Dynamics of a viscoelastic membrane with gradient constraint

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Abstract

Taking into account inertial and viscosity effects, we consider the dynamics of a two dimensional membrane subjected to an unilateral constraint on its deformation gradient. Specifically, due to the constitutive law, we assume that higher deformations lock the material, leading to the inequality $|\nabla u| \leq g$, where u denotes the displacement of the membrane and g is a certain positive threshold. We then introduce the concept of weak solutions to the associated wave equation, and prove the existence of them for any initial data and homogeneous Dirichlet boundary conditions. The presence of the gradient constraint provides the existence of a Lagrange multiplier λ related to the existence of a reaction term Υ, which corresponds to a strongly nonlinear term in the wave equation. We then extend the existence result to a weak form of the Neumann type boundary condition $\alpha u + \frac{\partial u}{\partial \nu} + \frac{\partial \dot{u}}{\partial \nu} + \Upsilon \cdot \nu = 0$, for any $\alpha \geq 0$, and we show that these solutions tend, as $\alpha \to \infty$, to a solution of the homogeneous Dirichlet constrained problem.

Key words: membrane, viscoelasticity, wave equation, strong damping, unilateral constraint, gradient constraint.

AMS (MOS) subject classification: 35L05, 74K15, 74D10, 47H05, 46A20.

1 Introduction

A linear model describing the damped vibrations $u = u(x, t)$ of a membrane subjected to external forces f in a bounded domain at instant t may be described by the equation

$$\ddot{u} - \Delta u - \Delta \dot{u} = f, \tag{1.1}$$

with suitable initial and boundary conditions. Here we denote $\dot{u} = \partial u/\partial t$, $\ddot{u} = \partial^2 u/\partial t^2$ and Δ is the usual Laplacian in the spacial variable x. It is well-known that the damping term $\Delta \dot{u}$, representing a viscoelastic effect, induces a dissipation in energy and a mathematical regularisation in the solution u.

In this work we are interested in studying this model subjected to the additional constraint

$$|\nabla u| \leq g, \tag{1.2}$$

for some given positive function g, representing a strain threshold which locks the membrane deformation. This is a special case of the constitutive law for "ideal locking materials" introduced by W.

Prager in 1957, and it was considered by Duvaut and Lions in 1972 [12, Chap.5.7], and by Demengel and Suquet [11] in the general stationary linearised elasticity framework. For recent works on locking materials type models see [5] and [25] and references therein. In fact, this problem, in the scalar case, corresponds to the equilibrium locked membrane, which displacement $u = u(x)$, with homogeneous Dirichlet boundary condition, satisfies the equation

$$-\Delta u - \text{ div } (\lambda \nabla u) = f. \tag{1.3}$$

Here $\lambda = \lambda(x)$ is a Lagrange multiplier, associated with the locking constraint, satisfying the unilateral conditions

$$\lambda \geq 0, \qquad |\nabla u| \leq g, \qquad \lambda(|\nabla u| - g) = 0. \tag{1.4}$$

Actually, this stationary problem is also the same mathematical model for the well-known elasto-plastic torsion problem, when u is the strain potential in two dimensions (see, e.g., [12, Section 5.6.6], or [16, Sections 1:6 and 8:4]). Although in this simple case of positive constants f and g, the regularity of the stationary solution allowed Brézis [8] to prove the existence of a unique bounded λ, the problem has also been considered in more general cases by several authors. In particular, recently in [3], a degenerate case of equation (1.3), corresponding to an equivalent weak formulation for the Monge-Kantorovich mass transfer problem was considered with $f \in L^2(\Omega)$ and $g \in L^\infty(\Omega)$, where λ is regarded as an element of the dual of $L^\infty(\Omega)$ (i.e. as a finite additive measure or as a charge). In fact, gradient type constraints arise naturally in other models for critical state problems in Mechanics and in Physics and a recent survey on the corresponding elliptic and parabolic problems can be found in [18].

In this work we give the first existence results of a weak solution and the corresponding generalised Lagrange multiplier, globally in time, for the dynamics of the locking viscoelastic membrane.

Although we still obtain weak solutions u with finite energy, the unilateral gradient constraint must be interpreted globally, with a multiplier Υ as an element of a subdifferential associated with the convex constraint. We also obtain a generalised Lagrangian multiplier λ, which cannot be interpreted in the point-wise sense but rather in the duality sense of L^∞. In this weak formulation, we obtain an energy inequality involving the data f, g, and the initial conditions.

Our main results (Theorem 2.2 and Theorem 2.8) are contained in Section 2.2 and state the existence of solutions (u, Υ) and (u, λ) to system (1.1)-(1.2) in suitable weak forms for the homogeneous Dirichlet problem (see Definitions 2.1 and 2.6). In Section 5 we give their corresponding similar weak formulations for the cases with Neumann and the Fourier type boundary conditions, and show existence of solutions (Theorem 5.4).

By combining a variant of the classical penalisation method proposed in [14, pag. 376] with appropriate a-priori estimates, obtained in Section 3, we use the duality techniques in Sobolev-Bochner spaces framework recently adopted in [7] for the damped wave equation with unilateral constraints (see also [21–23]).

More precisely, we make use of approximate functions u_ϵ, defined for all $\epsilon \in (0, 1)$, which are more regular and satisfy the damped wave equation with an additional reaction term coming from a penalised version of the constraint (1.2). Roughly speaking, the solutions u_ϵ to the wave equation with regularized penalisation should satisfy the equation

$$\ddot{u}_\epsilon - \Delta u_\epsilon - \Delta \dot{u}_\epsilon - \text{ div } (k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\nabla u_\epsilon) = f, \tag{1.5}$$

complemented with homogeneous Dirichlet boundary condition

$$u_\epsilon = 0 \text{ on } \partial\Omega. \tag{1.6}$$

Here $k_\epsilon(\cdot)$ is a suitable real valued function depending on the parameter ϵ (see (3.1) below).

Once we have stated and verified the existence of u_ϵ (Theorem 3.1), it is necessary to find suitable a-priori estimates for u_ϵ in order to pass to the limit as $\epsilon \to 0$. To prove that a limit u of u_ϵ is a solution as in Definitions 2.1 and 2.6, we have also to control the penalisation term

$$k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\nabla u_\epsilon.$$

This is shown to converge to a weak vector multiplier $\Upsilon \in \mathcal{H}'_\nabla$, where $\mathcal{H}_\nabla$ is a suitable Hilbert space (see Section 2). Extending the techniques of [3] for the elliptic problems, we show also that in the weak formulation with a more restrictive class of test functions we may replace Υ by $\lambda\nabla u$ for a charge $\lambda \in L^\infty(\Omega)'$, representing a generalised Lagrange multiplier.

These results and proofs are given in detail with a homogeneous Dirichlet boundary condition, but are, in Section 5, easily extended to Neumann and Fourier type boundary conditions (see Theorem 5.4 and Theorem 5.5). In the latter case, which depends on a positive parameter α, by using the techniques of [3] and [2], we are able to recover the solution of the Dirichlet problem as the limit $\alpha \to \infty$ (see Theorem 5.6), which corresponds to a kind of one parameter continuous dependence result for weak solutions.

2 Preliminaries and Main Results

2.1 Notation

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be an open bounded set with Lipschitz boundary having unit outer normal ν. The problem we will address is to find a real valued displacement $u = u(t,x) : (0,T) \times \Omega \to \mathbb{R}$, for a given time $T > 0$, which satisfies equation (1.1) and is subjected to the constraint (1.2). In order to specify our rigorous setting and introduce the concept of solution, we need some preliminaries. We will adopt the following notation

$$Q_T = (0,T) \times \Omega, \qquad \text{and } Q_t = (0,t) \times \Omega,$$

for any $t \in (0,T]$. Moreover we set

$$V = H_0^1(\Omega)$$

for the Dirichlet problem and

$$V = H^1(\Omega)$$

for the Neumann and Fourier problems. To shortcut the notation we denote by

$$\begin{aligned}
&\mathbb{L}^2(\Omega) := L^2(\Omega;\mathbb{R}^d), \qquad \mathbb{L}^2(\partial\Omega) := L^2(\partial\Omega;\mathbb{R}^d), \quad \mathbb{L}^\infty(\Omega) := L^\infty(\Omega;\mathbb{R}^d),\\
&\mathbb{L}^2(Q_t) := L^2(Q_t;\mathbb{R}^d), \qquad \mathbb{L}^\infty(Q_t) := L^\infty(Q_t;\mathbb{R}^d),
\end{aligned} \tag{2.1}$$

for any $t \in (0,T]$. We also introduce the symbol $\overline{L^2}(\Omega)$ to denote the subspace of $L^2(\Omega)$ consisting of function with null average, i.e. $\int_\Omega u(x)dx = 0$ if $u \in \overline{L^2}(\Omega)$. We denote by $\overline{V}$ the subspace of V consisting of functions with null mean value on Ω. We also need the following spaces

$$\begin{aligned}
&\mathcal{V} := H^1(0,T;L^2(\Omega)) \cap L^2(0,T;V),\\
&\overline{\mathcal{V}} := H^1(0,T;\overline{L^2}(\Omega)) \cap L^2(0,T;\overline{V}),\\
&\mathcal{H} := H^1(0,T;H^{-1}(\Omega;\mathbb{R}^d)) \cap L^2(0,T;\mathbb{L}^2(\Omega)),
\end{aligned} \tag{2.2}$$

and we denote by

$$\mathcal{H}_\nabla := \{F \in \mathcal{H} : F = \nabla v \text{ for some } v \in \mathcal{V}\}. \tag{2.3}$$

Notice that $\mathcal{H}_\nabla$ is a subspace of $\mathcal{H}$ and is a Hilbert space when endowed with the norm of $\mathcal{H}$. The counterparts of (2.2) and (2.3) in the case that the time T is replaced by $t \in (0,T)$, are $\mathcal{V}_t$, $\mathcal{H}_t$, and $\mathcal{H}_{\nabla,\mathrm{t}}$, respectively. Namely

$$\begin{aligned}
&\mathcal{V}_t := H^1(0,t;L^2(\Omega)) \cap L^2(0,t;V),\\
&\overline{\mathcal{V}}_t := H^1(0,t;\overline{L^2}(\Omega)) \cap L^2(0,t;\overline{V}),\\
&\mathcal{H}_t := H^1(0,t;H^{-1}(\Omega;\mathbb{R}^d)) \cap L^2(0,t;\mathbb{L}^2(\Omega)),
\end{aligned} \tag{2.4}$$

and

$$\mathcal{H}_{\nabla,\mathrm{t}} := \{F \in \mathcal{H}_t : F = \nabla v \text{ for some } v \in \mathcal{V}_t\}. \tag{2.5}$$

The scalar product in $L^2(\Omega)$ or $\mathbb{L}^2(\Omega)$ is denoted by $(\cdot,\cdot)$. The scalar product in $L^2(\partial\Omega)$ or $\mathbb{L}^2(\partial\Omega)$ is instead denoted by $(\cdot,\cdot)_{\partial\Omega}$. The duality between V' and V is noted by $\langle\cdot,\cdot\rangle$. Moreover we will need to employ the duality between $\mathcal{H}'_\nabla$ and $\mathcal{H}_\nabla$, denoted by $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. When we deal with the spaces $\mathcal{H}'_{\nabla,\mathrm{t}}$ and $\mathcal{H}_{\nabla,\mathrm{t}}$ the duality pairing is denoted by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_t$.

Eventually, we denote by $[\cdot,\cdot]$ both the duality between $\mathbb{L}^\infty(Q_T)'$ and $\mathbb{L}^\infty(Q_T)$, and the duality between $L^\infty(Q_T)'$ and $L^\infty(Q_T)$. When we work on Q_t instead of Q_T we employ the symbol $[\cdot,\cdot]_t$. We recall that an element $\sigma\in L^\infty(Q_T)'$, sometimes also called a charge, can be regarded as a finitely additive measure σ^*, with bounded total variation, which is also absolutely continuous with respect to the Lebesgue measure in Q_T and may be defined by a Radon integral

$$[\sigma,\phi]=\int_{Q_T}\phi d\sigma^*, \tag{2.6}$$

for all $\phi\in L^\infty(Q_T)$ (see [27, Chapter IV, Section 9, Example 5]).

Whenever $F\in\mathcal{H}_\nabla$ we can choose $v\in\overline{\mathcal{V}}$ (with null mean value on Ω) such that $F=\nabla v$. Thus we merely observe that the space $\mathcal{H}_\nabla$ coincides with

$$\{F\in\mathcal{H}: F=\nabla v \text{ for some } v\in\overline{\mathcal{V}}\}. \tag{2.7}$$

We invoke the following general fact which will be useful later (see [26, Proposition 1.2]). There is a constant $C>0$ depending on the domain Ω such that, for any $u\in\overline{L^2}(\Omega)$ it holds

$$\|u\|_{L^2}\le C\|\nabla u\|_{H^{-1}}. \tag{2.8}$$

This is applied to functions $F\in\mathcal{H}_\nabla$. Indeed, combining this with classical Poincaré inequality we find that

$$\|v\|_{\mathcal{V}}\le C\|F\|_{\mathcal{H}_\nabla}. \tag{2.9}$$

where $v\in\overline{\mathcal{V}}$ is such that $F=\nabla v$. We also need to introduce the space

$$X:=H^{-k-1}(\Omega), \tag{2.10}$$

where $k>1$ depends on d and is such that $L^1(\Omega)\subset\subset H^{-k}(\Omega)$ with continuous and compact embedding.

Let $g\in L^\infty(Q_T)$ be a positive function such that

$$g(x,t)\ge g_0>0 \quad \text{a.e. in } Q_T, \tag{2.11}$$

for some constant g_0. We introduce the operator J, defined for all $A\in\mathbb{L}^2(Q_T)$, as

$$J(A)=\int_{Q_T}K(|A(x,t)|^2-g(x,t)^2)dxdt=\begin{cases}0 & \text{if } |A|\le g \text{ a.e. in } Q_T,\\ +\infty & \text{otherwise,}\end{cases} \tag{2.12}$$

where

$$K(y):=\begin{cases}0 & \text{for } y\le 0,\\ +\infty & \text{for } y>0.\end{cases} \tag{2.13}$$

We set

$$\beta:=\partial J, \tag{2.14}$$

the classical subdifferential of J. To our scope, we need to consider the relaxation of the operator ∂J with respect to a weaker topology, namely we want to compute the subdifferential of J with respect to the duality between $\mathcal{H}'_\nabla$ and $\mathcal{H}_\nabla$. To this aim we first restrict J to the space $\mathcal{H}_\nabla$, and consider the restricted operator $J\llcorner_{\mathcal{H}_\nabla}$. We say that $G\in\mathcal{H}'_\nabla$ belongs to the subdifferential $\partial J\llcorner_{\mathcal{H}_\nabla}$ at $A\in\mathcal{H}_\nabla$, and we write $G\in\partial J\llcorner_{\mathcal{H}_\nabla}(A)$, if and only if, for all $B\in\mathcal{H}_\nabla$, it holds

$$J(B)-J(A)\ge\langle\!\langle G,B-A\rangle\!\rangle, \tag{2.15}$$

where we recall that $\langle\langle\cdot,\cdot\rangle\rangle$ represents the duality pairing between $\mathcal{H}'_\nabla$ and $\mathcal{H}_\nabla$. To simplify the notation we set

$$\beta_w := \partial J_{\llcorner \mathcal{H}_\nabla}.$$

For all $t \in (0,T]$ we can repeat the procedure above by defining J as in (2.12) with Q_t replacing Q_T. This will lead us to consider the the subdifferential of J restricted to the space $\mathcal{H}_{\nabla,t}$, which we denote by

$$\beta_{w,t} = \partial J_{\llcorner \mathcal{H}_{\nabla,t}}.$$

In Section 3.1 we will approximate β_w (and $\beta_{w,t}$ as well) by more regular operators.

2.2 Concept of weak solution — Dirichlet problem

We introduce first the concept of solution in the case $V = H^1_0(\Omega)$. Let $T > 0$, let $u_0, u_1 \in H^1_0(\Omega)$, and let a non-negative $g \in L^\infty(0,T;L^\infty(\Omega))$ be given.

Definition 2.1. A pair (u,Υ) with $u \in H^1(0,T;H^1_0(\Omega))$ and $\Upsilon \in \mathcal{H}'_\nabla$, is a weak solution to the constrained wave equation if the following properties hold:

(i) We have the following regularity

$$u \in W^{1,\infty}(0,T;L^2(\Omega)), \tag{2.16}$$
$$\dot u \in BV(0,T;X), \tag{2.17}$$

where X is the space introduced in (2.10). Moreover

$$\nabla u \in \mathbb{L}^\infty(Q_T), \tag{2.18}$$

and the function u accounts for the initial values

$$u(0) = u_0, \qquad \dot u(0) = u_1. \tag{2.19}$$

(ii) The following weak expression of the constrained wave equation holds

$$(\dot u(T),\varphi(T)) - (u_1,\varphi(0)) - \int_0^T (\dot u,\dot\varphi)ds + \int_0^T (\nabla u + \nabla \dot u, \nabla\varphi)ds + \langle\langle \Upsilon, \nabla\varphi\rangle\rangle = \int_0^T (f,\varphi)ds, \tag{2.20}$$

for all $\varphi \in \mathcal{V} = H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$, the function u satisfies the gradient constraint

$$|\nabla u| \le g \text{ a.e. in } \Omega, \tag{2.21}$$

and the term Υ satisfies

$$\Upsilon \in \beta_w(\nabla u). \tag{2.22}$$

In order to find a weak solution to the constrained wave equation we need to give suitable initial data. In the case of the homogeneous Dirichlet boundary condition, we shall require

$$u_0, u_1 \in V = H^1_0(\Omega) \tag{2.23}$$
$$\text{with } |\nabla u_0| \le g(0) \text{ a.e. in } \Omega. \tag{2.24}$$

We can now state our first main result:

Theorem 2.2. *Let $T > 0$, suppose $g \in W^{1,\infty}(0,T;L^\infty(\Omega))$ is such that*

$$g \ge g_0 > 0,$$

for some constant g_0, assume $f \in L^2(0,T;L^2(\Omega))$ and u_0, u_1 are as in (2.23) and (2.24). Then there exist $u \in H^1(0,T;H^1_0(\Omega))$ and $\Upsilon \in \mathcal{H}'_\nabla$ such that the pair (u,Υ) is a solution to the constrained wave equation with Dirichlet boundary condition in the sense of Definition 2.1.

As it will follow from the proof of Theorem 2.2, we may anticipate some remarks, the first one being a kind of local in time version of the weak formulation for the dynamics of the viscoelastic locked membrane.

Remark 2.3. For all $t \in (0,T)$ there exists $\Upsilon_t \in \mathcal{H}'_{\nabla,t}$ such that the following local version of the constrained wave equation holds

$$(\dot{u}(t), \varphi(t)) - (u_1, \varphi(0)) - \int_0^t (\dot{u}, \dot{\varphi}) ds + \int_0^t (\nabla u + \nabla \dot{u}, \nabla \varphi) ds + \langle\!\langle \Upsilon_t, \nabla \varphi \rangle\!\rangle_t = \int_0^t (f, \varphi) ds, \qquad (2.25)$$

for all $\varphi \in \mathcal{V}_t$. Moreover

$$\Upsilon_t \in \beta_{w,t}(\nabla u). \qquad (2.26)$$

Furthermore the reaction term Υ_t is compatible with Υ, in the sense that if $\varphi \in \mathcal{V}_t$ satisfies $\varphi(t) = 0$, then

$$\langle\!\langle \Upsilon, \widetilde{\varphi} \rangle\!\rangle = \langle\!\langle \Upsilon_t, \varphi \rangle\!\rangle_t, \qquad (2.27)$$

where $\widetilde{\varphi}$ is the extension to zero on $(t,T] \times \Omega$ of φ.

Remark 2.4. During the proof of Theorem 2.2 we will see that the reaction term Υ is the limit (as $\epsilon \to 0$) in $\mathcal{H}'_\nabla$ of a sequence

$$k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\nabla u_\epsilon \in \mathcal{H}'_\nabla, \qquad (2.28)$$

where $k_\epsilon(\cdot) : \mathbb{R} \to [0,\infty)$ is a suitable function depending on the parameter $\epsilon > 0$ (see (3.1) below). At the same time we will show that such a sequence is uniformly bounded in $\mathbb{L}^1(Q_T)$, which can be regarded as a subespace of $\mathbb{L}^\infty(Q_T)'$, and so we may consider that it also converges weakly* in this space to some $\Upsilon \in \mathbb{L}^\infty(Q_T)'$. Consequently, this Υ may also be represented by a finitely additive measure Υ^*, in the sense that

$$\langle\!\langle \Upsilon, \nabla\varphi \rangle\!\rangle = [\Upsilon, \nabla\varphi] = \int_{Q_T} \nabla\varphi \cdot d\Upsilon^*, \qquad (2.29)$$

for all $\varphi \in \mathcal{V} \cap L^\infty(0,T; W^{1,\infty}(\Omega))$.

Remark 2.5. We will also see that, likewise the sequence in (2.28), also the sequence

$$k_\epsilon(|\nabla u_\epsilon|^2 - g^2), \qquad (2.30)$$

will be uniformly bounded in $L^1(Q_T)$. This entails that there is, up to subsequences, $\lambda \in L^\infty(Q_T)'$ such that

$$k_\epsilon(|\nabla u_\epsilon|^2 - g^2) \rightharpoonup \lambda \text{ weakly* in } L^\infty(Q_T)'. \qquad (2.31)$$

The term λ plays the role of a Lagrange multiplier. The corresponding charge λ^* represents λ, in the sense that

$$[\lambda\nabla\psi, \nabla\varphi] = [\lambda, \nabla\psi \cdot \nabla\varphi] = \int_{Q_T} (\nabla\psi \cdot \nabla\varphi) d\lambda^*, \qquad (2.32)$$

for all $\varphi, \psi \in \mathcal{V} \cap L^\infty(0,T; W^{1,\infty}(\Omega))$.

In view of the previous remarks, we have to specify which is the relation between Υ and λ. This is clarified by our second main result, which is based on the new definition in terms of the displacement u and the Lagrange multiplier λ, and also provides an energy inequality for all $t \in (0,T)$.

Definition 2.6. A pair (u, λ) with $u \in H^1(0,T; H^1_0(\Omega))$, $\lambda \in L^\infty(Q_T)'$, is said to be a weak solution to the constrained wave equation if the following properties hold:

(i') Conditions (2.16), (2.17), and (2.18), hold, together with the initial condition (2.19).

(ii') The following weak expression of the wave equation holds

$$(\dot{u}(T), \varphi(T)) - (u_1, \varphi(0)) - \int_0^T (\dot{u}, \dot{\varphi}) ds + \int_0^T (\nabla u + \nabla \dot{u}, \nabla \varphi) ds + [\lambda, \nabla u \cdot \nabla \varphi] = \int_0^T (f, \varphi) ds, \tag{2.33}$$

for all $\varphi \in \mathcal{V} \cap L^\infty(0, T; W^{1,\infty}(\Omega))$. Moreover

$$|\nabla u| \leq g \text{ a.e. in } Q_T, \quad \lambda \geq 0 \quad \text{and} \quad \lambda(|\nabla u|^2 - g^2) = 0 \quad \text{in } L^\infty(Q_T)'. \tag{2.34}$$

(iii') For all $t \in (0, T)$ the following local version of the wave equation holds

$$(\dot{u}(t), \varphi(t)) - (u_1, \varphi(0)) - \int_0^t (\dot{u}, \dot{\varphi}) ds + \int_0^t (\nabla u + \nabla \dot{u}, \nabla \varphi) ds + [\lambda, \nabla u \cdot \nabla \varphi]_t = \int_0^t (f, \varphi) ds, \tag{2.35}$$

for all $\varphi \in \mathcal{V}_t \cap L^\infty(0, t; W^{1,\infty}(\Omega))$.

Remark 2.7. Let us comment on Definition 2.6. Note that λ is not defined as a distribution, but as a charge, i.e., an element in $L^\infty(Q_T)'$. Specifically, in point (iii') we have noted $[\cdot, \cdot]_t$ the duality between $L^\infty(Q_t)'$ and $L^\infty(Q_t)$. This is defined as

$$[\lambda, F]_t := [\lambda, \widetilde{F}],$$

where $\widetilde{F}$ is the extension of $F \in L^\infty(Q_t)$ to an element of $L^\infty(Q_T)$ by setting $\widetilde{F} = 0$ on $Q_T \setminus Q_t$.

As a consequence, if $\varphi \in \mathcal{V}_t \cap L^\infty(0, t; W^{1,\infty}(\Omega))$ is such that $\varphi(t) = 0$, and if $\widetilde{\varphi}$ denotes its extension to 0 on $Q_T \setminus Q_t$, then comparing (2.35) and (2.25) it is expected that

$$[\lambda, \nabla u \cdot \nabla \varphi]_t = \langle\!\langle \Upsilon_t, \varphi \rangle\!\rangle_t. \tag{2.36}$$

Nevertheless, note that the two definitions of solutions are not equivalent and in both cases the uniqueness of the solution is an open problem.

Indeed, this is the case, as it is established in our second main result, which implies that the energy of the system is not increasing if $f = 0$ and the threshold g is time independent.

Theorem 2.8. *Under the same assumptions of Theorem 2.2, there exists a weak solution (u, λ) to the constrained wave equation in the sense of Definition 2.6, which is related to a weak solution (u, Υ) in the sense of Definition 2.1 by the relation between λ and Υ given by*

$$[\lambda, \nabla u \cdot \nabla \varphi] = \langle\!\langle \Upsilon, \nabla \varphi \rangle\!\rangle, \tag{2.37}$$

for any $\varphi \in \mathcal{V} \cap L^\infty(0, T; W^{1,\infty}(\Omega))$. In addition the following energy inequality holds for a.e. $t \in (0, T]$,

$$\frac{1}{2}\|\dot{u}(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\nabla \dot{u}(s)\|_{L^2}^2 ds \leq \frac{1}{2}\|u_1\|_{L^2}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \int_0^t (f(s), \dot{u}(s)) ds - [\lambda, g\dot{g}]_t.$$

We conclude this section by observing that weak solutions provided by Theorem 2.2 and Theorem 2.8 also solve the same variational inequality version of (2.20) and of (2.33).

Remark 2.9. Let (u, Υ) (resp. (u, λ)) be a solution provided by Theorem 2.2 (resp. Theorem 2.8); then u satisfies the constraint $|\nabla u| \leq g$ a.e. in Q_T, and for all $\varphi \in \mathcal{V}$ with $|\nabla \varphi| \leq g$ a.e. in Q_T, by the definition of the subdifferential in (2.22), in the first case, and as a consequence of (2.34) in the second case, the following holds

$$(\dot{u}(T), u(T) - \varphi(T)) - \int_0^T (\dot{u}, \dot{u} - \dot{\varphi}) ds + \int_0^T (\nabla u + \nabla \dot{u}, \nabla u - \nabla \varphi) ds \leq$$
$$(u_1, u_0 - \varphi(0)) + \int_0^T (f, u - \varphi) ds. \tag{2.38}$$

3 The approximate problem

3.1 The penalisation term

Following the theory developed for elliptic and parabolic equations with unilateral contraints (see [14, Chapter 3, Section 5]) we introduce a penalisation operator in order to obtain, at the limit, a solution which satisfies the gradient contraint (see also [7], [21], [22], [23] for hyperbolic PDEs).

For any $\epsilon \in (0,1)$ we define

$$k_\epsilon(y) := \frac{1}{\epsilon}\frac{y^+}{\sqrt{y^2+1}}, \tag{3.1}$$

where $y^+ = y \vee 0$, $y \in \mathbb{R}$. The function $k_\epsilon : \mathbb{R} \to [0,+\infty)$ is continuous nondecreasing, assumes the value 0 on the set $(-\infty, 0]$, is strictly positive on $(0,+\infty)$, and bounded by $\frac{1}{\epsilon}$. As $\epsilon \searrow 0$ we have $k_\epsilon \nearrow k$, where

$$k(y) := \begin{cases} 0 & \text{for } y \le 0, \\ +\infty & \text{for } y > 0. \end{cases} \tag{3.2}$$

Let us denote by $K_\epsilon(y) := \int_0^y k_\epsilon(r)dr$, that is

$$K_\epsilon(y) = \frac{1}{\epsilon}\big(\sqrt{(y^+)^2+1} - 1\big).$$

The function K_ϵ is nonnegative and convex of class $C^1(\mathbb{R})$. As $\epsilon \searrow 0$ we have $K_\epsilon \nearrow K$, the function in (2.13) (which actually coincides with k).

Let g be the function introduced in (2.11). We define, for all $\epsilon \in (0,1)$, the operator

$$J_\epsilon(A) := \int_0^T \int_\Omega \frac{1}{2} K_\epsilon(|A(x,t)|^2 - g(x,t)^2) dxdt, \tag{3.3}$$

for any $A \in \mathbb{L}^2(Q_T)$. We say that $G \in \mathbb{L}^2(Q_T)$ belongs to the subdifferential ∂J_ϵ at $A \in \mathbb{L}^2(Q_T)$, and we write $G \in \partial J_\epsilon(A)$, if and only if for all $B \in \mathbb{L}^2(Q_T)$ we have

$$J_\epsilon(B) - J_\epsilon(A) \ge \int_{Q_T} G : (B - A) dxdt, \tag{3.4}$$

Notice that by definition of J_ϵ it turns out that

$$G \in \partial J_\epsilon(A) \quad \Rightarrow \quad G(x,t) = k_\epsilon(|A(x,t)|^2 - g(x,t)^2)A(x,t) \quad \text{for a.e. } (x,t) \in Q_T. \tag{3.5}$$

We now see how the operators J_ϵ approximate J, the operator defined in (2.12), as $\epsilon \to 0$. Note that since the operators K_ϵ are increasing as $\epsilon \searrow 0$, they converge pointwise to the limit function K defined in (2.13).

In particular J_ϵ converges pointwise to J as $\epsilon \to 0$. Now, applying [1, Theorem 3.20], we deduce that the approximate operators J_ϵ converge to J in the sense of Mosco. We set

$$\beta_\epsilon := \partial J_\epsilon, \tag{3.6}$$

the subdifferential of J_ϵ. Following the lines of [7] we conclude that the monotone operators β_ϵ are converging to β_w in the sense of graphs; namely,

$$\begin{aligned} &\forall (x,y) \text{ with } y \in \beta_w(x) \text{ there exist } (x_\epsilon, y_\epsilon) \text{ with } y_\epsilon \in \beta_\epsilon(x_\epsilon) \text{ such that} \\ &\qquad (x_\epsilon, y_\epsilon) \to (x,y) \text{ strongly in } \mathcal{H}_\nabla \times \mathcal{H}'_\nabla. \end{aligned} \tag{3.7}$$

As a consequence, if we prove that a sequence $(G_\epsilon, A_\epsilon) \in \mathcal{H}'_\nabla \times \mathcal{H}_\nabla$ with $G_\epsilon \in \beta_\epsilon(A_\epsilon)$ satisfies

$$\begin{aligned} &A_\epsilon \rightharpoonup A \text{ weakly in } \mathcal{H}_\nabla, \qquad G_\epsilon \rightharpoonup G \text{ weakly in } \mathcal{H}'_\nabla, \\ &\quad \text{and } \limsup_{\epsilon\to 0} \langle\!\langle G_\epsilon, A_\epsilon\rangle\!\rangle \le \langle\!\langle G, A\rangle\!\rangle, \end{aligned} \tag{3.8}$$

then we conclude [23, Lemma 2.4] that

$$G \in \beta_w(A). \tag{3.9}$$

3.2 The regularized problem

In this section we study the strongly damped wave equation with a regularized gradient constraint, i.e. we replace the full constraint (1.2) with a penalised version of it. More precisely, strong solutions u_ϵ to the wave equation with regularized penalisation should satisfy

$$\ddot{u}_\epsilon - \Delta u_\epsilon - \Delta \dot{u}_\epsilon - \text{ div } (k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\nabla u_\epsilon) = f. \tag{3.10}$$

We complement (3.10) with Dirichlet boundary condition

$$u_\epsilon = 0 \text{ on } \partial\Omega, \tag{3.11}$$

and initial data

$$u_\epsilon(0) = u_0, \quad \dot{u}_\epsilon(0) = u_1, \tag{3.12}$$
$$\text{and } u_0, u_1 \in V. \tag{3.13}$$

Actually, it is convenient to consider a slightly different equation than the strong formulation (3.10)-(3.11): we require that

$$\begin{aligned}(\dot{u}_\epsilon(t), \varphi(t)) - (u_1, \varphi(0)) - \int_0^t (\dot{u}_\epsilon, \dot{\varphi})ds + \int_0^t (\nabla u_\epsilon + \nabla \dot{u}_\epsilon, \nabla\varphi)ds \\ + \int_0^t (k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\nabla u_\epsilon, \nabla\varphi) = \int_0^t (f, \varphi)ds,\end{aligned} \tag{3.14}$$

for all $\varphi \in \mathcal{V}$ and $t \in (0, T]$.

For $\epsilon \in (0, 1)$, the following existence theorem for the regularized solutions holds.

Theorem 3.1. *Let $T > 0$ and let u_0, u_1 be as in* (3.12) *or* (3.13)*. Assume also $f \in L^2(0, T; L^2(\Omega))$ and $g \in W^{1,\infty}(0, T; L^\infty(\Omega))$. Then for all $\epsilon \in (0, 1)$ there exists a solution u_ϵ to* (3.14) *such that*

$$u_\epsilon \in W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; H_0^1(\Omega)), \tag{3.15}$$
$$\dot{u}_\epsilon \in H^1(0, T; H^{-1}(\Omega)). \tag{3.16}$$

Proof. We sketch the proof of Theorem 3.1, which is based on a standard time discretisation procedure: let $n \in \mathbb{N}$ be a positive integer, let $\tau := T/n$, and $t_k := k\tau$, $k = -1, 0, \ldots, n$. We define

$$u_{n,0} := u_0, \; u_{n,-1} := u_0 - \tau u_1,$$

and for all $k \geq 1$ we define recursively

$$u_{n,k} := \text{ argmin } \{F_{n,k}(u) : u \in V\}, \tag{3.17}$$

where

$$F_{n,k}(u) = \frac{1}{2}\|\frac{u - u_{n,k-1}}{\tau} - \frac{u_{n,k-1} - u_{n,k-2}}{\tau}\|^2_{L^2} + \frac{1}{2}\|\nabla u\|^2_{L^2} + \frac{\tau}{2}\|\frac{\nabla u - \nabla u_{n,k-1}}{\tau}\|^2_{L^2} + J_\epsilon(\nabla u) - (f(t_k), u),$$

which results convex and coercive. Notice the dependence of $u_{n,k}$ on ϵ. We have however dropped the label ϵ, for the reader convenience. As minimizer of $F_{n,k}$, $u_{n,k}$ satisfies the Euler-Lagrange equation

$$\begin{aligned}&\tau^{-1}(\frac{u_{n,k} - u_{n,k-1}}{\tau} - \frac{u_{n,k-1} - u_{n,k-2}}{\tau}, \varphi) + (\nabla u_{n,k}, \nabla\varphi) + (\frac{\nabla u_{n,k} - \nabla u_{n,k-1}}{\tau}, \nabla\varphi) \\ &(k_\epsilon(|\nabla u_{n,k}|^2 - g^2)\nabla u_{n,k}, \nabla\varphi) - (f(t_k), \varphi) = 0,\end{aligned} \tag{3.18}$$

for all $\varphi \in V$. Then one defines the piecewise affine interpolant $u_n : [-\tau, T] \to V$ by interpolating the values $u_{n,k}$ on the points t_k, $k = -1, \ldots, n$. Also, one set $v_{n,k} := \frac{u_{n,k} - u_{n,k-1}}{\tau}$, and define the piecewise affine function v_n by interpolating the values of $v_{n,k}$ on t_k. With the aid of the additional piecewise constant maps $\widehat{u}_n$ and $\widehat{v}_n$ (which equals $u_{n,k}$, $v_{n,k}$ on $[t_{k-1}, t_k)$, respectively), one puts

$\varphi = v_{n,k} := \frac{u_{n,k}-u_{n,k-1}}{\tau}$ in (3.18) and summing on $k = 0, \ldots, m$, $m \le n$, standard arguments allow to show the a-priori estimates

$$u_n \in H^1(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega)), \tag{3.19}$$

$$v_n \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V), \tag{3.20}$$

$$\int_\Omega K_\epsilon(|\nabla u_n|^2 - g^2)dx \in L^\infty(0,T). \tag{3.21}$$

The norms of the functions above are uniformly bounded by a constant independent of τ. On the other hand, thanks to the boundedness of k_ϵ by $\frac{1}{\epsilon}$, we infer

$$k_\epsilon(|\nabla u_n|^2 - g^2)\nabla u_n \in \mathbb{L}^2(Q_T), \tag{3.22}$$

uniformly with respect to τ. By (3.22) it follows that the operator

$$V \ni \varphi \mapsto \int_\Omega k_\epsilon(|\nabla u_n|^2 - g^2)\nabla u_n \cdot \nabla\varphi \, dx, \tag{3.23}$$

belongs to V', and hence by comparison in (3.18) we easily infer

$$\dot v_n \in L^2(0,T;V').$$

The estimates above allow to pass to the limit as $\tau \to 0$, obtaining the limiting functions u_ϵ and v_ϵ. Again, standard arguments show that the limits of u_n and $\widehat{u}_n$ coincide (likewise the limits of v_n and $\widehat{v}_n$) and also that $\dot u_\epsilon = v_\epsilon$. More precisely, we have

$$\begin{aligned}
&u_n \rightharpoonup u_\epsilon \text{ weakly in } H^1(0,T;V) \text{ and weakly star in } W^{1,\infty}(0,T;L^2(\Omega)),\\
&v_n \rightharpoonup \dot u_\epsilon \text{ weakly in } L^2(0,T;V) \cap H^1(0,T;V') \text{ and weakly star in } L^\infty(0,T;L^2(\Omega)),\\
&k_\epsilon(|\nabla u_n|^2 - g^2)\nabla u_n \rightharpoonup \Upsilon_\epsilon \text{ weakly in } \mathbb{L}^2(Q_T).
\end{aligned} \tag{3.24}$$

Moreover, since $k_\epsilon(|\nabla u_n|^2 - g^2)$ is uniformly bounded in $L^\infty(Q_T)$, we can assume that there is $\lambda_\epsilon \in L^\infty(Q_T)$ such that

$$k_\epsilon(|\nabla u_n|^2 - g^2) \rightharpoonup \lambda_\epsilon \text{ weakly star in } L^\infty(Q_T). \tag{3.25}$$

We also observe that, using Aubin-Lions Lemma, we infer

$$\begin{aligned}
&v_n(t) \to \dot u_\epsilon(t) \text{ strongly in } L^2(\Omega) \qquad \forall t \in [0,T],\\
&v_n \to \dot u_\epsilon \text{ strongly in } L^2(0,T;L^2(\Omega))
\end{aligned} \tag{3.26}$$

Now, if $\widehat{f}_n$ represents the piecewise constant interpolant of the values $f(t_k)$ on $[0,T]$, (3.18) might be written as

$$(\dot v_n(t), \varphi) + (\nabla u_n(t), \nabla\varphi) + (\nabla \dot u_n(t), \nabla\varphi) + (k_\epsilon(|\nabla u_n(t)|^2 - g^2(t))\nabla u_n(t), \nabla\varphi) = (\widehat{f}_n(t), \varphi), \tag{3.27}$$

for a.e. $t \in [0,T]$ and all $\varphi \in V$.

We can also use test functions $\varphi \in \mathcal{V}$, so that integrating (3.27) on $[0,T]$, we obtain

$$\begin{aligned}
&(v_n(T), \varphi) - (u_1, \varphi) - \int_0^T (v_n, \dot\varphi)dt + \int_0^T (\nabla u_n + \nabla \dot u_n, \nabla\varphi)dt\\
&+ \int_0^T (k_\epsilon(|\nabla u_n|^2 - g^2)\nabla u_n, \nabla\varphi)dt - \int_0^T (\widehat{f}_n, \varphi)dt = 0,
\end{aligned} \tag{3.28}$$

and passing to the limit as $\tau \to 0$ we get

$$(\dot u_\epsilon(T), \varphi) - (u_1, \varphi) - \int_0^T (\dot u_\epsilon, \dot\varphi)dt + \int_0^T (\nabla u_\epsilon + \nabla \dot u_\epsilon, \nabla\varphi)dt + \int_0^T (\Upsilon_\epsilon, \nabla\varphi)dt - \int_0^T (f, \varphi)dt = 0, \tag{3.29}$$

We have to identify Υ_ϵ. Putting $\varphi = u_n$ in (3.28) and letting $\tau \to 0$ we deduce

$$\limsup_{\epsilon\to 0} \int_0^T (k_\epsilon(|\nabla u_n|^2 - g^2)\nabla u_n, \nabla u_n)dt \le \int_0^T (f, u_\epsilon) - (\dot u_\epsilon(T), u_\epsilon(T)) + (u_1, u_0)$$
$$+ \int_0^T |\dot u_\epsilon|^2 dt - \int_0^T \int_\Omega |\nabla u_\epsilon|^2 dx dt - \frac{1}{2}\|\nabla u_\epsilon(T)\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2, \tag{3.30}$$

where we have used the convergences in (3.24) and (3.26). The right-hand side of the previous expression, by (3.29) with $\varphi = u_\epsilon$, equals $\int_0^T (\Upsilon_\epsilon, \nabla u_\epsilon)dt$, so that we infer

$$\limsup_{n\to\infty} \int_0^T (k_\epsilon(|\nabla u_n|^2 - g^2)\nabla u_n, \nabla u_n)dt \le \int_0^T (\Upsilon_\epsilon, \nabla u_\epsilon)dt,$$

which implies

$$\Upsilon_\epsilon \in \partial J_\epsilon(\nabla u_\epsilon).$$

But since K_ϵ is convex and of class C^1, from this we deduce

$$\Upsilon_\epsilon = k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\nabla u_\epsilon \qquad \text{a.e. in } Q_T.$$

In turn, from (3.29), we deduce (3.14), and the proof of Theorem 3.1 is complete. □

Note that, from (3.27), if $\varphi \in L^2(0,T;V)$ we can pass to the limit and get, equivalently,

$$\int_0^T \langle \ddot u_\epsilon, \varphi\rangle dt + \int_0^T (\nabla u_\epsilon + \nabla \dot u_\epsilon, \nabla\varphi)dt + \int_0^T (k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\nabla u_\epsilon, \nabla\varphi)dt = \int_0^T (f, \varphi)dt.$$

In addition, if we take any φ smooth and compactly supported in Q_T, we can integrate by parts the last expression and infer

$$\ddot u_\epsilon - \Delta u_\epsilon - \Delta \dot u_\epsilon - \operatorname{div}\,(k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\nabla u_\epsilon) = f, \tag{3.31}$$

as distributions in Q_T.

3.3 A priori estimates

Next, our strategy will be to consider solutions u_ϵ provided by the preceding theorem and to show that, as $\epsilon \to 0$, they converge to a weak solution of the constrained wave equation. To this aim we have first to enstablish some a-priori estimates independent of ϵ. We prove the following Lemma:

Lemma 3.2. *There is a constant $C > 0$ independent of ϵ such that for any $\epsilon \in (0,1)$ there holds*

$$\|u_\epsilon\|_{H^1(0,T;V)} \le C, \tag{3.32}$$
$$\|\dot u_\epsilon(t)\|_{L^2}^2 + \int_\Omega K_\epsilon(|\nabla u_\epsilon(t)|^2 - g(t)^2) \le C \quad \text{for all } t \in [0,T], \tag{3.33}$$
$$\|k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\|_{L^1(Q_T)} + \|k_\epsilon(|\nabla u|^2 - g^2)|\nabla u_\epsilon|\|_{L^1(Q_T)} + \|k_\epsilon(|\nabla u_\epsilon|^2 - g^2)|\nabla u_\epsilon|^2\|_{L^1(Q_T)} \le C, \tag{3.34}$$
$$\|k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\nabla u_\epsilon\|_{\mathcal{H}'_\nabla} \le C, \tag{3.35}$$
$$\|\ddot u_\epsilon\|_{L^1(0,T;X)} \le C, \tag{3.36}$$

where $X := H^{-k-1}(\Omega)$ and $k = k(d) \in \mathbb{N}$ is such that $L^1(\Omega;\mathbb{R}^d) \subset\subset H^{-k}(\Omega;\mathbb{R}^d)$ continuously and compactly.

Proof. In the next computations C represents a positive constant which might change from line to line. In order to shortcut the notation we will denote

$$\widehat K_\epsilon(t) := \frac{1}{2} K_\epsilon(|\nabla u_\epsilon(t)|^2 - g(t)^2), \quad \widehat k_\epsilon(t) := k_\epsilon(|\nabla u_\epsilon(t)|^2 - g(t)^2).$$

Step 1. Testing equation (3.14) by $\varphi = \dot{u}_\epsilon$ we get

$$\frac{1}{2}\|\dot{u}_\epsilon(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla u_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla \dot{u}_\epsilon(s)\|_{L^2}^2 ds + \int_\Omega \widehat{K}_\epsilon(t)dx$$
$$= \frac{1}{2}\|u_1\|_{L^2}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \int_\Omega \widehat{K}_\epsilon(0)dx + \int_0^t (f(s), \dot{u}_\epsilon(s))ds - \int_0^t \int_\Omega \widehat{k}_\epsilon(s)g(s)\dot{g}(s)dxds$$
$$\leq C + C\int_0^t \|\dot{u}_\epsilon(s)\|_{L^2}^2 ds + C\int_0^t \int_\Omega \widehat{k}_\epsilon(s)dxds. \tag{3.37}$$

Moreover, using that $g \geq g_0 > 0$,

$$\int_0^t \int_\Omega \widehat{k}_\epsilon(s)dxds \leq \frac{1}{g_0^2}\int_0^t \int_\Omega \widehat{k}_\epsilon(s)g(s)^2 dxds$$
$$= \frac{1}{g_0^2}\int_0^t \int_\Omega \widehat{k}_\epsilon(s)(g(s)^2 - |\nabla u_\epsilon(s)|^2)dxds + \frac{1}{g_0^2}\int_0^t \int_\Omega \widehat{k}_\epsilon(s)|\nabla u_\epsilon(s)|^2 dxds, \tag{3.38}$$

and setting $A_\epsilon := \{(x,s) \in Q_T : |\nabla u_\epsilon(x,s)|^2 - g(x,s)^2 \geq 0\}$, we also have

$$\frac{1}{g_0^2}\int_0^t \int_\Omega \widehat{k}_\epsilon(s)(g(s)^2 - |\nabla u_\epsilon(s)|^2)dxds = \frac{1}{g_0^2}\int_{A_\epsilon \cap Q_t} \widehat{k}_\epsilon(s)(g(s)^2 - |\nabla u_\epsilon(s)|^2)dxds \leq 0. \tag{3.39}$$

Therefore, from (3.37), we arrive at

$$\frac{1}{2}\|\dot{u}_\epsilon(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla u_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla \dot{u}_\epsilon(s)\|_{L^2}^2 ds + \int_\Omega \widehat{K}_\epsilon(t)dx$$
$$\leq C + C\int_0^t \|\dot{u}_\epsilon(s)\|_{L^2}^2 ds + C\int_0^t \int_\Omega \widehat{k}_\epsilon(s)|\nabla u_\epsilon(s)|^2 dsdx. \tag{3.40}$$

Testing the wave equation (3.14) with $\varphi = u_\epsilon$ instead we obtain

$$\int_0^t \int_\Omega \widehat{k}_\epsilon(s)|\nabla u_\epsilon(s)|^2 dxds = \int_0^t (f(s), u_\epsilon(s))ds + \int_0^t \|\dot{u}_\epsilon(s)\|_{L^2}^2 ds$$
$$- \int_0^t \|\nabla u_\epsilon(s)\|_{L^2}^2 ds - \frac{1}{2}\|\nabla u_\epsilon(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + (u_1, u_0) - (\dot{u}_\epsilon(t), u_\epsilon(t))$$
$$\leq C + C\int_0^t \|\dot{u}_\epsilon(s)\|_{L^2}^2 ds + \frac{\gamma}{2}\|\dot{u}_\epsilon(t)\|_{L^2}^2 + \frac{1}{2\gamma}\|u_\epsilon(t)\|_{L^2}^2, \tag{3.41}$$

where the constant $\gamma > 0$ is arbitrary, and by writing

$$\|u_\epsilon(t)\|_{L^2}^2 = \int_\Omega \big|u_0 + \int_0^t \dot{u}_\epsilon(s)ds\big|^2 dx \leq C + 2\int_\Omega (\int_0^t |\dot{u}_\epsilon(s)|ds)^2 dx$$
$$\leq C + 2\int_\Omega \big|T\int_0^t |\dot{u}_\epsilon(s)|^2 ds\big|dx \leq C + C\int_0^t \|\dot{u}_\epsilon(s)\|_{L^2}^2 ds, \tag{3.42}$$

we have estimated the term

$$\int_0^t (f(s), u_\epsilon(s))ds \leq C\int_0^t \|u_\epsilon(s)\|_{L^2} ds \leq C + C\int_0^t \|\dot{u}_\epsilon(s)\|_{L^2}^2 ds. \tag{3.43}$$

The constant C appearing in the previous estimates is independent of ϵ and depends on the external force f and the initial conditions u_0 and u_1. So, plugging (3.41) into (3.40) we infer

$$\frac{1}{2}\|\dot{u}_\epsilon(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla u_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla \dot{u}_\epsilon(s)\|_{L^2}^2 ds + \int_\Omega \widehat{K}_\epsilon(t)dx$$
$$\leq C + C\int_0^t \|\dot{u}_\epsilon(s)\|_{L^2}^2 ds + \frac{C\gamma}{2}\|\dot{u}_\epsilon(t)\|_{L^2}^2 + \frac{C}{2\gamma}\|u_\epsilon(t)\|_{L^2}^2. \tag{3.44}$$

Thus, after choosing $\gamma > 0$ small enough in (3.44), and using (3.42) again, we finally find a constant $C > 0$ independent of ϵ such that

$$C'\|\dot{u}_\epsilon(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla u_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla \dot{u}_\epsilon(s)\|_{L^2}^2 ds + \int_\Omega \widehat{K}_\epsilon(t)dx \leq C + C\int_0^t \|\dot{u}_\epsilon(s)\|_{L^2}^2 ds$$

where $C' > 0$ is a fixed constant independent of ϵ. This allows us to employ Gronwall Lemma providing the following estimates

$$u_\epsilon \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;V), \tag{3.45}$$

$$\int_\Omega \widehat{K}_\epsilon(\cdot)dx \in L^\infty(0,T) \tag{3.46}$$

uniformly with respect to $\epsilon \in (0,1)$. Furthermore, going back to (3.37) and (3.38), we also obtain

$$\widehat{k}_\epsilon \in L^1(Q_T), \tag{3.47}$$

$$\widehat{k}_\epsilon |\nabla u_\epsilon|^2 \in L^1(Q_T), \tag{3.48}$$

uniformly with respect to $\epsilon \in (0,1)$. From this and the inequality $2|\nabla u_\epsilon| \leq 1 + |\nabla u_\epsilon|^2$ we also get

$$\widehat{k}_\epsilon |\nabla u_\epsilon| \in L^1(Q_T), \tag{3.49}$$

uniformly with respect to $\epsilon \in (0,1)$. The first a-priori estimates are achieved.

Step 2. We will now prove that $\widehat{k}_\epsilon \nabla u_\epsilon \in \mathcal{H}'_\nabla$ uniformly with respect to ϵ. Indeed, let $F \in \mathcal{H}_\nabla$. We know there is $v \in \overline{\mathcal{V}}$ such that $F = \nabla v$, and by equation (3.14),

$$\begin{aligned}
&\langle\langle \widehat{k}_\epsilon \nabla u_\epsilon, F \rangle\rangle = \langle\langle \widehat{k}_\epsilon \nabla u_\epsilon, \nabla v \rangle\rangle = -(\dot{u}_\epsilon(T), v(T)) - (u_1, v(0)) + \int_0^T (\dot{u}_\epsilon, \dot{v})ds \\
&- \int_0^T (\nabla u_\epsilon + \nabla \dot{u}_\epsilon, \nabla v)ds + \int_0^t (f, v)ds \leq \\
&\leq \|\dot{u}_\epsilon(T)\|_{L^2}\|v(T)\|_{L^2} + C\|v(0)\|_{L^2} + \|\dot{u}_\epsilon\|_{L^2(L^2)}\|\dot{v}\|_{L^2(L^2)} + \|\nabla u_\epsilon\|_{H^1(L^2)}\|\nabla v\|_{L^2(L^2)} + C\|v\|_{L^2(L^2)} \\
&\leq C\|v\|_{\mathcal{V}} \leq \|F\|_{\mathcal{H}_\nabla},
\end{aligned} \tag{3.50}$$

where we have employed the uniform boundedness (3.45) and inequality (2.9). Hence

$$\widehat{k}_\epsilon \nabla u_\epsilon \in \mathcal{H}'_\nabla, \tag{3.51}$$

uniformly with respect to $\epsilon \in (0,1)$.

Step 3. Depending on the dimension d, there exists $k > 0$ such that $L^1(\Omega;\mathbb{R}^d) \subset\subset H^{-k}(\Omega;\mathbb{R}^d)$ continuously and compactly. We hence deduce that, if $h \in L^1(\Omega;\mathbb{R}^d)$, then div $h \in X$ where $X = H^{-k-1}(\Omega)$. In particular, we can now look at equation (3.31), and arguing by comparison, we infer

$$\ddot{u}_\epsilon \in L^1(0,T;X),$$

uniformly with respect to $\epsilon \in (0,1)$. □

4 Proof of the main results

We divide the proof of Theorem 2.2 in several steps.

4.1 Passage to the limit as $\epsilon \to 0$

In this Section we consider the solutions u_ϵ of the approximate problem and aim to pass to the limit as $\epsilon \to 0$. Notice that the uniform a-priori estimates provided in the previous section imply the following inclusion

$$\dot{u}_\epsilon \in L^2(0,T;V) \cap W^{1,1}(0,T;X), \tag{4.1}$$

uniformly with respect to $\epsilon \in (0,1)$. Estimate (4.1) entails that the sequence $\dot{u}_\epsilon$ is precompact in $L^2(0,T;L^2(\Omega))$ (this is a standard genealization of Aubin-Lions Lemma, see [20,24]). Moreover, since $\dot{u}_\epsilon \in BV(0,T;X)$ we conclude that, when we extract a suitable subsequence of $\epsilon \to 0$, thanks to a generalized Helly selection principle, the functions u_ϵ are converging pointwise for all $t \in [0,T]$ weakly in X.

We now extract a subsequence of $\epsilon \to 0$ such that, besides the previous convergence, also the following holds: there is a function $u \in H^1(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega))$ with $\dot{u} \in BV(0,T;X)$, and there is $\Upsilon \in \mathcal{H}'_\nabla$, such that

$$u_\epsilon \rightharpoonup u \text{ weakly in } H^1(0,T;V) \text{ and weakly star in } W^{1,\infty}(0,T;L^2(\Omega)), \tag{4.2}$$

$$\dot{u}_\epsilon \rightharpoonup \dot{u} \text{ weakly star in } BV(0,T;X), \tag{4.3}$$

$$\dot{u}_\epsilon \to \dot{u} \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{4.4}$$

$$\dot{u}_\epsilon(t) \rightharpoonup \dot{u}(t) \text{ weakly in } X \text{ for all } t \in [0,T], \tag{4.5}$$

$$\widehat{k}_\epsilon \nabla u_\epsilon \rightharpoonup \Upsilon \text{ weakly in } \mathcal{H}'_\nabla. \tag{4.6}$$

Moreover, for the same subsequence, we can assume there is some $\lambda \in L^\infty(Q_T)'$ with

$$\widehat{k}_\epsilon \rightharpoonup \lambda \text{ weakly star in } L^\infty(Q_T)'. \tag{4.7}$$

We are ready to prove the following:

Lemma 4.1. *The couple (u, Υ) satisfies*

$$(\dot{u}(T), \varphi(T)) - (u_1, \varphi(0)) - \int_0^T (\dot{u}, \dot{\varphi}) ds + \int_0^T (\nabla u + \nabla \dot{u}, \nabla \varphi) ds + \langle\langle \Upsilon, \nabla \varphi \rangle\rangle = \int_0^T (f, \varphi) ds, \tag{4.8}$$

for all $\varphi \in \mathcal{V}$. Moreover the limit function u satisfies

$$|\nabla u| \le g \text{ a.e. on } Q_T, \tag{4.9}$$

and it holds

$$\Upsilon \in \beta_w(\nabla u). \tag{4.10}$$

Proof. For all $t \in [0,T]$ we know that $\dot{u}_\epsilon(t)$ is uniformly bounded in $L^2(\Omega)$. In particular, convergence (4.5) implies that

$$\dot{u}_\epsilon(t) \rightharpoonup \dot{u}(t) \text{ weakly in } L^2(\Omega) \text{ for all } t \in [0,T]. \tag{4.11}$$

Using the definition of K_ϵ, estimate (3.46) says that there exists a constant $C > 0$ independent of ϵ such that

$$\int_0^T \int_\Omega \sqrt{\left((|\nabla u_\epsilon|^2 - g^2)^+\right)^2 + 1} - 1 \, dxdt \le \epsilon C. \tag{4.12}$$

Therefore the integrand is tending to 0 in $L^1(Q_T)$; up to subsequences,

$$\left(|\nabla u_\epsilon|^2 - g^2\right)^+ \to 0 \text{ a.e. on } Q_T. \tag{4.13}$$

Using that $K_\epsilon(y)$ has linear growth and is greater than $\frac{1}{\epsilon}(y-1)$, the dominated convergence theorem implies

$$\left(|\nabla u_\epsilon|^2 - g^2\right)^+ \to 0 \text{ strongly in } L^1(Q_T). \tag{4.14}$$

Standard lower-semicontinuity results give

$$\int_0^T \int_\Omega \left(|\nabla u|^2 - g^2\right)^+ dxdt \le \liminf_{\epsilon \to 0} \int_0^T \int_\Omega \left(|\nabla u_\epsilon|^2 - g^2\right)^+ dxdt = 0, \tag{4.15}$$

and therefore

$$|\nabla u| \leq g \text{ a.e. on } Q_T.$$

Now, the convergences (4.2)-(4.7), and (4.11) are sufficient to pass to the limit in the weak equation (3.14), entailing (4.8) for all $\varphi \in \mathcal{V}$. For all $\epsilon \in (0,1)$ we know that

$$\begin{aligned} \langle\langle \widehat{k}_\epsilon \nabla u_\epsilon, \nabla u_\epsilon \rangle\rangle &= -(\dot{u}_\epsilon(T), u_\epsilon(T)) + (u_1, u_0) + \int_0^T \|\dot{u}_\epsilon\|_{L^2}^2 ds \\ &- \int_0^T \|\nabla u_\epsilon(s)\|_{L^2}^2 ds - \frac{1}{2}\|\nabla u_\epsilon(T)\|_{L^2}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \int_0^T (f, u_\epsilon) ds, \end{aligned} \tag{4.16}$$

and taking the limsup as $\epsilon \to 0$ we get

$$\begin{aligned} \limsup_{\epsilon\to 0} \ \langle\langle \widehat{k}_\epsilon \nabla u_\epsilon, \nabla u_\epsilon \rangle\rangle &\leq -(\dot{u}(T), u(T)) + (u_1, u_0) + \int_0^T \|\dot{u}\|_{L^2}^2 ds - \int_0^T \|\nabla u(s)\|_{L^2}^2 ds \\ &- \frac{1}{2}\|\nabla u(T)\|_{L^2}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \int_0^T (f, u) ds = \langle\langle \Upsilon, \nabla u \rangle\rangle. \end{aligned} \tag{4.17}$$

The equality follows from (4.8) whereas, to obtain the inequality, we have exploited convergences (4.2)-(4.7), (4.11), and the standard lower semicontinuity property of the L^2-norms. This implies that

$$\Upsilon \in \beta_w(\nabla u).$$

□

Lemma 4.2. *Let λ be as in* (4.7)*, and let (u, Υ) be as in Lemma 4.1. Then λ satisfies* (2.34)*, and it is related with Υ by the condition*

$$[\lambda, \nabla u \cdot \nabla \varphi] = \langle\langle \Upsilon, \nabla \varphi \rangle\rangle \qquad \text{for all } \varphi \in \mathcal{V} \cap L^\infty(0,T; W^{1,\infty}(\Omega)). \tag{4.18}$$

Moreover

$$(\dot{u}(T), \varphi(T)) - (u_1, \varphi(0)) - \int_0^T (\dot{u}, \dot{\varphi}) ds + \int_0^T (\nabla u + \nabla \dot{u}, \nabla \varphi) ds + [\lambda, \nabla u \cdot \nabla \varphi] = \int_0^T (f, \varphi) ds, \tag{4.19}$$

for all $\varphi \in \mathcal{V} \cap L^\infty(0,T; W^{1,\infty}(\Omega))$.

Proof. We begin by observing that $\lambda \geq 0$ thanks to $\widehat{k}_\epsilon \geq 0$. Let $\varphi \in \mathcal{V} \cap L^\infty(0,T; W^{1,\infty}(\Omega))$, let us prove that

$$[\lambda, \nabla u \cdot \nabla \varphi] = \langle\langle \Upsilon, \nabla \varphi \rangle\rangle. \tag{4.20}$$

To prove (4.20) we observe that on the one hand

$$\lim_{\epsilon\to 0} \int_{Q_T} \widehat{k}_\epsilon \nabla u \cdot \nabla \varphi \, dxdt \to [\lambda, \nabla u \cdot \nabla \varphi],$$

so it suffices to show that in fact

$$\lim_{\epsilon\to 0} \int_{Q_T} \widehat{k}_\epsilon \nabla u \cdot \nabla \varphi \, dxdt \to \langle\langle \Upsilon, \nabla \varphi \rangle\rangle.$$

We write

$$\int_{Q_T} \widehat{k}_\epsilon \nabla u \cdot \nabla \varphi \, dxdt = \int_{Q_T} \widehat{k}_\epsilon \nabla u_\epsilon \cdot \nabla \varphi \, dxdt + \int_{Q_T} \widehat{k}_\epsilon (\nabla u - \nabla u_\epsilon) \cdot \nabla \varphi \, dxdt, \tag{4.21}$$

and since the first term in the right-hand side tends to $\langle\langle \Upsilon, \nabla \varphi \rangle\rangle$ we are left to prove that

$$\int_{Q_T} \widehat{k}_\epsilon (\nabla u - \nabla u_\epsilon) \cdot \nabla \varphi \, dxdt \to 0, \tag{4.22}$$

as $\epsilon \to 0$. We claim that

$$\int_{Q_T} \widehat{k}_\epsilon |\nabla u - \nabla u_\epsilon|^2 dxdt \to 0 \qquad \text{as } \epsilon \to 0. \tag{4.23}$$

From this it follows that

$$\int_{Q_T} \widehat{k}_\epsilon (\nabla u - \nabla u_\epsilon) \cdot \nabla \varphi \, dxdt \le \|\widehat{k}_\epsilon\|_{L^1(Q_T)} \|\widehat{k}_\epsilon^{1/2} |\nabla u - \nabla u_\epsilon|\|_{L^2(Q_T)} \|\nabla \varphi\|_{L^\infty(Q_T)} \to 0,$$

and (4.22) is proved. To show (4.23) we write

$$\begin{aligned} 0 \le \int_{Q_T} \widehat{k}_\epsilon |\nabla u - \nabla u_\epsilon|^2 dxdt &= \int_0^T (\widehat{k}_\epsilon (\nabla u_\epsilon - \nabla u), \nabla u_\epsilon - \nabla u) dt \\ &= \langle\!\langle \widehat{k}_\epsilon \nabla u_\epsilon, \nabla u_\epsilon \rangle\!\rangle - 2 \int_0^T (\widehat{k}_\epsilon \nabla u_\epsilon, \nabla u) dt + \int_0^T (\widehat{k}_\epsilon \nabla u, \nabla u) dt, \end{aligned} \tag{4.24}$$

so that passing to the limit as $\epsilon \to 0$, using (4.17), and the fact that the last term equals $[\widehat{k}_\epsilon, |\nabla u|^2]$, we infer

$$[\lambda, |\nabla u|^2] \ge \langle\!\langle \Upsilon, \nabla u \rangle\!\rangle. \tag{4.25}$$

We are left with proving the opposite inequality. To this aim, since $|\nabla u| \le g$ a.e. on Ω and $\lambda \ge 0$, we have

$$\begin{aligned} [\lambda, |\nabla u|^2] \le [\lambda, g^2] = \lim_{\epsilon \to 0} \int_{Q_T} \widehat{k}_\epsilon g^2 dxdt &\le \liminf_{\epsilon \to 0} \int_{Q_T} \widehat{k}_\epsilon |\nabla u_\epsilon|^2 dxdt \\ &\le \limsup_{\epsilon \to 0} \int_{Q_T} \widehat{k}_\epsilon |\nabla u_\epsilon|^2 dxdt \le \langle\!\langle \Upsilon, \nabla u \rangle\!\rangle. \end{aligned} \tag{4.26}$$

In the second inequality we have used that $\int_{Q_T} \widehat{k}_\epsilon (|\nabla u_\epsilon|^2 - g^2) dxdt \ge 0$, and (4.17) in the last inequality. In particular we infer

$$[\lambda, |\nabla u|^2] = \langle\!\langle \Upsilon, \nabla u \rangle\!\rangle, \tag{4.27}$$

and from (4.26) we also get

$$\lim_{\epsilon \to 0} \int_{Q_T} \widehat{k}_\epsilon (|\nabla u_\epsilon|^2 - g^2) dxdt = 0. \tag{4.28}$$

As a consequence, by writing

$$\frac{1}{\epsilon} \frac{y^2}{\sqrt{y^2+1}} = \frac{1}{\epsilon} (\sqrt{y^2+1} - \frac{1}{\sqrt{y^2+1}}) \ge \frac{1}{\epsilon} (\sqrt{y^2+1} - 1),$$

we infer from (4.28) that

$$\lim_{\epsilon \to 0} J_\epsilon (\nabla u_\epsilon) = 0. \tag{4.29}$$

This observation will be crucial in the proof of Theorem 2.8.

Going back to (4.24) we again pass to the limit and see that the right-hand side tends to zero, concluding (4.23). This also concludes the proof of (4.20).

It remains to show the last condition in (2.34). On the one hand we have that $\widehat{k}_\epsilon \ge 0$, and since $|\nabla u|^2 \le g^2$ it follows that

$$[\lambda, |\nabla u|^2 - g^2] = \lim_{\epsilon \to 0} \int_{Q_T} \widehat{k}_\epsilon (|\nabla u|^2 - g^2) dxdt \le 0.$$

Let us prove the opposite inequality. We know that $\int_{Q_T} \widehat{k}_\epsilon (|\nabla u_\epsilon|^2 - g^2) dxdt = 0$ by (4.28), and so

$$[\lambda, |\nabla u|^2 - g^2] \ge \limsup_{\epsilon \to 0} \int_{Q_T} \widehat{k}_\epsilon (|\nabla u_\epsilon|^2 - g^2) dxdt = 0,$$

where the first inequality follows from the fact that, thanks to (4.16), (4.17), and (4.27), we have

$$\limsup_{\epsilon\to 0}\int_{Q_T}\widehat{k}_\epsilon|\nabla u_\epsilon|^2dxdt = \limsup_{\epsilon\to 0}\langle\langle\widehat{k}_\epsilon\nabla u_\epsilon,\nabla u_\epsilon\rangle\rangle \leq \langle\langle\Upsilon,\nabla u\rangle\rangle = [\lambda,|\nabla u|^2].$$

It then follows that

$$[\lambda,|\nabla u|^2-g^2]=0, \tag{4.30}$$

which entails the last condition in (2.34), being $|\nabla u|+g\geq g_0>0$. To conclude (2.34) we have to prove that for all $\zeta\in L^\infty(Q_T)$ it holds

$$[\lambda(|\nabla u|^2-g^2),\zeta]=0. \tag{4.31}$$

To do this we use a Hölder inequality for charges (which may be obtained using Young inequality similarly to the Hölder inequality in L^p spaces and using Radon integral representation (2.6)) to write

$$\begin{aligned}
0 &\leq \big|[\lambda(g-|\nabla u|),\zeta]\big| \leq [\lambda(g-|\nabla u|),|\zeta|] \leq \big[\lambda(g^2-|\nabla u|^2),\frac{|\zeta|}{g+|\nabla u|}\big]\\
&\leq [\lambda(g^2-|\nabla u|^2),1]^{\frac12}\big[\lambda(g^2-|\nabla u|^2),\frac{|\zeta|^2}{(g+|\nabla u|)^2}\big]^{\frac12}\\
&= [\lambda,(g^2-|\nabla u|^2)]^{\frac12}\big[\lambda(g^2-|\nabla u|^2),\frac{|\zeta|^2}{(g+|\nabla u|)^2}\big]^{\frac12}=0,
\end{aligned} \tag{4.32}$$

where we have used that $0<g_0\leq g$, $g\geq|\nabla u|$ a.e. in Q_T, and (4.30). Lemma 4.2 is achieved. □

We further comment on some consequence of the previous proof. Passing to the liminf in (4.24) we can also conclude

$$\liminf_{\epsilon\to 0}\langle\langle\widehat{k}_\epsilon\nabla u_\epsilon,\nabla u_\epsilon\rangle\rangle \geq \langle\langle\Upsilon,\nabla u\rangle\rangle, \tag{4.33}$$

and this, together with (4.17) gives

$$\lim_{\epsilon\to 0}\langle\langle\widehat{k}_\epsilon\nabla u_\epsilon,\nabla u_\epsilon\rangle\rangle = \langle\langle\Upsilon,\nabla u\rangle\rangle. \tag{4.34}$$

In particular the inequality in (4.17) is an equality and using (4.16) we infer the following strong convergences

$$\int_0^T\|\nabla u_\epsilon(s)\|_{L^2}^2ds\to\int_0^T\|\nabla u(s)\|_{L^2}^2ds,$$

and

$$\|\nabla u_\epsilon(t)\|_{L^2}^2\to\|\nabla u(t)\|_{L^2}^2 \text{ for a.e. } t\in[0,T] \text{ and for } t=T.$$

That is

$$u_\epsilon\to u \text{ strongly in } L^2(0,T;V), \tag{4.35}$$

$$u_\epsilon(T)\to u(T) \text{ strongly in } V. \tag{4.36}$$

With this properties at disposal we are ready to conclude the proof of Theorem 2.2.

4.2 Proofs of Theorem 2.2 and Theorem 2.8

Proof of Theorem 2.2. We have to show that the pair (u,Υ) is a solution in the sense of Definition 2.1. Conditions (i) and (ii) are readily achieved by the results obtained in Lemma 4.1. Indeed (2.16) and (2.17) are obtained by (4.2) and (4.3), whereas, (2.18) follows from (4.9) and the initial data are satisfied by (4.2) and (4.11). To check point (ii) of Definition 2.1 we just invoke Lemma 4.1. Theorem 2.2 is proved. □

Let us comment on the local version of the weak equation (2.25). As a first observation, all the estimates above valid for the time interval $[0,T]$ are easily seen to be true on every subinterval $[0,t]$, $0<t<T$. Recalling the notation in (2.4) for the spaces $\mathcal{V}_t$, $\mathcal{H}_t$, and $\mathcal{H}_{\nabla,\mathrm{t}}$, we follow the lines of [7] from which it is easy to see that for all $t\in(0,T)$ there exists $\Upsilon_t\in\mathcal{H}'_{\nabla,\mathrm{t}}$ such that, for the same subsequence of the convergences (4.2)-(4.11), it holds

$$\widehat{k}_\epsilon \nabla u_\epsilon \llcorner_{Q_t} \rightharpoonup \Upsilon_t \text{ weakly in } \mathcal{H}'_{\nabla,\mathrm{t}}, \tag{4.37}$$

and

$$\Upsilon_t \in \beta_{w,t}(\nabla u), \tag{4.38}$$

where $\beta_{w,t}$ is the subdifferential of the functional J_t, obtained from J by integrating K over Q_t instead of Q_T. Namely,

$$J_t=\lim_{\epsilon\to 0} J_t^\epsilon, \qquad J_{\epsilon,t}(A):=\int_0^t\int_\Omega \frac{1}{2}K_\epsilon(|A(x,s)|^2-g(x,s)^2)dxds.$$

More precisely, we employ the wave equation (3.14) which is valid for any $t\in(0,T]$. Then we write

$$\langle\!\langle \widehat{k}_\epsilon\nabla u_\epsilon,\nabla\varphi\rangle\!\rangle_t=-(\dot u_\epsilon(t),\varphi(t))+(u_1,\varphi(0))+\int_0^t(\dot u_\epsilon,\dot\varphi)ds-\int_0^t(\nabla u_\epsilon+\nabla\dot u_\epsilon,\nabla\varphi)ds+\int_0^t(f,\varphi)ds,$$

and exploiting the convergences obtained so far we deduce that, for the same subsequence of $\epsilon\to 0$, the right-hand side tends to

$$-(\dot u(t),\varphi(t))+(u_1,\varphi(0))+\int_0^t(\dot u,\dot\varphi)ds-\int_0^t(\nabla u+\nabla\dot u,\nabla\varphi)ds+\int_0^t(f,\varphi)ds=:\langle\!\langle\Upsilon_t,\varphi\rangle\!\rangle_t.$$

The argument in (4.16) and (4.17) can be repeated achieving (4.38). From the last expression also equation (2.27) becomes evident. As for the case $t=T$ we pass to the limit in the weak equation (3.14) and (2.25) is inferred.

We are ready to prove Theorem 2.8.

Proof of Theorem 2.8. We have already proved the existence of λ in (4.7). Relation (2.37) follows from (4.18). We have to show that the pair (u,λ) is a weak solution in the sense of Definition 2.6. Condition (i') and (2.33) follows from Theorem 2.2 and relation (2.37). The conditions in (2.34) follow from Lemma 4.2, while condition (iii') follows from Lemma 4.2 and arguing as before for (2.25) (see Remark 2.3).

It remains to prove the energy inequality. From (4.29), we know that for a.e. $t\in[0,T]$ we have

$$\int_\Omega \widehat{K}_\epsilon(t)dx\to 0, \tag{4.39}$$

as $\epsilon\to 0$. Using (4.35)-(4.36) we pass to the limit the energy equality

$$\begin{aligned}&\frac{1}{2}\|\dot u_\epsilon(t)\|_{L^2}^2+\frac{1}{2}\|\nabla u_\epsilon(t)\|_{L^2}^2+\int_0^t\|\nabla\dot u_\epsilon(s)\|_{L^2}^2ds+\int_\Omega\widehat{K}_\epsilon(t)dx\\&=\frac{1}{2}\|u_0\|_{L^2}^2+\frac{1}{2}\|\nabla u_0\|_{L^2}^2+\int_0^t(f(s),\dot u_\epsilon(s))ds-\int_0^t\int_\Omega\widehat{k}_\epsilon(s)g(s)\dot g(s)dxds,\end{aligned}\tag{4.40}$$

arriving to

$$\begin{aligned}&\frac{1}{2}\|\dot u(t)\|_{L^2}^2+\frac{1}{2}\|\nabla u(t)\|_{L^2}^2+\int_0^t\|\nabla\dot u(s)\|_{L^2}^2ds\le\frac{1}{2}\|u_0\|_{L^2}^2+\frac{1}{2}\|\nabla u_0\|_{L^2}^2\\&+\int_0^t(f(s),\dot u(s))ds-\lim_{\epsilon\to0}\int_0^t\int_\Omega\widehat{k}_\epsilon(s)g(s)\dot g(s)dxds,\end{aligned}\tag{4.41}$$

for a.e. $t\in[0,T]$. The conclusion follows from the fact that, by (4.7), we have

$$\lim_{\epsilon\to0}\int_0^t\int_\Omega\widehat{k}_\epsilon(s)g(s)\dot g(s)dxds=[\lambda,g\dot g]_t.$$

□

5 Extension to Neumann type boundary conditions

In this section we aim to extend the previous results to a more general boundary condition for u. Specifically, for given $\alpha \in [0, +\infty)$ we would like to impose a Fourier boundary condition formally of the type

$$\alpha u + \frac{\partial u}{\partial \nu} + \frac{\partial \dot{u}}{\partial \nu} + \Upsilon \cdot \nu = 0 \text{ on } \partial\Omega, \tag{5.1}$$

if we had enough regularity on Υ. The homogeneous Neumann condition is inferred when $\alpha = 0$ whereas we can consider homogeneous Dirichlet condition setting formally $\alpha = +\infty$.

Moreover, in this section, we will redefine some notation; precisely we denote

$$V := H^1(\Omega).$$

Accordingly, all the notation in (2.2)-(2.5) are now redefined with this convention. The new hypothesis on the initial data will be

$$u(0) = u_0, \qquad \dot{u}(0) = u_1, \qquad u_0, u_1 \in H^1(\Omega). \tag{5.2}$$

We will look for solutions satisfying condition (5.1) in a weak sense. More precisely, we start from an approximate problem solution u_ϵ which satisfies (5.1) up to an error due to the presence of the penalisation term. The weak equation we are concerned with is the following

$$\begin{aligned}(\dot{u}_\epsilon(t), \varphi(t)) - (u_1, \varphi(0)) - \int_0^t (\dot{u}_\epsilon, \dot{\varphi}) ds + \int_0^t (\nabla u_\epsilon + \nabla \dot{u}_\epsilon, \nabla\varphi) ds \\ + \int_0^t \alpha (u_\epsilon, \varphi)_{\partial\Omega} ds + \int_0^t (k_\epsilon(|\nabla u_\epsilon|^2 - g^2)\nabla u_\epsilon, \nabla\varphi) ds = \int_0^t (f, \varphi) ds,\end{aligned} \tag{5.3}$$

for all $\varphi \in \mathcal{V}$ and $t \in (0, T]$. We can state the analogous of Theorem 3.1:

Theorem 5.1. *Let $T > 0$, let u_0, u_1 be as in* (5.2), *assume $f \in L^2(0, T; L^2(\Omega))$ and $g \in W^{1,\infty}(0, T; L^\infty(\Omega))$. Then for all $\epsilon \in (0, 1)$ there exists a solution u_ϵ to* (5.3) *with*

$$u_\epsilon \in W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)), \tag{5.4}$$

$$\dot{u}_\epsilon \in H^1(0, T; H^{-1}(\Omega)). \tag{5.5}$$

The proof of this result is identical to the one of Theorem 3.1, with the only difference that now we modify the definition of the functionals $F_{n,k}$. We set

$$u_{n,0} := u_0, \; u_{n,-1} := u_0 - \tau u_1,$$

and for all $k \geq 1$ we define recursively

$$u_{n,k} := \text{argmin}\, \{F^\alpha_{n,k}(u) : u \in V\}, \tag{5.6}$$

where

$$F^\alpha_{n,k}(u) = F_{n,k}(u) + \frac{\alpha}{2}\|u\|^2_{L^2(\partial\Omega)}. \tag{5.7}$$

Notice that for $\alpha = 0$ the functional $F^0_{n,k} = F_{n,k}$, and it is still coercive on V thanks to the presence of the inertial quadratic term. The Euler-Lagrange equation associated is

$$\begin{aligned}&\tau^{-1}\Big(\frac{u_{n,k} - u_{n,k-1}}{\tau} - \frac{u_{n,k-1} - u_{n,k-2}}{\tau}, \varphi\Big) + (\nabla u_{n,k}, \nabla\varphi) + \Big(\frac{\nabla u_{n,k} - \nabla u_{n,k-1}}{\tau}, \nabla\varphi\Big) \\ &(k_\epsilon(|\nabla u_{n,k}|^2 - g^2)\nabla u_{n,k}, \nabla\varphi) + \alpha(u_{n,k}, \varphi)_{\partial\Omega} - (f(t_k), \varphi) = 0,\end{aligned} \tag{5.8}$$

for all $\varphi \in \mathcal{V}$. Also in this case one puts $\varphi = v_{n,k} := \frac{u_{n,k}-u_{n,k-1}}{\tau}$ in (5.8) and sum on $k = 0, \ldots, m$, $m \leq n$. The interpolants u_n and v_n satisfy

$$\begin{aligned}
&u_n \in H^1(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega)),\\
&v_n \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V),\\
&v_n \in H^1(0,T;H^{-1}(\Omega)).
\end{aligned}$$

Again, standard arguments allow to pass to the limit as $\tau \to 0$, providing us with a solution of (5.3) (we omit the details, the proof being very similar to the one of Theorem 3.1).

Next, we fix initial data as in (5.2) and such that

$$|\nabla u_0| \leq g(0) \text{ a.e. on } \Omega. \tag{5.9}$$

We now discuss how to extend Theorem 2.2 to boundary conditions of the type of (5.1). First we introduce the concept of weak solution we look for.

Definition 5.2. A pair (u, Υ) with $u \in H^1(0,T;H^1(\Omega))$ and $\Upsilon \in \mathcal{H}'_\nabla$, is a weak solution to the constrained wave equation with Fourier type boundary condition if:

(i) Conditions (2.16), (2.17), and (2.18) hold, together with the initial condition (2.19).

(ii) The following weak expression of the wave equation holds

$$\begin{aligned}
&(\dot{u}(T), \varphi(T)) - (u_1, \varphi(0)) - \int_0^T (\dot{u}, \dot{\varphi}) ds + \int_0^T (\nabla u + \nabla \dot{u}, \nabla \varphi) ds\\
&+ \int_0^T \alpha(u, \varphi)_{\partial\Omega} ds + \langle\!\langle \Upsilon, \nabla \varphi \rangle\!\rangle = \int_0^T (f, \varphi) ds,
\end{aligned} \tag{5.10}$$

for all $\varphi \in \mathcal{V}$, and moreover (2.22) holds.

Also for this kind of solution we will prove the existence of λ as in Theorem 2.8, such that (2.37) holds. Thanks to this relation it is possible to show that the couple (u, λ) is a solution in the sense explained by the following:

Definition 5.3. A pair (u, λ) with $u \in H^1(0,T;H^1(\Omega))$, $\lambda \in L^\infty(Q_T)'$, is a weak solution to the constrained wave equation with Fourier type boundary conditions if the following properties hold:

(i') The function u satisfies (2.16), (2.17), (2.18), and the initial data (2.19).

(ii') The following expression of the wave equation holds

$$\begin{aligned}
&(\dot{u}(T), \varphi(T)) - (u_1, \varphi(0)) - \int_0^T (\dot{u}, \dot{\varphi}) ds + \int_0^T (\nabla u + \nabla \dot{u}, \nabla \varphi) ds\\
&+ \int_0^T \alpha(u, \varphi)_{\partial\Omega} ds + [\lambda, \nabla u \cdot \nabla \varphi] = \int_0^T (f, \varphi) ds,
\end{aligned} \tag{5.11}$$

for all $\varphi \in \mathcal{V} \cap L^\infty(0,T;W^{1,\infty}(\Omega))$, together with condition (2.34).

(iii') For all $t \in (0,T)$ the following local version of the wave equation holds

$$\begin{aligned}
&(\dot{u}(t), \varphi(t)) - (u_1, \varphi(0)) - \int_0^t (\dot{u}, \dot{\varphi}) ds + \int_0^t (\nabla u + \nabla \dot{u}, \nabla \varphi) ds\\
&+ \int_0^t \alpha(u, \varphi)_{\partial\Omega} ds + [\lambda, \nabla u \cdot \nabla \varphi]_t = \int_0^t (f, \varphi) ds,
\end{aligned} \tag{5.12}$$

for all $\varphi \in \mathcal{V}_t \cap L^\infty(0,t;W^{1,\infty}(\Omega))$.

We summarize the existence result for solutions as in Definition 5.2 and Definition 5.3 in the following:

Theorem 5.4. *Let $T > 0$, assume (5.2), suppose $g \in W^{1,\infty}(0,T;L^\infty(\Omega))$ be such that $g \geq g_0$, for some constant $g_0 > 0$, and assume $f \in L^2(0,T;L^2(\Omega))$. Then there exist $u \in H^1(0,T;H^1(\Omega))$, $\Upsilon \in \mathcal{H}'_\nabla$, and $\lambda \in L^\infty(Q_T)'$, such that (u,Υ) is a solution to the constrained wave equation with Neumann ($\alpha = 0$) or Fourier ($\alpha > 0$) boundary conditions in the sense of Definition (5.2), while (u,λ) is a solution in the sense of Definition 5.3. Also in this case λ and Υ are related by (2.37).*

Again, we have the following result on the energy of the system.

Theorem 5.5. *Let (u,λ) be a solution provided by Theorem 5.4. Then for all $t \in (0,T]$,*

$$\begin{aligned}
&\frac{1}{2}\|\dot u(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{2}\alpha\|u(t)\|_{L^2(\partial\Omega)}^2 + \int_0^t \|\nabla \dot u(s)\|_{L^2}^2 ds \\
&\leq \frac{1}{2}\|u_1\|_{L^2}^2 + \frac{1}{2}\alpha\|u_0\|_{L^2(\partial\Omega)}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \int_0^t (f(s),\dot u(s))ds - [\lambda, g\dot g]_t. \qquad (5.13)
\end{aligned}$$

The variational inequality formulation of the problem with Fourier type boundary conditions is the following. Let u be the solution of Theorem 5.4, then u satisfies $|\nabla u| \leq g$ a.e. in Q_T, and for all $\varphi \in \mathcal{V}$ with $|\nabla\varphi| \leq g$ a.e. in Q_T it holds

$$\begin{aligned}
&(\dot u(T), u(T) - \varphi(T)) - \int_0^T (\dot u, \dot u - \dot\varphi)dt + \int_0^T (\nabla u + \nabla \dot u, \nabla u - \nabla\varphi)dt \\
&+ \int_0^T \alpha(u, u-\varphi)_{\partial\Omega} dt \leq (u_1, u_0 - \varphi(0)) + \int_0^T (f, u-\varphi)dt. \qquad (5.14)
\end{aligned}$$

We now sketch the proofs of Theorems 5.4 and 5.5, which are completely similar to the proofs of Theorems 2.2 and 2.8.

We start with Lemma 3.2 that is achieved with identical estimates. Namely, testing (5.3) by $\dot u_\epsilon$ we infer

$$\begin{aligned}
&\frac{1}{2}\|\dot u_\epsilon(t)\|_{L^2}^2 + \frac{\alpha}{2}\|u_\epsilon(t)\|_{L^2(\partial\Omega)}^2 + \frac{1}{2}\|\nabla u_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla \dot u_\epsilon(s)\|_{L^2}^2 ds + \int_\Omega \widehat{K}_\epsilon(t)dx \\
&= \frac{1}{2}\|u_1\|_{L^2}^2 + \frac{\alpha}{2}\|u_0\|_{L^2(\partial\Omega)}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \int_\Omega \widehat{K}_\epsilon(0)dx + \int_0^t (f(s),\dot u_\epsilon(s))ds \\
&- \int_0^t\int_\Omega \widehat{k}_\epsilon(s)g(s)\dot g(s)dxds \leq C + C\int_0^t \|\dot u_\epsilon(s)\|_{L^2}^2 ds + C\int_0^t\int_\Omega \widehat{k}_\epsilon(s)dxds.
\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{1}{2}\|\dot u_\epsilon(t)\|_{L^2}^2 + \frac{\alpha}{2}\|u_\epsilon(t)\|_{L^2(\partial\Omega)}^2 + \frac{1}{2}\|\nabla u_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla \dot u_\epsilon(s)\|_{L^2}^2 ds + \int_\Omega \widehat{K}_\epsilon(t)dx \\
&\leq C + C\int_0^t \|\dot u_\epsilon(s)\|_{L^2}^2 ds + C\int_0^t\int_\Omega \widehat{k}_\epsilon(s)|\nabla u_\epsilon(s)|^2 dxds. \qquad (5.15)
\end{aligned}$$

The last term is estimated by

$$\begin{aligned}
&\int_0^t\int_\Omega \widehat{k}_\epsilon(s)|\nabla u_\epsilon(s)|^2 dxds = \int_0^t (f(s),u_\epsilon(s))ds + \int_0^t \|\dot u_\epsilon(s)\|_{L^2}^2 ds - \int_0^t \|\nabla u_\epsilon(s)\|_{L^2}^2 ds \\
&- \int_0^t \alpha\|u_\epsilon(s)\|_{L^2(\partial\Omega)}^2 ds - \frac{1}{2}\|\nabla u_\epsilon(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + (u_1,u_0) - (\dot u_\epsilon(t), u_\epsilon(t)) \\
&\leq C + C\int_0^t \|\dot u_\epsilon(s)\|_{L^2}^2 ds + \frac{\gamma}{2}\|\dot u_\epsilon(t)\|_{L^2}^2 + \frac{1}{2\gamma}\|u_\epsilon(t)\|_{L^2}^2, \qquad (5.16)
\end{aligned}$$

and we arrive to

$$\begin{aligned}
&C'\|\dot u_\epsilon(t)\|_{L^2}^2 + \frac{\alpha}{2}\|u_\epsilon(t)\|_{L^2(\partial\Omega)}^2 + \frac{1}{2}\|\nabla u_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla \dot u_\epsilon(s)\|_{L^2}^2 ds + 2\int_\Omega \widehat{K}_\epsilon(t)dx \\
&\leq C + C\int_0^t \|\dot u_\epsilon(s)\|_{L^2}^2 ds. \qquad (5.17)
\end{aligned}$$

Again Gronwall Lemma together with Poincaré inequality

$$\|u_\epsilon\|^2_{L^2} \le c\big(\|\nabla u_\epsilon\|^2_{L^2} + \|u_\epsilon\|^2_{L^2(\partial\Omega)}\big), \tag{5.18}$$

yields the same estimates as in Lemma 3.2. Specifically, the third, forth, and last estimate in this lemma are achieved identically as there, taking also into account the boundary terms, namely, for instance,

$$\begin{aligned}
&\langle\!\langle \widehat{k}_\epsilon \nabla u_\epsilon, F\rangle\!\rangle = \langle\!\langle \widehat{k}_\epsilon \nabla u_\epsilon, \nabla v\rangle\!\rangle = -(\dot u_\epsilon(T), v(T)) - (u_1, v(0)) + \int_0^T (\dot u_\epsilon, \dot v)dt + \int_0^T \alpha(u_\epsilon, v)_{\partial\Omega} dt \\
&- \int_0^T (\nabla u_\epsilon + \nabla \dot u_\epsilon, \nabla v)ds + \int_0^T (f, v)ds \le \\
&\le \|\dot u_\epsilon(T)\|_{L^2}\|v(T)\|_{L^2} + C\|v(0)\|_{L^2} + \|\dot u_\epsilon\|_{L^2(L^2)}\|\dot v\|_{L^2(L^2)} + \|\nabla u_\epsilon\|_{H^1(L^2)}\|\nabla v\|_{L^2(L^2)} + C\|v\|_{L^2(L^2)} \\
&\le C\|v\|_{\mathcal{V}} \le \|F\|_{\mathcal{H}_\nabla},
\end{aligned} \tag{5.19}$$

for all $F \in \mathcal{H}_\nabla$, with $F = \nabla v$.

Notice that convergence (4.2) entails also strong convergence of u_ϵ to u in $L^2(0,T;V)$, which in turn implies strong convergence of the boundary term $u_\epsilon \llcorner \partial\Omega$ to $u \llcorner \partial\Omega$. These convergences allows to replicate the arguments in (4.17) and (4.16). The rest of the proof of Theorem 5.4 is straightforward. Also Theorem 5.5 is easily achieved similarly to the proof of Theorem 2.8.

5.1 Limit as $\alpha \to +\infty$

In this section we show that a solution $(u^\alpha, \Upsilon^\alpha)$ provided by Theorem 5.4 tends, as $\alpha \to \infty$, to a solution (u, Υ) of Theorem 2.2. We have here introduced the label α to emphasize the dependence on the parameter $\alpha \in [0,\infty)$ appearing in Definition 5.2.

To prove this result, we have to make the assumption on the initial data that

$$u_0, u_1 \in H^1_0(\Omega).$$

Therefore we state the following:

Theorem 5.6. *Let $T > 0$, $u_0, u_1 \in H^1_0(\Omega)$ with $|\nabla u_0| \le g(0)$ a.e. on Ω, assume $g \in W^{1,\infty}(0,T;L^\infty(\Omega))$ and $f \in L^2(0,T;L^2(\Omega))$. Let $(u^\alpha, \Upsilon^\alpha, \lambda^\alpha)$ be a solution provided by Theorem 5.4. Then, as $\alpha \to \infty$, we have that there exists (u, Υ, λ) such that*

$$\begin{aligned}
&u^\alpha \rightharpoonup u \text{ weakly in } H^1(0,T;H^1(\Omega)) \\
&\Upsilon^\alpha \rightharpoonup \Upsilon \text{ weakly in } \mathcal{H}'_\nabla, \\
&\lambda^\alpha \rightharpoonup \lambda \text{ weakly star in } L^\infty(Q_T)',
\end{aligned}$$

and (u, Υ) is a solution as in Definition 2.1 and (u, λ) is a solution as in Definition 2.6.

Proof. We have to enstablish some a-priori estimates for $(u^\alpha, \Upsilon^\alpha)$ which are independent of α. We use the approximating sequence u^α_ϵ emploied in the proof of Theorem 5.4. Specifically, we go back to (5.17) and observe that the constant C is independent of α, thanks to the fact that u_0, u_1 are null on $\partial\Omega$. From this and following again the lines of the proof of Lemma 3.2, we easily arrive to the following:

Lemma 5.7. *There is a constant $C > 0$ independent of α and ϵ such that the estimates in Lemma 3.2 hold for u^α_ϵ replacing u_ϵ, and moreover*

$$\alpha\|u^\alpha_\epsilon(t)\|^2_{L^2(\partial\Omega)} \le C \text{ for all } t \in [0,T].$$

We hence take the limit as $\epsilon \to 0$ and obtain the following estimates for the pair $(u^\alpha, \Upsilon^\alpha)$ and λ^α. Namely

$$u^\alpha \in H^1(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;L^2), \tag{5.20}$$
$$\dot u^\alpha \in BV(0,T;X), \tag{5.21}$$
$$\Upsilon^\alpha \in \mathcal{H}'_\nabla, \tag{5.22}$$
$$\lambda^\alpha \in L^\infty(Q_T)', \tag{5.23}$$
$$\alpha^{1/2} u^\alpha \in L^2(0,T;L^2(\partial\Omega)), \tag{5.24}$$

and their norms in these spaces are equibounded by a constant independent of α. Hence, passing to the limit as $\alpha \to \infty$, we get, up to a subsequence, that there are u, Υ, λ with

$$u^\alpha \rightharpoonup u \text{ weakly in } H^1(0,T;H^1(\Omega)) \text{ and weakly star in } W^{1,\infty}(0,T;L^2(\Omega)), \tag{5.25}$$
$$\dot u^\alpha \rightharpoonup \dot u \text{ weakly star in } BV(0,T;X), \tag{5.26}$$
$$\dot u^\alpha \to \dot u \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{5.27}$$
$$\dot u^\alpha(t) \rightharpoonup \dot u(t) \text{ weakly in } L^2(\Omega) \text{ for all } t \in [0,T], \tag{5.28}$$
$$\Upsilon^\alpha \rightharpoonup \Upsilon \text{ weakly in } \mathcal{H}'_\nabla, \tag{5.29}$$
$$\lambda^\alpha \rightharpoonup \lambda \text{ weakly star in } L^\infty(Q_T)'. \tag{5.30}$$

Moreover, by (5.24),

$$u^\alpha \to 0 \text{ strongly in } L^2(0,T;L^2(\partial\Omega)),$$

and in particular $u^\alpha = 0$, a.e. on $(0,T) \times \partial\Omega$. In particular we infer that (2.20) is satisfied.

Also, from condition (i) of Definition 5.2 and from (2.18) valid for u^α we easily deduce that this condition is satisfied by u, and (i) of Definition 2.1 follows. In a similar way, following also the lines of the proof of Theorem 2.2 in [2], we see that the last condition in (ii) of Definition 2.1 is satisfied. It remains to show (2.22). We own to the argument in (4.16) and (4.17) using u^α in place of u_ϵ. The only difference between the present case and (4.16) is the appearence, in the right-hand side, of the term $-\int_0^T \alpha \|u^\alpha\|^2_{L^2(\partial\Omega)} dt \le 0$, which, being non-positive, leads to (4.17). As a consequence

$$\limsup_{\alpha\to\infty} \langle\!\langle \Upsilon^\alpha, \nabla u^\alpha \rangle\!\rangle \le \langle\!\langle \Upsilon, \nabla u \rangle\!\rangle, \tag{5.31}$$

which implies $\Upsilon \in \beta_w(\nabla u)$, and the proof that (u, Υ) is a solution as in Definition 2.1 is complete.

Let us now check that (u, λ) is a solution as in Definition 2.6. To this aim we follow the proof of Lemma 4.2. The fact that $\lambda \ge 0$ is straightforward. Let $\varphi \in \mathcal{V} \cap L^\infty(0,T;W^{1,\infty}(\Omega))$, and check that (4.18) holds. This is easy since we know that, for any α, it holds

$$[\lambda^\alpha, \nabla u^\alpha \cdot \nabla\varphi] = \langle\!\langle \Upsilon^\alpha, \nabla\varphi \rangle\!\rangle. \tag{5.32}$$

By (5.29) the right-hand side tends to $\langle\!\langle \Upsilon, \nabla\varphi \rangle\!\rangle$. We claim that

$$[\lambda^\alpha, \nabla u^\alpha \cdot \nabla\varphi] \to [\lambda, \nabla u \cdot \nabla\varphi], \tag{5.33}$$

which in turn will imply (2.37). To prove this, we first write

$$\begin{aligned}
0 &\le \liminf_{\alpha\to\infty} [\lambda^\alpha, |\nabla u^\alpha - \nabla u|^2] \le \limsup_{\alpha\to\infty} [\lambda^\alpha, |\nabla u^\alpha - \nabla u|^2] \\
&= \limsup_{\alpha\to\infty} \big([\lambda^\alpha, |\nabla u^\alpha|^2] + [\lambda^\alpha, |\nabla u|^2] - 2[\lambda^\alpha, \nabla u^\alpha \cdot \nabla u]\big) \\
&\le [\lambda, |\nabla u|^2] - \langle\!\langle \Upsilon, \nabla u \rangle\!\rangle \\
&\le [\lambda, g^2] - \langle\!\langle \Upsilon, \nabla u \rangle\!\rangle \le \liminf_{\alpha\to\infty} [\lambda^\alpha, g^2 - |\nabla u^\alpha|^2] = 0,
\end{aligned} \tag{5.34}$$

where we have used (5.31) twice, and that $[\lambda^\alpha, |\nabla u^\alpha|^2] = \langle\!\langle \Upsilon^\alpha, \nabla u^\alpha \rangle\!\rangle$ for all α. Hence we infer

$$\lim_{\alpha\to\infty} [\lambda^\alpha, |\nabla u^\alpha - \nabla u|^2] = 0,$$

and from this, since $\lambda^\alpha \geq 0$,

$$\lim_{\alpha\to 0}\left|[\lambda^\alpha, (\nabla u^\alpha - \nabla u)\cdot\nabla\varphi]\right| \leq \lim_{\alpha\to 0}[\lambda^\alpha, |\nabla u^\alpha - \nabla u||\nabla\varphi|] \\ \leq \lim_{\alpha\to 0}\Big([\lambda^\alpha, |\nabla u^\alpha - \nabla u|^2]^{\frac12}[\lambda^\alpha, |\nabla\varphi|^2]^{\frac12}\Big) = 0, \qquad (5.35)$$

so we conclude the claim (5.33). Hence (2.33) is obtained. The check of (2.35) is similar. It remains to prove the last condition in (2.34). First we see that from (5.34) it also follows that (5.31) is in fact an equality. Now, we know that

$$[\lambda^\alpha, |\nabla u^\alpha|^2 - g^2] = 0,$$

so it is sufficient to check that $\lim_{\alpha\to 0}[\lambda^\alpha, |\nabla u^\alpha|^2] = [\lambda, |\nabla u|^2]$. But this follows from the fact that (5.31) is an equality and since $\langle\!\langle \Upsilon, \nabla u\rangle\!\rangle = [\lambda, |\nabla u|^2] = [\lambda\nabla u, \nabla u]$, which also follows from (5.34). To conclude (2.34) we can now argue as in (4.32). □

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References

[1] H. Attouch, "Variational Convergence for Functions and Operators", Pitman, London, 1984.

[2] A. Azevedo, J. F. Rodrigues, L. Santos, *Lagrange multipliers for evolution problems with constraints on the derivatives*, St. Petersburg Math. J. **32**(3), (2021), 435-448.

[3] A. Azevedo, L. Santos, *Lagrange multipliers and transport densities*, J. Math. Pures Appl. **108**, (2017), 592–611.

[4] V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Spaces", Noordhoff, Leyden, 1976.

[5] B. Benešová, M. Kruzik, A. Schlömerkemper, *A note on locking materials and gradient polyconvexity*, Mathematical Models and Methods in Applied Sciences **28**(12), (2018), 2367–2401.

[6] V. Barbu, Y. Guo, M.A. Rammaha, and D. Toundykov, *Convex integrals on Sobolev spaces*, J. Convex Anal. J. Convex Anal.
19 (2012), 837–852.

[7] E. Bonetti, E. Rocca, R. Scala, G. Schimperna, *On the strongly damped wave equation with constraint*, Comm. Partial Differential Equations **42**(7), (2017), 1042–1064.

[8] H. Brézis, *Multiplicateur de Lagrange en torsion elasto-plastique*, Arch. Ration. Mech. Anal. **49**, (1972), 32–40.

[9] H. Brézis, *Intégrales convexes dans les espaces de Sobolev. (French)*, Israel J. Math., **13**, (1972), 9–23.

[10] H. Brézis, "Opérateurs Maximaux Monotones et Sémi-groupes de Contractions dans les Espaces de Hilbert", North-Holland Math. Studies **5**, North-Holland, Amsterdam, 1973.

[11] F. Demengel, P. Suquet, *On Locking Materials* Acta Applicandae Mathematicae **6**, (1986), 185–211.

[12] G. Duvaut, J.-L. Lions, "Inequalities in Mechanics and Physics", Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[13] C. Eck, J. Jarusek, M. Krbec, "Unilateral contact problems: variational methods and existence theorems", Pure and Applied Mathematics, monograph, Taylor Francis Group 2005.

[14] J.-L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires", Etudes mathématiques, Paris, Dunod, 1969.

[15] F. Miranda, J. F. Rodrigues, L. Santos, *Evolutionary quasi-variational and variational inequalities with constraints on the derivatives*, Adv. Nonlinear Anal. **9**(1), (2018), 250–277.

[16] J.F. Rodrigues, "Obstacle Problems in Mathematical Physics", North-Holland Mathematics Studies 134, 1987.

[17] J. F. Rodrigues, L. Santos, *A parabolic quasi-variational inequality arising in a superconductivity model*, Ann. Sc. Norm. Sup. Cl. Scie. **29**(1), (2000), 153–169.

[18] J. F. Rodrigues, L. Santos, *Variational and Quasi-Variational Inequalities with Gradient Type Constraints*, in "Topics in Applied Analysis and Optimisation", pages 319–361 (Eds. M. Hintermüller, J. F. Rodrigues). CIM Series in Mathematical Sciences, Springer Nature Switzerland AG 2019; https://doi.org/10.1007/978-3-030-33116-0_13, (2018).

[19] J. F. Rodrigues, L. Santos, *Quasivariational Solutions for First Order Quasilinear Equations with Gradient Constraint*, Arch. Rational Mech. Anal. **205**, (2012), 493–514.

[20] T. Roubicek, "Nonlinear Partial Differential Equations with Applications", Birkhäuser Springer Basel, 2005.

[21] R. Scala, G. Schimperna, *A contact problem for viscoelastic bodies with inertial effects and unilateral boundary constraints*, European J. of Appl. Math. **28**(1), (2017), 91–122.

[22] R. Scala, G. Schimperna, *On the viscous Cahn-Hilliard equation with singular potential and inertial term*, AIMS Mathematics **1**(1), (2016), 64–76.

[23] R. Scala, *A weak formulation for a rate-independent delamination evolution with inertial and viscosity effects subjected to unilateral constraint*, Interfaces Free Bound. **19**(1), (2017), 79–107.

[24] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. **146**, (1986), 65–96.

[25] Y. Şengül, *Viscoelasticity with limiting strain*, Discrete Contin. Dyn. Syst. Ser. S **14**(1), (2021), 57–70.

[26] R. Temam, "Navier-Stokes equations theory and numerical analysis", North-Holland Publishing Company, Amsterdam, New York, Oxford 1977.

[27] K. Yosida, "Functional Analysis", (sixth edition), Springer-Verlag, Berlin, Heidelberg, New York, 1980.