QUASISTATIC EVOLUTION FOR A MODEL IN STRAIN GRADIENT PLASTICITY

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Abstract. We prove the existence of a quasistatic evolution for a model in strain gradient plasticity proposed by Gurtin and Anand concerning isotropic, plastically irrotational materials under small deformations. This is done by means of the energetic approach to rate-independent evolution problems. Finally we study the asymptotic behavior of the evolution as the strain gradient length scales tend to zero recovering in the limit a quasistatic evolution in perfect plasticity.

Keywords : variational models, energy minimization, quasistatic evolution, higher order stresses, flow rule.

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Contents

1. Introduction

Since the early attempts of Aifantis [2], strain gradient plasticity models have been proposed in order to capture *phenomenologically* size effects in metals such as *strengthening* and *strain hardening*. These effects, which take place approximately at the scale of $500nm - 50\mu m$, cannot be modelled by conventional theories of plasticity. This fact led to the development of continuum theories of plasticity that incorporate size-dependence by accounting for strain gradients, namely the gradient of plastic strains. Following the classical papers by Nye [29] and by Ashby [3, 4], strain gradients induce *geometrically necessary dislocations*, and these dislocations together with *statistically stored dislocations* are the main responsible of size effects.

Several strain gradient theories, different from one another, have been recently proposed by different authors [10, 1, 12, 21, 11, 15, 16, 17, 18, 7, 20, 14]. In this paper we focus on the theory proposed by Gurtin and Anand [19]. In the context of small deformations, and in absence of plastic rotation, the strain gradient dependence enters the model via a microstress associated to the gradient of the plastic strain and by a free energy dependent of the macroscopic Burgers tensor.

Let $\Omega \subseteq \mathbb{R}^3$ be the reference configuration of the body. The strain $(\mathbf{E}u)_{ij} := (\partial_i u_j + \partial_j u_i)/2$ of the displacement $u : \Omega \to \mathbb{R}^3$ is decomposed as usual in the form

$$\mathbf{E}u = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{p}} \tag{1.1}$$

where $\mathbf{E}^{\mathrm{e}} \in M^{3\times 3}_{\mathrm{sym}}$ is the *elastic strain*, while $\mathbf{E}^{\mathrm{p}}$ is referred to as the *plastic strain*. It is assumed that $\mathbf{E}^{\mathrm{p}}$ has zero trace, i.e., $\mathbf{E}^{\mathrm{p}}$ belongs to the space of deviatoric matrices $M^{3\times 3}_D$. Beside the usual Cauchy stress $\mathbf{T}$ which satisfies the classical macroscopic force balance, the stress configuration of the system is described by *a second order* tensor $\mathbf{T}^{\mathrm{p}}$ and a *third order* tensor $\mathbb{K}^{\mathrm{p}}$ which satisfy the equilibrium condition

$$\mathbf{T}_D = \mathbf{T}^{\mathrm{p}} - \mathrm{div}\mathbb{K}^{\mathrm{p}}. \tag{1.2}$$

Here $\mathbf{T}_D$ denotes the deviatoric part of $\mathbf{T}$, i.e., $\mathbf{T}_D := \mathbf{T} - \frac{1}{3}\mathrm{tr}(\mathbf{T})Id$. The triple $(\mathbf{T}, \mathbf{T}^{\mathrm{p}}, \mathbb{K}^{\mathrm{p}})$ furnishes the internal power expenditure within a subbody $\mathcal{B} \subseteq \Omega$ by means of the relation

$$\mathcal{W}_{int}(\mathcal{B}) = \int_{\mathcal{B}} (\mathbf{T} : \dot{\mathbf{E}}^{\mathrm{e}} + \mathbf{T}^{\mathrm{p}} : \dot{\mathbf{E}}^{\mathrm{p}} + \mathbb{K}^{\mathrm{p}} : \nabla \dot{\mathbf{E}}^{\mathrm{p}})\, dx$$

where $(\dot{u}, \dot{\mathbf{E}}^{\mathrm{e}}, \dot{\mathbf{E}}^{\mathrm{p}})$ is a virtual velocity of the system. So $\mathbf{T}^{\mathrm{p}}$ and $\mathbb{K}^{\mathrm{p}}$ are higher order stresses conjugated to the plastic strain and its gradient. The balance equations for $\mathbf{T}$, $\mathbf{T}^{\mathrm{p}}$ and $\mathbb{K}^{\mathrm{p}}$ follow by equating the internal power expenditure to the power expenditure associated to the external loads. This entails also boundary conditions for the normal components of $\mathbf{T}$ and $\mathbb{K}^{\mathrm{p}}$ which are connected to the imposed traction and *micro-tractions* on parts of the boundary (see Section 3 for details).

The *free energy* of the system is a function of the elastic strain $\mathbf{E}^{\mathrm{e}}$ and of the macroscopic Burgers tensor $\mathbf{G} = \mathrm{curl}\mathbf{E}^{\mathrm{p}}$. In the separable quadratic isotropic case, it assumes the form

$$\psi = \mu|\mathbf{E}^{\mathrm{e}}_D|^2 + \frac{1}{2}k|\mathrm{tr}\mathbf{E}^{\mathrm{e}}|^2 + \frac{\mu L^2}{2}|\mathrm{curl}\mathbf{E}^{\mathrm{p}}|^2, \tag{1.3}$$

where μ and k are the elastic shear and bulk moduli, and L is an energetic length scale. The presence of $\mathrm{curl}\mathbf{E}^{\mathrm{p}}$ inside the free energy accounts for the incompatibility of the tensor field $\mathbf{E}^{\mathrm{p}}$, and so it is connected to the presence of *geometrically necessary dislocations* in Ω. By means of ψ, the energetic third order tensor $\mathbb{K}^{\mathrm{p}}_{\mathrm{en}}$ is defined as the symmetric-deviatoric part (in the first two subscripts) of $\frac{\partial \psi}{\partial \mathbf{G}}$. This entails a decomposition of $\mathbb{K}^{\mathrm{p}}$ into energetic and dissipative parts $\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}$ and $\mathbb{K}^{\mathrm{p}}_{\mathrm{en}}$ with

$$\mathbb{K}^{\mathrm{p}} = \mathbb{K}^{\mathrm{p}}_{\mathrm{diss}} + \mathbb{K}^{\mathrm{p}}_{\mathrm{en}}.$$

Let Ω be subject to body forces $f(t)$ and to traction forces $g(t)$ on a part $\partial_N\Omega$ of its boundary, with $t \in [0, T]$. Let $\partial\Omega$ be *microtraction-free*, i.e., *null power expenditure at the boundary* occurs (see Section 3 for details). Let us assume that a displacement $w(t)$ is imposed on $\partial_D\Omega := \partial\Omega \setminus \partial_N\Omega$. The laws governing the evolution $(u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t))$ of the system are obtained by the thermodynamical requirement

$$\dot{\psi}(\mathcal{B}) \le \mathcal{W}_{int}(\mathcal{B}),$$

where $\psi(\mathcal{B})$ is the free energy of the subbody $\mathcal{B}$ obtained integrating (1.3) over $\mathcal{B}$, and $\dot{\psi}(\mathcal{B})$ denotes its time derivative. In order to match such an inequality, Gurtin and Anand propose a flow rule involving $\dot{\mathbf{E}}^{\mathrm{p}}(t)$, $\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t)$, $\mathbf{T}^{\mathrm{p}}(t)$, $\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t)$, a *dissipative length scale* $l > 0$ and a hardening internal variable. This law reduces to the usual flow rules of classical plasticity when the length scales l and L are set to zero. In the *rate-independent regime*, and neglecting the hardening internal variable, it takes the form

$$\mathbf{T}^{\mathrm{p}}(t,x) = S_Y \frac{\dot{\mathbf{E}}^{\mathrm{p}}(t,x)}{d^p(t,x)}, \qquad \mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t,x) = S_Y \frac{l^2 \nabla \dot{\mathbf{E}}^{\mathrm{p}}(t,x)}{d^p(t,x)}. \tag{1.4}$$

Here $\dot{\mathbf{E}}^{\mathrm{p}}(t,x)$ and $\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t,x)$ denote the time derivative of $\mathbf{E}^{\mathrm{p}}(t,x)$ and $\nabla\mathbf{E}^{\mathrm{p}}(t,x)$ respectively, S_Y is the yield strength and

$$d^p(t,x) := \sqrt{|\dot{\mathbf{E}}^{\mathrm{p}}(t,x)|^2 + l^2|\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t,x)|^2}$$

is an *effective flow rate*. The higher order stresses $\mathbf{T}^{\mathrm{p}}(t)$ and $\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t)$ satisfy the *stress constraint*

$$\sqrt{|\mathbf{T}^{\mathrm{p}}(x)|^2 + l^{-2}|\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(x)|^2} \le S_Y, \tag{1.5}$$

and (1.4) is valid when relation (1.5) holds with equality, $(\dot{\mathbf{E}}^{\mathrm{p}}(t), \nabla\dot{\mathbf{E}}^{\mathrm{p}}(t)) = (0,0)$ otherwise. Notice that setting $l = L = 0$, we have $\mathbb{K}^{\mathrm{p}} = 0$, $\mathbf{T}^{\mathrm{p}} = \mathbf{T}_D$ and (1.4) reduces to the usual flow rule of von Mises type.

The aim of the paper is to provide an existence result of an evolution for the Gurtin-Anand model in the rate independent case without hardening. The case with positive hardening has been considered recently by Reddy, Ebobisse and McBride [30]. Adopting a primal formulation, they study the problem by means of variational inequalities in abstract Hilbert spaces. In the case without hardening, coercivity estimates fail, and the use of the abstract setting is no longer possible. This fact reflects what happens also at the level of classical plasticity, where perfect plasticity deserves an "ad hoc" treatment (see [31] and [8]).

Inspired by the recent paper of Dal Maso, DeSimone and Mora [8] concerning perfect plasticity, we recast the problem of the evolution for the Gurtin-Anand model in the framework of the energetic approach to rate-independent processes developed in [24, 25, 26, 27, 28].

Let us consider $\Omega \subseteq \mathbb{R}^N$ open, bounded and with Lipschitz boundary ($N \ge 3$). By means of variational arguments, we firstly construct a discretized in time evolution $(u_{k,i}, \mathbf{E}^{\mathrm{e}}_{k,i}, \mathbf{E}^{\mathrm{p}}_{k,i})$ relative to the nodes t^i_k of a subdivision $0 = t^k_0 < t^k_1 < \cdots < t^k_k = T$ of the time interval $[0,T]$ with step T/k.

In order to enforce variationally the stress constraint (1.5), we consider the function

$$\mathbf{E}^{\mathrm{p}} \mapsto S_Y \int_\Omega \sqrt{|\mathbf{E}^{\mathrm{p}}|^2 + l^2|\nabla\mathbf{E}^{\mathrm{p}}|^2}\, dx.$$

Since this map has linear growth in $\nabla\mathbf{E}^{\mathrm{p}}$, in order to perform direct minimization, we are naturally led to consider $\mathbf{E}^{\mathrm{p}}$ as a *function of bounded variation* $BV(\Omega; \mathrm{M}^{N\times N}_D)$ and to relax the functional to the form

$$\mathcal{H}(\mathbf{E}^{\mathrm{p}}) := S_Y \int_\Omega \sqrt{|\mathbf{E}^{\mathrm{p}}|^2 + l^2|\nabla\mathbf{E}^{\mathrm{p}}|^2}\, dx + l S_Y |D^s\mathbf{E}^{\mathrm{p}}|(\Omega),$$

where $D^s\mathbf{E}^{\mathrm{p}}$ denotes the singular part of the derivative of $\mathbf{E}^{\mathrm{p}}$.

The minimization problem we consider in order to construct $(u_{k,i}, \mathbf{E}^{\mathrm{e}}_{k,i}, \mathbf{E}^{\mathrm{p}}_{k,i})$ relative to the boundary displacement $w(t^i_k)$ once constructed $(u_{k,i-1}, \mathbf{E}^{\mathrm{e}}_{k,i-1}, \mathbf{E}^{\mathrm{p}}_{k,i-1})$ is the following:

$$\min_{(u,\mathbf{E}^{\mathrm{e}},\mathbf{E}^{\mathrm{p}})\mathcal{A}(w(t^i_k))} \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}) - \langle \mathcal{L}(t^i_k), u\rangle + \mathcal{H}(\mathbf{E}^{\mathrm{p}} - \mathbf{E}^{\mathrm{p}}_{k,i-1}). \tag{1.6}$$

Here $\mathcal{A}(w(t^i_k))$ is the class of admissible configurations for $w(t^i_k)$,

$$\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}) := \int_\Omega \left(\mu|\mathbf{E}^{\mathrm{e}}_D|^2 + \frac{1}{2}k|\mathrm{tr}\mathbf{E}^{\mathrm{e}}|^2\right) dx, \qquad \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}) := \frac{\mu L^2}{2}\int_\Omega |\mathrm{curl}\mathbf{E}^{\mathrm{p}}|^2\, dx,$$

$$\langle \mathcal{L}(t), u\rangle := \int_\Omega f(t)\cdot u\, dx + \int_{\partial_N\Omega} g(t)\cdot u\, d\mathcal{H}^{N-1},$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.

In order to have a well defined energy in (1.6), it suffices that the elastic strain $\mathbf{E}^{\mathrm{e}}$ and the Burgers tensor curl$\mathbf{E}^{\mathrm{p}}$ belong to the space of square integrable functions. As a consequence, the class $\mathcal{A}(t^i_k)$ turns out to be defined as the triples $(u, \mathbf{E}^{\mathrm{e}}, \mathbf{E}^{\mathrm{p}})$ with

$$u \in W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N), \quad \mathbf{E}^{\mathrm{e}} \in L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}}), \quad \mathbf{E}^{\mathrm{p}} \in BV(\Omega;\mathrm{M}^{N\times N}_D), \quad \mathrm{curl}\mathbf{E}^{\mathrm{p}} \in L^2(\Omega;\mathrm{M}^{N\times N}),$$

which satisfy the boundary condition $u = w(t^i_k)$ on $\partial_D\Omega$, and such that the compatibility condition (1.1) holds. Notice that the requirement $u \in W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$ follows by (1.1) and by the assumptions on $\mathbf{E}^{\mathrm{e}}$ and $\mathbf{E}^{\mathrm{p}}$ in view of Korn's inequality. We assume that $f(t) \in L^N(\Omega;\mathbb{R}^N)$ and $g(t) \in L^N(\partial_N\Omega;\mathbb{R}^N)$ so that the work $\mathcal{L}(t)$ of external forces turns out to be well defined. The displacement on $\partial_D\Omega$ is assumed to be given by the trace of a map in $W^{1,2}(\Omega;\mathbb{R}^N)$.

The minimum problem (1.6) admits solutions in $\mathcal{A}(w(t^i_k))$ provided that the external loads satisfy a suitable safe load condition (see (4.13)-(4.14)) which appears also in the study of evolutions in perfect plasticity. This condition entails some coercivity in BV for $\mathbf{E}^{\rm p}$ from the interaction between $\mathcal{H}(\mathbf{E}^{\rm p}-\mathbf{E}^{\rm p}_{k,i-1})$ and the linear term $\langle \mathcal{L}(t^i_k), u\rangle$. The existence of a solution for (1.6) follows by applying the direct method of the Calculus of Variations (Lemma 6.1).

The continuous in time evolution is obtained interpolating the discrete evolution $(u_{k,i}, \mathbf{E}^{\rm e}_{k,i}, \mathbf{E}^{\rm p}_{k,i})$ and sending $k \to +\infty$ (Section 7). If $w \in AC\left(0,T; W^{1,2}(\Omega;\mathbb{R}^N)\right)$, $f \in AC\left(0,T; L^N(\Omega;\mathbb{R}^N)\right)$, $g \in AC\left(0,T; L^\infty(\partial_N\Omega;\mathbb{R}^N)\right)$, and the safe load condition on f, g holds uniformly in time, we prove the convergence towards a quasistatic evolution $t \mapsto (u(t), \mathbf{E}^{\rm e}(t), \mathbf{E}^{\rm p}(t)) \in \mathcal{A}(w(t))$ which is absolutely continuous in time and which satisfies the following two conditions:

(a) Global minimality: for every $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}(w(t))$

$$\mathcal{Q}_1(\mathbf{E}^{\rm e}(t)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}(t)) - \langle \mathcal{L}(t), u(t)\rangle \le \mathcal{Q}_1(\mathbf{e}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{p}) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}(\mathbf{p}-\mathbf{E}^{\rm p}(t)); \tag{1.7}$$

(b) Energy balance:

$$\mathcal{E}(t) + \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\rm p}; 0, t) = \mathcal{E}(0) + \int_0^t \int_\Omega \mathbf{T}(\tau) : \mathbf{E}\dot{w}(\tau)\,dx\,d\tau - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\,d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\,d\tau \tag{1.8}$$

where $\mathbf{T}(t)$ is the Cauchy stress tensor, $\mathcal{E}(t) = \mathcal{Q}_1(\mathbf{E}^{\rm e}(t)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}(t)) - \langle \mathcal{L}(t), u(t)\rangle$, $\dot{\mathcal{L}}(t)$ is associated to $\dot{f}(t)$, $\dot{g}(t)$, and $\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\rm p}; 0, t)$ defined as

$$\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\rm p}; a, b) := \sup\left\{ \sum_{j=1}^k \mathcal{H}\left(\mathbf{E}^{\rm p}(t_j) - \mathbf{E}^{\rm p}(t_{j-1})\right) \,:\, a = t_0 < t_1 < \cdots < t_k = b \right\}$$

has the role of a *dissipation function*.

We refer to an evolution satisfying (a) and (b) as a *quasistatic evolution* for the Gurtin-Anand model (Definition 5.1).

The analysis of the global minimality condition (1.7) leads to the existence of higher order stresses $\mathbf{T}^{\rm p}(t)$, $\mathbb{K}^{\rm p}(t)$, $\mathbb{S}^{\rm p}(t)$ which together with the Cauchy stress $\mathbf{T}(t)$ satisfy the balance of internal and external powers in Ω

$$\int_\Omega \mathbf{T}(t) : \mathbf{e}\,dx + \int_\Omega \mathbf{T}^{\rm p}(t) : \mathbf{p}\,dx + \int_\Omega \mathbb{K}^{\rm p}(t) : \nabla\mathbf{p}\,dx + \langle \mathbb{S}^{\rm p}(t), D^s\mathbf{p}\rangle = \langle \mathcal{L}(t), v\rangle \tag{1.9}$$

for every virtual velocity $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}(0)$ (Lemma 8.1). Notice that a new higher order stress $\mathbb{S}^{\rm p}(t)$ conjugated to $D^s\mathbf{E}^{\rm p}$ appears from our approach: this is somehow natural since $D^s\mathbf{E}^{\rm p}$ is treated at the same level of $\nabla\mathbf{E}^{\rm p}$. The balance (1.9) entails the usual balance equation for the Cauchy stress (Proposition 8.2), the balance equation (1.2), the stress constraint (1.5), and the confinement $\|\mathbb{S}^{\rm p}(t)\| \le lS_Y$ for the singular stress $\mathbb{S}^{\rm p}(t)$ (Proposition 8.3).

The flow rule (1.4) follows from the analysis of the energy balance equality (1.8) (Proposition 8.8). It is also supplemented by a weak flow rule for the singular stress $\mathbb{S}^{\rm p}(t)$ (Proposition 8.7).

In Section 9, we study the asymptotic behavior of a quasistatic evolution for the Gurtin-Anand model when the length scales l, L vanish. As noted previously, by setting l, L equal to zero, the model reduces to the classical model of perfect plasticity of von Mises. Under suitable assumption on the initial configuration, we prove (Theorem 9.2) that the quasistatic evolution for the Gurtin-Anand model converges in a suitable sense to the evolution for elastic-perfectly plastic bodies in the framework proposed by Dal Maso, DeSimone and Mora [8]. The main difficulty we have to handle is the change in the mathematical setting of the problem, especially concerning the plastic strain. While in the strain gradient context $\mathbf{E}^{\rm p}$ is a BV function, in [8] it is modelled simply as a Radon measure.

The paper is organized as follows. In Section 2 we fix the notation and recall some basic tools we need from the theory of BV functions. In Section 3 we give a brief sketch of the Gurtin-Anand model, while in Section 4 we settle the mathematical framework we adopt in the analysis. The main results are stated in Section 5. The existence of a quasistatic evolution is obtained in Section 7 after exploiting the convergence of the discrete evolution constructed in Section 6. Section 8 is devoted to the proof of the balance equations and the flow rule. Finally Section 9 contains the asymptotic analysis as the strain gradient effects vanish.

2. Notation and preliminaries

In this section we recall some basic definitions and results employed in the rest of the paper.

Matrices. We will denote by $\mathrm{M}^{N\times N}$ the space of $N\times N$ matrices $\mathbf{A}=(a_{ij})$ with $a_{ij}\in\mathbb{R}$ endowed with the scalar product

$$\mathbf{A}:\mathbf{B}:=\sum_{i,j}a_{ij}b_{ij}. \tag{2.1}$$

The norm of $\mathbf{A}$ induced by the scalar product (2.1) is denoted by $|\mathbf{A}|$.

We will denote by $\mathrm{M}^{N\times N}_{\mathrm{sym}}$ the subspace of symmetric matrices, and by $\mathrm{M}^{N\times N}_{D}$ the subspace of $\mathrm{M}^{N\times N}_{\mathrm{sym}}$ of matrices $\mathbf{A}$ with zero trace, that is such that $\mathrm{tr}\mathbf{A}:=\sum_i a_{ii}=0$. Given $\mathbf{A}\in\mathrm{M}^{N\times N}_{\mathrm{sym}}$, we denote by $\mathbf{A}_D$ its projection on $\mathrm{M}^{N\times N}_{D}$, i.e.,

$$\mathbf{A}_D:=\mathbf{A}-\frac{1}{N}(\mathrm{tr}A)\mathbf{Id}, \tag{2.2}$$

where $\mathbf{Id}$ is the identity matrix.

The symmetrized gradient of an $\mathbb{R}^N$-valued function $u(x)$ is defined as

$$\mathbf{E}u:=\frac{\nabla u+\nabla u^T}{2},$$

where $(\nabla u)_{ij}=\frac{\partial u_i}{\partial x_j}$ is the gradient of u and ∇u^T denotes its transpose.

The gradient, the divergence and the curl of a $\mathrm{M}^{N\times N}$-valued function $\mathbf{A}(x)=(a_{ij}(x))$ are defined as

$$(\nabla\mathbf{A})_{ijk}:=\frac{\partial a_{ij}}{\partial x_k},\qquad (\mathrm{div}\mathbf{A})_i:=\sum_j\frac{\partial a_{ij}}{\partial x_j},\qquad (\mathrm{curl}\mathbf{A})_{ij}:=\sum_{p,q}\epsilon_{ipq}\frac{\partial a_{jq}}{\partial x_p},$$

where ϵ_{ipq} are the standard permutation symbols.

We will indicate by $\mathrm{M}^{N\times N\times N}$ the space of third order tensors $\mathbb{A}=(a_{ijk})$ with scalar product

$$\mathbb{A}:\mathbb{B}:=\sum_{i,j,k}a_{ijk}b_{ijk},$$

and $|\mathbb{A}|$ will denote the induced norm of $\mathbb{A}$.

We say that $\mathbb{A}=(a_{ijk})\in\mathrm{M}^{N\times N\times N}$ is *symmetric-deviatoric in its first two subscripts* if

$$a_{ijk}=a_{jik}\qquad\text{and}\qquad\sum_p a_{ppk}=0.$$

We write $\mathbb{A}\in\mathrm{M}^{N\times N\times N}_D$.

The divergence of a $\mathrm{M}^{N\times N\times N}$-valued function $\mathbb{A}(x)=(a_{ijk}(x))$ is given by

$$(\mathrm{div}\mathbb{A})_{ij}:=\sum_k\frac{\partial a_{ijk}}{\partial x_k}.$$

Functional spaces and measures. Given $\Omega\subseteq\mathbb{R}^N$ open and $1\le p<+\infty$, we will denote by $L^p(\Omega;\mathbb{R}^M)$ the space of p-summable functions on Ω with values in $\mathbb{R}^M$, and by $W^{1,p}(\Omega;\mathbb{R}^M)$ the usual Sobolev space of functions in $L^p(\Omega;\mathbb{R}^M)$ whose derivatives in the sense of distributions belong to L^p. Finally, $\mathcal{M}_b(\Omega;\mathbb{R}^M)$ will denote the space of $\mathbb{R}^M$-valued Radon measures on Ω, and for every $\mu\in\mathcal{M}_b(\Omega;\mathbb{R}^M)$ we will indicate by $|\mu|(\Omega)$ its total mass. We set $\|\mu\|_{\mathcal{M}_b(\Omega;\mathbb{R}^M)}:=|\mu|(\Omega)$. We refer the reader to [9] for the main properties concerning Sobolev spaces and Radon measures.

Let us recall some results from the theory of BV-functions. We refer the reader to [5] for an exhaustive treatment of the subject.

We say that $u \in BV(\Omega;\mathbb{R}^M)$ if $u \in L^1(\Omega;\mathbb{R}^M)$, and its distributional derivative Du is a vector-valued Radon measure on Ω. $BV(\Omega;\mathbb{R}^M)$ is a Banach space with respect to the norm

$$\|u\|_{BV(\Omega;\mathbb{R}^M)} := \|u\|_{L^1(\Omega;\mathbb{R}^M)} + |Du|(\Omega).$$

We will denote by $D^s u$ the singular part of Du with respect to the Lebesgue measure $\mathcal{L}^N$, and by ∇u the density of its absolutely continuous part.

We will say that a sequence $(u_n)_{n\in\mathbb{N}}$ in $BV(\Omega;\mathbb{R}^M)$ converges weakly* in $BV(\Omega;\mathbb{R}^M)$ to $u \in BV(\Omega;\mathbb{R}^M)$ if

$$\begin{aligned} &u_n \to u \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^M) \\ &Du_n \overset{*}{\rightharpoonup} Du \qquad \text{weakly* in } \mathcal{M}_b(\Omega;\mathbb{R}^M). \end{aligned} \tag{2.3}$$

The following compactness result holds: If Ω is bounded and with Lipschitz boundary, every bounded sequence in $BV(\Omega;\mathbb{R}^M)$ admits a subsequence converging weakly* in $BV(\Omega;\mathbb{R}^M)$.

Finally we will use throughout the paper the following embedding property of BV: If Ω is bounded and with Lipschitz boundary, then $BV(\Omega;\mathbb{R}^M)$ is continuously embedded into $L^q(\Omega;\mathbb{R}^M)$ for every $1 \le q \le \frac{N}{N-1}$, the embedding being compact for every $1 \le q < \frac{N}{N-1}$.

One-dimensional AC and BV functions with values in Banach spaces. Let X be a reflexive Banach space, or the dual of a separable Banach space. We denote by $BV(a,b;X)$ and $AC(a,b;X)$ the space of functions with bounded variations and the space of absolutely continuous functions from $[a,b]$ to X respectively. We refer the reader to [6] for the main properties of these spaces. We recall that the variation of $f \in BV(a,b;X)$ is defined as

$$\mathcal{V}(f;a,b) := \sup\left\{ \sum_{j=1}^{k} \|f(t_j) - f(t_{j-1})\|_X \,:\, a = t_0 < t_1 < \cdots < t_k = b \right\}. \tag{2.4}$$

If X is reflexive and $f \in AC(a,b;X)$, then the time derivative $\dot{f}(t)$ exists for a.e. $t \in [a,b]$. If X is the dual of a separable Banach space (and this is interesting when we consider the plastic strains), the time derivative $\dot{f}(t)$ exists as a weak-star limit for a.e. $t \in [a,b]$ (see [8, Theorem 7.1]).

We will often use the following generalization of Helly's theorem [8, Lemma 7.2]: if X is the dual of a separable Banach space, $(f_k)_{k\in\mathbb{N}}$ a sequence in $BV(a,b;X)$ with $\mathcal{V}(f_k;a,b)$ and $\|f_k(a)\|_X$ uniformly bounded, then there exist $f \in BV(a,b;X)$ and a subsequence $(f_{k_j})_{j\in\mathbb{N}}$ such that $f_k(t) \overset{*}{\rightharpoonup} f(t)$ weakly* in X for every $t \in [a,b]$.

3. The Gurtin-Anand model

In this section we quickly describe the Gurtin-Anand model [19] in strain gradient plasticity which describes the behavior of isotropic, plastically irrotational materials under small deformations. We present the *rate independent* case in which the internal hardening variable is neglected.

Let $\Omega \subseteq \mathbb{R}^N$ be the reference configuration of the body. The starting point of the theory is, as usual, the additive decomposition of the displacement strain $\mathbf{E}u = (\nabla u + \nabla u^T)/2$ into elastic and plastic parts

$$\mathbf{E}u = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{p}}. \tag{3.1}$$

The symmetric matrices $\mathbf{E}^{\mathrm{e}}$ and $\mathbf{E}^{\mathrm{p}}$ are referred to as the *elastic strain* and the *plastic strain* respectively. The plastic part $\mathbf{E}^{\mathrm{p}}$ is supposed to be unable to sustain volumetric changes, so that

$$\mathrm{tr}\mathbf{E}^{\mathrm{p}} = 0,$$

that is $\mathbf{E}^{\mathrm{p}} \in \mathrm{M}^{N\times N}_D$.

Higher order stresses and balance equations. Given a subbody $\mathcal{B} \subseteq \Omega$, besides the usual Cauchy stress $\mathbf{T} \in \mathrm{M}^{N\times N}_{\mathrm{sym}}$ conjugate to $\mathbf{E}^{\mathrm{e}}$, the analysis of its equilibrium involves also higher order stresses $\mathbf{T}^{\mathrm{p}} \in \mathrm{M}^{N\times N}_D$ and $\mathbb{K}^{\mathrm{p}} \in \mathrm{M}^{N\times N\times N}_D$ conjugate to $\mathbf{E}^{\mathrm{p}}$ and $\nabla\mathbf{E}^{\mathrm{p}}$ respectively. Given the rate like kinematical descriptors $(\dot u, \dot{\mathbf{E}}^{\mathrm{e}}, \dot{\mathbf{E}}^{\mathrm{p}})$, the power expenditure within $\mathcal{B}$ is given by

$$\mathcal{W}_{int}(\mathcal{B}) = \int_{\mathcal{B}} (\mathbf{T} : \dot{\mathbf{E}}^{\mathrm{e}} + \mathbf{T}^{\mathrm{p}} : \dot{\mathbf{E}}^{\mathrm{p}} + \mathbb{K}^{\mathrm{p}} : \nabla\dot{\mathbf{E}}^{\mathrm{p}})\, dx.$$

$\mathcal{W}_{int}(\mathcal{B})$ is balanced by the power expenditure of external forces

$$\mathcal{W}_{ext}(\mathcal{B}) = \int_{\partial\mathcal{B}} (t(\nu)\cdot \dot u + \mathbf{K}(\nu) : \dot{\mathbf{E}}^{\mathrm{p}})\, dA + \int_{\mathcal{B}} f\cdot \dot u\, dV,$$

where f is the external body force and $t(\nu)$ is the boundary traction (ν is the outward normal to $\mathcal{B}$) which are associated as usual to $\dot u$, while $\mathbf{K}(\nu) \in \mathrm{M}^{N\times N}_D$ is a *microtraction* associated to the plastic strain rate $\dot{\mathbf{E}}^{\mathrm{p}}$. The balance of power expenditures (that is $\mathcal{W}_{int}(\mathcal{B}) = \mathcal{W}_{ext}(\mathcal{B})$ for every subbody $\mathcal{B}$) leads to the equilibrium equations

$$-\mathrm{div}\mathbf{T} = f \qquad \text{and} \qquad \mathbf{T}^{\mathrm{p}} = \mathbf{T}_D + \mathrm{div}\mathbb{K}^{\mathrm{p}} \quad \text{in } \Omega,$$

where $\mathbf{T}_D$ is the deviatoric part of $\mathbf{T}$ as defined in (2.2). These equations are supplemented by boundary conditions for $\mathbf{T}$ and $\mathbb{K}^{\mathrm{p}}$. If traction forces g are present on a part $\partial_N\Omega$ of the boundary of Ω, we have as usual

$$\mathbf{T}\nu = g \qquad \text{on } \partial_N\Omega.$$

Concerning $\mathbb{K}^{\mathrm{p}}$, assuming *null microscopic power expenditure* at the boundary (see [19, Section 8]), we are led to the condition

$$\mathbb{K}^{\mathrm{p}}\nu = 0 \qquad \text{on } \partial\Omega.$$

The free energy. The *free energy* ψ is assumed to depend on $\mathbf{E}^{\mathrm{e}}$ and $\mathrm{curl}\mathbf{E}^{\mathrm{p}}$: in the quadratic separable case ψ has the form

$$\psi = \frac{1}{2}\mathbb{C}\mathbf{E}^{\mathrm{e}} : \mathbf{E}^{\mathrm{e}} + \frac{\mu L^2}{2}|\mathrm{curl}\mathbf{E}^{\mathrm{p}}|^2, \tag{3.2}$$

where $\mathbb{C}$ is the elastic tensor

$$\mathbb{C}\mathbf{E}^{\mathrm{e}} := 2\mu\mathbf{E}^{\mathrm{e}}_D + k(\mathrm{tr}\mathbf{E}^{\mathrm{e}})\mathbf{I} \tag{3.3}$$

with μ and k the elastic shear and bulk moduli. The constant $L > 0$ is an *energetic length scale.* The *energetic higher order stress tensor* $\mathbb{K}^{\mathrm{p}}_{\mathrm{en}}$ is then defined so that the identity

$$\mu L^2\mathrm{curl}\mathbf{E}^{\mathrm{p}} : \mathrm{curl}\mathbf{A} = \mathbb{K}^{\mathrm{p}}_{\mathrm{en}} : \nabla\mathbf{A} \tag{3.4}$$

holds for every $\mathrm{M}^{N\times N}$-valued function $\mathbf{A}$. In components we have

$$(\mathbb{K}^{\mathrm{p}}_{\mathrm{en}})_{jqp} := \mu L^2\left[\frac{\partial\mathbf{E}^{\mathrm{p}}_{jq}}{\partial x_p} - \frac{1}{2}\left(\frac{\partial\mathbf{E}^{\mathrm{p}}_{jp}}{\partial x_q} + \frac{\partial\mathbf{E}^{\mathrm{p}}_{qp}}{\partial x_j}\right) + \frac{1}{N}\delta_{jq}\sum_r \frac{\partial\mathbf{E}^{\mathrm{p}}_{rp}}{\partial x_r}\right], \tag{3.5}$$

where δ_{jq} is the usual Krönecker symbol. The stress $\mathbb{K}^{\mathrm{p}}$ is then additively decomposed in the following way

$$\mathbb{K}^{\mathrm{p}} = \mathbb{K}^{\mathrm{p}}_{\mathrm{diss}} + \mathbb{K}^{\mathrm{p}}_{\mathrm{en}}.$$

Admissibility of the stresses and the flow rule. Neglecting the hardening internal variable, i.e., if we are in the case without hardening nor softening, the admissibility for the stresses involved in the description of the behavior of Ω reads

$$\sqrt{|\mathbf{T}^{\mathrm{p}}(x)|^2 + l^{-2}|\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(x)|^2} \le S_Y, \tag{3.6}$$

where $l > 0$ is a *dissipative length scale*, and S_Y is a *yield constant.*

Assume now that body and traction forces vary with time, i.e., $f = f(t)$ and $g = g(t)$. The flow rule which drives the system requires that if $(\mathbf{T}^{\rm p}(t,x), \mathbb{K}^{\rm p}_{\rm diss}(t,x))$ is at the yield surface (that is (3.6) holds with equality) then

$$\begin{cases} \mathbf{T}^{\rm p}(t,x) = S_Y \dfrac{\dot{\mathbf{E}}^{\rm p}(t,x)}{\sqrt{|\dot{\mathbf{E}}^{\rm p}(t,x)|^2 + l^2|\nabla\dot{\mathbf{E}}^{\rm p}(t,x)|^2}} \\ \\ \mathbb{K}^{\rm p}_{\rm diss}(t,x) = S_Y \dfrac{l^2\nabla\dot{\mathbf{E}}^{\rm p}(t,x)}{\sqrt{|\dot{\mathbf{E}}^{\rm p}(t,x)|^2 + l^2|\nabla\dot{\mathbf{E}}^{\rm p}(t,x)|^2}}. \end{cases} \tag{3.7}$$

Here $\dot{\mathbf{E}}^{\rm p}(t,x)$ and $\nabla\dot{\mathbf{E}}^{\rm p}(t,x)$ denote the time derivative of $\mathbf{E}^{\rm p}(t,x)$ and $\nabla\mathbf{E}^{\rm p}(t,x)$ respectively. If $(\mathbf{T}^{\rm p}(t,x), \mathbb{K}^{\rm p}_{\rm diss}(t,x))$ is well inside the yield surface, then no plastic phenomena occurs, i.e., $(\dot{\mathbf{E}}^{\rm p}, \nabla\dot{\mathbf{E}}^{\rm p}) = (0,0)$. The flow rule (3.7) is a generalization of the von Mises flow rule in perfect plasticity (set $l = L = 0$, and note that $\mathbf{T}^p = \mathbf{T}_D$). It moreover implies that

$$\int_{\mathcal{B}} \dot{\psi}\, dV \le \mathcal{W}_{ext}(\mathcal{B}).$$

The previous inequality reflects the thermodynamical requirement that the increase in free energy of $\mathcal{B}$ is less than or equal to the power expended on $\mathcal{B}$.

4. Functional setting

In this section we state the precise mathematical framework we adopt to study quasistatic evolutions for the Gurtin-Anand model.

The reference configuration. Let the reference configuration be given by $\Omega \subseteq \mathbb{R}^N$, $N \ge 3$, bounded open set with Lipschitz boundary. Let $\partial\Omega$ be partitioned into two open (in the relative topology) disjoint sets $\partial_D\Omega$ and $\partial_N\Omega$ with the same boundary Γ such that $\mathcal{H}^{N-2}(\Gamma) < +\infty$.

Admissible configurations. Let the prescribed boundary displacement on $\partial_D\Omega$ be given by (the trace of) a Sobolev function $w \in W^{1,2}(\Omega;\mathbb{R}^N)$. An admissible configuration relative to the boundary datum w is given by a triple $(u, \mathbf{E}^{\rm e}, \mathbf{E}^{\rm p})$ such that

$$u \in W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N), \qquad \mathbf{E}^{\rm e} \in L^2(\Omega;\mathrm{M}^{N\times N}_{\rm sym}), \qquad \mathbf{E}^{\rm p} \in BV(\Omega;\mathrm{M}^{N\times N}_D) \tag{4.1}$$

such that

$$u = w \qquad \text{on } \partial_D\Omega, \tag{4.2}$$

$$\mathbf{E}u = \mathbf{E}^{\rm e} + \mathbf{E}^{\rm p}, \tag{4.3}$$

and

$$\mathrm{curl}\mathbf{E}^{\rm p} \in L^2(\Omega;\mathrm{M}^{N\times N}). \tag{4.4}$$

Equality (4.2) is intended in the sense of traces. Notice that, by the embedding properties of BV, (4.3) entails $\mathbf{E}u \in L^{\frac{N}{N-1}}(\Omega;\mathrm{M}^{N\times N}_{\rm sym})$; the requirement $u \in W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$ is then consistent with the regularity implied by Korn's inequality in view of the boundary condition (4.2). Let us denote by $\mathcal{A}(w)$ the family of admissible configurations for the boundary datum w, i.e.,

$$\mathcal{A}(w) := \{(u, \mathbf{E}^{\rm e}, \mathbf{E}^{\rm p}) \text{ such that (4.1) -(4.4) are satisfied}\}. \tag{4.5}$$

The free energy. The free energy of the configuration $(u, \mathbf{E}^{\rm e}, \mathbf{E}^{\rm p}) \in \mathcal{A}(w)$ is given according to (3.2) by

$$\Psi(\mathbf{E}^{\rm e}, \mathrm{curl}\mathbf{E}^{\rm p}) := \mathcal{Q}_1(\mathbf{E}^{\rm e}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}),$$

where

$$\mathcal{Q}_1(\mathbf{E}^{\rm e}) := \frac{1}{2}\int_\Omega \mathbb{C}\mathbf{E}^{\rm e} : \mathbf{E}^{\rm e}\, dx \tag{4.6}$$

and

$$\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}) := \frac{1}{2}\mu L^2 \int_\Omega |\mathrm{curl}\mathbf{E}^{\mathrm{p}}|^2\, dx. \tag{4.7}$$

Here $\mathbb{C}$ denotes the elasticity tensor (3.3): hence there exist $0 < \alpha_{\mathbb{C}} \le \beta_{\mathbb{C}} < +\infty$ such that for every $\mathbf{A} \in \mathrm{M}^{N\times N}_{\mathrm{sym}}$ we have

$$\alpha_{\mathbb{C}}|\mathbf{A}|^2 \le \mathbb{C}\mathbf{A} : \mathbf{A} \le \beta_{\mathbb{C}}|\mathbf{A}|^2. \tag{4.8}$$

The yield function $\mathcal{H}$. In order to get variationally the constraint for the higher order stresses according to (3.6), we are led to consider the *yield function*

$$\mathcal{H}(\mathbf{E}^{\mathrm{p}}) := S_Y \int_\Omega \sqrt{|\mathbf{E}^{\mathrm{p}}|^2 + l^2|\nabla\mathbf{E}^{\mathrm{p}}|^2}\, dx + lS_Y|D^s\mathbf{E}^{\mathrm{p}}|(\Omega) \tag{4.9}$$

defined for every $\mathbf{E}^{\mathrm{p}} \in BV(\Omega; \mathrm{M}^{N\times N}_D)$. Simple arguments on subadditive and positively one-homogeneous functions on measures (see [13]) show that $\mathcal{H}$ is the relaxation under the L^1-norm of the map

$$\mathbf{E}^{\mathrm{p}} \mapsto S_Y \int_\Omega \sqrt{|\mathbf{E}^{\mathrm{p}}|^2 + l^2|\nabla\mathbf{E}^{\mathrm{p}}|^2}\, dx$$

defined for a regular plastic strain $\mathbf{E}^{\mathrm{p}}$, which is connected to the effective flow rate proposed by Gurtin and Anand (see [19, Section 6.3]). As a consequence, $\mathcal{H}$ turns out to be naturally involved in an analysis which employs direct methods of the Calculus of Variations.

We will often use the lower semicontinuity of $\mathcal{H}$ along weakly* converging sequences, which is a direct consequence of the relaxation process through which $\mathcal{H}$ is obtained.

Lemma 4.1. *Let $(\mathbf{E}^{\mathrm{p}}_n)_{n\in\mathbb{N}}$ be a sequence in $BV(\Omega; \mathrm{M}^{N\times N}_D)$ such that*

$$\mathbf{E}^{\mathrm{p}}_n \overset{*}{\rightharpoonup} \mathbf{E}^{\mathrm{p}} \qquad \text{weakly}^* \text{ in } BV(\Omega; \mathrm{M}^{N\times N}_D)$$

for some $\mathbf{E}^{\mathrm{p}} \in BV(\Omega; \mathrm{M}^{N\times N}_D)$. Then we have

$$\mathcal{H}(\mathbf{E}^{\mathrm{p}}) \le \liminf_{n\to+\infty} \mathcal{H}(\mathbf{E}^{\mathrm{p}}_n).$$

Prescribed boundary displacements and body/traction forces. We assume that the prescribed boundary displacement on $\partial_D\Omega$ is given by (the trace of) a function $w(t,x)$ which is absolutely continuous in time with values in the Sobolev space $W^{1,2}(\Omega;\mathbb{R}^N)$, i.e.,

$$w \in AC\left(0,T; W^{1,2}(\Omega;\mathbb{R}^N)\right). \tag{4.10}$$

Moreover we assume that the prescribed body forces in Ω and traction forces on $\partial_N\Omega$ are given by

$$f \in AC\left(0,T; L^N(\Omega;\mathbb{R}^N)\right) \qquad \text{and} \qquad g \in AC\left(0,T; L^N(\partial_N\Omega;\mathbb{R}^N)\right). \tag{4.11}$$

For every $t \in [0,T]$ let us consider $\mathcal{L}(t) : W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ given by

$$\langle \mathcal{L}(t), u\rangle := \int_\Omega f(t)\cdot u\, dx + \int_{\partial_N\Omega} g(t)\cdot u\, d\mathcal{H}^{N-1}. \tag{4.12}$$

Here $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure, which is a generalization to arbitrary sets of the usual surface measure (see [9]). By means of Sobolev Embedding Theorem it is easily seen that $\mathcal{L}(t)$ is a continuous linear functional on $W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$.

Throughout the paper we will assume that the prescribed body and traction forces satisfy the following uniform *safe load condition*: we assume that for every $t \in [0,T]$ there exists $\rho(t) \in L^N(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}})$ such that

$$\begin{cases} -\mathrm{div}\rho(t) = f(t) & \text{in } \Omega \\ \rho(t)\nu = g(t) & \text{on } \partial_N\Omega \end{cases} \tag{4.13}$$

and there exists $\alpha > 0$ such that for every $\mathbf{A} \in \mathrm{M}^{N\times N}_D$ with $|\mathbf{A}| \le \alpha$ we have

$$|\mathbf{A} + \rho_D(t)| \le S_Y \qquad \text{a.e. in } \Omega. \tag{4.14}$$

Moreover we assume that $t \mapsto \rho(t)$ and $t \mapsto \rho_D(t)$ are absolutely continuous from $[0,T]$ to $L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}})$ and $L^\infty(\Omega; \mathrm{M}^{N\times N}_D)$ respectively. Notice that the trace condition in (4.13) is well defined in the dual of the traces on $\partial_N\Omega$ of $W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$ since ρ is an L^N-field with divergence in L^N. Moreover, for every $(u, \mathbf{E}^{\mathrm{e}}, \mathbf{E}^{\mathrm{p}}) \in \mathcal{A}(w)$ we have the following representation formula for $\mathcal{L}(t)$ (here we use $\mathcal{H}^{N-2}(\Gamma) < +\infty$):

$$\langle \mathcal{L}(t), u\rangle = -\langle \rho(t)\nu, w\rangle_{\partial_D\Omega} + \int_\Omega \rho(t) : \mathbf{E}^{\mathrm{e}}\, dx + \int_\Omega \rho_D(t) : \mathbf{E}^{\mathrm{p}}\, dx, \tag{4.15}$$

where the first term on the right-end side should be interpreted as the pairing between $H^{-1/2}(\partial_D\Omega;\mathbb{R}^N)$ and $H^{1/2}(\partial_D\Omega;\mathbb{R}^N)$.

Remark 4.2. Notice that for $\mathcal{L}(t)$ to be well defined in the dual of $W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$ it suffices to require $f(t) \in L^{N/2}(\Omega;\mathbb{R}^N)$. But in view of the safe load condition (4.13)-(4.14), $\rho(t)$ would only be an element of $L^{N/2}$ with divergence in $L^{N/2}$, so that its normal trace would be defined in the dual of the traces on $\partial\Omega$ of $W^{1,\frac{N}{N-2}}(\Omega;\mathbb{R}^N)$. Then the representation formula (4.15) would be no longer well defined (since $w \in W^{1,2}(\Omega;\mathbb{R}^N)$).

As a consequence of the safe load condition, we have the following coercivity estimate for $\mathcal{H}$.

Lemma 4.3. *For every $\mathbf{E}^{\mathrm{p}} \in BV(\Omega; \mathrm{M}^{N\times N}_D)$ we have*

$$\mathcal{H}(\mathbf{E}^{\mathrm{p}}) - \int_\Omega \rho_D(t) : \mathbf{E}^{\mathrm{p}}\, dx \ge \frac{\alpha}{2}\|\mathbf{E}^{\mathrm{p}}\|_{L^1(\Omega;\mathrm{M}^{N\times N}_D)} + \min\left\{l\frac{\alpha}{2}, lS_Y\right\}\|D\mathbf{E}^{\mathrm{p}}\|_{\mathcal{M}_b(\Omega;\mathrm{M}^{N\times N\times N}_D)}. \tag{4.16}$$

In particular there exists $\alpha_l > 0$ such that

$$\mathcal{H}(\mathbf{E}^{\mathrm{p}}) - \int_\Omega \rho_D(t) : \mathbf{E}^{\mathrm{p}}\, dx \ge \alpha_l \|\mathbf{E}^{\mathrm{p}}\|_{BV(\Omega;\mathrm{M}^{N\times N}_D)}. \tag{4.17}$$

Proof. Notice that by Hölder inequality we have

$$S_Y \int_\Omega \sqrt{|\mathbf{E}^{\mathrm{p}}|^2 + l^2|\nabla \mathbf{E}^{\mathrm{p}}|^2}\, dx \ge \sup_{(\tau_1,\tau_2)\in\mathcal{K}} \int_\Omega [\tau_1 : \mathbf{E}^{\mathrm{p}} + \tau_2 : \nabla\mathbf{E}^{\mathrm{p}}]\, dx$$

where

$$\mathcal{K} := \left\{(\tau_1,\tau_2) \in L^\infty(\Omega;\mathrm{M}^{N\times N}_D) \times L^\infty(\Omega;\mathrm{M}^{N\times N\times N}_D) \,:\, \sqrt{|\tau_1|^2 + l^{-2}|\tau_2|^2} \le S_Y \text{ a.e. in } \Omega\right\}.$$

We deduce that for every $(\tau_1,\tau_2) \in \mathcal{K}$

$$\mathcal{H}(\mathbf{E}^{\mathrm{p}}) - \int_\Omega \rho_D(t) : \mathbf{E}^{\mathrm{p}}\, dx \ge \int_\Omega [(\tau_1 - \rho_D(t)) : \mathbf{E}^{\mathrm{p}} + \tau_2 : \nabla\mathbf{E}^{\mathrm{p}}]\, dx + lS_Y|D^s\mathbf{E}^{\mathrm{p}}|(\Omega)$$

so that in view of (4.14) we get

$$\mathcal{H}(\mathbf{E}^{\mathrm{p}}) - \int_\Omega \rho_D(t) : \mathbf{E}^{\mathrm{p}}\, dx \ge \int_\Omega [\tilde\tau_1 : \mathbf{E}^{\mathrm{p}} + \tilde\tau_2 : \nabla\mathbf{E}^{\mathrm{p}}]\, dx + lS_Y|D^s\mathbf{E}^{\mathrm{p}}|(\Omega)$$

for every $\|\tilde\tau_1\|_{L^\infty(\Omega;\mathrm{M}^{N\times N}_D)} \le \frac{\alpha}{2}$ and $\|\tilde\tau_2\|_{L^\infty(\Omega;\mathrm{M}^{N\times N\times N}_D)} \le l\frac{\alpha}{2}$. We conclude that

$$\mathcal{H}(\mathbf{E}^{\mathrm{p}}) - \int_\Omega \rho_D(t) : \mathbf{E}^{\mathrm{p}}\, dx \ge \frac{\alpha}{2}\|\mathbf{E}^{\mathrm{p}}\|_{L^1(\Omega;\mathrm{M}^{N\times N}_D)} + l\frac{\alpha}{2}\|\nabla\mathbf{E}^{\mathrm{p}}\|_{L^1(\Omega;\mathrm{M}^{N\times N\times N}_D)} + lS_Y|D^s\mathbf{E}^{\mathrm{p}}|(\Omega)$$

so that (4.16) holds. Inequality (4.17) follows by choosing $\alpha_l := \min\left\{\frac{\alpha}{2}, l\frac{\alpha}{2}, lS_Y\right\}$. □

5. The main results

Let $T > 0$, and let w, f, g be the prescribed boundary displacements, body forces, and traction forces according to (4.10) and (4.11). We assume that f and g satisfy the uniform safe load condition (4.13)-(4.14).

We will denote by $\dot{w}(t)$, $\dot{f}(t)$ and $\dot{g}(t)$ the derivative at time $t \in [0, T]$ of w, f and g respectively. Notice that these derivatives exist for a.e. $t \in [0, T]$ since the maps are absolutely continuous with values in a reflexive Banach space. We will denote by $\dot{\mathcal{L}}(t)$ the external work associated to $\dot{f}(t)$ and $\dot{g}(t)$.

Given $\mathcal{H}$ as in (4.9), the $\mathcal{H}$-variation on $[a, b] \subseteq [0, T]$ of $t \mapsto \mathbf{E}^{\mathrm{p}}(t)$ is defined as

$$\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}; a, b) := \sup \left\{ \sum_{j=1}^{k} \mathcal{H}\left(\mathbf{E}^{\mathrm{p}}(t_j) - \mathbf{E}^{\mathrm{p}}(t_{j-1})\right) \,:\, a = t_0 < t_1 < \cdots < t_k = b \right\}. \tag{5.1}$$

The notion of quasistatic evolution for the Gurtin-Anand model is the following.

Definition 5.1 (Quasistatic evolution). *A map*

$$t \mapsto (u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t))$$

from $[0, T]$ *to* $W^{1,\frac{N}{N-1}}(\Omega; \mathbb{R}^N) \times L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}}) \times BV(\Omega; \mathrm{M}^{N\times N}_D)$ *is a quasistatic evolution for the Gurtin-Anand model if for every* $t \in [0, T]$ *we have* $(u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t)) \in \mathcal{A}(w(t))$ *and the following two conditions hold:*

(a) global stability*: for every* $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}(w(t))$

$$\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) - \langle \mathcal{L}(t), u(t)\rangle \le \mathcal{Q}_1(\mathbf{e}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{p}) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}(\mathbf{p} - \mathbf{E}^{\mathrm{p}}(t)); \tag{5.2}$$

(b) energy balance*: the function* $t \mapsto \mathbf{E}^{\mathrm{p}}(t)$ *has bounded variation from* $[0, T]$ *to* $BV(\Omega; \mathrm{M}^{N\times N}_D)$ *and*

$$\mathcal{E}(t) + \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}; 0, t) = \mathcal{E}(0) + \int_0^t \int_\Omega \mathbf{T}(\tau) : \mathbf{E}\dot{w}(\tau)\, dx\, d\tau - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\, d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\, d\tau, \tag{5.3}$$

where $\mathbf{T}(t) := \mathbb{C}\mathbf{E}^{\mathrm{e}}(t)$,

$$\mathcal{E}(t) := \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) - \langle \mathcal{L}(t), u(t)\rangle, \tag{5.4}$$

and $\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}; 0, t)$ *is defined in* (5.1).

Our first main result is the following existence theorem.

Theorem 5.2. *Let* $(u_0, \mathbf{E}^{\mathrm{e}}_0, \mathbf{E}^{\mathrm{p}}_0) \in \mathcal{A}(w(0))$ *satisfy the global stability condition*

$$\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_0) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_0) - \langle \mathcal{L}(0), u_0\rangle \le \mathcal{Q}_1(\mathbf{e}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{p}) - \langle \mathcal{L}(0), v\rangle + \mathcal{H}(\mathbf{p} - \mathbf{E}^{\mathrm{p}}_0)$$

for every $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}(w(0))$*. Then there exists a quasistatic evolution* $t \mapsto (u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t))$ *such that* $(u(0), \mathbf{E}^{\mathrm{e}}(0), \mathbf{E}^{\mathrm{p}}(0)) = (u_0, \mathbf{E}^{\mathrm{e}}_0, \mathbf{E}^{\mathrm{p}}_0)$*.*

Theorem 5.2 will be proved in Section 7 exploiting the convergence of a discrete in time evolution constructed through variational arguments in Section 6.

Our second main result shows that a quasistatic evolution satisfies the required constitutive equations, balance equations and the flow rule of the Gurtin and Anand model.

Theorem 5.3. *Let* $t \mapsto (u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t))$ *be a quasistatic evolution for the Gurtin-Anand model. Then the maps* $t \mapsto u(t)$, $t \mapsto \mathbf{E}^{\mathrm{e}}(t)$, $t \mapsto \mathbf{E}^{\mathrm{p}}(t)$, $t \mapsto \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)$ *are absolutely continuous from* $[0, T]$ *to* $W^{1,\frac{N}{N-1}}(\Omega; \mathbb{R}^N)$, $L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}})$, $BV(\Omega; \mathrm{M}^{N\times N}_D)$ *and* $L^2(\Omega; \mathrm{M}^{N\times N})$ *respectively. Moreover the following facts hold for every* $t \in [0, T]$*.*

(a) **Cauchy stress***:* $\mathbf{T}(t) = \mathbb{C}\mathbf{E}^{\mathrm{e}}(t)$ *satisfies the following balance equation*

$$\begin{cases} -\mathrm{div}\mathbf{T}(t) = f(t) & \text{in } \Omega \\ \mathbf{T}(t)\nu = g(t) & \text{on } \partial_N \Omega. \end{cases} \tag{5.5}$$

(b) **Higher order stresses**: *there exist* $\mathbf{T}^{\rm p}(t) \in L^\infty(\Omega; \mathrm{M}^{N\times N}_D)$, $\mathbb{K}^{\rm p}(t) \in L^2(\Omega; \mathrm{M}^{N\times N\times N}_D)$ *and* $\mathbb{K}^{\rm p}_{\rm diss}(t) \in L^\infty(\Omega; \mathrm{M}^{N\times N\times N}_D)$ *such that defining* $\mathbb{K}^{\rm p}_{\rm en}(t)$ *as in* (3.5) *starting from* $\mathbf{E}^{\rm p}(t)$ *and setting* $\mathbf{T}_D(t) := (\mathbf{T}(t))_D$, *we have*

$$\mathbb{K}^{\rm p}(t) = \mathbb{K}^{\rm p}_{\rm en}(t) + \mathbb{K}^{\rm p}_{\rm diss}(t) \quad \text{in } \Omega,$$

$$\begin{cases} \mathbf{T}^{\rm p}(t) = \mathbf{T}_D(t) + \mathrm{div}\mathbb{K}^{\rm p}(t) & \text{in } \Omega \\ \mathbb{K}^{\rm p}(t)\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.6}$$

and

$$\sqrt{|\mathbf{T}^{\rm p}(t)|^2 + l^{-2}|\mathbb{K}^{\rm p}_{\rm diss}(t)|^2} \le S_Y \qquad \text{a.e. in } \Omega. \tag{5.7}$$

(c) **The flow rule**: *if* $\dot{\mathbf{E}}^{\rm p}(t)$ *exists, and* $x \in \Omega$ *is a Lebesgue point for* $\dot{\mathbf{E}}^{\rm p}(t), \nabla\dot{\mathbf{E}}^{\rm p}(t), \mathbf{T}^{\rm p}(t)$, $\mathbb{K}^{\rm p}_{\rm diss}(t)$, *the flow rule* (3.7) *is satisfied.*

Notice that the normal trace which appears in (5.5) is well defined in $H^{-1/2}(\partial\Omega; \mathbb{R}^N)$ since $\mathbf{T}(t)$ is an L^2-field with divergence in L^2. Similarly, the normal trace in (5.6) is well defined in $H^{-1/2}(\partial\Omega; \mathbb{R}^{N\times N})$ because $\mathbb{K}^{\rm p}(t)$ is an L^2-field (by definition of $\mathbb{K}^{\rm p}_{\rm en}(t)$ and by the constraint (5.7) for $\mathbb{K}^{\rm p}_{\rm diss}(t)$) with divergence in L^2 (by the balance equation (5.6) and by the constraint (5.7) for $\mathbf{T}^{\rm p}(t)$).

In Section 9 we will analyse the behavior of a quasistatic evolution as the length scales l and L go to zero, i.e., when the strain gradient effects vanish. We will prove (Theorem 9.2) that the quasistatic evolution converges to an evolution for perfect plasticity according to the framework recently proposed by Dal Maso, DeSimone and Mora in [8].

6. The discrete in time evolution

In this section we construct a discretized in time evolution for the Gurtin-Anand model employing a step by step minimization procedure. The convergence of this approximated evolution to a quasistatic evolution for the Gurtin-Anand model as the time step discretization goes to zero will be proved in the next section.

Let $k \in \mathbb{N}$, $k \ge 1$, and let us set $t^i_k := \frac{i}{k}T$ for every $i = 0, 1, \dots, k$. Let us set

$$u_{k,0} := u_0 \qquad \mathbf{E}^{\rm e}_{k,0} := \mathbf{E}^{\rm e}_0 \qquad \mathbf{E}^{\rm p}_{k,0} := \mathbf{E}^{\rm p}_0$$

where $(u_0, \mathbf{E}^{\rm e}_0, \mathbf{E}^{\rm p}_0) \in \mathcal{A}(w(0))$ is the initial configuration of the system given by Theorem 5.2.

Supposing to have constructed $(u_{k,i}, \mathbf{E}^{\rm e}_{k,i}, \mathbf{E}^{\rm p}_{k,i}) \in \mathcal{A}(w(t^i_k))$, let $(u_{k,i+1}, \mathbf{E}^{\rm e}_{k,i+1}, \mathbf{E}^{\rm p}_{k,i+1}) \in \mathcal{A}(w(t^{i+1}_k))$ $(i = 0, \dots, k-1)$ be a solution of the following minimization problem

$$\min_{(u,\mathbf{E}^{\rm e},\mathbf{E}^{\rm p}) \in \mathcal{A}(w(t^{i+1}_k))} \mathcal{Q}_1(\mathbf{E}^{\rm e}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}) - \langle \mathcal{L}(t^{i+1}_k), u\rangle + \mathcal{H}(\mathbf{E}^{\rm p} - \mathbf{E}^{\rm p}_{k,i}). \tag{6.1}$$

The existence of a solution for problem (6.1) is established in the following lemma.

Lemma 6.1. *Problem* (6.1) *admits a solution.*

Proof. The result follows by applying the direct method of the Calculus of Variations. In fact, let

$$(u_n, \mathbf{E}^{\rm e}_n, \mathbf{E}^{\rm p}_n) \in \mathcal{A}(w(t^{i+1}_k))$$

be a minimizing sequence for (6.1). By comparison with $(w(t^{i+1}_k), \mathbf{E}w(t^{i+1}_k), 0)$ we get

$$\begin{aligned} &\mathcal{Q}_1(\mathbf{E}^{\rm e}_n) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_n) - \langle \mathcal{L}(t^{i+1}_k), u_n\rangle + \mathcal{H}(\mathbf{E}^{\rm p}_n - \mathbf{E}^{\rm p}_{k,i}) \\ &\qquad\qquad \le \mathcal{Q}_1(\mathbf{E}w(t^{i+1}_k)) - \langle \mathcal{L}(t^{i+1}_k), w(t^{i+1}_k)\rangle + \mathcal{H}(\mathbf{E}^{\rm p}_{k,i}) := C. \end{aligned}$$

By the representation formula (4.15) for $\mathcal{L}(t^{i+1}_k)$ we deduce that

$$\begin{aligned} &\mathcal{Q}_1(\mathbf{E}^{\rm e}_n) - \int_\Omega \rho(t^{i+1}_k) : \mathbf{E}^{\rm e}_n \, dx + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_n) + \mathcal{H}(\mathbf{E}^{\rm p}_n - \mathbf{E}^{\rm p}_{k,i}) - \int_\Omega \rho_D(t^{i+1}_k) : (\mathbf{E}^{\rm p}_n - \mathbf{E}^{\rm p}_{k,i}) \, dx \\ &\qquad\qquad \le C + \int_\Omega \rho_D(t^{i+1}_k) : \mathbf{E}^{\rm p}_{k,i} \, dx - \langle \rho(t^{i+1}_k)\nu, w(t^{i+1}_k)\rangle_{\partial_D\Omega}. \end{aligned}$$

By the coercivity of $\mathcal{Q}_1$, $\mathcal{Q}_2$ in L^2 and by (4.17) we get

$$\|\mathbf{E}^{\rm e}_n\|^2_{L^2(\Omega;\mathrm{M}^{N\times N}_{\rm sym})} + \|\mathrm{curl}\mathbf{E}^{\rm p}_n\|^2_{L^2(\Omega;\mathrm{M}^{N\times N})} + \|\mathbf{E}^{\rm p}_n - \mathbf{E}^{\rm p}_{k,i}\|_{BV(\Omega;\mathrm{M}^{N\times N}_D)} \le C_1$$

for some $C_1 > 0$. Up to a subsequence we may assume that

$$\mathbf{E}^{\rm e}_n \rightharpoonup \mathbf{E}^{\rm e} \qquad \text{weakly in } L^2(\Omega;\mathrm{M}^{N\times N}_{\rm sym})$$

and

$$\mathbf{E}^{\rm p}_n \overset{*}{\rightharpoonup} \mathbf{E}^{\rm p} \qquad \text{weakly}^* \text{ in } BV(\Omega;\mathrm{M}^{N\times N}_D).$$

As a consequence we get $\mathrm{curl}\mathbf{E}^{\rm p} \in L^2(\Omega;\mathrm{M}^{N\times N})$ and that

$$\mathrm{curl}\mathbf{E}^{\rm p}_n \rightharpoonup \mathrm{curl}\mathbf{E}^{\rm p} \qquad \text{weakly in } L^2(\Omega;\mathrm{M}^{N\times N}).$$

By the compatibility $\mathbf{E}u_n = \mathbf{E}^{\rm e}_n + \mathbf{E}^{\rm p}_n$ and by the embedding $BV(\Omega;\mathrm{M}^{N\times N}_D) \hookrightarrow L^{\frac{N}{N-1}}(\Omega;\mathrm{M}^{N\times N}_D)$ we get that $(\mathbf{E}u_n)_{n\in\mathbb{N}}$ is bounded in $L^{\frac{N}{N-1}}(\Omega;\mathrm{M}^{N\times N}_{\rm sym})$. In view of the boundary condition $u_n = w_{k,i+1}$ on $\partial_D\Omega$, Korn's inequality implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$. Up to a further subsequence we can thus suppose that

$$u_n \rightharpoonup u \qquad \text{weakly in } W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$$

for some $u \in W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$ with $u = w(t^{i+1}_k)$ on $\partial_D\Omega$. We clearly have that $(u,\mathbf{E}^{\rm e},\mathbf{E}^{\rm p}) \in \mathcal{A}(w(t^{i+1}_k))$, and by lower semicontinuity ($\mathcal{Q}_1$, $\mathcal{Q}_2$ are quadratic, $\mathcal{L}(t^{i+1}_k)$ is linear, and $\mathcal{H}$ is lower semicontinuous by Lemma 4.1) we deduce that

$$\begin{aligned}\mathcal{Q}_1(\mathbf{E}^{\rm e}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}) - \langle \mathcal{L}(t^{i+1}_k), u\rangle + \mathcal{H}(\mathbf{E}^{\rm p} - \mathbf{E}^{\rm p}_{k,i})\\ \le \liminf_{n\to+\infty}\Big[\mathcal{Q}_1(\mathbf{E}^{\rm e}_n) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_n) - \langle \mathcal{L}(t^{i+1}_k), u_n\rangle + \mathcal{H}(\mathbf{E}^{\rm p}_n - \mathbf{E}^{\rm p}_{k,i})\Big].\end{aligned}$$

We conclude that $(u,\mathbf{E}^{\rm e},\mathbf{E}^{\rm p})$ is a minimizer for problem (6.1), so that the proof is concluded. □

The discretized in time evolution is obtained interpolating the data obtained by the minimization procedure described above. Let us set for $t^i_k \le t < t^{i+1}_k$

$$w_k(t) := w(t^i_k) \qquad \text{and} \qquad \mathcal{L}_k(t) := \mathcal{L}(t^i_k).$$

We collect the main properties of the discretized in time evolution (essential for the passage to the limit as the time step discretization goes to zero) in the following proposition.

Proposition 6.2. *There exists a map* $t \mapsto (u_k(t),\mathbf{E}^{\rm e}_k(t),\mathbf{E}^{\rm p}_k(t))$ *with* $t \in [0,T]$, *such that* $(u_k(0),\mathbf{E}^{\rm e}_k(0),\mathbf{E}^{\rm p}_k(0)) = (u_0,\mathbf{E}^{\rm e}_0,\mathbf{E}^{\rm p}_0)$ *and such that the following facts hold.*

(a) $(u_k(t),\mathbf{E}^{\rm e}_k(t),\mathbf{E}^{\rm p}_k(t)) \in \mathcal{A}(w_k(t))$ *for every* $t\in[0,T]$, *and for every* $(v,\mathbf{e},\mathbf{p}) \in \mathcal{A}(w_k(t))$ *we have*

(6.2) $$\mathcal{Q}_1(\mathbf{E}^{\rm e}_k(t)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_k(t)) - \langle \mathcal{L}_k(t), u_k(t)\rangle \le \mathcal{Q}_1(\mathbf{e}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{p}) - \langle \mathcal{L}_k(t), v\rangle + \mathcal{H}(\mathbf{E}^{\rm p}_k(t) - \mathbf{p}).$$

(b) *Setting*

$$\mathcal{E}_k(t) := \mathcal{Q}_1(\mathbf{E}^{\rm e}_k(t)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_k(t)) - \langle \mathcal{L}_k(t), u_k(t)\rangle$$

we have for every $t^i_k \le t < t^{i+1}_k$

(6.3) $$\begin{aligned}\mathcal{E}_k(t) + \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\rm p}_k;0,t) &\le \mathcal{Q}_1(\mathbf{E}^{\rm e}_0) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_0) - \langle \mathcal{L}(0), u_0\rangle\\ &+ \int_0^{t^i_k}\int_\Omega \mathbb{C}\mathbf{E}^{\rm e}_k(\tau) : \mathbf{E}\dot w(\tau)\,dx\,d\tau - \int_0^{t^i_k}\langle \dot{\mathcal{L}}(\tau), u_k(\tau)\rangle\,d\tau - \int_0^{t^i_k}\langle \mathcal{L}_k(\tau), \dot w(\tau)\rangle\,d\tau + e_k,\end{aligned}$$

where $e_k \to 0$ *as* $k\to+\infty$ *and* $\mathcal{D}_{\mathcal{H}}$ *is defined in* (5.1).

(c) *There exists a constant* C *independent of* $k\in\mathbb{N}$ *and* $t\in[0,T]$ *such that*

(6.4) $$\|u_k(t)\|_{W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)} + \|\mathbf{E}^{\rm e}_k(t)\|_{L^2(\Omega;\mathrm{M}^{N\times N}_{\rm sym})} + \|\mathrm{curl}\mathbf{E}^{\rm p}_k(t)\|_{L^2(\Omega;\mathrm{M}^{N\times N})} + \mathcal{V}(\mathbf{E}^{\rm p}_k;0,t) \le C,$$

where $\mathcal{V}(\mathbf{E}^{\rm p}_k;0,t)$ *is the total variation of* $\mathbf{E}^{\rm p}_k$ *on* $[0,t]$ *defined in* (2.4).

Proof. For every $t_k^i \le t < t_k^{i+1}$ let us set

$$u_k(t) := u_{k,i}, \qquad \mathbf{E}^{\rm e}_k(t) := \mathbf{E}^{\rm e}_{k,i} \qquad \text{and} \qquad \mathbf{E}^{\rm p}_k(t) := \mathbf{E}^{\rm p}_{k,i},$$

where $(u_{k,j}, \mathbf{E}^{\rm e}_{k,j}, \mathbf{E}^{\rm p}_{k,j}) \in \mathcal{A}(w(t_k^j))$ are a solution of the minimization problems (6.1). The minimality property (6.2) follows immediately by the subadditivity of $\mathcal{H}$.

Let us prove (6.3). By construction, comparing $(u_{k,j}, \mathbf{E}^{\rm e}_{k,j}, \mathbf{E}^{\rm p}_{k,j})$ with

$$(u_{k,j-1} + w(t_k^j) - w(t_k^{j-1}), \mathbf{E}^{\rm e}_{k,j-1} + \mathbf{E}w(t_k^j) - \mathbf{E}w(t_k^{j-1}), \mathbf{E}^{\rm p}_{k,j-1}) \in \mathcal{A}(w(t_k^j))$$

we get

$$\begin{aligned}(6.5)\quad &\mathcal{Q}_1(\mathbf{E}^{\rm e}_{k,j}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_{k,j}) + \mathcal{H}(\mathbf{E}^{\rm e}_{k,j} - \mathbf{E}^{\rm e}_{k,j-1}) - \langle \mathcal{L}(t_k^j), u_{k,j}\rangle \\ &\le \mathcal{Q}_1(\mathbf{E}^{\rm e}_{k,j-1} + \mathbf{E}w(t_k^j) - \mathbf{E}w(t_k^{j-1})) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_{k,j-1}) - \langle \mathcal{L}(t_k^j), u_{k,j-1} + w(t_k^j) - w(t_k^{j-1})\rangle \\ &= \mathcal{Q}_1(\mathbf{E}^{\rm e}_{k,j-1}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_{k,j-1}) - \langle \mathcal{L}(t_k^{j-1}), u_{k,j-1}\rangle + \int_{t_k^{j-1}}^{t_k^j}\int_\Omega \mathbb{C}\mathbf{E}^{\rm e}_k(\tau) : \mathbf{E}\dot w(\tau)\,dx\,d\tau \\ &\qquad - \int_{t_k^{j-1}}^{t_k^j} \langle \dot{\mathcal{L}}(\tau), u_k(\tau)\rangle\,d\tau - \int_{t_k^{j-1}}^{t_k^j} \langle \mathcal{L}_k(\tau), \dot w(\tau)\rangle\,d\tau + \delta_{k,j}\end{aligned}$$

where

$$\delta_{k,j} := \mathcal{Q}_1(\mathbf{E}w(t_k^j) - \mathbf{E}w(t_k^{j-1})) - \langle \mathcal{L}(t_k^j) - \mathcal{L}(t_k^{j-1}), w(t_k^j) - w(t_k^{j-1})\rangle.$$

Summing up from $j=1$ to $j=i$ we get

$$\begin{aligned}\mathcal{E}_k(t) + \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\rm p}_k; 0, t) \le \mathcal{E}_k(0) &+ \int_0^{t_k^i}\int_\Omega \mathbb{C}\mathbf{E}^{\rm e}_k(\tau) : \mathbf{E}\dot w(\tau)\,dx\,d\tau \\ &- \int_0^{t_k^i} \langle \dot{\mathcal{L}}(\tau), u_k(\tau)\rangle\,d\tau - \int_0^{t_k^i} \langle \mathcal{L}_k(\tau), \dot w(\tau)\rangle\,d\tau + \sum_{j=1}^i \delta_{k,j}\end{aligned}$$

Since

$$\begin{aligned}\delta_{k,j} \le \frac{\beta_{\mathbb{C}}}{k}&\int_{t_k^{j-1}}^{t_k^j} \|\mathbf{E}\dot w(\tau)\|^2_{L^2(\Omega;\mathrm{M}^{N\times N}_{\rm sym})}\,d\tau \\ &+ \sup_j \|\mathcal{L}(t_k^j) - \mathcal{L}(t_k^{j-1})\|_{(W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N))^*} \int_{t_k^{j-1}}^{t_k^j} \|\dot w(\tau)\|_{W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)}\,d\tau,\end{aligned}$$

setting $e_k := \sum_{j=1}^k \delta_{k,j}$, we get $e_k \to 0$ as $k \to +\infty$. Since

$$\mathcal{E}_k(0) = \mathcal{Q}_1(\mathbf{E}^{\rm e}_0) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_0) - \langle \mathcal{L}(0), u_0\rangle,$$

inequality (6.3) follows.

Let us prove (6.4). Using the safe load condition on $f$and g, by (4.15) we can rewrite the first inequality of (6.5) in the following form

$$\begin{aligned}\mathcal{Q}_1(\mathbf{E}^{\rm e}_{k,j}) - \int_\Omega \rho(t_k^j) : \mathbf{E}^{\rm e}_{k,j}\,dx + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_{k,j}) + \mathcal{H}(\mathbf{E}^{\rm e}_{k,j} - \mathbf{E}^{\rm e}_{k,j-1}) - \int_\Omega \rho_D(t_k^j) : \mathbf{E}^{\rm p}_{k,j}\,dx \\ \le \mathcal{Q}_1(\mathbf{E}^{\rm e}_{k,j-1} + \mathbf{E}w(t_k^j) - \mathbf{E}w(t_k^{j-1})) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_{k,j-1}) \\ - \int_\Omega \rho(t_k^j) : \mathbf{E}^{\rm e}_{k,j-1}\,dx - \int_\Omega \rho_D(t_k^j) : \mathbf{E}^{\rm p}_{k,j-1}\,dx\end{aligned}$$

so that

$$\mathcal{Q}_1(\mathbf{E}^{\rm e}_{k,j}) - \int_\Omega \rho(t^j_k) : \mathbf{E}^{\rm e}_{k,j}\,dx + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_{k,j}) + \mathcal{H}(\mathbf{E}^{\rm e}_{k,j} - \mathbf{E}^{\rm e}_{k,j-1})$$
$$- \int_\Omega \rho_D(t^j_k) : (\mathbf{E}^{\rm p}_{k,j} - \mathbf{E}^{\rm p}_{k,j-1})\,dx \le \mathcal{Q}_1(\mathbf{E}^{\rm e}_{k,j-1}) - \int_\Omega \rho(t^{j-1}_k) : \mathbf{E}^{\rm e}_{k,j-1}\,dx + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_{k,j-1})$$
$$+ \int_{t^{j-1}_k}^{t^j_k}\int_\Omega \mathbb{C}\mathbf{E}^{\rm e}_k(\tau) : \mathbf{E}\dot w(\tau)\,dx\,d\tau - \int_{t^{j-1}_k}^{t^j_k}\int_\Omega \dot\rho(\tau) : \mathbf{E}^{\rm e}_k(\tau)\,dx\,d\tau + \tilde\delta_{k,j},$$

where

$$\tilde\delta_{k,j} := \mathcal{Q}_1(\mathbf{E}w(t^j_k) - \mathbf{E}w(t^{j-1}_k)) \le \frac{\beta_{\mathbb{C}}}{k}\int_{t^{j-1}_k}^{t^j_k} \|\mathbf{E}\dot w(\tau)\|^2_{L^2(\Omega;\mathrm{M}^{N\times N}_{\rm sym})}\,d\tau.$$

Summing up from 0 to i we have

$$\mathcal{Q}_1(\mathbf{E}^{\rm e}_{k,i}) - \int_\Omega \rho(t^i_k) : (\mathbf{E}^{\rm e}_{k,i} - \mathbf{E}w(t^i_k))\,dx + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_{k,i})$$
$$+ \sum_{j=0}^{i}\left[\mathcal{H}(\mathbf{E}^{\rm e}_{k,j} - \mathbf{E}^{\rm e}_{k,j-1}) - \int_\Omega \rho_D(t^j_k) : (\mathbf{E}^{\rm p}_{k,j} - \mathbf{E}^{\rm p}_{k,j-1})\,dx\right]$$
$$\le \mathcal{Q}_1(\mathbf{E}^{\rm e}_{k,0}) - \int_\Omega \rho(0) : (\mathbf{E}^{\rm e}_{k,0} - \mathbf{E}w_{k,0})\,dx + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_{k,0})$$
$$+ \int_0^{t^i_k}\int_\Omega \mathbb{C}\mathbf{E}^{\rm e}_k(\tau) : \mathbf{E}\dot w(\tau)\,dx\,d\tau - \int_0^{t^i_k}\int_\Omega \dot\rho(\tau) : (\mathbf{E}^{\rm e}_k(\tau) - \mathbf{E}w_k(\tau))\,dx\,d\tau + \tilde e_k,$$

where $\tilde e_k := \sum_{j=0}^k \tilde\delta_{k,j} \to 0$ as $k \to +\infty$. Since by (4.17) we have

$$\sum_{j=0}^{i}\left[\mathcal{H}(\mathbf{E}^{\rm e}_{k,j} - \mathbf{E}^{\rm e}_{k,j-1}) - \int_\Omega \rho_D(t^j_k) : (\mathbf{E}^{\rm p}_{k,j} - \mathbf{E}^{\rm p}_{k,j-1})\,dx\right] \ge \alpha_l \sum_{j=0}^{i} \|\mathbf{E}^{\rm e}_{k,j} - \mathbf{E}^{\rm e}_{k,j-1}\|_{BV(\Omega;\mathrm{M}^{N\times N}_D)},$$

we deduce that

$$\mathcal{Q}_1(\mathbf{E}^{\rm e}_k(t)) - \int_\Omega \rho(t^i_k) : (\mathbf{E}^{\rm e}_k(t) - \mathbf{E}w_k(t))\,dx + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}_k(t)) + \alpha_l\mathcal{V}(\mathbf{E}^{\rm e}_k;0,t) \tag{6.6}$$
$$\le C_1 + \int_0^{t^i_k}\int_\Omega \mathbb{C}\mathbf{E}^{\rm e}_k(\tau) : \mathbf{E}\dot w(\tau)\,dx\,d\tau - \int_0^{t^i_k}\int_\Omega \dot\rho(\tau) : (\mathbf{E}^{\rm e}_k(\tau) - \mathbf{E}w_k(\tau))\,dx\,d\tau$$

for some $C_1 > 0$ independent of k and t. Since $\mathcal{Q}_1(\mathbf{E}^{\rm e}_k(t))$ is quadratic, we get that $\|\mathbf{E}^{\rm e}_k(t)\|_{L^2}$ is uniformly bounded in k and t. Hence from (6.6) we deduce also that $\|\mathrm{curl}\mathbf{E}^{\rm p}_k(t)\|_{L^2}$ and $\mathcal{V}(\mathbf{E}^{\rm p}_k;0,t)$ are uniformly bounded with respect to k and t. Since $u_k(t) = w_k(t)$ on $\partial_D\Omega$, by Korn's inequality we have also that $u_k(t)$ is uniformly bounded in $W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$ with respect to k and t. The proof of (6.4) is thus concluded. □

7. Existence of a quasistatic evolution and approximation results

In this section we prove that the discrete evolution $t \mapsto (u_k(t), \mathbf{E}^{\rm e}_k(t), \mathbf{E}^{\rm p}_k(t))$ given by Proposition 6.2 converges (in a suitable sense) as $k \to +\infty$ to a quasistatic evolution for the Gurtin-Anand model. This will be done in Lemma 7.1, Lemma 7.2 and Lemma 7.3. Theorem 5.2 will thus follow combining these lemmas.

Lemma 7.1. *There exists a subsequence of $t \mapsto (u_k(t), \mathbf{E}^{\rm e}_k(t), \mathbf{E}^{\rm p}_k(t))$ (still denoted by the same symbol), and a map $t \mapsto (u(t), \mathbf{E}^{\rm e}(t), \mathbf{E}^{\rm p}(t))$ with $(u(0), \mathbf{E}^{\rm e}(0), \mathbf{E}^{\rm p}(0)) = (u_0, \mathbf{E}^{\rm e}_0, \mathbf{E}^{\rm p}_0)$ and such that for every $t \in [0,T]$*

$$(u(t), \mathbf{E}^{\rm e}(t), \mathbf{E}^{\rm p}(t)) \in \mathcal{A}(w(t)),$$

$$u_k(t) \rightharpoonup u(t) \qquad \textit{weakly in } W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N), \tag{7.1}$$

$$\mathbf{E}^{\rm e}_k(t) \rightharpoonup \mathbf{E}^{\rm e}(t) \qquad \textit{weakly in } L^2(\Omega;\mathrm{M}^{N\times N}_{\rm sym}), \tag{7.2}$$

$$\mathbf{E}^{\mathrm{p}}_k(t) \overset{*}{\rightharpoonup} \mathbf{E}^{\mathrm{p}}(t) \qquad \textit{weakly}^* \textit{ in } BV(\Omega; \mathrm{M}^{N\times N}_D) \tag{7.3}$$

and

$$\mathrm{curl}\mathbf{E}^{\mathrm{p}}_k(t) \rightharpoonup \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t) \qquad \textit{weakly in } L^2(\Omega; \mathrm{M}^{N\times N}). \tag{7.4}$$

Moreover, $t \mapsto \mathbf{E}^{\mathrm{p}}(t)$ has bounded variation, and there exists $C > 0$ such that for every $t \in [0,T]$

$$\|u(t)\|_{W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)} + \|\mathbf{E}^{\mathrm{e}}(t)\|_{L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}})} + \|\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)\|_{L^2(\Omega;\mathrm{M}^{N\times N})} + \mathcal{V}(\mathbf{E}^{\mathrm{p}};0,t) \le C. \tag{7.5}$$

Finally for every $t \in [0,T]$ and for every $(v,\mathbf{e},\mathbf{p}) \in \mathcal{A}(w(t))$ the following global stability condition holds

$$\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) - \langle \mathcal{L}(t), u(t)\rangle \le \mathcal{Q}_1(\mathbf{e}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{p}) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}(\mathbf{p} - \mathbf{E}^{\mathrm{p}}(t)). \tag{7.6}$$

Proof. By Proposition 6.2 we have

$$\|u_k(t)\|_{W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)} + \|\mathbf{E}^{\mathrm{e}}_k(t)\|_{L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}})} + \|\mathrm{curl}\mathbf{E}^{\mathrm{p}}_k(t)\|_{L^2(\Omega;\mathrm{M}^{N\times N})} + \mathcal{V}(\mathbf{E}^{\mathrm{p}}_k;0,t) \le C \tag{7.7}$$

for some C independent of k and t. Since $\mathbf{E}^{\mathrm{p}}_k(0) = \mathbf{E}^{\mathrm{p}}_0$ and $\mathcal{V}(\mathbf{E}^{\mathrm{p}}_k;0,T) \le C$, the existence of $\mathbf{E}^{\mathrm{p}} \in BV(0,T;BV(\Omega;\mathrm{M}^{N\times N}_D))$ such that (7.3) holds (up to a subsequence) follows by applying the generalized version of Helly's Theorem proved in [8, Lemma 7.2] (notice that BV can be seen as the dual of a separable Banach space in such a way that the associated convergence with respect to the weak star topology is precisely the weak star convergence defined in (2.3)).

Since weak star convergence in BV implies strong convergence in L^1, by (7.7) we deduce that $\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t) \in L^2(\Omega;\mathrm{M}^{N\times N})$, and that (7.4) holds.

Let us fix $t \in [0,T]$. In view of the coercivity estimate (7.7), we may assume that there exist $\tilde{u} \in W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$, $\widetilde{\mathbf{E}^{\mathrm{e}}} \in L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}})$ and a subsequence k_j (depending a priori on t) such that

$$u_{k_j}(t) \rightharpoonup \tilde{u} \qquad \text{weakly in } W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$$

and

$$\mathbf{E}^{\mathrm{e}}_{k_j}(t) \rightharpoonup \widetilde{\mathbf{E}^{\mathrm{e}}} \qquad \text{weakly in } L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}}). \tag{7.8}$$

It follows easily that $(\tilde{u}, \widetilde{\mathbf{E}^{\mathrm{e}}}, \mathbf{E}^{\mathrm{p}}(t)) \in \mathcal{A}(w(t))$. We claim that for every $(v,\mathbf{e},\mathbf{p}) \in \mathcal{A}(w(t))$ we have

$$\mathcal{Q}_1(\widetilde{\mathbf{E}^{\mathrm{e}}}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) - \langle \mathcal{L}(t), \tilde{u}\rangle \le \mathcal{Q}_1(\mathbf{e}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{p}) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}(\mathbf{p} - \mathbf{E}^{\mathrm{p}}(t)). \tag{7.9}$$

Notice that in view of (7.9), it turns out that $\tilde{u}$ and $\widetilde{\mathbf{E}^{\mathrm{e}}}$ are uniquely determined by $\mathbf{E}^{\mathrm{p}}(t)$. In fact the pair $(\tilde{u}, \widetilde{\mathbf{E}^{\mathrm{e}}})$ minimizes the convex functional $(v,\mathbf{e}) \mapsto \mathcal{Q}_1(\mathbf{e}) - \langle \mathcal{L}(t), v\rangle$ on the convex set $K := \{(v,\mathbf{e}) : (v,\mathbf{e},\mathbf{E}^{\mathrm{p}}(t)) \in \mathcal{A}(w(t))\}$. Since the functional is strictly convex in $\mathbf{e}$, $\widetilde{\mathbf{E}^{\mathrm{e}}}$ is uniquely determined, and so is $\tilde{u}$ in view of Korn's inequality. Setting $u(t) := \tilde{u}$ and $\mathbf{E}^{\mathrm{e}}(t) := \widetilde{\mathbf{E}^{\mathrm{e}}}$, we get that (7.1) and (7.2) hold (without passing to a further subsequence).

In view of (7.7) we deduce that (7.5) holds. Finally, the global stability is given precisely by (7.9).

In order to conclude the proof, we need to prove claim (7.9). Let us set

$$v_j := v - \tilde{u} + u_{k_j}(t), \qquad \mathbf{e}_j := \mathbf{e} - \widetilde{\mathbf{E}^{\mathrm{e}}} + \mathbf{E}^{\mathrm{e}}_{k_j}(t) \qquad \text{and} \qquad \mathbf{p}_j := \mathbf{p} - \mathbf{E}^{\mathrm{p}}(t) + \mathbf{E}^{\mathrm{p}}_{k_j}(t).$$

We have $(v_j, \mathbf{e}_j, \mathbf{p}_j) \in \mathcal{A}(w_{k_j}(t))$. By (6.2) we have that

$$\begin{aligned}
&\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_{k_j}(t)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_{k_j}(t)) - \langle \mathcal{L}_{k_j}(t), u_{k_j}(t)\rangle \\
&\qquad \le \mathcal{Q}_1(\mathbf{e}_j) + \mathcal{Q}_2(\mathrm{curl}\mathbf{p}_j) - \langle \mathcal{L}_{k_j}(t), v_j\rangle + \mathcal{H}(\mathbf{p}_j - \mathbf{E}^{\mathrm{p}}_{k_j}(t)) \\
&= \mathcal{Q}_1(\mathbf{e} - \widetilde{\mathbf{E}^{\mathrm{e}}} + \mathbf{E}^{\mathrm{e}}_{k_j}(t)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{p} - \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t) + \mathrm{curl}\mathbf{E}^{\mathrm{p}}_{k_j}(t)) - \langle \mathcal{L}_{k_j}(t), v - \tilde{u} + u_{k_j}(t)\rangle + \mathcal{H}(\mathbf{p} - \mathbf{E}^{\mathrm{p}}(t))
\end{aligned}$$

so that we get

$$0 \le \mathcal{Q}_1(\mathbf{e} - \widetilde{\mathbf{E}^{\mathrm{e}}}) + \int_\Omega \mathbb{C}(\mathbf{e} - \widetilde{\mathbf{E}^{\mathrm{e}}}) : \mathbf{E}^{\mathrm{e}}_{k_j}(t)\, dx$$
$$+ \mathcal{Q}_2(\mathrm{curl}\mathbf{p} - \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) + \mu L^2 \int_\Omega (\mathrm{curl}\mathbf{p} - \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) : \mathrm{curl}\mathbf{E}^{\mathrm{p}}_{k_j}(t)\, dx$$
$$- \langle \mathcal{L}_{k_j}(t), v - \tilde{u}\rangle + \mathcal{H}(\mathbf{p} - \mathbf{E}^{\mathrm{p}}(t)).$$

Letting $j \to +\infty$, in view of (7.8), (7.3), (7.4) and since $t \mapsto \mathcal{L}(t)$ is absolutely continuous with values in $(W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N))^*$ we obtain

$$0 \le \mathcal{Q}_1(\mathbf{e} - \widetilde{\mathbf{E}^{\mathrm{e}}}) + \int_\Omega \mathbb{C}(\mathbf{e} - \widetilde{\mathbf{E}^{\mathrm{e}}}) : \widetilde{\mathbf{E}^{\mathrm{e}}}\, dx$$
$$+ \mathcal{Q}_2(\mathrm{curl}\mathbf{p} - \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) + \mu L^2 \int_\Omega (\mathrm{curl}\mathbf{p} - \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) : \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)\, dx$$
$$- \langle \mathcal{L}(t), v - \tilde{u}\rangle + \mathcal{H}(\mathbf{p} - \mathbf{E}^{\mathrm{p}}(t)).$$

Adding to both sides the term $\mathcal{Q}_1(\widetilde{\mathbf{E}^{\mathrm{e}}}) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) - \langle \mathcal{L}(t), \tilde{u}\rangle$, we obtain precisely (7.9), so that the proof is concluded. ☐

We have the following estimate from above for the total energy.

Lemma 7.2. *Let $t \mapsto (u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t))$ be the evolution given by Lemma 7.1. Then for every $t \in [0,T]$ we have*

$$\mathcal{E}(t) + \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}; 0, t) \le \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_0) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_0) - \langle \mathcal{L}(0), u_0\rangle + \int_0^t \int_\Omega \mathbb{C}\mathbf{E}^{\mathrm{e}}(\tau) : \mathbf{E}\dot{w}(\tau)\, dx\, d\tau$$
$$- \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\, d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\, d\tau, \tag{7.10}$$

where $\mathcal{E}(t)$ and $\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}; 0, t)$ are defined in (5.4) *and* (5.1) *respectively.*

Proof. Let us fix $t \in [0,T]$. By (6.3) we have

$$\mathcal{E}_k(t) + \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}_k; 0, t) \le \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_0) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_0) - \langle \mathcal{L}(0), u_0\rangle + \int_0^{t^i_k} \int_\Omega \mathbb{C}\mathbf{E}^{\mathrm{e}}_k(\tau) : \mathbf{E}\dot{w}(\tau)\, dx\, d\tau$$
$$- \int_0^{t^i_k} \langle \dot{\mathcal{L}}(\tau), u_k(\tau)\rangle\, d\tau - \int_0^{t^i_k} \langle \mathcal{L}_k(\tau), \dot{w}(\tau)\rangle\, d\tau + e_k \tag{7.11}$$

where $e_k \to 0$ as $k \to +\infty$. In view of (7.2), (7.4), (7.1) and since $\mathcal{L}_k(t) \to \mathcal{L}(t)$ strongly in $\left(W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)\right)^*$, we get by lower semicontinuity

$$\mathcal{E}(t) \le \liminf_{k\to+\infty} \mathcal{E}_k(t).$$

Moreover, by (7.3) and the lower semicontinuity of $\mathcal{H}$ with respect to the weak star convergence in BV, the very definition of $\mathcal{D}_{\mathcal{H}}$ implies that

$$\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}; 0, t) \le \liminf_{k\to+\infty} \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}_k; 0, t).$$

By Lebesgue Dominate Convergence we get as $k \to +\infty$

$$\int_0^{t^i_k} \int_\Omega \mathbb{C}\mathbf{E}^{\mathrm{e}}_k(\tau) : \mathbf{E}\dot{w}(\tau)\, dx\, d\tau - \int_0^{t^i_k} \langle \dot{\mathcal{L}}(\tau), u_k(\tau)\rangle\, d\tau - \int_0^{t^i_k} \langle \mathcal{L}_k(\tau), \dot{w}(\tau)\rangle\, d\tau$$
$$\to \int_0^t \int_\Omega \mathbb{C}\mathbf{E}^{\mathrm{e}}(\tau) : \mathbf{E}\dot{w}(\tau)\, dx\, d\tau - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\, d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\, d\tau.$$

Then (7.10) follows passing to the limit in (7.11). ☐

The following estimate from below for the total energy holds.

Lemma 7.3. *Let $t \mapsto (u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t))$ be the evolution given by Lemma 7.1. Then for every $t \in [0,T]$ we have*

$$\mathcal{E}(t) + \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}; 0, t) \geq \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_0) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_0) - \langle \mathcal{L}(0), u_0\rangle + \int_0^t \int_\Omega \mathbb{C}\mathbf{E}^{\mathrm{e}}(\tau) : \mathbf{E}\dot{w}(\tau)\, dx\, d\tau - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\, d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\, d\tau, \tag{7.12}$$

where $\mathcal{E}(t)$ and $\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}; 0, t)$ are defined in (5.4) *and* (5.1) *respectively.*

Proof. Let $t \in [0,T]$, $h \geq 1$, and let us set $s_h^i := \frac{i}{h} t$ for $i = 0, 1, \dots h$. By the global stability condition (7.6), comparing $(u(s_h^j), \mathbf{E}^{\mathrm{e}}(s_h^j), \mathbf{E}^{\mathrm{p}}(s_h^j))$ with

$$\left(u(s_h^{j+1}) - w(s_h^{j+1}) + w(s_h^j), \mathbf{E}^{\mathrm{e}}(s_h^{j+1}) - \mathbf{E}w(s_h^{j+1}) + \mathbf{E}w(s_h^j), \mathbf{E}^{\mathrm{p}}(s_h^{j+1})\right) \in \mathcal{A}(w(s_h^j))$$

we get

$$\begin{aligned} &\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(s_h^{j+1}) - \mathbf{E}w(s_h^{j+1}) + \mathbf{E}w(s_h^j)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(s_h^{j+1})) \\ &\qquad - \langle \mathcal{L}(s_h^j), u(s_h^{j+1}) - w(s_h^{j+1}) + w(s_h^j)\rangle + \mathcal{H}(\mathbf{E}^{\mathrm{p}}(s_h^{j+1}) - \mathbf{E}^{\mathrm{p}}(s_h^j)) \\ &\qquad\qquad \geq \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(s_h^j)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(s_h^j)) - \langle \mathcal{L}(s_h^j), u(s_h^j)\rangle \end{aligned}$$

which can be rewritten in the following form

$$\begin{aligned} &\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(s_h^{j+1})) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(s_h^{j+1})) - \langle \mathcal{L}(s_h^{j+1}), u(s_h^{j+1})\rangle + \mathcal{H}(\mathbf{E}^{\mathrm{p}}(s_h^{j+1}) - \mathbf{E}^{\mathrm{p}}(s_h^j)) \geq \\ &\quad \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(s_h^j)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(s_h^j)) - \langle \mathcal{L}(s_h^j), u(s_h^j)\rangle + \int_{s_h^j}^{s_h^{j+1}} \int_\Omega \mathbb{C}\overline{\mathbf{E}^{\mathrm{e}}_h}(s) : \mathbf{E}\dot{w}(s)\, dx\, ds \\ &\quad - \int_{s_h^j}^{s_h^{j+1}} \langle \dot{\mathcal{L}}(s), \bar{u}_h(s)\rangle\, ds - \int_{s_h^j}^{s_h^{j+1}} \langle \overline{\mathcal{L}_h}(s), \dot{w}(s)\rangle\, ds + \bar{\delta}_{h,j} \end{aligned} \tag{7.13}$$

where for $s_h^j < s \leq s_h^{j+1}$ we set

$$\overline{u_h}(s) := u(s_h^{j+1}), \qquad \overline{\mathbf{E}^{\mathrm{e}}_h}(s) := \mathbf{E}^{\mathrm{e}}(s_h^{j+1}), \qquad \overline{\mathbf{E}^{\mathrm{p}}_h}(s) := \mathbf{E}^{\mathrm{p}}(s_h^{j+1}), \qquad \overline{\mathcal{L}_h}(s) := \mathcal{L}(s_h^{j+1})$$

and

$$\bar{\delta}_{h,j} := -\mathcal{Q}_1(\mathbf{E}w(s_h^{j+1}) - \mathbf{E}w(s_h^j)) - \langle \mathcal{L}(s_h^{j+1}) - \mathcal{L}(s_h^j), w(s_h^{j+1}) - w(s_h^j)\rangle.$$

Summing up in (7.13) from 0 to $h-1$ we get

$$\begin{aligned} &\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t)) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) - \langle \mathcal{L}(t), u(t)\rangle + \sum_{j=0}^{h-1} \mathcal{H}(\mathbf{E}^{\mathrm{p}}(s_h^{j+1}) - \mathbf{E}^{\mathrm{p}}(s_h^j)) \\ &\qquad \geq \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_0) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_0) - \langle \mathcal{L}(0), u_0\rangle \\ &\qquad + \int_0^t \int_\Omega \mathbb{C}\overline{\mathbf{E}^{\mathrm{e}}_h}(s) : \mathbf{E}\dot{w}(s)\, dx\, ds - \int_0^t \langle \dot{\mathcal{L}}(s), \overline{u_h}(s)\rangle\, ds - \int_0^t \langle \overline{\mathcal{L}_h}(s), \dot{w}(s)\rangle\, ds + \bar{e}_h \end{aligned}$$

where $\bar{e}_h \to 0$ as $h \to +\infty$. By the very definition of $\mathcal{D}_{\mathcal{H}}$ we get

$$\mathcal{E}(t) + \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}; 0, t) \geq \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_0) + \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_0) - \langle \mathcal{L}(0), u_0\rangle + \int_0^t \int_\Omega \mathbb{C}\overline{\mathbf{E}^{\mathrm{e}}_h}(s) : \mathbf{E}\dot{w}(s)\, dx\, ds - \int_0^t \langle \dot{\mathcal{L}}(s), \overline{u_h}(s)\rangle\, ds - \int_0^t \langle \overline{\mathcal{L}_h}(s), \dot{w}(s)\rangle\, ds + \bar{e}_h. \tag{7.14}$$

Since $\mathbf{E}^{\mathrm{p}} \in BV(0,T; BV(\Omega; \mathrm{M}^{N\times N}_D))$, we have that $\mathbf{E}^{\mathrm{p}}(t)$ is continuous in time with respect to the strong norm in $BV(\Omega; \mathrm{M}^{N\times N}_D)$ up to a countable set in $[0,T]$. Let $s \in [0,T]$ be a continuity point of $\mathbf{E}^{\mathrm{p}}$, and let $s_n \to s$. Then

$$\mathbf{E}^{\mathrm{e}}(s_n) \rightharpoonup \mathbf{E}^{\mathrm{e}}(s) \qquad \text{weakly in } L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}}) \tag{7.15}$$

and

$$u(s_n) \rightharpoonup u(s) \qquad \text{weakly in } W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N). \tag{7.16}$$

In fact up to a subsequence we have that

$$\mathbf{E}^{\mathrm{e}}(s_n) \rightharpoonup \widetilde{\mathbf{E}^{\mathrm{e}}} \qquad \text{weakly in } L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}})$$

and

$$u(s_n) \rightharpoonup \tilde{u} \qquad \text{weakly in } W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$$

with $(\tilde{u},\widetilde{\mathbf{E}^{\mathrm{e}}},\mathbf{E}^{\mathrm{p}}(s)) \in \mathcal{A}(w(s))$. Given $(v,\mathbf{e},\mathbf{E}^{\mathrm{p}}(s)) \in \mathcal{A}(w(s))$, by the global stability condition (7.6), comparing $(u(s_n),\mathbf{E}^{\mathrm{e}}(s_n),\mathbf{E}^{\mathrm{p}}(s_n))$ with $(v-w(s)+w(s_n),\mathbf{e}-\mathbf{E}w(s)+\mathbf{E}w(s_n),\mathbf{E}^{\mathrm{p}}(s))$, and taking into account the continuity of $\mathcal{H}$ with respect to the BV-norm we obtain that $(\tilde{u},\widetilde{\mathbf{E}^{\mathrm{e}}})$ is a minimizer of the convex functional $(v,\mathbf{e}) \mapsto \mathcal{Q}_1(\mathbf{e}) - \langle \mathcal{L}(s), v\rangle$ on the convex set $\mathcal{K} := \{(v,\mathbf{e}) : (v,\mathbf{e},\mathbf{E}^{\mathrm{p}}(s)) \in \mathcal{A}(w(s))\}$. By uniqueness of the minimizer, we have that $\tilde{u} = u(s)$ and $\widetilde{\mathbf{E}^{\mathrm{e}}} = \mathbf{E}^{\mathrm{e}}(s)$, so that (7.15) and (7.16) follow.

By (7.15) and (7.16) we have that for a.e. every $s \in [0,t]$

$$\overline{\mathbf{E}^{\mathrm{e}}_h}(s) \rightharpoonup \mathbf{E}^{\mathrm{e}}(s) \qquad \text{weakly in } L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}}) \tag{7.17}$$

and

$$\overline{u_h}(s) \rightharpoonup u(s) \qquad \text{weakly in } W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N). \tag{7.18}$$

Taking into account that for every $s \in [0,T]$

$$\overline{\mathcal{L}_h}(s) \to \mathcal{L}(s) \qquad \text{strongly in } \left(W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)\right)^*,$$

in view of (7.17) and (7.18), passing to the limit in (7.14) we get by Dominated Convergence (take into account (7.5)) that (7.12) follows. □

We are in a position to prove Theorem 5.2. Indeed, the evolution $t \mapsto (u(t),\mathbf{E}^{\mathrm{e}}(t),\mathbf{E}^{\mathrm{p}}(t))$ given by Lemma 7.1 is a quasistatic evolution for the Gurtin-Anand model because it satisfies the global stability condition in view of (7.6), and it satisfies the energy balance because of (7.10) and (7.12).

The convergence of the discrete in time evolution to the continuous one can be improved in the following way.

Proposition 7.4. *Let $t \mapsto (u(t),\mathbf{E}^{\mathrm{e}}(t),\mathbf{E}^{\mathrm{p}}(t))$ be the quasistatic evolution for the Gurtin-Anand model given by Lemma 7.1. Then for every $t \in [0,T]$ we have for $k \to +\infty$*

$$\mathcal{E}_k(t) \to \mathcal{E}(t) \tag{7.19}$$

and

$$\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}_k;0,t) \to \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}};0,t). \tag{7.20}$$

In particular we get that

$$\mathbf{E}^{\mathrm{e}}_k(t) \to \mathbf{E}^{\mathrm{e}}(t) \qquad \textit{strongly in } L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}}) \tag{7.21}$$

$$\mathrm{curl}\mathbf{E}^{\mathrm{p}}_k(t) \to \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t) \qquad \textit{strongly in } L^2(\Omega;\mathrm{M}^{N\times N}) \tag{7.22}$$

so that for every $t \in [0,T]$

$$\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_k(t)) \to \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t)), \qquad \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_k(t)) \to \mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)) \tag{7.23}$$

and

$$\langle \mathcal{L}_k(t), u_k(t)\rangle \to \langle \mathcal{L}(t), u(t)\rangle. \tag{7.24}$$

Proof. Notice that by lower semicontinuity we have for every $t \in [0,T]$

$$\mathcal{E}(t) \le \liminf_{k\to+\infty} \mathcal{E}_k(t). \tag{7.25}$$

Moreover, by the lower semicontinuity of $\mathcal{H}$ with respect to the weak star convergence, and by the very definition of $\mathcal{D}_{\mathcal{H}}$, we deduce that for every $t \in [0,T]$

$$\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}};0,t) \le \liminf_{k\to+\infty} \mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}_k;0,t). \tag{7.26}$$

By (6.3) and (7.12) we get that

$$\mathcal{E}(t)+\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}};0,t)\le \liminf_{k\to+\infty}\left(\mathcal{E}_k(t)+\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}_k;0,t)\right)\le \limsup_{k\to+\infty}\left(\mathcal{E}_k(t)+\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}_k;0,t)\right)$$

$$\le \limsup_{k\to+\infty}\Bigg[\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_0)+\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_0)-\langle\mathcal{L}(0),u_0\rangle$$

$$+\int_0^{t_k^i}\int_\Omega \mathbb{C}\mathbf{E}^{\mathrm{e}}_k(\tau):\mathbf{E}\dot w(\tau)\,dx\,d\tau-\int_0^{t_k^i}\langle\dot{\mathcal{L}}(\tau),u_k(\tau)\rangle\,d\tau-\int_0^{t_k^i}\langle\mathcal{L}_k(\tau),\dot w(\tau)\rangle\,d\tau+e_k\Bigg]$$

$$=\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_0)+\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_0)-\langle\mathcal{L}(0),u_0\rangle+\int_0^t\int_\Omega \mathbb{C}\mathbf{E}^{\mathrm{e}}(\tau):\mathbf{E}\dot w(\tau)\,dx\,d\tau$$

$$-\int_0^t\langle\dot{\mathcal{L}}(\tau),u(\tau)\rangle\,d\tau-\int_0^t\langle\mathcal{L}(\tau),\dot w(\tau)\rangle\,d\tau\le\mathcal{E}(t)+\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}};0,t).$$

We conclude that for every $t\in[0,T]$

$$\lim_{k\to+\infty}\left(\mathcal{E}_k(t)+\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}}_k;0,t)\right)=\mathcal{E}(t)+\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}};0,t).$$

From (7.25) and (7.26) we deduce that (7.19) and (7.20) hold. Since by lower semicontinuity

$$\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t))\le\liminf_{k\to+\infty}\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_k(t))\qquad\text{and}\qquad\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t))\le\liminf_{k\to+\infty}\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_k(t)),$$

while

$$\langle\mathcal{L}_k(t),u_k(t)\rangle\to\langle\mathcal{L}(t),u(t)\rangle,$$

from (7.19) we deduce that (7.23) and (7.24) hold. In particular (7.21) and (7.22) follow, and the proof is concluded. □

8. Balance equations and the flow rule

This section is devoted to the proof of Theorem 5.3, that is, we prove that a quasistatic evolution $t\mapsto(u(t),\mathbf{E}^{\mathrm{e}}(t),\mathbf{E}^{\mathrm{p}}(t))$ for the Gurtin-Anand model satisfies the prescribed balance equations and the flow rule.

We need the following lemma.

Lemma 8.1. *For every $t\in[0,T]$ there exist $\mathbf{T}^{\mathrm{p}}(t)\in L^\infty(\Omega;\mathrm{M}^{N\times N}_D)$, $\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t)\in L^\infty(\Omega;\mathrm{M}^{N\times N\times N}_D)$ and $\mathbb{S}^{\mathrm{p}}(t)\in\left(\mathcal{M}_b(\Omega;\mathrm{M}^{N\times N\times N}_D)\right)^*$ such that for every $(\mathbf{A},\mathbb{B},\mathbb{L})\in L^1(\Omega;\mathrm{M}^{N\times N}_D)\times L^1(\Omega;\mathrm{M}^{N\times N\times N}_D)\times\mathcal{M}_b(\Omega;\mathrm{M}^{N\times N\times N}_D)$*

$$\left|\int_\Omega\mathbf{T}^{\mathrm{p}}(t):\mathbf{A}\,dx+\int_\Omega\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t):\mathbb{B}\,dx+\langle\mathbb{S}^{\mathrm{p}}(t),\mathbb{L}\rangle\right|\le S_Y\int_\Omega\sqrt{|\mathbf{A}|^2+l^2|\mathbb{B}|^2}\,dx+lS_Y|\mathbb{L}|(\Omega),\tag{8.1}$$

and such that for every $(v,\mathbf{e},\mathbf{p})\in\mathcal{A}(0)$

$$\int_\Omega\mathbf{T}(t):\mathbf{e}\,dx+\mu L^2\int_\Omega\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t):\mathrm{curl}\mathbf{p}\,dx-\langle\mathcal{L}(t),v\rangle=-\int_\Omega\mathbf{T}^{\mathrm{p}}(t):\mathbf{p}\,dx-\int_\Omega\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t):\nabla\mathbf{p}\,dx-\langle\mathbb{S}^{\mathrm{p}}(t),D^s\mathbf{p}\rangle.\tag{8.2}$$

In particular, setting $\mathbb{K}^{\mathrm{p}}(t):=\mathbb{K}^{\mathrm{p}}_{\mathrm{en}}(t)+\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t)$ with $\mathbb{K}^{\mathrm{p}}_{\mathrm{en}}(t)$ defined in (3.4)-(3.5) *starting from $\mathbf{E}^{\mathrm{p}}(t)$, for every $(v,\mathbf{e},\mathbf{p})\in\mathcal{A}(0)$ the following identity holds*

$$\int_\Omega\mathbf{T}(t):\mathbf{e}\,dx+\int_\Omega\mathbf{T}^{\mathrm{p}}(t):\mathbf{p}\,dx+\int_\Omega\mathbb{K}^{\mathrm{p}}(t):\nabla\mathbf{p}\,dx+\langle\mathbb{S}^{\mathrm{p}}(t),D^s\mathbf{p}\rangle=\langle\mathcal{L}(t),v\rangle.\tag{8.3}$$

Proof. Let us fix $t\in[0,T]$. From the global stability condition (5.2), for every $(v,\mathbf{e},\mathbf{p})\in\mathcal{A}(0)$ and $\varepsilon\in\mathbb{R}$ we get that

$$\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t))+\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t))-\langle\mathcal{L}(t),u(t)\rangle\le\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t)+\varepsilon\mathbf{e})+\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)+\varepsilon\mathrm{curl}\mathbf{p})-\langle\mathcal{L}(t),u(t)+\varepsilon v\rangle+\mathcal{H}(\varepsilon\mathbf{p})$$

so that

$$\mathcal{Q}_1(\mathbf{E}^{\rm e}(t)+\varepsilon\mathbf{e})+\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}(t)+\varepsilon\mathrm{curl}\mathbf{p})-\varepsilon\langle\mathcal{L}(t),v\rangle+\mathcal{H}(\varepsilon\mathbf{p})\geq\mathcal{Q}_1(\mathbf{E}^{\rm e}(t))+\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\rm p}(t)).$$

Taking the left and right derivative for $\varepsilon=0$ we get

$$\int_\Omega\mathbb{C}\mathbf{E}^{\rm e}(t):\mathbf{e}\,dx+\mu L^2\int_\Omega\mathrm{curl}\mathbf{E}^{\rm p}(t):\mathrm{curl}\mathbf{p}\,dx-\langle\mathcal{L}(t),v\rangle+\mathcal{H}(\mathbf{p})\geq 0$$

and

$$\int_\Omega\mathbb{C}\mathbf{E}^{\rm e}(t):\mathbf{e}\,dx+\mu L^2\int_\Omega\mathrm{curl}\mathbf{E}^{\rm p}(t):\mathrm{curl}\mathbf{p}\,dx-\langle\mathcal{L}(t),v\rangle-\mathcal{H}(-\mathbf{p})\leq 0$$

so that, since $\mathbf{T}(t):=\mathbb{C}\mathbf{E}^{\rm e}(t)$,

$$\left|\int_\Omega\mathbf{T}(t):\mathbf{e}\,dx+\mu L^2\int_\Omega\mathrm{curl}\mathbf{E}^{\rm p}(t):\mathrm{curl}\mathbf{p}\,dx-\langle\mathcal{L}(t),v\rangle\right|\leq\mathcal{H}(\mathbf{p}).$$

The previous inequality shows that the linear functional on $\mathcal{A}(0)$

$$(v,\mathbf{e},\mathbf{p})\mapsto\int_\Omega\mathbb{C}\mathbf{E}^{\rm e}(t):\mathbf{e}\,dx+\mu L^2\int_\Omega\mathrm{curl}\mathbf{E}^{\rm p}(t):\mathrm{curl}\mathbf{p}\,dx-\langle\mathcal{L}(t),v\rangle$$

depends indeed only on $\mathbf{p}$.

Let $X\subseteq L^1(\Omega;\mathrm{M}^{N\times N}_D)\times L^1(\Omega;\mathrm{M}^{N\times N\times N}_D)\times\mathcal{M}_b(\Omega;\mathrm{M}^{N\times N\times N}_D)$ be the linear subspace generated by $\{(\mathbf{p},\nabla\mathbf{p},D^s\mathbf{p})\,:\,(v,\mathbf{e},\mathbf{p})\in\mathcal{A}(0)$ for some $v\in W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N),\mathbf{e}\in L^2(\Omega;\mathrm{M}^{N\times N}_{\rm sym})\}$. By applying Hahn-Banach theorem we deduce that the linear functional

$$\varphi(\mathbf{p},\nabla\mathbf{p},D^s\mathbf{p}):=\int_\Omega\mathbf{T}(t):\mathbf{e}\,dx+\mu L^2\int_\Omega\mathrm{curl}\mathbf{E}^{\rm p}(t):\mathrm{curl}\mathbf{p}\,dx-\langle\mathcal{L}(t),v\rangle \tag{8.4}$$

on the linear space X can be extended in a continuous way to the entire space $L^1(\Omega;\mathrm{M}^{N\times N}_D)\times L^1(\Omega;\mathrm{M}^{N\times N\times N}_D)\times\mathcal{M}_b(\Omega;\mathrm{M}^{N\times N\times N}_D)$ in such a way

$$|\varphi(\mathbf{A},\mathbb{B},\mathbb{L})|\leq S_Y\int_\Omega\sqrt{|\mathbf{A}|^2+l^2|\mathbb{B}|^2}\,dx+lS_Y|\mathbb{L}|(\Omega) \tag{8.5}$$

for every $(\mathbf{A},\mathbb{B},\mathbb{L})\in L^1(\Omega;\mathrm{M}^{N\times N}_D)\times L^1(\Omega;\mathrm{M}^{N\times N\times N}_D)\times\mathcal{M}_b(\Omega;\mathrm{M}^{N\times N\times N}_D)$. By representing φ, in view of (8.5) and (8.4), we obtain that there exist $\mathbf{T}^{\rm p}(t)\in L^\infty(\Omega;\mathrm{M}^{N\times N}_D)$, $\mathbb{K}^{\rm p}_{\rm diss}(t)\in L^\infty(\Omega;\mathrm{M}^{N\times N\times N}_D)$, and $\mathbb{S}^{\rm p}(t)\in\big(\mathcal{M}_b(\Omega;\mathrm{M}^{N\times N\times N}_D)\big)^*$ such that (8.1) and (8.2) hold. Finally, (8.3) follows by (8.2) in view of the very definition of $\mathbb{K}^{\rm p}_{\rm en}(t)$. □

The following Proposition concerns the balance equation for the Cauchy stress.

Proposition 8.2 (**Balance equations for the Cauchy stress**). *For every $t\in[0,T]$ we have*

$$\begin{cases}-\mathrm{div}\mathbf{T}(t)=f(t) & \text{in }\Omega\\ \mathbf{T}(t)\nu=g(t) & \text{on }\partial_N\Omega.\end{cases} \tag{8.6}$$

Proof. Let $v\in C^\infty(\overline{\Omega},\mathbb{R}^N)$ such that $v=0$ on $\partial_D\Omega$. Choosing $(v,\mathbf{E}v,0)\in\mathcal{A}(0)$ in (8.3) we deduce

$$\int_\Omega\mathbf{T}(t):\mathbf{E}v\,dx=\langle\mathcal{L}(t),v\rangle. \tag{8.7}$$

Then clearly $-\mathrm{div}\mathbf{T}(t)=f(t)$ in the sense of distributions in Ω. Since $\mathbf{T}(t)\in L^2(\Omega;\mathrm{M}^{N\times N}_{\rm sym})$ and its divergence belongs in particular to $L^2(\Omega;\mathbb{R}^N)$, we have that the normal trace of $\mathbf{T}(t)$ on $\partial\Omega$ is well defined as an element of $H^{-1/2}(\partial\Omega;\mathbb{R}^N)$. Integrating by parts in (8.7) we get immediately the second relation of (8.6). □

Concerning the higher order stresses, the following result holds.

Proposition 8.3 (**The higher order stresses**). *For every $t\in[0,T]$ let $\mathbf{T}^{\rm p}(t)$, $\mathbb{K}^{\rm p}_{\rm diss}(t)$, $\mathbb{K}^{\rm p}(t)$ and $\mathbb{S}^{\rm p}(t)$ be as in Lemma 8.1. Then*

$$\begin{cases}\mathbf{T}^{\rm p}(t)=\mathbf{T}_D(t)+\mathrm{div}\mathbb{K}^{\rm p}(t) & \text{in }\Omega\\ \mathbb{K}^{\rm p}(t)\nu=0 & \text{on }\partial\Omega,\end{cases} \tag{8.8}$$

where $\mathbf{T}_D(t) := (\mathbf{T}(t))_D$ *denotes the deviatoric part of the Cauchy stress.*

Moreover, the higher order stresses $\mathbf{T}^{\rm p}(t)$, $\mathbb{K}^{\rm p}_{\rm diss}(t)$ *satisfy the constraint*

$$\sqrt{|\mathbf{T}^{\rm p}(t,x)|^2 + l^{-2}|\mathbb{K}^{\rm p}_{\rm diss}(t,x)|^2} \le S_Y \qquad \text{for a.e. } x \in \Omega, \tag{8.9}$$

while the stress $\mathbb{S}^{\rm p}(t)$ *satisfies*

$$\|\mathbb{S}^{\rm p}(t)\|_{(\mathcal{M}_b(\mathrm{M}_D^{N\times N\times N}))^*} \le lS_Y. \tag{8.10}$$

Proof. The stress constraints (8.9) and (8.10) follow by choosing $(\mathbf{A}, \mathbb{B}, 0)$ and $(0, 0, \mathbb{L})$ respectively in (8.1).

Let us come to (8.8). Let $\mathbf{p} \in C^\infty(\overline{\Omega}, \mathrm{M}_D^{N\times N})$, so that in particular $(0, -\mathbf{p}, \mathbf{p}) \in \mathcal{A}(0)$. Then (8.3) yields

$$-\int_\Omega \mathbf{T}(t) : \mathbf{p}\, dx + \int_\Omega \mathbf{T}^{\rm p}(t) : \mathbf{p}\, dx + \int_\Omega \mathbb{K}^{\rm p}(t) : \nabla\mathbf{p}\, dx = 0.$$

Since $\mathbf{p}$ takes values in the space of deviatoric matrices, we can replace $\mathbf{T}(t)$ by $\mathbf{T}_D(t)$ so that we obtain

$$\int_\Omega (\mathbf{T}^{\rm p}(t) - \mathbf{T}_D(t)) : \mathbf{p}\, dx + \int_\Omega \mathbb{K}^{\rm p}(t) : \nabla\mathbf{p}\, dx = 0. \tag{8.11}$$

We conclude that the first relation of (8.8) holds. As a consequence, in view of (8.9) and the definition of $\mathbb{K}^{\rm p}_{\rm en}(t)$, we have that $\mathbb{K}^{\rm p}(t) \in L^2(\Omega; \mathrm{M}_D^{N\times N\times N})$ with divergence in $L^2(\Omega; \mathrm{M}_D^{N\times N})$, so that its normal trace on $\partial\Omega$ is well defined as an element of $H^{-1/2}(\partial\Omega; \mathbb{R}^{N\times N})$. Integrating by parts in (8.11) we obtain also the second relation of (8.8), and the proof is concluded. □

Remark 8.4. Note that relation (8.3) represents the balance of internal and external power expenditures on the whole body Ω (see Section 3). Due to our variational approach which requires $\mathbf{E}^{\rm p}(t) \in BV(\Omega; \mathrm{M}_D^{N\times N})$ so that $D\mathbf{E}^{\rm p}(t)$ has also a singular part, a stress $\mathbb{S}^{\rm p}(t)$ associated to $D^s\mathbf{E}^{\rm p}(t)$ appears in the balance. In order to get a balance equation for a subbody $\mathcal{B} \subset\subset \Omega$ with sufficiently smooth boundary, we can reason as follows. Let us assume to be in the physical case $N = 3$. As a consequence, admissible displacements v turn out to belong to $L^3(\Omega; \mathbb{R}^3)$.

Let $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}(0)$ be such that $\mathbf{p}$ belongs also to $L^2(\Omega; M_D^{3\times 3})$, and let $\varphi \in C_c^\infty(\Omega)$. By (8.3) we can write

$$\begin{aligned}
&\int_\Omega \mathbf{T}(t) : (\varphi\mathbf{e})\, dx + \int_\Omega \mathbf{T}^{\rm p}(t) : (\varphi\mathbf{p})\, dx + \int_\Omega \mathbb{K}^{\rm p}(t) : (\varphi\nabla\mathbf{p})\, dx + \langle \mathbb{S}^{\rm p}(t), \varphi D^s\mathbf{p}\rangle \\
&= \int_\Omega \mathbf{T}(t) : (\varphi\mathbf{e} + v \odot \nabla\varphi)\, dx + \int_\Omega \mathbf{T}^{\rm p}(t) : (\varphi\mathbf{p})\, dx + \int_\Omega \mathbb{K}^{\rm p}(t) : \nabla(\varphi\mathbf{p})\, dx + \langle \mathbb{S}^{\rm p}(t), D^s(\varphi\mathbf{p})\rangle \\
&\qquad - \int_\Omega \mathbf{T}(t) : (v \odot \nabla\varphi)\, dx - \int_\Omega \mathbb{K}^{\rm p}(t) : (\mathbf{p} \otimes \nabla\varphi)\, dx \\
&\qquad = \langle \mathcal{L}(t), \varphi v\rangle - \int_\Omega \mathbf{T}(t) : (v \odot \nabla\varphi)\, dx - \int_\Omega \mathbb{K}^{\rm p}(t) : (\mathbf{p} \otimes \nabla\varphi)\, dx,
\end{aligned} \tag{8.12}$$

where the last equality follows since $(\varphi v, \varphi\mathbf{e} + v \odot \nabla\varphi, \varphi\mathbf{p}) \in \mathcal{A}(0)$ (we use $\mathbf{p} \in L^2(\Omega; M_D^{3\times 3})$ to ensure $\mathrm{curl}(\varphi\mathbf{p}) \in L^2(\Omega; M^{3\times 3})$). Here $(v \odot \nabla\varphi)_{i,j} = (v_i\partial_j\varphi + v_j\partial_i\varphi)/2$. As a consequence we have that the distribution

$$\varphi \mapsto -\int_\Omega \mathbf{T}(t) : (v \odot \nabla\varphi)\, dx - \int_\Omega \mathbb{K}^{\rm p}(t) : (\mathbf{p} \otimes \nabla\varphi)\, dx$$

turns out to be a measure $\mu \in \mathcal{M}_b(\Omega)$. Moreover, considering the measure $\eta \in \mathcal{M}_b(\Omega; \mathbb{R}^3)$ given by

$$\int_\Omega \psi\, d\eta := \int_\Omega \mathbf{T}(t) : (v \odot \psi)\, dx + \int_\Omega \mathbb{K}^{\rm p}(t) : (\mathbf{p} \otimes \psi)\, dx, \qquad \psi \in C_c^\infty(\Omega; \mathbb{R}^3)$$

we get immediately that $\mathrm{div}\eta = \mu$. According to [22], for every subset $\mathcal{B} \subset\subset \Omega$ with sufficiently smooth boundary we have that η admits normal trace $\eta \cdot \nu$ on $\partial\mathcal{B}$ defined as an element of the dual of $C^1(\partial\mathcal{B})$, in such a way that the following Gauss-Green formula holds

$$\int_{\mathcal{B}} d(\mathrm{div}\eta) = \langle \eta\cdot\nu, 1_{\partial\mathcal{B}}\rangle.$$

Let us denote formally $\eta\cdot\nu$ by $[\mathbf{T}(t)\nu\cdot v + \mathbb{K}^{\mathrm{p}}(t)\nu : \mathbf{p}]$, and let $[\mathbb{K}^{\mathrm{p}}(t) : \nabla\mathbf{p} + \mathbb{S}^{\mathrm{p}}(t) : D^s\mathbf{p}]$ be the measure such that

$$\int_\Omega \varphi\, d[\mathbb{K}^{\mathrm{p}}(t) : \nabla\mathbf{p} + \mathbb{S}^{\mathrm{p}}(t) : D^s\mathbf{p}] = \int_\Omega \mathbb{K}^{\mathrm{p}}(t) : (\varphi\nabla\mathbf{p})\, dx + \langle \mathbb{S}^{\mathrm{p}}(t), \varphi D^s\mathbf{p}\rangle.$$

By (8.12) we can write choosing $\varphi = 1_{\mathcal{B}}$

$$\int_{\mathcal{B}} \mathbf{T}(t) : \mathbf{e}\, dx + \int_{\mathcal{B}} \mathbf{T}^{\mathrm{p}}(t) : \mathbf{p}\, dx + [\mathbb{K}^{\mathrm{p}}(t) : \nabla\mathbf{p} + \mathbb{S}^{\mathrm{p}}(t) : D^s\mathbf{p}](\mathcal{B}) = \int_{\mathcal{B}} f(t)\cdot v\, dx + \langle [\mathbf{T}(t)\nu\cdot v + \mathbb{K}^{\mathrm{p}}(t)\nu : \mathbf{p}], 1_{\partial\mathcal{B}}\rangle \tag{8.13}$$

which is a weak form for the balance of power expenditures for the subbody $\mathcal{B}$ relative to the virtual velocity $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}(0)$.

In order to obtain the balance of powers for $\mathcal{B}$ relative to a general virtual velocity $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}(0)$ (without the restriction $\mathbf{p} \in L^2(\Omega; M^{3\times 3}_D)$) one can proceed by approximation obtaining a weaker form for (8.13). Let $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}(0)$, and let $(v_n, \mathbf{e}_n, \mathbf{p}_n) \in \mathcal{A}(0)$ be such that $\mathbf{p}_n \in L^2(\Omega; M^{3\times 3}_D)$, $v_n \to v$ strongly in $W^{1,3/2}(\Omega;\mathbb{R}^3)$, $\mathbf{e}_n \to \mathbf{e}$ strongly in $L^2(\Omega; M^{3\times 3}_{\mathrm{sym}})$, $\mathbf{p}_n \to \mathbf{p}$ strictly in $BV(\Omega; M^{3\times 3}_D)$ and $\mathrm{curl}\mathbf{p}_n \to \mathrm{curl}\mathbf{p}$ strongly in $L^2(\Omega; M^{3\times 3})$. Up to a subsequence, there exists a measure $[\mathbb{K}^{\mathrm{p}}(t) : \nabla\mathbf{p} + \mathbb{S}^{\mathrm{p}}(t) : D^s\mathbf{p}] \in \mathcal{M}_b(\Omega)$ such that for every $\varphi \in C^\infty_c(\Omega)$

$$\lim_{n\to+\infty} \left[\int_\Omega \mathbb{K}^{\mathrm{p}}(t) : (\varphi\nabla\mathbf{p}_n)\, dx + \langle \mathbb{S}^{\mathrm{p}}(t), \varphi D^s\mathbf{p}_n\rangle\right] = \int_\Omega \varphi\, d[\mathbb{K}^{\mathrm{p}}(t) : \nabla\mathbf{p} + \mathbb{S}^{\mathrm{p}}(t) : D^s\mathbf{p}].$$

Moreover, by (8.12) written for $(v_n, \mathbf{e}_n, \mathbf{p}_n)$ (using $\|\varphi\|_\infty \le C\|\nabla\varphi\|_\infty$ and by a simple application of Hahn-Banach theorem) there exists a measure $[\mathbb{K}^{\mathrm{p}}(t) : \mathbf{p}] \in \mathcal{M}_b(\Omega;\mathbb{R}^3)$ such that

$$\lim_{n\to+\infty} \int_\Omega \mathbb{K}^{\mathrm{p}}(t) : (\mathbf{p}_n \otimes \nabla\varphi)\, dx = \int_\Omega \nabla\varphi\, d[\mathbb{K}^{\mathrm{p}}(t) : \mathbf{p}].$$

We conclude that the following equality holds for every $\varphi \in C^\infty_c(\Omega)$

$$\int_\Omega \mathbf{T}(t) : (\varphi\mathbf{e})\, dx + \int_\Omega \mathbf{T}^{\mathrm{p}}(t) : (\varphi\mathbf{p})\, dx + \int_\Omega \varphi\, d[\mathbb{K}^{\mathrm{p}}(t) : \nabla\mathbf{p} + \mathbb{S}^{\mathrm{p}}(t) : D^s\mathbf{p}] = \langle \mathcal{L}(t), \varphi v\rangle - \int_\Omega \mathbf{T}(t) : (v \odot \nabla\varphi)\, dx - \int_\Omega \nabla\varphi\, d[\mathbb{K}^{\mathrm{p}}(t) : \mathbf{p}].$$

Reasoning as before, the measure $\eta \in \mathcal{M}_b(\Omega;\mathbb{R}^3)$ such that

$$\int_\Omega \psi\, d\eta = \int_\Omega \mathbf{T}(t) : (v \odot \psi)\, dx + \int_\Omega \psi\, d[\mathbb{K}^{\mathrm{p}}(t) : \mathbf{p}], \qquad \psi \in C^\infty_c(\Omega;\mathbb{R}^3)$$

is such that $\mathrm{div}\eta \in \mathcal{M}_b(\Omega)$. Denoting formally by $[\mathbf{T}(t)\nu\cdot v + \mathbb{K}^{\mathrm{p}}(t)\nu : \mathbf{p}]$ the normal trace of η on $\partial\mathcal{B}$, we can write choosing $\varphi = 1_{\mathcal{B}}$

$$\int_{\mathcal{B}} \mathbf{T}(t) : \mathbf{e}\, dx + \int_{\mathcal{B}} \mathbf{T}^{\mathrm{p}}(t) : \mathbf{p}\, dx + [\mathbb{K}^{\mathrm{p}}(t) : \nabla\mathbf{p} + \mathbb{S}^{\mathrm{p}}(t) : D^s\mathbf{p}](\mathcal{B}) = \int_{\mathcal{B}} f(t)\cdot v\, dx + \langle [\mathbf{T}(t)\nu\cdot v + \mathbb{K}^{\mathrm{p}}(t)\nu : \mathbf{p}], 1_{\partial\mathcal{B}}\rangle$$

which is the required weak form for the balance of power expenditures on $\mathcal{B}$. If $(u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t))$ and $(v, \mathbf{e}, \mathbf{p})$ are sufficiently regular, such a balance reduces to the usual one in which normal traces are taken in a classical sense: in such a case, $\mathbb{S}^{\mathrm{p}}(t)$ clearly disappears, and we come back to the original formulation of Gurtin and Anand.

In order to prove the flow rule, we need the following regularity result.

Proposition 8.5. *The maps* $t \mapsto u(t), t \mapsto \mathbf{E}^{\mathrm{e}}(t), t \mapsto \mathbf{E}^{\mathrm{p}}(t), t \mapsto \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)$ *are absolutely continuous from* $[0,T]$ *to* $W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$, $L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}})$, $BV(\Omega;\mathrm{M}^{N\times N}_D)$ *and* $L^2(\Omega;\mathrm{M}^{N\times N})$ *respectively. Moreover, for a.e.* $t\in[0,T]$ *we have*

$$\|\dot u(t)\|_{W^{1,\frac{N}{N-1}}} \le C_1(w,\rho,\alpha,\mathbb{C},l)[\|\dot\rho(t)\|_{L^2}+\|\dot\rho_D(t)\|_{L^\infty}+\|\mathbf{E}\dot w(t)\|_{L^2}] \tag{8.14}$$

$$\|\dot{\mathbf{E}}^{\mathrm{e}}(t)\|_{L^2} \le C_2(w,\rho,\alpha,\mathbb{C})[\|\dot\rho(t)\|_{L^2}+\|\dot\rho_D(t)\|_{L^\infty}+\|\mathbf{E}\dot w(t)\|_{L^2}] \tag{8.15}$$

$$\|\dot{\mathbf{E}}^{\mathrm{p}}(t)\|_{BV} \le C_3(w,\rho,\alpha,\mathbb{C},l)[\|\dot\rho(t)\|_{L^2}+\|\dot\rho_D(t)\|_{L^\infty}+\|\mathbf{E}\dot w(t)\|_{L^2}] \tag{8.16}$$

$$\|\mathrm{curl}\dot{\mathbf{E}}^{\mathrm{p}}(t)\|_{L^2} \le C_4(w,\rho,\alpha,\mathbb{C},L)[\|\dot\rho(t)\|_{L^2}+\|\dot\rho_D(t)\|_{L^\infty}+\|\mathbf{E}\dot w(t)\|_{L^2}], \tag{8.17}$$

where ρ *and* α *appear in the uniform safe load condition* (4.13)-(4.14).

Finally, we have that $t\mapsto u(t)$ *and* $t\mapsto \mathbf{E}^{\mathrm{p}}(t)$ *are absolutely continuous from* $[0,T]$ *to* $W^{1,1}(\Omega;\mathbb{R}^N)$ *and* $L^1(\Omega;\mathrm{M}^{N\times N}_D)$ *respectively, and for a.e.* $t\in[0,T]$

$$\|\dot u(t)\|_{W^{1,1}} \le C_5(w,\rho,\alpha,\mathbb{C})[\|\dot\rho(t)\|_{L^2}+\|\dot\rho_D(t)\|_{L^\infty}+\|\mathbf{E}\dot w(t)\|_{L^2}] \tag{8.18}$$

$$\|\dot{\mathbf{E}}^{\mathrm{p}}(t)\|_{L^1} \le C_6(w,\rho,\alpha,\mathbb{C})[\|\dot\rho(t)\|_{L^2}+\|\dot\rho_D(t)\|_{L^\infty}+\|\mathbf{E}\dot w(t)\|_{L^2}]. \tag{8.19}$$

Proof. The proof relies heavily on [8, Theorem 5.2]. We exploit the calculations in our context since we aim to understand the precise dependence on the material length scales l and L of the norms involved in the statement.

Let $t_1,t_2\in[0,T]$ with $t_1<t_2$. Since by the very definition of $\mathcal{D}_{\mathcal{H}}$ we have $\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}};t_1,t_2)\ge \mathcal{H}(\mathbf{E}^{\mathrm{p}}(t_2)-\mathbf{E}^{\mathrm{p}}(t_1))$, by the energy balance (5.3) we may write

$$\begin{aligned}\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t_2))-\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t_1))+\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_2))-\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_1))+\mathcal{H}(\mathbf{E}^{\mathrm{p}}(t_2)-\mathbf{E}^{\mathrm{p}}(t_1))\\ -\langle\mathcal{L}(t_2),u(t_2)\rangle+\langle\mathcal{L}(t_1),u(t_1)\rangle \le \int_{t_1}^{t_2}\int_\Omega \mathbf{T}(\tau):\mathbf{E}\dot w(\tau)\,dx\,d\tau\\ -\int_{t_1}^{t_2}\langle\dot{\mathcal{L}}(\tau),u(\tau)\rangle\,d\tau-\int_{t_1}^{t_2}\langle\mathcal{L}(\tau),\dot w(\tau)\rangle\,d\tau.\end{aligned} \tag{8.20}$$

Let us consider $(v,\mathbf{e},\mathbf{p})\in\mathcal{A}(0)$ such that

$$v:=u(t_2)-u(t_1)-(w(t_2)-w(t_1)),\quad \mathbf{e}:=\mathbf{E}^{\mathrm{e}}(t_2)-\mathbf{E}^{\mathrm{e}}(t_1)-(\mathbf{E}w(t_2)-\mathbf{E}w(t_1)),$$

and $\mathbf{p}:=\mathbf{E}^{\mathrm{p}}(t_2)-\mathbf{E}^{\mathrm{p}}(t_1)$. By combining (8.1) and (8.2), we deduce

$$\begin{aligned}-\int_\Omega \mathbf{T}(t_1):(\mathbf{E}^{\mathrm{e}}(t_2)-\mathbf{E}^{\mathrm{e}}(t_1)-(\mathbf{E}w(t_2)-\mathbf{E}w(t_1)))\,dx\\ -\mu L^2\int_\Omega \mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_1):(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_2)-\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_1))\,dx\\ +\langle\mathcal{L}(t_1),u(t_2)-u(t_1)-(w(t_2)-w(t_1))\rangle\le\mathcal{H}(\mathbf{E}^{\mathrm{p}}(t_2)-\mathbf{E}^{\mathrm{p}}(t_1)).\end{aligned} \tag{8.21}$$

Inserting (8.21) into (8.20), and taking into account (4.15) we obtain

$$\begin{aligned}&\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t_2)-\mathbf{E}^{\mathrm{e}}(t_1))+\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_2)-\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_1))\le\\ &\quad\int_{t_1}^{t_2}\int_\Omega\mathbf{T}(\tau):\mathbf{E}\dot w(\tau)\,dx\,d\tau-\int_{t_1}^{t_2}\langle\dot{\mathcal{L}}(\tau),u(\tau)\rangle\,d\tau-\int_{t_1}^{t_2}\langle\mathcal{L}(\tau),\dot w(\tau)\rangle\,d\tau\\ &\quad+\langle\mathcal{L}(t_2)-\mathcal{L}(t_1),u(t_2)\rangle-\int_\Omega\mathbf{T}(t_1):(\mathbf{E}w(t_2)-\mathbf{E}w(t_1))\,dx+\langle\mathcal{L}(t_1),w(t_2)-w(t_1)\rangle\\ &=\int_{t_1}^{t_2}\int_\Omega(\mathbf{T}(\tau)-\mathbf{T}(t_1)):\mathbf{E}\dot w(\tau)\,dx\,d\tau-\int_{t_1}^{t_2}\langle\dot{\mathcal{L}}(\tau),u(\tau)-u(t_2)\rangle\,d\tau-\int_{t_1}^{t_2}\langle\mathcal{L}(\tau)-\mathcal{L}(t_1),\dot w(\tau)\rangle\,d\tau\\ &=\int_{t_1}^{t_2}\int_\Omega(\mathbf{T}(\tau)-\mathbf{T}(t_1)):\mathbf{E}\dot w(\tau)\,dx\,d\tau-\int_{t_1}^{t_2}\int_\Omega\dot\rho(\tau):(\mathbf{E}^{\mathrm{e}}(\tau)-\mathbf{E}^{\mathrm{e}}(t_2))\,dx\,d\tau\\ &\quad-\int_{t_1}^{t_2}\int_\Omega\dot\rho_D(\tau):(\mathbf{E}^{\mathrm{p}}(\tau)-\mathbf{E}^{\mathrm{p}}(t_2))\,dx\,d\tau-\int_{t_1}^{t_2}\int_\Omega(\rho(\tau)-\rho(t_1)):\mathbf{E}\dot w(\tau)\,dx\,d\tau.\end{aligned}$$

By the coercivity estimate (4.8) for the elasticity tensor $\mathbb{C}$ we deduce

$$\begin{aligned}(8.22)\quad & \alpha_{\mathbb{C}}\|\mathbf{E}^{\mathrm{e}}(t_2)-\mathbf{E}^{\mathrm{e}}(t_1)\|^2_{L^2}+\frac{\mu L^2}{2}\|\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_2)-\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_1)\|^2_{L^2}\\ &\le \beta_{\mathbb{C}}\int_{t_1}^{t_2}\|\mathbf{E}^{\mathrm{e}}(\tau)-\mathbf{E}^{\mathrm{e}}(t_1)\|_{L^2}\|\mathbf{E}\dot{w}(\tau)\|_{L^2}\,d\tau+\int_{t_1}^{t_2}\|\dot{\rho}\|_{L^2}\|\mathbf{E}^{\mathrm{e}}(\tau)-\mathbf{E}^{\mathrm{e}}(t_2)\|_{L^2}\,d\tau\\ &\quad+\int_{t_1}^{t_2}\|\dot{\rho}_D(\tau)\|_{L^\infty}\|\mathbf{E}^{\mathrm{p}}(\tau)-\mathbf{E}^{\mathrm{p}}(t_2)\|_{L^1}\,d\tau+\int_{t_1}^{t_2}\|\rho(\tau)-\rho(t_1)\|_{L^2}\|\mathbf{E}\dot{w}(\tau)\|_{L^2}\,d\tau.\end{aligned}$$

By (4.16) we have for $t_1\le s\le t_2$

$$\begin{aligned}(8.23)\quad & \frac{\alpha}{2}\|\mathbf{E}^{\mathrm{p}}(t_2)-\mathbf{E}^{\mathrm{p}}(s)\|_{L^1}+\alpha_l\|D\mathbf{E}^{\mathrm{p}}(t_2)-D\mathbf{E}^{\mathrm{p}}(s)\|_{\mathcal{M}_b}\\ &\le \mathcal{H}\left(\mathbf{E}^{\mathrm{p}}(t_2)-\mathbf{E}^{\mathrm{p}}(s)\right)-\int_\Omega \rho_D(t_2):\left(\mathbf{E}^{\mathrm{p}}(t_2)-\mathbf{E}^{\mathrm{p}}(s)\right)dx\end{aligned}$$

where $\alpha_l:=\min\{l\frac{\alpha}{2},lS_Y\}$. Combining (8.23) and (8.20) with $t_1=s$ and using (4.15) we obtain

$$\begin{aligned}(8.24)\quad & \frac{\alpha}{2}\|\mathbf{E}^{\mathrm{p}}(t_2)-\mathbf{E}^{\mathrm{p}}(s)\|_{L^1}+\alpha_l\|D\mathbf{E}^{\mathrm{p}}(t_2)-D\mathbf{E}^{\mathrm{p}}(s)\|_{\mathcal{M}_b}\\ &\le \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(s))-\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t_2))+\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(s))-\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_2))\\ &\quad+\langle\mathcal{L}(t_2),u(t_2)\rangle-\langle\mathcal{L}(s),u(s)\rangle+\int_s^{t_2}\int_\Omega \mathbf{T}(\tau):\mathbf{E}\dot{w}(\tau)\,dx\,d\tau\\ &\quad-\int_s^{t_2}\langle\dot{\mathcal{L}}(\tau),u(\tau)\rangle\,d\tau-\int_s^{t_2}\langle\mathcal{L}(\tau),\dot{w}(\tau)\rangle\,d\tau\\ &\quad-\int_\Omega \rho_D(t_2)(\mathbf{E}^{\mathrm{p}}(t_2)-\mathbf{E}^{\mathrm{p}}(s))\,dx\\ &\le \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(s))-\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t_2))+\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(s))-\mathcal{Q}_2(\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_2))\\ &\quad+\int_\Omega \rho(t_2):(\mathbf{E}^{\mathrm{e}}(t_2)-\mathbf{E}^{\mathrm{e}}(s))\,dx+\int_\Omega(\rho(t_2)-\rho(s)):\mathbf{E}^{\mathrm{e}}(s)\,dx\\ &\quad+\int_\Omega(\rho_D(t_2)-\rho_D(s)):\mathbf{E}^{\mathrm{p}}(s)\,dx\\ &\quad-\int_s^{t_2}\int_\Omega\left\{\dot{\rho}(\tau):\mathbf{E}^{\mathrm{e}}(\tau)+\dot{\rho}_D(\tau):\mathbf{E}^{\mathrm{p}}(\tau)-(\mathbf{T}(\tau)-\rho(\tau)):\mathbf{E}\dot{w}(\tau)\right\}\,dx\,d\tau.\end{aligned}$$

Notice that

$$\sup_\tau\|\rho(\tau)\|_{L^2},\quad \sup_\tau\|\rho_D(\tau)\|_{L^\infty},$$

and

$$\sup_\tau\|\mathbf{E}^{\mathrm{e}}(\tau)\|_{L^2},\quad \sup_\tau\|\mathbf{E}^{\mathrm{p}}(\tau)\|_{L^1},\quad \sup_\tau\|\mathrm{curl}\mathbf{E}^{\mathrm{p}}(\tau)\|_{L^2}$$

are finite (in fact $t\mapsto\mathbf{E}^{\mathrm{p}}(t)$ has bounded variation, while for $\mathbf{E}^{\mathrm{e}}(t)$ and $\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t)$ we can use the energy balance (5.3)). From (8.24) we obtain for every $t_1\le s\le t_2$

$$\begin{aligned}(8.25)\quad & \frac{\alpha}{2}\|\mathbf{E}^{\mathrm{p}}(t_2)-\mathbf{E}^{\mathrm{p}}(s)\|_{L^1}+\alpha_l\|D\mathbf{E}^{\mathrm{p}}(t_2)-D\mathbf{E}^{\mathrm{p}}(s)\|_{\mathcal{M}_b}\\ &\le C_1\left(\|\mathbf{E}^{\mathrm{e}}(t_2)-\mathbf{E}^{\mathrm{e}}(s)\|_{L^2}+\sqrt{\frac{\mu}{2}}L\|\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t_2)-\mathrm{curl}\mathbf{E}^{\mathrm{p}}(s)\|_{L^2}+\int_s^{t_2}\psi(\tau)\,d\tau\right),\end{aligned}$$

where

$$(8.26)\qquad \psi(\tau):=\|\dot{\rho}(\tau)\|_{L^2}+\|\dot{\rho}_D(\tau)\|_{L^\infty}+\|\mathbf{E}\dot{w}(\tau)\|_{L^2}$$

and C_1 depends on ρ, $\sup_\tau \|\mathbf{E}^{\rm e}(\tau)\|_{L^2}$, $\sup_\tau \|\mathbf{E}^{\rm p}(\tau)\|_{L^1}$, $\sup_\tau L\|\mathrm{curl}\mathbf{E}^{\rm p}(\tau)\|_{L^2}$ and the elasticity tensor $\mathbb{C}$. By (8.22) we conclude

$$\begin{aligned}\alpha_{\mathbb{C}}\|\mathbf{E}^{\rm e}(t_2)-\mathbf{E}^{\rm e}(t_1)\|^2_{L^2}+\frac{\mu L^2}{2}\|\mathrm{curl}\mathbf{E}^{\rm p}(t_2)-\mathrm{curl}\mathbf{E}^{\rm p}(t_1)\|^2_{L^2}\\ \le C_2\left(\|\mathbf{E}^{\rm e}(t_2)-\mathbf{E}^{\rm e}(t_1)\|_{L^2}+\sqrt{\frac{\mu}{2}}L\|\mathrm{curl}\mathbf{E}^{\rm p}(t_2)-\mathrm{curl}\mathbf{E}^{\rm p}(t_1)\|_{L^2}\right)\int_{t_1}^{t_2}\psi(\tau)\,d\tau\\ +C_2\int_{t_1}^{t_2}\psi(\tau)\left(\|\mathbf{E}^{\rm e}(\tau)-\mathbf{E}^{\rm e}(t_1)\|_{L^2}+\sqrt{\frac{\mu}{2}}L\|\mathrm{curl}\mathbf{E}^{\rm p}(\tau)-\mathrm{curl}\mathbf{E}^{\rm p}(t_1)\|_{L^2}\right)d\tau\\ +C_2\left(\int_{t_1}^{t_2}\psi(\tau)\,d\tau\right)^2,\end{aligned}$$

where C_2 depends also on α. By Cauchy's inequality we obtain

$$\begin{aligned}\|\mathbf{E}^{\rm e}(t_2)-\mathbf{E}^{\rm e}(t_1)\|^2_{L^2}+\frac{\mu L^2}{2}\|\mathrm{curl}\mathbf{E}^{\rm p}(t_2)-\mathrm{curl}\mathbf{E}^{\rm p}(t_1)\|^2_{L^2}\\ \le C_3\int_{t_1}^{t_2}\psi(\tau)\left((\|\mathbf{E}^{\rm e}(\tau)-\mathbf{E}^{\rm e}(t_1)\|_{L^2}+\sqrt{\frac{\mu}{2}}L\|\mathrm{curl}\mathbf{E}^{\rm p}(\tau)-\mathrm{curl}\mathbf{E}^{\rm p}(t_1)\|_{L^2}\right)d\tau\\ +C_3\left(\int_{t_1}^{t_2}\psi(\tau)\,d\tau\right)^2.\end{aligned}$$

By means of a Gronwall type Lemma [8, Lemma 5.3] we get in particular that

$$\|\mathbf{E}^{\rm e}(t_2)-\mathbf{E}^{\rm e}(t_1)\|_{L^2}+\sqrt{\frac{\mu}{2}}L\|\mathrm{curl}\mathbf{E}^{\rm p}(t_2)-\mathrm{curl}\mathbf{E}^{\rm p}(t_1)\|_{L^2}\le C_4\int_{t_1}^{t_2}\psi(\tau)\,d\tau, \tag{8.27}$$

where C_4 depends on ρ, α, $\sup_\tau \|\mathbf{E}^{\rm e}(\tau)\|_{L^2}$, $\sup_\tau \|\mathbf{E}^{\rm p}(\tau)\|_{L^1}$, $\sup_\tau L\|\mathrm{curl}\mathbf{E}^{\rm p}(\tau)\|_{L^2}$ and the elasticity tensor $\mathbb{C}$. As a consequence we get that $t\mapsto \mathbf{E}^{\rm e}(t)$ and $t\mapsto \mathrm{curl}\mathbf{E}^{\rm p}(t)$ are absolutely continuous from $[0,T]$ to $L^2(\Omega;\mathrm{M}^{N\times N}_{\rm sym})$ and $L^2(\Omega;\mathrm{M}^{N\times N})$ respectively. By (8.25), we get that $t\mapsto \mathbf{E}^{\rm p}(t)$ is absolutely continuous from $[0,T]$ to $BV(\Omega;\mathrm{M}^{N\times N}_D)$.

Let us now come to the proof of (8.14)-(8.17). By the energy balance (5.3), and by the very definition of $\mathcal{H}$ we deduce that

$$\|\mathbf{E}^{\rm p}(t)\|_{L^1}\le C_5\left(1+\int_0^t(1+\psi(\tau))\|\mathbf{E}^{\rm e}(\tau)\|_{L^2}\,d\tau+\int_0^t(1+\psi(\tau))\|\mathbf{E}^{\rm p}(\tau)\|_{L^1}\,d\tau\right),$$

where ψ is as in (8.26) and C_5 depends only on the initial conditions and on $w,\rho,\alpha,\mathbb{C}$. By means of classical Gronwall lemma and taking the sup in t we obtain

$$\sup_{t\in[0,T]}\|\mathbf{E}^{\rm p}(t)\|_{L^1}\le C_6\left(1+\sup_{t\in[0,T]}\|\mathbf{E}^{\rm e}(t)\|_{L^2}\right).$$

By the energy balance (5.3) we conclude that $\sup_{t\in[0,T]}\|\mathbf{E}^{\rm e}(t)\|_{L^2}$ is bounded uniformly independently of l and L, so that the same holds for $\sup_{t\in[0,T]}\|\mathbf{E}^{\rm p}(t)\|_{L^1}$ and $\sup_{t\in[0,T]}L\|\mathrm{curl}\mathbf{E}^{\rm p}(t)\|_{L^2}$. By (8.27) we conclude that (8.15) and (8.17) hold. Inequalities (8.16) and (8.19) follow by (8.25). Finally the absolute continuity of $t\mapsto u(t)$ from $[0,T]$ to $W^{1,\frac{N}{N-1}}(\Omega;\mathbb{R}^N)$ and inequality (8.14) follow from the compatibility condition $\mathbf{E}u(t):=\mathbf{E}^{\rm e}(t)+\mathbf{E}^{\rm p}(t)$ and Korn's inequality. Inequality (8.18) follows in a similar way. □

Remark 8.6. Notice that the constants $C_1,\dots,C_6$ of Proposition 8.5 depend also on the initial condition $(u_0,\mathbf{E}^{\rm e}_0,\mathbf{E}^{\rm p}_0)$. More precisely, from the previous proof it can be evicted that they depend on $|\mathcal{E}(0)|$ and $\|\mathbf{E}^{\rm p}_0\|_{L^1(\Omega;\mathrm{M}^{N\times N}_D)}$.

We are now in a position to prove the flow rule for the Gurtin-Anand model. Let us start with the following weak form.

Proposition 8.7 (Weak form of the flow rule). *For a.e. $t\in[0,T]$ the following facts hold.*

(a) *For every* $(\mathbf{A},\mathbb{B})\in L^\infty(\Omega;\mathrm{M}_D^{N\times N})\times L^\infty(\Omega;\mathrm{M}_D^{N\times N\times N})$ *such that* $\sqrt{|\mathbf{A}(x)|^2+l^{-2}|\mathbb{B}(x)|^2}\le S_Y$ *for a.e.* $x\in\Omega$ *we have*

$$\int_\Omega(\mathbf{T}^{\mathrm{p}}(t)-\mathbf{A}):\dot{\mathbf{E}}^{\mathrm{p}}(t)\,dx+\int_\Omega(\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t)-\mathbb{B}):\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t)\,dx\ge 0. \tag{8.28}$$

(b) *For every* $\mathbb{L}\in(\mathcal{M}_b(\Omega;\mathrm{M}_D^{N\times N\times N})^*$ *with* $\|\mathbb{L}\|_{(\mathcal{M}_b(\Omega;\mathrm{M}_D^{N\times N\times N}))^*}\le lS_Y$ *we have*

$$\langle\mathbb{S}^{\mathrm{p}}(t)-\mathbb{L},D^s\dot{\mathbf{E}}^{\mathrm{p}}(t)\rangle\ge 0. \tag{8.29}$$

Proof. Since by Proposition 8.5 the map $t\mapsto\mathbf{E}^{\mathrm{p}}(t)$ is absolutely continuous from $[0,T]$ to $BV(\Omega;\mathrm{M}_D^{N\times N})$, by [8, Theorem 7.1] we obtain

$$\mathcal{D}_{\mathcal{H}}(\mathbf{E}^{\mathrm{p}};0,t)=\int_0^t\mathcal{H}(\dot{\mathbf{E}}^{\mathrm{p}}(\tau))\,d\tau.$$

Then differentiating the energy balance equation (5.3) we obtain for a.e. $t\in[0,T]$

$$\int_\Omega\mathbf{T}(t):\dot{\mathbf{E}}^{\mathrm{e}}(t)\,dx+\mu L^2\int_\Omega\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t):\mathrm{curl}\dot{\mathbf{E}}^{\mathrm{p}}(t)\,dx-\langle\dot{\mathcal{L}}(t),u(t)\rangle-\langle\mathcal{L}(t),\dot u(t)\rangle+\mathcal{H}(\dot{\mathbf{E}}^{\mathrm{p}}(t))$$
$$=\int_\Omega\mathbf{T}(t):\mathbf{E}\dot w(t)\,dx-\langle\dot{\mathcal{L}}(t),u(t)\rangle-\langle\mathcal{L}(t),\dot w(t)\rangle$$

so that

$$\mathcal{H}(\dot{\mathbf{E}}^{\mathrm{p}}(t))=-\int_\Omega\mathbf{T}(t):(\dot{\mathbf{E}}^{\mathrm{e}}(t)-\mathbf{E}\dot w(t))\,dx-\mu L^2\int_\Omega\mathrm{curl}\mathbf{E}^{\mathrm{p}}(t):\mathrm{curl}\dot{\mathbf{E}}^{\mathrm{p}}(t)\,dx$$
$$+\langle\mathcal{L}(t),\dot u(t)-\dot w(t)\rangle.$$

Since $(\dot u(t)-\dot w(t),\dot{\mathbf{E}}^{\mathrm{e}}(t)-\mathbf{E}\dot w(t),\dot{\mathbf{E}}^{\mathrm{p}}(t))\in\mathcal{A}(0)$, by (8.2) we get

$$\mathcal{H}(\dot{\mathbf{E}}^{\mathrm{p}}(t))=\int_\Omega\mathbf{T}^{\mathrm{p}}(t):\dot{\mathbf{E}}^{\mathrm{p}}(t)\,dx+\int_\Omega\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t):\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t)\,dx+\langle\mathbb{S}^{\mathrm{p}}(t),D^s\dot{\mathbf{E}}^{\mathrm{p}}(t)\rangle. \tag{8.30}$$

Now recall that

$$\mathcal{H}(\dot{\mathbf{E}}^{\mathrm{p}}(t))=\mathcal{F}(\dot{\mathbf{E}}^{\mathrm{p}}(t),\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t),D^s\dot{\mathbf{E}}^{\mathrm{p}}(t)):=S_Y\int_\Omega\sqrt{|\mathbf{E}^{\mathrm{p}}|^2+l^2|\nabla\mathbf{E}^{\mathrm{p}}|^2}\,dx+lS_Y|D^s\mathbf{E}^{\mathrm{p}}|(\Omega).$$

Since $\mathcal{F}:L^1(\Omega;\mathrm{M}_D^{N\times N})\times L^1(\Omega;\mathrm{M}_D^{N\times N\times N})\times\mathcal{M}_b(\Omega;\mathrm{M}_D^{N\times N\times N})\to[0,+\infty[$ is continuous (with respect to the strong norm), we have $\mathcal{F}(\dot{\mathbf{E}}^{\mathrm{p}}(t),\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t),D^s\dot{\mathbf{E}}^{\mathrm{p}}(t))=\mathcal{F}^{**}(\dot{\mathbf{E}}^{\mathrm{p}}(t),\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t),D^s\dot{\mathbf{E}}^{\mathrm{p}}(t))$, where $*$ denotes the Fenchel transformation. Moreover, we have that $\mathcal{F}^*$ is the indicator function of the set

$$\mathcal{K}:=\{(\mathbf{A},\mathbb{B},\mathbb{L})\in L^\infty(\Omega;\mathrm{M}_D^{N\times N})\times L^\infty(\Omega;\mathrm{M}_D^{N\times N\times N})\times(\mathcal{M}_b(\Omega;\mathrm{M}_D^{N\times N\times N}))^*:$$
$$\sqrt{|\mathbf{A}|^2+l^{-2}|\mathbb{B}|^2}\le S_Y\text{ a.e. in }\Omega\text{ and }\|\mathbb{L}\|_{\mathcal{M}_b^*}\le lS_Y\}.$$

As a consequence, by (8.30) we deduce that for every $(\mathbf{A},\mathbb{B},\mathbb{L})\in\mathcal{K}$ we have

$$\int_\Omega(\mathbf{T}^{\mathrm{p}}(t)-\mathbf{A}):\dot{\mathbf{E}}^{\mathrm{p}}(t)\,dx+\int_\Omega(\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t)-\mathbb{B}):\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t)\,dx+\langle\mathbb{S}^{\mathrm{p}}(t)-\mathbb{L},D^s\dot{\mathbf{E}}^{\mathrm{p}}(t)\rangle\ge 0.$$

Choosing $\mathbb{L}=\mathbb{S}^{\mathrm{p}}(t)$, which is possible in view of the constraint (8.10), we obtain (8.28). Inequality (8.29) follows by choosing $\mathbf{A}=\mathbf{T}^{\mathrm{p}}(t)$ and $\mathbb{B}=\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t)$. □

Let us now prove that the weak flow rule (8.28) for the higher order stresses $\mathbf{T}^{\mathrm{p}}(t)$ and $\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t)$ reduces under suitable regularity assumptions to the usual flow rule given by Gurtin and Anand.

Proposition 8.8 (Flow rule). *Let* $t\in[0,T]$ *be such that* $\dot{\mathbf{E}}^{\mathrm{p}}(t)$ *and* $\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t)$ *exist, and let* $x\in\Omega$ *be a Lebesgue point for* $\mathbf{T}^{\mathrm{p}}(t)$, $\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t)$, $\dot{\mathbf{E}}^{\mathrm{p}}(t)$ *and* $\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t)$. *Then if*

$$\sqrt{|\mathbf{T}^{\mathrm{p}}(t,x)|^2+l^{-2}|\mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}(t,x)|^2}<S_Y$$

we have

$$(\dot{\mathbf{E}}^{\mathrm{p}}(t,x),\nabla\dot{\mathbf{E}}^{\mathrm{p}}(t,x))=(0,0), \tag{8.31}$$

while if

$$\sqrt{|\mathbf{T}^{\rm p}(t,x)|^2 + l^{-2}|\mathbb{K}^{\rm p}_{\rm diss}(t,x)|^2} = S_Y$$

we have

$$\begin{cases} \mathbf{T}^{\rm p}(t,x) = S_Y \dfrac{\dot{\mathbf{E}}^{\rm p}(t,x)}{\sqrt{|\dot{\mathbf{E}}^{\rm p}(t,x)|^2 + l^2|\nabla\dot{\mathbf{E}}^{\rm p}(t,x)|^2}} \\[2ex] \mathbb{K}^{\rm p}_{\rm diss}(t,x) = S_Y \dfrac{l^2\nabla\dot{\mathbf{E}}^{\rm p}(t,x)}{\sqrt{|\dot{\mathbf{E}}^{\rm p}(t,x)|^2 + l^2|\nabla\dot{\mathbf{E}}^{\rm p}(t,x)|^2}}. \end{cases} \tag{8.32}$$

Proof. Let K be the convex set defined as

$$K := \{(\mathbf{A},\mathbb{B}) \in \mathrm{M}^{N\times N}_D \times \mathrm{M}^{N\times N\times N}_D : \sqrt{|\mathbf{A}|^2 + l^{-2}|\mathbb{B}|^2} \le S_Y\}.$$

Let π_K denote the projection onto K, and let π^1_K, π^2_K be its components. Let $(\mathbf{A},\mathbb{B}) \in K$, $\varepsilon > 0$, and let us set

$$\mathcal{C}^\varepsilon_{\mathbf{A},\mathbb{B}} := (\mathbf{T}^{\rm p}(t) + \varepsilon(\mathbf{A} - \mathbf{T}^{\rm p}(t,x)), \mathbb{K}^{\rm p}_{\rm diss}(t) + \varepsilon(\mathbb{B} - \mathbb{K}^{\rm p}_{\rm diss}(t,x))) \\ \in L^\infty(\Omega;\mathrm{M}^{N\times N}_D) \times L^\infty(\Omega;\mathrm{M}^{N\times N\times N}_D).$$

For every $r > 0$ let us set

$$\mathbf{F} := \begin{cases} \pi^1_K\left(\mathcal{C}^\varepsilon_{\mathbf{A},\mathbb{B}}\right) & \text{in } B(x,r) \\ \mathbf{T}^{\rm p}(t) & \text{outside } B(x,r) \end{cases}$$

and

$$\mathbb{G} := \begin{cases} \pi^2_K\left(\mathcal{C}^\varepsilon_{\mathbf{A},\mathbb{B}}\right) & \text{in } B(x,r) \\ \mathbb{K}^{\rm p}_{\rm diss}(t) & \text{outside } B(x,r). \end{cases}$$

Since $(\mathbf{F},\mathbb{G})$ are admissible for the weak flow rule (8.28), we obtain

$$\frac{1}{r^N}\left[\int_{B(x,r)} (\mathbf{T}^{\rm p}(t) - \mathbf{F}) : \dot{\mathbf{E}}^{\rm p}(t)\,dx + \int_{B(x,r)} (\mathbb{K}^{\rm p}_{\rm diss}(t) - \mathbb{G}) : \nabla\dot{\mathbf{E}}^{\rm p}(t)\,dx\right] \ge 0.$$

Since π_K is a Lipschitz mapping, we have that x is also a Lebesgue point for $\pi_K(\mathcal{C}^\varepsilon_{\mathbf{A},\mathbb{B}})$ with Lebesgue value

$$\pi_K(\mathbf{T}^{\rm p}(t,x) + \varepsilon(\mathbf{A} - \mathbf{T}^{\rm p}(t,x)), \mathbb{K}^{\rm p}_{\rm diss}(t,x) + \varepsilon(\mathbb{B} - \mathbb{K}^{\rm p}_{\rm diss}(t,x))).$$

Sending $r \to 0$, and considering $0 < \varepsilon < 1$, in view of the convexity of K we obtain

$$(\mathbf{A} - \mathbf{T}^{\rm p}(t,x)) : \dot{\mathbf{E}}^{\rm p}(t,x) + (\mathbb{B} - \mathbb{K}^{\rm p}_{\rm diss}(t,x)) : \nabla\dot{\mathbf{E}}^{\rm p}(t,x) \le 0.$$

Since the previous inequality holds for every $(\mathbf{A},\mathbb{B}) \in K$, we deduce that $(\dot{\mathbf{E}}^{\rm p}(t,x), \nabla\dot{\mathbf{E}}^{\rm p}(t,x))$ belongs to the normal cone to K at $(\mathbf{T}^{\rm p}(t,x), \mathbb{K}^{\rm p}_{\rm diss}(t,x))$. In particular, if $(\mathbf{T}^{\rm p}(t,x), \mathbb{K}^{\rm p}_{\rm diss}(t,x)) \in \mathrm{int}K$, we get that (8.31) holds, while if $(\mathbf{T}^{\rm p}(t,x), \mathbb{K}^{\rm p}_{\rm diss}(t,x)) \in \partial K$, (8.32) follows. □

Remark 8.9. A strong form for the flow rule (8.29) could be obtained following the arguments of [8, Theorem 6.2]. Notice that, in view of the presence of a singular part for $D\mathbf{E}^{\rm p}(t)$ and of its associated stress, plasticity can develop also when $\|\mathbb{S}^{\rm p}\|_{\mathcal{M}_b(\Omega;\mathrm{M}^{N\times N\times N}_D)} = lS_Y$ and $\sqrt{|\mathbf{T}^{\rm p}(t,x)|^2 + l^{-2}|\mathbb{K}^{\rm p}_{\rm diss}(t,x)|^2} < S_Y$.

9. Asymptotic analysis as $l \to 0$ and $L \to 0$

In this section we want to understand the behavior of a quasistatic evolution for the Gurtin-Anand model as the length scales l, L vanish. Our goal is to prove that the quasistatic evolution converges in a suitable sense to an evolution for perfect plasticity. The result is somehow natural, since the strain gradient effects vanish.

More precisely, we prove under suitable assumptions the convergence to a quasistatic evolution for linearly elastic-perfectly plastic bodies recently proposed by Dal Maso, DeSimone and Mora [8]. The main mathematical problem we have to face in order to prove such a convergence is that the functional setting of the problem changes, in particular for what concerns the plastic strains. In fact in the strain gradient context, the plastic strain is a BV function (since its gradient enters in the equations), while in [8] it is modelled only as a Radon measure in $\Omega\cup\partial_D\Omega$. Similar problems occur for the displacements, in view of the compatibility condition.

In Section 9.1 we briefly recall the model for quasistatic evolution in perfect plasticity recently proposed in [8]. Section 9.2 is devoted to the proof of the convergence result (Theorem 9.2).

9.1. **The Dal Maso-DeSimone-Mora model for perfect plasticity.** Let us briefly recall the model for quasistatic evolution in perfect plasticity recently proposed in [8]. We formulate the results in the particular form we need for our asymptotic problem, using the notation of the previous sections.

Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 3$) be open bounded, let $\partial_D\Omega$ and $\partial_N\Omega$ have the same boundary Γ (relative to $\partial\Omega$), and let us assume that

$$\partial\Omega \text{ and } \Gamma \text{ are of class } C^2. \tag{9.1}$$

Given $w \in W^{1,2}(\Omega;\mathbb{R}^N)$, the class of admissible configurations for the boundary datum w is given by

$$\mathcal{A}_{pp}(w) := \big\{(u, \mathbf{E}^{\mathrm{e}}, \mathbf{E}^{\mathrm{p}}) \in BD(\Omega) \times L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \partial_D\Omega; \mathrm{M}^{N\times N}_D) : \\ \mathbf{E}u = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{p}} \text{ in } \Omega, \mathbf{E}^{\mathrm{p}} = (w-u)\odot\nu\, d\mathcal{H}^{N-1} \text{ on } \partial_D\Omega\big\}.$$

Here $BD(\Omega)$ denotes the space of functions with bounded deformation on Ω,

$$BD(\Omega) := \{u \in L^1(\Omega;\mathbb{R}^N) : \mathbf{E}u \in \mathcal{M}_b(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}})\},$$

which is a Banach space with respect to the norm $\|u\|_{BD(\Omega)} := \|u\|_{L^1(\Omega;\mathbb{R}^N)} + \|\mathbf{E}u\|_{\mathcal{M}_b(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}})}$. We refer the reader to [32] for the main properties of $BD(\Omega)$. The term $(w-u)$ on $\partial_D\Omega$ is intended in the sense of traces. Finally the subscripts "pp" stand for "perfect plasticity".

Given $\mathbf{E}^{\mathrm{p}} \in \mathcal{M}_b(\Omega\cup\partial_D\Omega; \mathrm{M}^{N\times N}_D)$, we set

$$\mathcal{H}_{pp}(\mathbf{E}^{\mathrm{p}}) := S_Y|\mathbf{E}^{\mathrm{p}}|(\Omega\cup\partial_D\Omega),$$

while for $\mathbf{E}^{\mathrm{e}} \in L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}})$ we consider $\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}})$ as defined in (4.6).

Let $t \in [0,T]$, and let the boundary displacement be given by

$$w \in AC\left(0,T; W^{1,2}(\Omega;\mathbb{R}^N)\right). \tag{9.2}$$

Let the body and traction forces be given by

$$f \in AC\left(0,T; L^N(\Omega;\mathbb{R}^N)\right) \qquad \text{and} \qquad g \in AC\left(0,T; L^\infty(\partial_N\Omega;\mathbb{R}^N)\right), \tag{9.3}$$

and let us denote by $\mathcal{L}(t)$ the associated work as in (4.12). Let us assume that f, g satisfy the uniform safe load condition (4.13)-(4.14). We can simply suppose as in [8] that $t \mapsto \rho(t)$ is absolutely continuous from $[0,T]$ to $L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}})$, since in view of the regularity of Ω we get $\rho(t) \in L^N(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}})$ by the embedding result [22, Proposition 2.5].

Given an initial configuration

$$(u_0, \mathbf{E}^{\mathrm{e}}_0, \mathbf{E}^{\mathrm{p}}_0) \in \mathcal{A}_{pp}(w(0)),$$

a quasistatic evolution $t \mapsto (u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t))$ in the sense of Dal Maso-DeSimone-Mora [8] is a map from $[0,T]$ to $BD(\Omega) \times L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega\cup\partial_D\Omega; \mathrm{M}^{N\times N}_D)$ with $(u(0), \mathbf{E}^{\mathrm{e}}(0), \mathbf{E}^{\mathrm{p}}(0)) = (u_0, \mathbf{E}^{\mathrm{e}}_0, \mathbf{E}^{\mathrm{p}}_0)$ and such that for every $t \in [0,T]$ the following facts hold:

(a) $(u(t), \mathbf{E}^{\rm e}(t), \mathbf{E}^{\rm p}(t)) \in \mathcal{A}_{pp}(w(t))$;

(b) Global stability: for every $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}(w(t))$

$$\mathcal{Q}_1(\mathbf{E}^{\rm e}(t)) - \langle \mathcal{L}(t), u(t)\rangle \le \mathcal{Q}_1(\mathbf{e}) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}_{pp}(\mathbf{p} - \mathbf{E}^{\rm p}(t));$$

(b) Energy balance: the function $t \mapsto \mathbf{E}^{\rm p}(t)$ has bounded variation from $[0,T]$ to $\mathcal{M}_b(\Omega \cup \partial_D\Omega; {\rm M}^{N\times N}_D)$ and

$$\mathcal{E}_{pp}(t) + \mathcal{D}_{pp}(\mathbf{E}^{\rm p}; 0, t) = \mathcal{E}_{pp}(0) + \int_0^t \int_\Omega \mathbf{T}(\tau) : \mathbf{E}\dot{w}(\tau)\, dx\, d\tau - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\, d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\, d\tau,$$

where $\mathbf{T}(t) := \mathbb{C}\mathbf{E}^{\rm e}(t)$,

$$\mathcal{E}_{pp}(t) := \mathcal{Q}_1(\mathbf{E}^{\rm e}(t)) - \langle \mathcal{L}(t), u(t)\rangle$$

and $\mathcal{D}_{pp}(\mathbf{E}^{\rm p}; 0, t) := S_Y \mathcal{V}(\mathbf{E}^{\rm p}; 0, t)$.

In order to prove the convergence result of the next section, we need to recall the pairing between stress and strain which gives a useful representation of the work $\mathcal{L}(t)$ similar to (4.15). Following [8, Section 2], for every $t \in [0,t]$ and for every $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}(w(t))$ it is possible to define the measure $[\rho_D(t) : \mathbf{p}] \in \mathcal{M}_b(\Omega \cup \partial_D\Omega)$ such that

$$\langle \mathcal{L}(t), v\rangle = -\langle \rho(t)\nu, w(t)\rangle_{\partial_D\Omega} + \int_\Omega \rho(t) : \mathbf{e}\, dx + [\rho_D(t) : \mathbf{p}](\Omega \cup \partial_D\Omega), \tag{9.4}$$

and such that for every $\varphi \in C^1(\overline{\Omega})$

$$\int_{\Omega\cup\partial_D\Omega} \varphi\, d[\rho_D(t) : \mathbf{p}] = \langle \mathcal{L}(t), \varphi v\rangle + \langle \rho(t)\nu, \varphi w(t)\rangle_{\partial_D\Omega} - \int_\Omega \rho(t) : \varphi\mathbf{e}\, dx - \int_\Omega \rho(t) : [\nabla\varphi \odot v]\, dx. \tag{9.5}$$

A similar pairing $[\dot\rho_D(t) : \mathbf{p}] \in \mathcal{M}_b(\Omega \cup \partial_D\Omega)$ can also be defined (for a.e. $t \in [0,T]$), so that (9.4) and (9.5) hold with $\dot\rho_D(t)$, $\dot\rho(t)$ and $\dot{\mathcal{L}}(t)$ in place of $\rho_D(t)$, $\rho(t)$ and $\mathcal{L}(t)$.

9.2. **The convergence result as $l, L \to 0$.** Let $\Omega \subseteq \mathbb{R}^N$ satisfy (9.1) and let w, f, g be as in (9.2) and (9.3): notice that these data are admissible for an evolution for the Gurtin and Anand model. Let us assume that f, g satisfy the uniform safe load condition (4.13)-(4.14).

Let us consider $l_n \to 0$ and $L_n \to 0$, and let us denote by

$$t \mapsto (u_n(t), \mathbf{E}^{\rm e}_n(t), \mathbf{E}^{\rm p}_n(t))$$

a quasistatic evolution for the Gurtin-Anand model relative to the data w, f, g and the material length scales $l = l_n$ and $L = L_n$. Let us denote by $\mathcal{Q}_2^n$, $\mathcal{H}_n$ and $\mathcal{E}_n$ the energies corresponding to $\mathcal{Q}_2$, $\mathcal{H}$ and $\mathcal{E}$ respectively.

Let us assume that the initial configuration $(u_n(0), \mathbf{E}^{\rm e}_n(0), \mathbf{E}^{\rm p}_n(0))$ is such that there exist $u_0 \in BD(\Omega)$, $\mathbf{E}^{\rm e}_0 \in L^2(\Omega; {\rm M}^{N\times N}_{\rm sym})$ and $\mathbf{E}^{\rm p}_0 \in \mathcal{M}_b(\Omega; {\rm M}^{N\times N}_D)$ with

$$u_n(0) \stackrel{*}{\rightharpoonup} u_0 \qquad \text{weakly}^* \text{ in } BD(\Omega), \tag{9.6}$$

$$\mathbf{E}^{\rm e}_n(0) \rightharpoonup \mathbf{E}^{\rm e}_0 \qquad \text{weakly in } L^2(\Omega; {\rm M}^{N\times N}_{\rm sym}) \tag{9.7}$$

and

$$\mathbf{E}^{\rm p}_n(0) \stackrel{*}{\rightharpoonup} \mathbf{E}^{\rm p}_0 \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega; {\rm M}^{N\times N}_D). \tag{9.8}$$

Recall that weak star convergence in $BD(\Omega)$ is given by weak convergence in L^1 for the functions and weak star convergence in the sense of measures for the symmetrized gradients.

Let us assume moreover that convergence for the initial free energies holds, that is

$$\mathcal{Q}_1(\mathbf{E}^{\rm e}_n(0)) + \mathcal{Q}_2^n({\rm curl}\mathbf{E}^{\rm p}_n(0)) \to \mathcal{Q}_1(\mathbf{E}^{\rm e}_0). \tag{9.9}$$

We have the following compactness result.

Lemma 9.1. *Let us assume that* $(u_n(0), \mathbf{E}^{\mathrm{e}}_n(0), \mathbf{E}^{\mathrm{p}}_n(0))$ *satisfy* (9.6)-(9.9)*. There exist*

$$u \in AC(0,T;BD(\Omega)), \qquad \mathbf{E}^{\mathrm{e}} \in AC(0,T;L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}}))$$

and

$$\mathbf{E}^{\mathrm{p}} \in AC(0,T;\mathcal{M}_b(\Omega \cup \partial_D\Omega;\mathrm{M}^{N\times N}_D))$$

such that, up to a subsequence, for every $t \in [0,T]$

$$u_n(t) \overset{*}{\rightharpoonup} u(t) \qquad \text{weakly}^* \text{ in } BD(\Omega), \tag{9.10}$$

$$\mathbf{E}^{\mathrm{e}}_n(t) \rightharpoonup \mathbf{E}^{\mathrm{e}}(t) \qquad \text{weakly in } L^2(\Omega;\mathrm{M}^{N\times N}_{\mathrm{sym}}), \tag{9.11}$$

and, setting $\mathbf{E}^{\mathrm{p}}_n(t) = 0$ *on* $\partial_D\Omega$*,*

$$\mathbf{E}^{\mathrm{p}}_n(t) \overset{*}{\rightharpoonup} \mathbf{E}^{\mathrm{p}}(t) \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega \cup \partial_D\Omega;\mathrm{M}^{N\times N}_D). \tag{9.12}$$

Moreover for every $t \in [0,T]$

$$(u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t)) \in \mathcal{A}_{pp}(w(t)). \tag{9.13}$$

Proof. Let B be an open ball in $\mathbb{R}^N$ such that $\overline{\Omega} \subseteq B$, and let us set $\widetilde{\Omega} := B \setminus \overline{\partial_N\Omega}$. For every $t \in [0,T]$ let us consider $\tilde{u}_n(t) \in W^{1,\frac{N}{N-1}}(\widetilde{\Omega};\mathbb{R}^N)$, $\tilde{\mathbf{E}}^{\mathrm{e}}_n(t) \in L^2(\widetilde{\Omega};\mathrm{M}^{N\times N}_{\mathrm{sym}})$, $\tilde{\mathbf{E}}^{\mathrm{p}}_n(t) \in BV(\widetilde{\Omega};\mathrm{M}^{N\times N}_D)$ defined as

$$\tilde{u}_n(t) := \begin{cases} u_n(t) & \text{in } \Omega \\ w(t) & \text{in } \widetilde{\Omega} \setminus \Omega, \end{cases}$$

$$\tilde{\mathbf{E}}^{\mathrm{e}}_n(t) := \begin{cases} \mathbf{E}^{\mathrm{e}}_n(t) & \text{in } \Omega \\ \mathbf{E}w(t) & \text{in } \widetilde{\Omega} \setminus \Omega \end{cases}$$

and

$$\tilde{\mathbf{E}}^{\mathrm{p}}_n(t) := \begin{cases} \mathbf{E}^{\mathrm{p}}_n(t) & \text{in } \Omega \\ 0 & \text{in } \widetilde{\Omega} \setminus \Omega. \end{cases}$$

By (9.6), (9.8) and (9.9) and in view of Remark 8.6 and Proposition 8.5, we deduce that $t \mapsto \tilde{u}_n(t)$, as a map from $[0,T]$ to $BD(\widetilde{\Omega})$, has a variation which is uniformly bounded independently on n. More precisely, the sequence $(u_n)_{n\in\mathbb{N}}$ is equi-absolutely continuous. The same holds for $t \mapsto \tilde{\mathbf{E}}^{\mathrm{e}}_n(t)$ and $t \mapsto \tilde{\mathbf{E}}^{\mathrm{p}}_n(t)$ considered as maps from $[0,T]$ to $L^2(\widetilde{\Omega};\mathrm{M}^{N\times N}_{\mathrm{sym}})$ and $L^1(\widetilde{\Omega};\mathrm{M}^{N\times N}_D)$ respectively.

Recall that $BD(\widetilde{\Omega})$ can be seen as a dual space, with associated weak star convergence given precisely by the weak star convergence in BD previously defined.

Then considering $BD(\widetilde{\Omega})$ as a dual space and $L^1(\widetilde{\Omega};\mathrm{M}^{N\times N}_D)$ as a subspace of $\mathcal{M}_b(\widetilde{\Omega};\mathrm{M}^{N\times N}_D)$, we may apply the generalized version of Helly's theorem [8, Lemma 7.2] to obtain

$$\tilde{u} \in AC(0,T;BD(\widetilde{\Omega})), \qquad \tilde{\mathbf{E}}^{\mathrm{e}} \in AC(0,T;L^2(\widetilde{\Omega};\mathrm{M}^{N\times N}_{\mathrm{sym}}))$$

and

$$\tilde{\mathbf{E}}^{\mathrm{p}} \in AC(0,T;\mathcal{M}_b(\widetilde{\Omega};\mathrm{M}^{N\times N}_D))$$

such that, up to a subsequence, for every $t \in [0,T]$

$$\tilde{u}_n(t) \overset{*}{\rightharpoonup} \tilde{u}(t) \qquad \text{weakly}^* \text{ in } BD(\widetilde{\Omega}), \tag{9.14}$$

$$\tilde{\mathbf{E}}^{\mathrm{e}}_n(t) \rightharpoonup \tilde{\mathbf{E}}^{\mathrm{e}}(t) \qquad \text{weakly in } L^2(\widetilde{\Omega};\mathrm{M}^{N\times N}_{\mathrm{sym}}) \tag{9.15}$$

and

$$\tilde{\mathbf{E}}^{\mathrm{p}}_n(t) \overset{*}{\rightharpoonup} \tilde{\mathbf{E}}^{\mathrm{p}}(t) \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\widetilde{\Omega};\mathrm{M}^{N\times N}_D). \tag{9.16}$$

We have clearly that for every $t \in [0,T]$

$$\tilde{u}(t) = w(t), \quad \tilde{\mathbf{E}}^{\mathrm{e}}(t) = \mathbf{E}w(t), \quad \tilde{\mathbf{E}}^{\mathrm{p}}(t) = 0 \qquad \text{on } \widetilde{\Omega} \setminus \overline{\Omega}.$$

Let us denote by $u(t)$ and $\mathbf{E}^{\mathrm{e}}(t)$ the restrictions of $\tilde{u}(t)$ and $\tilde{\mathbf{E}}^{\mathrm{e}}(t)$ to Ω, and let $\mathbf{E}^{\mathrm{p}}(t)$ denote the restriction of $\tilde{\mathbf{E}}^{\mathrm{p}}(t)$ to $\Omega \cup \partial_D\Omega$.

Relations (9.10) and (9.11) follow directly from (9.14) and (9.15). By (9.16), and taking into account that $\tilde{\mathbf{E}}^{\mathrm{p}}_n(t) = 0$ outside $\overline{\Omega}$, we obtain (9.12).

From the compatibility condition

$$\tilde{\mathbf{E}}u_n(t) = \tilde{\mathbf{E}}^{\mathrm{e}}_n(t) + \tilde{\mathbf{E}}^{\mathrm{p}}_n(t)$$

we deduce that in the limit we have

$$\tilde{\mathbf{E}}u(t) = \tilde{\mathbf{E}}^{\mathrm{e}}(t) + \tilde{\mathbf{E}}^{\mathrm{p}}(t)$$

so that

$$\mathbf{E}^{\mathrm{p}}(t) \llcorner \partial_D\Omega = \tilde{\mathbf{E}}^{\mathrm{p}}(t) \llcorner \partial_D\Omega = (w(t) - u(t)) \odot \nu \, d\mathcal{H}^{N-1} \llcorner \partial_D\Omega,$$

where $u(t)$ is intended in the sense of traces on $\partial_D\Omega$. We deduce that (9.13) holds, and the proof is concluded. ☐

The main theorem of the section is the following asymptotic result.

Theorem 9.2. *Let $t \mapsto (u_{l,L}(t), \mathbf{E}^{\mathrm{e}}_{l,L}(t), \mathbf{E}^{\mathrm{p}}_{l,L}(t))$ be a quasistatic evolution for the Gurtin-Anand model such that the initial configuration satisfies conditions (9.6)-(9.9) for $l, L \to 0$. Then for every $l_n \to 0$ and $L_n \to 0$, there exist a subsequence $(l_{n_j}, L_{n_j})_{j\in\mathbb{N}}$ and a quasistatic evolution $t \mapsto (u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t))$ for perfect plasticity in the sense of [8] such that setting*

$$u_j := u_{l_{n_j}, L_{n_j}}, \qquad \mathbf{E}^{\mathrm{e}}_j := \mathbf{E}^{\mathrm{e}}_{l_{n_j}, L_{n_j}}, \qquad \mathbf{E}^{\mathrm{p}}_j := \mathbf{E}^{\mathrm{p}}_{l_{n_j}, L_{n_j}}$$

for every $t \in [0,T]$ we have

$$u_j(t) \overset{*}{\rightharpoonup} u(t) \qquad \text{weakly}^* \text{ in } BD(\Omega), \tag{9.17}$$

$$\mathbf{E}^{\mathrm{e}}_j(t) \to \mathbf{E}^{\mathrm{e}}(t) \qquad \text{strongly in } L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}}), \tag{9.18}$$

and

$$\mathbf{E}^{\mathrm{p}}_j(t) \overset{*}{\rightharpoonup} \mathbf{E}^{\mathrm{p}}(t) \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega \cup \partial_D\Omega; \mathrm{M}^{N\times N}_D). \tag{9.19}$$

In particular for every $t \in [0,T]$

$$\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_j(t)) \to \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t)) \qquad \text{and} \qquad \mathcal{Q}_2^{n_j}(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_j(t)) \to 0, \tag{9.20}$$

so that convergence for the free energy holds.

Proof. We divide the proof in several steps.

Step 1: Compactness and admissibility. By Lemma 9.1 there exist a subsequence n_j,

$$u \in AC(0,T; BD(\Omega)), \qquad \mathbf{E}^{\mathrm{e}} \in AC(0,T; L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}}))$$

and

$$\mathbf{E}^{\mathrm{p}} \in AC(0,T; \mathcal{M}_b(\Omega \cup \partial_D\Omega; \mathrm{M}^{N\times N}_D))$$

such that setting $u_j := u_{n_j}$, $\mathbf{E}^{\mathrm{e}}_j := \mathbf{E}^{\mathrm{e}}_{n_j}$ and $\mathbf{E}^{\mathrm{p}}_j := \mathbf{E}^{\mathrm{p}}_{n_j}$, for every $t \in [0,T]$ relations (9.17) and (9.19) hold,

$$\mathbf{E}^{\mathrm{e}}_j(t) \rightharpoonup \mathbf{E}^{\mathrm{e}}(t) \qquad \text{weakly in } L^2(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}}), \tag{9.21}$$

and $(u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t)) \in \mathcal{A}_{pp}(w(t))$, so that the triple $(u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t))$ is admissible. Finally, from the energy balance (5.3), and by the assumptions for $t = 0$, we deduce that for every $t \in [0,T]$

$$\mathcal{Q}_2^{n_j}(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_j(t)) \le C \tag{9.22}$$

for some constant C independent of j and t.

Step 2: Global stability. Let us fix $t \in [0,T]$. In order to prove that $(u(t), \mathbf{E}^{\mathrm{e}}(t), \mathbf{E}^{\mathrm{p}}(t)) \in \mathcal{A}_{pp}(w(t))$ satisfies the global stability condition

$$\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t)) - \langle \mathcal{L}(t), u(t)\rangle \le \mathcal{Q}_1(\mathbf{e}) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}_{pp}(\mathbf{p} - \mathbf{E}^{\mathrm{p}}(t)) \tag{9.23}$$

for every $(v, \mathbf{e}, \mathbf{p}) \in \mathcal{A}_{pp}(w(t))$, in view of [8, Theorem 3.6] it suffices to prove that the Cauchy stress $\mathbf{T}(t) = \mathbb{C}\mathbf{E}^{\mathrm{e}}(t)$ satisfies the equilibrium conditions

$$\begin{cases} -\mathrm{div}\mathbf{T}(t) = f(t) & \text{in } \Omega \\ \mathbf{T}(t)\nu = g(t) & \text{on } \partial_N\Omega \end{cases} \tag{9.24}$$

and the constraint

$$|\mathbf{T}_D(t,x)| \le S_Y \qquad \text{for a.e. } x \in \Omega, \tag{9.25}$$

where $\mathbf{T}_D(t) := (\mathbf{T}(t))_D$.

Equation (9.24) follows from the equilibrium equation for the Cauchy stress $\mathbf{T}_j(t) = \mathbb{C}\mathbf{E}^{\rm e}_j(t)$ given by (8.6) in view of the weak convergence of $\mathbf{T}_j(t)$ to $\mathbf{T}(t)$ which comes from (9.21).

In order to prove (9.25), let us consider the corresponding constraint in the strain gradient context given by (8.9). Let $\mathbb{K}^{\rm p}_{diss,j}(t)$, $\mathbb{K}^{\rm p}_{en,j}(t)$, $\mathbb{K}^{\rm p}_j(t) = \mathbb{K}^{\rm p}_{diss,j}(t) + \mathbb{K}^{\rm p}_{en,j}(t)$ and $\mathbf{T}^{\rm p}_j(t)$ the higher order stresses associated to $(u_j(t), \mathbf{E}^{\rm e}_j(t), \mathbf{E}^{\rm p}_j(t))$. Notice that, in view of (3.5) and of (9.22) we get

$$\mathbb{K}^{\rm p}_{en,j}(t) \to 0 \qquad \text{strongly in } L^2(\Omega; \mathrm{M}^{N\times N\times N}_D). \tag{9.26}$$

Moreover by (8.8) and (8.9) we have that

$$\mathbf{T}^{\rm p}_j(t) = (\mathbf{T}_j(t))_D + \mathrm{div}\mathbb{K}^{\rm p}_j(t), \tag{9.27}$$

and

$$\sqrt{|\mathbf{T}^{\rm p}_j(t,x)|^2 + l_{n_j}^{-2}|\mathbb{K}^{\rm p}_{diss,j}(t,x)|^2} \le S_Y \qquad \text{for a.e. } x \in \Omega.$$

In particular we have that $(\mathbf{T}^{\rm p}_j(t))_{j\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega; \mathrm{M}^{N\times N}_D)$ and

$$\mathbb{K}^{\rm p}_{diss,j}(t) \to 0 \qquad \text{strongly in } L^\infty(\Omega; \mathrm{M}^{N\times N\times N}_D).$$

By (9.26) we conclude that $\mathbb{K}^{\rm p}_j(t) \to 0$ strongly in $L^2(\Omega; \mathrm{M}^{N\times N\times N}_D)$. Notice that from (9.27) we deduce that $\mathrm{div}\mathbb{K}^{\rm p}_j(t)$ is bounded in $L^2(\Omega; \mathrm{M}^{N\times N}_D)$. We obtain

$$\mathrm{div}\mathbb{K}^{\rm p}_j(t) \rightharpoonup 0 \qquad \text{weakly in } L^2(\Omega; \mathrm{M}^{N\times N}_D)$$

so that in view of (9.27) and (9.21)

$$\mathbf{T}^{\rm p}_j(t) \rightharpoonup \mathbf{T}_D(t) \qquad \text{weakly in } L^2(\Omega; \mathrm{M}^{N\times N}_D). \tag{9.28}$$

Since $\mathbf{T}^{\rm p}_j(t) \in \mathcal{K} := \{\mathbf{A} \in L^2(\Omega; \mathrm{M}^{N\times N}_D) : |\mathbf{A}| \le S_Y \text{ a.e. in } \Omega\}$, and $\mathcal{K}$ is closed in the weak topology of $L^2(\Omega; \mathrm{M}^{N\times N}_D)$, by (9.28) we deduce that (9.25) holds. Hence (9.23) follows, and Step 2 is concluded.

Step 3: Energy balance and conclusion. Since $t \mapsto u_j(t)$ is absolutely continuous from $[0,T]$ to $W^{1,\frac{N}{N-1}}(\Omega; \mathbb{R}^N)$, integrating by parts we can write the energy balance (5.3) in the following form

$$\begin{aligned} &\mathcal{Q}_1(\mathbf{E}^{\rm e}_j(t)) + \mathcal{Q}_2^{n_j}(\mathrm{curl}\mathbf{E}^{\rm p}_j(t)) + \mathcal{D}_{\mathcal{H}_{n_j}}(\mathbf{E}^{\rm p}_j; 0, t) - \int_0^t \langle \mathcal{L}(\tau), \dot{u}_j(\tau)\rangle \, d\tau \\ &\quad = \mathcal{Q}_1(\mathbf{E}^{\rm e}_j(0)) + \mathcal{Q}_2^{n_j}(\mathrm{curl}\mathbf{E}^{\rm p}_j(0)) + \int_0^t \int_\Omega \mathbf{T}_j(\tau) : \mathbf{E}\dot{w}(\tau)\, dx\, d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle \, d\tau. \end{aligned} \tag{9.29}$$

We claim that for every $t \in [0,T]$

$$\begin{aligned} \liminf_{j\to+\infty} \left[\mathcal{D}_{\mathcal{H}_{n_j}}(\mathbf{E}^{\rm p}_j; 0, t) - \int_0^t \langle \mathcal{L}(\tau), \dot{u}_j(\tau)\rangle \, d\tau \right] &\ge \mathcal{D}_{\mathcal{H}_{pp}}(\mathbf{E}^{\rm p}; 0, t) - \int_0^t \langle \mathcal{L}(\tau), \dot{u}(\tau)\rangle \, d\tau \\ &= \mathcal{D}_{\mathcal{H}_{pp}}(\mathbf{E}^{\rm p}; 0, t) - \langle \mathcal{L}(t), u(t)\rangle + \langle \mathcal{L}(0), u(0)\rangle + \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle \, d\tau. \end{aligned} \tag{9.30}$$

Passing to the limit in (9.29), by (9.21), (9.9) and (9.30) we get for every $t \in [0,T]$

$$\begin{aligned} &\mathcal{Q}_1(\mathbf{E}^{\rm e}(t)) - \langle \mathcal{L}(t), u(t)\rangle + \mathcal{D}_{\mathcal{H}_{pp}}(\mathbf{E}^{\rm p}; 0, t) \le \mathcal{Q}_1(\mathbf{E}^{\rm e}(0)) - \langle \mathcal{L}(0), u(0)\rangle \\ &\quad + \int_0^t \int_\Omega \mathbf{T}(\tau) : \mathbf{E}\dot{w}(\tau)\, dx\, d\tau - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle \, d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle \, d\tau. \end{aligned}$$

In view of the global stability condition (9.23), by [8, Theorem 4.7] we have that also the opposite inequality holds, so that the energy balance follows. From the previous steps, we conclude that $t \mapsto (u(t), \mathbf{E}^{\rm e}(t), \mathbf{E}^{\rm p}(t))$ is a quasistatic evolution according to Dal Maso, DeSimone and Mora [8].

By (9.29), (9.30) and the energy balance, we deduce that for every $t \in [0,T]$ we have

$$\mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}_j(t)) \to \mathcal{Q}_1(\mathbf{E}^{\mathrm{e}}(t)) \qquad \text{and} \qquad \mathcal{Q}_2^{n_j}(\mathrm{curl}\mathbf{E}^{\mathrm{p}}_j(t)) \to 0$$

so that (9.20) holds. In view of (9.21), we conclude that (9.18) follows.

Let us prove claim (9.30). Recall that by [8, Theorem 7.1] we have the following representation of the dissipation

$$\mathcal{D}_{\mathcal{H}_{n_j}}(\mathbf{E}^{\mathrm{p}}_j; 0, t) = \int_0^t \mathcal{H}_{n_j}(\dot{\mathbf{E}}^{\mathrm{p}}_j(\tau))\, d\tau.$$

From the representation (4.15) we get

$$\mathcal{D}_{\mathcal{H}_{n_j}}(\mathbf{E}^{\mathrm{p}}_j; 0, t) - \int_0^t \langle \mathcal{L}(\tau), \dot{u}_j(\tau)\rangle\, d\tau = \int_0^t \left[\mathcal{H}_{n_j}(\dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)) - \int_\Omega \rho_D(\tau) : \dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)\, dx\right] d\tau - \int_0^t \int_\Omega \rho(\tau) : \dot{\mathbf{E}}^{\mathrm{e}}_j(\tau)\, dx\, d\tau + \int_0^t \langle \rho(\tau)\nu, \dot{w}(\tau)\rangle_{\partial_D \Omega}\, d\tau. \tag{9.31}$$

Moreover for every $0 \le \tau \le t$

$$\mathcal{H}_{n_j}(\dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)) - \int_\Omega \rho_D(\tau) : \dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)\, dx \ge \int_\Omega \left[S_Y |\dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)| - \rho_D(\tau) : \dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)\right] dx,$$

and the integrand of the right-end side is positive in view of the safe load condition (4.14). Let $\varphi \in C^1(\overline{\Omega})$ with $0 \le \varphi \le 1$ and $\varphi = 0$ near $\overline{\partial_N \Omega}$. Applying again the representation result [8, Theorem 7.1] for the dissipation $\mathcal{D}_{\mathcal{H}_{pp}}$ we conclude

$$\begin{aligned}
&\liminf_{j\to+\infty} \int_0^t \left[\mathcal{H}_{n_j}(\dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)) - \int_\Omega \rho_D(\tau) : \dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)\, dx\right] \\
&\quad \ge \liminf_{j\to+\infty} \int_0^t \int_\Omega \left[S_Y |\dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)| - \rho_D(\tau) : \dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)\right] dx\, d\tau \\
&\quad \ge \liminf_{j\to+\infty} \int_0^t \int_\Omega \left[S_Y |\varphi\dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)| - \rho_D(\tau) : \varphi\dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)\right] dx\, d\tau \\
&\quad = \liminf_{j\to+\infty} \left[\mathcal{D}_{\mathcal{H}_{pp}}(\varphi\mathbf{E}^{\mathrm{p}}_j; 0, t) - \int_0^t \int_\Omega \rho_D(\tau) : \varphi\dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)\, dx\, d\tau\right].
\end{aligned} \tag{9.32}$$

By the very definition of $\mathcal{D}_{\mathcal{H}_{pp}}$ and by (9.19), it is easy to see that

$$\liminf_{j\to+\infty} \mathcal{D}_{\mathcal{H}_{pp}}(\varphi\mathbf{E}^{\mathrm{p}}_j; 0, t) \ge \mathcal{D}_{\mathcal{H}_{pp}}(\varphi\mathbf{E}^{\mathrm{p}}; 0, t). \tag{9.33}$$

On the other hand, the absolute continuity of $t \mapsto \mathbf{E}^{\mathrm{p}}_j(t)$ implies that

$$\int_0^t \int_\Omega \rho_D(\tau) : \varphi\dot{\mathbf{E}}^{\mathrm{p}}_j(\tau)\, dx\, d\tau = \int_\Omega \rho_D(t) : \varphi\mathbf{E}^{\mathrm{p}}_j(t)\, dx - \int_\Omega \rho_D(0) : \varphi\mathbf{E}^{\mathrm{p}}_j(0)\, dx - \int_0^t \int_\Omega \dot{\rho}_D(\tau) : \varphi\mathbf{E}^{\mathrm{p}}_j(\tau)\, dx\, d\tau. \tag{9.34}$$

Integrating by parts, for a.e. $\tau \in [0,t]$ we have

$$\int_\Omega \dot{\rho}_D(\tau) : \varphi\mathbf{E}^{\mathrm{p}}_j(\tau)\, dx = \langle \dot{\mathcal{L}}(\tau), \varphi u_j(\tau)\rangle + \langle \dot{\rho}(\tau)\nu, w(\tau)\rangle_{\partial_D\Omega} - \int_\Omega \dot{\rho}(\tau) : \varphi\mathbf{E}^{\mathrm{e}}_j(\tau)\, dx - \int_\Omega \dot{\rho}(\tau) : [\nabla\varphi \odot u_j(\tau)]\, dx.$$

In view of the embedding result [22, Proposition 2.5], we get $\dot{\rho}(\tau) \in L^N(\Omega; \mathrm{M}^{N\times N}_{\mathrm{sym}})$ for a.e. $\tau \in [0,t]$. By (9.21) and (9.17), and since $\varphi = 0$ near $\partial_N \Omega$, we deduce for a.e. $\tau \in [0,t]$

$$\begin{aligned}
\lim_{j\to+\infty} \int_\Omega \dot{\rho}_D(\tau) : \varphi\mathbf{E}^{\mathrm{p}}_j(\tau)\, dx &= \langle \dot{\mathcal{L}}(\tau), \varphi u(\tau)\rangle + \langle \dot{\rho}(\tau)\nu, w(\tau)\rangle_{\partial_D\Omega} - \int_\Omega \dot{\rho}(\tau) : \varphi\mathbf{E}^{\mathrm{e}}(\tau)\, dx \\
&\quad - \int_\Omega \dot{\rho}(\tau) : [\nabla\varphi \odot u(\tau)]\, dx = \int_{\Omega\cup\partial_D\Omega} \varphi\, d[\dot{\rho}_D(\tau) : \mathbf{E}^{\mathrm{p}}(\tau)],
\end{aligned} \tag{9.35}$$

where $[\dot\rho_D(\tau):\mathbf{E}^{\rm p}(\tau)]$ is the measure defined in the previous subsection, and the last equality follows by (9.5) (with $\dot\rho_D$ in place of ρ_D). Similarly we obtain

$$\lim_{j\to+\infty}\int_\Omega \rho_D(t):\varphi\mathbf{E}^{\rm p}_j(t)\,dx=\int_{\Omega\cup\partial_D\Omega}\varphi\,d[\rho_D(t):\mathbf{E}^{\rm p}(t)] \tag{9.36}$$

and

$$\lim_{j\to+\infty}\int_\Omega \rho_D(0):\varphi\mathbf{E}^{\rm p}_j(0)\,dx=\int_{\Omega\cup\partial_D\Omega}\varphi\,d[\rho_D(0):\mathbf{E}^{\rm p}(0)]. \tag{9.37}$$

Letting $\varphi\to 1_{\Omega\cup\partial_D\Omega}$ we obtain from (9.32), (9.33), (9.34) and (9.35)-(9.37)

$$\begin{aligned}\liminf_{j\to+\infty}\int_0^t\Big[\mathcal{H}_{n_j}(\dot{\mathbf{E}}^{\rm p}_j(\tau))-\int_\Omega\rho_D(\tau):\dot{\mathbf{E}}^{\rm p}_j(\tau)\,dx\Big]&\ge\mathcal{D}_{\mathcal{H}_{pp}}(\mathbf{E}^{\rm p};0,t)\\ -[\rho_D(t):\mathbf{E}^{\rm p}(t)](\Omega)+[\rho_D(0):\mathbf{E}^{\rm p}(0)](\Omega\cup\partial_D\Omega)&+\int_0^t[\dot\rho_D(\tau):\mathbf{E}^{\rm p}(\tau)](\Omega\cup\partial_D\Omega)\,d\tau\\ &=\mathcal{D}_{\mathcal{H}_{pp}}(\mathbf{E}^{\rm p};0,t)-\int_0^t[\rho_D(\tau):\dot{\mathbf{E}}^{\rm p}(\tau)](\Omega\cup\partial_D\Omega)\,d\tau.\end{aligned} \tag{9.38}$$

In conclusion passing to the limit in (9.31), by (9.38) and (9.21) we get

$$\begin{aligned}&\liminf_{j\to+\infty}\Big[\mathcal{D}_{\mathcal{H}_{n_j}}(\mathbf{E}^{\rm p}_j;0,t)-\int_0^t\langle\mathcal{L}(\tau),\dot u_j(\tau)\rangle\,d\tau\Big]\\ &\quad\ge\mathcal{D}_{\mathcal{H}_{pp}}(\mathbf{E}^{\rm p};0,t)-\int_0^t[\rho_D(\tau):\dot{\mathbf{E}}^{\rm p}(\tau)](\Omega\cup\partial_D\Omega)\,d\tau-\int_0^t\int_\Omega\rho(\tau):\dot{\mathbf{E}}^{\rm e}(\tau)\,dx\,d\tau\\ &\qquad+\int_0^t\langle\rho(\tau)\nu,\dot w(\tau)\rangle_{\partial_D\Omega}\,d\tau=\mathcal{D}_{\mathcal{H}_{pp}}(\mathbf{E}^{\rm p};0,t)-\int_0^t\langle\mathcal{L}(\tau),\dot u(\tau)\rangle\,d\tau,\end{aligned}$$

where the last equality comes from the integration by parts (9.4). We deduce that claim (9.30) holds, and the proof is concluded. □

References

[1] Acharya A., Bassani J.L.: Incompatibility and crystal plasticity. *J. Mech. Phys. Solids* **48**, (2000), 1565–1595.
[2] Aifantis, E.C.: On the microstructural origin of certain inelastic models. *Trans. ASME J. Eng. Mater. Technol.***106**, 326–330.
[3] Ashby M.F.: The deformation of plastically non-homogeneous alloys. *Philos. Mag.* **21** (1970), 399–424.
[4] Ashby M.F.: The deformation of plastically non-homogeneous alloys. In Kelly A., Nicholson R.B. (Eds.), *Strengthening methods in Crystals*, 137–192, Elsevier, Amsterdam, 1971.
[5] Ambrosio L., Fusco N., Pallara D.: *Functions of bounded variations and Free Discontinuity Problems.* Clarendon Press, Oxford, 2000.
[6] Brezis H.: *Operateurs maximaux monotones et semi-groups de contractions dans les espaces de Hilbert.* North-Holland, Amsterdam-London; American Elsevier, New York, 1973.
[7] Cermelli P., Gurtin M.E., On the characterization of geometrically necessary dislocations in finite plasticity. *J. Mech. Phys. Solids* **49**, (2000), 1539–1568.
[8] Dal Maso G., DeSimone A., Mora M.G.: Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Arch. Ration. Mech. Anal.* **180** (2006), no. 2, 237–291.
[9] Evans L. C., Gariepy R. F.: *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
[10] Fleck N.A., Hutchinson J.W.: Strain gradient plasticity. *Adv. Appl. Mech.* **33** (1997), 295–361.
[11] Fleck N.A., Hutchinson J.W.: A reformulation of strain gradient plasticity. *J. Mech. Phys. Solids.* **49** (2001), 2245–2271.
[12] Gao H., Huang Y., Nix W. D., Hutchinson J. W.: Mechanism-based strain gradient plasticity. I. Theory. *J. Mech. Phys. Solids* **47** (1999), no. 6, 1239–1263.
[13] Goffman C., Serrin J.: Sublinear functions of measures and variational integrals. *Duke Math. J.* **31** (1964), 159–178.
[14] Gudmundson P.: A unified treatment of strain gradient plasticity. *J. Mech. Phys. Solids* **52** (2004), 1379–1406.
[15] Gurtin M.E.: On the plasticity of single crystals: free energy, microforces, plastic strain gradients. *J. Mech. Phys. Solids* **48** (2000), no. 5, 989–1036.
[16] Gurtin M.E.: A gradient theory of single-crystal viscoplasticity that accounts for geometrically necessary dislocations. *J. Mech. Phys. Solids* **50** (2002), no. 1, 5–32.

[17] Gurtin M.E.: On a framework for small-deformation viscoplasticity: free energy, microforces, strain gradients. *Int. J. Plasticity* **19** (2003), 47–90.
[18] Gurtin M.E.: A gradient theory of small-deformation isotropic plasticity that accounts for the Burgers vector and for dissipation due to plastic spin. *J. Mech. Phys. Solids* **52** (2004), no. 11, 2545–2568.
[19] Gurtin M.E., Anand L.: A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. I. Small deformations. *J. Mech. Phys. Solids* **53** (2005), no. 7, 1624–1649.
[20] Gurtin M.E., Needleman A.: Boundary conditions in small-deformation, single-crystal plasticity that account for the Burgers vector. *J. Mech. Phys. Solids* **53**, (2005) 1–31.
[21] Huang Y., Gao H., Nix W.D., Hutchinson J.W.: Mechanism-based strain gradient plasticity-II. Analysis. *J. Mech. Phys. Solids* **48** (2000), 99–128.
[22] Kohn R., Temam R.: Dual spaces of stresses and strains, with applications to Hencky plasticity. *Appl. Math. Optim.* **10** (1983), no. 1, 1–35.
[23] Mainik A., Mielke A.: Existence results for energetic models for rate-independent systems. *Calc. Var. Partial Differential Equations* **22** (2005), no. 1, 73–99.
[24] Mielke A.: *Evolution of rate-independent systems.* Evolutionary equations. Vol. II, 461–559, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005.
[25] Mielke A.: *Analysis of energetic models for rate-independent materials.* Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 817–828, Higher Ed. Press, Beijing, 2002.
[26] Mielke A., Theil F.: A mathematical model for rate independent phase transformations with hysteresis. *Proceedings of the Workshop on "Models of Continuum Mechanics in Analysis and Engeneering"*, H.-D. Alber, R. Balean and R. Farwig, editors, Shaker-Verlag, 117–129, 1999.
[27] Mielke A., Theil F.: On rate-independent hysteresis models. *NoDEA Nonlinear Differential Equations Appl.* **11** (2004), no. 2, 151–189.
[28] Mielke A., Theil F., Levitas V.: A variational formulation of rate-independent phase transformations using an extremum principle. *Arch. Ration. Mech. Anal.* **162** (2002), no. 2, 137–177.
[29] Nye J.F.: Some geometrical relations in dislocated crystals. *Acta Metall.* **1** (1953), 153–162.
[30] Reddy B. Daya, Ebobisse F., McBride A.: Well-posedness of a model of strain gradient plasticity for plastically irrotational materials. *Int. J. Plasticity* in press.
[31] Suquet P.-M.: Sur les quations de la plasticit: existence et rgularit des solutions. (French) [On the equations of plasticity: existence and regularity of solutions] *J. Mcanique* **20** (1981), no. 1, 3–39.
[32] Temam R.: *Problèmes mathématiques en plasticité.* Méthodes Mathématiques de l'Informatique, 12. Gauthier-Villars, Montrouge, 1983.

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