

Mean value formulas for classical solutions to uniformly parabolic equations in divergence form

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Abstract

We prove surface and volume mean value formulas for classical solutions to uniformly parabolic equations in divergence form. We emphasize that our results only rely on the classical theory, and our arguments follow the lines used in the original theory of harmonic functions. We provide two proofs relying on two different formulations of the divergence theorem, one stated for sets with *almost C^1 -boundary*, the other stated for sets with finite perimeter.

1 Introduction

Let Ω be an open subset of \mathbb{R}^{N+1} . We consider classical solutions u to the equation $\mathcal{L}u = f$ in Ω , where \mathcal{L} is a parabolic operator in divergence form defined for $z = (x, t) \in \mathbb{R}^{N+1}$ as follows

$$\mathcal{L}u(z) := \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}(z) \frac{\partial u}{\partial x_j}(z) \right) + \sum_{i=1}^N b_i(z) \frac{\partial u}{\partial x_i}(z) + c(z)u(z) - \frac{\partial u}{\partial t}(z). \quad (1.1)$$

In the following we use the notation $A(z) := (a_{ij}(z))_{i,j=1,\dots,N}$, $b(z) := (b_1(z), \dots, b_N(z))$ and we write $\mathcal{L}u$ in the short form

$$\mathcal{L}u(z) := \operatorname{div} (A(z) \nabla_x u(z)) + \langle b(z), \nabla_x u(z) \rangle + c(z)u(z) - \frac{\partial u}{\partial t}(z). \quad (1.2)$$

Here div , ∇_x and $\langle \cdot, \cdot \rangle$ denote the divergence, the gradient and the inner product in \mathbb{R}^N , respectively. We assume that the matrix $A(z)$ is symmetric and that the coefficients of the operator \mathcal{L} are Hölder continuous functions with respect to the parabolic distance. This means that there exist two constants $M > 0$ and $\alpha \in]0, 1]$, such that

$$|c(x, t) - c(y, s)| \leq M \left(|x - y|^\alpha + |t - s|^{\alpha/2} \right), \quad (1.3)$$

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for every $(x, t), (y, s) \in \mathbb{R}^{N+1}$. We require that the above condition is satisfied not only by c , but also by $a_{ij}, \frac{\partial a_{ij}}{\partial x_i}, b_i, \frac{\partial b_i}{\partial x_i}$, for $i, j = 1, \dots, N$, with the same constants M and α . We finally assume that the coefficients of \mathcal{L} are bounded and that \mathcal{L} is uniformly parabolic, *i.e.*, there exist two constants λ, Λ , with $0 < \lambda < \Lambda$, such that

$$\lambda|\xi|^2 \leq \langle A(z)\xi, \xi \rangle \leq \Lambda|\xi|^2, \quad \left| \frac{\partial a_{ij}}{\partial x_i} \right| \leq \Lambda, \quad |b_i(z)| \leq \Lambda, \quad |c(z)| \leq \Lambda, \quad (1.4)$$

for every $\xi \in \mathbb{R}^N$, for every $z \in \mathbb{R}^{N+1}$, and for $i, j = 1, \dots, N$. Under the above assumptions, the classical parametrix method provides us with the existence of a fundamental solution Γ . In Section 2 we shall quote from the monograph of Friedman [11] the results we need for our purposes.

The main achievements of this note are some mean value formulas for the solutions to $\mathcal{L}u = f$ that are written in terms of the level and super-level sets of the fundamental solution Γ . We extend previous results of Fabes and Garofalo [8] and Garofalo and Lanconelli [12] in that we weaken the regularity requirement on the coefficients of \mathcal{L} that in [8, 12] are assumed to be C^∞ smooth. As applications of the mean value formulas we give an elementary proof of the parabolic strong maximum principle. We note that the conditions on the functions $\frac{\partial a_{ij}}{\partial x_i}$'s are needed in order to deal with classical solutions to the adjoint equation $\mathcal{L}^*v = 0$, as the mean value formulas rely on the divergence theorem applied to the function $(\xi, \tau) \mapsto \Gamma(x, t, \xi, \tau)$.

We introduce some notation in order to state our main results. For every $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ and for every $r > 0$, we set

$$\begin{aligned} \psi_r(z_0) &:= \left\{ z \in \mathbb{R}^{N+1} \mid \Gamma(z_0; z) = \frac{1}{r^N} \right\}, \\ \Omega_r(z_0) &:= \left\{ z \in \mathbb{R}^{N+1} \mid \Gamma(z_0; z) > \frac{1}{r^N} \right\}. \end{aligned} \quad (1.5)$$

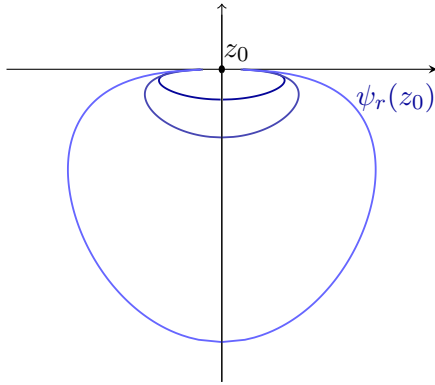


FIG.1 - $\psi_r(z_0)$ for three different values of r .

Similarly to the elliptic case, we call $\psi_r(z_0)$ and $\Omega_r(z_0)$ respectively the *parabolic sphere* and the *parabolic ball* with radius r and “center” at (x_0, t_0) . Note that, unlike the elliptic setting, z_0 belongs to the topological boundary of $\Omega_r(z_0)$. Because of the properties of the fundamental solution of uniformly parabolic operators, the parabolic balls $\Omega_r(z_0)$ are bounded sets and shrink

to the center z_0 as $r \rightarrow 0$. We finally introduce the following kernels

$$\begin{aligned} K(z_0; z) &:= \frac{\langle A(z) \nabla_x \Gamma(z_0; z), \nabla_x \Gamma(z_0; z) \rangle}{|\nabla_{(x,t)} \Gamma(z_0; z)|}, \\ M(z_0; z) &:= \frac{\langle A(z) \nabla_x \Gamma(z_0; z), \nabla_x \Gamma(z_0; z) \rangle}{\Gamma(z_0; z)^2}. \end{aligned} \quad (1.6)$$

Here $\nabla_x \Gamma(z_0; z)$ and $|\nabla_{(x,t)} \Gamma(z_0; z)|$ denote the gradient with respect to the space variable x and the norm of the gradient with respect to the variables (x, t) of Γ , respectively. Moreover, we agree to set $K(z_0; z) = 0$ whenever $\nabla_{(x,t)} \Gamma(z_0; z) = 0$. In the following, \mathcal{H}^N denotes the N -dimensional Hausdorff measure. The first achievements of this note are the following mean value formulas.

Theorem 1.1 *Let Ω be an open subset of \mathbb{R}^{N+1} , and let u be a classical solution to $\mathcal{L}u = f$ in Ω . Then, for every $z_0 \in \Omega$ and for almost every $r > 0$ such that $\Omega_r(z_0) \subset \Omega$ we have*

$$\begin{aligned} u(z_0) &= \int_{\psi_r(z_0)} K(z_0; z) u(z) d\mathcal{H}^N(z) + \int_{\Omega_r(z_0)} f(z) \left(\frac{1}{r^N} - \Gamma(z_0; z) \right) dz, \\ u(z_0) &= \frac{1}{r^N} \int_{\Omega_r(z_0)} M(z_0; z) u(z) dz + \frac{N}{r^N} \int_0^r \left(\varrho^{N-1} \int_{\Omega_\varrho(z_0)} f(z) \left(\frac{1}{\varrho^N} - \Gamma(z_0; z) \right) dz \right) d\varrho. \end{aligned}$$

The second statement holds for every $r > 0$ such that $\Omega_r(z_0) \subset \Omega$.

Note that $\frac{1}{r^N} - \Gamma(z_0; z) < 0$ in the set $\Omega_r(z_0)$, because of its very definition (1.5). This fact, together with the non-negativity of the kernels (1.6) will be used in the sequel to obtain the strong maximum principle from Theorem 1.1.

We next put Theorem 1.1 in its context. It restores the mean value formulas first proved by Pini in [15] for the heat equation $\partial_t u = \partial_x^2 u$, then by Watson in [20] for the heat equation in several space variables. We also recall the mean value formulas first proved by Fabes and Garofalo in [8] for the equation $\mathcal{L}u = 0$, then extended by Garofalo and Lanconelli [12] to the equation $\mathcal{L}u = f$, where the operator \mathcal{L} has the form (1.1) and its coefficients are assumed to be C^∞ smooth. This extra regularity assumption on the coefficients of \mathcal{L} is due to the fact that the mean value formula relies on the divergence theorem applied to the parabolic sphere $\psi_r(z_0)$. Since the explicit expression of the fundamental solution Γ is not available when the coefficients of \mathcal{L} are variable, the authors of [8] and [12] rely on the Sard theorem (see [17]) which guarantees that $\psi_r(z_0)$ is a manifold for *almost every* positive r , provided that the fundamental solution Γ is $N + 1$ times differentiable. The smoothness of the coefficients of the operator \mathcal{L} is used in [8] and [12] in order to have the needed regularity on Γ .

The main goal of this note is the restoration of natural regularity hypotheses for the existence of classical solutions to $\mathcal{L}u = f$. These assumptions can be further weakened, since the existence of a fundamental solution has been proved for operators with Dini continuous coefficients. We prefer to keep our treatment in the usual setting of Hölder continuous functions for the sake of simplicity. The unnecessary regularity conditions on the coefficients of \mathcal{L} can be removed in two ways. Following an approach close to the classical one, it is possible to rely on a result due to Dubovickiĭ [7] (see also Bojarski, Hajłasz, and Strzelecki [3]) which allows to reduce the regularity requirement on Γ in order to apply a generalized divergence theorem for *sets with*

almost C^1 boundary. This is presented in Section 3 and applied in Section 4. The other approach relies on geometric measure theory and is presented in the last section: we show how the proof of Theorem 1.1 can be modified relying on the generalized divergence theorem proved by De Giorgi [5, 6] in the framework of finite perimeter sets. As said before, this deep theory is not necessary in the present context, but it is more flexible and its generalization to Carnot groups (where the analogue of Dubovickii's Theorem is not available) will allow us to extend the results of the present paper to *degenerate* parabolic operators. We have presented the application to uniformly parabolic operators to pave the way to this generalization, which will be the subject of a forthcoming paper.

The mean value formulas stated in Theorem 1.1 provide us with a simple proof of the strong maximum (minimum) principle for the operator \mathcal{L} when $c = 0$. Note that, in this case, the constant function $u(x, t) = 1$ is a solution to $\mathcal{L}u = 0$, so that the mean value formula gives $\frac{1}{r^N} \int_{\Omega_r(z_0)} M(z_0; z) dz = 1$. In order to state this result we first introduce the notion of *attainable set*. We say that a curve $\gamma : [0, T] \rightarrow \mathbb{R}^{N+1}$ is \mathcal{L} -*admissible* if it is absolutely continuous and

$$\dot{\gamma}(s) = (\dot{x}_1(s), \dots, \dot{x}_N(s), -1)$$

for almost every $s \in [0, T]$, with $\dot{x}_1, \dots, \dot{x}_N \in L^2([0, T])$.

Definition 1.2 *Let Ω be any open subset of \mathbb{R}^{N+1} , and let $z_0 \in \Omega$. The attainable set is*

$$\mathcal{A}_{z_0}(\Omega) = \left\{ z \in \Omega \mid \text{there exists an } \mathcal{L} \text{ - admissible curve } \gamma : [0, T] \rightarrow \Omega \text{ such that } \gamma(0) = z_0 \text{ and } \gamma(T) = z \right\}.$$

Whenever there is no ambiguity on the choice of the set Ω we denote $\mathcal{A}_{z_0} = \mathcal{A}_{z_0}(\Omega)$.

Proposition 1.3 *Let Ω be any open subset of \mathbb{R}^{N+1} , and suppose that $c = 0$. let $z_0 = (x_0, t_0) \in \Omega$ and let u be a classical solution to $\mathcal{L}u = f$. If $u(z_0) = \max_{\Omega} u$ and $f \geq 0$ in Ω , then*

$$u(z) = u(z_0) \quad \text{for every } z \in \overline{\mathcal{A}_{z_0}(\Omega)}.$$

In particular, as a consequence, we find $f(z) = 0$ for every $z \in \overline{\mathcal{A}_{z_0}(\Omega)}$.

The analogous result holds true if $u(z_0) = \min_{\Omega} u$ and $f \leq 0$ in Ω .

In the remaining part of this introduction we focus on some modified mean value formulas useful in the proof of parabolic Harnack inequality. As already noticed, the main difficulty one encounters in the proof of the Harnack inequality is due to the unboundedness of the kernels introduced in (1.6). In order to overcome this issue, we can rely on the idea introduced by Kupcov in [13], and developed by Garofalo and Lanconelli in [12] in the case of parabolic operators with smooth coefficients. This method provides us with some bounded kernels and gives us a useful tool for a direct proof of the Harnack inequality. We outline here the procedure. Let m be a positive integer, and let u be a solution to $\mathcal{L}u = f$ in \mathbb{R}^{N+1} . We set

$$\tilde{u}(x, y, t) := u(x, t), \quad \tilde{f}(x, y, t) := f(x, t), \quad (x, y, t) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R},$$

and we note that

$$\widetilde{\mathcal{L}} \tilde{u}(x, y, t) = \tilde{f}(x, y, t) \quad \widetilde{\mathcal{L}} = \mathcal{L} + \sum_{j=1}^m \frac{\partial^2}{\partial^2 y_j} = \mathcal{L} + \Delta_y.$$

Moreover, if Γ and K_m denote fundamental solutions of \mathcal{L} and of the heat equation in \mathbb{R}^m , respectively, then the function

$$\tilde{\Gamma}(\xi, \eta, \tau; x, y, t) = \Gamma(\xi, \tau; x, t) K_m(\eta, \tau; y, t)$$

is a fundamental solution of $\widetilde{\mathcal{L}}$. Then, integrating with respect to y in the mean value formulas of Theorem 1.1, applied to \tilde{u} and to the operator $\widetilde{\mathcal{L}}$, gives new kernels, that are bounded whenever $m > 2$. We introduce further notations.

$$\begin{aligned} \Omega_r^{(m)}(z_0) &:= \left\{ z \in \mathbb{R}^{N+1} \mid (4\pi(t_0 - t))^{-m/2} \Gamma(z_0; z) > \frac{1}{r^{N+m}} \right\}, \\ N_r(z_0; z) &:= 2\sqrt{t_0 - t} \sqrt{\log \left(\frac{r^{N+m}}{(4\pi(t_0 - t))^{m/2}} \Gamma(z_0; z) \right)}, \\ M_r^{(m)}(z_0; z) &:= \omega_m N_r^m(z_0; z) \left(M(z_0; z) + \frac{m}{m+2} \cdot \frac{N_r^2(z_0; z)}{4(t_0 - t)^2} \right), \\ W_r^{(m)}(z_0; z) &:= \omega_m N_r^m(z_0; z) - \omega_m \frac{m}{2} \Gamma(z_0, z) \cdot \tilde{\gamma} \left(\frac{m}{2}; \frac{N_r^2(z_0; z)}{4(t_0 - t)} \right), \end{aligned} \tag{1.7}$$

where $M(z_0; z)$ is the kernel introduced in (1.6), ω_m denotes the volume of the m -dimensional unit ball and $\tilde{\gamma}$ is the lower incomplete gamma function

$$\tilde{\gamma}(s; w) := \int_0^w \tau^{s-1} e^{-\tau} d\tau.$$

Note that the function $N(z_0, z)$ is well defined for every $z \in \Omega_r^{(m)}(z_0)$, as the argument of the logarithm is positive, and that we did not point out the dependence of N_r on the space dimension m to avoid a possible confusion with its powers appearing in the definitions of $M_r^{(m)}$ and $W_r^{(m)}$.

Proposition 1.4 *Let Ω be an open subset of \mathbb{R}^{N+1} , and let u be a classical solution to $\mathcal{L}u = f$ in Ω . Then, for every $z_0 \in \Omega$ and for every $r > 0$ such that $\Omega_r^{(m)}(z_0) \subset \Omega$ we have*

$$u(z_0) = \frac{1}{r^N} \int_{\Omega_r^{(m)}(z_0)} M_r^{(m)}(z_0; z) u(z) dz + \frac{N+m}{r^{N+m}} \int_0^r \left(\varrho^{N+m-1} \int_{\Omega_\varrho^{(m)}(z_0)} W_\varrho^{(m)}(z_0; z) f(z) dz \right) d\varrho.$$

We conclude this introduction with some comments about our main results. Mean value formulas don't require the uniqueness of the fundamental solution Γ . In Section 2 we recall the main results we need on the existence of a fundamental solution together with some known facts about its uniqueness. We also recall in Proposition 2.1 an asymptotic bound of Γ which allows us to use a direct procedure in a part of the proof of Theorem 1.1. An alternative and more general approach has been introduced by Cupini and Lanconelli in [4], where a wide family of differential operators with smooth coefficients is considered.

2 Fundamental solution

In this Section we recall some notation and some known result on the classical theory of uniformly parabolic equations that will be used in the sequel. Points of \mathbb{R}^{N+1} are denoted by $z = (x, t)$, $\zeta = (\xi, \tau)$ and Ω denotes an open subset of \mathbb{R}^{N+1} .

Let u be a real valued function defined on Ω . We say that u belongs to $C^{2,1}(\Omega)$ if $u, \frac{\partial u}{\partial x_j}, \frac{\partial^2 u}{\partial x_i \partial x_j}$ for $i, j = 1, \dots, N$ and $\frac{\partial u}{\partial t}$ are continuous functions, it belongs to $C^{2+\alpha, 1+\alpha/2}(\Omega)$ if u and all the derivatives of u listed above belong to the space $C^\alpha(\Omega)$ of the Hölder continuous functions defined by (1.3). A function u belongs to $C_{\text{loc}}^\alpha(\Omega)$ ($C_{\text{loc}}^{2+\alpha, 1+\alpha/2}(\Omega)$, respectively) if it belongs to $C^\alpha(K)$ (resp. $C^{2+\alpha, 1+\alpha/2}(K)$) for every compact set $K \subset \Omega$. Let f be a continuous function defined on Ω . We say that $u \in C^{2,1}(\Omega)$ is a classical solution to $\mathcal{L}u = f$ in Ω if the equation (1.1) is satisfied at every point $z \in \Omega$.

According to Friedman [11], we say that a fundamental solution Γ for the operator \mathcal{L} is a function $\Gamma = \Gamma(z; \zeta)$ defined for every $(z; \zeta) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ with $t > \tau$, which satisfies the following conditions:

1. For every $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ the function $\Gamma(\cdot; \zeta)$ belongs to $C^{2,1}(\mathbb{R}^N \times]\tau, +\infty[)$ and is a classical solution to $\mathcal{L}\Gamma(\cdot; \zeta) = 0$ in $\mathbb{R}^N \times]\tau, +\infty[$;
2. for every $\varphi \in C_c(\mathbb{R}^N)$ the function

$$u(z) = \int_{\mathbb{R}^N} \Gamma(z; \xi, \tau) \varphi(\xi) d\xi,$$

is a classical solution to the Cauchy problem

$$\begin{cases} \mathcal{L}u = 0, & z \in \mathbb{R}^N \times]\tau, +\infty[\\ u(\cdot, \tau) = \varphi & \text{in } \mathbb{R}^N. \end{cases}$$

Note that u is defined for $t > \tau$, then the above identity is understood as follows: for every $\xi \in \mathbb{R}^N$ we have $\lim_{(x,t) \rightarrow (\xi, \tau)} u(x, t) = \varphi(\xi)$. We also point out that the two above conditions do not guarantee the uniqueness of the fundamental solution. A uniqueness result is obtained in the proof of Theorem 15 of [11] under the further assumptions that $\Gamma(x, t; \xi, \tau) \rightarrow 0$ as $|x| \rightarrow +\infty$ and $|\partial_{x_j} \Gamma(x, t; \xi, \tau)| \rightarrow 0$ as $|x| \rightarrow +\infty$, for $j = 1, \dots, N$, uniformly with respect to t varying in bounded intervals of the form $]\tau, \tau + T]$. As we shall see in the following, estimates (2.12) and (2.13) hold for the fundamental solution Γ built by the parametrix method, thus Γ is the unique fundamental solution satisfying the above asymptotic conditions.

We outline here the parametrix method for the construction of a fundamental solution Γ of \mathcal{L} . We first note that, if the matrix A in the operator \mathcal{L} is constant, then the fundamental solution of \mathcal{L} is explicitly known

$$\Gamma_A(z; \zeta) = \frac{1}{\sqrt{(4\pi(t-\tau))^N \det A}} \exp\left(-\frac{\langle A^{-1}(x-\xi), x-\xi \rangle}{4(t-\tau)}\right), \quad (2.1)$$

and moreover the *reproduction property* holds:

$$\Gamma_A(z; \zeta) = \int_{\mathbb{R}^N} \Gamma_A(x, t; y, s) \Gamma_A(y, s; \xi, \tau) dy, \quad (2.2)$$

for every $z = (x, t), \zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ and $s \in \mathbb{R}$ with $\tau < s < t$. A direct computation shows that, for every $T > 0$, and $\Lambda^+ > \Lambda$ as in (1.4), there exists a positive constant $C^+ = C^+(\lambda, \Lambda, \Lambda^+, T)$ such that

$$\left| \frac{\partial \Gamma_A}{\partial x_j}(z; \zeta) \right| \leq \frac{C^+}{\sqrt{t-\tau}} \Gamma^+(z; \zeta), \quad \left| \frac{\partial^2 \Gamma_A}{\partial x_i \partial x_j}(z; \zeta) \right| \leq \frac{C^+}{t-\tau} \Gamma^+(z; \zeta) \quad (2.3)$$

for any $i, j = 1, \dots, N$ and for every $z, \zeta \in \mathbb{R}^{N+1}$ such that $0 < t - \tau \leq T$. Here the function

$$\Gamma^+(z; \zeta) = \frac{1}{(\Lambda + 4\pi(t - \tau))^{N/2}} \exp\left(-\frac{|x - \xi|^2}{4\Lambda(t - \tau)}\right), \quad (2.4)$$

is the fundamental solution of $\Lambda^+ \Delta - \frac{\partial}{\partial t}$. The *parametrix* Z for \mathcal{L} is defined as

$$Z(z; \zeta) := \Gamma_{A(\zeta)}(z; \zeta) = \frac{1}{\sqrt{(4\pi(t - \tau))^N \det A(\zeta)}} \exp\left(-\frac{\langle A(\zeta)^{-1}(x - \xi), x - \xi \rangle}{4(t - \tau)}\right). \quad (2.5)$$

More specifically, for every fixed $(\xi, \tau) \in \mathbb{R}^{N+1}$, $Z(\cdot; \xi, \tau)$ is the fundamental solution of the operator \mathcal{L}_ζ obtained by *freezing* the coefficients a_{ij} 's of the operator \mathcal{L} at the point ζ :

$$\mathcal{L}_\zeta := \operatorname{div}(A(\zeta) \nabla_x) - \frac{\partial}{\partial t}. \quad (2.6)$$

Note that

$$\mathcal{L}Z(z; \zeta) := \operatorname{div}[(A(z) - A(\zeta)) \nabla_x Z(z; \zeta)], \quad (2.7)$$

which vanishes as $z \rightarrow \zeta$, by the continuity of the matrix A . The fundamental solution Γ for \mathcal{L} is obtained from Z by an iterative procedure. We define the sequence of functions $(\mathcal{L}Z)_1(z; \zeta) := \mathcal{L}Z(z; \zeta)$,

$$(\mathcal{L}Z)_{k+1}(z; \zeta) := \int_\tau^t \left(\int_{\mathbb{R}^N} (\mathcal{L}Z)_k(x, t; y, s) \mathcal{L}Z(y, s; \xi, \tau) dy \right) ds, \quad k \in \mathbb{N}. \quad (2.8)$$

Note that estimates (2.3) also apply to Z then, by using the Hölder continuity of the coefficients of \mathcal{L} , we obtain

$$|\mathcal{L}Z(z; \zeta)| \leq \frac{\tilde{C}}{(t - \tau)^{1-\alpha/2}} \Gamma^+(z; \zeta),$$

for a positive constant \tilde{C} depending on $\lambda, \Lambda, \Lambda^+, T$ and on the constant M in (1.3). This inequality and the reproduction property (2.2) applied to Γ^+ imply that, for every $k \geq 2$, the integral that defines $(\mathcal{L}Z)_k$ converges and

$$|(\mathcal{L}Z)_k(z; \zeta)| \leq \frac{(\Gamma_E(\alpha/2) \tilde{C})^k}{\Gamma_E(\alpha k/2) (t - \tau)^{1-k\alpha/2}} \Gamma^+(z; \zeta), \quad k \in \mathbb{N},$$

where Γ_E denotes the Euler's Gamma function. Theorem 8 in [11, Chapter 1] states that, under the assumption that the coefficients $a_{ij}, \frac{\partial a_{ij}}{\partial x_i}, b_i, \frac{\partial b_i}{\partial x_i}$, for $i, j = 1, \dots, N$ and c belong to the space $C^\alpha(\mathbb{R}^N \times]T_0, T_1[)$ with $T_0 < T_1$ and satisfy (1.4), the series

$$\Gamma(z; \zeta) := Z(z; \zeta) + \sum_{k=1}^{\infty} \int_\tau^t \left(\int_{\mathbb{R}^N} Z(x, t; y, s) (\mathcal{L}Z)_k(y, s; \xi, \tau) dy \right) ds \quad (2.9)$$

converges in $\mathbb{R}^N \times]T_0, T_1[$ and it turns out that its sum Γ is a fundamental solution for \mathcal{L} . We next list some properties of the function Γ defined in (2.9). We mainly refer to the monograph [11] by Friedman.

1. Theorem 8 in [11]: for every $\zeta \in \mathbb{R}^{N+1}$ the function $\Gamma(\cdot; \zeta)$ belongs to $C^{2,1}(\mathbb{R}^N \times]\tau, +\infty[)$ and it is a classical solution to $\mathcal{L}\Gamma = 0$ in $\mathbb{R}^N \times]\tau, +\infty[$.
2. Theorem 9 in [11]: for every bounded functions $\varphi \in C(\mathbb{R}^N)$ and $f \in C^\alpha(\mathbb{R}^N \times]\tau, T_1])$, with $T_0 < \tau < T_1$, the function

$$u(z) = \int_{\mathbb{R}^N} \Gamma(z; \zeta) \varphi(\xi) d\xi - \int_{\tau}^t \left(\int_{\mathbb{R}^N} \Gamma(x, t; \xi, s) f(\xi, s) d\xi \right) ds$$

is a classical solution to the Cauchy problem

$$\begin{cases} \mathcal{L}u = f, & z \in \mathbb{R}^N \times]\tau, +\infty[\\ u(\cdot, \tau) = \varphi & \text{in } \mathbb{R}^N. \end{cases} \quad (2.10)$$

3. Theorem 15 in [11]: The function $\Gamma^*(z; \zeta) := \Gamma(\zeta; z)$ is the fundamental solution of the transposed operator \mathcal{L}^* acting on a suitably smooth function v as follows

$$\mathcal{L}^*v(z) := \operatorname{div}(A(z)\nabla_x v(z)) - \langle b(z), \nabla_x v(z) \rangle + (c(z) - \operatorname{div} b(z))v(z) + \frac{\partial u}{\partial t}(z). \quad (2.11)$$

4. Inequalities (6.10) and (6.11) in [11]: for every positive T and $\Lambda^+ > \Lambda$ there exists a positive constant C^+ such that

$$\Gamma(z; \zeta) \leq C^+ \Gamma^+(z; \zeta), \quad (2.12)$$

for every $z = (x, t), \zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ with $0 < t - \tau < T$. Moreover, the following bounds for the derivatives hold

$$\begin{aligned} \left| \frac{\partial \Gamma}{\partial x_j}(z; \zeta) \right| &\leq \frac{C^+}{\sqrt{t - \tau}} \Gamma^+(z; \zeta), & \left| \frac{\partial^2 \Gamma}{\partial x_i \partial x_j}(z; \zeta) \right| &\leq \frac{C^+}{t - \tau} \Gamma^+(z; \zeta), \\ \left| \frac{\partial \Gamma}{\partial \xi_j}(z; \zeta) \right| &\leq \frac{C^+}{\sqrt{t - \tau}} \Gamma^+(z; \zeta), & \left| \frac{\partial^2 \Gamma}{\partial \xi_i \partial \xi_j}(z; \zeta) \right| &\leq \frac{C^+}{t - \tau} \Gamma^+(z; \zeta), \end{aligned} \quad (2.13)$$

for any $i, j = 1, \dots, N$ and for every $z, \zeta \in \mathbb{R}^{N+1}$ with $0 < t - \tau < T$.

We recall that the monograph [11] also contains an existence and uniqueness result for the Cauchy problem under the assumptions that the functions φ and f in the Cauchy problem (2.10) do satisfy the following growth condition:

$$|\varphi(x)| + |f(z)| \leq C_0 \exp(h|x|^2) \quad \text{for every } x \in \mathbb{R}^N \text{ and } t \in]\tau, T_1],$$

for some positive constants C_0 and h . The reproduction property (2.2) for Γ holds as a direct consequence of the uniqueness of the solution to the Cauchy problem. We also have

$$e^{-\Lambda(t-\tau)} \leq \int_{\mathbb{R}^N} \Gamma(x, t; \xi, \tau) d\xi \leq e^{\Lambda(t-\tau)}$$

for every $(x, t), (\xi, \tau) \in \mathbb{R}^{N+1}$ with $\tau < t$, where Λ is the constant introduced in (1.4).

We conclude this section by quoting a statement on the asymptotic behavior of fundamental solutions, which in the stochastic theory is referred to as *large deviation principle*. In our setting

it is useful in the description of the *parabolic ball* $\Omega_r(z_0)$ introduced in (1.5). The first large deviation theorem is due to Varhadhan [18, 19], who considers parabolic operators \mathcal{L} whose coefficients only depend on x and are Hölder continuous. It states that

$$4(t - \tau) \log(\Gamma(x, t; \xi, \tau)) \longrightarrow -d^2(x, \xi) \quad \text{as } t \rightarrow \tau, \quad (2.14)$$

uniformly with respect to x, ξ varying on compact sets. Here $d(x, \xi)$ denotes the Riemannian distance (induced by the matrix A) of x and ξ . Several extensions of the large deviation principle are available in literature, under different assumption on the regularity of the coefficients of \mathcal{L} . Azencott considers in [2] operators with smooth coefficients and proves more accurate estimates for the asymptotic behavior of $\log(\Gamma(x, t; \xi, \tau))$. Garofalo and Lanconelli prove an analogous result by using purely PDEs methods in [12]. We recall here a version of this result which is suitable for our purposes.

Proposition 2.1 [THEOREM 1.2 IN [16]] *For every $\eta \in]0, 1[$ there exists $C_\eta > 0$ such that*

$$(1 - \eta)Z(z; \zeta) \leq \Gamma(z; \zeta) \leq (1 + \eta)Z(z; \zeta) \quad (2.15)$$

for every $z, \zeta \in \mathbb{R}^{N+1}$ such that $Z(z; \zeta) > C_\eta$.

We finally prove a simple consequence of Proposition 2.1 that will be used in the following. We introduce some further notation in order to give its statement. We first note that the function Γ^* can be built by using the parametrix method, starting from the expression of the parametrix relevant to \mathcal{L}^* , that is

$$Z^*(z; \zeta) := \Gamma_{A(\zeta)}^*(z; \zeta) = \frac{1}{\sqrt{(4\pi(\tau - t))^N \det A(\zeta)}} \exp\left(-\frac{\langle A(\zeta)^{-1}(x - \xi), x - \xi \rangle}{4(\tau - t)}\right). \quad (2.16)$$

We set

$$\Omega_r^*(z_0) := \left\{ z \in \mathbb{R}^{N+1} \mid Z^*(z; z_0) \geq \frac{2}{r^N} \right\}, \quad (2.17)$$

and we point out that its explicit expression is:

$$\begin{aligned} \Omega_r^*(z_0) = & \left\{ (x, t) \in \mathbb{R}^{N+1} \mid \langle A^{-1}(z_0)(x - x_0), x - x_0 \rangle \leq \right. \\ & \left. - 4(t_0 - t) \left(\log\left(\frac{2}{r^N}\right) + \frac{1}{2} \log(\det A(z_0)) + \frac{N}{2} \log(4\pi(t_0 - t)) \right) \right\}. \end{aligned} \quad (2.18)$$

We have

Lemma 2.2 *There exists a positive constant r^* such that*

$$\Omega_r^*(z_0) \subset \Omega_r(z_0)$$

for every $z_0 \in \mathbb{R}^{N+1}$ and $r \in]0, r^]$.*

PROOF. As said before, the function Γ^* can be built by using the parametrix Z^* defined in (2.16). In particular, Proposition 2.1 applies to Γ^* . Then, if we apply the estimate (2.15) with $\eta = \frac{1}{2}$ and we use (2.11), we find that there exists $C^* > 0$ such that

$$\frac{1}{2}Z^*(\zeta; z_0) \leq \Gamma(z_0; \zeta) \leq \frac{3}{2}Z^*(\zeta; z_0)$$

for every $z_0, \zeta \in \mathbb{R}^{N+1}$ such that $Z^*(\zeta; z_0) > C^*$. The claim then follows from (1.5) and (2.17) by choosing $r^* := \left(\frac{2}{C^*}\right)^{1/N}$. \square

3 A generalized divergence theorem

Let Ω be an open subset of \mathbb{R}^n , and let $\Phi \in C^1(\Omega; \mathbb{R}^n)$. The classical divergence formula reads

$$\int_E \operatorname{div} \Phi \, dz = - \int_{\partial E} \langle \nu, \Phi \rangle \, d\mathcal{H}^{n-1}, \quad (3.1)$$

where E is a bounded set such that $\overline{E} \subset \Omega$ and its boundary is C^1 .

We are interested in the situation in which E is the *super-level* set of a real valued function $F \in C^1(\Omega)$, that is $E = \{F > y\}$ for some $y \in \mathbb{R}$. At every point $z \in \partial E$ such that $\nabla F(z) \neq 0$ the *inner* unit normal vector $\nu = \nu(z)$ appearing in (3.1) is defined as $\nu(z) = \frac{1}{|\nabla F(z)|} \nabla F(z)$ and ∂E is a C^1 manifold in a neighborhood of z . But, if we denote

$$\operatorname{Crit}(F) := \{z \in \mathbb{R}^n : \nabla F = 0\},$$

the set of *critical points* and $F(\operatorname{Crit}(F))$ the set of *critical values* of F , under our hypotheses we cannot apply the classical Sard theorem to state that “for almost every $y \in \mathbb{R}$ the level set $\{F = y\}$ is globally a C^1 manifold”. Indeed, Whitney proves in [21] that there exist functions $F \in C^1(\Omega)$ having the property that $\{F = y\} \cap \operatorname{Crit}(F)$ is not empty for every y . Therefore, the purpose of this section is to discuss a version of (3.1) when the boundary of E is C^1 up to a closed set of null Hausdorff measure and to see how it can be applied in our framework. We first introduce the class of sets with the relevant regularity and state the corresponding divergence formula. We draw this definition and the following theorem from [14, Section 9.3].

Definition 3.1 *An open set $E \subset \mathbb{R}^n$ has almost C^1 -boundary if there is a closed set $M_0 \subset \partial E$ with $\mathcal{H}^{n-1}(M_0) = 0$ such that, for every $z_0 \in M = \partial E \setminus M_0$ there exist $s > 0$ and $F \in C^1(B(z_0, s))$ with the property that*

$$\begin{aligned} B(z_0, s) \cap E &= \{z \in B(z_0, s) : F(z) > 0\}, \\ B(z_0, s) \cap \partial E &= \{z \in B(z_0, s) : F(z) = 0\} \end{aligned}$$

and $\nabla F(z) \neq 0$ for every $z \in B(z_0, s)$. We call M the regular part of ∂E (note that M is a C^1 -hypersurface). The inner unit normal to E is the continuous vector field $\nu \in C^0(M; \mathbb{S}^{n-1})$ given by

$$\nu(z) = \frac{\nabla F(z)}{|\nabla F(z)|}, \quad z \in B(z_0, s) \cap M.$$

Let us state the divergence theorem for sets with almost C^1 -boundary.

Theorem 3.2 *If $E \subset \mathbb{R}^n$ is an open set with almost C^1 -boundary and M is the regular part of its boundary, then for every $\Phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ the following equality holds*

$$\int_E \operatorname{div} \Phi \, dz = - \int_M \langle \nu, \Phi \rangle \, d\mathcal{H}^{n-1}. \quad (3.2)$$

If $F \in C^1(\Omega)$ and $E = \{F > y\}$ for some $y \in \mathbb{R}$, we can apply Theorem 3.2 thanks to the following result due to A. Ya. Dubovickii [7], that generalizes Sard’s theorem.

Theorem 3.3 (DUBOVICKII) *Assume that \mathcal{N}^n and \mathcal{M}^m are two smooth Riemannian manifolds of dimension n and m , respectively. Let $F : \mathcal{N}^n \rightarrow \mathcal{M}^m$ be a function of class C^k . Set $s = n - m - k + 1$, then for \mathcal{H}^m -a.e. $y \in \mathcal{M}^m$*

$$\mathcal{H}^s(\{F = y\} \cap \text{Crit}(F)) = 0. \quad (3.3)$$

Notice that if $m = k = 1$ and $\mathcal{M}^m = \mathbb{R}$, then $s = n - 1$ and for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$ the critical part of $\{F = y\}$ is an \mathcal{H}^{n-1} null set, while its regular part is an $(n - 1)$ -manifold of class C^1 . In other words, $\{F = y\}$ is a set with almost C^1 -boundary and we cannot apply the classical divergence theorem (3.1), but rather Theorem 3.2.

Summarizing, we have the following result, that immediately follows from the above discussion.

Proposition 3.4 *Let Ω be an open subset of \mathbb{R}^n and let $F \in C^1(\Omega; \mathbb{R})$. Then, for \mathcal{H}^1 -almost every $y \in \mathbb{R}$, we have:*

$$\int_{\{F > y\}} \text{div } \Phi \, dz = - \int_{\{F=y\} \setminus \text{Crit}(F)} \langle \nu, \Phi \rangle \, d\mathcal{H}^{n-1}, \quad \forall \Phi \in C_c^1(\Omega; \mathbb{R}^n),$$

where $\nu = \frac{\nabla F}{|\nabla F|}$.

PROOF. By Dubovickii Theorem 3.3 for \mathcal{H}^1 -almost every $y \in \mathbb{R}$ the set $\{F > y\}$ has almost C^1 -boundary, hence Theorem 3.2 applies. Moreover, as F is continuous, for any such y we have $\partial\{F > y\} \subset \{F = y\}$, $\mathcal{H}^{n-1}(\{F = y\} \setminus \partial\{F > y\}) = 0$ and the regular part of $\partial\{F > y\}$ is $\{F = y\} \setminus \{\nabla F = 0\}$ and has full \mathcal{H}^{n-1} measure. \square

In order to prove Theorem 1.1 we apply Proposition 3.4 to the super-level set $\Omega_r(z_0)$ of the fundamental solution $\Gamma(z_0, \cdot)$ of \mathcal{L} . Then, as explained in the Introduction, we have to cut at a time less than t_0 to avoid the singularity of the kernels at z_0 . Therefore, we specialize Proposition 3.4 as follows.

Proposition 3.5 *Let $G \in C^1(\mathbb{R}^{N+1} \setminus \{(x_0, t_0)\}; \mathbb{R})$. Then for \mathcal{H}^1 -almost every $w, \varepsilon \in \mathbb{R}$*

$$\int_{\{G > w\} \cap \{t < t_0 - \varepsilon\}} \text{div } \Phi \, dz = - \int_{(\{G=w\} \setminus \text{Crit}(G)) \cap \{t < t_0 - \varepsilon\}} \langle \nu, \Phi \rangle \, d\mathcal{H}^N + \int_{\{G > w\} \cap \{t = t_0 - \varepsilon\}} \langle e, \Phi \rangle \, d\mathcal{H}^N,$$

for every $\Phi \in C_c^1(\Omega; \mathbb{R}^{N+1})$, where $\nu = \frac{\nabla G}{|\nabla G|}$ and $e = (0, \dots, 0, 1)$.

PROOF. Notice that for \mathcal{H}^1 -a.e. $w \in \mathbb{R}$ the level set $\{G > w\}$ has almost- C^1 boundary and fix such a value. Let S be the \mathcal{H}^N -negligible singular set of $\partial\{G > w\}$: by Fubini theorem, for \mathcal{H}^1 -a.e. $\varepsilon > 0$ the set $S \cap \{t = t_0 - \varepsilon\}$ is in turn \mathcal{H}^{N-1} -negligible, and out of this set the unit normal is given \mathcal{H}^N -a.e. by ν in $\{G = w\} \setminus \text{Crit}(G) \cap \{t < t_0 - \varepsilon\}$ and by e in $\{G > w\} \cap \{t = t_0 - \varepsilon\}$. Therefore, Proposition 3.4 applies with $n = N + 1$, $\Omega = \mathbb{R}^{N+1} \setminus \{(x_0, t_0)\}$,

$$F(x, t) = (G(x, t) - w) \wedge (t - t_0 + \varepsilon),$$

$y = 0$ and the set

$$\Sigma = (\partial\{G > w\} \cap \{t < t_0 - \varepsilon\} \cap \text{Crit}(G)) \cup (\{G = w\} \cap \{t = t_0 - \varepsilon\})$$

is \mathcal{H}^N -negligible. \square

The last result we need to prove Theorem 1.1 is the coarea formula for Lipschitz functions. We refer to [9, 3.2.12] or [1], Theorem 2.93 and formula (2.74) for the proof.

Theorem 3.6 (COAREA FORMULA FOR LIPSCHITZ FUNCTIONS) *Let $G : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Lipschitz function, and let g be a non-negative measurable function. Then*

$$\int_{\mathbb{R}^n} g(z) |\nabla G(z)| dz = \int_{\mathbb{R}} \left(\int_{\{G=y\}} g(z) d\mathcal{H}^{n-1}(z) \right) dy. \quad (3.4)$$

4 Proof of the mean value formulas and maximum principle

In this Section we give the proof of the mean value formulas and of the strong maximum principle.

PROOF. OF THEOREM 1.1. Let Ω be an open subset of \mathbb{R}^{N+1} , and let u be a classical solution to $\mathcal{L}u = f$ in Ω . Let $z_0 = (x_0, t_0) \in \Omega$ and let $r_0 > 0$ be such that $\overline{\Omega_{r_0}(z_0)} \subset \Omega$. We prove our claim by applying Proposition 3.5 with $G(z) = \Gamma(z_0; z)$ and $w = \frac{1}{r^N}$, where $r \in]0, r_0]$ is such that the statement of Proposition 3.5 holds true with $w = \frac{1}{r^N}$, and $\varepsilon := \varepsilon_k$ for some monotone sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ such that $\varepsilon_k \rightarrow 0$ as $k \rightarrow +\infty$ (see Figure 2).

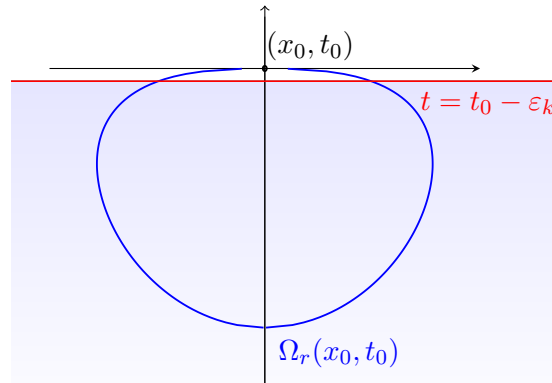


FIG.2 - The set $\Omega_r(x_0, t_0) \cap \{t < t_0 - \varepsilon_k\}$.

For this choice of r , we set $v(z) := \Gamma(z_0; z) - \frac{1}{r^N}$, and we note that

$$u(z)\mathcal{L}^*v(z) - v(z)\mathcal{L}u(z) = \operatorname{div}_x(u(z)A(z)\nabla_x v(z) - v(z)A(z)\nabla_x u(z)) - \operatorname{div}_x(u(z)v(z)b(z)) + \partial_t(u(z)v(z)) \quad (4.1)$$

for every $z \in \Omega \setminus \{z_0\}$. We then recall that $\mathcal{L}^*v = 0$ and $\mathcal{L}u = f$ in $\Omega \setminus \{z_0\}$. Then (4.1) can be written as follows

$$-v(z)f(z) = \operatorname{div} \Phi(z), \quad \Phi(z) := (uA\nabla_x v - vA\nabla_x u - uvb, uv)(z).$$

We then apply Proposition 3.5 to the set $\Omega_r(z_0) \cap \{t < t_0 - \varepsilon_k\}$ and we find

$$\begin{aligned} \int_{\Omega_r(z_0) \cap \{t < t_0 - \varepsilon_k\}} -v(z)f(z) dz &= - \int_{\psi_r(z_0) \setminus \operatorname{Crit}(\Gamma) \cap \{t < t_0 - \varepsilon_k\}} \langle \nu, \Phi \rangle d\mathcal{H}^N + \\ &\quad \int_{\Omega_r(z_0) \cap \{t = t_0 - \varepsilon_k\}} \langle e, \Phi \rangle d\mathcal{H}^N, \end{aligned} \quad (4.2)$$

where $\nu(z) = \frac{\nabla_{(x,t)}\Gamma(z_0,z)}{|\nabla_{(x,t)}\Gamma(z_0,z)|}$ and $e = (0, \dots, 0, 1)$. We next let $k \rightarrow +\infty$ in the above identity. As f is continuous on $\overline{\Omega_r(z_0)}$ and $v \in L^1(\Omega_r(z_0))$, we find

$$\lim_{k \rightarrow +\infty} \int_{\Omega_r(z_0) \cap \{t < t_0 - \varepsilon_k\}} -v(z)f(z) dz = \int_{\Omega_r(z_0)} -v(z)f(z) dz. \quad (4.3)$$

We next consider the last integral in the right hand side of (4.2). We have $\langle e, \Phi \rangle(z) = u(z)v(z)$, then

$$\int_{\Omega_r(z_0) \cap \{t=t_0 - \varepsilon_k\}} \langle e, \Phi \rangle d\mathcal{H}^N = \int_{\mathcal{I}_r^k(z_0)} u(x, t_0 - \varepsilon_k) \left(\Gamma(x_0, t_0; x, t_0 - \varepsilon_k) - \frac{1}{r^N} \right) dx, \quad (4.4)$$

where we have denoted

$$\mathcal{I}_r^k(z_0) := \left\{ x \in \mathbb{R}^N \mid (x, t_0 - \varepsilon_k) \in \overline{\Omega_r(z_0)} \right\}.$$

We next prove that the right hand side of (4.4) tends to $u(z_0)$ as $k \rightarrow +\infty$. Since Γ is the fundamental solution to \mathcal{L} we have

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^N} \Gamma(x_0, t_0; x, t_0 - \varepsilon_k) u(x, t_0 - \varepsilon_k) dx = u(x_0, t_0),$$

then, being u continuous on $\overline{\Omega_r(z_0)}$, we only need to show that

$$\lim_{k \rightarrow +\infty} \mathcal{H}^N(\mathcal{I}_r^k(z_0)) = 0, \quad \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^N \setminus \mathcal{I}_r^k(z_0)} \Gamma(x_0, t_0; x, t_0 - \varepsilon_k) dx = 0. \quad (4.5)$$

With this aim, we note that the upper bound (2.12) and (2.4) imply

$$\mathcal{I}_r^k(z_0) \subset \left\{ x \in \mathbb{R}^N \mid |x - x_0|^2 \leq 4\Lambda^+ \varepsilon_k \left(\log(C^+ r^N) - \frac{N}{2} \log(4\pi\Lambda^+ \varepsilon_k) \right) \right\}.$$

The first assertion of (4.5) is then a plain consequence of the above inclusion. In order to prove the second statement in (4.5), we rely on Lemma 2.2. We let $r_0 := \min(r, r^*)$, so that

$$\Omega_{r_0}^*(z_0) \subset \Omega_{r_0}(z_0) \subset \Omega_r(z_0),$$

thus

$$\begin{aligned} \mathbb{R}^N \setminus \mathcal{I}_r^k(z_0) &\subset \left\{ x \in \mathbb{R}^N \mid Z^*(x, t_0 - \varepsilon_k; x_0, t_0) \leq \frac{2}{r_0^N} \right\} \\ &= \left\{ x \in \mathbb{R}^N \mid \langle A(z_0)(x - x_0), x - x_0 \rangle \right. \\ &\quad \left. \geq -4\varepsilon_k \left(\log\left(\frac{2}{r_0^N}\right) + \frac{1}{2} \log(\det A(z_0)) + \frac{N}{2} \log(4\pi\varepsilon_k) \right) \right\}. \end{aligned}$$

By using again (2.12), the above inclusion, and the change of variable $x = x_0 + 2\sqrt{\Lambda^+ \varepsilon_k} \xi$, we find

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \mathcal{I}_r^k(z_0)} \Gamma(x_0, t_0; x, t_0 - \varepsilon_k) dx &\leq C^+ \int_{\mathbb{R}^N \setminus \mathcal{I}_r^k(z_0)} \Gamma^+(x_0, t_0; x, t_0 - \varepsilon_k) dx \\ &\leq \frac{C^+}{\pi^{N/2}} \int \left\{ \langle A(z_0)\xi, \xi \rangle \geq -\frac{1}{\Lambda^+} \left(\log\left(\frac{2}{r_0^N}\right) + \frac{1}{2} \log(\det A(z_0)) + \frac{N}{2} \log(4\pi\varepsilon_k) \right) \right\} \exp(-|\xi|^2) d\xi. \end{aligned}$$

The second assertion of (4.5) then follows. Thus, we have shown that

$$\lim_{k \rightarrow +\infty} \int_{\Omega_r(z_0) \cap \{t=t_0-\varepsilon_k\}} \langle e, \Phi \rangle d\mathcal{H}^N = u(z_0). \quad (4.6)$$

We are left with the first integral in the right hand side of (4.2). We preliminarily note that its limit, as $k \rightarrow +\infty$, does exist. Moreover, for every $z \in \psi_r(z_0)$ we have $v(z) = 0$, then $\Phi(z) = (u(z)A(z)\nabla_x v(z), 0)$, so that

$$\int_{\psi_r(z_0) \setminus \text{Crit}(\Gamma) \cap \{t < t_0 - \varepsilon_k\}} \langle \nu, \Phi \rangle d\mathcal{H}^N = \int_{\psi_r(z_0) \setminus \text{Crit}(\Gamma) \cap \{t < t_0 - \varepsilon_k\}} u(x, t) K(z_0; z) d\mathcal{H}^N,$$

where

$$K(z_0; z) = \frac{\langle A(z)\nabla_x \Gamma(z_0; z), \nabla_x \Gamma(z_0; z) \rangle}{|\nabla_{(x,t)} \Gamma(z_0; z)|}$$

is the kernel defined in (1.6). Note that K is non-negative and, if we consider the function $u = 1$ and we let $k \rightarrow +\infty$, we find

$$\lim_{k \rightarrow +\infty} \int_{\psi_r(z_0) \setminus \text{Crit}(\Gamma) \cap \{t < t_0 - \varepsilon_k\}} K(z_0; z) d\mathcal{H}^N = \int_{\psi_r(z_0) \setminus \text{Crit}(\Gamma)} K(z_0; z) d\mathcal{H}^N < +\infty.$$

Thus, if u is a classical solution to $\mathcal{L}u = 0$, we obtain

$$\lim_{k \rightarrow +\infty} \int_{\psi_r(z_0) \setminus \text{Crit}(\Gamma) \cap \{t < t_0 - \varepsilon_k\}} \langle \nu, \Phi \rangle d\mathcal{H}^N = \int_{\psi_r(z_0) \setminus \text{Crit}(\Gamma)} K(z_0; z) u(z) d\mathcal{H}^N. \quad (4.7)$$

We recall that Dubovickii's theorem implies that $\mathcal{H}^N(\psi_r(z_0) \cap \text{Crit}(\Gamma)) = 0$ for \mathcal{H}^1 almost every r , so that we can equivalently write

$$\lim_{k \rightarrow +\infty} \int_{\psi_r(z_0) \setminus \text{Crit}(\Gamma) \cap \{t < t_0 - \varepsilon_k\}} \langle \nu, \Phi \rangle d\mathcal{H}^N = \int_{\psi_r(z_0)} K(z_0; z) u(z) d\mathcal{H}^N. \quad (4.8)$$

The proof of the first assertion of Theorem 1.1 then follows by using (4.3), (4.6) and (4.8) in (4.2).

The proof of the second assertion of Theorem 1.1 is a direct consequence of the first one and of the coarea formula stated in Theorem 3.6. Indeed, fix a positive r as above, multiply by $\frac{N}{r}$ and integrate over $]0, r[$. We find

$$\begin{aligned} \frac{N}{r^N} \int_0^r \varrho^{N-1} u(z_0) d\varrho &= \frac{N}{r^N} \int_0^r \varrho^{N-1} \left(\int_{\psi_\varrho(z_0)} K(z_0; z) u(z) d\mathcal{H}^N(x, t) \right) d\varrho + \\ &\quad \frac{N}{r^N} \int_0^r \varrho^{N-1} \left(\int_{\Omega_r(z_0)} f(z) \left(\frac{1}{r^N} - \Gamma(z_0; z) \right) dx dt \right) d\varrho. \end{aligned} \quad (4.9)$$

The right hand side of the above inequality equals $u(z_0)$, while the last term agrees with the last term appearing in the statement of Theorem 1.1. In order to conclude the proof we only need to show that

$$\begin{aligned} \int_0^r \varrho^{N-1} \left(\int_{\left\{ \Gamma(z_0; z) = \frac{1}{\varrho^N} \right\}} K(z_0; z) u(z) d\mathcal{H}^N(z) \right) d\varrho &= \\ \frac{1}{N} \int_0^r \varrho^{N-1} \int_{\Omega_\varrho(z_0)} M(z_0; z) u(z) dz. \end{aligned} \quad (4.10)$$

In order to prove (4.9), we substitute $y = \frac{1}{\varrho^N}$ in the left hand side of (4.10) and we recall the definition of the kernel K . We find

$$\begin{aligned} & \int_0^r \varrho^{N-1} \left(\int_{\{\Gamma(z_0; z) = \frac{1}{\varrho^N}\}} \frac{\langle A(z) \nabla_x \Gamma(z_0; z), \nabla_x \Gamma(z_0; z) \rangle}{|\nabla_{(x,t)} \Gamma(z_0; z)|} u(z) d\mathcal{H}^N(z) \right) d\varrho \\ &= \frac{1}{N} = \int_{\frac{1}{r^N}}^{+\infty} \frac{1}{y^2} \left(\int_{\{\Gamma(z_0; z) = y\}} \frac{\langle A(z) \nabla_x \Gamma(z_0; z), \nabla_x \Gamma(z_0; z) \rangle}{|\nabla_{(x,t)} \Gamma(z_0; z)|} u(z) d\mathcal{H}^N(z) \right) dy \\ &= \frac{1}{N} \int_{\frac{1}{r^N}}^{+\infty} \left(\int_{\{\Gamma(z_0; z) = y\}} \frac{\langle A(z) \nabla_x \Gamma(z_0; z), \nabla_x \Gamma(z_0; z) \rangle}{\Gamma^2(z_0; z) |\nabla_{(x,t)} \Gamma(z_0; z)|} u(z) d\mathcal{H}^N(z) \right) dy. \end{aligned} \quad (4.11)$$

We conclude the proof of (4.10) by applying the coarea formula stated in Theorem 3.6. \square

PROOF. OF PROPOSITION 1.3. We first note that, as a direct consequence of our assumption $c = 0$, we have that $\mathcal{L} 1 = 0$, then Theorem 1.1 yields

$$\frac{1}{\varrho^N} \int_{\Omega_{\varrho}(z_1)} M(z_1; z) dz = 1, \quad \forall z_1 \in \Omega, \varrho > 0 \text{ such that } \overline{\Omega_{\varrho}(z_1)} \subset \Omega.$$

We claim that, if $u(z_1) = \max_{\Omega} u$, then

$$u(z) = u(z_1) \quad \text{for every } z \in \overline{\Omega_{\varrho}(z_1)}. \quad (4.12)$$

By using again Theorem 1.1 and the above identity we obtain

$$\begin{aligned} 0 &= \frac{1}{\varrho^N} \int_{\Omega_{\varrho}(z_1)} M(z_1; z) ((u(z) - u(z_1))) dz \\ &\quad + \frac{N}{\varrho^N} \int_0^{\varrho} \left(s^{N-1} \int_{\Omega_s(z_1)} f(z) \left(\frac{1}{s^N} - \Gamma(z_1; z) \right) dz \right) ds \leq 0, \end{aligned}$$

since $f \geq 0$ and $u(z) \leq u(z_1)$, being $u(z_1) = \max_{\Omega} u$. We have also used the fact that $M(z_1; z) \geq 0$ and $\Gamma(z_1; z) \geq \frac{1}{s^N}$ for every $z \in \Omega_s(z_1)$. Hence, $M(z_1; z)((u(z) - u(z_1))) = 0$ for \mathcal{H}^{N+1} almost every $z \in \Omega_{\varrho}(z_1)$. As already noticed, Dubovickii's theorem implies that $\mathcal{H}^N(\psi_s(z_1) \cap \text{Crit}(\Gamma)) = 0$, for almost every $s \in]0, \varrho]$, then $M(z_1; z) \neq 0$ for \mathcal{H}^{N+1} almost every $z \in \Omega_{\varrho}(z_1)$. As a consequence $u(z) = u(z_1)$ for \mathcal{H}^{N+1} almost every $z \in \Omega_{\varrho}(z_1)$, and the claim (4.12) follows from the continuity of u .

We are in position to conclude the proof of Proposition 1.3. Let z be a point of $\mathcal{A}_{z_0}(\Omega)$, and let $\gamma : [0, T] \rightarrow \Omega$ be an \mathcal{L} -admissible path such that $\gamma(0) = z_0$ and $\gamma(T) = z$. We will prove that $u(\gamma(t)) = u(z_0)$ for every $t \in [0, T]$. Let

$$I := \{t \in [0, T] \mid u(\gamma(s)) = u(z_0) \text{ for every } s \in [0, t]\}, \quad \bar{t} := \sup I.$$

Clearly, $I \neq \emptyset$ as $0 \in I$. Moreover I is closed, because of the continuity of u and γ , then $\bar{t} \in I$. We now prove by contradiction that $\bar{t} = T$.

Suppose that $\bar{t} < T$. Let $z_1 := \gamma(\bar{t})$ and note that $z_1 \in \Omega$, $u(z_1) = \max_{\Omega} u$. We aim to show that there exist positive constants r_1 and s_1 such that $\overline{\Omega_{r_1}(z_1)} \subset \Omega$ and

$$\gamma(\bar{t} + s) \in \Omega_{r_1}(z_1) \quad \text{for every } s \in [0, s_1]. \quad (4.13)$$

As a consequence of (4.12) we obtain $u(\gamma(\bar{t} + s)) = u(z_1) = u(z_0)$ for every $s \in [0, s_1]$, and this contradicts the definition of \bar{t} .

The proof of (4.13) is a consequence of Lemma 2.2. It is not restrictive to assume that $r_1 \leq r^*$, then it is sufficient to show that there exists a positive s_1 such that

$$\gamma(\bar{t} + s) \in \Omega_{r_1}^*(z_1) \quad \text{for every } s \in [0, s_1]. \quad (4.14)$$

Recall the definition of $\gamma(\bar{t} + s) = (x(\bar{t} + s), t(\bar{t} + s))$. We have $\gamma(\bar{t}) = z_1 = (x_1, t_1)$, $t(\bar{t} + s) = t_1 - s$ and, for every positive s

$$\begin{aligned} |x(s + \bar{t}) - x_1| &= \left| \int_0^s \dot{x}(\bar{t} + s) ds \right| \leq \int_0^s |\dot{x}(\bar{t} + s)| ds \\ &\leq \left(\int_0^s |\dot{x}(\bar{t} + s)|^2 ds \right)^{1/2} s^{1/2} \leq \|\dot{x}\|_{L^2([0, T])} \sqrt{s}, \end{aligned}$$

then

$$\langle A^{-1}(z_1)(x(\bar{t} + s) - x_1), x(\bar{t} + s) - x_1 \rangle \leq s \cdot \|A^{-1}(z_1)\| \cdot \|\dot{x}\|_{L^2([0, T])}^2.$$

By using the above inequality in (2.18) we see that there exists a positive constant s_1 such that (4.14) holds. This proves (4.13), and then $u(z) = u(z_0)$ for every $z \in \mathcal{A}_{z_0}(\Omega)$. By the continuity of u we conclude that $u(z) = u(z_0)$ for every $z \in \overline{\mathcal{A}_{z_0}(\Omega)}$. Eventually, since u is constant in $\overline{\mathcal{A}_{z_0}(\Omega)}$ and $c = 0$, we conclude that $\mathcal{L}u = 0$. \square

PROOF. OF PROPOSITION 1.4. Let m be a positive integer, and let u be a solution to $\mathcal{L}u = f$ in $\Omega \subset \mathbb{R}^{N+1}$. As said in the Introduction, we set

$$\tilde{u}(x, y, t) := u(x, t), \quad \tilde{f}(x, y, t) := f(x, t),$$

for every $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}$ such that $(x, t) \in \Omega$, and we note that

$$\widetilde{\mathcal{L}} \tilde{u}(x, y, t) = \tilde{f}(x, y, t) \quad \widetilde{\mathcal{L}} := \mathcal{L} + \sum_{j=1}^m \frac{\partial^2}{\partial^2 y_j}.$$

Moreover, the function

$$\tilde{\Gamma}(x_0, y_0, t_0; x, y, t) := \Gamma(x_0, t_0; x, t) \cdot \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\left(\frac{|y_0 - y|^2}{4(t_0 - t)}\right)$$

is a fundamental solution of $\widetilde{\mathcal{L}}$. We then use $\tilde{\Gamma}$ to represent the solution u in accordance with Theorem 1.1 as follows

$$\begin{aligned} u(z_0) = \tilde{u}(x_0, y_0, t_0) &= \frac{(4\pi)^{-m/2}}{r^{N+m}} \int_{\tilde{\Omega}_r(x_0, y_0, t_0)} \tilde{M}(x_0, y_0, t_0; x, y, t) u(x, t) dx dy dt + \\ &\quad \frac{N+m}{r^{N+m}} \int_0^r \left(\varrho^{N+m-1} \int_{\tilde{\Omega}_r(x_0, y_0, t_0)} f(x, t) \left(\frac{1}{(4\pi)^{m/2} \varrho^{N+m}} - \tilde{\Gamma}(x_0, y_0, t_0; x, y, t) \right) dx dy dt \right) d\varrho. \end{aligned}$$

where $\tilde{\Omega}_r(x_0, y_0, t_0)$ is the parabolic ball relevant to $\tilde{\Gamma}$ and

$$\tilde{M}(x_0, y_0, t_0; x, y, t) = M(x_0, t_0; x, t) + \frac{|y_0 - y|^2}{4(t_0 - t)^2}.$$

The proof is accomplished by integrating the above identity with respect to the variable y . \square

5 An approach relying on sets of finite perimeter

In this section we present another approach to the generalized divergence theorem, relying on De Giorgi's theory of perimeters, see [5, 6] or [1, 14], and we show how this leads to a slightly different proof of Theorem 1.1. This approach requires more prerequisites than that used in Section 3, but, as explained in the Introduction, is more flexible and avoids the Dubovickii theorem. In this section, if μ is a Borel measure and E is a Borel set, we use the notation $\mu \llcorner E(B) = \mu(E \cap B)$. As before, $C_c^1(\Omega)$ denotes the set of C^1 functions compactly supported in the open set $\Omega \subset \mathbb{R}^n$.

Definition 5.1 (FUNCTION OF BOUNDED VARIATION) *Let $u \in L^1(\Omega)$; we say that u is a function of bounded variation in Ω if its distributional derivative $Du = (D_1u, \dots, D_nu)$ is an \mathbb{R}^n -valued Radon measure in Ω , i.e., if*

$$\int_{\Omega} u \frac{\partial \varphi}{\partial z_i} dz = - \int_{\Omega} \varphi dD_iu, \quad \forall \varphi \in C_c^1(\Omega), \quad i = 1, \dots, n$$

or, in vectorial form,

$$\int_{\Omega} u \operatorname{div} \Phi dz = - \sum_{i=1}^n \int_{\Omega} \Phi_i dD_iu = - \int_{\Omega} \langle \Phi, Du \rangle, \quad \forall \Phi \in C_c^1(\Omega; \mathbb{R}^n). \quad (5.1)$$

The vector space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$. The variation $V(u, \Omega)$ of u in Ω is defined by:

$$V(u, \Omega) := \sup \left\{ \int_{\Omega} u \operatorname{div} \Phi dz : \Phi \in C_c^1(\Omega; \mathbb{R}^n), \|\Phi\|_{\infty} \leq 1 \right\}.$$

We recall that $V(u, \Omega) = |Du|(\Omega) < \infty$ for any $u \in BV(\Omega)$, where $|Du|$ denotes the total variation of the measure Du . We also recall that if $u \in C^1(\Omega)$ then

$$V(u, \Omega) = \int_{\Omega} |\nabla u| dz.$$

When the function u is the characteristic functions χ_E of some measurable set, its variation is said *perimeter of E* .

Definition 5.2 (SETS OF FINITE PERIMETER) *Let E be an \mathcal{L}^n -measurable subset of \mathbb{R}^n . For any open set $\Omega \subset \mathbb{R}^n$ the perimeter of E in Ω is denoted by $P(E, \Omega)$ and it is the variation of χ_E in Ω , i.e.,*

$$P(E, \Omega) := \sup \left\{ \int_{\Omega} \operatorname{div} \Phi dz : \Phi \in C_c^1(\Omega; \mathbb{R}^n), \|\Phi\|_{\infty} \leq 1 \right\}.$$

We say that E is a set of finite perimeter in Ω if $P(E, \Omega) < \infty$.

Obviously, several properties of the perimeter of E can be stated in terms of the variation of χ_E . In particular, if $\mathcal{L}^n(E \cap \Omega)$ is finite, then $\chi_E \in L^1(\Omega)$ and E has finite perimeter in Ω if and only if $\chi_E \in BV(\Omega)$ and $P(E, \Omega) = |D\chi_E|(\Omega)$. Both the notations $|D\chi_E|(B)$ and $P(E, B)$, B

Borel, are used to denote the total variation measure of χ_E on a Borel set B and we say that E is a set of locally finite perimeter in Ω if $P(E, K) < \infty$ for every compact set $K \subset \Omega$. Finally, formula (5.1) looks like a divergence theorem:

$$\int_E \operatorname{div} \Phi \, dz = - \int_{\Omega} \langle \Phi, D\chi_E \rangle, \quad \forall \Phi \in C_c^1(\Omega; \mathbb{R}^n), \quad (5.2)$$

but it becomes more readable if some precise information is given on the set where the measure $D\chi_E$ is concentrated. Therefore, we introduce the notions of *reduced boundary* and of *density* and recall the structure theorem for sets with finite perimeter due to E. De Giorgi, see [6] and [1, Theorem 3.59], and the characterization due to H. Federer.

Definition 5.3 (REDUCED BOUNDARY) *Let Ω be an open subset of \mathbb{R}^n and let E be a set of locally finite perimeter in Ω . We say that $z \in \Omega$ belongs to the reduced boundary $\mathcal{F}E$ of E if $|D\chi_E|(B_\varrho(z)) > 0$ for every $\varrho > 0$ and the limit*

$$\nu_E(z) := \lim_{\varrho \rightarrow 0^+} \frac{D\chi_E(B_\varrho(z))}{|D\chi_E|(B_\varrho(z))}$$

exists in \mathbb{R}^n and satisfies $|\nu_E(z)| = 1$. The function $\nu_E : \mathcal{F}E \rightarrow \mathbb{S}^{n-1}$ is Borel continuous and it is called the generalized (or measure-theoretic) inner normal to E .

Notice that the reduced boundary is a subset of the topological boundary. The Besicovitch differentiation theorem, see e.g. [1, Theorem 2.22], yields $D\chi_E = \nu_E |D\chi_E|$, and $|D\chi_E|(\Omega \setminus \mathcal{F}E) = 0$, hence (5.2) becomes

$$\int_E \operatorname{div} \Phi \, dz = - \int_{\mathcal{F}E} \langle \nu_E, \Phi \rangle \, d|D\chi_E|, \quad \forall \Phi \in C_c^1(\Omega; \mathbb{R}^N). \quad (5.3)$$

The relation between the topological boundary and the reduced boundary can be further analyzed by introducing the notion of *density* of a set at a given point.

Definition 5.4 (POINTS OF DENSITY α) *For every $\alpha \in [0, 1]$ and every \mathcal{L}^n -measurable set $E \subset \mathbb{R}^n$ we denote by $E^{(\alpha)}$ the set*

$$E^{(\alpha)} = \left\{ z \in \mathbb{R}^n : \lim_{\varrho \rightarrow 0^+} \frac{\mathcal{L}^n(E \cap B_\varrho(z))}{\mathcal{L}^n(B_\varrho(z))} = \alpha \right\}.$$

Thus $E^{(\alpha)}$, which turns out to be a Borel set, is the set of all points where E has density α . The sets $E^{(0)}$ and $E^{(1)}$ are called *the measure-theoretic exterior* and *interior* of E and, in general, strictly contain the topological exterior and interior of the set E , respectively. We recall the well known *Lebesgue's density theorem*, that asserts that for every \mathcal{L}^n -measurable set $E \subset \mathbb{R}^n$

$$\mathcal{L}^n(E \triangle E^{(1)}) = 0, \quad \mathcal{L}^n((\mathbb{R}^n \setminus E) \triangle E^{(0)}) = 0,$$

i.e., the density of E is 0 or 1 at \mathcal{L}^n -almost every point in \mathbb{R}^n . This notion allows to introduce the *essential* or *measure-theoretic* boundary of E as $\partial^* E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$, which is contained in the topological boundary and contains the reduced boundary. Finally, the De Giorgi structure

theorem says that $|DX_E| = \mathcal{H}^{n-1} \llcorner \mathcal{F}E$ and a deep result due to Federer (see [9, 4.5.6] or [1, Theorem 3.61]) states that if E has finite perimeter in \mathbb{R}^n then

$$\mathcal{F}E \subset E^{(1/2)} \subset \partial^* E \quad \text{and} \quad \mathcal{H}^{n-1}(\mathbb{R}^n \setminus (E^{(0)} \cup \mathcal{F}E \cup E^{(1)})) = 0$$

hence, in particular, ν_E is defined \mathcal{H}^{n-1} -a.e. in $\partial^* E$. Notice also (see [1, Theorem 3.62]) that if $\mathcal{H}^{n-1}(\partial E) < \infty$ then E has finite perimeter. The results of De Giorgi and Federer imply that if E is a set of finite perimeter in Ω then $DX_E = \nu_E \mathcal{H}^{n-1} \llcorner \mathcal{F}E$ and the divergence theorem (5.3) can be rewritten in the form:

$$\int_E \operatorname{div} \Phi \, dz = - \int_{\mathcal{F}E} \langle \nu_E, \Phi \rangle \, d\mathcal{H}^{n-1} = - \int_{\partial^* E} \langle \nu_E, \Phi \rangle \, d\mathcal{H}^{n-1}, \quad \forall \Phi \in C_c^1(\Omega; \mathbb{R}^n), \quad (5.4)$$

much closer to the classical formula (3.1). Indeed, the only difference is that the inner normal and the boundary are understood in a measure-theoretic sense and not in the topological one; in particular, for a generic set of finite perimeter, $\mathcal{F}E$ needs not to be closed and ν_E needs not to be continuous. Moreover, ν_E is defined \mathcal{H}^{n-1} -a.e. in $\partial^* E$.

Let us see now how we can rephrase the results of Section 3 in terms of perimeters and how we can modify the proof of Theorem 1.1. We first recall the Fleming–Rischel formula (see [10] or [1, Theorem 3.40]), *i.e.*, the coarea formula for BV functions.

Theorem 5.5 (COAREA FORMULA IN BV) *For any open set $\Omega \subset \mathbb{R}^n$ and $G \in L_{\text{loc}}^1(\Omega)$ one has*

$$V(G, \Omega) = \int_{\mathbb{R}} P(\{z \in \Omega : G(z) > y\}, \Omega) \, dy.$$

In particular, if $G \in BV(\Omega)$ the set $\{G > y\}$ has finite perimeter in Ω for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$ and

$$|DG|(B) = \int_{\mathbb{R}} |DX_{\{G>y\}}|(B) \, dy, \quad DG(B) = \int_{\mathbb{R}} DX_{\{G>y\}}(B) \, dy, \quad \forall B \in \mathcal{B}(\Omega).$$

Now we are ready to state the analogue of Proposition 3.4 and to prove Theorem 1.1 again.

Proposition 5.6 *Let Ω be an open subset of \mathbb{R}^n and let $F \in BV(\Omega; \mathbb{R}) \cap C(\Omega; \mathbb{R})$. Then, for \mathcal{H}^1 -almost every $y \in \mathbb{R}$, we have:*

$$\int_{\{F>y\}} \operatorname{div} \Phi \, dz = - \int_{\partial^* \{F>y\}} \langle \nu, \Phi \rangle \, d\mathcal{H}^{n-1}, \quad \forall \Phi \in C_c^1(\Omega; \mathbb{R}^n), \quad (5.5)$$

where ν is the generalized inner normal to $\{F > y\}$.

PROOF. By Theorem 5.5, the set $\{F > y\}$ has finite perimeter in Ω for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$, hence we may apply (5.4) with $E = \{F > y\}$ and conclude. \square

As in Section 3, we have to cut the integration domain: therefore, we study the intersection between the super-level set of a generic function $G \in BV(\Omega; \mathbb{R}) \cap C(\Omega; \mathbb{R})$ and a half-space $H_t = \{x \in \mathbb{R}^n : \langle x, e \rangle < t\}$, for some $e \in \mathbb{S}^{n-1}$, $t \in \mathbb{R}$. First, we present a general formula that characterizes the intersection of two sets of finite perimeter for which we refer to Maggi's book, see [14, Theorem 16.3].

Theorem 5.7 (INTERSECTION OF SETS OF FINITE PERIMETER) *If A and B are sets of locally finite perimeter in Ω , and we let*

$$\{\nu_A = \nu_B\} = \{x \in \mathcal{F}A \cap \mathcal{F}B : \nu_A(x) = \nu_B(x)\},$$

then $A \cap B$ is a set of locally finite perimeter in Ω , with

$$D\chi_{A \cap B} = D\chi_A \llcorner B^{(1)} + D\chi_B \llcorner A^{(1)} + \nu_A \mathcal{H}^{n-1} \llcorner \{\nu_A = \nu_B\}. \quad (5.6)$$

In the case in which B is a half-space, formula (5.6) can be greatly simplified; indeed we can prove the following corollary.

Corollary 5.8 (INTERSECTIONS WITH A HALF-SPACE) *Let E be a set of locally finite perimeter in Ω and let $H_t = \{z \in \mathbb{R}^n : \langle z, e \rangle < t\}$ for some $e \in \mathbb{S}^{n-1}$, $t \in \mathbb{R}$. Then, for every $t \in \mathbb{R}$, $E \cap H_t$ is a set of locally finite perimeter in Ω and moreover, for \mathcal{H}^1 -almost every $t \in \mathbb{R}$,*

$$D\chi_{E \cap H_t} = D\chi_E \llcorner H_t - e \mathcal{H}^{n-1} \llcorner (E \cap \{\langle x, e \rangle = t\}).$$

PROOF. The half-space H_t is clearly a set of locally finite perimeter in Ω for every $t \in \mathbb{R}$, and for every $t \in \mathbb{R}$ we have, $H_t^{(1)} = H_t$, $\mathcal{F}H_t = \partial H_t = \{\langle x, e \rangle = t\}$ and $\nu_{H_t} \equiv -e$. Then, applying Theorem 5.7 we see that $E \cap H_t$ is a set of locally finite perimeter in Ω for every $t \in \mathbb{R}$ and (5.6) reads

$$D\chi_{E \cap H_t} = D\chi_E \llcorner H_t - e \mathcal{H}^{n-1} \llcorner (E^{(1)} \cup \{\nu_E = \nu_{H_t}\}).$$

Since by Fubini theorem

$$0 = \mathcal{L}^n(E \Delta E^{(1)}) = \int_{\mathbb{R}} \mathcal{H}^{n-1} \left((E \Delta E^{(1)}) \cap \{\langle x, e \rangle = t\} \right) dt,$$

for \mathcal{H}^1 -a.e. $t \in \mathbb{R}$ we have

$$\mathcal{H}^{n-1} \left((E \Delta E^{(1)}) \cap \{\langle x, e \rangle = t\} \right) = 0.$$

Therefore,

$$D\chi_{H_t} \llcorner E = D\chi_{H_t} \llcorner E^{(1)} = -e \mathcal{H}^{n-1} \llcorner (E \cap \{\langle x, e \rangle = t\}) = -e \mathcal{H}^{n-1} \llcorner (E \cap \{\nu_E = \nu_{H_t}\})$$

for \mathcal{H}^1 -a.e. $t \in \mathbb{R}$ and the thesis follows. \square

The following corollary allows us to perform (with some modifications) the last part of the proof of our main result.

Corollary 5.9 *Let $\Omega = \mathbb{R}^{N+1} \setminus \{(z_0)\}$, $G \in BV(\Omega; \mathbb{R}) \cap C(\Omega; \mathbb{R})$, $H_t = \{z \in \mathbb{R}^{N+1} : \langle z, e \rangle < t\}$ for some $e \in \mathbb{S}^N$, $t \in \mathbb{R}$. Then, for \mathcal{H}^1 -almost every $w \in \mathbb{R}$ and for every $t < t_0$ the set $E \cap H_t$ has locally finite perimeter in Ω and*

$$\int_{\{G > w\} \cap H_t} \operatorname{div} \Phi \, dz = - \int_{\partial^* \{G > w\} \cap H_t} \langle \nu, \Phi \rangle \, d\mathcal{H}^N + \int_{\{G > w\} \cap \{\langle x, e \rangle = t\}} \langle e, \Phi \rangle \, d\mathcal{H}^N,$$

for every $\Phi \in C_c^1(\Omega; \mathbb{R}^N)$, where ν is the generalized inner normal to $\partial^(\{G > w\})$. In particular, if $e = (0, \dots, 0, 1)$, for every $\varepsilon > 0$*

$$\int_{\{G > w\} \cap \{t < t_0 - \varepsilon\}} \operatorname{div} \Phi \, dz = - \int_{\partial^* \{G > w\} \cap \{t < t_0 - \varepsilon\}} \langle \nu, \Phi \rangle \, d\mathcal{H}^N + \int_{\{G > w\} \cap \{t = t_0 - \varepsilon\}} \langle e, \Phi \rangle \, d\mathcal{H}^N.$$

Notice that the difference between Proposition 3.5 and Corollary 5.9 is that in the former we can exclude the set of critical points of G from the surface integral, thanks to Dubovickii theorem, and we know that ν is given by the normalized gradient of G *everywhere* in the integration set, whereas in the latter we don't need to know any estimate on the size of $\text{Crit}(G)$ and ν is defined \mathcal{H}^N -a.e. on the integration set (still coinciding with the normalized gradient of G out of $\text{Crit}(G)$, of course). First, notice that we apply Corollary 5.9 to $G(z) = \Gamma(z_0; z)$, which is $C^1(\Omega)$, hence Lipschitz on bounded sets. As a consequence, $\partial\{G > w\} \subseteq \{G = w\}$ and comparing the coarea formulas (3.4) and (5.5), we deduce that $\mathcal{H}^N(\{G = w\} \setminus \partial^*\{G > w\}) = 0$ for \mathcal{H}^1 -a.e. w . Let us see how this entails modifications of the proof of Theorem 1.1: the proof goes in the same vein until (4.7), (4.8), which in the present context are replaced by

$$\begin{aligned} \lim_{k \rightarrow +\infty} \int_{\psi_r(z_0) \cap \{t < t_0 - \varepsilon_k\}} \langle \nu, \Phi \rangle d\mathcal{H}^N &= \int_{\psi_r(z_0)} K(z_0; z) u(z) d\mathcal{H}^N \\ &= \int_{\psi_r(z_0) \setminus \text{Crit}(\Gamma)} K(z_0; z) u(z) d\mathcal{H}^N \end{aligned}$$

where the first equality follows from Corollary 5.9, as explained, and the last equality follows from the fact that the kernel K vanishes in $\text{Crit}(\Gamma)$. The rest of the proof needs no modifications.

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