|_{\mathscr{H}^1_0} + \int_0^T\left\|\frac{\partial \boldsymbol{J}_0}{\partial t}\right\|_{L^2}^2 d\tau\right).\end{aligned} \tag{3.7}$$

The energy estimates obtained during the procedure imply the following fact.

Theorem 3.3. *Let $\Omega\subset\mathbb{R}^3$ satisfy (2.4), let $\sigma\in L^\infty(\Omega;\mathbb{R}^{3\times3})$ and let $\mu\in C^1(\mathbb{R}^3)$ be such that (2.5) holds. Given $\boldsymbol{J}_0\in L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ and $\boldsymbol{H}_0\in L^2(\Omega)$, with $\boldsymbol{\nabla}\cdot(\mu\boldsymbol{H}_0)=0$ in Ω, there exists a unique weak solution $\boldsymbol{H}\in L^2(0,T;\mathscr{H}^1_0)$, with $\partial_t\boldsymbol{H}\in L^2(0,T;(\mathscr{H}^1_0)^*)$, and we have*

$$\|\boldsymbol{H}\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{H}\|_{L^2(0,T;\mathscr{H}^1_0)} \le c_1\left(\|\boldsymbol{H}_0\|_{L^2} + \|\boldsymbol{J}_0\|_{L^2(0,T;L^2(\Omega))}\right). \tag{3.8}$$

If also $\boldsymbol{H}_0\in\mathscr{H}^1_0$ and $\boldsymbol{J}_0\in W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))$, then $\boldsymbol{H}$ is a strong solution and

$$\|\boldsymbol{H}\|_{W^{1,2}(0,T;L^2(\Omega))} + \|\boldsymbol{H}\|_{L^\infty(0,T;\mathscr{H}^1_0)} \le c_2\left(\|\boldsymbol{H}\|_{\mathscr{H}^1_0} + \left\|\frac{\partial \boldsymbol{J}_0}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}\right). \tag{3.9}$$

Eventually, if in addition $\sigma^{-1}\boldsymbol{\nabla}\times\boldsymbol{H}_0\in\mathscr{H}^1_0$ then

$$\|\boldsymbol{H}\|_{W^{1,\infty}(0,T;L^2(\Omega))} + \|\boldsymbol{H}\|_{W^{1,2}(0,T;\mathscr{H}^1_0)} \le c_3\left(\|\sigma^{-1}\boldsymbol{\nabla}\times\boldsymbol{H}_0\|_{\mathscr{H}^1_0} + \left\|\frac{\partial \boldsymbol{J}_0}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}\right). \tag{3.10}$$

In the estimates, the constants c_1, c_2, c_3 depend on σ_0, T, and Ω, only.

Proof. Uniqueness follows at once by the estimates. The existence of a field solving the weak equation, with the first estimate, is a consequence of (3.4) and of a routine compactness argument. In case $\boldsymbol{H}_0\in\mathscr{H}^1_0$, we can use (3.6), too: as a consequence, we obtain the second estimate; also, choosing a test function of the form $\phi(\boldsymbol{x},t)=\boldsymbol{\psi}(\boldsymbol{x})h(t)$, with $h\in C^1_0(0,T)$, in the weaker equation and integrating by parts, we deduce the stronger equation for almost all times $t\in(0,T)$ because of the arbitrariness of h. Eventually, under the additional assumption that $\boldsymbol{\nabla}\times(\sigma^{-1}\boldsymbol{\nabla}\times\boldsymbol{H}_0)\in\mathscr{H}^1_0$ we can use also (3.7) and that implies the last statement. ∎

REMARK 3.4. If the initial data satisfy the assumption $\boldsymbol{\nabla} \times (\sigma^{-1}\boldsymbol{\nabla} \times \boldsymbol{H}_0) \in \mathscr{H}^1_0$, then by Aubin-Lions Lemma the convergence of the Galerkin method holds, at least, is in the following sense:

$$\lim_{m\to\infty} \int_0^T \|\boldsymbol{H}_m(\cdot,t) - \boldsymbol{H}(\cdot,t)\|^2_{L^2}\, dt = 0$$

The case of heterogeneous media with conductivities that include discontinuities is interesting in applications. The following regularity-related result is proved in [15, Theorem 4.1]. The proof presented there is based on the method introduced in [2], which combines elliptic Campanato-type estimates and the classical De Giorgi-Nash regularity with the relevant Helmholtz decompositions.

Theorem 3.5 (regularity)**.** *Let $\Omega \subset \mathbb{R}^3$ satisfy (2.4), let $\sigma \in L^\infty(\Omega;\mathbb{R}^{3\times 3})$ and let $\mu \in C^1(\mathbb{R}^3)$ be such that (2.5) holds. Then, there exists $\alpha_0 \in (0, \frac{1}{2}]$, only depending on σ_0, such that for every $\alpha \in (0,\alpha_0]$ the following holds: for every $\mathbf{H}_0 \in C^{0,\alpha}(\overline{\Omega};\mathbb{R}^3)$ and for every $\boldsymbol{J}_0 \in L^2(0,T;C^{0,\alpha}(\overline{\Omega};\mathbb{R}^3))$, if $(\mathbf{E},\mathbf{H})$ is a weak solution of (1.6), then $\mathbf{H} \in L^2(0,T\,;C^{0,\alpha}(\overline{\Omega}\,;\mathbb{R}^3))$, and we have*

$$\|\mathbf{H}\|_{L^2(0,T;C^{0,\alpha}(\overline{\Omega};\mathbb{R}^3))} \le C\Big[\|\mathbf{H}_0\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{R}^3)} + \|\mathbf{H}\|_{W^{1,2}(0,T;\mathscr{H}^1_0)} + \|\boldsymbol{J}_0\|_{L^2(0,T;C^{0,\alpha}(\overline{\Omega};\mathbb{R}^3))}\Big], \tag{3.11}$$

where the constant C depends on σ_0, T, and on Ω, only.

REMARK 3.6. If the initial data satisfy also the assumption that $\sigma^{-1}\boldsymbol{\nabla} \times \boldsymbol{H}_0 \in \mathscr{H}^1_0$ then

$$\|\mathbf{H}\|_{L^\infty(0,T;C^{0,\alpha}(\overline{\Omega};\mathbb{R}^3))} \le C\Big[\|\mathbf{H}_0\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{R}^3)} + \|\mathbf{H}\|_{W^{1,\infty}(0,T;\mathscr{H}^1_0)} + \|\boldsymbol{J}_0\|_{L^\infty(0,T;C^{0,\alpha}(\overline{\Omega};\mathbb{R}^3))}\Big].$$

The appropriate assumptions on the conductivity coefficients σ do include the possibility of discontinuities, as said. Nonetheless, it can however be legit to consider simplified situations in which the singularities are concentrated along smooth geometric objects.

For example, a relevant situation is that of an heterogeneous isotropic medium described by a piecewise constant conductivity, jumping across a plane. In this simpler case, the magneto-quasistatic field solves a system of three scalar heat equations mildly coupled by the presence in the right hand side of the components of a given forcing term: in each level set of σ, we have

$$\mu\sigma\frac{\partial \boldsymbol{H}}{\partial t} - \boldsymbol{\nabla}^2\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{J}_0\,. \tag{3.12}$$

In (3.12), $\boldsymbol{\nabla}^2$ denotes the vector-valued Laplace operator defined componentwise by

$$\boldsymbol{\nabla}^2\boldsymbol{\psi} = \begin{pmatrix} \partial^2_{xx}f + \partial^2_{yy}f + \partial^2_{zz}f \\ \partial^2_{xx}g + \partial^2_{yy}g + \partial^2_{zz}g \\ \partial^2_{xx}h + \partial^2_{yy}h + \partial^2_{zz}h \end{pmatrix}, \qquad \text{for all } \boldsymbol{\psi} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}.$$

Theorem 3.7 (Transmission conditions)**.** *Let $r > 0$, let $R > 0$, and let $\boldsymbol{e} \in \mathbb{R}^3$ and let*

$$\Omega = \big\{\boldsymbol{x} \in \mathbb{R}^3 \colon |\boldsymbol{x}\cdot\boldsymbol{e}| \le R\,,\ |\boldsymbol{x} - (\boldsymbol{x}\cdot\boldsymbol{e})\boldsymbol{e}| \le r\big\}\,,$$

$$\Omega_+ = \big\{\boldsymbol{x} \in \Omega \,:\, \boldsymbol{x}\cdot\boldsymbol{e} > 0\big\}\,, \qquad \text{and} \qquad \Omega_- = \big\{\boldsymbol{x} \in \Omega \,:\, \boldsymbol{x}\cdot\boldsymbol{e} < 0\big\}\,.$$

Let $\sigma_+,\sigma_- > 0$, let $\sigma = \sigma_+\cdot 1_{\Omega_+} + \sigma_-\cdot 1_{\Omega_-}$, and let $[\![\cdot]\!]$ denote the jump across $\Sigma = \{\boldsymbol{x} \in \mathbb{R}^3 \colon \boldsymbol{x}\cdot\boldsymbol{e} = 0\}$ in the sense of traces. Then, for every $\boldsymbol{H}_0 \in \mathscr{H}^1_0$ and for every $\boldsymbol{J}_0 \in L^2(0,T;L^2(\Omega;\mathbb{R}^3))$, a vector field $\boldsymbol{H}$ is a weak solution of (1.6) if, and only if, it solves (3.12) in $\Omega_+ \cup \Omega_-$ and the transmission conditions

$$[\![\boldsymbol{H}]\!] = 0 \tag{3.13a}$$

$$[\![\boldsymbol{e} \times (\sigma^{-1}\boldsymbol{\nabla} \times \boldsymbol{H})]\!] = 0 \tag{3.13b}$$

hold along Σ for a.e. $0 \le t \le T$.

Proof. Because $\boldsymbol{H}_0 \in \mathscr{H}^1_0$, a vector field $\boldsymbol{H}$ is a weak solution if and only if it is a strong one, i.e., (3.1) holds for all $\boldsymbol{\psi} \in \mathscr{H}^1_0$. In that case, choosing first test fields $\boldsymbol{\psi}$ with support either contained in Ω_+ or in Ω_- we see that equations (1.6a) and (1.6b) hold in $\Omega_+ \cup \Omega_- = \Omega \setminus \Sigma$. Combining those equations with (3.1) when the support $\boldsymbol{\psi}$ has non-empty intersection with Σ, after an integration by parts we arrive at

$$\int_\Sigma \boldsymbol{\psi} \cdot \boldsymbol{e} \times \left(\mu \frac{\partial \boldsymbol{H}}{\partial t} \right) dS = 0\,, \qquad \int_\Sigma \boldsymbol{\psi} \cdot \boldsymbol{e} \times (\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{H})\, dS = 0\,.$$

As $\boldsymbol{\psi}$ can be any element of $\mathscr{H}^1_0$, we infer that

$$\left[\!\!\left[\boldsymbol{e} \times \frac{\partial \boldsymbol{H}}{\partial t} \right]\!\!\right] = 0\,, \qquad [\![\boldsymbol{e} \times (\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{H})]\!] = 0\,,$$

and (3.13b) is proved. Because, by assumption, $\boldsymbol{H}_0$ must not jump across Σ, the first identity in the latter implies $[\![\boldsymbol{e} \times \boldsymbol{H}]\!] = 0$. That $[\![\boldsymbol{e} \cdot \boldsymbol{H}]\!] = 0$ too, follows from (1.6c) by divergence theorem. This proves also (3.13a) and concludes the proof. □

Remark 3.8. When in force for solutions, condition (3.13a) is valid not just in the sense of traces, because $\boldsymbol{H}$ is continuous across Σ in view of Theorem 3.5. On the other hand, condition (3.13b) limits the regularity of $\boldsymbol{H}$ in the scale of Hölder spaces, and indicates that it must not be continuously differentiable: any jump in the coefficient σ must be compensated by the curl of $\boldsymbol{H}$.

4. Hyperbolic estimates and their singular limit

We devote this section to some comments on the asymptotic behaviour of electro-magnetic fields in media with vanishing dielectricity. This issue, in the case of homogeneous media, has been already considered in literature, even for quasi-linear models: we refer the interested reader to [17]. The problem in heterogeneous media does not give rise to particular additional issues, at least if the conducivity is described by a smooth function; for sake of simplicity, in the present section we will limit our attention to this simpler case.

Let $\boldsymbol{E}_\varepsilon$, $\boldsymbol{H}_\varepsilon$ solve the complete set of Maxwell equations (1.4) under the boundary conditions (1.9). From (1.4a), we arrive at the estimate

$$\int_0^T \|\boldsymbol{E}_\varepsilon\|^2_{L^2(\sigma)}\, dt + \max_{0 \le t \le T} \left[\|\boldsymbol{E}_\varepsilon\|^2_{L^2(\varepsilon)} + \|\boldsymbol{H}_\varepsilon\|^2_{L^2(\mu)} \right] \le C \int_0^T \|\boldsymbol{J}_0\|^2_{L^2}\, dt + 2L^2_{0\varepsilon} \tag{4.1}$$

where the constant C is independent of ε, and $L_{0\varepsilon} = \|\boldsymbol{E}_{0\varepsilon}\|_{L^2} + \|\boldsymbol{H}_{0\varepsilon}\|_{L^2}$ where $\boldsymbol{E}_{0\varepsilon}$ and $\boldsymbol{H}_{0\varepsilon}$ are the data involved in the initial conditions (1.5).

Differentiating in time equations (1.4a), multiplying the first one by $\partial_t \boldsymbol{E}_\varepsilon$ and the second one by $\partial_t \boldsymbol{H}_\varepsilon$, integrating by parts, and using (1.9), by a similar argument we may also get

$$\int_0^T \left\| \frac{\partial \boldsymbol{E}_\varepsilon}{\partial t} \right\|^2_{L^2(\sigma)} dt + \max_{0 \le t \le T} \left[\left\| \frac{\partial \boldsymbol{E}_\varepsilon}{\partial t} \right\|^2_{L^2(\varepsilon)} + \left\| \frac{\partial \boldsymbol{H}_\varepsilon}{\partial t} \right\|^2_{L^2(\mu)} \right] \le C \int_0^T \left\| \frac{\partial \boldsymbol{J}_0}{\partial t} \right\|^2_{L^2} dt + 2M^2_{0\varepsilon}\,, \tag{4.2}$$

with a constant C that is also independent of ε. In this case, the right hand side involves the quantity $M_{0\varepsilon} = \|\partial_t \boldsymbol{E}_\varepsilon(0)\|_{L^2} + \|\partial_t \boldsymbol{H}_\varepsilon(0)\|_{L^2}$, which does not depend on the data directly, but rather through the solutions.

Note that both $L_{0\varepsilon}$ and $M_{0\varepsilon}$ depend on ε. When considering the limit as $\varepsilon \to 0^+$ it is therefore natural to dictate some additional requirement on the initial data so as to provide uniform bounds for these quantities. In particular, a uniform bound for $L_{0\varepsilon}$ in (4.1) is a minimal requirement. This would be ensured, for example, by conditions

$$\boldsymbol{H}_{0\varepsilon} \in \mathscr{H}^n_0\,, \quad \boldsymbol{E}_{0\varepsilon} \in \mathscr{H}^n_0 \tag{4.3}$$

and

$$A_0 := \limsup_{\varepsilon\to 0^+} \Big(\|\boldsymbol{H}_{0\varepsilon}\|_{\mathscr{H}^n_0} + \|\boldsymbol{E}_{0\varepsilon}\|_{\mathscr{H}^n_0} \Big) < +\infty\,. \tag{4.4}$$

In fact, if (4.3) and (4.4) hold then we further have that

$$\liminf_{\varepsilon\to 0^+}\ \|\boldsymbol{H}_{0\varepsilon} - \boldsymbol{H}_0\|_{\mathscr{H}^{n-1}_0} = 0\,, \tag{4.5}$$

for a suitable $\boldsymbol{H}_0 \in \mathscr{H}^n_0$.

Here and henceforth, we make use of a fixed $\rho \in C^\infty_0(\mathbb{R}^3)$, with $0 \le \rho \le 1$ and $\int \rho(\boldsymbol{x})\, d\boldsymbol{x} = 1$, assumed to be even and have compact support in the unit ball, and for every $\delta \in (0,1)$ we set $\rho_\delta(\boldsymbol{x}) = \delta^{-3}\rho(\boldsymbol{x}/\delta)$, for all $\boldsymbol{x} \in \mathbb{R}^3$. For every single $\delta > 0$ and for all (scalar, vector-, or tensor-valued) functions (or distributions) $\boldsymbol{u}$ we shall denote by

$$\boldsymbol{u} \star \rho_\delta(\boldsymbol{x}) = \int \rho_\delta(\boldsymbol{x} - \boldsymbol{y})\boldsymbol{u}(\boldsymbol{y})\, d\boldsymbol{y}\,, \qquad \boldsymbol{x} \in \mathbb{R}^3\,,$$

the (componentwise) convolution product of $\boldsymbol{u}$ by the mollifier ρ_δ. We recall that $\boldsymbol{u} * \rho_\delta \in C^\infty(\mathbb{R}^3)$, and that $\boldsymbol{u} \in \mathscr{H}^n_0$ implies that $\boldsymbol{u} \star \rho_\delta \to \boldsymbol{u}$, as $\delta \to 0^+$, strongly in $\mathscr{H}^n_0$.

Theorem 4.1 (weak convergence)**.** *Let $\Omega \subset \mathbb{R}^3$ satisfy (2.4), let σ be a smooth function, satisfying (2.5a), with $\|\sigma\|_{C^3(\overline{\Omega};\mathbb{R}^{3\times 3})} \le \sigma_0^{-1}$, and let μ be a positive constant, with $\mu > \sigma_0$. Let $n \in \{1,2,3\}$, let $\boldsymbol{J}_0 \in W^{1,2}(0,T;\mathscr{H}^n_0)$, and let $\boldsymbol{H}_0 \in \mathscr{H}^n_0$. For all $\varepsilon > 0$, let $(\boldsymbol{E}_{0\varepsilon}, \boldsymbol{H}_{0\varepsilon})$ satisfy (4.3) and (4.4), and let $(\boldsymbol{E}_\varepsilon, \boldsymbol{H}_\varepsilon)$ be a solution of Maxwell's equations (1.4) with boundary conditions (1.9) and initial conditions (1.5). Then, as $\varepsilon \to 0^+$,*

$$\boldsymbol{H}_\varepsilon \overset{*}{\rightharpoonup} \boldsymbol{H} \qquad \text{weakly-}* \text{ in } L^\infty(0,T;\mathscr{H}^n_0), \tag{4.6a}$$

$$\boldsymbol{E}_\varepsilon \rightharpoonup \boldsymbol{E} \qquad \text{weakly in } L^2(0,T;\mathscr{H}^n_0), \tag{4.6b}$$

$$\frac{\partial \boldsymbol{H}_\varepsilon}{\partial t} \rightharpoonup \frac{\partial \boldsymbol{H}}{\partial t} \qquad \text{weakly in } L^2(0,T;\mathscr{H}^{n-1}_0), \tag{4.6c}$$

where $(\boldsymbol{E}, \boldsymbol{H})$ is the solution of (1.6) with boundary conditions (1.9) and initial conditions (1.7).

Proof. Let $s = (s_1, s_2, s_3)$ be a multi-index of lentgth $|s| = s_1 + s_2 + s_3 \le n$. We use notation $\partial^s = \frac{\partial^{s_1}}{\partial x_1^{s_1}} \frac{\partial^{s_2}}{\partial x_1^{s_2}} \frac{\partial^{s_3}}{\partial x_1^{s_3}}$. Let $\delta > 0$, and let $\boldsymbol{J}_s = \partial^s \boldsymbol{J}_0 \star \rho_\delta$. Then $\boldsymbol{E}_s = \partial^s \boldsymbol{E}_\varepsilon \star \rho_\delta$ and $\boldsymbol{H}_s = \partial^s \boldsymbol{H}_\varepsilon \star \rho_\delta$ are weak solutions of the system

$$\begin{cases} \boldsymbol{\nabla} \times \boldsymbol{H}_s - \varepsilon \partial_t \boldsymbol{E}_s - \sigma_0 \boldsymbol{E}_s = \boldsymbol{f}_s & \text{in } \Omega \times (0,T), \\ \mu \dfrac{\partial \boldsymbol{H}_s}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{E}_s = 0\,, & \text{in } \Omega \times (0,T), \\ \boldsymbol{n} \times \boldsymbol{H}_s = 0\,, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.7}$$

with the source being defined by $\boldsymbol{f}_s = \boldsymbol{J}_s - (\sigma - \sigma_0)\boldsymbol{\psi}_s - \boldsymbol{R}_s$, where

$$\boldsymbol{R}_s = \sum_{\substack{\beta \le s \\ \beta \ne s}} \binom{s}{\beta} (\partial^{s-\beta}\sigma \partial^\beta \boldsymbol{\psi}) \star \rho_\delta\,.$$

As a consequence of the energy identity

$$\frac{d}{dt}\left(\varepsilon\|\boldsymbol{E}_s\|_{L^2}^2 + \|\boldsymbol{H}_s\|_{L^2}^2\right) + 2\,(\sigma\boldsymbol{E}_s\,,\boldsymbol{E}_s)_{L^2} = 2\,(\boldsymbol{J}_s + \boldsymbol{R}_s\,,\boldsymbol{E}_s)_{L^2}$$

after a finite recursion argument we arrive at

$$\sum_{|s|\le n}\left[\frac{d}{dt}\left(\varepsilon\|\boldsymbol{E}_s\|_{L^2}^2 + \|\boldsymbol{H}_s\|_{L^2}^2\right) + \|\boldsymbol{E}_s\|_{L^2}^2\right] \le c_1\|\boldsymbol{J}_s\|_{\mathscr{H}_0^n}^2$$

where c_1 depends on σ_0 and on n, only. We recall that for every $\boldsymbol{\psi}\in\mathscr{H}_0^n$

$$c_2^{-1}\|\boldsymbol{\psi}\|_{\mathscr{H}_0^n}^2 \le \sum_{|\alpha|\le n}\|\partial^s\boldsymbol{\psi}\|_{L^2}^2 \le c_2\|\boldsymbol{\psi}\|_{\mathscr{H}_0^n}^2$$

where c_2 only depends on n. Hence, in view of (4.4),

$$\sum_{|s|\le n}\left[\sup_{0\le t\le T}\left(\varepsilon\|\boldsymbol{E}_s\|_{L^2}^2 + \|\boldsymbol{H}_s\|_{L^2}^2\right) + \int_0^T\|\boldsymbol{E}_s\|_{L^2}^2\,dt\right] \le 2c_1\int_0^T\|\boldsymbol{J}\|_{\mathscr{H}_0^n}^2\,dt + 2c_2A_0^2\,.$$

where c_2 is an absolute constant. Thus, by setting

$$c_3 = \sqrt{2c_2\left(c_1\int_0^T\|\boldsymbol{J}\|_{\mathscr{H}_0^n}^2\,dt + A_0^2\right)}\,,$$

the mapping that takes every pair of fields $(\boldsymbol{\varphi},\boldsymbol{\psi})$, with $\boldsymbol{\varphi}(0)=\boldsymbol{E}_\varepsilon$ and $\boldsymbol{\psi}(0)=\boldsymbol{H}_\varepsilon$, to the solution of system (1.4), with (1.9), subject to the initial conditions (1.5), maps the space

$$\left\{(\boldsymbol{\varphi},\boldsymbol{\psi})\in\mathcal{X}:\ \|\sqrt{\varepsilon}\boldsymbol{\varphi}\|_{L^\infty(0,T;\mathscr{H}_0^n)} + \|\boldsymbol{\psi}\|_{L^2(0,T;\mathscr{H}_0^n)} + c_1\|\boldsymbol{\varphi}\|_{L^2(0,T;\mathscr{H}_0^n)} \le c_3\right\}\,, \tag{4.8}$$

where $\mathcal{X}=C([0,T];\mathscr{H}_0^n\times\mathscr{H}_0^n)\cap C^1([0,T];\mathscr{H}_0^{n-1}\times\mathscr{H}_0^{n-1})$, into itself. Also, it is easily seen that this mapping is a contraction if we equip the vector space (4.8) with the distance induced by the norm

$$(\boldsymbol{\varphi},\boldsymbol{\psi})\longmapsto\|(\sqrt{\varepsilon}\boldsymbol{\varphi}\,,\boldsymbol{\psi})\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3))}\,.$$

By uniqueness, we infer the estimate

$$\limsup_{\varepsilon\to0}\left[\sup_{0\le t\le T}\left(\varepsilon\|\boldsymbol{E}_\varepsilon\|_{\mathscr{H}_0^n}^2 + \|\boldsymbol{H}_\varepsilon\|_{\mathscr{H}_0^n}^2\right) + \int_0^T\|\boldsymbol{E}_\varepsilon\|_{\mathscr{H}_0^n}^2\,dt\right] \le 2c_1\,c_2\int_0^T\|\boldsymbol{J}\|_{\mathscr{H}_0^n}^2\,dt + 2c_2\cdot A_0^2\,.$$

By Banach-Alaouglu theorem, we deduce that (4.6) holds, along a suitable sequence $\varepsilon_j\to0^+$, for an appropriate limit $(\boldsymbol{E},\boldsymbol{H})$. The fact that the limit solves (1.6) with (1.9) is a consequence. By uniqueness, (4.6) holds then for any sequence $\varepsilon_j\to0^+$, as desired. □

Remark 4.2. As a consequence of the proof of Theorem 4.1 and of the second equation in (1.4a), there exists a constant $c>0$, depending only σ_0, such that the solution satisfies the estimate

$$\sup_{0\le t\le T}\left(\varepsilon\|\boldsymbol{E}_\varepsilon\|_{\mathscr{H}_0^3}^2 + \|\boldsymbol{H}_\varepsilon\|_{\mathscr{H}_0^3}^2\right) + \int_0^T\|\boldsymbol{E}_\varepsilon\|_{\mathscr{H}_0^3}^2\,dt + \int_0^T\left\|\frac{\partial\boldsymbol{H}_\varepsilon}{\partial t}\right\|_{\mathscr{H}_0^2}^2\,dt \le c^2 \tag{4.9}$$

for all $\varepsilon\in(0,c^{-1})$, provided that

$$\int_0^T\|\boldsymbol{J}\|_{\mathscr{H}_0^n}^2\,dt + A_0^2 \le c\,.$$

The material above suggests one to consider the formal expansion of solutions of the hyperbolic Maxwell system (1.4)

$$\boldsymbol{E}_\varepsilon = \boldsymbol{E} + \sqrt{\varepsilon}\boldsymbol{\varphi} + o(\sqrt{\varepsilon})\,, \qquad \boldsymbol{H}_\varepsilon = \boldsymbol{H} + \sqrt{\varepsilon}\boldsymbol{\psi} + o(\sqrt{\varepsilon})\,,$$

in which the solution of the magneto-quasistatic Maxwell system (1.6) is the first term. We see that the higher order term $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ is provided, formally, by equations

$$\begin{cases} \boldsymbol{\nabla} \times \boldsymbol{\psi} - \sigma\boldsymbol{\varphi} = \sqrt{\varepsilon}\partial_t \boldsymbol{E}\,, \\ \boldsymbol{\nabla} \times \boldsymbol{\varphi} + \mu \dfrac{\partial \boldsymbol{\psi}}{\partial t} = 0\,, \\ \boldsymbol{n} \times \boldsymbol{\psi} = 0\,. \end{cases} \tag{4.10}$$

In particular, by applying the parabolic estimate (3.8) of Theorem 3.3 to the system (4.10), and by combining the result with (4.9), we see that for the hope of giving rigour to the expansion to be legit it becomes relevant to consider the limit

$$\lim_{\varepsilon\to 0^+} \frac{\boldsymbol{H}_{0\varepsilon} - \boldsymbol{H}_0}{\sqrt{\varepsilon}}\,.$$

The existence of the limit, with respect to an appropriate topology, may be useful in order to give an initial value to $\boldsymbol{\psi}$ and solve (4.10). This is related to the following theorem.

Theorem 4.3 (singular convergence). *Under the assumptions made in Theorem 4.1, and under the additional assumption that*

$$\|\boldsymbol{H}_{0\varepsilon} - \boldsymbol{H}_0\|_{\mathscr{H}^1_0} = O(\sqrt{\varepsilon})\,, \qquad \text{as } \varepsilon \to 0^+, \tag{4.11}$$

as $\varepsilon \to 0^+$ we have

$$\|\boldsymbol{H}_\varepsilon - \boldsymbol{H}\|_{L^\infty(0,T;\mathscr{H}^1_0)} = O(\sqrt{\varepsilon}) \qquad \text{and} \qquad \|\boldsymbol{E}_\varepsilon - \boldsymbol{E}\|_{L^2(0,T;\mathscr{H}^1_0)} = O(\sqrt{\varepsilon})\,.$$

Moreover, for every $T_0 \in (0, T)$

$$\sup_{t\in[T_0,T]} \|\boldsymbol{E}_\varepsilon(t) - \boldsymbol{E}(t)\|_{\mathscr{H}^1_0} = O(\varepsilon^{1/4})\,, \qquad \text{as } \varepsilon \to 0^+.$$

Proof. In view of Remark 4.2, after the limit procedure we know that

$$\boldsymbol{E} \in C(0, T; \mathscr{H}^2_0) \cap W^{1,2}(0, T; \mathscr{H}^1_0)\,, \tag{4.12}$$

and that

$$\limsup_{\varepsilon\to 0^+} \sup_{t\in[0,T]} \|\boldsymbol{E}_\varepsilon(t)\|_{\mathscr{H}^2_0} \le c_1\,, \qquad \int_0^T \|\partial_t \boldsymbol{E}(t)\|^2_{\mathscr{H}^1_0}\, dt \le c_2\,. \tag{4.13}$$

for appropriate positive constants c_1 and c_2. Setting $\boldsymbol{\xi}_\varepsilon = \boldsymbol{E}_\varepsilon - \boldsymbol{E}$ and $\boldsymbol{\chi}_\varepsilon = \boldsymbol{H}_\varepsilon - \boldsymbol{H}$, we see that $\boldsymbol{\eta}_\varepsilon = \boldsymbol{\nabla} \times \boldsymbol{\xi}_\varepsilon$ and $\boldsymbol{\zeta}_\varepsilon = \boldsymbol{\nabla} \times \boldsymbol{\chi}_\varepsilon$ solve the system

$$\begin{cases} \boldsymbol{\nabla} \times \boldsymbol{\zeta}_\varepsilon - \sigma\boldsymbol{\eta}_\varepsilon - \varepsilon \dfrac{\partial \boldsymbol{\eta}_\varepsilon}{\partial t} = \varepsilon \boldsymbol{\nabla} \times \dfrac{\partial \boldsymbol{E}}{\partial t}\,, & \text{in } \Omega \times (0, T), \\ \mu \dfrac{\partial \boldsymbol{\zeta}_\varepsilon}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{\eta}_\varepsilon = 0 & \text{in } \Omega \times (0, T), \\ \boldsymbol{n} \times \boldsymbol{\zeta}_\varepsilon = 0 & \text{on } \partial\Omega \times (0, T), \end{cases}$$

whence if follows that

$$\frac{1}{2}\frac{d}{dt}\Big(\varepsilon\|\boldsymbol{\eta}_\varepsilon\|^2_{L^2} + \mu\|\boldsymbol{\zeta}_\varepsilon\|^2_{L^2}\Big) + (\sigma\boldsymbol{\eta}_\varepsilon, \boldsymbol{\eta}_\varepsilon)_{L^2} = -\varepsilon\left(\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{E}}{\partial t}, \boldsymbol{\eta}_\varepsilon\right)\,. \tag{4.14}$$

Then, thanks to (4.13), from this energy identity we deduce that

$$\sup_{t\in[0,T]}\left(\varepsilon\|\boldsymbol{\xi}_\varepsilon\|^2_{\mathscr{H}^1_0}+\mu\|\boldsymbol{\chi}_\varepsilon\|^2_{\mathscr{H}^1_0}\right)+\sigma_0\int_0^T\|\boldsymbol{\xi}_\varepsilon\|_{\mathscr{H}^1_0}\,dt\le c_2\cdot\varepsilon+\varepsilon\|\boldsymbol{\xi}_\varepsilon(0)\|^2_{\mathscr{H}^1_0}+\|\boldsymbol{\chi}_\varepsilon(0)\|^2_{\mathscr{H}^1_0}\,.$$

Using now (4.11) and recalling the first inequality in (4.13), we infer

$$\sup_{t\in[0,T]}\left(\varepsilon\|\boldsymbol{\xi}_\varepsilon\|^2_{\mathscr{H}^1_0}+\|\boldsymbol{\chi}_\varepsilon\|^2_{\mathscr{H}^1_0}\right)+\int_0^T\|\boldsymbol{\xi}_\varepsilon\|_{\mathscr{H}^1_0}\,dt\le c_4\cdot(c_1+c_2+c_3)\cdot\varepsilon$$

where the constant $c_4>0$ is independent of ε, which ends the first part of the proof.

In order to prove the last statement, we use (4.12) and (4.13) to estimate the right hand side in identity (4.14). By doing so, we find a constant $c_5>0$, independent of ε, such that

$$\varepsilon e^{-\frac{t}{\varepsilon}}\frac{d}{dt}\left(e^{\frac{t}{\varepsilon}}\|\boldsymbol{\xi}_\varepsilon(t)\|^2_{\mathscr{H}^1_0}\right)+\frac{d}{dt}\|\boldsymbol{\chi}_\varepsilon(t)\|^2_{\mathscr{H}^1_0}=\frac{d}{dt}\left(\varepsilon\|\boldsymbol{\xi}_\varepsilon\|^2_{\mathscr{H}^1_0}+\|\boldsymbol{\chi}_\varepsilon\|^2_{\mathscr{H}^1_0}\right)+\|\boldsymbol{\xi}_\varepsilon\|^2_{\mathscr{H}^1_0}\le c_5\varepsilon\|\partial_t\boldsymbol{E}\|_{\mathscr{H}^1_0}\,.$$

Integrating in time, by Theorem 4.3 we arrive at

$$e^{t/\varepsilon}\|\boldsymbol{\xi}_\varepsilon(t)\|^2_{\mathscr{H}^1_0}\le\|\boldsymbol{\xi}_\varepsilon(0)\|^2_{\mathscr{H}^1_0}+c_5\int_0^t e^{\tau/\varepsilon}\|\partial_t\boldsymbol{E}(\tau)\|_{\mathscr{H}^1_0}\,d\tau\,.$$

By (4.13) and Cauchy-Schwartz inequality, the latter implies

$$e^{t/\varepsilon}\|\boldsymbol{\xi}_\varepsilon(t)\|^2_{\mathscr{H}^1_0}\le\|\boldsymbol{\xi}_\varepsilon(0)\|^2_{\mathscr{H}^1_0}+c_5\sqrt{\tfrac{c_2}{2}}\cdot e^{t/\varepsilon}\sqrt{\varepsilon}\tag{4.15}$$

whence the conclusion. ☐

Theorem 4.4 (non-singular convergence)**.** *Under the assumptions of Theorem 4.3, and assuming furthermore that*

$$\|\boldsymbol{\nabla}\times\boldsymbol{H}_{0\varepsilon}-\sigma\boldsymbol{E}_{0\varepsilon}\;-\boldsymbol{J}_0(0)\|_{\mathscr{H}^1_0}=O(\sqrt{\varepsilon})\,,\qquad \text{as } \varepsilon\to 0^+,\tag{4.16}$$

we have

$$\sup_{t\in[0,T]}\|\boldsymbol{E}_\varepsilon(t)-\boldsymbol{E}(t)\|_{L^2(\Omega;\mathbb{R}^2)}=O(\sqrt{\varepsilon})\,,\qquad \text{as } \varepsilon\to 0^+.$$

Proof. By triangle inequality, (4.11) and (4.16) imply

$$\|\sigma\boldsymbol{E}(0)-\sigma\boldsymbol{E}_{0\varepsilon}\|_{\mathscr{H}^1_0}=O(\varepsilon^{1/2})\,,\qquad \text{as } \varepsilon\to 0^+.$$

Then, repeating the proof of Theorem 4.3 all the way up to (4.15), we finally obtain

$$\|\boldsymbol{E}_\varepsilon(t)-\boldsymbol{E}(t)\|^2_{\mathscr{H}^1_0}\le C_1e^{-t/\varepsilon}\varepsilon+C_2\varepsilon^{1/2}\,,$$

for suitable constants $C_1,C_2>0$. ☐

5. A forward model with singular sources

We introduce in this section a model that suites the case of sources very concentrated in limited regions, rather than being spread over a wide area. To this aim, we consider sources described by vector-valued (current) distributions $\boldsymbol{J}_0\in\mathcal{E}'(\mathbb{R}^3\times\mathbb{R})^3$ with compact support in $\Omega\times(0,T)$. In this case, solutions to the magneto-quasistatic Maxwell equations (1.6) are understood in the sense of distributions, viz.

$$\int\int_\Omega\boldsymbol{H}\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\sigma^{-1}\boldsymbol{\varphi})\,d\boldsymbol{x}\,dt-\int\int_\Omega\mu\boldsymbol{H}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t}\,d\boldsymbol{x}\,dt=\langle\boldsymbol{J}_0\,,\boldsymbol{\varphi}\rangle\,,\tag{5.1}$$

for all $\boldsymbol{\varphi}\in C_0^\infty(\Omega\times(0,T))$, where $\langle\cdot\,,\cdot\rangle$ denotes the distributional duality pairing.

5.1. **Subsurface sources.** In particular, $\boldsymbol{J}_0$ may be concentrated along, and tangent to, a given surface Σ. If in addition it belongs to the space defined in (2.1), then formally a vector field $\boldsymbol{H} \in W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))$ solving Equation (1.8) is such that

$$\mu \frac{\partial \boldsymbol{H}}{\partial t} + \boldsymbol{\nabla} \times \left(\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{H}\right) = \boldsymbol{\nabla} \times (\sigma^{-1} \boldsymbol{J}_0)\,, \qquad \text{in } \mathcal{D}'((\Omega \setminus \Sigma) \times (0,T)) \tag{5.2}$$

together with the transmission conditions

$$[\![\boldsymbol{n} \times \boldsymbol{H}]\!] = \boldsymbol{J}_0 \qquad \text{across } \Sigma. \tag{5.3}$$

To see that (5.3) holds, we introduce the electric field by setting $\boldsymbol{E} = \sigma^{-1}(\boldsymbol{\nabla} \times \boldsymbol{H} - \boldsymbol{J}_0)$. Then

$$\int_\Omega \boldsymbol{H} \cdot \boldsymbol{\nabla} \times \boldsymbol{\varphi}\, d\boldsymbol{x} - \int_\Omega \sigma \boldsymbol{E} \cdot \boldsymbol{\varphi}\, d\boldsymbol{x} = \int_\Sigma \boldsymbol{J}_0 \cdot \boldsymbol{\varphi}\, d\Sigma \tag{5.4}$$

for any arbitrary test field $\boldsymbol{\varphi} \in C_0^\infty(\Omega;\mathbb{R}^3)$. Considering a region $D \subset \Omega$, with $\overline{D} \subset \Omega$, that is split into un upper part D^+ and a lower one D^- by Σ, the validity of the previous equation for all field supported in $D \setminus \Sigma$ implies that (1.6a) and (1.6b) hold locally in D^+ and in D^-. Thus, we have

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{H} - \sigma \boldsymbol{E} = 0\,, &\qquad \text{in } (\Omega \setminus \Sigma) \times (0,T), \\ \mu \frac{\partial \boldsymbol{H}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{E} = 0\,, &\qquad \text{in } \Omega \times (0,T). \end{aligned} \tag{5.5}$$

Therefore, if in (5.4) we choose $\boldsymbol{\varphi}$ with support intersecting Σ, then integrating by parts and using the first equation in (5.5) we get

$$\int_\Sigma \boldsymbol{\varphi} \cdot [\![\boldsymbol{H} \times \boldsymbol{n}]\!]\, d\Sigma = \int_\Sigma \boldsymbol{\varphi} \cdot \boldsymbol{J}_0\, d\Sigma\,,$$

and that implies (5.3) because $\boldsymbol{\varphi}$ can have any arbitrary trace along Σ.

5.2. **Signals in free homogeneous space.** Incidentally, we consider a fictitious model in which signals are free to propagate in a free space, without boundary conditions, filled with a homogeneous isotropic medium.

We fix $\kappa > 0$ and we recall that the function $\Gamma_\kappa \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$ defined by

$$\Gamma_\kappa(\boldsymbol{x}, t) = \begin{cases} \left(\dfrac{\kappa}{4\pi t}\right)^{\frac{3}{2}} e^{-\frac{\kappa}{4t}(x^2+y^2+z^2)}\,, & \text{if } \boldsymbol{x} = (x,y,z) \in \mathbb{R}^3 \text{ and } t > 0, \\ 0 & \text{if } \boldsymbol{x} = (x,y,z) \in \mathbb{R}^3 \text{ and } t \leq 0, \end{cases} \tag{5.6}$$

solves

$$\kappa \frac{\partial}{\partial t} \Gamma - \nabla^2 \Gamma = 0\,, \qquad \text{in } \mathbb{R}^3 \times (0,+\infty),$$

where ∇^2 denotes the Laplace operator, with

$$\lim_{t \to 0^+} \int_{\mathbb{R}^3} \Gamma(x,t) \phi(x)\, dx = \phi(0)\,.$$

As a consequence, given constants $\mu > 0$ and $\overline{\sigma} > 0$, a particular solution of

$$\mu \cdot \overline{\sigma} \cdot \frac{\partial}{\partial t} \boldsymbol{H} - \boldsymbol{\nabla}^2 \boldsymbol{H}_{\sigma_0} = \boldsymbol{\nabla} \times \boldsymbol{J}_0\,, \qquad \text{in } \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}),$$

is provided us setting $\boldsymbol{H}_{\sigma_0} = \Gamma_\kappa \star (\boldsymbol{\nabla} \times \boldsymbol{J}_0)$, where $\kappa = \mu \cdot \overline{\sigma}$ and the vector valued convolution in space-time is defined for all $u \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$ and for all $\boldsymbol{T} \in \mathcal{E}'(\mathbb{R}^3 \times \mathbb{R})^3$ componentwise, according to the formula

$$u \star \boldsymbol{T}(\boldsymbol{x}, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} u(\boldsymbol{x} - \boldsymbol{y}, t - s) \boldsymbol{T}(y, s)\, d\boldsymbol{y}\, ds\,.$$

We make the assumption that the source is concentrated in a region where the electric conductivity introduced in (2.5a) is isotropic and homogeneous, say $\sigma(x) = \sigma_0$ at all points x belonging to the support of $\boldsymbol{J}_0$. Then, by the material above the auxiliary system

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{H}_\kappa - \overline{\sigma}\, \boldsymbol{E}_\kappa &= \boldsymbol{J}_0\,, \qquad \text{in } \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}), \\ \boldsymbol{\nabla} \times \boldsymbol{E}_\kappa + \mu \frac{\partial}{\partial t} \boldsymbol{H}_\kappa &= 0\,, \qquad \text{in } \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}), \end{aligned}$$

is equipped with a solution $(\boldsymbol{E}_\kappa, \boldsymbol{H}_\kappa) \in C^\infty((\mathbb{R}^3 \times \mathbb{R}) \setminus \operatorname{supp}(\boldsymbol{J}_0))$ given by

$$\boldsymbol{H}_\kappa = \Gamma_\kappa \star (\boldsymbol{\nabla} \times \boldsymbol{J}_0)\,, \tag{5.7a}$$

$$\boldsymbol{E}_\kappa = \frac{1}{\overline{\sigma}} (\boldsymbol{\nabla} \times \boldsymbol{H}_\kappa - \boldsymbol{J}_0)\,. \tag{5.7b}$$

Eventually, since $\boldsymbol{\nabla} \cdot \boldsymbol{J}_0 = 0$ we have

$$\boldsymbol{\nabla} \times \boldsymbol{H}_\kappa = \Gamma_\kappa \star (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{J}_0) = \Gamma_\kappa \star (-\boldsymbol{\nabla}^2 \boldsymbol{J}_0) = -(\nabla^2 \Gamma_\kappa) \star \boldsymbol{J}_0\,. \tag{5.8}$$

5.3. **Renormalised signals.** Now we consider the signals propagating, virtually, in the medium defined by difference between the real medium and the fictitious one that was introduced in the previous section.

Given a solution of (1.6) in the sense of distributions, the fields defined by difference setting $\boldsymbol{u} = \boldsymbol{H} - \boldsymbol{H}_\kappa$ and $\boldsymbol{v} = \boldsymbol{E} - \boldsymbol{E}_\kappa$ solve a system of the form

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{u} - \sigma \boldsymbol{v} &= \boldsymbol{f}\,, \\ \boldsymbol{\nabla} \times \boldsymbol{v} + \mu \frac{\partial}{\partial t} \boldsymbol{u} &= 0\,, \end{aligned} \tag{5.9}$$

in $\mathcal{D}'(\Omega \times (0,T))$. Equivalently, the magnetic part $\boldsymbol{u}$ is dictated by the magneto-quasistatic problem

$$\mu \frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{\nabla} \times (\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{u}) = \boldsymbol{f} \tag{5.10}$$

In both (5.9) and (5.10), we set

$$\boldsymbol{f}(x) = \begin{cases} \mu(\overline{\sigma} - \sigma) \dfrac{\partial \Gamma_\kappa}{\partial t} \star \boldsymbol{J}_0 & \text{in } \{\sigma \not\equiv \sigma_0\} \\ 0 & \text{in } \{\sigma \equiv \sigma_0\} \end{cases} \tag{5.11}$$

By [3, Appendix A.1], the boundary values $\boldsymbol{u} \times \boldsymbol{n} = -\boldsymbol{H}_\kappa \times \boldsymbol{n}$ belong to the space $H^{-\frac{1}{2}}(\operatorname{div}_\tau; \partial\Omega)$, defined in as in (2.1) with $\Sigma = \partial\Omega$, because their tangential divergence equals $-\boldsymbol{\nabla} \cdot \boldsymbol{H}_\kappa \times \boldsymbol{n}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{H}_\kappa = \Gamma_\kappa \star \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{J}_0) = 0$. Hence they extend to a suitable function $\boldsymbol{F}_\kappa \in L^2(0, T; \mathscr{H}^1_0)$. Setting $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{F}_\kappa$ we then have

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{w} - \sigma \boldsymbol{v} &= \boldsymbol{f} + \boldsymbol{\nabla} \times \boldsymbol{F}_\kappa\,, \qquad \text{in } \Omega \times (0,T), \\ \boldsymbol{\nabla} \times \boldsymbol{v} + \mu \frac{\partial \boldsymbol{w}}{\partial t} &= 0\,, \qquad \text{in } \Omega \times (0,T), \\ \boldsymbol{w} \times \boldsymbol{n} &= 0\,, \qquad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{5.12}$$

Equivalently, we have that $\boldsymbol{w} \in L^2(0,T;\mathscr{H}^1_0)$ and

$$\mu \frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{\nabla} \times \left(\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{w}\right) = \boldsymbol{\nabla} \times \left(\sigma^{-1} \left(\boldsymbol{f} + \boldsymbol{\nabla} \times \boldsymbol{F}_\kappa\right)\right). \tag{5.13}$$

Since we are assuming the singular source $\boldsymbol{J}_0$ to be supported in the level set $\{\sigma(x) = \overline{\sigma}\}$, (5.11) implies $\boldsymbol{f} \in L^\infty(\Omega \times (0,T))$, whence it follows that $\boldsymbol{f} \in L^2(0,T;L^2(\Omega;\mathbb{R}^3))$, because Ω is bounded. Hence, by Theorem 3.3, there exists a unique $\boldsymbol{w} \in W^{1,2}(0,T;\mathscr{H}^1_0)$ solving (5.13), with $\boldsymbol{w}(0) = 0$. Then, setting $\boldsymbol{v} = \sigma^{-1}\left(\boldsymbol{\nabla} \times w - \boldsymbol{f} + \boldsymbol{\nabla} \times \boldsymbol{F}_\kappa\right)$, we have

$$\begin{aligned} &\int_\Omega \boldsymbol{w} \cdot \boldsymbol{\nabla} \times \boldsymbol{\varphi}\, dx - \int_\Omega \sigma \boldsymbol{v} \cdot \boldsymbol{\varphi}\, dx = \int_\Omega (\boldsymbol{f} + \boldsymbol{\nabla} \times \boldsymbol{F}_\kappa) \cdot \boldsymbol{\varphi}\, dx \\ &\int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla} \times \boldsymbol{\psi}\, dx + \int_\Omega \mu \frac{\partial}{\partial t} \boldsymbol{w} \cdot \boldsymbol{\psi}\, dx = 0\,, \end{aligned} \tag{5.14}$$

for all $\boldsymbol{\varphi} \in \mathscr{H}^1$, for all $\boldsymbol{\psi} \in \mathscr{H}^1_0$, and for a.e. $t \in (0,T)$. We thus have proved the following.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^3$ satisfy (2.4), let $\mu, \overline{\sigma}$ be positive constants and set $\kappa = \mu \cdot \overline{\sigma}$. Let σ be a piecewise constant positive bounded function, and let $\boldsymbol{H}_0 \in \mathscr{H}^1_0$. Let $\boldsymbol{J} \in \mathcal{E}'(\mathbb{R}^3 \times \mathbb{R})$ be a vector-valued distribution with support contained in the level set $\{\sigma(x) = \overline{\sigma}\}$. Then there exist a unique solution $(\boldsymbol{E}, \boldsymbol{H})$ of (1.6) in $\mathcal{D}'(\Omega \times (0,T))$, that takes the form $\boldsymbol{E} = \boldsymbol{E}_\kappa + \boldsymbol{v}$, and*

$$\boldsymbol{H} = \boldsymbol{H}_\kappa + \boldsymbol{F}_\kappa + \boldsymbol{w}\,, \tag{5.15}$$

where

- *(i) $(\boldsymbol{E}_\kappa, \boldsymbol{H}_\kappa)$ is defined as in (5.7),*
- *(ii) $\boldsymbol{F}_\kappa$ is any element of $L^2(0,T;\mathscr{H}^1_0)$ agreeing with $-\boldsymbol{H}_\kappa$ on $\partial\Omega \times (0,T)$, and*
- *(iii) $\boldsymbol{w}$ is the solution of (5.13), with $\boldsymbol{f}$ being defined by (5.11), under the initial condition $\boldsymbol{w}(0) = 0$.*

Theorem 5.1 provides a split formula (5.15) that can be used to compute the values of the magnetic field $\boldsymbol{H}$ due to the singular source $\boldsymbol{J}_0$. The first two summands in the right hand side are the result of representation formulas involving convolution with explicit kernels, see (5.7a). The extra term $\boldsymbol{w}$ is the solution of the magneto-quasistatic system (5.14), with a diffuse source (also defined by convolution, see (5.11)), that falls in the theory discussed in Section 3.

We now report some basic strategy for the methods in numerical approximation of $\boldsymbol{w}$. The following material does not make, by any means, any pretence of completeness. For a modern and comprehensive exposition of the topic, we refer the interested reader to the book [3]. For an introduction to the more general magneto-hydrodynamic setting we mention the nice treatise [20].

5.4. **Curl-conforming elements.** For $k \in \{0,1,2,3\}$, the *standard k-simplex* in $\mathbb{R}^3$ is the closed convex hull $\triangle^k$ of $\{0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$. Any non-degenerate affine image of $\triangle^k$ is called an *affine k-simplex*. Given $k \in \{1,2,3\}$, the convex hull of any proper nonempty subset of the $k+1$ points that define $\triangle^k$ is called a *face* of $\triangle^k$ and we say that F is a *face of the affine k-simplex* $\triangle$ if $F = T_\triangle(F^k)$, where $T_\triangle$ is the unique affine map with $T_\triangle(\triangle^k) = \triangle$ and F^k is a face of $\triangle^k$; if that happens, clearly F itself is a $(k-1)$-simplex. An *affine simplical complex* $\mathcal{K}$ in $\mathbb{R}^3$ is comprised of pairwise disjoint simplices such that every face of a simplex of $\mathcal{K}$ also belongs to $\mathcal{K}$, with the intersection of any two simplices in $\mathcal{K}$ being a face to each of them.

Any subfamily of simplices of $\mathcal{K}$ containing the faces of all of its elements is also an affine simplicial complex and is called a subcomplex of $\mathcal{K}$; the one that contains all simplices of $\mathcal{K}$ having dimension at most k is called the *k-skeleton* of $\mathcal{K}$.

An *affine simplicial triangulation* $\mathcal{T}$ of a polytope Q in $\mathbb{R}^3$ is a simplicial complex whose union gives Q. Formal finite linear combinations of k-simplices in $\mathcal{T}$ with integer coefficients are called *k-chains* and form a module $\mathcal{C}_k(\mathcal{T};\mathbb{Z})$ over the ring $\mathbb{Z}$ of integers. Group homomorphisms from $\mathcal{C}_k(\mathcal{T};\mathbb{Z})$ to $\mathbb{R}$ are called *k-cochains* and form an abelian group denoted by $\mathcal{C}^k(\mathcal{T};\mathbb{Z})$.

We introduce a set of piecewise linear 1-forms setting

$$\mathbb{P}(\mathcal{T}) = \Big\{ \omega \in C(\Omega\,;(\mathbb{R}^3)^*) \colon \forall \text{ 3-simplex } \triangle \in \mathcal{T},\ \exists\ \alpha,\beta \in (\mathbb{R}^3)^* \text{ with } T_\triangle^*\omega = \alpha + \star(\beta\wedge\xi)\ \Big\}$$

where $T_\triangle^*\omega$ stands for the pull-back of ω under the mapping $T_\triangle$, $\star$ is Hodge operator and ξ is the differential form defined by $\xi(x) = x_1dx_1 + x_2dx_2 + x_3dx_3$. We associate every $\omega \in \mathbb{P}(\mathcal{T})$ with the 1-cochain defined on 1-simplices by $\mathrm{e} \mapsto \int_\mathrm{e} \omega$ and extended to $C_1(\mathcal{T};\mathbb{Z})$ by linearity. This describes an algebraic homomorphism from $\mathbb{P}(\mathcal{T})$ into $\mathcal{C}^1(\mathcal{T};\mathbb{Z})$. Indeed, given a 3-simplex $\triangle \in \mathcal{T}$ and two covectors $\alpha,\beta \in (\mathbb{R}^3)^*$, assuming $\int_\mathrm{e}(\alpha+\star\beta\wedge\xi) = 0$ for all of six edges e of $\triangle$ implies that $\alpha = \beta = 0$. Then, the knowledge of $\boldsymbol{\varphi} \in \mathbb{P}(\mathcal{T})$ is equivalent to that of all the *degrees of freedom*

$$|\det T_\triangle| \int_\mathrm{e} \Big(\boldsymbol{\varphi}_{|\triangle} \circ T_\triangle\Big)\cdot\boldsymbol{\tau}\,, \qquad \text{for all 3-simplices } \triangle \in \mathcal{T}, \tag{5.16}$$

with $\boldsymbol{\tau}$ being the vector associated to the edge e.

The mapping that takes any differential form $\boldsymbol{\varphi}$ having precise values on segments to the element $\omega_{\boldsymbol{\varphi}} \in \mathbb{P}(\mathcal{T})$ with the same degrees of freedom is called *interpolation* and introduces an error

$$\|\boldsymbol{\varphi} - \omega_{\boldsymbol{\varphi}}\|_{H\Lambda^1(\Omega)} \le \max_{\triangle\in\mathcal{T}} |\det T_\triangle|^{\frac{1}{3}}\, \|\boldsymbol{\varphi}\|_{H\Lambda^1(\Omega)}$$

(see [3]). In the simpler situation in which $\Omega = Q$ is a convex polytope, we assume $\mathcal{T}$ to be an affine simplicial triangulation that is the union of many subcomplices, all consisting of $3! = 6$ affine 3-simplices. In each subcomplex, one simplex $\triangle$ is obtained from $\triangle^3$ by a translation and a homotetic transformation of ratio $h = |\det T_\triangle|^{\frac{1}{3}}$. If $\omega \in C^{0,\alpha}\Lambda^1(\overline{\Omega})$ and we call $\boldsymbol{\psi}[\omega]$ an element of $L^\infty\Lambda^1(\Omega)$ that agrees with ω on $\triangle$, then

$$\|\omega - \boldsymbol{\psi}[\omega]\|_{L^\infty\Lambda^1(\Omega)} \le ch^\alpha\,.$$

Also, given any $\omega_1,\omega_2 \in C^{0,\alpha}\Lambda^1(\overline{\Omega}) \cap \mathbb{P}(\mathcal{T})$, with $T_\triangle^*\omega_i = \alpha_i^\triangle + \star(\beta_i^\triangle \wedge \xi)$, for $i = 1,2$ and for all $\triangle \in \mathcal{T}$, we have

$$(\sigma^{-1}d\boldsymbol{\psi}[\omega_1]\,, d\boldsymbol{\psi}[\omega_2])_{L^2\Lambda^1} \sim \frac{h}{6}\sum_{\triangle\in\mathcal{T}} \beta_1^\triangle\cdot\beta_2^\triangle\,.$$

Remark 5.2. Every vector field $\boldsymbol{w}$ with precise values on segments is associated uniquely to a differential form that has well defined degrees of freedom. For example, formula (5.16) makes sense if $\boldsymbol{\varphi} \in C(\Omega)^3$, or else if $\boldsymbol{\varphi} \in H^s(\Omega)^3 \cap \boldsymbol{\nabla}\times L^p(\Omega)^3$ with $s > \frac{1}{2}$ and $p > 2$. If $\boldsymbol{w}$ is the remainder term in (5.15), then the corresponding form belongs to $C^{0,\alpha}\Lambda^1(\overline{\Omega})$ by Theorem 3.5. This material suggests a convenient approximation of $\boldsymbol{w}$, with error $O(h^\alpha)$.

6. Inverse source problems

6.1. **Nonuniqueness of volume currents.** It is known since the work of Helmholtz that the reconstruction of electric sources from tangential boundary measurements is generally an ill-posed problem. For the existence of non-radiating volume source currents in time-harmonic regime, we refer to [1] in the complete hyperbolic setting and to [4] in the eddy current case. There are little differences between the problem considered in the literature and the one surveyed in these pages, but we present however some details on this topic. An expedient integral identity holds: denoting by Θ^* the adjoint mapping to the linear "Dirichlet-to-Neumann" map

$$\boldsymbol{n} \times \boldsymbol{H} \longmapsto \boldsymbol{n} \times \left(\sigma^{-1}\, \boldsymbol{\nabla} \times \boldsymbol{H}\right)$$

we see that the forward parabolic equation (1.8) with initial condition $\boldsymbol{H}(0) = 0$, implies the identity

$$\int_0^T \int_\Omega \boldsymbol{H} \cdot \mathcal{P}^*(\boldsymbol{\xi})\, d\boldsymbol{x}\, dt + \int_0^T \left\langle \sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{\xi} + \Theta^*(\boldsymbol{\xi})\,, \boldsymbol{n} \times \boldsymbol{H} \right\rangle dt = \int_0^T \int_\Omega \boldsymbol{\nabla} \times (\sigma^{-1} \boldsymbol{J}_0) \cdot \boldsymbol{\xi}\, dx\, dt\,, \tag{6.1}$$

for all smooth vector fields $\boldsymbol{\xi}$ such that $\boldsymbol{\xi}(T) = 0$. We introduced the backward parabolic operator

$$\mathcal{P}^*(\boldsymbol{\xi}) = -\mu \partial_t \boldsymbol{\xi} + \boldsymbol{\nabla} \times \left(\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{\xi}\right). \tag{6.2}$$

Thus, tangential boundary measurements will be blind to any source that is orthogonal, in a suitable fractional Sobolev space, to the kernel of the adjoint operator $\mathcal{P}^*$ with vanishing final data

$$\mathcal{K}(\Omega) = \left\{ \boldsymbol{\xi} \in W^{1,2}(0,T; \mathscr{H}^1_0) : \ \ \mathcal{P}^*(\boldsymbol{\xi}) = 0\,,\ \boldsymbol{\xi}(T) = 0 \right\}.$$

For a more precise appreciation of this point, it is convenient to introduce the space $\mathscr{S}(\Omega)$ of *radiating sources*, consisting of the closure in $L^2(0,T; L^2(\Omega;\mathbb{R}^3))$ of the vector space

$$\left\{ \boldsymbol{\xi} \in \mathcal{K}(\Omega) : \ \ \boldsymbol{\xi} = \nabla \times (\sigma^{-1} \boldsymbol{v}) \text{ for some } \boldsymbol{v} \in L^2(0,T; L^2(\Omega;\mathbb{R}^3)) \right\}.$$

It can be seen that

$$\mathscr{N}(\Omega) = \{ \boldsymbol{g} \in L^2(0,T; L^2(\Omega;\mathbb{R}^3)) : \ \ (\boldsymbol{g}\,, \boldsymbol{f})_{L^2} = 0\,,\ \text{for all } \boldsymbol{f} \in \mathscr{S}(\Omega) \}$$

is not a trivial space. $\mathscr{N}(\Omega)$ is said to collect *non-radiating sources* because of the following result.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^3$ satisfy (2.4), let $\mu > 0$, let $\sigma \in L^\infty(\Omega;\mathbb{R}^{3\times 3})$, and assume that (2.5) holds. Let $\boldsymbol{J}_0 \in L^2(0,T; L^2(\Omega;\mathbb{R}^3))$, with $\nabla \times (\sigma^{-1} \boldsymbol{J}_0) = \boldsymbol{f} + \boldsymbol{g}$ where $\boldsymbol{f} \in \mathscr{S}(\Omega)$ and $\boldsymbol{g} \in \mathscr{N}(\Omega)$, and let $(\boldsymbol{E}, \boldsymbol{H})$ be the solution of (1.6), with (1.9), subject to the initial condition $\boldsymbol{H}(0) = 0$. Then*

- *(i) the knowledge of $\boldsymbol{n} \times \boldsymbol{H}$ on $\partial\Omega \times (0,T)$ uniquely determines $\boldsymbol{f}$;*
- *(ii) if $\boldsymbol{f} = 0$, then $\boldsymbol{n} \times \boldsymbol{H} = 0$ on $\partial\Omega \times (0,T)$.*

Proof. If $\boldsymbol{n} \times \boldsymbol{H} = 0$ on $\partial\Omega \times (0,T)$, then by (6.1) we have $(\boldsymbol{f}\,, \boldsymbol{\xi})_{L^2} = 0$ for all $\boldsymbol{\xi} \in \mathscr{S}(\Omega)$ and that implies (i). As for (ii), we note that, for every element $\boldsymbol{\eta}$ of the trace space defined by (2.1) with $\Sigma = \partial\Omega$, the backward-in-time parabolic problem

$$\begin{cases} \mathcal{P}^*(\boldsymbol{\xi}) = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{\xi}(T) = 0 & \text{in } \Omega \times \{T\}, \\ \sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{\xi} + \Theta^*(\boldsymbol{\xi}) = \boldsymbol{\eta} & \text{on } \partial\Omega \times (0,T), \end{cases}$$

has a unique solution, for which (6.1) implies

$$\int_0^T \langle \boldsymbol{\eta}\,, \boldsymbol{n} \times \boldsymbol{H} \rangle = 0\,.$$

Since $\boldsymbol{\eta}$ was arbitrary, we obtain that $\boldsymbol{n} \times \boldsymbol{H} = 0$. ∎

6.2. **Uniqueness for atomic sources.** In view of non-uniqueness of volume currents, when considering the inverse problem one is lead to postulate a priori assumptions on the structure of the source. In particular, we restrict our attention to sources that satisfy precise properties related to

- the geometry of their support;
- the vector-valued information on their orientation
- the scalar information about their intensity as currents.

The simplest situation is related to a source with a trivial orientation, which brings one to consider scalar equations, rather than systems. Also, we may assume a uniform electric current density. Then, we are led to the consider the following equation

$$\begin{aligned} \mu_0\sigma_0\frac{\partial u}{\partial t}(\boldsymbol{x},t) - \nabla^2 u(\boldsymbol{x},t) &= 0\,, \qquad (\boldsymbol{x},t)\in\mathbb{R}^3\times(0,T) \\ u(\boldsymbol{x},0) &= \sum_{j=1}^{N}\delta(\boldsymbol{x}-\boldsymbol{a}_j)\,, \qquad \text{in } \mathbb{R}^3, \end{aligned} \tag{6.3}$$

where the positive integer N and the points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$ are unknown, and δ is Dirac's delta.

Toy Problem. Given points $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^2\times\{0\}$ in general position, and given a sequence $(t_j)_j \subset (0,T)$ **determine** N and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$ from the knowledge of $u(\boldsymbol{x},t_1)$, $u(\boldsymbol{x},t_2)$, $u(\boldsymbol{x},t_3)$, etc. assuming that (6.3) holds.

The toy problem is easier to consider if we further assume that $N=1$. Then, given $\boldsymbol{a}$ and $\boldsymbol{b}$ in the horizontal plane, the corresponding solutions of (6.3) are given by

$$u(\boldsymbol{x},t) = \Gamma(\boldsymbol{x}-\boldsymbol{a}, \mu\sigma_0 t)\,, \qquad v(\boldsymbol{x},t) = \Gamma(\boldsymbol{x}-\boldsymbol{b}, \mu\sigma_0 t)\,,$$

where Γ is as in (5.6) with $\kappa = \mu_0\sigma_0$. Then, by assumption, the three points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ are at the same euclidean distance r to $\boldsymbol{a}$: in particular, as in Figure 6.2, the point $\boldsymbol{a}$ belongs to the vertical circle obtained intersecting the spheres of radius r centred at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively. If $\boldsymbol{x}_3$

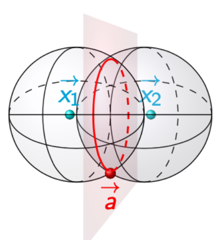

Figure 1. The case $N=1$

is not on the line joining $\boldsymbol{x}_1$ and $\boldsymbol{x}_3$, then the sphere of radius centred at $\boldsymbol{x}_3$ hits the circle exactly at two points, among which the relevant one is $\boldsymbol{a}$.

If $N\geq 1$ is unknown, we assume that there exist two unordered sets of points $A=\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N\}$ and $B=\{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_M\}$ and we denote u, v the signals generated by A, B, respectively. The limit as $t\to\infty$ in the following identity between analytic functions

$$\sum_{j=1}^{N} e^{-\mu_0\sigma_0\frac{|\boldsymbol{x}_1-\boldsymbol{a}_j|^2}{4t}} = \sum_{j=1}^{M} e^{-\mu_0\sigma_0\frac{|\boldsymbol{x}_1-\boldsymbol{b}_j|^2}{4t}} \tag{6.4}$$

gives $N = M$. Before sending $t \to 0^+$, we may arrange the points in increasing distance to, say, the first reference point $\boldsymbol{x}_1$

$$|\boldsymbol{a}_1 - \boldsymbol{x}_1| \leq |\boldsymbol{a}_2 - \boldsymbol{x}_1| \leq \ldots \leq |\boldsymbol{a}_N - \boldsymbol{x}_1|$$
$$|\boldsymbol{b}_1 - \boldsymbol{x}_1| \leq |\boldsymbol{b}_2 - \boldsymbol{x}_1| \leq \ldots \leq |\boldsymbol{b}_N - \boldsymbol{x}_1|$$

Denoting by n_1 (resp, m_1) the maximal integer n (resp. m) for which $|\boldsymbol{a}_n - \boldsymbol{x}_1| = |\boldsymbol{a}_1 - \boldsymbol{x}_1| =: R_1$ (resp., for which $|\boldsymbol{b}_m - \boldsymbol{x}_1| = |\boldsymbol{b}_1 - \boldsymbol{x}_1| =: S_1$), we have

$$\exp\left[-\mu_0\sigma_0\frac{R_1^2}{4t}\right](n_1 + o(1)) = \exp\left[-\mu_0\sigma_0\frac{S_1^2}{4t}\right](m_1 + o(1)), \qquad \text{as } t \to 0^+,$$

whence it follows that $R_1 = S_1$ and $n_1 = m_1$. Then from (6.4) we have arrived at

$$\sum_{j=n_1+1}^{N} e^{-\mu_0\sigma_0\frac{|\boldsymbol{x}_1-\boldsymbol{a}_j|^2}{4t}} = \sum_{j=n_1+1}^{N} e^{-\mu_0\sigma_0\frac{|\boldsymbol{x}_1-\boldsymbol{b}_j|^2}{4t}}. \tag{6.5}$$

We can repeat this argument starting, this time, from (6.5) rather than from (6.4). By a finite descent, we conclude that there exist $k \leq N$, k positive integers $n_1 < \ldots < n_k$, and k positive numbers $R_1, \ldots, R_k$ such that all points $\boldsymbol{a}_j$ and $\boldsymbol{b}_j$, with $n_\ell < j \leq n_{\ell+1}$, are at the same distance R_ℓ to $\boldsymbol{x}_1$, for $\ell = 1, \ldots, k-1$.

Also, we can arrive at similar conclusions arguing as done above except for replacing $\boldsymbol{x}_1$ by either of $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$. Therefore, for every single $j \in \{1, \ldots, N\}$ we have

$$|\boldsymbol{x}_1 - \boldsymbol{a}_j| = |\boldsymbol{x}_2 - \boldsymbol{a}_j| = |\boldsymbol{x}_3 - \boldsymbol{a}_j| \tag{6.6}$$

As the points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ are in general position by assumption, as seen previously in the case of a single source (6.6) determines $\boldsymbol{a}_j$. Thus, $A = B$.

REMARK 6.2. For the purposes of applications, it would be interesting to consider the inverse problem under the "dipole assumption" in the case of non-constant conductivities: we do not consider this issue here.

6.3. **Uniqueness of subsurface currents.** It might be relevant to consider sources that are concentrated along surfaces such as fault planes, and to reconstruct them from boundary measurements. We mention a result that applies to media described by smooth constitutive coefficients. Uniqueness in this case results are available for surface currents that are a priori concentrated on the boundaries of subdomains: the following result, for example, requires the knowledge of a continuous set of measurements, that holds for sources of the "separable" form

$$\boldsymbol{J}_0(\boldsymbol{x}, t) = h(t)\boldsymbol{f}(\boldsymbol{x}). \tag{6.7}$$

Theorem 6.3. *Let Ω satisfy (2.4), and let μ and σ be smooth positive functions satisfying (2.5). Let $B \subset \Omega$ be a connected open set with a Lipschitz regular boundary Σ. Assume that $\boldsymbol{J}_0 \in W^{1,2}(0,T; L^2(\Omega))$ is of the form (6.7) and that $\boldsymbol{H}$ is a solution of (1.6), with (1.9), subject to the initial conditions $\boldsymbol{H}(0) = \boldsymbol{H}_0$. Then, the knowledge of*

$$m(t, \boldsymbol{\psi}) = \int_\Sigma \mu\boldsymbol{H} \times \boldsymbol{\nu} \cdot \boldsymbol{\psi}\, d\Sigma \tag{6.8}$$

for all $t \in [0, T]$ and for all $\boldsymbol{\psi} \in \mathscr{H}_0^1$, uniquely determines $\boldsymbol{J}_0$.

Proof. To replicate the proof done in [23] in hyperbolic setting, one needs the estimates (3.8), (3.9) for the magnetic field and the higher order estimate

$$\|\boldsymbol{H}\|_{W^{1,\infty}(0,T;\mathscr{H}^1_0)} + \|\boldsymbol{H}\|_{W^{1,2}(0,T;\mathscr{H}^2_0)} \le c(\boldsymbol{\psi}) \left(\int_0^T h^2\, dt \right)^{\frac{1}{2}}$$

that holds under the additional assumption that σ is smooth (for, see the proof of Theorem 4.1). □

6.4. **Inverse source problems with controllability.** A different class of inverse source problems presuppose the complete knowledge of the initial and final state of the system, and the full knowledge of the initial value of the source. For the following result, we refer to [24] (which focus on the electric field, with unessential changes).

Theorem 6.4. *Let Ω satisfy (2.4), let μ, σ be positive functions satisfying (2.5), let $\boldsymbol{H}_0 \in L^2(\Omega)$, and let $\boldsymbol{J}_0 \in W^{1,2}(0,T;L^2(\Omega))$ and let $\boldsymbol{H} \in L^2(0,T;\mathscr{H}^1_0)$, with $\partial_t \boldsymbol{H} \in L^2(0,T,(\mathscr{H}^1_0)^*)$, be a weak solution of (1.6) and (1.9), with (1.7). Then, there exists $T_0 \in (0,T)$ such that, for every $\tau < T_0$, the knowledge of the final state $\boldsymbol{H}(\tau)$ and of the initial source $\boldsymbol{J}_0(\boldsymbol{x},0)$ determine uniquely $\boldsymbol{J}_0$.*

Acknowledgments. The uniqueness for atomic sources presented in Section 6.2 was illustrated to the author by Sergio Vessella in private communication. Some results commented in this paper were presented in Cetraro, Italy in Summer 2019, during the CIME-EMS Summer School in applied mathematics "Applied Mathematical Problems in Geophysics". The author is grateful to Massimo Chiappini, Roberto Carluccio, and Cesidio Bianchi from the Istituto Nazionale di Geofisica e Vulcanologia INGV for their interest in this research, and the Istituto Nazionale di Alta Matematica INDAM is acknowledged for the financial support.

References

[1] Albanese R.; Monk P.B.: The inverse source problem for Maxwell's equations in magnetoencelography. *SIAM J. Appl. Math.* 2006 **62** pp. 1369–82. 5, 19

[2] Alberti, G.S.: Hölder regularity for Maxwell's equations under minimal assumptions on the coefficients, *Calc. Var. Partial Differential Equations*, **57** (2018), pp. 1–11. 10

[3] Alonso Rodriguez, A.; Valli, A.: *Eddy Current Approximation of Maxwell Equations*, Springer-Verlag Italia, Milan, 2010. 17, 18, 19

[4] Alonso Rodriguez, A.; Camaño, J.; Valli, A.: Inverse source problems for eddy current equations. *Inverse problems* **28** 2012. 5, 19

[5] Brown, B. M.; Marletta, M.; Reyes, J.M.: Uniqueness for an inverse problem in electromagnetism with partial data, *Journ. of Diff. Equations* **260** (2016) no. 8, pp. 6525–6547. 5

[6] Athanasiou, M.A.; Anagnostopoulos, G. C.; Iliopoulos, A. C.;Pavlos, G. P.; David C. N.: Enhanced ULF radiation observed by DEMETER two months around the strong 2010 Haiti earthquake *Nat. Hazards Earth Syst. Sci.*, 11, 1091–1098, 2011. 3

[7] Bakun, W.H. et ali: Implications for prediction and hazard assessment from the 2004 Parkfield earthquake. *Nature* **437**, pages 969–974 (2005). 2

[8] Bernardi A.; Fraser-Smith, A.C.;McGill, P.R.; Villard Jr., O.G.: ULF magnetic field measurements near the epicenter of the Ms 7.1 Loma Prieta earthquake. *Physics of the Earth and Planetary Interiors* **68**, Issues 1-2, August 1991, Pages 45-63

[9] Costabel, M.: A coercive bilinear form for Maxwell's equations, *J. Math. An. Appl.* **157** (1991), pp. 527–541. 6

[10] Draganov,A. B.; Inan, U.S.; Taranenko, Yu. N.: ULF magnetic signatures at the Earth's surface due to ground waterflow, *Geophys. Res. Lett.* **18**, 1127-1130. 3

[11] Duvaut G.; Lions, J.L.: *Inequalities in Mathematical Physics*, Springer-Verlag Berlin, Heidelberg, 1976. 6

[12] Evans L.C.: *Partial Differential Equations.* AMS Graduate Studies in Mathematics, 2010. 6

[13] Fenoglio,M.A. ;Johnston, M.J.S.;Byerlee, J.D.: Magnetic and electric fields associated with changes in high pore pressure in fault zone - application to the Loma Prieta ULF emissions.*Proc. of Workshop LXIII, Menlo Park, CA*, 262-278, 1994. 3

[14] Frasier-Smith, Bernardi, McGill, Ladd, Helliwell, Villard Jr. Low Frequency magnetic field measurement near the epicenter of the M_s 7.1 Loma Prieta earthquake. *Geophys. Res. Letters* **17** (1990) no.9 pp.1465–1468. 2

[15] Francini, E.; Franzina, G.; Vessella, S. Existence and regularity for eddy currents system with non-smooth conductivity. *SIAM J. Math. Anal.* 2020, 52(2), pp. 2134–2157. 6, 10

[16] Johnston,M. J. S.; Müller, R.J.; Sasai, Y.: Magnetic field observations in the near-field: the 28 June 1992 M_w 7.3 Landers ,California, Earthquake, *Bull. Seism. Soc. Am.*, 8, 792-798, 1994 3

[17] Milani, A.: The Quasi-Stationary Maxwell Equations as Singular Limit of the Complete Equations: the Quasi-Linear Case. *Journ. of Math. Anal. and Appl.* **102** (1984), pp. 251–274. 11

[18] Molchanov, Hayakawa, Generation o fULF electromagnetic emission by microfracturing, *Geophysical rev lett.*, Vol. 22, no. 22, pp. 3091-3094. 3

[19] Okaji, T.: Strong unique continuation property for time harmonic Maxwell equations. *J. Math. Soc. Japan* **54** (2002) pp. 89-122.

[20] Pagliantini, C.: *Computational magnetohydrodynamics with discrete differential forms*. PhD Thesis, ETH Zürich. 18

[21] Palangio, P.; Di Lorenzo, C.; Masci, F.; Di Persio, M.: The study of the electromagnetic anomalies linked with the Earth's crustal activity in the frequency band [0.001 Hz–100 kHz]. *Nat. Hazards Earth Syst. Sci.*, 7, 507–511, 2007 2, 4

[22] Sebastiani G.; and Malagnini L.: Forecasting the Next Parkfield Mainshock on the San Andreas Fault (California). *J. Ecol. and Nat. Resour.* **4** (2020) no. 6. 2

[23] Slodička, M.; Galba, M.: Recovery of a time dependent source from a surface measurement in Maxwell's equations. *Comp. and Math. with Appl.* **71** (2016) 368–380. 5, 23

[24] Stratis, I.G.; Yannacopoulos, A.N.: Some remarks on a class of inverse problems reelate to the parabolic approximation to the Maxwell equations: a controllability approach. *Math. Meth. in Appl. Sci.* **38** (2015) 3866–3878. 5, 23

[25] Tatham, R. H.; McCormack, M.D.: *Multicomponent Seismology in Petroleum Exploration.* Society of Exploration Geophysicists, 1991. 2

[26] Thomas, J.N. ; Love, J.J.;. Johnston, M.J.S.: On the reported magnetic precursor of the 1989 Loma Prieta earthquake. *Physics of the Earth and Planetary Interiors* 173 (2009) 207–215. 2

[27] Walker, N.; Kadirkamanathan, V.; Pokhotelov, O. A.: Changes in the ultra-low frequency wave field during the precursorphase to the Sichuan earthquake: DEMETER observations *Ann. Geophys.*, **31**, 1597–1603, 2013. 3