BV FUNCTIONS AND SETS OF FINITE PERIMETER ON CONFIGURATION SPACES

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Abstract. This paper contributes to foundations of the geometric measure theory in the infinite dimensional setting of the configuration space over the Euclidean space $\mathbb{R}^n$ equipped with the Poisson measure π. We first provide a rigorous meaning and construction of the m-codimensional Poisson measure —formally written as "$(\infty - m)$-dimensional Poisson measure"— on the configuration space. We then show that our construction is consistent with potential analysis by establishing the absolute continuity with respect to Bessel capacities. Secondly, we introduce three different definitions of BV functions based on the variational, relaxation and the semigroup approaches, and prove the equivalence of them. Thirdly, we construct perimeter measures and introduce the notion of the reduced boundary. We then prove that the perimeter measure can be expressed by the 1-codimensional Poisson measure restricted on the reduced boundary, which is a generalisation of De Giorgi's identity to the configuration space. Finally, we construct the total variation measures for BV functions, and prove the Gauß–Green formula.

Contents

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1. Introduction

The purpose of this paper is to contribute to the foundation of the geometric measure theory in the infinite-dimensional setting of the *configuration space (without multiplicity)* over the standard Euclidean space $\mathbb{R}^n$:

$$\Upsilon(\mathbb{R}^n) := \left\{\gamma = \sum_{i\in\mathbb{N}} \delta_{x_i} : \gamma(K) < \infty \text{ for every compact } K \subset \mathbb{R}^n,\ \gamma(\{x\}) \in \{0,1\},\ \forall x \in \mathbb{R}^n\right\}.$$

We equip it with the vague topology τ_v, the L^2-transportation (extended) distance d_Υ, which stems from the optimal transport problem, and the Poisson measure π whose intensity measure is the Lebesgue measure $\mathbf{L}^n$ on $\mathbb{R}^n$ (see Section 2 for more details). The resulting *topological (extended) metric measure structure* $(\Upsilon(\mathbb{R}^n), \tau_v, \mathsf{d}_\Upsilon, \pi)$ has fundamental importance for describing dynamical systems of infinite particles, hence it has been extensively investigated in the last years [1, 2, 23, 24, 31] (see also references therein) by a number of research groups both from the fields of statistical physics and probability. In spite of its importance for the applications to statistical physics and stochastic analysis, the theory of BV functions and sets of finite perimeters on $\Upsilon(\mathbb{R}^n)$ has – so far – never been investigated in any literature that we are aware of, due to its nature of infinite-dimenisonality and non-linearity.

The theory of BV functions and sets of finite perimeters beyond the standard Euclidean space has seen a thriving development in the last years, see [3, 4, 5, 6, 7, 9, 10, 13, 18, 35] and references therein. However, all of these results do not cover the configuration space $\Upsilon(\mathbb{R}^n)$ due to its intrinsic infinite dimensionality and the fact that the canonical (extended) distance function can be infinite on sets of π-positive measure. This property causes several pathological phenomena (see details in [23]):

- the extended distance d_Υ is not continuous with respect the vague topology τ_v;
- d_Υ-metric balls are negligible with respect to the Poisson measure π;

- d_Υ–Lipschitz functions are not necessarily π-measurable;
- the Riesz–Markov-Kakutani's representation theorem does not hold.

The latter makes it difficult to construct total variation measures supporting the Gauß–Green formula by means of the standard functional analytic technique.

In the setting of infinite-dimensional spaces, the study of the geometric measure theory has been pioneered by Feyel–de la Pradelle [26], Fukushima [28], Fukushima–Hino [29] and Hino [30] in the Wiener space. In [26], they constructed the finite-codimensional Gauß–Hausdorff measure in the Wiener space and investigated its relation to capacities. In [28] and [29], they developed the theory of BV functions and constructed perimeter measures, and prove the Gauß–Green formula. Based on these results, Hino introduced in [30] a notion of the reduced boundary and investigated relations between the one-codimensional Hausdorff–Gauß measure and the perimeter measures. Further fine properties were investigated by Ambrosio-Figalli [11], Ambrosio-Figalli-Runa [12], Ambrosio-Miranda-Pallara [15, 16], Ambrosio-Maniglia-Miranda-Pallara [14]. The notion of BV functions has been studied also in a Gelfand triple by Röckner–Zhu–Zhu [38, 39, 40]. All of the aforementioned results rely heavily on the linear structure of the Wiener space or the Hilbert space, which is used to perform finite-dimensional approximations. However, the configuration space does not have a linear structure and there is no chance to apply similar techniques.

1.1. **Non-linear dimension reduction and overview of the main results.** To overcome the difficulties explained above, we develop a *non-linear dimensional reduction* tailored for the configuration space $\Upsilon(\mathbb{R}^n)$. A key observation is that $\Upsilon(B_r)$, the configuration space over the Euclidean closed metric balls B_r centered at the origin o with radius $r > 0$, is essentially finite dimensional. More precisely, due to the compactness of B_r, $\Upsilon(B_r)$ can be written as the disjoint union $\sqcup_{k\in\mathbb{N}}\Upsilon^k(B_r)$ of the k-particle configuration spaces $\Upsilon^k(B_r)$, each of which is isomorphic to the quotient space of the k-product space $B_r^{\times k}$ by the symmetric group. In light of this observation, the main task is to lift the geometric measure theory in $\Upsilon(B_r)$ to the infinite-dimensional space $\Upsilon(\mathbb{R}^n)$ by finite-dimensional approximations.

Based on this strategy, we first construct the m-codimensional Poisson measure on the configuration space with much care of the measurability issues (Theorem 3.7 and Definition 3.8), and study its relation to $(1,p)$-Bessel capacities (Theorem 4.3). Secondly, we introduce three different definitions of BV functions based on the variational, relaxation and the semigroup approaches, and prove the equivalence of them (Theorem 5.17). In the process of showing the equivalence of these three definitions, we develop the p-Bakry–Émery inequality (Theorem 5.15) for the heat semigroup on $\Upsilon(\mathbb{R}^n)$ for $1 < p < \infty$, which was previously known only for $p = 2$ in Erbar–Huesmann [24]. Thirdly, we construct perimeter measures and introduce the notion of the reduced boundary in Section 6. We then prove that the perimeter measure can be expressed by the 1-codimensional Poisson measure restricted on the reduced boundary (Theorem 6.15). Fourthly, we construct the total variation measures for BV functions and prove the Gauß–Green formula (Theorem 7.7).

We now explain each result in more detail.

1.2. **m-codimensional Poisson measure.** The first achievement of this paper is the construction of the m-codimensional Poisson measure on $\Upsilon(\mathbb{R}^n)$. Since $\Upsilon(\mathbb{R}^n)$ is infinite-dimensional, it is formally written as

$$\text{“}(\infty - m)\text{-dimensional Poisson measure”},$$

which is an ill-defined object without justification. An important remark is that, in the case of finite-dimensional spaces, usually the construction of finite-codimensional measures builds upon covering arguments, which heavily rely on the volume doubling property of the ambient measure. However, this property does not hold in the configuration space $\Upsilon(\mathbb{R}^n)$.

To overcome this difficulty we construct the m-codimensional Poisson measure on $\Upsilon(\mathbb{R}^n)$ by passing to the limit of finite dimensional approximations obtained by using the m-codimensional Poisson measure on $\Upsilon(B_r)$. The key step in the construction is to prove *the monotonicity* of these finite dimensional approximations with respect to the radius r, allowing us to find a unique limit measure.

More in details, based on the decomposition $\Upsilon(B_r) = \sqcup_{k\in\mathbb{N}}\Upsilon^k(B_r)$, we build $\rho^m_{\Upsilon(B_r)}$, *the spherical Hausdorff measure of codimension* m in $\Upsilon(B_r)$, by summing the m-codimensional spherical Hausdorff measure $\rho^{m,k}_{\Upsilon(B_r)}$ on the k-particle configuration space $\Upsilon^k(B_r)$, which is obtained by the quotient measure of the m-codimensional spherical Hausdorff measure on the k-product space $B_r^{\times k}$ with a suitable renormalization corresponding to the Poisson measure. *The localised m-codimensional Poisson measure* ρ^m_r of a set $A \subset \Upsilon(\mathbb{R}^n)$ is then obtained by averaging the $\rho^m_{\Upsilon(B_r)}$-measure of sections of A with the Poisson measure $\pi_{B_r^c}$ on $\Upsilon(B_r^c)$, i.e.

$$\rho^m_r(A) := \int_{\Upsilon(B_r^c)} \rho^m_{\Upsilon(B_r)}(\{\gamma \in \Upsilon(B_r) \, : \gamma + \eta \in A\}) d\pi_{B_r^c}(\eta)\,.$$

We prove that ρ^m_r is well-defined on Borel sets (indeed, we prove it for all Suslin sets), and that it is monotone increasing with respect to r (Theorem 3.7 and Definition 3.8). In particular, we can define *the m-codimensional Poisson measure* as

$$\rho^m := \lim_{r\to\infty} \rho^m_r\,.$$

We refer the readers to Section 3 for the detailed construction of ρ^m.

1.3. **Bessel capacity.** In Section 4, we compare the m-codimensional Poisson measure ρ^m and $\mathrm{Cap}_{\alpha,p}$, the *Bessel capacity*, proving that zero capacity sets are ρ^m negligible provided $\alpha p > m$ (Theorem 4.3). This result, that is well-known in the case of finite-dimensional spaces, proves that our m-codimensional Poisson measure ρ^m behaves coherently with the potential analytic structure of $\Upsilon(\mathbb{R}^n)$. To prove it, we introduce the (α, p)-Bessel capacity $\mathrm{Cap}^{\Upsilon(B_r)}_{\alpha,p}$ on $\Upsilon(B_r)$ and the localised (α, p)-Bessel capacity $\mathrm{Cap}^r_{\alpha,p}$ on $\Upsilon(\mathbb{R}^n)$ based on the localisation argument of the L^p-heat semigroup $\{T_t\}$ on $\Upsilon(\mathbb{R}^n)$. We prove that $\mathrm{Cap}_{\alpha,p}$ is approximated by $\mathrm{Cap}^r_{\alpha,p}$ as $r \to \infty$, hence we can obtain the sought conclusion by lifting the corresponding result for $\rho^m_{\Upsilon(B_r)}$ and $\mathrm{Cap}^{\Upsilon(B_r)}_{\alpha,p}$ in $\Upsilon(B_r)$ (see Proposition 4.14). We refer the readers to Section 4 for the detailed arguments.

A remarkable application of the result above is Corollary 7.4, which establishes a fundamental relation between the potential theory and the theory of BV functions. It says that, if $\mathrm{Cap}_{1,2}(A) = 0$ then $|\mathsf{D}F|(A) = 0$ for any $F \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))\cap L^2(\Upsilon(\mathbb{R}^n), \pi)$, where $|\mathsf{D}F|$ is the total variation measure (Definition 7.2) and $\mathrm{BV}(\Upsilon(\mathbb{R}^n))$ is the space of BV functions (Definition 5.18). The latter result will be fundamental for applications of our theory to stochastic analysis, which will be the subject of a forthcoming paper.

1.4. **Functions of bounded variations and Cacioppoli sets.** In the second part of this paper we develop the theory of BV functions and sets of finite perimeter in $\Upsilon(\mathbb{R}^n)$. In Section 5 we propose three different notions of functions with bounded variation. The first one follows the classical *variational approach*, the second one is built upon the *relaxation approach*, while the third one relies on the regularisation properties of the *heat semigroup*. It turns out that they are all equivalent, as shown in Section 5.5, and the resulting class is denoted by $\mathrm{BV}(\Upsilon(\mathbb{R}^n))$. For any function $F \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))$ we are able to define a total variation measure $|\mathsf{D}F|$ and to prove a *Gauß-Green formula* (see Theorem below). It is worth remarking that in our infinite dimesional setting we cannot rely on the Riesz–Markov–Kakutani's representation theorem due to the lack of local compactness. In particular, the construction of the total variation measure is not straightforward and requires several work. We follow an unusual path to show its existence: we first develop the theory of *sets with finite perimeter* relying on the non-linear dimension reduction, and after we employ the *coarea formula* to build the total variation measure of a BV function as a sovraposition of perimeter measures.

Sets of finite perimeter are introduced as those Borel sets E such that $\chi_E \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, where χ_E denotes the indicator function on E. In Section 6, we study their structure by means of the non-linear reduction approach, a part of which utilizes a strategy inspired by Hino [30] for the study of Wiener spaces. The key result in this regard is Proposition 5.5 saying that if E is of finite perimeter then the projection $E_{\eta,r} := \{\gamma \in \Upsilon(B_r) : \gamma + \eta \in E\}$ has finite *localized total variation* in B_r, for $\pi_{B_r^c}$-a.e. $\eta \in B_r^c$. Hence, we can reduce the problem to the study of sections that are sets with finite perimeter on $\Upsilon(B_r)$. As already remarked, the latter is essentially a finite dimensional space, so we can appeal to classical tools of the geometric measure theory to attack the problem.

The *reduced boundary* $\partial^* E$ of a set of finite perimeter $E \subset \Upsilon(\mathbb{R}^n)$ is then defined in terms of the reduced boundary of the sections $E_{\eta,r}$, through a limit procedure. The resulting object allows representing the perimeter measure as

$$\|E\| = \rho^1|_{\partial^* E} ,$$

which is a generalization of the identity proven by E. De Giorgi [21, 22] in the Euclidean setting.

It is worth stressing that our approach to the BV theory significantly deviates from the standard one. Indeed, as already explained, we define the total variation measure $|\mathsf{D}F|$ of a function $F \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))$ by imposing the validity of the coarea formula. More precisely, we show that a.e. level set $\{F > t\}$ is of finite perimeter and we set

$$|\mathsf{D}F| := \int_{-\infty}^{\infty} \big\|\{F > t\}\big\| dt ,$$

taking advantage of the fact that the perimeter measure $\|\{F > t\}\|$ has been already defined using finite dimensional approximations. The reason for this non-standard treatment is that we are not able to build directly $|\mathsf{D}F|$ through a finite dimensional approximation, since the latter does not have a simple expression in terms of 1-codimensional Poisson measure ρ^1 restricted to a suitable subset. Our approach is, however, consistent with the standard one, as shown in Corollary 7.3 and in Theorem 7.7.

In the statement below we summarize the main results in Section 6 and Section 7 concerning functions of bounded variations and a sets of finite perimeter. We denote by $\mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ the space of cylinder vector fields on $\Upsilon(\mathbb{R}^n)$ and by $(T\Upsilon, \langle\cdot,\cdot\rangle_{T\Upsilon})$ the tangent bundle on $\Upsilon(\mathbb{R}^n)$ with the pointwise inner product $\langle\cdot,\cdot\rangle_{T\Upsilon}$ (see Section 2.5).

Theorem (Theorems 6.15, 7.7)**.** *For $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, there exists a unique positive finite measure $|\mathsf{D}F|$ on $\Upsilon(\mathbb{R}^n)$ and a unique $T\Upsilon$-valued Borel measurable function σ on $\Upsilon(\mathbb{R}^n)$ so that $|\sigma|_{T\Upsilon} = 1$ $|\mathsf{D}F|$-a.e., and*

$$\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma\rangle_{T\Upsilon} d|\mathsf{D}F| , \quad \forall V \in \mathrm{CylV}(\Upsilon) .$$

If, furthermore, $F = \chi_E$, then

$$|\mathsf{D}\chi_E| = \rho^1|_{\partial^* E}.$$

As indicated in [28, 29] in the case of the Wiener space, the aforementioned theorem in the configuration space $\Upsilon(\mathbb{R}^n)$ has significant applications to stochastic analysis, in particular regarding the singular boundary theory of infinite-particle systems of Brownian motions on $\mathbb{R}^n$, which will be addressed in a forthcoming paper.

1.5. **Structure of the paper.** In Section 2 we collect all the needed preliminary results regarding the geometric-analytic structure of the configuration space, Suslin sets and measurability of sections. In Section 3 we construct the m-codimensional measure whose relation with the Bessel capacity is studied Section 4. Section 5 is devoted to the study of BV functions. We introduce three different notion and prove their equivalence. In Section 6 we introduce and study sets of finite perimeter. We build the notion of reduced boundary and the perimeter measure, and we show integration by parts formulae. In Section 7.1 we introduce the total variation measure of functions with bounded variations by employing the coarea formula. We also show the validity of the Gauß-Green type integration by parts formula.

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2. Preliminaries

2.1. **Notational convention.** In this paper, the bold fonts $\mathbf{S}^m$, $\mathbf{L}^n$, $\mathbf{T}_t, \ldots$ are mainly used for objects in the product space $\mathbb{R}^{n\times k}$ or vector-valued objects, while the serif fonts $\mathsf{S}, \mathsf{D}, \ldots$ are used for objects in the quotient space $\mathbb{R}^{n\times k}/\mathfrak{S}_k$ with respect to the k-symmetric group $\mathfrak{S}_k$ or for objects in the configuration space $\Upsilon(\mathbb{R}^n)$.

The lower-case fonts $f, g, h, v, w, \ldots$ are used for functions on the base space $\mathbb{R}^n$, while the upper-case fonts $F, G, H, V, W, \ldots$ are used for functions on the configuration space $\Upsilon(\mathbb{R}^n)$.

We denote by χ_E the indicator function on E, i.e., $\chi_E = 1$ on E and $\chi_E = 0$ on E^c. Let $\Omega \subset \mathbb{R}^n$ be a closed domain. We denote by $C_0^\infty(\Omega)$ the space of smooth functions with compact support on $\Omega \setminus \partial\Omega$, while $C_c^\infty(\Omega)$ denotes the space of compactly supported smooth functions on Ω. Notice that $f \in C_c^\infty(\Omega)$ does not necessarily vanish at the boundary, and $C_c^\infty(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$.

2.2. **Configuration spaces.** Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $B_r := B_r(o) \subset \mathbb{R}^n$ be the closed ball with radius $r > 0$ centered at the origin o. Let δ_x denote the point measure at $x \in \mathbb{R}^n$, i.e. $\delta_x(A) = 1$ if and only if $x \in A$. We denote by $\Upsilon(\mathbb{R}^n)$ the *configuration space over* $\mathbb{R}^n$ *without multiplicity*, i.e. the set of all locally finite point measures γ on $\mathbb{R}^n$ so that $\gamma(\{x\}) \in \{0, 1\}$ for any $x \in \mathbb{R}^n$:

$$\Upsilon(\mathbb{R}^n) := \Big\{\gamma = \sum_{i\in\mathbb{N}} \delta_{x_i} : \gamma(K) < \infty \text{ for every compact } K \subset \mathbb{R}^n,\ \gamma(\{x\}) \in \{0,1\}\ \forall x \in \mathbb{R}^n\Big\}.$$

Let $\Upsilon(A)$ denote the configuration space over a subset $A \subset \mathbb{R}^n$, and $\Upsilon^k(A)$ denote the space of k-configurations on a subset A, i.e. $\Upsilon^k(A) = \{\gamma \in \Upsilon(A) : \gamma(A) = k\}$. We equip $\Upsilon(\mathbb{R}^n)$ with the vague topology τ_v, i.e., $\gamma_n \in \Upsilon(\mathbb{R}^n)$ converges to $\gamma \in \Upsilon(\mathbb{R}^n)$ in τ_v if and only if $\gamma_n(f) \to \gamma(f)$ for any $f \in C_c(\mathbb{R}^n)$. For a subset $A \subset \mathbb{R}^n$, we equip $\Upsilon(A)$ with the relative topology as a subset in $\Upsilon(\mathbb{R}^n)$. Let $\mathscr{B}(\Upsilon(A), \tau_v)$ denote the Borel σ-algebra associated with the vague topology τ_v. For a set $A \subset \mathbb{R}^n$, let $\mathrm{pr}_A : \Upsilon(\mathbb{R}^n) \to \Upsilon(A)$ be the projection defined by the restriction of configurations on A, i.e. $\mathrm{pr}_A(\gamma) = \gamma|_A$.

Given $A \subset \mathbb{R}^n$, an open or closed domain, we denote by π_A the *Poisson measure* on $\Upsilon(A)$ whose intensity measure is the Lebesgue measure restricted on A, namely, π_A is the unique Borel probability measure so that, for all $f \in C_c(A)$, the following holds

$$\int_{\Upsilon(A)} e^{f^\star} d\pi_A = \exp\Big\{\int_A (e^f - 1) d\mathbf{L}^n(x)\Big\}. \tag{2.1}$$

Here $\mathbf{L}^n$ denotes the n-dimensional Lebesgue measure. See [32] for a reference of the expression (2.1). We write $\pi = \pi_{\mathbb{R}^n}$. Note that π_A coincides with the push-forwarded measure $\pi_A = (\mathrm{pr}_A)_\# \pi$. Let

$$\mathrm{diag}_k := \{(x)_{1\le i\le m} \in (\mathbb{R}^n)^{\times k} : \exists i, j \text{ s.t. } x_i = x_j\},$$

denote the diagonal set in $(\mathbb{R}^n)^{\times k}$, and let $\mathfrak{S}_k$ denote the k-symmetric group. For any set $A \subset \mathbb{R}^n$, we identify

$$\Upsilon^k(A) \cong (A^{\times k} \setminus \mathrm{diag}_k)/\mathfrak{S}_k, \quad k \in \mathbb{N}.$$

Let $\mathsf{s}_k : A^{\times k} \setminus \mathrm{diag}_k \to \Upsilon^k(A)$ be the canonical projection with respect to $\mathfrak{S}_k$, i.e. $\mathsf{s}_k : (x_i)_{1\le i\le k} \mapsto \sum_{i=1}^k \delta_{x_i}$. We say that a function $f : \sqcup_{k=1}^\infty (\mathbb{R}^n)^{\times k} \to \mathbb{R}$ is *symmetric* iff $f(\mathbf{x}_{\sigma_k}) = f(\mathbf{x}_k)$ with $\mathbf{x}_{\sigma_k} := (x_{\sigma_k(1)}, \dots, x_{\sigma_k(k)})$ for any permutation $\sigma_k \in \mathfrak{S}_k$ and any $k \in \mathbb{N}$.

For $\mathbf{x}_k, \mathbf{y}_k \in A^{\times k}$ with $\mathsf{s}_k(\mathbf{x}_k) = \gamma \in \Upsilon^k(A)$ and $\mathsf{s}_k(\mathbf{y}_k) = \eta \in \Upsilon^k(A)$, define *the L^2-transportation distance* $\mathsf{d}_{\Upsilon^k}(\gamma, \eta)$ on $\Upsilon^k(A)$ by the quotient metric w.r.t. $\mathfrak{S}_k$:

$$\mathsf{d}_{\Upsilon^k}(\gamma, \eta) = \inf_{\sigma_k \in \mathfrak{S}_k} |\mathbf{x}_{\sigma_k} - \mathbf{y}_k|_{\mathbb{R}^{nk}} \,. \tag{2.2}$$

Here $|\mathbf{x}_k - \mathbf{y}_k|_{\mathbb{R}^{nk}}$ denotes the standard Euclidean distance in $\mathbb{R}^{nk}$.

2.3. **Spherical Hausdorff measure.** Let (X, d) be a metric space and n be the Hausdorff dimension of X. For $m \le n$, define the *m-dimensional spherical Hausdorff measure* $\mathbf{S}_X^m$ on X:

(2.3)

$$\mathbf{S}_X^m(A) := \lim_{\varepsilon\to 0} \mathbf{S}_{X,\varepsilon}^m(A) := \lim_{\varepsilon\to 0} \inf\Big\{\sum_{i\in\mathbb{N}} \mathsf{diam}(B_i)^m : \ B_i \text{ open ball with } \mathsf{diam}(B_i) < \varepsilon,\ A \subset \sum_{i\in\mathbb{N}} B_i\Big\}.$$

Here $\mathsf{diam}(B_i) = \sup\{d(x,y) : x, y \in B_i\}$ denotes the diameter of B_i. We call $\mathbf{S}_{X,\varepsilon}^m$ the *m-dimensional ε-Hausdorff measure.* If $X = \mathbb{R}^n$, we simply write $\mathbf{S}^m$ and $\mathbf{S}_\varepsilon^m$ instead of $\mathbf{S}_{\mathbb{R}^n}^m$ and $\mathbf{S}_{\mathbb{R}^n,\varepsilon}^m$ respectively.

Remark 2.1 (Comparison with the standard Hausdorff measure)**.** In the case of $m < n$, the spherical Hausdorff measure $\mathbf{S}_X^m$ does not coincide with the standard Hausdorff measure in general, the latter is smaller since it is defined allowing *all non-empty coverings* instead of *open balls*. In the case of $m = n$ and $X = \mathbb{R}^n$, however, $\mathbf{S}^m$ coincides with the standard n-dimensional Hausdorff measure and also with the n-dimensional Lebesgue measure ([25, 2.10.35]). Note that $\mathbf{S}^m$ is a Borel measure, but not σ-finite for $m < n$.

For a bounded set $A \subset \mathbb{R}^n$, let $\mathbf{S}^n|_A$ be the spherical Hausdorff measure restricted on A. The spherical Hausdorff measure $(\mathbf{S}^n|_A)^{\otimes k}$ on $A^{\times k}$ can be push-forwarded to the k-configuration space $\Upsilon^k(A)$ by the projection map s_k, i.e.

$$\mathsf{S}_A^k := \frac{1}{k!}(\mathsf{s}_k)_\#(\mathbf{S}^n|_A)^{\otimes k} \,.$$

It is immediate by construction to see that S_A^k is the spherical Hausdorff measure on $\Upsilon^k(A)$ induced by the L^2-transportation distance d_{Υ^k} up to constant multiplication. We introduce the m-codimensional spherical Hausdorff measure and the m-codimensional ε-spherical Hausdorff measure on $\Upsilon^k(A)$ as follows

$$\mathsf{S}_A^{m,k} = \frac{1}{k!}(\mathsf{s}_k)_\#(\mathbf{S}^{nk-m}|_{A^{\times k}}), \quad \mathsf{S}_{A,\varepsilon}^{m,k} = \frac{1}{k!}(\mathsf{s}_k)_\#(\mathbf{S}_\varepsilon^{nk-m}|_{A^{\times k}}). \tag{2.4}$$

One can immediately see that $\mathsf{S}_{A,\varepsilon}^{m,k}$ is (up to constant multiplication) the m-codimensional ε-spherical Hausdorff measure on $\Upsilon^k(A)$ associated with the L^2-transportation distance d_{Υ^k}.

2.4. **Regularity of the spherical Hausdorff measures.** In this section, we prove the upper semi-continuity of the ε-spherical Hausdorff measure on sections of compact sets, which will be of use in Section 3.

Proposition 2.2. *Let (X, d_X), (Y, d_Y) be metric spaces, and $K \subset X \times Y$ be a compact set. Then, the map $Y \ni y \mapsto \mathbf{S}_{X,\varepsilon}^m(K^y)$ is upper semi-continuous. Here, $K^y := \{x \in X : (x, y) \in K\}$.*

Proof. Let us fix $y \in Y$ and a sequence $y_n \to y$. The family of compact sets $(K_{y_n} \times \{y_n\})_{n\in\mathbb{N}} \subset K$ is precompact with respect to the Hausdorff topology in K endowed with the product metric (e.g., [19, Theorem 7.3.8]). In particular, we can take a (non-relabeled) subsequence so that $K^{y_n} \times \{y_n\} \to \bar{K} \times \{y\} \subset K$, as $n \to \infty$ in the Hausdorff topology, and $\bar{K} \subset K^y$ by the definition of K^y.

Let us fix $\delta > 0$ and a family of open balls $B_1, \dots, B_\ell \subset X$ with radius smaller than $\varepsilon(1-\delta) > 0$ so that

$$\bar{K} \subset \bigcup_{i=1}^{\ell} B_i,$$

and

$$\mathbf{S}^m_{X,\varepsilon}(\bar{K}) \geq c(m) \sum_{i=1}^{\ell} r_i^m - \delta. \tag{2.5}$$

Here $c(m)$ denotes the constant depending on m such that $\mathsf{vol}_m(B_i) = c(m) r_i^m$. Note that we can always take $\ell = \ell(\delta)$ to be finite for any $\delta > 0$ by the compactness of $\bar{K}$. Let $\underline{r} = \underline{r}(\delta) := \min\{r_i : 1 \leq i \leq l(\delta)\} > 0$ be the minimum radius among $\{B_i\}_{1 \leq i \leq l}$.

We claim that there exists $\bar{k} = \bar{k}(\delta) \in \mathbb{N}$ so that $K^{y_{n_k}} \subset \cup_{i=1}^{\ell} B(x_i, \frac{1}{1-\delta} r_i)$ for any $k \geq \bar{k}$. Here x_i and r_i are the centre and the radius of B_i.

Indeed, by the Hausdorff convergence of $K^{y_{n_k}}$ to $\bar{K}$, there exists $\bar{k} := \bar{k}(\delta) \in \mathbb{N}$ such that, for any $k > \bar{k}$, it holds that $K^{y_{n_k}} \subset B_{\underline{r}\delta}(\bar{K})$. Here, $B_{\underline{r}\delta}(\bar{K})$ denotes the $\underline{r}\delta$-tubular neighbourhood of $\bar{K}$ in X, i.e., $B_{\underline{r}\delta}(\bar{K}) := \{x \in X : d(x, \bar{K}) < \underline{r}\delta\}$. Hence, for any $z \in K^{y_{n_k}}$, we can always find $x \in \bar{K}$ such that $d(x,z) < \underline{r}\delta$. By noting $x \in B_i$ for some $i = 1, \dots, \ell$, we conclude $z \in B(x_i, \frac{1}{1-\delta} r_i)$ by noting $\frac{1}{1-\delta} r_i - r_i = \frac{\delta}{1-\delta} r_i \geq \delta \underline{r}$.

By using the claim in the previous paragraph, the monotonicity $\mathbf{S}^m_{X,a} \geq \mathbf{S}^m_{X,b}$ whenever $a \leq b$, and (2.5), we obtain that

$$\mathbf{S}^m_{X,\varepsilon}(K^{y_{n_k}}) \leq c(m)(1+\delta)^m \sum_{i=1}^{\ell} r_i^m \leq (1+\delta)^m \mathbf{S}^m_{X,\varepsilon(1-\delta)}(\bar{K}) + \delta(1+\delta)^m c(m), \tag{2.6}$$

for any $k \geq \bar{k}(\delta)$. By taking $\delta \to 0$ after taking $k \to \infty$, we conclude that

$$\limsup_{k \to \infty} \mathbf{S}^m_{X,\varepsilon}(K^{y_{n_k}}) \leq \mathbf{S}^m_{X,\varepsilon}(\bar{K}) \leq \mathbf{S}^m_{X,\varepsilon}(K^y), \tag{2.7}$$

which is the sought conclusion. □

2.5. **Differential structure on configuration spaces.** In this section $\Omega \subset \mathbb{R}^n$ will be denoting either a closed domain with smooth boundary or the whole Euclidean space $\mathbb{R}^n$. Below we review the natural differential structure of $\Upsilon(\Omega)$, obtained by lifting the Euclidean one on Ω. We follow closely the presentation in [2].

Cylinder functions, vector fields and divergence. We introduce a class of test functions on the configuration space $\Upsilon(\mathbb{R}^n)$.

Definition 2.3 (Cylinder functions)**.** We define the class of *cylinder functions* as

$$\mathrm{CylF}(\Upsilon(\Omega)) := \left\{\Phi(f_1^\star, \dots, f_k^\star) \,:\, \Phi \in C_b^\infty(\mathbb{R}^k),\, f_i \in C_c^\infty(\Omega),\, k \in \mathbb{N}\right\}, \tag{2.8}$$

where $f^\star(\gamma) := \int_\Omega f d\gamma$ for any $\gamma \in \Upsilon(\Omega)$. We call f_i *inner function* and Φ *outer function.*

The tangent space $T_\gamma \Upsilon(\Omega)$ at $\gamma \in \Upsilon(\Omega)$ is identified with the Hilbert space of measurable γ-square-integrable vector fields $V : \Omega \to T(\mathbb{R}^n)$ equipped with the scalar product

$$\langle V^1, V^2 \rangle_{T\Upsilon} = \int_\Omega \langle V^1(x), V^2(x) \rangle_{T\mathbb{R}^n} d\gamma(x)\,.$$

We define the tangent bundle of $\Upsilon(\Omega)$ by $T\Upsilon(\Omega) := \sqcup_{\gamma \in \Upsilon(\Omega)} T_\gamma \Upsilon(\Omega)$.

Definition 2.4 (Cylinder vector fields)**.** We define two classes of *cylinder vector fields* as

$$\mathrm{CylV}(\Upsilon(\Omega)) := \left\{ V(\gamma,x) = \sum_{i=1}^{k} F_i(\gamma) v_i(x) : F_i \in \mathrm{CylF}(\Upsilon(\Omega)),\ v_i \in C_c^\infty(\Omega;\mathbb{R}^n), k \in \mathbb{N} \right\},$$

$$\mathrm{CylV}_0(\Upsilon(\Omega)) := \left\{ V(\gamma,x) = \sum_{i=1}^{k} F_i(\gamma) v_i(x) : F_i \in \mathrm{CylF}(\Upsilon(\Omega)),\ v_i \in C_0^\infty(\Omega;\mathbb{R}^n), k \in \mathbb{N} \right\}.$$

Notice that $\mathrm{CylV}_0(\Upsilon(\Omega)) \subset \mathrm{CylV}(\Upsilon(\Omega))$, and $\mathrm{CylV}_0(\Upsilon(\Omega)) = \mathrm{CylV}(\Upsilon(\Omega))$ when $\Omega = \mathbb{R}^n$.

Given $p \in [1,\infty)$, we introduce the space of p-integrable vector fields $L^p(T\Upsilon(\Omega),\pi)$ as the completion of $\mathrm{CylV}(\Upsilon(\Omega))$ with respect to the norm

$$\|V\|^p_{L^p(T\Upsilon)} := \int_{\Upsilon(\Omega)} |V(\gamma)|^p_{T_\gamma\Upsilon} d\pi(\gamma).$$

The latter is well defined because $\|V\|_{L^p(T\Upsilon)} < \infty$ for any $V \in \mathrm{CylV}(\Upsilon(\Omega))$ as the following proposition shows.

Proposition 2.5. *For any* $1 \le p < \infty$ *it holds*

$$\|V\|_{L^p(T\Upsilon)} < \infty \quad \textit{for any } V \in \mathrm{CylV}(\Upsilon(\Omega)). \tag{2.9}$$

Moreover, $\mathrm{CylV}_0(\Upsilon(\Omega))$ *is dense in* $L^p(T\Upsilon(\Omega),\pi)$*.*

Proof. Let $V(\gamma,x) = \sum_{i=1}^k F_i(\gamma)v_i(x)$. Then, we have that

$$\int_{\Upsilon(\Omega)} |V|^p_{T_\gamma\Upsilon(\Omega)} d\pi(\gamma) \le \max_{1\le i\le k} \|F\|_{L^\infty} \sum_{i,j=1}^{k} \int_{\Upsilon(\Omega)} \left(\int_\Omega |v_i||v_j| d\gamma \right)^{p/2} d\pi(\gamma).$$

By the exponential integrability implied by (2.1), we obtain that the function $\gamma \mapsto G(\gamma) := \int_\Omega |v_i||v_j|d\gamma$ is $L^p(\Upsilon(\Omega),\pi_\Omega)$ for any $1 \le p < \infty$, which concludes the first assertion.

The density of $\mathrm{CylV}_0(\Upsilon(\Omega))$ in $L^p(T\Upsilon(\Omega),\pi)$ follows from the density of $C_0^\infty(\Omega;\mathbb{R}^n)$ in $L^p(\Omega;\mathbb{R}^n)$. More precisely, we check that for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ and $\varepsilon > 0$ there exists $W \in \mathrm{CylV}_0(\Upsilon(\Omega))$ such that $\int_{\Upsilon(\Omega)} |V - W|^p_{T_\gamma\Upsilon} d\pi \le \varepsilon$. To this aim we write $V = \sum_{i=1}^k F_i v_i$ and pick $w_i \in C_0^\infty(\Omega)$ such that $\sum_{i=1}^k \|v_i - w_i\|_{L^p} < \varepsilon$ and set $W := \sum_{i=1}^k F_i w_i$. It is straightforward to see that W satisfies the needed estimate. Noting that $L^p(T\Upsilon(\Omega),\pi)$ is defined as the completion of $\mathrm{CylV}(\Upsilon(\Omega))$ with respect to the norm $\|V\|_{L^p(T\Upsilon)}$, the proof is complete. □

Definition 2.6 (Directional derivatives. [2, Def. 3.1])**.** Let $F = \Phi(f_1^*,\ldots,f_k^*) \in \mathrm{CylF}(\Upsilon(\Omega))$ and $v \in C_0^\infty(\Omega,\mathbb{R}^n)$. We denote by ϕ the flow associated to v, i.e.

$$\frac{d}{dt}\phi_t(x) = v(\phi_t(x)), \quad \phi_0(x) = x \in \Omega.$$

The *directional derivative* $\nabla_v F(\gamma) \in T_\gamma\Upsilon(\Omega)$ is defined by

$$\nabla_v F(\gamma) := \frac{d}{dt} F(\phi_t(\gamma))\Big|_{t=0},$$

where $\phi_t(\gamma) := \sum_{x\in\gamma} \delta_{\phi_t(x)}$.

Definition 2.7 (Gradient of cylinder functions. [2, Def. 3.3])**.** The gradient $\nabla_{\Upsilon(\Omega)}F$ of $F \in \mathrm{CylF}(\Upsilon(\Omega))$ at $\gamma \in \Upsilon(\Omega)$ is defined as the unique vector field $\nabla_{\Upsilon(\Omega)}F$ so that

$$\nabla_v F(\gamma) = \langle \nabla_{\Upsilon(\Omega)} F, v \rangle_{T_\gamma\Upsilon(\Omega)}, \quad \forall \gamma \in \Upsilon(\Omega),\ \forall v \in C_0^\infty(\Omega,\mathbb{R}^n).$$

By the expression (2.8), the gradient $\nabla_{\Upsilon(\Omega)}F$ can be written as

$$\nabla_{\Upsilon(\Omega)} F(\gamma) = \sum_{i=1}^{k} \partial_i \Phi(f_1^\star,\ldots,f_k^\star)(\gamma) \nabla f_i \in T_\gamma\Upsilon(\Omega). \tag{2.10}$$

We set $\nabla := \nabla_{\Upsilon(\mathbb{R}^n)}$.

Notice that $\nabla_{\Upsilon(\Omega)} F \in \mathrm{CylV}(\Upsilon(\Omega))$ for any $F \in \mathrm{CylF}(\Upsilon(\Omega))$ by (2.10). In particular, for any $F \in \mathrm{CylF}(\Omega)$, it holds that $\nabla_{\Upsilon(\Omega)} F \in L^p(T\Upsilon(\Omega), \pi)$ for any $1 \le p < \infty$ by Proposition 2.5.

Remark 2.8 (Ampleness of L^∞-vector fields)**.** By Proposition 2.5, $\mathrm{CylV}_0(\Upsilon(\Omega)) \subset L^p(T\Upsilon(\Omega), \pi)$ for any $p \in [1, \infty)$, while the inclusion is false for $p = \infty$. See [23, Example 4.35] for a counterexample. However, $\mathrm{CylV}_0(\Upsilon(\Omega))$ can be approximated by the subspace of bounded cylinder vector fields with respect to the pointwise convergence and the $L^p(T\Upsilon)$ convergence for $1 \le p < \infty$. Indeed, given $\varepsilon > 0$ and $V = \sum_{i=1}^k F_i(\gamma) v_i(x) \in \mathrm{CylV}_0(\Upsilon(\Omega))$ it holds

$$|V|^2_{T_\gamma \Upsilon} = \sum_{i,j=1}^k F_i(\gamma) F_j(\gamma) \int_{\Upsilon(\Omega)} v_i(x) \cdot v_j(x) d\gamma(x) \,, \qquad \frac{1}{1 + \varepsilon |V|^2_{T_\gamma \Upsilon}} \in \mathrm{CylF}(\Upsilon(\Omega)) \,,$$

hence

$$V_\varepsilon := \frac{1}{1 + \varepsilon |V|^2_{T_\gamma \Upsilon(\Omega)}} V \in \mathrm{CylV}(\Upsilon(\Omega)) \,.$$

Finally, notice that for any $\gamma \in \Upsilon(\Omega)$ it holds

$$|V - V_\varepsilon|_{T_\gamma \Upsilon(\Omega)} = \varepsilon \frac{|V|^3_{T_\gamma \Upsilon(\Omega)}}{1 + \varepsilon |V|^2_{T_\gamma \Upsilon(\Omega)}} \le \varepsilon |V|^3_{T_\gamma \Upsilon(\Omega)} \to 0 \,, \quad \text{as } \varepsilon \to 0 \,.$$

Moreover, for any $1 \le p < \infty$ we have

$$\|V - V_\varepsilon\|_{L^p(T\Upsilon)} \le \varepsilon \|V\|^3_{L^{3p}(T\Upsilon)} \to 0 \,, \quad \text{as } \varepsilon \to 0 \,.$$

We now define the adjoint operator of the gradient $\nabla_{\Upsilon(\Omega)}$.

Definition 2.9 (Divergence. [2, Def. 3.5])**.** Let $p > 1$. We say that $V \in L^p(T\Upsilon(\Omega), \pi)$ is in the domain $\mathcal{D}(\nabla^*_{\Upsilon(\Omega)})$ of the divergence if there exists a unique function $\nabla^*_{\Upsilon(\Omega)} V \in L^p(\Upsilon(\Omega), \pi)$ such that

$$\int_{\Upsilon(\Omega)} \langle V, \nabla_{\Upsilon(\Omega)} F \rangle_{T_\gamma \Upsilon} d\pi(\gamma) = - \int_{\Upsilon(\Omega)} F(\nabla^*_{\Upsilon(\Omega)} V) d\pi \,, \quad \forall F \in \mathrm{CylF}(\Upsilon(\Omega)). \tag{2.11}$$

We set $\nabla^* = \nabla^*_{\Upsilon(\mathbb{R}^n)}$.

Proposition 2.10. *Any $V(\gamma, x) = \sum_{i=1}^k F_i(\gamma) v_i(x) \in \mathrm{CylV}_0(\Upsilon(\Omega))$ belongs to the domain $\mathcal{D}(\nabla^*_{\Upsilon(\Omega)})$ of the divergence and the following identity holds*

$$\nabla^*_{\Upsilon(\Omega)} V(\gamma) = \sum_{i=1}^k \nabla_{v_i} F_i(\gamma) + \sum_{i=1}^k F_i(\gamma) (\nabla^* v_i)^\star \gamma \,. \tag{2.12}$$

In particular, $\nabla^*_{\Upsilon(\Omega)} V \in L^p(\Upsilon(\Omega), \pi)$ for any $p \in [1, \infty)$.

Proof. Let $r > 0$ be such that $\mathrm{supp}(v_i) \subset \Omega_r := \{x \in \Omega : d(x, \partial\Omega) > r\}$. For any $\varepsilon < r/2$ we define $\phi_\varepsilon \in C_0^\infty(\Omega)$ satisfying $\phi = 1$ on Ω_ε. For any $i = 1, \dots, k$ we write $F_i = \Phi_i(f^*_{1,i}, \dots, f^*_{m_i,i})$ and set $F_i^\varepsilon := \Phi_i((\phi_\varepsilon f_{1,i})^*, \dots, (\phi_\varepsilon f_{m_i,i})^*)$. Observe that $V_\varepsilon := \sum_{i=1}^k F_i^\varepsilon(\gamma) v_i \in \mathrm{CylV}_0(\Upsilon(\Omega))$ and $V_\varepsilon \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$. By [2, Prop. 3.1] it holds

$$\begin{aligned} \nabla^*_{\Upsilon(\Omega)} V_\varepsilon(\gamma) = \nabla^*_{\Upsilon(\mathbb{R}^n)} V_\varepsilon(\gamma) &= \sum_{i=1}^k \nabla_{v_i} F_i^\varepsilon(\gamma) + \sum_{i=1}^k F_i^\varepsilon(\gamma) (\nabla^* v_i)^\star \gamma \\ &= \sum_{i=1}^k \nabla_{v_i} F_i(\gamma) + \sum_{i=1}^k F_i^\varepsilon(\gamma) (\nabla^* v_i)^\star \gamma \,. \end{aligned}$$

Here we used that $F_i = F_i^\varepsilon$ on any γ concentrated on the support of v_i. The sought conclusion (2.12) follows from the observation that $F_i^\varepsilon \to F_i$ in $L^p(\Upsilon(\Omega), \pi)$ and $V_\varepsilon \to V$ in $L^p(T\Upsilon(\mathbb{R}^n), \pi)$ combined with (2.11). The last assertion is then a direct consequence from Proposition 2.5 and (2.12). □

Sobolev spaces. We now introduce the $(1,p)$-Sobolev space. The operator

$$\nabla_{\Upsilon(\Omega)} : \mathrm{CylF}(\Upsilon(\Omega)) \subset L^p(\Upsilon(\Omega),\pi) \to \mathrm{CylV}(\Upsilon(\Omega)) \tag{2.13}$$

is densely defined and closable. The latter fact is a direct consequence of the integration by parts formula (2.12). Indeed, we observe that, if $F_n \in \mathrm{CylF}(\Upsilon(\Omega))$, $F_n \to 0$ in $L^p(\Upsilon(\Omega),\pi)$, and $\nabla_{\Upsilon(\Omega)} F_n \to W$ in $L^p(T\Upsilon(\Omega)),\pi)$, then for any $V \in \mathrm{CylV}_0(\Upsilon(\Omega))$, it holds

$$\int_{\Upsilon(\Omega)} \langle V, W\rangle_{T_\gamma\Upsilon} d\pi(\gamma) = \lim_{n\to\infty} \int_{\Upsilon(\Omega)} \langle V, \nabla_{\Upsilon(\Omega)} F_n\rangle_{T_\gamma\Upsilon} d\pi(\gamma) = -\lim_{n\to\infty} \int_{\Upsilon(\Omega)} (\nabla^*_{\Upsilon(\Omega)} V) F_n d\pi(\gamma) = 0\,,$$

yielding $W = 0$ as a consequence of the density of $\mathrm{CylV}_0(\Upsilon(\Omega))$ in $L^p(T\Upsilon(\Omega)),\pi)$ by Proposition 2.5. The above argument justifies the following definition.

Definition 2.11 ($H^{1,p}$-Sobolev spaces)**.** Let $1 < p < \infty$. We define $H^{1,p}(\Upsilon(\Omega),\pi)$ as the closure of $\mathrm{CylF}(\Upsilon(\Omega))$ in $L^p(\Upsilon(\Omega),\pi)$ with respect to the following $(1,p)$-Sobolev norm:

$$\|F\|^p_{H^{1,p}(\Upsilon(\Omega))} := \|F\|^p_{L^p(\Upsilon(\Omega))} + \|\nabla_{\Upsilon(\Omega)} F\|^p_{L^p(\Upsilon(\Omega))}\,.$$

We set $\|F\|_{H^{1,p}} := \|F\|_{H^{1,p}(\Upsilon(\mathbb{R}^n))}$. When $p = 2$, we write the corresponding Dirichlet form (i.e., a closed form satisfying the unit contraction property [34, Def. 4.5]) by

$$\mathcal{E}_{\Upsilon(\Omega)}(F,G) := \int_{\Upsilon(\Omega)} \big\langle \nabla_{\Upsilon(\Omega)} F, \nabla_{\Upsilon(\Omega)} G\big\rangle_{T_\gamma\Upsilon(\Omega)} d\pi(\gamma), \quad F, G \in H^{1,2}(\Upsilon(\Omega),\pi)\,.$$

We set $\mathcal{E} := \mathcal{E}_{\Upsilon(\mathbb{R}^n)}$.

Remark 2.12 (The case of $p = 1$)**.** As is indicated by (2.12), it is not true in general that $\nabla^*_{\Upsilon(\Omega)} V \in L^\infty(\Upsilon(\Omega),\pi)$ since arbitrarily many finite particles can be concentrated on the supports of inner functions of $F \in \mathrm{CylF}(\Upsilon(\Omega))$ and vector fields v_i. See [23, Example 4.35] for more details. Due to this fact, the standard integration by part argument for the closability of the operator $\nabla_{\Upsilon(\Omega)} : \mathrm{CylF}(\Upsilon(\Omega)) \to \mathrm{CylV}(\Upsilon(\Omega)) \subset L^p(T\Upsilon(\Omega))$ does not work in the case of $p = 1$. For this reason, we restricted the definition of the $H^{1,p}$-Sobolev spaces to the case $1 < p < \infty$ in Definition 2.11.

Once the closed form $\mathcal{E}_{\Upsilon(\Omega)}$ on $L^2(\Upsilon(\Omega),\pi)$ is constructed, one can define the infinitesimal generator on $L^2(\Upsilon(\Omega),\pi)$ as the unique non-positive definite self-adjoint operator by general theory of functional analysis.

Definition 2.13 (Laplace operator [2, Theorem 4.1])**.** *The $L^2(\Upsilon(\Omega),\pi)$-Laplace operator $\Delta_{\Upsilon(\Omega)}$* with domain $\mathcal{D}(\Delta_{\Upsilon(\Omega)})$ is defined as the unique non-positive definite self-adjoint operator $\Delta_{\Upsilon(\Omega)}$ so that

$$\mathcal{E}_{\Upsilon(\Omega)}(F,G) = -\int_{\Upsilon(\Omega)} (\Delta_{\Upsilon(\Omega)} F) G d\pi, \quad F \in \mathcal{D}(\Delta_{\Upsilon(\Omega)}),\ G \in \mathcal{D}(\mathcal{E}_{\Upsilon(\Omega)})\,.$$

In the case of $\Omega = \mathbb{R}^n$, employing (2.10) and (2.12), one can compute that

$$\Delta_{\Upsilon(\mathbb{R}^n)} F := \nabla^*_{\Upsilon(\mathbb{R}^n)} \nabla_{\Upsilon(\mathbb{R}^n)} F, \quad \forall F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))\,.$$

Let $\{T_t^{\Upsilon(\Omega)}\}$ and $\{G_\alpha^{\Upsilon(\Omega)}\}$ be the L^2-contraction strongly continuous Markovian semigroup and the resolvent, respectively, corresponding to the energy $\mathcal{E}_{\Upsilon(\Omega)}$. We set $G_\alpha := G_\alpha^{\Upsilon(\mathbb{R}^n)}$ and $T_t := T_t^{\Upsilon(\mathbb{R}^n)}$. By the Markovian property and Riesz–Thorin Interpolation Theorem, the L^2-operators $T_t^{\Upsilon(\Omega)}$ and $\{G_\alpha^{\Upsilon(\Omega)}\}$ can be uniquely extended to the L^p-contraction strongly continuous Markovian semigroup and resolvent, respectively, for any $1 \le p < \infty$ (see e.g. [42, Section 2, p. 70]).

2.6. **Product semigroups and exponential cylinder functions.** In this section, we relate the finite-product semigroup on $\Omega^{\times k}$ and the semigroup on $\Upsilon^k(\Omega)$ when $\Omega \subset \mathbb{R}^n$ is a bounded closed domain with smooth boundary. To this aim we introduce a class of test functions, which is suitable to compute the semigroups.

Definition 2.14 (Exponential cylinder functions. [2, (4.12)])**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded closed domain with smooth boundary. A class $\mathrm{ECyl}(\Upsilon(\Omega))$ of *exponential cylinder functions* is defined as the vector space spanned by

$$\left\{\exp\left\{\log(1+f)^\star\right\} \,:\, f \in \mathcal{D}(\Delta_\Omega) \cap L^2(\Omega)\,, \Delta_\Omega f \in L^1(\Omega),\ -\delta \le f \le 0 \text{ for some } \delta \in (0,1)\right\}\,.$$

Here $(\Delta_\Omega, \mathcal{D}(\Delta_\Omega))$ denotes the L^2-Neumann Laplacian on Ω.

The space $\mathrm{ECyl}(\Upsilon(\Omega))$ is dense in $L^p(\Upsilon(\Omega))$ for any $1 \le p < \infty$ (see [2, p. 479]). Noting that Δ_Ω is essentially self-adjoint on the core $C_c^\infty(\Omega) \cap \{\frac{\partial f}{\partial n} = 0 \text{ in } \partial\Omega\}$, where $\frac{\partial}{\partial n}$ is the normal derivative on $\partial\Omega$, and the corresponding L^2-semigroup $\{T_t^\Omega\}$ is conservative, we can apply the same argument in the proof of [2, Prop. 4.1] to obtain the following: $T_t^{\Upsilon(\Omega)}\mathrm{ECyl}(\Upsilon(\Omega)) \subset \mathrm{ECyl}(\Upsilon(\Omega))$ and

$$T_t^{\Upsilon(\Omega)} \exp\left\{\log(1+f)^\star\right\} = \exp\left\{\log\left(1 + (T_t^\Omega f)\right)^\star\right\}. \tag{2.14}$$

Let $T_t^{\Omega,\otimes k}$ be the k-tensor semigroup of T_t^Ω, i.e. the unique semigroup in $L^p(\Omega^k)$ satisfying

$$T_t^{\Omega,\otimes k} f(x_1, \dots, x_k) := T_t^\Omega f_1(x_1) \cdots T_t^\Omega f_k(x_k) \tag{2.15}$$

whenever $f(x_1, \dots, x_k) = f_1(x_1) \cdot \ldots \cdot f_k(x_k)$ where $f_i \in L^\infty(\Omega)$ for any $i = 1, \dots, k$.

Proposition 2.15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded closed domain with smooth boundary and $1 \le p < \infty$. For $F \in L^p(\Upsilon^k(\Omega), \pi_\Omega|_{\Upsilon^k(\Omega)})$, it holds*

$$T_t^{\Omega,\otimes k}(F \circ \mathsf{s}_k) = (T_t^{\Upsilon^k(\Omega)} F) \circ \mathsf{s}_k, \quad \mathbf{S}^{kn}_{\Omega^{\times k}}\text{-a.e.} \tag{2.16}$$

Proof. Since $\mathrm{ECyl}(\Upsilon^k(\Omega))$ is dense in $L^p(\Upsilon^k(\Omega))$ for any $1 \le p < \infty$, it suffices to show (2.16) only for $F \in \mathrm{ECyl}(\Upsilon^k(\Omega))$. Furthermore, we can reduce the argument to the case $F = \exp\{\log(1+f)^\star\}$ by using the linearity of semigroups. From (2.15) and (2.14) we get

$$\begin{aligned}
T_t^{\Omega,\otimes k}(F \circ \mathsf{s}_k)(x_1, \dots, x_k) &= T_t^{\Omega,\otimes k}(\exp\{\log(1+f)^\star\} \circ \mathsf{s}_k)(x_1, \dots, x_k) \\
&= T_t^{\Omega,\otimes k}\Big(\prod_{i=1}^k (1+f)(\cdot_i)\Big)(x_1, \dots, x_k) \\
&= \prod_{i=1}^k \Big(1 + T_t^\Omega f(x_i)\Big) \\
&= \exp\Big\{\log\big(1 + (T_t^\Omega f)^\star\big)\Big\} \circ \mathsf{s}_k(x_1, \dots, x_k) \\
&= T_t^{\Upsilon(\Omega)} \exp\{\log(1+f)^\star\} \circ \mathsf{s}_k(x_1, \dots, x_k)\,.
\end{aligned}$$

The proof is complete. □

2.7. **Suslin sets.** Let X be a set. We denote by $\mathbb{N}^\mathbb{N}$ the space of all infinite sequences $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers. For $\phi \in \mathbb{N}^\mathbb{N}$, we write $\phi|_l \in \mathbb{N}^l$ for the restriction of ϕ to the first l elements, i.e., $\phi_l := (\phi_i : 1 \le i \le l)$. Let $\mathcal{S} := \cup_{l\in\mathbb{N}}\mathbb{N}^l$, and for $\sigma \in \mathcal{S}$, we denote the length of the sequence σ by $\#\sigma := \#\{\sigma_i\}$. Let $\mathscr{E} \subset 2^X$ be a family of subsets in X. We write $\mathcal{S}(\mathscr{E})$ for the family of sets expressible in the following form:

$$\bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{l\ge 1} E_{\phi|_l},$$

for some family $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ in $\mathscr{E}$. A family $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ is called *Suslin scheme*; the corresponding set $\cup_{\phi\in\mathbb{N}^\mathbb{N}} \cap_{l\ge 1} E_{\phi|_l}$ is its *kernel*; the operation

$$\{E_\sigma\}_{\sigma\in\mathcal{S}} \mapsto \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{l\ge 1} E_{\phi|_l},$$

is called *Suslin's operation.* Thus, $\mathcal{S}(\mathscr{E})$ is the family of sets generated from sets in $\mathscr{E}$ by Suslin's operation, whose element is called an $\mathscr{E}$-Suslin set. It is well-known that $\mathcal{S}(\mathscr{E})$ is closed under Suslin's operation ([43], and e.g., [27, 421D Theorem]).

Assume that

$$(X,\tau) \text{ is a Polish space, } \mathsf{c} \text{ is a Choque capacity on } X,\ \mu \text{ is a bounded Borel measure,}$$
$$\mathscr{E} := \mathcal{C}(X) := \{C : \text{closed set in } X\}, \tag{2.17}$$

and $A \in \mathcal{S}(\mathscr{E})$ is simply called a *Suslin set.* We refer readers to, e.g., [27, 432I Definition] for the definition of Choque capacity. If E_σ is compact for all $\sigma \in \mathcal{S}$, we call $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ a *compact* Suslin scheme. We say that $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ is regular if $E_\sigma \subset E_\tau$ whenever $\#\tau \le \#\sigma$ and $\sigma_i \le \tau_i$ for any $i < \#\sigma$ ([27, 421X (n) & 422H Theorem (b)]).

Remark 2.16. Under the assumption (2.17), the following hold:

(i) Every Borel set is a Suslin set, i.e., $\mathcal{B}(\tau) \subset \mathcal{S}(\mathscr{E})$ (e.g., [17, 6.6.7. Corollary]);

(ii) Every Suslin set is μ-measurable, i.e., $\mathcal{S}(\mathscr{E}) \subset \overline{\mathcal{B}}^{\mu}(\tau)$ (e.g., [27, 431B Corollary]);

(iii) Let A be a Suslin set in X. Then, A is the kernel of a *compact regular* Suslin scheme $\{E_\sigma\}_{\sigma\in\mathcal{S}}$. Furthermore, it holds that

$$\mathsf{c}(A) = \sup_{\psi\in\mathbb{N}^{\mathbb{N}}} \mathsf{c}(A_\psi), \quad A_\psi = \bigcup_{\phi\le\psi}\bigcap_{l\ge1} E_{\phi|_l}, \tag{2.18}$$

whereby $\phi \le \psi$ means that $\phi(l) \le \psi(l)$ for all $l \in \mathbb{N}$ (e.g., [27, 423B Theorem & the proof of 432J Theorem]). By the regularity of $\{E_\sigma\}_{\sigma\in\mathcal{S}}$, (2.18) can be reduced to the following form:

$$\mathsf{c}(A) = \sup_{\psi\in\mathbb{N}^{\mathbb{N}}} \mathsf{c}(A_\psi), \quad A_\psi = \bigcap_{l\ge1} E_{\psi|_l}, \quad \psi \in \mathbb{N}^{\mathbb{N}}; \tag{2.19}$$

(iv) A subset $A \subset X$ is Suslin iff A is analytic iff A is K-analytic ([27, 423E Theorem (b)]).

3. Finite-codimensional Poisson measures

In this section, we construct finite-codimensional Poisson measures on $\Upsilon(\mathbb{R}^n)$. As a first step we prove measurability results for sections of *Suslin* subsets of the configuration space.

3.1. Measurability of sections of Suslin sets.

Let $B \subset \mathbb{R}^n$. For $A \subset \Upsilon(\mathbb{R}^n)$ and $\eta \in \Upsilon(B)$, the section $A_{\eta,B} \subset \Upsilon(B^c)$ of A at η is defined as

$$A_{\eta,B} = \{\gamma \in \Upsilon(B^c) : \gamma + \eta \in A\}. \tag{3.1}$$

The subset of $A_{\eta,B}$ consisting of k-particle space $\Upsilon^k(B^c)$ is denoted by $A^k_{\eta,B} := A_{\eta,B} \cap \Upsilon^k(B^c)$. To shorten notation we often write $A_{\eta,r}$ in place of A_{η,B_r^c}, where B_r is the closed ball centered at the origin.

Lemma 3.1. *Let $B \subset \mathbb{R}^n$ be a Borel set. If A is Suslin in $\Upsilon(\mathbb{R}^n)$ then $A^k_{\eta,B}$ is Suslin in $\Upsilon^k(B^c)$ for any $\eta \in \Upsilon(B)$, $k \in \mathbb{N}$ and $r > 0$.*

Proof. We can express $A_{\eta,B} = \mathrm{pr}_{B^c}\big(\mathrm{pr}_B^{-1}(\eta) \cap A\big)$. The set $\mathrm{pr}_B^{-1}(\eta) \cap A$ is Suslin in $\Upsilon(\mathbb{R}^n)$ whenever A is Suslin. Set $\Upsilon_{\eta,B}(\mathbb{R}^n) = \mathrm{pr}_B^{-1}(\eta) \cap \Upsilon(\mathbb{R}^n)$, which is Suslin. The map $\mathrm{pr}_{B^c} : \Upsilon_{\eta,B}(\mathbb{R}^n) \to \Upsilon(B^c)$ is continuous by definition. Thus, $A_{\eta,B}$ is the continuous image $\mathrm{pr}_{B^c}\big(\mathrm{pr}_B^{-1}(\eta) \cap A\big)$ of the Suslin set $\mathrm{pr}_B^{-1}(\eta) \cap A$ in the Suslin Hausdorff space $\Upsilon_{\eta,B}(\mathbb{R}^n)$. Hence, $A_{\eta,B}$ is Suslin ([27, 423B Proposition (b) & 423E Theorem (b)]). By noting that $A^k_{\eta,B} = A_{\eta,B} \cap \Upsilon^k(B^c)$, and that $\Upsilon^k(B^c)$ is Borel in $\Upsilon(B^c)$, we conclude that $A^k_{\eta,B}$ is Suslin. □

Lemma 3.2. *Let $B \subset \mathbb{R}^n$ be an open set. Let $A \subset \Upsilon(\mathbb{R}^n)$ be the kernel of compact Suslin's scheme $\{E_\sigma\}_{\sigma\in\mathcal{S}}$, i.e., $A = \cup_{\phi\in\mathbb{N}^{\mathbb{N}}} \cap_{l\ge1} E_{\phi|_l}$ with E_σ compact for any $\sigma \in \mathcal{S}$. Then, $A_{\eta,B}$ is the kernel of compact Suslin's scheme $\{(E_\sigma)_{\eta,r}\}_{\sigma\in\mathcal{S}}$.*

Proof. By expressing $(E_\sigma)_{\eta,B} = \mathrm{pr}_{B^c}\big(\Upsilon_{\eta,B}(\mathbb{R}^n) \cap E_\sigma\big)$, where $\Upsilon_{\eta,B}(\mathbb{R}^n) = \mathrm{pr}_B^{-1}(\eta) \cap \Upsilon(\mathbb{R}^n)$, we see that $(E_\sigma)_{\eta,B}$ is compact since $\Upsilon_{\eta,B}(\mathbb{R}^n)$ is closed, E_σ is compact by the hypothesis, pr_{B^c} is continuous on $\Upsilon_{\eta,B}(\mathbb{R}^n)$ and every continuous image of compact sets is compact. To see that $A_{\eta,B}$ is the kernel of $\{(E_\sigma)_{\eta,r}\}_{\sigma\in\mathcal{S}}$,

$$A_{\eta,B} = p_r\big(\Upsilon_{\eta,B}(\mathbb{R}^n) \cap A\big) = p_r\left(\Upsilon_{\eta,B}(\mathbb{R}^n) \cap \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{l\geq 1} E_{\phi|_l}\right) = p_r\left(\bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{l\geq 1} \Upsilon_{\eta,B}(\mathbb{R}^n) \cap E_{\phi|_l}\right)$$
$$= \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{l\geq 1} p_r\big(\Upsilon_{\eta,B}(\mathbb{R}^n) \cap E_{\phi|_l}\big) = \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{l\geq 1} (E_\sigma)_{\eta,B}\,.$$

The proof is complete. □

3.2. **Localised finite-codimensional Poisson measures.** In this section, we construct *a localised* version of the m-codimensional Poisson measure ρ_r^m, which will be used to construct the m-condimensional Poisson measure by taking the limit for $r \to \infty$. We also show that Suslin sets are contained in the domain of the finite-codimensional Poisson measure.

Let $A \subset \Upsilon(\mathbb{R}^n)$ be a Suslin subset. By Lemma 3.1, the set $A^k_{\eta,r} = A^k_{\eta,B_r^c}$ is Suslin, thus also analytic by (iv) of Remark 2.16. Since $\mathsf{S}^{m,k}_{B_r}$ is an outer measure on $\Upsilon^k(B_r)$ by construction, $\mathsf{S}^{m,k}_{B_r}$ is a Choque capacity. Therefore, all analytic sets are contained in the completion of the domain of $\mathsf{S}^{m,k}_{B_r}$ (see e.g. [27, 432J]), hence $\mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r})$ is well-defined. We define the domain $\mathscr{D}^m$ of the m-codimensional measures by

$$\mathscr{D}^m := \bigcap_{r>0} \mathscr{D}^m_r, \tag{3.2}$$

where the localised domain $\mathscr{D}^m_r$ is defined by

$$\mathscr{D}^m_r := \{A \subset \Upsilon(\mathbb{R}^n) \,:\, \text{the map } \Upsilon(B_r^c) \ni \eta \mapsto \mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r}) \text{ is } \pi_{B_r^c}\text{-measurable for every } k\}\,.$$

We first introduce the m-codimensional Poisson measure on the configuration space $\Upsilon(B_r)$ over the ball B_r.

Definition 3.3. The *m-codimensional Poisson measure* $\rho^m_{\Upsilon(B_r)}$ *on* $\Upsilon(B_r)$ is defined as

$$\rho^m_{\Upsilon(B_r)}(A) := e^{-\mathbf{S}^n(B_r)} \sum_{k=1}^{\infty} \mathsf{S}^{m,k}_{B_r}(A^k) \quad \text{for any Suslin set } A \text{ in } \Upsilon(B_r)\,, \tag{3.3}$$

where $A^k = A \cap \Upsilon^k(B_r)$.

Remark 3.4. Notice that $\rho^0_{\Upsilon(B_r)} = \pi_{B_r}$, in other words the 0-codimension Poisson measure $\rho^0_{\Upsilon(B_r)}$ on $\Upsilon(B_r)$ is the Poisson measure π_{B_r} on $\Upsilon(B_r)$. It can be shown by noting that the m-dimensional spherical Hausdorff measure $\mathbf{S}^m$ and the n-dimensional Lebesgue measure $\mathbf{L}^n$ coincide when $m = n$ (see Remark 2.1).

We introduce the localised m-codimensional Poisson measure on $\Upsilon(\mathbb{R}^n)$ by averaging the m-codimensional Poisson measure $\rho^m_{\Upsilon(B_r)}$ by means of $\pi_{B_r^c}$.

Definition 3.5. The *localised m-codimensional Poisson measure* ρ^m_r *on* $\Upsilon(\mathbb{R}^n)$ is defined by

$$\rho^m_r(A) = \int_{\Upsilon(B_r^c)} \rho^m_{\Upsilon(B_r)}(A_{\eta,r}) d\pi_{B_r^c}(\eta), \quad \forall A \in \mathscr{D}^m. \tag{3.4}$$

Before investigating the main properties of ρ^m_r, we check that sufficiently many sets are contained in $\mathscr{D}^m$, i.e. we show that all Suslin sets are contained in the domain $\mathscr{D}^m$ for $m \leq n$.

Proposition 3.6. *Any Suslin set in $\Upsilon(\mathbb{R}^n)$ is contained in $\mathscr{D}^m$ for $m \leq n$.*

Proof. Let $A \subset \Upsilon(\mathbb{R}^n)$ be a Suslin set. Let $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ be a Suslin scheme whose kernel is A. Noting that $\Upsilon(B_r^c)$ is Polish, by applying (i) of Remark 2.16 with $X = \Upsilon(B_r^c)$ and $\mu = \pi_{B_r^c}$, any Suslin set is $\pi_{B_r^c}$-measurable. Hence, it suffices to show that every super-level set $\{\eta : \mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r}) > a\}$ is Suslin for any $a \in \mathbb{R}$, $r > 0, k \in \mathbb{N}$ and $m \le n$. Note that $A^k_{\eta,r}$ is Suslin by Lemma 3.1, whence the expression $\{\eta : \mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r}) > a\}$ is well-defined as was discussed in the paragraph before (3.2).

Since $\Upsilon(\mathbb{R}^n)$ is Polish, by using (iii) in Remark 2.16, we may assume that $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ is a compact regular Suslin scheme. By Lemma 3.2 and $\Upsilon(B_r) = \sqcup_{k\in\mathbb{N}}\Upsilon^k(B_r)$, we see that $A^k_{\eta,r} \subset \Upsilon^k(B_r)$ is the kernel of the compact regular Suslin scheme $\{(E_\sigma)^k_{\eta,r}\}_{\sigma\in\mathcal{S}}$, whereby $(E_\sigma)^k_{\eta,r} := (E_\sigma)_{\eta,B_r^c} \cap \Upsilon^k(B_r)$. Since $\mathsf{S}^{m,k}_{B_r}$ is an outer measure on $\Upsilon^k(B_r)$ by construction, $\mathsf{S}^{m,k}_{B_r}$ is a Choque capacity on $\Upsilon^k(B_r)$. Hence, by applying (2.19) in (iii) of Remark 2.16 with $X = \Upsilon^k(B_r)$ and $\mathsf{c} = \mathsf{S}^{m,k}_{B_r}$, we obtain that

$$\mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r}) = \sup_{\psi\in\mathbb{N}^\mathbb{N}} \mathsf{S}^{m,k}_{B_r}((A^k_{\eta,r})_\psi), \quad (A^k_{\eta,r})_\psi = \bigcap_{l\ge1}(E_{\psi|_l})^k_{\eta,r}, \quad \psi \in \mathbb{N}^\mathbb{N}.$$

Thus, noting the monotonicity $\mathsf{S}^{m,k}_{B_r,\varepsilon} \le \mathsf{S}^{m,k}_{B_r,\delta}$ $(\delta \le \varepsilon)$ of the ε-Hausdorff measure defined in (2.3), the super-level set $\{\eta : \mathsf{S}^{m,k}_r(A^k_{\eta,r}) > a\}$ can be expressed in the following way:

$$\{\eta : \mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r}) > a\} = \bigcup_{\varepsilon>0}\bigcup_{\psi\in\mathbb{N}^\mathbb{N}} \{\eta : \mathsf{S}^{m,k}_{B_r,\varepsilon}\big((A^k_{\eta,r})_\psi\big) > a\}.$$

Since the space $\mathcal{S}(\mathscr{E})$ of Suslin sets is closed under Suslin's operation, it suffices to show that $\{\eta : \mathsf{S}^{m,k}_{B_r,\varepsilon}\big((A^k_{\eta,r})_\psi\big) > a\}$ is Suslin.

We equip $\Upsilon^k(B_r)$ with the L^2-transportation distance d_{Υ^k} as defined in (2.2), and equip $\Upsilon(B_r^c)$ with some distance d generating the vague topology. By Proposition 2.2 and noting that $(A^k_{\eta,r})_\psi$ is compact and that $\mathsf{S}^{m,k}_{B_r,\varepsilon}$ is (up to constant multiplication) the m-codimensional ε-spherical Hausdorff measure on $\Upsilon^k(B_r)$ associated with d_{Υ^k}, we conclude that $\{\eta : \mathsf{S}^{m,k}_{B_r,\varepsilon}\big((A^k_{\eta,r})_\psi\big) > a\}$ is open in $\Upsilon(B_r^c)$ for any $a \in \mathbb{R}$, $r > 0, k \in \mathbb{N}$ and $m \le n$. The proof is complete. □

3.3. **Finite-codimensional Poisson Measures.** In this section, we construct the m-codimensional Poisson measure on $\Upsilon(\mathbb{R}^n)$, which is the first main result of this paper. By Proposition 3.6, the set function ρ^m_r given in (3.4) turned out to be well-defined in the sense that the space $\mathcal{S}(\mathscr{E})$ of all Susline sets in $\Upsilon(\mathbb{R}^n)$ are contained in its domain $\mathscr{D}^m$. We show the following monotonicity result which allows us to pass to the limit of ρ^m_r as $r \to \infty$.

Theorem 3.7. *The map $r \mapsto \rho^m_r(A)$ is monotone non-decreasing for any $A \in \mathcal{S}(\mathscr{E})$.*

The proof of Theorem 3.7 is given at the end of this section. We can now introduce the *m-codimensional Poisson measure* on $\Upsilon(\mathbb{R}^n)$ as the monotone limit of ρ^m_r on the space $\mathcal{S}(\mathscr{E})$ of Suslin sets:

$$\rho^m(A) = \lim_{r\to\infty}\rho^m_r(A), \quad \forall A \in \mathcal{S}(\mathscr{E}). \tag{3.5}$$

Definition 3.8 (m-codimensional Poisson Measure)**.** Let $\mathfrak{D}^m$ be the completion of $\mathcal{S}(\mathscr{E})$ with respect to ρ^m. The measure $(\rho^m, \mathfrak{D}^m)$ is called the *m-codimensional Poisson measure* on $\Upsilon(\mathbb{R}^n)$.

Remark 3.9. We give two remarks below:

(i) Note $\rho^0 = \pi$, i.e. 0-codimensional Poisson measure ρ^0 on $\Upsilon(\mathbb{R}^n)$ is the Poisson measure π on $\Upsilon(\mathbb{R}^n)$ by noting that the m-dimensional spherical Hausdorff measure $\mathbf{S}^m$ and the n-dimensional Lebesgue measure $\mathbf{L}^n$ coincide when $m = n$ (see Remark 2.1).

(ii) The construction of ρ^m, a priori, depends on the choice of the exhaustion $\{B_r\} \subset \mathbb{R}^n$. However, in Proposition 3.13, we will see that it is not the case.

The rest of this section is devoted to the proof of Theorem 3.7. Let us begin with a definition.

Definition 3.10 (Section of Functions, Multi-Section)**.** Let $M, N \subset \mathbb{R}^n$ be two disjoint sets and $L = M \sqcup N$. For every $F : \Upsilon(L) \to \mathbb{R}$ and $\xi \in \Upsilon(M)$, define $F_{\xi,M} : \Upsilon(N) \to \mathbb{R}$ as

$$F_{\xi,M}(\zeta) := F(\zeta + \xi), \quad \zeta \in \Upsilon(N). \tag{3.6}$$

For a set $A \subset \Upsilon(\mathbb{R}^n)$, let $A_{\xi,\eta,M,N}$ denote the *multi-section* both at $\xi \in \Upsilon(M)$ and $\zeta \in \Upsilon(N)$:

$$A_{\xi,\zeta,M,N} := \{\gamma \in \Upsilon(L^c) : \gamma + \xi + \zeta \in A\}, \quad \text{and} \quad A^k_{\xi,\zeta,M,N} = A_{\xi,\zeta,M,N} \cap \Upsilon^k(L^c). \tag{3.7}$$

Lemma 3.11. *Let A be a Suslin set in $\Upsilon(\mathbb{R}^n)$. Let $M, N \subset \mathbb{R}^n$ be two disjoint Borel sets. Set $L = M \sqcup N$. Let $F : \Upsilon(L) \to \mathbb{R}$ be defined by $\gamma \mapsto F(\gamma) := \mathsf{S}^{m,k}_{L^c}(A^k_{\gamma,L})$. Then,*

$$F_{\xi,M}(\zeta) = \mathsf{S}^{m,k}_{L^c}(A^k_{\zeta,\xi,N,M}), \quad \forall \xi \in \Upsilon(M),\ \forall \zeta \in \Upsilon(N). \tag{3.8}$$

Proof. The set $A^k_{\zeta,\xi,N,M}$ is Suslin by the same argument as in Lemma 3.1. Thus, $\mathsf{S}^{m,k}_{L^c}(A^k_{\zeta,\xi,N,M})$ is well-defined. By Definition 3.10, we have that

$$F_{\xi,M}(\zeta) = F(\zeta + \xi) = \mathsf{S}^{m,k}_{L^c}(A^k_{\zeta+\xi,L}) = \mathsf{S}^{m,k}_{L^c}\big(\{\gamma \in \Upsilon(L^c) : \gamma + \xi + \zeta \in A\}\big) = \mathsf{S}^{m,k}_{L^c}(A^k_{\zeta,\xi,N,M})\,.$$

The proof is complete. □

The next lemma is straightforward by the independence of the Poisson measures π_M and π_N.

Lemma 3.12. *Suppose the same notation M, N and L as in Lemma 3.11. For any bounded measurable function G on $\Upsilon(L)$,*

$$\int_{\Upsilon(L)} G(\eta)d\pi_L(\eta) = \int_{\Upsilon(N)} \int_{\Upsilon(M)} G_{\xi,M}(\zeta)d\pi_M(\xi)d\pi_N(\zeta). \tag{3.9}$$

Proof of Theorem 3.7. Let $\mathcal{A}_{r,\varepsilon} := B_{r+\varepsilon} \backslash B_r$ be the annulus of width ε and radius r. Fix $A \in \mathcal{S}(\mathscr{E})$, $r > 0$, $\varepsilon > 0$ and $\zeta \in \Upsilon(B^c_{r+\varepsilon})$. We claim that

$$\mathsf{S}^{m,k}_{B_{r+\varepsilon}}(A^k_{\zeta,B^c_{r+\varepsilon}}) \geq \sum_{j=0}^{k} \int_{\Upsilon(\mathcal{A}_{r,\varepsilon})} \mathsf{S}^{m,k}_{B_r}(A^{k-j}_{\zeta,\xi,B^c_{r+\varepsilon},\mathcal{A}_{r,\varepsilon}})d\mathsf{S}^j_{\mathcal{A}_{r,\varepsilon}}(\xi). \tag{3.10}$$

Let us first show how (3.10) concludes the proof. For simplicity of notation, we set $M = \mathcal{A}_{r,\varepsilon}$, $N = B^c_{r+\varepsilon}$ and $L = M \sqcup N$. Then, (3.10) is reformulated as follows:

$$\mathsf{S}^{m,k}_{N^c}(A^k_{\zeta,N}) \geq \sum_{j=0}^{k} \int_{\Upsilon(M)} \mathsf{S}^{m,k}_{L^c}(A^{k-j}_{\zeta,\xi,N,M})d\mathsf{S}^j_M(\xi).$$

Then, by using Lemma 3.12 and Lemma 3.11 we deduce

$$\begin{aligned}
\rho^m_r(A) &= e^{-\mathbf{S}^n(L^c)} \sum_{k=0}^{\infty} \int_{\Upsilon(L)} \mathsf{S}^{m,k}_{L^c}(A^k_{\eta,L})d\pi_L(\eta) \\
&= e^{-\mathbf{S}^n(L^c)} \sum_{k=0}^{\infty} \int_{\Upsilon(N)} \int_{\Upsilon(M)} \big(\mathsf{S}^{m,k}_{L^c}(A^k_{\zeta,L})\big)_{\xi,M} d\pi_M(\xi)d\pi_N(\zeta) \\
&= e^{-\mathbf{S}^n(L^c)} \sum_{k=0}^{\infty} \int_{\Upsilon(N)} \int_{\Upsilon(M)} \mathsf{S}^{m,k}_{L^c}(A^k_{\zeta,\xi,N,M})d\pi_M(\xi)d\pi_N(\zeta) \\
&= e^{-\mathbf{S}^n(L^c)} e^{-\mathbf{S}^n(M)} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \int_{\Upsilon(N)} \int_{\Upsilon(M)} \mathsf{S}^{m,k-j}_{L^c}(A^{k-j}_{\zeta,\xi,N,M})d\mathsf{S}^j_M(\xi)d\pi_N(\zeta) \\
&\leq e^{-\mathbf{S}^n(N^c)} \sum_{k=0}^{\infty} \int_{\Upsilon(N)} \mathsf{S}^{m,k}_{N^c}(A^k_{\zeta,N})d\pi_N(\zeta) \\
&= \rho^m_{r+\varepsilon}(A)\,.
\end{aligned}$$

To show (3.10), it is enough to verify that, for any bounded measurable function F on $\Upsilon(\mathbb{R}^n)$,

$$\int_{\Upsilon(N^c)} F_{\zeta,N}(\gamma) d\mathsf{S}^{m,k}_{N^c}(\gamma) \geq \sum_{j=0}^{k} \int_{\Upsilon(M)} \int_{\Upsilon(L^c)} (F_{\zeta,N})_{\xi,M}(\gamma) d\mathsf{S}^{m,k-j}_{L^c}(\gamma) d\mathsf{S}^{j}_{M}(\xi). \tag{3.11}$$

By the definition of $\mathsf{S}^{m,k}_{N^c}$, the L.H.S. of (3.11) can be deduced as follows:

$$\int_{\Upsilon(N^c)} F_{\zeta,N}(\gamma) d\mathsf{S}^{m,k}_{N^c}(\gamma) = \frac{1}{k!} \int_{(N^c)^{\otimes k}} (F_{\zeta,N} \circ \mathbf{s}_k)(\mathbf{x}_k) d\mathbf{S}^{nk-m}_{N^c}(\mathbf{x}_k),$$

whereby $\mathbf{x}_k := (x_0, \ldots, x_{k-1})$ and $\mathbf{x}_0 = x_0$. Furthermore, by the definition of $(F_{\zeta,N})_{\xi,M}$, the R.H.S. of (3.11) can be deduced as follows:

$$\begin{aligned}
&\int_{\Upsilon(M)} \int_{\Upsilon(L^c)} (F_{\zeta,N})_{\xi,M}(\gamma) d\mathsf{S}^{m,k-j}_{L^c}(\gamma) d\mathsf{S}^{j}_{M}(\xi) \\
&= \int_{\Upsilon(M)} \int_{\Upsilon(L^c)} (F_{\zeta,N})(\gamma+\xi) d\mathsf{S}^{m,k-j}_{L^c}(\gamma) d\mathsf{S}^{j}_{M}(\xi) \\
&= \frac{1}{j!(k-j)!} \int_{M^{\times j}} \int_{(L^c)^{\times(k-j)}} (F_{\zeta,N} \circ \mathbf{s}_k)(\mathbf{x}_{k-j}, \mathbf{y}_j) d\mathbf{S}^{n(k-j)-m}_{L^c}(\mathbf{x}_{k-j}) d\mathbf{S}^{nj}_{M}(\mathbf{y}_j),
\end{aligned}$$

whereby $(\mathbf{x}_{k-j}, \mathbf{y}_j) = (x_0, \ldots, x_{k-j-1}, y_0, \ldots, y_{j-1})$. Hence, in order to conclude (3.11), it suffices to show the following inequality: for any bounded measurable symmetric function f on $(\mathbb{R}^n)^{\times k}$,

$$\int_{B^{\times k}_{r+\varepsilon}} f(\mathbf{x}_k) d\mathbf{S}^{nk-m}_{B_{r+\varepsilon}}(\mathbf{x}_k) \geq \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \int_{B_r^{\times(k-j)}} \int_{\mathcal{A}^{\times j}_{r,\varepsilon}} f(\mathbf{x}_{k-j}, \mathbf{y}_j) d\mathbf{S}^{n(k-j)-m}_{\mathcal{A}_{r,\varepsilon}}(\mathbf{x}_{k-j}) d\mathbf{S}^{nj}_{B_r}(\mathbf{y}_j).$$

By using the symmetry of f and a simple combinatorial argument, we obtain

$$\int_{B^{\times k}_{r+\varepsilon}} f(\mathbf{x}_k) d\mathbf{S}^{nk-m}_{B_{r+\varepsilon}}(\mathbf{x}_k) = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \int_{B_r^{\times(k-j)}} \int_{\mathcal{A}^{\times j}_{r,\varepsilon}} f(\mathbf{x}_{k-j}, \mathbf{y}_j) d\mathbf{S}^{nk-m}_{B_{r+\varepsilon}}(\mathbf{x}_{k-j}, \mathbf{y}_j),$$

while [25, 2.10.27, p. 190] implies

$$\begin{aligned}
&\int_{B_r^{\times(k-j)}} \int_{\mathcal{A}^{\times j}_{r,\varepsilon}} f(\mathbf{x}_{k-j}, \mathbf{y}_j) d\mathbf{S}^{nk-m}_{B_{r+\varepsilon}}((\mathbf{x}_{k-j}), (\mathbf{y}_j)) \\
&\geq \int_{B_r^{\times(k-j)}} \int_{\mathcal{A}^{\times j}_{r,r+\varepsilon}} f(\mathbf{x}_{k-j}, \mathbf{y}_j) d\mathbf{S}^{n(k-j)-m}_{\mathcal{A}_{r,\varepsilon}}((\mathbf{x}_{k-j})) d\mathbf{S}^{nj}_{B_r}(\mathbf{y}_j).
\end{aligned}$$

The proof is complete. □

3.4. **Independence of ρ^m from the exhaustion.** So far we have built the m-codimensional measure ρ^m by passing to the limit a sequence of finite dimensional measures ρ^m_r. The latter have been constructed by relying on the exhaustion $\{B_r : r > 0\}$ of $\mathbb{R}^n$. Hence, a priori, ρ^m depends on the chosen exhaustion. In this subsection we make a remark that actually it is not the case.

Let $\Omega \subset \mathbb{R}^n$ be a compact set. Following closely the proof in section 3.3 we can prove that

$$\rho^m_\Omega(A) := e^{-\mathbf{S}^n(\Omega)} \sum_{k=1}^{\infty} \int_{\Upsilon(\Omega^c)} \mathsf{S}^{m,k}_{\Omega}(A^k_{\eta,\Omega^c}) \; d\pi_{B_r^c}(\eta). \tag{3.12}$$

is well defined for any Suslin set A.

The next proposition can be proven by arguing as in Theorem 3.7. We omit the proof.

Proposition 3.13 (Independence of Exhaustion)**.** *Let $0 < r < R < \infty$ and $\Omega \subset \mathbb{R}^n$ be a compact subset satisfying $B_r \subset \Omega \subset B_R$. Then*

$$\rho^m_r(A) \leq \rho^m_\Omega(A) \leq \rho^m_R(A)\,, \quad \text{for any Suslin set } A\,. \tag{3.13}$$

In particular ρ^m does not depend on the choice of the exhaustion.

4. Bessel capacity and finite-codimensional Poisson measure

In this section, we discuss an important relation between *Bessel capacities* and finite-codimensional Poisson measures ρ^m. This will play a significant role to develop fundamental relations between potential theory and theory of BV functions in Section 5 and Section 7.

Definition 4.1 (Bessel operator)**.** *Let $\alpha > 0$ and $1 \le p < \infty$. We set*

$$B_{\alpha,p} := \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-t} t^{\alpha/2-1} T_t^{(p)} dt \,, \tag{4.1}$$

where $T_t^{(p)}$ is the L^p-heat semigroup, see Section 2.5.

Notice that $B_{\alpha,p}$ is well defined for $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ and satisfies

$$\|B_{\alpha,p} F\|_{L^p} \le \|F\|_{L^p} \,, \tag{4.2}$$

due to the contractivity of $T_t^{(p)}$ in $L^p(\Upsilon(\mathbb{R}^n), \pi)$.

Definition 4.2 (Bessel capacity)**.** Let $\alpha > 0$ and $1 \le p < \infty$. The (α, p)-*Bessel capacity* is defined as

$$\mathrm{Cap}_{\alpha,p}(E) := \inf\{\|F\|_{L^p}^p : B_{\alpha,p} F \ge 1 \text{ on } E,\ F \ge 0\} \,, \tag{4.3}$$

for any $E \subset \Upsilon(\mathbb{R}^n)$.

We are now ready to state the main theorem of this section.

Theorem 4.3. *Let $\alpha p > m$. Then, $\mathrm{Cap}_{\alpha,p}(E) = 0$ implies $\rho^m(E) = 0$ for any $E \in \mathcal{S}(\mathscr{E})$.*

Let us briefly explain the heuristic idea of proof. In view of the identities

$$\rho^m(E) = \lim_{r\to\infty} \rho^m_r(E) \,,$$

$$\rho^m_r(E) = e^{-\mathbf{S}^n(B_r)} \sum_{k=1}^{\infty} \int_{\Upsilon(B_r^c)} \mathsf{S}^{m,k}_{B_r}(E^k_{\eta,r}) d\pi_{B_r^c}(\eta) \,,$$

it is enough to prove that $\mathsf{S}^{m,k}_{B_r}(E^k_{\eta,r}) = 0$ for $\pi_{B_r^c}$-a.e. η, all $k \in \mathbb{N}$ and $r > 0$. This, along with the implication

$$\mathrm{Cap}_{\alpha,p}(E) = 0 \implies \mathrm{Cap}^{\eta,r}_{\alpha,p}(E^k_{\eta,r}) = 0, \quad \text{for } \pi_{B_r^c}\text{-a.e. } \eta \text{ and all } k \in \mathbb{N} \text{ and } r > 0 \,, \tag{4.4}$$

where $\mathrm{Cap}^{\eta,r}_{\alpha,p}$ is the Bessel (α, p)-capacity on $\Upsilon^k(B_r)$, reduces the problem to a finite dimensional one. More precisely we will show that

$$\mathrm{Cap}^{\eta,r}_{\alpha,p}(E^k_{\eta,r}) = 0 \implies \mathsf{S}^{m,k}_{B_r}(E^k_{\eta,r}) = 0 \,,$$

which is a standard result of finite dimensional spaces.

In the rest of this section, we make the aforementioned idea rigorous. The key point is to show (4.4), for which we introduce localisations of functional analytic objects in Section 4.1 and Section 4.2. We then introduce *localised Bessel operators* and *localised Bessel capacities* in Section 4.3.

4.1. **Localization of sets and functions.**

Lemma 4.4. *Let $A \subset \Upsilon(\mathbb{R}^n)$ be a π-measurable set. Let $B \subset \mathbb{R}^n$ be a Borel set. Then, $A_{\eta,B}$ is π_{B^c}-measurable for π_B-a.e. $\eta \in \Upsilon(B)$. Moreover, if $\pi(A) = 0$, then $\pi_{B^c}(A_{\eta,B}) = 0$ for a.e. $\eta \in \Upsilon(B)$.*

Proof. By hypothesis, there exist Borel sets $\underline{A} \subset A \subset \overline{A}$ so that $\pi(\overline{A} \setminus \underline{A}) = 0$. By (i) in Remark 2.16, $\underline{A}$ and $\overline{A}$ are Suslin. By Lemma 3.1, $\underline{A}_{\eta,B}$ and $\overline{A}_{\eta,B}$ are Suslin. By the standard disintegration argument as in Lemma 3.12, it holds that

$$0 = \pi(\overline{A} \setminus \underline{A}) = \int_{\Upsilon(B)} \pi_{B^c}((\overline{A} \setminus \underline{A})_{\eta,B}) d\pi_B(\eta). \tag{4.5}$$

Therefore, there exists a π_B-measurable set $\Omega \subset \Upsilon(B)$ so that $\pi_{B^c}((\overline{A} \setminus \underline{A})_{\eta,B}) = 0$ for any $\eta \in \Omega$. By noting that $\underline{A}_{\eta,B} \subset A_{\eta,B} \subset \overline{A}_{\eta,B}$, we concluded that $A_{\eta,B}$ is π_B-measurable since, up to π_B negligible sets, it coincides with a Suslin set and every Suslin set is π_B-measurable by (ii) in Remark 2.16. The proof of the first assertion is complete.

If $\pi(A) = 0$ the disintegration

$$0 = \pi(A) = \int_{\Upsilon(B)} \pi_{B^c}(A_{\eta,B}) d\pi_B(\eta)\,, \tag{4.6}$$

immediately gives the second assertion. □

Corollary 4.5. *Let $A \subset \Upsilon(\mathbb{R}^n)$ be a π-measurable set, $B \subset \mathbb{R}^n$ a Borel set, and let g be a π-measurable function on $\Upsilon(\mathbb{R}^n)$ with $g \geq 1$ π-a.e. on A. Then, for π_B-a.e. η it holds*

$$g_{\eta,B} \geq 1, \quad \pi_{B^c}\text{-a.e. on } A_{\eta,B}\,. \tag{4.7}$$

Proof. Taking $\widetilde{A} = A \setminus \{g \geq 1\}$ and applying Lemma 4.4 with $\widetilde{A}$ in place of A, we obtain the conclusion. □

Lemma 4.6. *Let $1 \leq p < \infty$ and $r > 0$. Let $F^n, F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ such that $F^n \to F$ in $L^p(\Upsilon(\mathbb{R}^n), \pi)$ as $n \to \infty$. Then, there exists a subsequence (non-relabelled) of (F^n) and a measurable set $A_r \subset \Upsilon(\mathbb{R}^n)$ so that $\pi_{B_r^c}(A_r) = 1$ and*

$$F^n_{\eta,r} \to F_{\eta,r}, \quad \text{in } L^p(\pi_{B_r}), \text{ for any } \eta \in A_r.$$

Note that $F_{\eta,r} := F_{\eta,B_r^c}$ was defined in Definition 3.10.

Proof. By Lemma 3.12, we have that

$$\int_{\Upsilon(B_r^c)} \left(\int_{\Upsilon(B_r)} |F^n_{\eta,r} - F_{\eta,r}|^p d\pi_{B_r} \right) d\pi_{B_r^c}(\eta) = \int_{\Upsilon(\mathbb{R}^n)} |F^n - F|^p d\pi \to 0, \quad \text{as } n \to \infty\,. \tag{4.8}$$

In particular, up to subsequence $\int_{\Upsilon(B_r)} |F^n_{\eta,r} - F_{\eta,r}|^p d\pi_{B_r} \to 0$ for $\pi_{B_r^c}$-a.e. η, which completes the proof. □

4.2. **Localisation of energies, resolvents and semigroups.** In this section, we localise differential operators and related objects introduced in Section 2.5.

Let $r > 0$. The *localised energy* $(\mathcal{E}_r, \mathcal{D}(\mathcal{E}_r))$ is defined by the closure of the following pre-Dirichlet form

$$\mathcal{E}_r(F) = \int_{\Upsilon(\mathbb{R}^n)} |\nabla_r F|^2_{T\Upsilon} d\pi \quad F \in \mathrm{CylF}(\Upsilon), \tag{4.9}$$

where

$$\nabla_r F(\gamma, x) := \chi_{B_r}(x) \nabla F(\gamma, x)\,. \tag{4.10}$$

The closability follows from [36, Lemma 2.2], or [23, the first paragraph of the proof of Thm. 3.48]. We denote by $\{G^r_\alpha\}_{\alpha>0}$ and $\{T^r_t\}_{t>0}$ the L^2-resolvent operator and the semigroup associated with $(\mathcal{E}_r, \mathcal{D}(\mathcal{E}_r))$, respectively. The next result is taken from [36, Lemma 2.2] or [23, the second paragraph of the proof of Thm. 3.48].

Proposition 4.7. *The form $(\mathcal{E}_r, \mathcal{D}(\mathcal{E}_r))$ is monotone non-decreasing in r, i.e. for any $s \leq r$,*

$$\mathcal{D}(\mathcal{E}_r) \subset \mathcal{D}(\mathcal{E}_s), \quad \mathcal{E}_s(F) \leq \mathcal{E}_r(F), \quad F \in \mathcal{D}(\mathcal{E}_r).$$

Furthermore, $\lim_{r\to\infty} \mathcal{E}_r(F) = \mathcal{E}(F)$ *for* $F \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$.

The next proposition shows the monotonicity property for the resolvent operator G^r_α and the semigroup T^r_t.

Proposition 4.8. *The resolvent operator $\{G^r_\alpha\}_\alpha$ and the semigroup $\{T^r_t\}_t$ are monotone non-increasing on non-negative functions, i.e.,*

$$G^r_\alpha F \le G^s_\alpha F, \quad T^r_t F \le T^s_t F, \quad \text{for any nonnegative } F \in L^2(\Upsilon,\pi), \quad s \le r. \tag{4.11}$$

Furthermore, $\lim_{r\to\infty} G^r_\alpha F = G_\alpha F$ *and* $\lim_{r\to\infty} T^r_t F = T_t F$ *for* $F \in L^2(\Upsilon,\pi)$ *and* $\alpha, t > 0$.

Proof. Thanks to the identity

$$G^r_\alpha = \int_0^\infty e^{-\alpha t} T^r_t \, dt \,,$$

it suffices to show (4.11) only for T^r_t. By a direct application of [37, Theorem 3.3] and the monotonicity of the Dirichlet form in Proposition 4.7, we obtain the monotonicity of the semigroup. The second part of the statement follows from the monotone convergence $\mathcal{E}_r \uparrow \mathcal{E}$ combined with [33, S.14, p.372]. □

The next proposition is taken from [36, (4.1), Prop. 4.1] or [23, Proposition 3.45].

Proposition 4.9. *For any $F \in \mathrm{CylF}(\Upsilon)$, $\eta \in \Upsilon(B^c_r)$ and $r > 0$, it holds*

$$\mathcal{E}_r(F) = \int_{\Upsilon(B^c_r)} \mathcal{E}_{\Upsilon(B_r)}(F_{\eta,r}) d\pi_{B^c_r}(\eta) \,, \tag{4.12}$$

where $\mathcal{E}_{\Upsilon(B_r)}$ is the energy on $\Upsilon(B_r)$ defined in Definition 2.11, and $F_{\eta,r} = F_{\eta,B^c_r}$ was introduced in Definition 3.10.

Note that $(\mathcal{E}_{\Upsilon(B_r)}, \mathrm{CylF}(\Upsilon(B_r)))$ is closable and the closure is denoted by $(\mathcal{E}_{\Upsilon(B_r)}, H^{1,2}(\Upsilon(B_r),\pi))$, see Definition 2.11. Recall that $\{G^{\Upsilon(B_r)}_\alpha\}_\alpha$ and $\{T^{\Upsilon(B_r)}_t\}$ denote the L^2-resolvent operator and the semigroup corresponding to $(\mathcal{E}_{\Upsilon(B_r)}, H^{1,2}(\Upsilon(B_r),\pi))$.

We now provide the relation between $\{G^r_\alpha\}_{\alpha>0}$, $\{T^r_t\}_{t>0}$ and $\{G^{\Upsilon(B_r)}_\alpha\}_\alpha$, $\{T^{\Upsilon(B_r)}_t\}$.

Proposition 4.10. *Let $\alpha > 0$, $t > 0$, and $r > 0$ be fixed. Then, for any bounded measurable function F, it holds that*

$$G^r_\alpha F(\gamma) = G^{\Upsilon(B_r)}_\alpha F_{\gamma|_{B^c_r},r}(\gamma|_{B_r}) \,, \tag{4.13}$$

$$T^r_t F(\gamma) = T^{\Upsilon(B_r)}_t F_{\gamma|_{B^c_r},r}(\gamma|_{B_r}) \,, \tag{4.14}$$

for π-a.e. $\gamma \in \Upsilon(\mathbb{R}^n)$.

Proof. Let $F, H \in \mathrm{CylF}(\Upsilon)$. Set $R^r_\alpha F(\cdot) := G^{\Upsilon(B_r)}_\alpha F_{\cdot|_{B^c_r},r}(\cdot|_{B_r})$. By definition,

$$\big(R^r_\alpha F\big)_{\eta,r}(\cdot) = G^{\Upsilon(B_r)}_\alpha F_{\eta,r}(\cdot), \quad \pi_{B_r}\text{-a.e. on } \Upsilon(B_r) \text{ for } \pi_{B^c_r}\text{-a.e. } \eta. \tag{4.15}$$

By Proposition 4.9, we get

$$\begin{aligned}
\mathcal{E}_r(R^r_\alpha F, H) &= \int_{\Upsilon(\mathbb{R}^n)} \mathcal{E}_{\Upsilon(B_r)}((R^r_\alpha F)_{\eta,r}, H_{\eta,r}) d\pi(\eta) \\
&= \int_{\Upsilon(\mathbb{R}^n)} \mathcal{E}_{\Upsilon(B_r)}(G^{\Upsilon(B_r)}_\alpha F_{\eta,r}, H_{\eta,r}) d\pi(\eta) \\
&= \int_{\Upsilon(\mathbb{R}^n)} \left(\int_{\Upsilon(B_r)} (F_{\eta,r} - \alpha G^{\Upsilon(B_r)}_\alpha F_{\eta,r}) H_{\eta,r} d\pi_{B_r} \right) d\pi_{B^c_r}(\eta) \\
&= \int_{\Upsilon(\mathbb{R}^n)} (F - \alpha R^r_\alpha F) H \; d\pi \,,
\end{aligned} \tag{4.16}$$

where the second line follows from (4.15), and in the third line we used the fundamental equality $-\Delta G^{\Upsilon(B_r)}_\alpha F_{\eta,r} + \alpha G^{\Upsilon(B_r)}_\alpha F_{\eta,r} = F_{\eta,r}$. Since the resolvent operator G^r_α is the unique L^2-bounded operator satisfying

$$\mathcal{E}_r(G^r_\alpha F, H) = \int_{\Upsilon(\mathbb{R}^n)} (F - \alpha G^r_\alpha F) H \; d\pi \,,$$

we conclude that $G^r_\alpha F(\gamma) = R^r_\alpha F(\gamma) = G^{\Upsilon(B_r)}_\alpha F_{\gamma|_{B_r^c},r}(\gamma|_{B_r})$ π-a.e. γ. Thus the proof of (4.13) is complete.

The second conclusion (4.14) follows from the identity

$$\int_0^\infty e^{-\alpha t} T^r_t F(\gamma) dt = G^r_\alpha F(\gamma) dt = G^{\Upsilon(B_r)}_\alpha F_{\gamma|_{B_r^c},r}(\gamma|_{B_r}) = \int_0^\infty e^{-\alpha t} T^{\Upsilon(B_r)}_t F_{\gamma|_{B_r^c},r}(\gamma|_{B_r}) dt,$$

and the injectivity of the Laplace transform on continuous functions in t. □

Remark 4.11. Although Proposition 4.10 provides the statement only for the L^2-semigroups and resolvents, it is straightforward to extend it to the L^p-semigroups and resolvents for any $1 \le p < \infty$.

4.3. **Localized Bessel operators.** Let $B^r_{\alpha,p}$ and $B^{\Upsilon(B_r)}_{\alpha,p}$ be the (α,p)-Bessel operators corresponding to $\{T^r_t\}_{t>0}$ and $\{T^{\Upsilon(B_r)}_t\}_{t>0}$, respectively defined in the analogous way as in (4.1). The corresponding (α,p)-Bessel capacities are denoted by $\mathrm{Cap}^r_{\alpha,p}$ and $\mathrm{Cap}^{\Upsilon(B_r)}_{\alpha,p}$ defined in the analogous way as in (4.3)

Lemma 4.12. $\mathrm{Cap}^r_{\alpha,p}(E) \le \mathrm{Cap}_{\alpha,p}(E)$ *for any* $E \subset \Upsilon(\mathbb{R}^n)$ *and* $r > 0$.

Proof. It suffices to show that $B^r_{\alpha,p}F \le B_{\alpha,p}F$ for any $F \ge 0$ with $F \in L^p(\Upsilon, \pi)$, which immediately follows from Proposition 4.8 and (4.1). □

Lemma 4.13. *If* $\mathrm{Cap}_{\alpha,p}(E) = 0$, *then* $\mathrm{Cap}^{\Upsilon(B_r)}_{\alpha,p}(E_{\eta,r}) = 0$ *for* $\pi_{B_r^c}$*-a.e.* η *and any* $r > 0$.

Proof. By Lemma 4.12 we may assume $\mathrm{Cap}^r_{\alpha,p}(E) = 0$ for any $r > 0$. Let $\{F_n\} \subset L^p(\Upsilon, \pi)$ be a sequence so that $F_n \ge 0$, $B^r_{\alpha,p}F_n \ge 1$ on E, and $\|F_n\|^p_{L^p} \to 0$. By Lemma 4.5, $(F_n)_{\eta,r} \ge 0$ for $\pi_{B_r^c}$-a.e. η. Furthermore, by Lemma 4.6, there exists $A_r \subset \Upsilon(B_r^c)$ and a (non-relabelled) subsequence $(F_n)_{\eta,r}$ so that $\pi_{B_r^c}(A_r) = 1$, and for every $\eta \in A_r$,

$$(F_n)_{\eta,r} \to 0, \quad \text{in } L^p(\Upsilon(B_r), \pi_{B_r}) \,. \tag{4.17}$$

By Proposition 4.10 and Remark 4.11, we have that

$$\begin{aligned}(B^r_{\alpha,p}F_n)_{\eta,r} &= \left(\frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-t} t^{\alpha/2-1} T^r_t F_n dt\right)_{\eta,r} \\ &= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-t} t^{\alpha/2-1} \big(T^r_t F_n\big)_{\eta,r} dt \\ &= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-t} t^{\alpha/2-1} T^{\Upsilon(B_r)}_t (F_n)_{\eta,r} dt \\ &= B^{\Upsilon(B_r)}_{\alpha,p}(F_n)_{\eta,r}. \end{aligned} \tag{4.18}$$

Note that we dropped the specification of p in the semigroups for the notational simplicity in (4.18).

Since $B^r_{\alpha,p}F_n \ge 1$ on E, by applying Corollary 4.5, we obtain that $(B^r_{\alpha,p}F_n)_{\eta,r} \ge 1$ on $E_{\eta,r}$ for $\pi_{B_r^c}$-a.e. η. Thus, by (4.18), $B^{\Upsilon(B_r)}_{\alpha,p}(F_n)_{\eta,r} \ge 1$ on $E_{\eta,r}$ for $\pi_{B_r^c}$-a.e. η. By (4.17), we conclude that $\mathrm{Cap}^{\Upsilon(B_r)}_{\alpha,p}(E_{\eta,r}) = 0$ for $\pi_{B_r^c}$-a.e. η and any $r > 0$. The proof is complete. □

4.4. **Finite-dimensional counterpart.** In this section, we develop the finite-dimensional counterpart of Theorem 4.3. The goal is to prove the following proposition.

Proposition 4.14. *Let* $\alpha p > m$. *If* $\mathrm{Cap}^{\Upsilon^k(B_r)}_{\alpha,p}(E) = 0$, *then* $\mathsf{S}^{m,k}_{B_r}(E) = 0$ *for any* $k \in \mathbb{N}$.

Proof. Recall that $T^{\Omega,\otimes k}_t$ is the k-tensor semigroup of T^Ω_t as defined in (2.15). Let $B^{B_r^{\times k}}_{\alpha,p}$ be the corresponding Bessel operator defined analogously as in (4.1), and $\mathrm{Cap}^{B_r^{\times k}}_{\alpha,p}$ be the corresponding (α,p)-capacity.

Let $\{F_m\} \subset L^p(\Upsilon(B_r), \pi_{B_r})$ be a sequence so that $F_m \geq 0$ and $B_{\alpha,p}^{\Upsilon^k(B_r)} F_m \geq 1$ on $E \subset \Upsilon^k(E)$, and $\|F_m\|_{L^p} \to 0$. By Proposition 2.15 and the definition of Bessel operator, we have

$$B_{\alpha,p}^{\Upsilon^k(B_r)} F_m \circ \mathsf{s}_k = B_{\alpha,p}^{B_r^{\times k}} (F_m \circ \mathsf{s}_k)\,,$$

hence $F_m \circ \mathsf{s}_k \geq 0$, $B_{\alpha,p}^{B_r^{\times k}} (F_m \circ \mathsf{s}_k) \geq 1$ on $\mathsf{s}_k^{-1}(E)$. Furthermore,

$$\|F_m \circ \mathsf{s}_k\|_{L^p(B_r^{\times k})} = C(k,n,r)\|F_m\|_{L^p(\Upsilon^k(B_r))} \to 0\,, \quad \text{as } m \to \infty\,,$$

where $C(k,n,r) > 0$ comes from the constant appearing in front of the Hausdorff measure in the definition of π_{B_r}. This implies that $\mathrm{Cap}_{\alpha,p}^{B_r^{\times k}}(\mathsf{s}_k^{-1}(E)) = 0$. We can now rely on standard capacity estimates in the Euclidean setting (see, e.g. [44, Theorem 2.6.16]) to conclude that $\mathbf{S}^{nk-m}(\mathsf{s}_k^{-1}(E)) = 0$. Recalling (2.4), we have that

$$\mathsf{S}_{B_r}^{m,k}(E) = \frac{1}{k!}(\mathsf{s}_k)_\# \mathbf{S}^{nk-m}(E) = \frac{1}{k!}\mathbf{S}^{nk-m}(\mathsf{s}_k^{-1}(E)) = 0\,.$$

The proof is complete. □

4.5. **Proof of Theorem 4.3.** Let $E \in \mathcal{S}(\mathscr{E})$ such that $\mathrm{Cap}_{\alpha,p}(E) = 0$. Thanks to Lemma 4.13 we have $\mathrm{Cap}_{\alpha,p}^{\Upsilon(B_r)}(E) = 0$ for any $r > 0$, hence $\mathsf{S}_{B_r}^{m,k}(E_{\eta,r}^k) = 0$ for any $k \in \mathbb{N}$ as a consequence of Proposition 4.14. It implies

$$\rho_r^m(E) = e^{-\mathbf{S}^n(B_r)} \sum_{k=1}^{\infty} \int_{\Upsilon(B_r^c)} \mathsf{S}_{B_r}^{m,k}(E_{\eta,r}^k) d\pi_{B_r^c}(\eta) = 0\,,$$

for any $r > 0$. Recalling that $\rho_r^m(E) \uparrow \rho^m(E)$ by (3.5), we obtain the sought conclusion.

5. BV Functions

In this section, we introduce functions of bounded variations (called *BV functions*) on $\Upsilon(\mathbb{R}^n)$ following three different approaches: the variational approach (§5.1), the relaxation approach (§5.2), and the semigroup approach (§5.3). In Section 5.5, we prove that they all coincide.

5.1. **Variational approach.** Let us begin by introducing a class of BV functions through integration by parts. We then discuss localisation properties.

Definition 5.1 (BV functions I: variational approach)**.** Let $\Omega \subset \mathbb{R}^n$ be either a closed domain with smooth boundary or $\mathbb{R}^n$. For $F \in \cup_{p>1} L^p(\Upsilon(\Omega), \pi)$, we define the total variation as

$$\mathcal{V}_{\Upsilon(\Omega)}(F) := \sup\left\{\int_{\Upsilon(\mathbb{R}^n)} (\nabla_{\Upsilon(\Omega)}^* V) F d\pi : V \in \mathrm{CylV}_0(\Upsilon(\Omega)),\ |V|_{T\Upsilon(\Omega)} \leq 1\right\}. \tag{5.1}$$

We set $\mathcal{V}(F) := \mathcal{V}_{\Upsilon(\mathbb{R}^n)}(F)$. We say that F *is BV in the variational sense* if $\mathcal{V}(F) < \infty$.

Remark 5.2. The assumption $F \in \cup_{p>1} L^p(\Upsilon(\Omega), \pi)$ plays an important role in Definition 5.1, ensuring that $\int_{\Upsilon(\Omega)} (\nabla^* V) F d\pi$ is well defined for any $V \in \mathrm{CylV}(\Upsilon(\Omega))$. Indeed, one can easily prove that $\nabla_{\Upsilon(\Omega)}^* V \in \cup_{1 \leq p < \infty} L^p(\Upsilon(\Omega), \pi)$ for any $V \in \mathrm{CylV}_0(\Upsilon(\Omega))$, but it is not $L^\infty(\Upsilon(\Omega), \pi)$ in general.

Remark 5.3. As was shown in Remark 2.8, the set of $V \in \mathrm{CylV}_0(\Upsilon(\Omega))$ with $|V|_{T\Upsilon} \leq 1$ is not empty and dense in $\mathrm{CylV}_0(\Upsilon(\Omega))$ with respect to the point-wise topology and the $L^p(T\Upsilon)$ topology for $1 \leq p < \infty$.

In order to localize the total variation we employ a family of cylinder vector fields concentrated on B_r, for some $r > 0$.

Definition 5.4. For $F \in \cup_{p>1} L^p(\Upsilon(\mathbb{R}^n), \pi)$, we define the *localized total variation* as

$$\mathcal{V}_r(F) := \sup\left\{ \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F d\pi : V \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon(\mathbb{R}^n)} \le 1 \right\}, \tag{5.2}$$

where

$$\mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n)) := \left\{ V(\gamma, x) = \sum_{i=1}^k F_i(\gamma) v_i(x) : F_i \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n)),\ v_i \in C_0^\infty(B_r; \mathbb{R}^n), k \in \mathbb{N} \right\}.$$

The next result shows that $\mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}) < \infty$ for $\pi_{B_r^c}$-a.e. η whenever $\mathcal{V}_r(F) < \infty$. It is the key step to perform our nonlinear dimension reduction. Indeed it allows to reduce the study of BV functions on $\Upsilon(\mathbb{R}^n)$ to their sections, which live on the finite dimensional space $\Upsilon(B_r)$.

Proposition 5.5. *Let $r > 0$ and $p > 1$. For $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ with $\mathcal{V}_r(F) < \infty$, it holds*

$$\int_{\Upsilon(B_r^c)} \mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}) d\pi_{B_r^c}(\eta) = \mathcal{V}_r(F). \tag{5.3}$$

Let us begin with a simple technical lemma.

Lemma 5.6. *Let $r > 0$. For all $V \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n))$, and $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ it holds*

$$\int_{\Upsilon(B_r^c)} \left(\int_{\Upsilon(B_r)} F_{\eta,r}(\gamma) \nabla^*_{\Upsilon(B_r)} V_{\eta,r}(\gamma) d\pi_{B_r}(\gamma) \right) d\pi_{B_r^c}(\eta) = \int_{\Upsilon(\mathbb{R}^n)} F \nabla^* V d\pi. \tag{5.4}$$

Proof of Lemma 5.6. Recall that for $r > 0$ and $\eta \in \Upsilon(B_r^c)$ we have $V_{\eta,r} \in \mathrm{CylV}_0(B_r)$. By the divergence formula (2.12) and the disintegration Lemma 3.12, we have that

$$\begin{aligned}
&\int_{\Upsilon(B_r^c)} \left(\int_{\Upsilon(B_r)} F_{\eta,r}(\gamma) \nabla^*_{\Upsilon(B_r)} V_{\eta,r}(\gamma) d\pi_{B_r}(\gamma) \right) d\pi_{B_r^c}(\eta) \\
&= -\int_{\Upsilon(B_r^c)} \left(\int_{\Upsilon(B_r)} F_{\eta,r}(\gamma) \Big(\sum_{i=1}^k \nabla_{v_i} (F_i)_{\eta,r}(\gamma) + \sum_{i=1}^k (F_i)_{\eta,r}(\gamma) (\nabla^* v_i)^\star(\gamma) \Big) d\pi_{B_r}(\gamma) \right) d\pi_{B_r^c}(\eta) \\
&= -\int_{\Upsilon(B_r^c)} \int_{\Upsilon(B_r)} \left(F \Big(\sum_{i=1}^k \nabla_{v_i} F_i + \sum_{i=1}^k F_i (\nabla^* v_i)^\star \Big) \right)_{\eta,r} (\gamma) d\pi_{B_r}(\gamma) d\pi_{B_r^c}(\eta) \\
&= -\int_{\Upsilon(B_r^c)} F \Big(\sum_{i=1}^k \nabla_{v_i} F_i + \sum_{i=1}^k F_i (\nabla^* v_i)^\star \Big) d\pi \\
&= \int_{\Upsilon(\mathbb{R}^n)} F \nabla^* V d\pi.
\end{aligned}$$

The proof is complete. ∎

Proof of Proposition 5.5. We first prove that

$$\int_{\Upsilon(B_r^c)} \mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}) d\pi_{B_r^c}(\eta) \ge \mathcal{V}_r(F). \tag{5.5}$$

Let $V_i \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n))$ with $|V_i|_{T\Upsilon} \le 1$ so that

$$\mathcal{V}_r(F) = \lim_{i \to \infty} \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V_i) F d\pi.$$

Observe that $(V_i)_{\eta,r} \in \mathrm{CylV}_0(\Upsilon(B_r))$, then by definition of $\mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r})$ we get

$$\int_{\Upsilon(B_r)} ((\nabla^* V_i) F)_{\eta,r} d\pi_{B_r} = \int_{\Upsilon(B_r)} (\nabla^*_{\Upsilon(B_r)} (V_i)_{\eta,r}) F_{\eta,r} d\pi_{B_r} \le \mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}), \quad \forall i \in \mathbb{N}.$$

Therefore, by Lemma 5.6,

$$\begin{aligned}\mathcal{V}_r(F) &= \lim_{i\to\infty}\int_{\Upsilon(\mathbb{R}^n)}(\nabla^* V_i)F d\pi\\ &= \lim_{i\to\infty}\int_{\Upsilon(B_r^c)}\int_{\Upsilon(B_r)}(\nabla^*_{\Upsilon(B_r)}(V_i)_{\eta,r})F_{\eta,r}d\pi_{B_r}d\pi_{B_r^c}(\eta)\\ &\le \int_{\Upsilon(B_r^c)}\mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r})d\pi_{B_r^c}(\eta)\,,\end{aligned}$$

which completes the proof of (5.5).

Let us now pass to the proof of the opposite inequality

$$\int_{\Upsilon(B_r^c)}\mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r})d\pi_{B_r^c}(\eta)\le\mathcal{V}_r(F)\,. \tag{5.6}$$

We divide it into three steps.

Step 1. We show the existence of $\{V_i : i\in\mathbb{N}\}\subset \mathrm{CylV}_0(\Upsilon(B_r))$ such that $|V_i|_{T\Upsilon}\le 1$ and

$$\mathcal{V}_{\Upsilon(B_r)}(G)=\sup_{i\in\mathbb{N}}\int_{\Upsilon(B_r)}(\nabla^*_{\Upsilon(B_r)}V_i)G d\pi_{B_r}\,, \tag{5.7}$$

for any $G\in\cup_{p>1}L^p(\Upsilon(B_r),\pi)$.

First we observe that there exists $\mathcal{F}F:=\{G_i : i\in\mathbb{N}\}\subset\mathrm{CylF}(\Upsilon(B_r))$ such that any cylinder function can be approximated strongly in $H^{1,q}(\Upsilon(B_r))$ for any $q<\infty$, by elements of $\mathcal{F}F$. Let $D\subset C_0^\infty(B_r;\mathbb{R}^n)$ be a countable dense subset, w.r.t. the C^1-norm: $\|v\|_{C^1}:=\|\nabla v\|_{L^\infty}+\|v\|_{L^\infty}$. We define the countable family

$$\mathcal{F}V:=\left\{\beta V(\gamma,x)\phi_\alpha(|V|_{T_\gamma\Upsilon}) : V(\gamma,x)=\sum_{i=1}^m w_i(x)G_i(\gamma),\ \alpha,\beta\in\mathbb{Q}^+,\ m\in\mathbb{N},\ w_i\in D,\ G_i\in\mathcal{F}F\right\}\,,$$

where $\phi_\alpha\in C^\infty([0,\infty))$ satisfies $0\le\phi_\alpha\le 1$, $|\phi'_\alpha|\le 2/\alpha$ and $\phi_\alpha(t)=1$ on $[0,1+\alpha]$, $\phi(t)=0$ on $[1+2\alpha,\infty)$.

Fix $\delta>0$, $q\in[1,\infty)$ and $V\in\mathrm{CylV}_0(\Upsilon(B_r))$ with $|V|_{T_\gamma\Upsilon}\le 1$. To prove (5.7) it suffices to show that there exists $W\in\mathcal{F}V$ with $|W|_{T\Upsilon}\le 1$ such that $\|\nabla^*(V-W)\|_{L^q}\le\delta$.

Fix $t\in(q,2q)$ and $\varepsilon\in(0,1/9)$. Letting $V=\sum_{i=1}^m F_i v_i\in\mathrm{CylV}_0(\Upsilon(B_r))$, we pick $G_i\in\mathcal{F}F$ and $w_i\in D$ such that

$$\sum_{i=1}^m\big(\|v_i-w_i\|_{C^1}+\|F_i-G_i\|_{L^t(\Upsilon(B_r))}+\|\nabla(F_i-G_i)\|_{L^t(\Upsilon(B_r))}\big)<\varepsilon, \tag{5.8}$$

and consider $\bar W:=\sum_{i=1}^m w_iG_i$. By using the divergence formula (2.12), we can obtain that

$$\int_{\Upsilon(B_r)}|\nabla^*(\bar W-V)|^t d\pi_{B_r}+\int_{\Upsilon(B_r)}\big||\bar W|_{T_\gamma\Upsilon}-|V|_{T_\gamma\Upsilon}\big|^t d\pi_{B_r}\le C\varepsilon^t\,, \tag{5.9}$$

where $C=\max\{\|w_i\|_{C^1},\|G_i\|_{L^t(\Upsilon(B_r))},\|\nabla G_i\|_{L^t(\Upsilon(B_r))} : 1\le i\le m\}$ does not depend on ε. We assume without loss of generality that $\varepsilon,\varepsilon^{\frac{1}{10t}}\in\mathbb{Q}$ and set

$$W:=(1-2\varepsilon^{\frac{1}{10t}})\phi_{\varepsilon^{\frac{1}{10t}}}\big(|\bar W|^2_{T_\gamma\Upsilon}\big)\bar W\in\mathcal{F}V\,, \tag{5.10}$$

which satisfies

$$|W|_{T_\gamma\Upsilon}=(1-2\varepsilon^{\frac{1}{10t}})\phi_{\varepsilon^{\frac{1}{10t}}}\big(|\bar W|^2_{T_\gamma\Upsilon}\big)|\bar W|_{T_\gamma\Upsilon}\le(1-2\varepsilon^{\frac{1}{10t}})(1+2\varepsilon^{\frac{1}{10t}})\le 1\,.$$

We now check that $\|\nabla^*(V-W)\|_{L^q}\le\delta$. From the identity

$$\nabla^* W=(1-2\varepsilon^{\frac{1}{10t}})\phi_{\varepsilon^{\frac{1}{10t}}}\big(|\bar W|^2_{T_\gamma\Upsilon}\big)(\nabla^*\bar W)-2(1-2\varepsilon^{\frac{1}{10t}})\phi'_{\varepsilon^{\frac{1}{10t}}}\big(|\bar W|^2_{T_\gamma\Upsilon}\big)|\bar W|^2_{T_\gamma\Upsilon}\,,$$

and the inequality

$$\left|\phi'_{\varepsilon^{\frac{1}{10t}}}\left(|\bar W|^2_{T_\gamma\Upsilon}\right)\right||\bar W|^2_{T_\gamma\Upsilon} \le 2\varepsilon^{-\frac{1}{10t}}\chi_{\{1+\varepsilon^{\frac{1}{10t}}\le|\bar W|^2_{T_\gamma\Upsilon}\le 1+2\varepsilon^{\frac{1}{10t}}\}}|\bar W|^2_{T_\gamma\Upsilon} \le 5\varepsilon^{-\frac{1}{10t}}\chi_{\{|\bar W|^2_{T_\gamma\Upsilon}\ge 1+\varepsilon^{\frac{1}{10t}}\}}\,,$$

we obtain

$$\begin{aligned}\|\nabla^*(W-\bar W)\|_{L^q} &\le \left\|\left((1-2\varepsilon^{\frac{1}{10t}})\phi_{\varepsilon^{\frac{1}{10t}}}(|\bar W|^2_{T_\gamma\Upsilon})-1\right)(\nabla^*\bar W)\right\|_{L^q} + 5\varepsilon^{-\frac{1}{10t}}\left\|\chi_{\{|\bar W|^2_{T_\gamma\Upsilon}\ge 1+\varepsilon^{\frac{1}{10t}}\}}\right\|_{L^q}\\ &\le 5\varepsilon^{\frac{1}{10t}}\|\nabla^*\bar W\|_{L^q} + \left\|\chi_{\{|\bar W|^2_{T_\gamma\Upsilon}\ge 1+\varepsilon^{\frac{1}{10t}}\}}(\nabla^*\bar W)\right\|_{L^q} + 5\varepsilon^{-\frac{1}{10t}}\left\|\chi_{\{|\bar W|^2_{T_\gamma\Upsilon}\ge 1+\varepsilon^{\frac{1}{10t}}\}}\right\|_{L^q}\\ &\le C\Big(\|\nabla^*\bar W\|_{L^t},t,q\Big)\left(\varepsilon^{\frac{1}{10t}}+\varepsilon^{-\frac{1}{10t}}\left\|\chi_{\{|\bar W|^2_{T_\gamma\Upsilon}\ge 1+\varepsilon^{\frac{1}{10t}}\}}\right\|_{L^t}\right),\end{aligned}\tag{5.11}$$

where we estimated $\|\chi_{\{|\bar W|^2_{T_\gamma\Upsilon}\ge 1+\varepsilon^{\frac{1}{10t}}\}}(\nabla^*\bar W)\|_{L^q}$ by means of the Hölder inequality and using that $t<2q$. The Chebyshev inequality and (5.9) give

$$\begin{aligned}\left\|\chi_{\{|\bar W|^2_{T_\gamma\Upsilon}\ge 1+\varepsilon^{\frac{1}{10t}}\}}\right\|_{L^t} &\le \left\|\chi_{\{|\bar W|_{T_\gamma\Upsilon}\ge 1+\varepsilon^{\frac{1}{3t}}\}}\right\|_{L^t} \le \left\|\chi_{\{||\bar W|_{T_\gamma\Upsilon}-|V|_{T_\gamma\Upsilon}|\ge\varepsilon^{\frac{1}{3t}}\}}\right\|_{L^t}\\ &\le \left(\varepsilon^{-\frac{1}{3t}}\left\||\bar W|_{T_\gamma\Upsilon}-|V|_{T_\gamma\Upsilon}\right\|_{L^1}\right)^{1/t} \le C\varepsilon^{\frac{1}{t}-\frac{1}{3t^2}} \le C\varepsilon^{\frac{1}{2t}} \quad (\varepsilon<1),\end{aligned}$$

where $C=\max\{\|w_i\|_{C^1},\|G_i\|_{L^1(\Upsilon(B_r))},\|\nabla G_i\|_{L^1(\Upsilon(B_r))} : 1\le i\le m\}$ is independent of ε. Therefore, we conclude

$$\begin{aligned}\|\nabla^*(W-V)\|_{L^q} &\le \|\nabla^*(W-\bar W)\|_{L^q}+\|\nabla^*(\bar W-V)\|_{L^q}\\ &\le C(\varepsilon^{\frac{1}{10t}}+\varepsilon^{\frac{1}{5t}})+\varepsilon\le\delta\,,\end{aligned}$$

provided ε is small enough. The proof of (5.7) is complete.

Step 2. We conclude the proof of (5.6).

Note that the map $\gamma\mapsto F(\gamma)\nabla^*_{\Upsilon(B_r)}V(\gamma|_{B_r})$ is π-measurable. Furthermore, by Lemma 4.4, F_{η,B_r^c} is π_{B_r}-measurable and the map

$$\Upsilon(B_r^c)\ni\eta\mapsto\int_{\Upsilon(B_r)}(\nabla^*_{\Upsilon(B_r)}V)F_{\eta,B_r^c}d\pi_{B_r}$$

is $\pi_{B_r^c}$-measurable. Therefore, the map $\eta\mapsto\mathcal V_{\Upsilon(B_r)}(F_{\eta,B_r^c})$ is $\pi_{B_r^c}$-measurable.

Fix now $\varepsilon>0$ and define a sequence $\{C_j : j\in\mathbb N\}$ of subsets in $\Upsilon(B_r^c)$ so that $C_0=\emptyset$, and

$$\begin{aligned}C_j := \Big\{\eta\in\Upsilon(B_r^c) :\ & F_{\eta,r}\text{ is }\ \pi_{B_r}\text{-measurable and,}\\ &\int_{\Upsilon(B_r)}(\nabla^*_{\Upsilon(B_r)}V_j)F_{\eta,r}d\pi_{B_r}\ge(1-\varepsilon)\mathcal V_{\Upsilon(B_r)}(F_{\eta,r})\wedge\varepsilon^{-1}\Big\}\setminus\bigcup_{i=1}^{j-1}C_i\,,\end{aligned}$$

where the family $\{V_i : i\in\mathbb N\}$ has been built in Step 1.

Then, C_j is $\pi_{B_r^c}$-measurable for any j and $\pi_{B_r^c}(\Upsilon(B_r^c)\setminus\cup_{j=1}^\infty C_j)=0$. Set

$$W_n^\eta(\gamma):=W_n(\gamma+\eta):=\sum_{j=1}^n V_j(\gamma)\chi_{C_j}(\eta),\quad \gamma\in\Upsilon(B_r),\ \eta\in\Upsilon(B_r^c)\,.$$

We approximate χ_{C_j} by $\{F_j^i\}_{i\in\mathbb N}\subset\mathrm{CylF}(\Upsilon(B_r^c))$ with $|F_j^i|\le 1$ in the strong $L^{p'}(\Upsilon(B_r^c),\pi_{B_r^c})$ topology, where $\frac{1}{p'}+\frac{1}{p}=1$. Thus, setting $W_n^i(\gamma+\eta):=\sum_{j=1}^n V_j(\gamma)F_j^i(\eta)$, we see that

$$\int_{\Upsilon(B_r^c)}\|\nabla^*_{\Upsilon(B_r)}(W_n-W_n^i)(\cdot+\eta)\|_{L^{p'}(\Upsilon(B_r))}d\pi_{B_r^c}(\eta)\to 0\quad\text{as } i\to\infty\,.$$

Notice that $W_n^i \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n))$, hence

$$\lim_{i\to\infty}\int_{\Upsilon(B_r^c)}\left(\int_{\Upsilon(B_r)}(\nabla^*_{\Upsilon(B_r)}W_n^i(\cdot+\eta))f_{\eta,r}d\pi_{B_r}\right)d\pi_{B_r^c}(\eta) \tag{5.12}$$
$$=\int_{\Upsilon(B_r^c)}\int_{\Upsilon(B_r)}(\nabla^*_{\Upsilon(B_r)}W_n^\eta)f_{\eta,r}d\pi_{B_r}\pi_{B_r^c}(\eta)$$
$$=\int_{\Upsilon(B_r^c)}\left(\sum_{j=1}^n\chi_{C_j}(\eta)\int_{\Upsilon(B_r)}(\nabla^*_{\Upsilon(B_r)}V_j)f_{\eta,r}d\pi_{B_r}\right)d\pi_{B_r^c}(\eta)$$
$$\geq(1-\varepsilon)\int_{\Upsilon(B_r^c)}\left(\sum_{j=1}^n\chi_{C_j}(\eta)\mathcal{V}_{\Upsilon(B_r)}(f_{\eta,r})\wedge\varepsilon^{-1}\right)d\pi_{B_r^c}$$
$$=(1-\varepsilon)\int_{\cup_{j=1}^n C_j}\mathcal{V}_{\Upsilon(B_r)}(f_{\eta,r})\wedge\varepsilon^{-1}d\pi_{B_r^c}(\eta).$$

By Lemma 5.6,

$$\int_{\Upsilon(\mathbb{R}^n)}(\nabla^*W_n^i)fd\pi=\left(\int_{\Upsilon(B_r)}(\nabla^*_{\Upsilon(B_r)}W_n^i(\cdot+\eta))f_{\eta,r}d\pi_{B_r}\right)d\pi_{B_r^c}(\eta)\,, \tag{5.13}$$

which along with (5.12) gives the claimed inequality by letting $i\to\infty$ and $n\to\infty$. □

5.2. **Relaxation approach.** In this subsection we introduce a second notion of functions with bounded variations. We rely on a relaxation approach.

Definition 5.7 (BV functions II: Relaxation)**.** Let $F\in L^1(\Upsilon(\mathbb{R}^n))$, we define *the total variation of F* by

$$|\mathsf{D}_*F|(\Upsilon(\mathbb{R}^n)):=\inf\{\liminf_{n\to\infty}\|\nabla F_n\|_{L^1(T\Upsilon)}\,:\,F_n\to F\text{ in }\;L^1(\Upsilon(\mathbb{R}^n))\,,\,F_n\in\mathrm{CylF}(\Upsilon(\mathbb{R}^n))\}\,. \tag{5.14}$$

If $|\mathsf{D}_*F|(\Upsilon(\mathbb{R}^n))<\infty$, we say that F has finite *relaxed total variation.*

Definition 5.8 (Total variation pre-measure)**.** If $|\mathsf{D}_*F|(\Upsilon(\mathbb{R}^n))<\infty$, we define a map

$$|\mathsf{D}_*F|:\{G\in\mathrm{CylF}(\Upsilon(\mathbb{R}^n))\,:\,G\text{ is nonnegative}\}\to\mathbb{R}\,,$$

$$|\mathsf{D}_*F|[G]:=\inf\left\{\liminf_{n\to\infty}\int_{\Upsilon(\mathbb{R}^n)}G|\nabla F_n|_{T\Upsilon}d\pi\,:\,F_n\to F\text{ in }\;L^1(A,\pi)\,,\,F_n\in\mathrm{CylF}(\Upsilon(\mathbb{R}^n))\right\}\,. \tag{5.15}$$

Notice that $|\mathsf{D}_*F|[G]\leq\|G\|_{L^\infty}|\mathsf{D}_*F|$ and $|\mathsf{D}_*F|[G_1+G_2]\geq|\mathsf{D}_*F|[G_1]+|\mathsf{D}_*F|[G_2]$. By construction, $|\mathsf{D}_*F|[G]$ is the lower semi-continuous envelope of the functional $\mathrm{CylF}(\Upsilon(\mathbb{R}^n))\ni F\mapsto\int_{\Upsilon(\mathbb{R}^n)}G|\nabla F|_{T\Upsilon}d\pi$. Therefore, the map $F\mapsto|\mathsf{D}_*F|(G)$ is lower semi-continuous with respect to the L^1-convergence for any nonnegative $G\in\mathrm{CylF}(\Upsilon(\mathbb{R}^n))$.

It will be shown in Corollary 7.4 that $|\mathsf{D}_*F|$ is represented by a finite measure $|\mathsf{D}F|$, i.e.

$$|\mathsf{D}_*F|[G]=\int_{\Upsilon(\mathbb{R}^n)}Gd|\mathsf{D}F|\quad\text{for any nonnegative }G\in\mathrm{CylF}(\Upsilon(\mathbb{R}^n))\,.$$

Remark 5.9. Our definitions above deviate slightly from the ones considered in the literature of BV functions on metric measure spaces [10, 35].

Given a metric measure structure (X,d,m) and $F\in L^1(X,m)$, usually the total variation $|\mathsf{D}_*F|(X)$ is defined by considering regularization sequences $F_n\in\mathrm{Lip}_b(X)$ that are Lipschitz and bounded. The supports of the latter are fine enough to prove that, for any open set $A\subset X$, the map

$$|\mathsf{D}_* F|(A) := \inf\{\liminf_{n\to\infty} \|\nabla F_n\|_{L^1(A)} \ : \ F_n \to F \text{ in } \ L^1(A,m)\,,\ F_n \in \mathrm{Lip}_b(X)\}\,, \tag{5.16}$$

is a restriction to open set of a finite measure, provided $|\mathsf{D}_* F|(X) < \infty$.

To capture the geometric-analytic structure of the configuration space $\Upsilon(\mathbb{R}^n)$, which is an extended metric measure structure, is natural to replace Lipschitz functions by Cylinder functions as a class of test objects. The supports of the latter, however, are coarser than open sets, hence it is not natural to define the total variation measure via (5.16). Definition 5.8 provides the natural replacement of (5.16) in our setting.

5.3. **Heat semigroup approach.** In this subsection we present the third approach to BV functions. We employ the heat semigroup to define the total variation of a function $F \in L^p(\Upsilon(\mathbb{R}^n))$, $p > 1$.

Proposition 5.10. *Let $F \in \cup_{p>1} L^p(\Upsilon(\mathbb{R}^n), \pi)$. Then $\|\nabla T_t F\|_{L^1} < \infty$ for any $t > 0$ and the following limit exists*

$$\mathcal{T}(F) := \lim_{t\to 0} \|\nabla T_t F\|_{L^1}\,. \tag{5.17}$$

Definition 5.11 (BV functions III: Heat semigroup)**.** A function $F \in \cup_{p>1} L^p(\Upsilon(\mathbb{R}^n), \pi)$ is *BV in the sense of the heat semigroup* if $\mathcal{T}(F) < \infty$. We define *the total variation of* F by $\mathcal{T}(F)$.

To prove Proposition 5.10, we need the *Bakry-Émery inequality* with exponent $q = 1$, i.e. for any $t, s > 0$, $F \in \cup_{p>1} L^p(\Upsilon(\mathbb{R}^n), \pi)$, it holds

$$\int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t F| d\pi < \infty\,, \qquad |\nabla T_{t+s} F| \le T_t |\nabla T_s F| \quad \pi\text{-a.e.}\,. \tag{5.18}$$

The inequality (5.18) will be proven in Corollary 5.15 in Section 5.4. Let us now use it to show Proposition 5.10.

Proof of Proposition 5.10. Let $F \in L^p(\Upsilon, \pi)$ for $p > 1$. By (5.18), we see that

$$\|\nabla T_t F\|_{L^1} \le \liminf_{s\to 0} \|\nabla T_{t+s} F\|_{L^1} \le \liminf_{s\to 0} \|\nabla T_s F\|_{L^1}\,.$$

By taking $\limsup_{t\to 0}$, we obtain $\limsup_{t\to 0} \|\nabla T_t F\|_{L^1} \le \liminf_{s\to 0} \|\nabla T_s F\|_{L^1}$, which concludes the proof. □

5.4. **p-Bakry-Émery inequality.** In order to complete the proof of Proposition 5.10, we show the p-Bakry-Émery inequality for the *Hodge heat flow*, which implies in turn the scalar version (5.18). It will play a significant role also in the proof of Theorem 5.17.

Definition 5.12 (Hodge Laplacian)**.** Let us consider $V = \sum_{k=1}^m F_k v_k \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ with $F_k = \Phi_k((f_1^k)^\star, \dots, (f_\ell^k)^\star)$. Define Hodge Laplacian of V as

$$\begin{aligned} \Delta_H V(\gamma, x) := &\sum_{k=1}^m \sum_{i,j=1}^\ell \frac{\partial^2 \Phi_k}{\partial x_i \partial x_j}\big((f_1^k)^\star\gamma, \dots, (f_\ell^k)^\star\gamma\big)\big\langle \nabla f_i^k, \nabla f_j^k\big\rangle_{T_\gamma \Upsilon} v_k(x) \\ &+ \sum_{k=1}^m \sum_{i=1}^\ell \frac{\partial \Phi_k}{\partial x_i}\big((f_1^k)^\star\gamma, \dots, (f_\ell^k)^\star\gamma\big)(\Delta f_k)^\star\gamma v_k(x) \\ &+ \sum_{k=1}^m \Phi_k\big((f_1^k)^\star\gamma, \dots, (f_\ell^k)^\star\gamma\big)\Delta_H v_k(x), \end{aligned} \tag{5.19}$$

where $\Delta_H v_k$ is the Hodge Laplacian of $v_k \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$. It turns out that $\Delta_H V$ does not depend on the choice of both the representative of V and the inner and outer functions of F_k (see [1, Theorem 3.5]).

We define the corresponding energy functional:

$$\mathcal{E}_H(V,W) := \langle -\Delta_H V, W\rangle_{L^2(T\Upsilon,\pi)} = \int_{\Upsilon(\mathbb{R}^n)} \mathbf{\Gamma}^\Upsilon(V,W)d\pi\,, \quad V,W \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))\,, \tag{5.20}$$

where $\mathbf{\Gamma}^\Upsilon$ denotes the square field operator associated with Δ_H. By [1, Theorem 3.5], the form $\mathcal{E}_H$ is closable on $\mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ and the corresponding closure is denoted by $\mathcal{D}(\mathcal{E}_H)$ and the corresponding (Friedrichs) extension of $\mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ is denoted by $\mathcal{D}(\Delta_H)$. Let $\{\mathbf{T}_t\}$ denote the corresponding L^2-semigroup. By a general theory of functional analysis, it holds that

$$\mathbf{T}_t V \in \mathcal{D}(\mathcal{E}_H)\,, \quad \text{for any } t \geq 0 \text{ and } V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))\,. \tag{5.21}$$

The following intertwining property holds.

Theorem 5.13. *$\nabla T_t F = \mathbf{T}_t \nabla F$ for any $t \geq 0$ and for any $F \in W^{1,2}(\Upsilon(\mathbb{R}^n),\pi)$.*

Proof. Using the commutation $\nabla\Delta = \Delta_H\nabla$ of the operators at the level of the base space $\mathbb{R}^n$, we can show $\nabla\Delta F = \Delta_H\nabla F$ for any $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ by the representation (5.19). By the essential self-adjointness of $(\Delta, \mathrm{CylF}(\Upsilon(\mathbb{R}^n)))$ (see [2, Theorem 5.3]), and applying [41, Theorem 2.1] with $\mathcal{D} = \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$, $A = \Delta$, $\hat{A} = \Delta_H$, $\hat{T}_t = \mathbf{T}_t$, $R = 0$, we conclude the sought conclusion. □

Theorem 5.14. *Let $F \in \mathcal{D}(\mathcal{E}_H)$. Then $|\mathbf{T}_t F|_{T\Upsilon} \leq T_t|F|_{T\Upsilon}$ π-a.e. for any $t \geq 0$. In particular $\mathbf{T}_t$ can be extended to the L^p-velocity fields $L^p(T\Upsilon,\pi)$ for any $1 \leq p < \infty$.*

Proof. By the Weitzenböck formula [1, Theorem 3.7] on $\Upsilon(\mathbb{R}^n)$, we can express $\Delta_H = \nabla^*\nabla + R^\Upsilon$, where R^Υ is the lifted curvature tensor from the base space $\mathbb{R}^n$. Using the flatness $\mathbb{R}^n$ we can easily deduce $R^\Upsilon = 0$.

Now, setting $\mathbf{\Gamma}(V,W) := \mathbf{\Gamma}^\Upsilon(V,W) + 2R^\Upsilon(V,W) = \mathbf{\Gamma}^\Upsilon(V,W)$ we can apply [42, Theorem 3.1] (see the proof of [42, Theorem 3.1] for $p=1$) and [42, Proposition 3.5], to get the sought conclusion of the first assertion.

We now prove the second assertion. Let $V \in L^p(T\Upsilon)$, the density of cylinder vector fields gives the existence of a sequence $V_n \in \mathrm{CylF}(\Upsilon) \subset \mathcal{D}(\mathcal{E}_H)$ such that $|V_n - V|_{T\Upsilon} \to 0$ in $L^p(\Upsilon(\mathbb{R}^n),\pi)$ as $n \to \infty$. We can define

$$\mathbf{T}_t V := \lim_{n\to\infty} \mathbf{T}_t V_n\,. \tag{5.22}$$

The existence of the limit follows from

$$|\mathbf{T}_t V_n - \mathbf{T}_t V_m|_{T\Upsilon} \leq T_t|V_n - V_m|_{T\Upsilon}\,, \tag{5.23}$$

as well as the independence of the limit from the approximating sequence $(V_n)_{n\in\mathbb{N}}$. □

Theorem 5.15 (p-Bakry-Émery estimate)**.** *Let $p > 1$. The following assertions hold:*

(i) *$T_t : H^{1,p}(\Upsilon(\mathbb{R}^n),\pi) \to H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$ is a continuous operator for any $t > 0$.*

(ii) *For any $F \in H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$,*

$$|\nabla T_t F|^p_{T\Upsilon} \leq T_t|\nabla F|^p_{T\Upsilon} \quad \pi\text{-a.e.}\,. \tag{5.24}$$

(iii) *Let $1 < p \leq 2$. For any $F \in L^p(\Upsilon(\mathbb{R}^n),\pi)$ it holds that*

$$\|\nabla T_t F\|_{L^p} \leq C(p)t^{-1/2}\|F\|_{L^p}, \quad \forall t > 0. \tag{5.25}$$

In particular, $T_t : L^p(\Upsilon(\mathbb{R}^n),\pi) \to H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$ is a continuous operator for any $t > 0$.

(iv) *For any $t,s > 0$, $F \in L^p(\Upsilon(\mathbb{R}^n),\pi)$, it holds that $\|\nabla T_t F\|_{L^1(\Upsilon(\mathbb{R}^n))} < \infty$ and*

$$|\nabla T_{t+s} F|_{T\Upsilon} \leq T_t|\nabla T_s F|_{T\Upsilon} \quad \pi\text{-a.e.}\,. \tag{5.26}$$

Proof. **(i).** By Theorem 5.13 and Theorem 5.14, for any $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ it holds that

$$|\nabla T_t F|_{T\Upsilon} = |\mathbf{T}_t \nabla F|_{T\Upsilon} \leq T_t|\nabla F|_{T\Upsilon} \quad \pi\text{-a.e.}\,. \tag{5.27}$$

A simple application of Jensen's inequality to (5.27) gives

$$|\nabla T_t F|^p_{T\Upsilon} \le T_t|\nabla F|^p_{T\Upsilon}\,, \quad \text{for any } F \in \mathrm{CylF}(\Upsilon) \text{ and } p \ge 1\,. \tag{5.28}$$

Let $F_n \in \mathrm{CylF}(\Upsilon)$ be a $H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$-Cauchy sequence. Then, by (5.28) and the invariance $\pi(T_t f) = \pi(f)$,

$$\int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t(F_n - F_m)|^p_{T\Upsilon} d\pi \le \int_{\Upsilon(\mathbb{R}^n)} T_t|\nabla(F_n - F_m)|^p_{T\Upsilon} d\pi = \int_{\Upsilon(\mathbb{R}^n)} |\nabla(F_n - F_m)|^p_{T\Upsilon} d\pi \to 0. \tag{5.29}$$

Since $H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$ is the closure of $\mathrm{CylF}(\Upsilon)$ w.r.t. the norm $\|\nabla\cdot\|_{L^p(\Upsilon,\pi)} + \|\cdot\|_{L^p(\Upsilon,\pi)}$, by (5.29), the operator T_t is extended to $H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$ continuously. The proof of the first assertion is complete.

(ii). Let $F \in H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$ and take $F_n \in \mathrm{CylF}(\Upsilon)$ converging to F in $H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$. Then, by the lower semi-continuity of $|\nabla\cdot|^p_{T\Upsilon}$ w.r.t. the L^p-strong convergence, the continuity of the L^p-semigroup T_t and the inequality (5.28), we obtain

$$|\nabla T_{t+s}F|^p_{T\Upsilon} = |\nabla T_t T_s F|^p_{T\Upsilon} \le \liminf_{n\to\infty} |\nabla T_t T_s F_n|^p_{T\Upsilon} \le \liminf_{n\to\infty} T_t|\nabla T_s F_n|^p_{T\Upsilon} \le T_t|\nabla T_s F|^p_{T\Upsilon}.$$

Here the last equality follows from the assertion (i).

(iii). Let $p > 1$ be fixed. For any $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ satisfying $F \ge 0$, it holds

$$p(p-1)\int_0^t \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_s F|^2_{T\Upsilon} |T_s F|^{p-2} d\pi ds = \int_{\Upsilon(\mathbb{R}^n)} |F|^p d\pi - \int_{\Upsilon(\mathbb{R}^n)} |T_t F|^p d\pi \le \int_{\Upsilon(\mathbb{R}^n)} |F|^p d\pi\,.$$

It follows by studying first $\frac{d}{dt}\int_{\Upsilon(\mathbb{R}^n)} |T_t(F+\varepsilon)| d\pi$, and after letting $\varepsilon \to 0$.

By the contraction property of T_t, we obtain

$$\begin{aligned}\int_0^t \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_s F|^p_{T\Upsilon} d\pi ds &\le \left(\int_0^t \int_{\Upsilon(\mathbb{R}^n)} |T_s F|^p d\pi ds\right)^{\frac{2-p}{2}} \left(\int_0^t \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_s F|^2_{T\Upsilon} |T_s F|^{p-2} d\pi ds ds\right)^{\frac{p}{2}} \\ &\le C t^{\frac{2-p}{2}} \|F\|^p_{L^p}\,.\end{aligned}$$

We now employ the Bakry-Émery inequality (5.28) combined with the contraction property of T_t to show that $s \to \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_s F|^p_{T\Upsilon} d\pi$ is nonincreasing, which yields

$$t\int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t F|^p_{T\Upsilon} d\pi \le \int_0^t \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_s F|^p_{T\Upsilon} d\pi ds \le C t^{\frac{2-p}{2}} \|F\|^p_{L^p}\,. \tag{5.30}$$

This implies our conclusion for cylinder functions. We extended it to any $F \in L^p(\Upsilon(\mathbb{R}^n),\pi)$ by means of a density argument. Indeed, given $F \in L^p(\Upsilon(\mathbb{R}^n),\pi)$, we can find $F_n \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ such that $F_n \to F$ in L^p. The continuity of the semigroup T_t gives $T_t F_n \to T_t F$ in L^p, while the lower semicontinuity of the functional $G \to \int_{\Upsilon(R^n)} |\nabla G|^p_{T\Upsilon(\mathbb{R}^n)} d\pi$ with respect to the L^p convergence for $p > 1$ yields

$$\int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t F|^p_{T\Upsilon} d\pi \le \liminf_{n\to\infty} \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t F_n|^p_{T\Upsilon} d\pi \le C t^{-1/2} \|F\|_{L^p}\,.$$

(iv). Note that the assertion in the case of $1 < p \le 2$ implies the one in the case of $p > 2$ by $L^p(\Upsilon(\mathbb{R}^n),\pi) \subset L^q(\Upsilon(\mathbb{R}^n),\pi)$ whenever $1 \le q \le p$. Thus, we only need to prove it in the case of $1 < p \le 2$. Let $F \in L^p(\Upsilon(\mathbb{R}^n),\pi)$. Then, by the assertion (iii), $T_s F \in H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$. Take G_n converging to $T_s F$ in $H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$. Then, up to taking a subsequence from $\{G_n\}$, and by making use of (5.27), we conclude that

$$|\nabla T_{t+s}F|_{T\Upsilon} = |\nabla T_t T_s F|_{T\Upsilon} = \lim_{n\to\infty} |\nabla T_t G_n|_{T\Upsilon} \le \lim_{n\to\infty} T_t|\nabla G_n|_{T\Upsilon} = T_t|\nabla T_s F|_{T\Upsilon}\,.$$

The proof is complete. □

Remark 5.16. In [24], the 2-Bakry-Émery estimate was proved in the case of the configuration space over a complete Riemannian manifold with Ricci curvature bound. For the purpose of the current paper, however, we need a stronger estimate, i.e., the p-Bakry-Émery estimate (5.24) for arbitrary $1 < p < \infty$ and also the regularity estimate (5.25) of the heat semigroup, both of which do not follow only from the 2-Bakry-Émery inequality.

5.5. **Equivalence of BV functions.** In Section 5, we introduced the three different definitions (the variational/the relaxation/the semigroup approaches) of BV functions. In this section we show that the three different definitions of BV functions are equivalent and we introduce a universal definition of BV functions for $L^2(\Upsilon(\mathbb{R}^n), \pi)$-functions.

Theorem 5.17 (Equivalence of BV functions)**.** *Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi)$. Then,*

$$\mathcal{V}(F) = |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) = \mathcal{T}(F)\,.$$

The proof of Theorem 5.17 will be given later in this section. Thanks to Theorem 5.17, we can introduce a universal definition of BV functions for $L^2(\Upsilon(\mathbb{R}^n), \pi)$-functions.

Definition 5.18 (BV functions)**.** *A function $F \in L^2(\Upsilon(\mathbb{R}^n))$ belongs to $\mathrm{BV}(\Upsilon(\mathbb{R}^n))$ if*

$$\mathcal{V}(F) = |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) = \mathcal{T}(F) < \infty\,.$$

We prepare several lemmas for the proof of Theorem 5.17.

Lemma 5.19. *For any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ and $t \geq 0$ it holds*

$$(\nabla^* \mathbf{T}_t V) = T_t(\nabla^* V)\,. \tag{5.31}$$

In particular $(\nabla^* \mathbf{T}_t V) \in L^p(\Upsilon(\mathbb{R}^n))$ for any $1 < p < \infty$.

Proof. Let $F \in \mathrm{CylF}(\Upsilon)$. By the π-symmetry of T_t and Theorem 5.13, we have that

$$\begin{aligned}\int_{\Upsilon(\mathbb{R}^n)} F\, T_t(\nabla^* V) d\pi &= \int_{\Upsilon(\mathbb{R}^n)} T_t F\, (\nabla^* V) d\pi = -\int_{\Upsilon(\mathbb{R}^n)} \langle V(\gamma,\cdot), \nabla T_t F(\gamma)\rangle_{T\Upsilon} d\pi \\ &= -\int_{\Upsilon(\mathbb{R}^n)} \langle V(\gamma,\cdot), \mathbf{T}_t \nabla F(\gamma)\rangle_{T\Upsilon} d\pi = -\int_{\Upsilon(\mathbb{R}^n)} \langle \mathbf{T}_t V(\gamma,\cdot), \nabla F(\gamma)\rangle_{T\Upsilon} d\pi \\ &= \int_{\Upsilon(\mathbb{R}^n)} F(\nabla^* \mathbf{T}_t V) d\pi,\end{aligned}$$

which immediately implies (5.31). The proof is complete. □

Let us now introduce $\mathbf{D}^p(T\Upsilon(\mathbb{R}^n), \pi)$, the space of vector fields with divergence in $L^p(\Upsilon(\mathbb{R}^n), \pi)$, as the closure of $\mathrm{CylV}(\Upsilon(\mathbb{R}^n)) \subset L^p(T\Upsilon(\mathbb{R}^n))$ with respect to the norm $\|V\|_{L^p} + \|\nabla^* V\|_{L^p}$.

In the case $p = 2$, we have the following inclusion

$$\mathcal{D}(\mathcal{E}_H) \subset \mathbf{D}^2(T\Upsilon(\mathbb{R}^n), \pi)\,, \tag{5.32}$$

as a consequence of the inequality $\|\nabla^* V\|_{L^2} \leq \mathcal{E}_H(V, V)$ for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$.

Lemma 5.20. *Let $1 < p < \infty$ and $1 < p' < \infty$ such that $1/p + 1/p' = 1$. If $F \in L^{p'}(\Upsilon(\mathbb{R}^n), \pi)$ then*

$$\mathcal{V}(F) = \sup\left\{\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F d\pi : V \in \mathbf{D}^p(T\Upsilon(\mathbb{R}^n), \pi),\ |V|_{T\Upsilon} \leq 1\right\}. \tag{5.33}$$

Proof. Let $V \in \mathbf{D}^p(T\Upsilon(\mathbb{R}^n), \pi)$ with $|V|_{T\Upsilon} \leq 1$, to conclude the proof we just need to build a sequence $(W_n)_{n\in\mathbb{N}} \subset \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ such that $|W_n| \leq 1$ and $\|\nabla^* V - \nabla^* W_n\|_{L^p} \to 0$ as $n \to \infty$. To that aim we first consider a sequence $V_n \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ such that $\|V - V_n\|_{L^p} + \|\nabla^* V - \nabla^* V_n\|_{L^p} \to 0$ as $n \to \infty$, which exists by definition. We now define W_n by cutting V_n of as we did in (5.10) in the proof of Proposition 5.5. □

Proof of Theorem 5.17. We first show the inequality $\mathcal{V}(F) \le |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$ in the general case of $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ with $1 < p \le \infty$.

Let $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ for some $p > 1$ and $|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) < \infty$. Let $F_n \in \mathrm{CylF}(\Upsilon)$ such that $F_n \to F$ in $L^1(\Upsilon(\mathbb{R}^n))$ and $\|\nabla F_n\|_{L^1(T\Upsilon)} \to |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$. Let $F_{n,M} := (F_n \vee -M) \wedge M$ and $F_M := (F \vee -M) \wedge M$. Then, $F_{n,M} \to F_M$ in $L^1(\Upsilon(\mathbb{R}^n), \pi)$ and $\|\nabla F_{n,M}\|_{L^1(T\Upsilon)} \le \|\nabla F_n\|_{L^1(T\Upsilon)}$. Thus, $\limsup_{n\to\infty} \|\nabla F_{n,M}\|_{L^1(T\Upsilon)} \le |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$. By the integration by parts formula (2.11), it holds

$$\int_{\Upsilon(\mathbb{R}^n)} F_{n,M} \nabla^* V d\pi = -\int_{\Upsilon(\mathbb{R}^n)} \langle V, \nabla F_{n,M}\rangle_{T\Upsilon} d\pi \le \|\nabla F_{n,M}\|_{L^1(T\Upsilon)} \le \|\nabla F_n\|_{L^1(T\Upsilon)} ,$$

for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ with $|V|_{T\Upsilon} \le 1$. By taking a (non-relabelled) subsequence from $\{F_{n,M}\}$ so that $F_{n,M} \to F_M$ π-a.e., and using the dominated convergence theorem (notice that $|F_{n,M}\nabla^* V| \le M|\nabla^* V| \in L^1(\Upsilon(\mathbb{R}^n), \pi)$ uniformly in n), we obtain that

$$\int_{\Upsilon(\mathbb{R}^n)} F_M \nabla^* V d\pi = \lim_{n\to\infty} \int_{\Upsilon(\mathbb{R}^n)} F_{n,M} \nabla^* V d\pi \le \liminf_{n\to\infty} \|\nabla F_n\|_{L^1(T\Upsilon)} \le |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)),$$

for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ with $|V|_{T\Upsilon} \le 1$. Since $F_M \to F$ in $L^p(\Upsilon(\mathbb{R}^n), \pi)$ as $M \to \infty$ by the hypothesis $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$, we conclude $\mathcal{V}(F) \le |\mathsf{D}F|(\Upsilon(\mathbb{R}^n))$.

We now show the inequality $|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) \le \mathcal{V}(F) < \infty$. Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi)$. Set $F_n = T_{1/n} F \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$. By the symmetry of $\mathbf{T}_t$ in $L^2(T\Upsilon, \pi)$ and Lemma 5.19, we have that, for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ with $|V|_{T\Upsilon} \le 1$, it holds

$$\int_{\Upsilon(\mathbb{R}^n)} F_n \nabla^* V d\pi = \int_{\Upsilon(\mathbb{R}^n)} T_{1/n}(\nabla^* V) F d\pi = \int_{\Upsilon(\mathbb{R}^n)} \nabla^*(\mathbf{T}_{1/n} V) F d\pi. \tag{5.34}$$

The inclusion (5.21) and (5.32) imply that $\mathbf{T}_{1/n} V \in \mathbf{D}^2(T\Upsilon(\mathbb{R}^n), \pi)$, while Theorem 5.14 ensures that $|\mathbf{T}_{1/n} V|_{T\Upsilon} \le T_{1/n}|V|_{T\Upsilon} \le 1$. Therefore, we can apply Lemma 5.20 to (5.34) to obtain $\|\nabla F_n\|_{L^1} \le \mathcal{V}(F)$. Since $F_n \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$ and $\mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ is dense in $H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$, we have $|\mathsf{D}_* F_n|(\Upsilon(\mathbb{R}^n)) \le \|\nabla F_n\|_{L^1}$ by definition. By the lower semi-continuity $|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$ with respect to the L^2-convergence, it holds

$$|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) \le \liminf_{n\to\infty} |\mathsf{D}_* F_n|(\Upsilon(\mathbb{R}^n)) \le \liminf_{n\to\infty} \|\nabla F_n\|_{L^1(T\Upsilon,\pi)} \le \mathcal{V}(F).$$

This conclude the proof of $\mathcal{V}(F) = |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$.

We now prove $\mathcal{T}(F) \le |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$. Let $F_n \in \mathrm{CylF}(\Upsilon)$ such that $F_n \to F$ in $L^1(\Upsilon(\mathbb{R}^n), \pi)$ and $\|\nabla F_n\|_{L^1(T\Upsilon)} \to |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$. Then, by the 1-Bakry–Émery inequality (5.27) on cylinder functions,

$$\|\nabla T_t F\|_{L^1} \le \liminf_{n\to\infty} \|\nabla T_t F_n\|_{L^1} \le \liminf_{n\to\infty} \|\nabla F_n\|_{L^1} = |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)).$$

Thus, $\mathcal{T}(F) \le |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$.

Finally we prove $\mathcal{V}(F) \le \mathcal{T}(F)$. For $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ and $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ with $|V|_{T\Upsilon} \le 1$, we have that

$$\int_{\Upsilon(\mathbb{R}^n)} T_t F \nabla^* V d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle \nabla T_t F, V\rangle d\pi \le \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t F|_{T\Upsilon} d\pi.$$

Since $T_t F \to F$ in $L^p(\Upsilon(\mathbb{R}^n), \pi)$, we obtain that

$$\int_{\Upsilon(\mathbb{R}^n)} F \nabla^* V d\pi \le \lim_{t\to 0} \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t F|_{T\Upsilon} d\pi.$$

Thus, we conclude $\mathcal{V}(F) \le \mathcal{T}(F)$. The proof is complete. □

Remark 5.21. The proof of all the inequalities except $|\mathsf{D}_*F|(\Upsilon(\mathbb{R}^n)) \le \mathcal{V}(F)$ remains true for any $1 < p < \infty$. In order to prove the inequality $|\mathsf{D}_*F|(\Upsilon(\mathbb{R}^n)) \le \mathcal{V}(F)$ in full generality following the same strategy we need show that $\mathbf{T}_t V \in \mathbf{D}^p(T\Upsilon(\mathbb{R}^n), \pi)$ for $1 < p < \infty$ and $V \in \mathrm{CylV}(T\Upsilon)$. This should follow, for instance, from the L^p–boundedness of vector-valued Riesz transforms, and will be addressed in a future work.

6. Sets of finite perimeter

In this section we introduce and study the notion of *set with finite perimeter*. Let us begin with a definition

Definition 6.1 (Sets of finite perimeters)**.** Let $\Omega \subset \mathbb{R}^n$ be either a closed domain or the Euclidean space $\mathbb{R}^n$. A Borel set $E \subset \Upsilon(\Omega)$ is said to have finite perimeter if $\mathcal{V}_{\Upsilon(\Omega)}(\chi_E) < \infty$.

We refer the reader to Definition 5.1 for the introduction of the total variation $\mathcal{V}_{\Upsilon(\Omega)}(\cdot)$.

6.1. Sets of finite perimeter in $\Upsilon(B_r)$.

We first develop the necessary theory in the configuration space $\Upsilon(B_r)$, in which every argument essentially comes down to finite-dimensional geometric analysis since only finitely many particles are allowed to belong to B_r.

Let us recall the decomposition $\Upsilon(B_r) = \bigsqcup_{k\ge 0} \Upsilon^k(B_r)$, where $(\Upsilon^k(B_r), \mathsf{d}_{\Upsilon^k}, \pi^k_{B_r})$ is the k-particle configuration space $\Upsilon^k(B_r)$ over B_r equipped with the L^2-transportation distance d_{Υ^k} and $\pi^k_{B_r} := \pi_{B_r}|_{\Upsilon^k(B_r)}$. We introduce the reduced boundary in $\Upsilon(B_r)$.

Definition 6.2 (Reduced boundary in $\Upsilon(B_r)$)**.** Fix $r > 0$. Given $E \subset \Upsilon(B_r)$, set $E^k := E \cap \Upsilon^k(B_r)$ and define

$$\partial^*_{\Upsilon(B_r)} E := \bigsqcup_{k\ge 0} \partial^*_{\Upsilon^k(B_r)} E^k ,$$

$$\partial^*_{\Upsilon^k(B_r)} E^k := \left\{ \gamma \in \Upsilon^k(B_r) \,:\, \limsup_{s\to 0} \frac{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma) \cap E^k)}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma))} > 0, \quad \limsup_{s\to 0} \frac{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma) \setminus E^k)}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma))} > 0 \right\} ,$$

where $\mathsf{B}^k_s(\gamma)$ denotes the metric ball of radius $s > 0$ centered at $\gamma \in \Upsilon^k(B_r)$ w.r.t. d_{Υ^k}.

We can readily show that the m-codimensional Hausdorff measure $\rho^m_{\Upsilon^k(B_r)}$ w.r.t. d_{Υ^k} coincides with the push-forwarded measure of the m-codimensional spherical Hausdorff measure $\rho^m_{B_r^{\times k}}$ on $B_r^{\times k}$ w.r.t. the quotient map $\mathbf{s}_k$:

$$\rho^m_{\Upsilon^k(B_r)} = (\mathbf{s}_k)_\# \rho^m_{B_r^{\times k}} = (\mathbf{s}_k)_\# \mathbf{S}^{nk-m}_{B_r^{\times k}}, \tag{6.1}$$

where $\mathbf{S}^{nk-m}_{B_r^{\times k}}$ is the m-codimensional spherical Hausdorff measure on $B_r^{\times k}$ and $\mathbf{s}_k$ is the quotient map $B_r^{\times k} \to \Upsilon^k(B_r)$ as defined in Section 2. Having this in mind, we prove the following Gauß–Green formula in $\Upsilon(B_r)$.

Proposition 6.3 (Gauß–Green formula in $\Upsilon(B_r)$)**.** *Fix $r > 0$. If $E \subset \Upsilon(B_r)$ is a set of finite perimeter then there exists a vector field $\sigma_E : \Upsilon(B_r) \to T\Upsilon(B_r)$ such that $|\sigma_E|_{T\Upsilon(B_r)} = 1$ $\rho^1_{\Upsilon(B_r)}$-a.e. on $\partial^*_{\Upsilon(B_r)} E$, and*

$$\int_E (\nabla^* V) d\pi_{B_r} = \int_{\partial^*_{\Upsilon(B_r)} E} \langle V, \sigma_E \rangle d\rho^1_{\Upsilon(B_r)} \quad \textit{for any } V \in \mathrm{CylV}(\Upsilon(B_r)). \tag{6.2}$$

Moreover $\mathcal{V}_{\Upsilon(B_r)}(\chi_E) = \rho^1_{\Upsilon(B_r)}(\partial^*_{\Upsilon(B_r)} E)$.

Proof. Exploiting the decomposition $\Upsilon(B_r) = \bigsqcup_{k\ge 0} \Upsilon^k(B_r)$, where each $\Upsilon^k(B_r)$ is a connected component, we reduce our analysis to the study of $E^k := E \cap \Upsilon^k(B_r)$.

Set $\mathbf{E}^k := \mathbf{s}_k^{-1}(E^k)$. Given

$$V = \sum_{k=1}^m \Phi(f^*_{1,k}, \dots, f^*_{n_k,k}) v_k \in \mathrm{CylV}(\Upsilon(B_r)) ,$$

we can define $\mathbf{V} \in C_0^\infty(B_r^{\times k}; \mathbb{R}^{nk})$ as

$$\mathbf{V}(x_1, \dots, x_k) = \sum_{k=1}^{m} \Phi(f_{1,k}(x_1) + \dots + f_{n_k,k}(x_k), \dots, f_{n_k,k}(x_1) + \dots + f_{n_k,k}(x_k)) v_k(x_1, \dots, x_k)\,.$$

Notice that $|\mathbf{V}|_{\mathbb{R}^{nk}} \leq 1$ whenever $|V|_{T\Upsilon} \leq 1$. It is now immediate to see that $\mathbf{E}^k$ is of finite perimeter on $B_r^{\times k}$. Thus, standard results og geometric measure theory on the Euclidean space $\mathbb{R}^{nk}$ (see e.g., [44, Thm. 5.8.2]), we obtain

$$\int_{\mathbf{E}^k} (\nabla^* \mathbf{V}) d\mathbf{S}^{nk}_{B_r^{\times k}} = \int_{\partial^*_{B_r^{\times k}} \mathbf{E}^k} \langle \mathbf{V}, \sigma_{\mathbf{E}^k} \rangle d\rho^1_{B_r^{\times k}}. \quad \text{for any } \mathbf{V} \in C_0^\infty(B_r^{\times k}; \mathbb{R}^{nk})\,. \tag{6.3}$$

Here $\sigma_{\mathbf{E}^k}$ is a vector field $\sigma_{\mathbf{E}^k} : B_r^{\times k} \to \mathbb{R}^{nk}$ such that $|\sigma_{\mathbf{E}^k}|_{\mathbb{R}^{nk}} = 1$ $\rho^1_{B_r^{\times k}}$-a.e. on $\partial^*_{B_r^{\times k}} \mathbf{E}^k$. By passing to the quotient by means of the map $\mathbf{s}_k$ in both sides of (6.3) and using (6.1), we get the sought conclusion. □

Remark 6.4. An alternative proof of Proposition 6.3 can be given by employing the theory of RCD spaces (see [8] and references therein). Indeed $(\Upsilon^k(B_r), \mathsf{d}^k, \pi^k_{B_r})$ is an RCD$(0, kn)$ and E^k is of finite perimeter. Hence we can apply [18, Theorem 2.2] to get the integration by parts formula, written in terms of the total variation measure $|D\chi_{E^k}|$. From [9, Corollary 4.7] we deduce the identity $|D\chi_{E^k}| = \rho^1_{\Upsilon(B_r)}|_{\partial^*_{\Upsilon^k(B_r)} E^k}$.

Let us now put a measurability statement. The proof follows by arguing exactly in the same way as in the proof of Proposition 3.6, thus, we omit the proof.

Lemma 6.5. *Fix $r > 0$. If $F : \Upsilon(\mathbb{R}^n) \to \mathbb{R}$ is a Borel function, then*

$$\Upsilon(B_r^c) \ni \eta \to \int_{\Upsilon(B_r)} F_{\eta,r} d\rho^1_{\Upsilon(B_r)} \quad \textit{is } \pi_{B_r^c}\textit{-measurable}\,.$$

6.2. **Sets of finite perimeter on $\Upsilon(\mathbb{R}^n)$.** We now study sets of finite perimeter on the configuration space $\Upsilon(\mathbb{R}^n)$ by employing the already developed theory for the space $\Upsilon(B_r)$. The main idea is to reduce a set $E \subset \Upsilon(\mathbb{R}^n)$ to its sections $E_{\eta,r} \subset \Upsilon(B_r)$ and apply the results for sets of finite perimeter in $\Upsilon(B_r)$, combined with the disintegration argument. We finally let $r \to \infty$ to recover the information on the perimeter of the original set E.

Let us begin by introducing the definition of the reduced boundary in $\Upsilon(\mathbb{R}^n)$.

Definition 6.6 (Reduced boundary in $\Upsilon(\mathbb{R}^n)$)**.** *Let $E \subset \Upsilon(\mathbb{R}^n)$ be a Borel set. For any $r > 0$ we set*

$$\partial^*_r E := \{\gamma \in \Upsilon(\mathbb{R}^n) \,:\, \gamma|_{B_r} \in \partial^*_{\Upsilon(B_r)} E_{\gamma|_{B_r^c}, r}\}\,. \tag{6.4}$$

The reduced boundary of E is defined as follows

$$\partial^* E := \liminf_{i \to \infty,\, i \in \mathbb{N}} \partial^*_i E = \bigcup_{i>0} \bigcap_{j>i,\, j \in \mathbb{N}} \partial^*_j E\,. \tag{6.5}$$

Remark 6.7. We defined $\partial^* E$ by taking the liminf along the sequence $\{\partial^*_i E\}_{i \in \mathbb{N}}$. This choice is completely arbitrary and, as we will see in the sequel (cf. Theorem 6.15), if we change the defining sequence, then the reduced boundary can change, but only up to an $\|E\|$-negligible set, where $\|E\|$ is the perimeter measure that will be defined later. Thus, the reduced boundary is well-defined up to $\|E\|$-negligible sets.

Notice that, for any $\eta \in \Upsilon(B_r^c)$ it holds

$$(\partial^*_r E)_{\eta,r} = \partial^*_{\Upsilon(B_r)} E_{\eta,r}\,. \tag{6.6}$$

Lemma 6.8. *If E is a Borel subset of $\Upsilon(\mathbb{R}^n)$, then $\partial^*_r E$ and $\partial^* E$ are Borel.*

Proof. Since $\partial^* E = \liminf_{r\to\infty} \partial_r^* E$, it suffices to show the Borel measurability of $\partial_r^* E$ for any $r > 0$.

Step 1: We prove the following statement: for any $k \in \mathbb{N}$ and $s > 0$ the function

$$\{\gamma \in \Upsilon(\mathbb{R}^n) : \gamma(B_r) = k\} \ni \gamma \mapsto \frac{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}))} \tag{6.7}$$

is Borel.

Since the Borel measurability of the map $\gamma \mapsto \pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}))$ is easy, we only give a proof of the Borel measurability of the map $\gamma \mapsto \pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})$.

Let us identify $\{\gamma \in \Upsilon(\mathbb{R}^n) : \gamma(B_r) = k\} \simeq \Upsilon^k(B_r) \times \Upsilon(B_r^c)$. It allows us to introduce the product topology τ_p on $\{\gamma \in \Upsilon(\mathbb{R}^n) : \gamma(B_r) = k\}$, that is coarser than the vague topology τ_v as a consequence of the following observation: since B_r^c is open, the vague topology τ_v on $\Upsilon(B_r^c)$ coincides with the relative topology induced by $\Upsilon(\mathbb{R}^n)$. Thus, it suffices to see that the vague topology on $\Upsilon(B_r)$ is coarser than the relative topology induced by $\Upsilon(\mathbb{R}^n)$. For this purpose, we only need to show that, for any $\phi \in C_c(B_r)$ (note that ϕ does not necessarily vanish at the boundary of B_r), there exists an extension $\widetilde{\phi} \in C_c(\mathbb{R}^n)$ so that $\widetilde{\phi} = \phi$ on B_r. Given $\phi \in C_c(B_r)$, we take $\Phi \in C(\mathbb{R}^n)$ which is the extension of ϕ to $\mathbb{R}^n$ given by the Tietz extension theorem. Let us now pick $\kappa \in C_c(\mathbb{R}^n)$ such that $\kappa = 1$ on B_r and $\kappa = 0$ on B_{2r}^c. Then, it holds $\widetilde{\phi} := \kappa\Phi \in C_c(\mathbb{R}^n)$ and $\widetilde{\phi} = \phi$ in B_r, which concludes the sought statement.

By the inclusion $\tau_p \subset \tau_v$ of the topologies, we have the inclusion of the corresponding Borel σ-algebras $\mathscr{B}(\tau_p) \subset \mathscr{B}(\tau_v)$. Since the map

$$\Upsilon^k(B_r) \times \Upsilon^k(B_r) \times \Upsilon(B_r^c) \ni (\gamma_1, \gamma_2, \eta) \to \chi_E(\gamma_1 + \eta)\chi_{\mathsf{B}^k_s(\gamma_2)}(\gamma_1)\,, \tag{6.8}$$

is $\mathscr{B}(\tau_p)$-measurable, it is also $\mathscr{B}(\tau_v)$-measurable. Hence, Fubini's theorem gives that

$$\Upsilon^k(B_r)\times\Upsilon(B_r^c) \ni (\gamma_2, \eta) \to \int_{\Upsilon^k(B_r)} \chi_E(\gamma_1+\eta)\chi_{\mathsf{B}^k_s(\gamma_2)}(\gamma_1)d\pi^k_{\mathsf{B}_r}(\gamma_1) = \pi^k_{B_r}(\mathsf{B}^k_s(\gamma_2|_{B_r})\cap E^k_{\eta,r})\,, \tag{6.9}$$

is $\mathscr{B}(\tau_v)$-measurable as well.

Step 2: Fix $k \in \mathbb{N}$ and set

$$A_1^{k,r} := \left\{\gamma \in \Upsilon(\mathbb{R}^n) : \limsup_{s\to 0} \frac{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}_s(\gamma|_{B_r}))} > 0\right\}\,,$$

$$A_2^{k,r} := \left\{\gamma \in \Upsilon(\mathbb{R}^n) : \limsup_{j\to \infty} \frac{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}))} > 0\right\}\,.$$

Then $A_1^{k,r} = A_2^{k,r}$.

Observe that $A_2^{k,r} \subset A_1^{k,r}$. The converse inequality follows from the following observation. If $2^{-j} \le s \le 2^{-j+1}$ then

$$\begin{aligned}
\frac{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}))} &\ge \frac{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}))}\,\frac{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}))}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}))} \\
&\ge C(k,n)\frac{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}))}\,,
\end{aligned}$$

where we used the estimate $C(n,k)^{-1}e^{-\mathbf{L}^n(B_r)}s^{nk} \le \pi^k_{B_r}(\mathsf{B}^k_s(\gamma)) \le C(n,k)e^{-\mathbf{L}^n(B_r)}s^{nk}$ for any $s < r/5$, $\gamma \in \Upsilon(B_r)$ and some constant $C(n,k) \ge 1$ depending only on n and k. Indeed, the latter estimate can be obtained by the following observation: letting $\gamma = \{x_1, \dots, x_k\}$, we have

$$B_r^{\times k} \cap \mathbf{s}_k^{-1}(\mathsf{B}^k_s(\gamma)) = B_r^{\times k} \cap \bigcup_{\sigma_k \in \mathfrak{S}_k} B_s(\mathbf{x}_{\sigma_k})\,,$$

hence
$$\pi^k_{B_r}(\mathsf{B}^k_s(\gamma)) = \frac{e^{-\mathbf{L}^n(B_r)}}{k!}\mathbf{L}^{kn}(B_r^{\times k}\cap \mathbf{s}_k^{-1}(\mathsf{B}^k_s(\gamma))) \le e^{-\mathbf{L}^n(B_r)}C(n,k)s^{nk}\,,$$
recall that $\mathbf{L}^n$ denotes the n-dimensional Lebesgue measure. The opposite inequality follows from
$$\begin{aligned}\pi^k_{B_r}(\mathsf{B}^k_s(\gamma)) &= \frac{e^{-\mathbf{L}^n(B_r)}}{k!}\mathbf{L}^{kn}(B_r^{\times k}\cap \mathbf{s}_k^{-1}(\mathsf{B}^k_s(\gamma)))\\ &\ge \frac{e^{-\mathbf{L}^n(B_r)}}{k!}\mathbf{L}^{kn}(B_r^{\times k}\cap B_s(\mathbf{x}_{\sigma_k})) \ge e^{-\mathbf{L}^n(B_r)}C(n,k)s^{nk}\,.\end{aligned}$$

Step 3: Let us conclude the proof. Thanks to Step 1 and Step 2 we know that $A_1^{k,r}$ is Borel for any $k\in\mathbb{N}$ and $r>0$. The same arguments as in Step 1 and Step 2 apply to the Borel measurability for the following set:

$$\left\{\gamma\in\Upsilon(\mathbb{R}^n)\,:\,\limsup_{s\to 0}\frac{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r})\setminus E^k_{r,\gamma|_{B_r^c}})}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}))}>0\right\}\,, \tag{6.10}$$

hence $\partial_r^* E$ is a Borel set. □

6.3. **Perimeter measures.** In this subsection, based on the variational approach, we introduce the perimeter measure $\|E\|$ for a set $E\subset\Upsilon(\mathbb{R}^n)$ satisfying $\mathcal{V}(\chi_E)<\infty$. In order to construct $\|E\|$, we first introduce a localised perimeter measure $\|E\|_r$ on $\Upsilon(\mathbb{R}^n)$, and show the monotonicity of $\|E\|_r$ as $r\to\infty$.

Definition 6.9. For any Borel set $E\subset\Upsilon(\mathbb{R}^n)$ with $\mathcal{V}_r(\chi_E)<\infty$, we define
$$\|E\|_r := \rho^1_{\Upsilon(B_r)}|_{(\partial^*_r E)_{\eta,r}}\otimes\pi_{B_r^c}(\eta)\quad\text{on }\Upsilon(\mathbb{R}^n)\,, \tag{6.11}$$
which is equivalently defines as follows: for any bounded Borel measurable function F on $\Upsilon(\mathbb{R}^n)$,
$$\int_{\Upsilon(\mathbb{R}^n)} F d\|E\|_r := \int_{\Upsilon(B_r^c)}\left(\int_{\Upsilon(B_r)} F_{\eta,r}d\rho^1_{\Upsilon(B_r)}|_{\partial^*_{\Upsilon(B_r)}E_{\eta,r}}\right)d\pi_{B_r^c}(\eta). \tag{6.12}$$

Lemma 6.10. *Let $r>0$. For any Borel set $E\subset\Upsilon(\mathbb{R}^n)$ with $\mathcal{V}(\chi_E)<\infty$, $\|E\|_r$ is a well-defined finite Borel measure.*

Proof. Let us first show that $\|E\|_r$ is well-defined. The map $\gamma\mapsto F_{\eta,r}(\gamma)$ is $\rho^1_{\Upsilon(B_r)}|_{\partial^* E_{\eta,r}}$-measurable by Lemma 3.1. On account of the definition (6.12), we only need to show that the map
$$\Upsilon(B_r^c)\ni\eta\to\int_{\Upsilon(B_r)} F_{\eta,r}d\rho^1_{\Upsilon(B_r)}|_{\partial^*_{\Upsilon(B_r)}E_{\eta,r}}\,, \tag{6.13}$$
is $\pi_{B_r^c}$-measurable for any Borel function $F:\Upsilon(\mathbb{R}^n)\to\mathbb{R}$. To show it, we use (6.6) and rewrite
$$\int_{\partial^*_{\Upsilon(B_r)}E_{\eta,r}} F_{\eta,r}\,d\rho^1_{\Upsilon(B_r)} = \int_{(\partial^*_r E)_{\eta,r}} F_{\eta,r}\,d\rho^1_{\Upsilon(B_r)} = \int_{\Upsilon(B_r)}(\chi_{\partial^*_r E}F)_{\eta,r}\,d\rho^1_{\Upsilon(B_r)}\,.$$
Now, the claimed conclusion follows from Lemma 6.5 by observing that $\chi_{\partial^*_r E}F$ is a Borel function.

The finiteness of the measure $\|E\|_r$ is immediate by Proposition 6.3 and Proposition 5.5, indeed
$$\|E\|_r(\Upsilon(\mathbb{R}^n)) = \int_{\Upsilon(B_r^c)}\mathcal{V}_{\Upsilon(B_r)}((\chi_E)_{\eta,r})d\pi_{B_r^c}(\eta) = \mathcal{V}_r(\chi_E)\le\mathcal{V}(\chi_E)<\infty.$$
□

Lemma 6.11. *Let $r>0$. For any Borel set $E\subset\Upsilon(\mathbb{R}^n)$ with $\mathcal{V}_r(\chi_E)<\infty$, there exists a vector field $\sigma_{E,r}:\Upsilon(\mathbb{R}^n)\to T\Upsilon(\mathbb{R}^n)$ such that*

(i) *$\sigma_{E,r}(\gamma)\in T_\gamma\Upsilon(\mathbb{R}^n)$ satisfies $\sigma_{E,r}(\gamma,x)=0$ for $x\in B_r^c$;*

(ii) $|\sigma_{E,r}|_{T\Upsilon} = 1$, $\|E\|_r$-*a.e.;*

(iii) *for any* $V \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n))$,

$$\int_E (\nabla^* V) d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r \,. \tag{6.14}$$

(iv) $\mathcal{V}_r(\chi_E) = \|E\|_r(\Upsilon(\mathbb{R}^n))$, *and for any nonnegative function* $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ *it holds*

$$\int_{\Upsilon(\mathbb{R}^n)} F d\|E\|_r = \sup\left\{\int_E (\nabla^* FV) d\pi \,:\, V \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n)),\, |V|_{T\Upsilon} \le 1\right\}. \tag{6.15}$$

Proof. By Proposition 5.5, there exists a measurable set $\Omega_r \subset \Upsilon(B_r^c)$ so that $\pi_{B_r^c}(\Omega_r) = 1$ and $\mathcal{V}_{\Upsilon(B_r)}(\chi_{E_{\eta,r}}) < \infty$ for every $\eta \in \Omega_r$. By Proposition 6.3, for every $\eta \in \Omega_r$, there exists a unique $T\Upsilon(B_r)$-valued Borel measurable map $\sigma_{\eta,r}$ on $\Upsilon(B_r)$ so that $|\sigma_{\eta,r}|_{T\Upsilon(B_r)} = 1$ $\rho^1_{\Upsilon(B_r)}|_{\partial^* E_{\eta,r}}$-a.e., and

$$\int_{E_{\eta,r}} (\nabla^* V_{\eta,r}) d\pi_{B_r} = \int_{\partial^*_{\Upsilon(B_r)} E_{\eta,r}} \langle V_{\eta,r}, \sigma_{\eta,r}\rangle_{T\Upsilon(B_r)} d\rho^1_{\Upsilon(B_r)}, \quad \forall V \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n))\,. \tag{6.16}$$

Note that, in the above, we used $V_{\eta,r} \in \mathrm{CylV}_0(\Upsilon(B_r))$ whenever $V \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n))$. By taking the integral with respect to $\pi_{B_r^c}$, and arguing as in Proposition 4.9 we obtain

$$\begin{aligned} \int_E (\nabla^* V) d\pi &= \int_{\Upsilon(B_r^c)} \int_{E_{\eta,r}} (\nabla_r^* V_{\eta,r}) d\pi_{B_r} d\pi_{B_r^c}(\eta) \\ &= \int_{\Upsilon(B_r^c)} \int_{\partial^*_{\Upsilon(B_r)} E_{\eta,r}} \langle V_{\eta,r}, \sigma_{\eta,r}\rangle_{T\Upsilon(B_r)} d\rho^1_{\Upsilon(B_r)} d\pi_{B_r^c}(\eta)\,. \end{aligned} \tag{6.17}$$

Note that the map $\eta \mapsto \int_{\partial^*_{\Upsilon(B_r)} E_{\eta,r}} \langle V_{\eta,r}, \sigma_{\eta,r}\rangle_{T\Upsilon(B_r)} d\rho^1_{\Upsilon(B_r)}$ is $\pi_{B_r^c}$-measurable since, in view of (6.16), it is equal to a $\pi_{B_r^c}$-measurable function, and therefore, the argument (6.17) is justified. For $\gamma \in \Upsilon(\mathbb{R}^n)$ we define

$$\sigma_{E,r}(\gamma) := \begin{cases} \sigma_{\gamma|_{B_r^c}, r}(\gamma|_{B_r}) & \text{if } \gamma|_{B_r^c} \in \Omega_r, \\ \sigma_r(\gamma) = 0 & \text{otherwise}\,. \end{cases} \tag{6.18}$$

Let us now observe that, for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$, we have

$$(\langle V, \sigma_{E,r}\rangle_{T\Upsilon(\mathbb{R}^n)})_{\eta,r} = \langle V_{\eta,r}, \sigma_{\eta,r}\rangle_{T\Upsilon(B_r)}\,. \tag{6.19}$$

By combining the definition (6.11) of $\|E\|_r$ with (6.17), (6.18) and (6.19), we deduce the assertion (iii).

The assertion (i) follows from the definition (6.18), and the assertion (ii) follows from

$$(|\sigma_{E,r}|_{T\Upsilon(\mathbb{R}^n)})_{\eta,r} = |\sigma_{\eta,r}| = 1, \quad \rho^1_{\Upsilon(B_r)}|_{\partial^* E_{\eta,r}}\text{-a.e.}.$$

We now prove (iv). We first prove the equality $\mathcal{V}_r(\chi_E) = \|E\|_r(\Upsilon(\mathbb{R}^n))$. From (iii) and (ii) we deduce

$$\begin{aligned} \mathcal{V}_r(\chi_E) &= \sup\left\{\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) f d\pi : V \in \mathrm{CylV}^r(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon(\mathbb{R}^n)} \le 1\right\} \\ &= \sup\left\{\int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r : V \in \mathrm{CylV}^r(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon(\mathbb{R}^n)} \le 1\right\} \\ &\le \|E\|_r(\Upsilon(\mathbb{R}^n))\,. \end{aligned}$$

Furthermore, Proposition 5.5 and Lemma 6.3 imply

$$\mathcal{V}_r(\chi_E) \ge \int_{\Upsilon(B_r^c)} \mathcal{V}_{\Upsilon(B_r)}((\chi_E)_{\eta,r}) d\pi_{B_r^c}(\eta) = \int_{\Upsilon(B_r^c)} \rho^1_{\Upsilon(B_r)}(\partial^*_{\Upsilon(B_r)} E_{\eta,r}) d\pi_{B_r^c}(\eta) = \|E\|_r(\Upsilon(\mathbb{R}^n))\,. \tag{6.20}$$

Thus, the proof of the equality $\mathcal{V}_r(\chi_E) = \|E\|_r(\Upsilon(\mathbb{R}^n))$ is complete.

Let us finally address (6.15). From the equality $\mathcal{V}_r(\chi_E) = \|E\|_r(\Upsilon(\mathbb{R}^n))$, we deduce the existence of a sequence $V_k \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n))$ such that $|V_k|_{T\Upsilon} \le 1$, and

$$\lim_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} \langle V_k, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r = \int_{\Upsilon(\mathbb{R}^n)} d\|E\|_r\,,$$

hence,

$$\begin{aligned}\lim_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} |V_k - \sigma_{E,r}|^2_{T\Upsilon} d\|E\|_r &= \lim_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} (|V_k|^2_{T\Upsilon} + |\sigma_{E,r}|^2_{T\Upsilon} - 2\langle V_k, \sigma_{E,r}\rangle_{T\Upsilon}) d\|E\|_r \\ &\le \lim_{k\to\infty} 2 \int_{\Upsilon(\mathbb{R}^n)} (1 - \langle V_k, \sigma_{E,r}\rangle_{T\Upsilon}) d\|E\|_r = 0\,.\end{aligned}$$

Therefore, for any $F \in \mathrm{CylF}(\Upsilon)$

$$\lim_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} F\langle V_k, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r = \int_{\Upsilon(\mathbb{R}^n)} F\, d\|E\|_r\,,$$

in particular, by making use of (6.14) with $V = FV_k$, it holds that

$$\int_{\Upsilon(\mathbb{R}^n)} F\, d\|E\|_r \le \sup\left\{ \int_E (\nabla^* FV) d\pi \,:\, V \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n)),\, |V|_{T\Upsilon} \le 1 \right\}. \tag{6.21}$$

The converse inequality follows form $|\sigma_{E,r}|_{T\Upsilon} = 1$ $\|E\|_r$-a.e. and the fact that F is nonnegative:

$$\int_E (\nabla^* FV) d\pi = \int_{\Upsilon(\mathbb{R}^n)} F\langle V_k, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r \le \int_{\Upsilon(\mathbb{R}^n)} |F| d\|E\|_r = \int_{\Upsilon(\mathbb{R}^n)} F d\|E\|_r\,. \tag{6.22}$$

The proof is complete. □

Corollary 6.12. *If $\mathcal{V}(\chi_E) < \infty$, then $r \mapsto \|E\|_r(A)$ is monotone non-decreasing for every Borel measurable set A.*

Proof. In view of the density of cylinder functions on $L^2(\Upsilon(\mathbb{R}^n), \pi)$ it is enough to check that

$$r \to \int_{\Upsilon(\mathbb{R}^n)} F d\|E\|_r \quad \text{is non-decreasing}\,,$$

for any nonnegative $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$. The latter conclusion easily follows from (6.15) and the inclusion $\mathrm{CylV}_0^s(\Upsilon(\mathbb{R}^n)) \subset \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n))$ for $s \le r$. □

By the monotonicity of $r \mapsto \|E\|_r$ in Corollary 6.12, we may define the limit measure as follows:

Definition 6.13 (Perimeter measure)**.** Given $E \subset \Upsilon(\mathbb{R}^n)$ with $\mathcal{V}(\chi_E) < \infty$, we define the perimeter measure as

$$\|E\|(A) := \lim_{r\to\infty} \|E\|_r(A) \quad \text{for any Borel set } A\,. \tag{6.23}$$

We finally obtain the Gauß–Green formula for the perimeter measure $\|E\|$.

Theorem 6.14 (Gauß–Green formula for $\|E\|$)**.** *For any Borel set $E \subset \Upsilon(\mathbb{R}^n)$ with $\mathcal{V}(\chi_E) < \infty$, there exists a unique element $\sigma_E \in L^2(T\Upsilon, \|E\|)$ such that $|\sigma_E|_{T\Upsilon} = 1$ $\|E\|$-a.e. and*

$$\int_E \nabla^* V d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_E\rangle_{T\Upsilon} d\|E\| \quad \forall V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n)). \tag{6.24}$$

Proof. We assume without loss of generality that $\|E\|(\Upsilon(\mathbb{R}^n)) = 1$. Note that, for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$, there exists $r > 0$ so that $V \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n))$. Thus, by (iii) in Lemma 6.11, for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$, there exists $r > 0$ and $\sigma_{E,r} : \Upsilon(\mathbb{R}^n) \to T\Upsilon$ so that $|\sigma_{E,r}| = 1$ $\|E\|$-a.e., and

$$\begin{aligned}\int_E \nabla^* V d\pi &= \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r \\ &\le \|V\|_{L^2(T\Upsilon,\|E\|_r)} \\ &\le \|V\|_{L^2(T\Upsilon,\|E\|)} . \end{aligned} \tag{6.25}$$

The last inequality followed from the monotonicity in Corollary 6.12. In particular, denoting by $H \subset L^2(T\Upsilon, \|E\|)$ the closure of $\mathrm{CylV}(\Upsilon)$ in $L^2(T\Upsilon, \|E\|)$, the linear operator L defined below

$$L : H \to \mathbb{R}\,, \quad H \ni V \mapsto L(V) := \int_E \nabla^* V d\pi\,, \tag{6.26}$$

is a well-defined continuous operator on the Hilbert space H and satisfies $\|L\| \le 1$. Therefore, the Riesz representation theorem in the Hilbert space H gives the existence of $\sigma_E \in H$ so that

$$\|\sigma_E\|_{L^2(T\Upsilon,\|E\|)} \le 1\,, \qquad \int_E \nabla^* V d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma\rangle d\|E\| \quad \forall V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))\,. \tag{6.27}$$

It suffices to show that $|\sigma|_{T\Upsilon} = 1$ $\|E\|$-a.e. By (iv) in Lemma 6.11 and Corollary 6.12, we deduce that

$$\begin{aligned} 1 = \|E\|(\Upsilon(\mathbb{R}^n)) &= \lim_{r\to\infty} \|E\|_r(\Upsilon(\mathbb{R}^n)) = \lim_{r\to\infty} \mathcal{V}_r(\chi_E) \\ &= \lim_{r\to\infty} \sup_{V \in \mathrm{CylV}_r^r(\Upsilon(\mathbb{R}^n)),\, |V|_{T\Upsilon}\le 1} \int_E \nabla^* V d\pi \le \int_{\Upsilon(\mathbb{R}^n)} |\sigma|_{T\Upsilon} d\|E\| \le \|\sigma\|_{L^2(T\Upsilon,\|E\|)} \le 1\,, \end{aligned}$$

which yields $\|\sigma\|_{L^1(T\Upsilon,\|E\|)} = \|\sigma\|_{L^2(T\Upsilon,\|E\|)} = 1$. Therefore $|\sigma|_{T\Upsilon} = 1$ $\|E\|$-a.e. as a consequence of the characterisation of the equality in Jensen's inequality. The proof is complete. □

6.4. **Perimeters and one-codimensional Poisson measures.** In this subsection, we prove one of the main results in this paper. Namely, the perimeter measure $\|E\|$ based on the variational approach (Definition 6.13) coincides with the 1-codimensinal Poisson measure ρ^1 (Definition 3.8) restricted on the reduced boundary $\partial^* E$ of E (Definition 6.6).

Theorem 6.15. *Let $E \subset \Upsilon(\mathbb{R}^n)$ be a set with $\mathcal{V}(\chi_E) < \infty$. Then,*

$$\|E\| = \rho^1|_{\partial^* E}.$$

Before giving the proof, we prove a lemma.

Lemma 6.16. *Let $E \subset \Upsilon(\mathbb{R}^n)$ be a set with $\mathcal{V}(\chi_E) < \infty$. Then, for any $r > 0$, $\varepsilon > 0$, it holds*

$$(\partial_r^* E)_{\eta,r} \subset (\partial_{r+\varepsilon}^* E)_{\eta,r} \quad \text{up to } \rho^1_{\Upsilon(B_r)}\text{-negligible sets for } \pi_{B_r^c}\text{-a.e. } \eta. \tag{6.28}$$

Namely, there exists a measurable set $\Omega_{r,\varepsilon} \subset \Upsilon(\mathbb{R}^n)$ so that $\pi_{B_r^c}(\Omega_{r,\varepsilon}) = 1$ and for any $\eta \in \Omega_{r,\varepsilon}$, it holds that

$$\rho^1_{\Upsilon(B_r)}\Big((\partial_r^* E)_{\eta,r} \setminus \partial_{r+\varepsilon}^* E)_{\eta,r}\Big) = 0\,. \tag{6.29}$$

Proof. By (6.6) and the definition (6.11) of the perimeter measure $\|E\|_r$, we see that

$$\begin{aligned} \infty > \|E\|(A) \ge \|E\|_{r+\varepsilon}(A) &= \int_{\Upsilon(B^c_{r+\varepsilon})} \rho^1_{\Upsilon(B_{r+\varepsilon})}(\partial^*_{\Upsilon(B_{r+\varepsilon})} E_{\eta,r+\varepsilon} \cap A_{\eta,r+\varepsilon}) d\pi_{B^c_{r+\varepsilon}}(\eta) \\ &= \int_{\Upsilon(B^c_{r+\varepsilon})} \rho^1_{\Upsilon(B_{r+\varepsilon})}\big((\partial^*_{r+\varepsilon} E)_{\eta,r+\varepsilon} \cap A_{\eta,r+\varepsilon}\big) d\pi_{B^c_{r+\varepsilon}}(\eta) \\ &= \int_{\Upsilon(B^c_{r+\varepsilon})} \rho^1_{\Upsilon(B_{r+\varepsilon})}\big((\partial^*_{r+\varepsilon} E \cap A)_{\eta,r+\varepsilon}\big) d\pi_{B^c_{r+\varepsilon}}(\eta)\,. \end{aligned} \tag{6.30}$$

By the monotonicity $\|E\|_{r+\varepsilon}(A) \geq \|E\|_r(A)$ in Corollary 6.12, we obtain that

$$\int_{\Upsilon(B^c_{r+\varepsilon})} \rho^1_{\Upsilon(B_{r+\varepsilon})}\big((\partial^*_{r+\varepsilon}E\cap A)_{\eta,r+\varepsilon}\big)d\pi_{B^c_{r+\varepsilon}}(\eta) \geq \int_{\Upsilon(B^c_r)} \rho^1_{\Upsilon(B_r)}\big((\partial^*_r E\cap A)_{\eta,r}\big)d\pi_{B^c_r}(\eta).$$

Taking $A = \Upsilon(\mathbb{R}^n)\setminus \partial^*_{r+\varepsilon}E$, we have that

$$\begin{aligned} 0 &= \int_{\Upsilon(B^c_{r+\varepsilon})} \rho^1_{\Upsilon(B_{r+\varepsilon})}\big((\partial^*_{r+\varepsilon}E\cap A)_{\eta,r+\varepsilon}\big)d\pi_{B^c_{r+\varepsilon}}(\eta) \\ &\geq \int_{\Upsilon(B^c_r)} \rho^1_{\Upsilon(B_r)}\big((\partial^*_r E\cap A)_{\eta,r}\big)d\pi_{B^c_r}(\eta)\,. \end{aligned}$$

Thus, $\rho^1_{\Upsilon(B_r)}\big((\partial^*_r E\cap A)_{\eta,r}\big) = 0$ for $\pi_{B^c_r}$-a.e. η, which implies that

$$\big(\partial^*_r E \cap (\Upsilon(\mathbb{R}^n)\setminus \partial^*_{r+\varepsilon}E)\big)_{\eta,r} = (\partial^*_r E)_{\eta,r} \setminus \big((\partial^*_{r+\varepsilon}E)_{\eta,r}\cap(\partial^*_r E)_{\eta,r}\big)$$

is $\rho^1_{\Upsilon(B_r)}$-negligible for $\pi_{B^c_r}$-a.e. η. The proof is complete. □

Proof of Theorem 6.15. Fix $r>0$ and $\eta\in\Upsilon(B^c_r)$. It holds

$$(\partial^* E)_{\eta,r} := \left(\bigcup_{i>0}\bigcap_{j>i}\partial^*_j E\right)_{\eta,r} = \bigcup_{i>0}\bigcap_{j>i}(\partial^*_j E)_{\eta,r}\,. \tag{6.31}$$

The monotonicity formula (6.29) in Lemma 6.16 gives the existence of $\Omega_{r,j}\subset\Upsilon(\mathbb{R}^n)$ so that $\pi_{B^c_r}(\Omega_{r,j}) = 1$, and for any $\eta\in\Omega_{r,j}$

$$(\partial^*_r E)_{\eta,r}\subset(\partial^*_j E)_{\eta,r}\quad j\geq r\quad\text{up to a } \rho^1_{\Upsilon(B_r)}\text{-negligible set}\,.$$

Take $\Omega_r = \cap_{j\geq r, j\in\mathbb{N}}\Omega_{r,j}$. Then $\pi_{B^c_r}(\Omega_r)=1$, and by using (6.6), we obtain that for any $\eta\in\Omega_r$,

$$\partial^*_{\Upsilon(B_r)}E_{\eta,r} = (\partial^*_r E)_{\eta,r}\subset(\partial^* E)_{\eta,r}\quad\text{up to a } \rho^1_{\Upsilon(B_r)}\text{-negligible set}\,.$$

This implies that for any Borel set $A\subset\Upsilon(\mathbb{R}^n)$,

$$\rho^1_{\Upsilon(B_r)}(\partial^*_{\Upsilon(B_r)}E_{\eta,r}\cap A_{\eta,r}) \leq \rho^1_{\Upsilon(B_r)}((\partial^* E\cap A)_{\eta,r}),\quad\forall\eta\in\Omega_r.$$

Thus, by noting that $\pi_{B^c_r}(\Omega_r)=1$ and recalling Definition 6.13, Definition 3.8, we obtain

$$\begin{aligned} \|E\|(A) &:= \lim_{r\to\infty}\|E\|_r(A) \\ &= \lim_{r\to\infty}\int_{\Upsilon(B^c_r)} \rho^1_{\Upsilon(B_r)}(\partial^*_{\Upsilon(B_r)}E_{\eta,r}\cap A_{\eta,r})d\pi_{B^c_r}(\eta) \\ &\leq \lim_{r\to\infty}\int_{\Upsilon(B^c_r)} \rho^1_{\Upsilon(B_r)}((\partial^* E\cap A)_{\eta,r})d\pi_{B^c_r}(\eta) \\ &= \rho^1(A\cap\partial^* E)\,. \end{aligned}$$

In order to conclude the proof, it is enough to check that

$$\|E\|(\Upsilon(\mathbb{R}^n)) \geq \rho^1(\partial^* E)\,. \tag{6.32}$$

Indeed, given any Borel set A, by making use of the already proven inequality $\|E\|\leq\rho^1|_{\partial^* E}$, we obtain

$$\|E\|(\Upsilon(\mathbb{R}^n)) = \|E\|(A)+\|E\|(A^c) \leq \rho^1(A\cap\partial^* E)+\rho^1(A^c\cap\partial^* E) = \rho^1(\partial^* E)\leq\|E\|(\Upsilon(\mathbb{R}^n))\,.$$

Thus, $\|E\|(A)+\|E\|(A^c) = \rho^1(A\cap\partial^* E)+\rho^1(A^c\cap\partial^* E)$ for any Borel set A. Assume that there exists a Borel set A so that $\|E\|(A) < \rho^1(A\cap\partial^* E)$. Since $\|E\|\leq\rho^1(\cdot\cap\partial^* E)$, it implies

$$\|E\|(A)+\|E\|(A^c) < \rho^1(A\cap\partial^* E)+\rho^1(A^c\cap\partial^* E),$$

which is a contradiction.

We now prove (6.32). Let $s < r$. By recalling Definitions 6.9, 3.5 of $\|E\|_r$ and ρ^1_r respectively and using the monotonicity of ρ^1_r in Theorem 3.7, we have

$$\|E\|_r(\Upsilon(\mathbb{R}^n)) = \int_{\Upsilon(B_r^c)} \rho^1_{\Upsilon(B_r)}((\partial^*_r E)_{\eta,r}) d\pi_{B_r^c}(\eta) = \rho^1_r(\partial^*_r E) \geq \rho^1_s(\partial^*_r E)\,,$$

hence

$$\|E\|(\Upsilon(\mathbb{R}^n)) = \lim_{i\to\infty} \|E\|_i(\Upsilon(R^n)) \geq \liminf_{i\to\infty} \rho^1_s(\partial^*_i E) \geq \rho^1_s(\liminf_{i\to\infty} \partial^*_i E) = \rho_s(\partial^* E)\,.$$

Passing to the limit $s \to \infty$, we conclude (6.32). The proof is complete. □

7. Total variation and Gauß–Green formula

In this section, we prove a relation between the coarea with respect to the perimeter measure $\|E\|$ and the variation $|\mathsf{D}_* F|$ obtained via relaxation of Cylinder functions. As an application, we introduce the total variation *measure* $|\mathsf{D}F|$ for BV functions F, and prove the Gauß–Green formula.

7.1. Total variation measures via coarea formula.

Recall that, for $F \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, the map $\mathrm{CylF}(\Upsilon(\mathbb{R}^n)) \ni G \mapsto |\mathsf{D}_* F|[G]$ is defined by the relaxation approach in Definition 5.7. The main result of this subsection is the following formula:

Theorem 7.1. *Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$. Then,*

$$\mathcal{V}(\chi_{\{F>t\}}) < \infty \quad a.e.\ t \in \mathbb{R}, \tag{7.1}$$

and the following formula holds:

$$\int_{-\infty}^{\infty} \left(\int_{\Upsilon(\mathbb{R}^n)} G d\big\|\{F > t\}\big\| \right) dt = |\mathsf{D}_* F|[G], \quad \text{for any nonnegative } G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))\,. \tag{7.2}$$

The proof of Theorem 7.1 will be given later in this section. Before discussing the proof, we study several consequences of Theorem 7.1. By (7.1), the left-hand side of (7.2) makes sense with $G \equiv 1$ since the right-hand side $|\mathsf{D}_* F|[1] < \infty$ is finite due to $F \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))$ and Theorem 5.17. This leads us to provide the following definition of the total variation measure.

Definition 7.2 (Total variation measure)**.** For $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, define *the total variation measure* $|\mathsf{D}F|$ as follows:

$$|\mathsf{D}F| := \int_{-\infty}^{\infty} \big\|\{F > t\}\big\| dt. \tag{7.3}$$

We now investigate relations between the total variation measure $|\mathsf{D}\chi_E|$ and the perimeter measure $\|E\|$ defined in Definition 6.13 and the $(1,2)$-capacity $\mathrm{Cap}_{1,2}$ defined in Definition 4.3.

Corollary 7.3 (Total variation and perimeters)**.** *Let $E \subset \Upsilon(\mathbb{R}^n)$ satisfy $|\mathsf{D}\chi_E|(\Upsilon(\mathbb{R}^n)) < \infty$. Then,*

$$|\mathsf{D}\chi_E| = \|E\| \quad \textit{as measures}\,.$$

.

Proof. By Theorem 5.17, $\mathcal{V}(\chi_E) < \infty$ and $\|E\|$ is well-defined. Noting that

$$\{\chi_E > t\} = \begin{cases} \Upsilon(\mathbb{R}^n) & t \leq 0; \\ E & 0 < t \leq 1; \\ \emptyset & t > 1, \end{cases}$$

and $\|\Upsilon(\mathbb{R}^n)\| = 0$ and $\|\emptyset\| = 0$, we obtain that

$$|\mathsf{D}\chi_E|(A) = \int_{-\infty}^{\infty} \big\|\{\chi_E > t\}\big\|(A)dt = 0 + \|E\|(A) + 0 = \|E\|(A) \quad \text{for any Borel set A}\,.$$

The proof is complete. □

Corollary 7.4 (Total variation and capacity)**.** *Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$. For any Borel set $A \subset \Upsilon(\mathbb{R}^n)$,*

$$\mathrm{Cap}_{1,2}(A) = 0 \implies |\mathsf{D}F|(A) = 0.$$

Proof. Let $\mathrm{Cap}_{1,2}(A) = 0$. By Theorem 7.1, and Theorem 6.15, we can write

$$|\mathsf{D}F|(A) = \int_{-\infty}^{\infty} \|\{F > t\}\|(A)dt = \int_{-\infty}^{\infty} \rho^1(\partial^*\{F > t\} \cap A)dt\,, \tag{7.4}$$

hence it suffices to show that $\rho^1(\partial^*\{F > t\} \cap A) = 0$. This follows from the absolute continuity of ρ^1 with respect to $\mathrm{Cap}_{1,2}$ obtained in Theorem 4.3. □

7.2. **Proof of Theorem 7.1.** This subsection is devoted to the proof of Theorem 7.1. Let us begin with two propositions.

Proposition 7.5. *Let $E \subset \Upsilon(\mathbb{R}^n)$ be a set with $\mathcal{V}(\chi_E) < \infty$. Then, for any nonnegative function $G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ it holds*

$$\int_{\Upsilon(\mathbb{R}^n)} Gd\|E\| = \sup\left\{\int_E (\nabla^* GV)d\pi \,:\, V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n)),\, |V|_{T\Upsilon} \le 1\right\}. \tag{7.5}$$

In particular, the following hold:

(i) *if $F_k \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$, and $F_k \to \chi_E$ in $L^1(\Upsilon(\mathbb{R}^n), \pi)$ as $k \to \infty$, then*

$$\liminf_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_k|_{T\Upsilon} d\pi \ge \int_{\Upsilon(\mathbb{R}^n)} Gd\|E\|\,, \quad \textit{for any nonnegative } G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))\,;$$

(ii) *if $\chi_{E_k} \to \chi_E$ in $L^1(\Upsilon(\mathbb{R}^n), \pi)$ as $k \to \infty$, where $(E_k)_k$ are sets of finite perimeter, then*

$$\liminf_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} Gd\|E_k\| \ge \int_{\Upsilon(\mathbb{R}^n)} Fd\|E\|\,, \quad \textit{for any nonnegative } G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))\,.$$

Proof. Fix $\varepsilon > 0$. We pick $r > 0$ such that $\int_{\Upsilon(\mathbb{R}^n)} Gd\|E\|_r \ge \int_{\Upsilon(\mathbb{R}^n)} Gd\|E\| - \varepsilon$. From (6.15) we deduce the existence of $V \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n))$ with $|V|_{T\Upsilon} \le 1$ such that $\int_E (\nabla^* GV)d\pi \ge \int_{\Upsilon(\mathbb{R}^n)} Gd\|E\|_r - \varepsilon$, yielding

$$\int_{\Upsilon(\mathbb{R}^n)} Gd\|E\| \le \int_E (\nabla^* GV)d\pi + 2\varepsilon \le \sup\left\{\int_E (\nabla^* GV)d\pi \,:\, V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n)),\, |V|_{T\Upsilon} \le 1\right\} + 2\varepsilon\,.$$

By taking $\varepsilon \to 0$, the one inequality is proved.

We now prove the converse inequality. Take a representative $G = \Phi(f_1^\star, \ldots, f_k^\star)$ and take $r > 0$ so that $\cup_{i=1}^k \mathrm{supp}[f_i] \subset B_r$. By the divergence formula (2.12), we can easily see

$$\begin{aligned}&\sup\left\{\int_E (\nabla^* GV)d\pi \,:\, V \in \mathrm{CylV}_0^r(\Upsilon(\mathbb{R}^n)),\, |V|_{T\Upsilon} \le 1\right\}\\ &= \sup\left\{\int_E (\nabla^* GV)d\pi \,:\, V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n)),\, |V|_{T\Upsilon} \le 1\right\}.\end{aligned}$$

By combining it with the formula (6.15) and the monotonicity of $r \mapsto \|E\|_r$ in Corollary 6.12, the converse inequality is proved.

Let us now prove (i) and (ii). Fix $\varepsilon > 0$. By Theorem 6.14, we can take $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ such that $|V|_{T\Upsilon} \le 1$ and

$$\int_E (\nabla^* GV) d\pi \ge \int_{\Upsilon(\mathbb{R}^n)} G d\|E\| - \varepsilon\,.$$

Let k_j be a subsequence such that $\lim_{j\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_{k_j}|_{T\Upsilon} d\pi = \liminf_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_k|_{T\Upsilon} d\pi$, it holds

$$\begin{aligned}\int_{\Upsilon(\mathbb{R}^n)} G d\|E\| - \varepsilon \le \int_E (\nabla^* GV) d\pi = \lim_{j\to\infty} \int_{\Upsilon(\mathbb{R}^n)} F_{k_j} (\nabla^* GV) d\pi = \lim_{j\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G\langle \nabla F_{k_j}, V\rangle_{T\Upsilon} d\pi \\ \le \liminf_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_k|_{T\Upsilon} d\pi\,.\end{aligned}$$

Furthermore, by using Theorem 6.14 with V being GV, we deduce that

$$\begin{aligned}\int_{\Upsilon(\mathbb{R}^n)} G d\|E\| - \varepsilon \le \int_E (\nabla^* GV) d\pi = \lim_{j\to\infty} \int_{E_{k_j}} (\nabla^* GV) d\pi = \lim_{j\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G\langle V, \sigma_{E_{k_j}}\rangle_{T\Upsilon} d\|E_{k_j}\| \\ \le \liminf_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G d\|E_{k_j}\|\,.\end{aligned}$$

The proof is complete. □

Proposition 7.6. *For any $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ it holds*

(7.6)
$$\int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} G\, d\big\|\{F > t\}\big\| dt = \int_{\Upsilon(\mathbb{R}^n)} G\, |\nabla F|_{T\Upsilon} d\pi\,, \quad \textit{for any nonnegative } G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))\,.$$

Proof. The map

$$\mathbb{R} \ni t \to m(t) := \int_{\{F>t\}} G|\nabla F|_{T\Upsilon} d\pi \tag{7.7}$$

is monotone and finite since $|\nabla F|_{T\Upsilon} \in L^1(\Upsilon(\mathbb{R}^n))$. Let $t \in \mathbb{R}$ be a point on which the map $t \mapsto m(t)$ is differentiable and set

$$g_\varepsilon(s) := \begin{cases} 1 & s \le t \\ \varepsilon^{-1}(t-s)+1 & t \le s \le t+\varepsilon \\ 0 & s > t+\varepsilon\,. \end{cases} \tag{7.8}$$

Notice that $g_\varepsilon \circ F \to \chi_{\{F>t\}}$ in $L^p(\Upsilon(\mathbb{R}^n))$ for any $p \in [1,\infty)$ as $\varepsilon \to 0$. Indeed,

$$\int_{\Upsilon(\mathbb{R}^n)} |g_\varepsilon \circ F - \chi_{\{F>t\}}|^p d\pi \le 2^p \pi(\{t \le F \le t+\varepsilon\}) \to 0, \quad \text{as } \varepsilon \to 0\,. \tag{7.9}$$

Standard calculus rules give

$$\int_{\Upsilon(\mathbb{R}^n)} G|\nabla(g_\varepsilon \circ F)|_{T\Upsilon} d\pi \le \varepsilon^{-1} \int_{\{t<F\le t+\varepsilon\}} G|\nabla F|_{T\Upsilon} d\pi \le \frac{m(t+\varepsilon)-m(t)}{\varepsilon}\,, \tag{7.10}$$

while (7.5) in Proposition 7.5 implies

$$\int_{\Upsilon(\mathbb{R}^n)} G\, d\|\{F>t\}\| \le \liminf_{\varepsilon\to 0} \int_{\Upsilon(\mathbb{R}^n)} G|\nabla(g_\varepsilon \circ F)|_{T\Upsilon} d\pi = m'(t)\,. \tag{7.11}$$

Since m is differentiable for a.e. $t \in \mathbb{R}$, the one inequality comes by integrating (7.11).

Let us prove the converse inequality. Let $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ such that $|V|_{T\Upsilon} \leq 1$. Then, by Theorem 6.14, we deduce

$$\begin{aligned}\int_{\Upsilon(\mathbb{R}^n)} F(\nabla^* GV) d\pi &= \int_{-\infty}^{\infty} \int_{\{F>t\}} (\nabla^* GV) d\pi dt \\ &\leq \int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} G d\|\{F>t\}\| dt\,,\end{aligned}$$

which easily yields the sought conclusion. □

Proof of Theorem 7.1. Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi)$ such that $|\mathsf{D}F|(\Upsilon(\mathbb{R}^n)) < \infty$ and $G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ be nonnegative. By definition there exists a sequence $(F_n) \subset \mathrm{CylF}(\Upsilon)$ such that $F_n \to F$ in $L^1(\Upsilon(\mathbb{R}^n), \pi)$ and $\int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_n|_{T\Upsilon} d\pi \to |\mathsf{D}_* F|[G]$. From Proposition 7.6 we get

$$\int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} G d\|\{F_n > t\}\| dt = \int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_n|_{T\Upsilon} d\pi\,, \tag{7.12}$$

and passing to the limit for $n \to \infty$ we deduce

$$\int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} G d\|\{F > t\}\| dt \leq |\mathsf{D}_* F|[G]\,, \tag{7.13}$$

as a consequence of (ii) in Proposition 7.5 and Fatous' Lemma. In particular $\{F > t\}$ is of finite perimeter for a.e.-$t \in \mathbb{R}$.

Let us now fix $\varepsilon > 0$ and consider $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ such that $|V|_{T\Upsilon} \leq 1$ and $\mathcal{V}(F) - \varepsilon \leq \int_{\Upsilon(\mathbb{R}^n)} F(\nabla^* V) d\pi$. By Theorem 6.14, we have

$$\begin{aligned}|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) - \varepsilon = \mathcal{V}(F) - \varepsilon &\leq \int_{\Upsilon(\mathbb{R}^n)} F(\nabla^* V) d\pi = \int_{-\infty}^{\infty} \int_{\{F>t\}} (\nabla^* V) d\pi dt \\ &\leq \int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} d\|\{F>t\}\| dt\,,\end{aligned}$$

which easily yields

$$\int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} d\|\{F > t\}\| dt \geq |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) = |\mathsf{D}_* F|[1]\,.$$

The sought conclusion follows now by recalling that $|\mathsf{D}_* F|[G_1 + G_2] \geq |\mathsf{D}_* F|[G_1] + |\mathsf{D}_* F|[G_2]$ and by the same argument in the paragraph after (6.32). Indeed,

$$\begin{aligned}|\mathsf{D}_* F|[G] + |\mathsf{D}_* F|[1-G] \leq |\mathsf{D}_* F|[1] &\leq \int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} d\|\{F > t\}\| dt \\ &= \int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} G d\|\{F>t\}\| dt + \int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} (1-G) d\|\{F>t\}\| dt \\ &\leq |\mathsf{D}_* F|[G] + |\mathsf{D}_* F|[1-G]\,,\end{aligned}$$

for any $0 \leq G \leq 1$, $G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$. The proof is complete. □

7.3. **Gauß–Green formula.** We prove the Gauß–Green formula, which is one of the main results in this paper.

Theorem 7.7 (Gauß–Green formula)**.** *For $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, there exists a unique element $\sigma_F \in L^2(T\Upsilon, |\mathsf{D}F|)$ such that $|\sigma_F|_{T\Upsilon} = 1$ $|\mathsf{D}F|$-a.e., and*

$$\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_F \rangle_{T\Upsilon} d|\mathsf{D}F|, \quad \forall V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))\,. \tag{7.14}$$

Proof. We assume without loss of generality that $|\mathsf{D}F|(\Upsilon(\mathbb{R}^n)) = 1$. By Theorem 6.14 and Theorem 7.1, it holds that

$$\begin{aligned}
\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F d\pi &= \int_{-\infty}^{\infty} \int_{\{F>t\}} (\nabla^* V) d\pi dt \\
&= \int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_{\{F>t\}} \rangle_{T\Upsilon} d \big\| \{F > t\} \big\| dt \\
&\le \int_{\Upsilon(\mathbb{R}^n)} |V|_{T\Upsilon} d|\mathsf{D}F| \\
&\le \|V\|_{L^2(T\Upsilon, |\mathsf{D}F|)}, \qquad (7.15)
\end{aligned}$$

for any $V \in \mathrm{CylV}(\Upsilon)$. In particular, denoting by $H \subset L^2(T\Upsilon, |\mathsf{D}F|)$ the closure of $\mathrm{CylV}(\Upsilon)$ in $L^2(T\Upsilon, |\mathsf{D}F|)$, the map L defined by

$$L : H \to \mathbb{R}, \quad H \ni V \mapsto L(V) := \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F d\pi, \tag{7.16}$$

is a well-defined continuous operator on the Hilbert space H and satisfies $\|L\| \le 1$. Therefore, the Riesz representation theorem on the Hilbert space H gives the existence of $\sigma_F \in H$ so that

$$\|\sigma_F\|_{L^2(T\Upsilon, |\mathsf{D}F|)} \le 1, \qquad \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_F \rangle d|\mathsf{D}F| \quad \forall V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n)). \tag{7.17}$$

From Theorem 5.17 and Theorem 7.1, we deduce

$$\begin{aligned}
1 = |\mathsf{D}F|(\Upsilon(\mathbb{R}^n)) = |\mathsf{D}_* F|[1] = \mathcal{V}(F) &= \sup_{V \in \mathrm{CylV}, |V|_{T\Upsilon} \le 1} \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F d\pi \\
\le \int_{\Upsilon(\mathbb{R}^n)} |\sigma_F|_{T\Upsilon} d|\mathsf{D}F| &\le \|\sigma_F\|_{L^2(T\Upsilon, |\mathsf{D}F|)} \le 1,
\end{aligned}$$

which yields $\|\sigma_F\|_{L^1(T\Upsilon, |\mathsf{D}F|)} = \|\sigma_F\|_{L^2(T\Upsilon, |\mathsf{D}F|)} = 1$, and therefore $|\sigma_F|_{T\Upsilon} = 1$ $|\mathsf{D}F|$-a.e. as a consequence of the characterization of the equality in Jensen's inequality. The proof is complete.

□

7.4. BV **and Sobolev functions.** In this subsection, we discuss the consistency of the just developed theory of BV functions with the $(1,2)$-Sobolev space $H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$.

Proposition 7.8. *Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$. Suppose $|\mathsf{D}F| \ll \pi$ with $|\mathsf{D}F| = H \cdot \pi$ and $H \in L^2(\Upsilon(\mathbb{R}^n), \pi)$. Then $F \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$ and*

$$H = |\nabla F|, \qquad \sigma_F = \frac{\nabla F}{|\nabla F|} \cdot \chi_{\{|\nabla F| \neq 0\}},$$

where σ_F is the unique element in $L^2(T\Upsilon, |\mathsf{D}F|)$ in the Gauß–Green formula (7.14).

Proof. By Theorem 7.7 and recalling $\mathbf{T}_t V \in \mathcal{D}(\mathcal{E}_H) \subset \mathbf{D}^2(T\Upsilon(\mathbb{R}^n), \pi)$ for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n), \pi)$ by (5.32), the approximation of $\mathbf{T}_t V$ by $\mathrm{CylV}(\Upsilon(\mathbb{R}^n), \pi)$ implies that

$$\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* G) F d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle G, \sigma_F \rangle_{T\Upsilon} F d\pi \quad \forall G \in \mathbf{T}_t \mathrm{CylV}(\Upsilon(\mathbb{R}^n)) \quad \forall t > 0, \tag{7.18}$$

where $\mathbf{T}_t \mathrm{CylV}(\Upsilon(\mathbb{R}^n)) := \{G = \mathbf{T}_t F : F \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))\}$ for $t > 0$. By Lemma 5.19 and the π-symmetry of T_t, for any $U \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$, setting $G = \mathbf{T}_t U$, we obtain

$$\begin{aligned}
\int_{\Upsilon(\mathbb{R}^n)} \langle U, \nabla T_t F \rangle d\pi &= \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* U) T_t F d\pi = \int_{\Upsilon(\mathbb{R}^n)} T_t(\nabla^* U) F d\pi = \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* G) F d\pi \\
&= \int_{\Upsilon(\mathbb{R}^n)} \langle G, \sigma_F \rangle_{T\Upsilon} d|\mathsf{D}F| = \int_{\Upsilon(\mathbb{R}^n)} \langle G, \sigma_F \rangle_{T\Upsilon} H d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle U, \mathbf{T}_t(H\sigma_F) \rangle_{T\Upsilon} d\pi.
\end{aligned}$$

Thus, $\mathbf{T}_t(H\sigma_F) = \nabla T_t F$. Letting $t \to 0$, $\mathbf{T}_t(H\sigma_F)$ converges to $H\sigma_F$ in $L^2(T\Upsilon, \pi)$, which implies that $\nabla T_t F$ converges to $H\sigma_F$ in $L^2(T\Upsilon(\mathbb{R}^n), \pi)$. Since $T_t F \to F$ in $L^2(\Upsilon(\mathbb{R}^n), \pi)$, we conclude that $F \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$, and $\nabla F = H\sigma_F$. Therefore, $H \cdot \pi = |\mathsf{D}F| = |\nabla F| \cdot \pi$, and

$$\sigma_F = \frac{\nabla F}{H} \chi_{\{H \neq 0\}} = \frac{\nabla F}{|\nabla F|} \chi_{\{|\nabla F| \neq 0\}} .$$

The proof is complete. □

References

[1] S. Albeverio, A. Daletskii, E. Lytvynov, Laplace operators on differential forms over configuration spaces, *Journal of Geometry and Physics* **37** (2001), 15–46.

[2] S. Albeverio, Yu-G. Kondratiev, M. Röckner, Analysis and geometry on configuration spaces. *J. Funct. Anal.*, **154**(2): (1998), 444–500.

[3] R.P. Alonso, F. Baudoin, L. Chen, L.G Roger, N. Shanmugalingam, A. Teplyaev, Besov class via heat semigroup on Dirichlet spaces I: Sobolev type inequalities, *J. Funct. Anal.* **278**, (2020) no. 11, 108459, 48.

[4] R.P. Alonso, F. Baudoin, L. Chen, L.G Roger, N. Shanmugalingam, A. Teplyaev, Besov class via heat semigroup on Dirichlet spaces II: BV functions and Gaussian heat kernel estimates, *Calc. Var. Partial Differential Equations* **59**, (2020) no. 3, Paper No. 103, 32.

[5] R.P. Alonso, F. Baudoin, L. Chen, L.G Roger, N. Shanmugalingam, A. Teplyaev, Besov class via heat semigroup on Dirichlet spaces III: BV functions and sub-Gaussian heat kernel estimates, *Calc. Var. Partial Differential Equations* **60**, (2020) no. 5, Paper No. 170, 38.

[6] L. Ambrosio, Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces, *Adv. Math.*, **159** (2001), 51–67.

[7] L. Ambrosio, Fine properties of sets of finite perimeter in doubling metric measure spaces, *Set-Valued Anal.*, **10** (2002), 111–128.

[8] L. Ambrosio, Calculus, heat flow and curvature-dimension bounds in metric measure spaces, *Proceedings of the ICM 2018*, (2018).

[9] L. Ambrosio, E. Brué, D. Semola, Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in RCD spaces. *Geom. Funct. Anal.* **29**: (2019), 949–1001.

[10] L. Ambrosio, S. Di Marino, Equivalent definitions of BV space and of total variation on metric measure spaces. *J. Funct. Anal.*, **266** (2014), 4150–4188.

[11] L. Ambrosio, A. Figalli, Surface measures and convergence of the Ornstein-Uhlenbeck semigroup in Wiener spaces, *Ann. Fac. Sci. Toulouse Math.*, **20**, (2011) 407–438.

[12] L. Ambrosio, A. Figalli, E. Runa, On sets of finite perimeter in Wiener spaces: reduced boundary and convergence to halfspaces, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, **24** (2013), 111–122.

[13] L. Ambrosio, R. Ghezzi, V. Magnani, BV functions and sets of finite perimeter in sub-Riemannian manifolds, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **32** (2015), 489–517.

[14] L. Ambrosio, S. Maniglia, M. Miranda Jr, D. Pallara, BV functions in abstract Wiener spaces, *J. Funct. Anal.*, **258** (2010), 785—13.

[15] L. Ambrosio, M. Miranda Jr, D. Pallara, Some fine properties of BV functions on Wiener spaces, *Anal. Geom. Metr. Spaces*, **3** (2015), 212–230.

[16] L. Ambrosio, M. Miranda Jr, D. Pallara, Sets with finite perimeter in Wiener spaces, perimeter measure and boundary rectifiability, *Discrete Contin. Dyn. Syst.*, **28** (2010), 591–606.

[17] V.I. Bogachev, *Measure Theory.* Springer-Verlag, Berlin, 2007.

[18] E. Brué, E. Pasqualetto, D. Semola, Rectifiability of the reduced boundary for sets of finite perimeter over RCD(K, N) spaces. Preprint arXiv:1909.00381.

[19] D. Burago, Yu. Burago, S. Ivanov, *A Course in Metric Geometry,* Graduate Studies in Mathematics **33**, AMS, 2001.

[20] E.B. Davies, *Heat kernels and spectral theory*, Cambridge University Press, 1989.

[21] E. De Giorgi, Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni, *Ann. Mat. Pura Appl.* (4) **36** (1954), 191–213.

[22] E. De Giorgi, Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni, *Ricerche Mat.* 4 (1955), 95 − 113.

[23] L. Dello Schiavo, and K. Suzuki, Configuration Spaces Over Singular Spaces I − Dirichlet Form and Metric Measure Geometry − Preprint arXiv:2109.03192.

[24] M. Erbar, M. Huesmann, Curvature bounds for configuration spaces. *Calc. Var.*, **54** (2015), 307 − 430.

[25] H. Federer, *Geometric measure theory*, Springer, Grundleherender Math. Wissenschaften, **153** (1969).
[26] D. Feyel and A. de la Pradelle, Hausdorff Measure on the Wiener Space, *Potential Anal.* **1** 177–189, (1992).
[27] D. H. Fremlin, *Measure Theory* – Volume IV, Torres Fremlin (ed.), 2003.
[28] M. Fukushima, BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space, *J. Funct. Anal.* **174** (2000), 227 – 249.
[29] M. Fukushima, M. Hino, On the space of BV functions and a related stochastic calculus in infinite dimensions, *J. Funct. Anal.* **183** (2001), 245–268.
[30] M. Hino, Sets of finite perimeter and the Hausdorff–Gauss measure on the Wiener space, *J. Funct. Anal.* **258**, (5) (2010), 1656–1681.
[31] Yu-G. Kondratiev, E.W. Lytvynov, M. Röckner. The Heat Semigroup on Configuration Spaces. *Publ. Res. Inst. Math. Sci.*, **39** (2002), 1–48.
[32] I. M. Gelfand, N. Ya. Vilenkin, *Some Applications of Harmonic Analysis. Rigged Hilbert Spaces,* Academic Press, New York, (1964).
[33] M. Reed, B. Simon, *Methods of Modern Mathematical Physics: Functional Analysis*, Vol. 1, Academic Press, San Diego, (1977).
[34] Z.-M. Ma, M. Röckner, *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Graduate Studies in Mathematics. Springer. (1992)
[35] M. Miranda Jr., Functions of bounded variation on "good" metric spaces, *J. Math. Pures Appl.*, **82** (2003), 975–1004.
[36] H. Osada. Dirichlet Form Approach to Infinite-Dimensional Wiener Processes with Singular Interactions. *Comm. Math. Phys.*, **176** (1996), 117–131.
[37] El-Maati Ouhabaaz, Invariance of Closed Convex Sets and Domination Criteria for Semigroups, *Potential Analysis* **5** (1996), 611–625.
[38] M. Röckner, R-C. Zhu, X-C. Zhu, BV functions in a Gelfand triple and the stochastic reflection problem on a convex set of a Hilbert space, *Comptes Rendus Mathematique* **348** (2010) 21–22, 1175–1178.
[39] M. Röckner, R-C. Zhu, X-C. Zhu, The stochastic reflection problem on an infinite dimensional convex set and BV functions in a Gelfand triple, *Ann Probab.* **40**(4), (2012) 1759–1794 .
[40] M. Röckner, R-C. Zhu, and X-C. Zhu, BV functions in a Gelfand triple for differentiable measure and its applications, *Forum Math* **27** (2015) 1657–1687 .
[41] I. Shigekawa. *Defective intertwining property and generator domain*, *J. Funct. Anal.*, **239** (2006), 357–374.
[42] I. Shigekawa, L^p contraction semigroups for vector valued functions, *J. Funct. Anal.*, **147** (1997), 69–108.
[43] M. Suslin, 'Sur une définition des ensembles mesurables B sans nombres infinis', C.R.Acad. Sci. (Paris) **164** (1917) 88–91.
[44] W.Ziemer. *Weakly Differentiable Functions.* Springer, (1989).