

VARIATIONAL ANALYSIS OF THE J_1 - J_2 - J_3 MODEL: A NON-LINEAR LATTICE VERSION OF THE AVILES-GIGA FUNCTIONAL

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ABSTRACT. We study the variational limit of the frustrated J_1 - J_2 - J_3 spin model on the square lattice in the vicinity of the ferromagnet/helimagnet transition point as the lattice spacing vanishes. We carry out the Γ -convergence analysis of proper scalings of the energy and we characterize the optimal cost of a chirality transition in BV proving that the system is asymptotically driven by a discrete version of a non-linear perturbation of the Aviles-Giga energy functional.

Keywords: Γ -convergence, frustrated spin systems, chirality transitions, Aviles-Giga functional, eikonal equation

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1. INTRODUCTION

Low-energy states of two-dimensional magnetic compounds feature a large variety of complex magnetic patterns. The emergence of some of these structures is usually the result of a number of competing interactions whose relative weight may drastically change with the length scale. From the physical point of view the resulting unconventional magnetic order often corresponds to a rich phase diagram. The experimental community has recently made great progresses in unveiling critical properties of such phase diagrams. Besides, in the statistical mechanics community there has been a quest for elementary lattice spin models that would reproduce some of the most surprising geometric patterns of low-energy states introducing a minimal number of parameters in the model (see [27] and the references therein for a recent overview on this topic). One of the key features of such energetic models is the frustration mechanism, that is, roughly speaking, the presence of conflicting interatomic forces that prevent the energy of every pair of interacting spins to be simultaneously minimized. In the recent years, several examples of frustrated spin models have been investigated from a variational perspective, *cf.* [1, 29, 30, 21, 31, 15, 17, 23, 9, 10]. As these examples show, the presence of frustration in a lattice spin system depends on both the topological properties of the lattice and the symmetry properties of the interaction potentials.

In this paper we are going to investigate a model in which frustration originates from the competition of ferromagnetic (F) and antiferromagnetic (AF) interactions. This model is known as the J_1 - J_2 - J_3 F-AF classical spin model on the square lattice (see, *e.g.*, [49]). To

each configuration of two-dimensional unitary spins on the square lattice, namely $u: \mathbb{Z}^2 \rightarrow \mathbb{S}^1$, we associate the energy

$$E(u) = -J_1 \sum_{|\sigma-\sigma'|=1} u^\sigma \cdot u^{\sigma'} + J_2 \sum_{|\sigma-\sigma''|=\sqrt{2}} u^\sigma \cdot u^{\sigma''} + J_3 \sum_{|\sigma-\sigma'''|=2} u^\sigma \cdot u^{\sigma'''}$$

where J_1 , J_2 , and J_3 are positive constants (the interaction parameters of the model) and for every lattice point $\sigma \in \mathbb{Z}^2$ we let u^σ denote the value of the spin variable u at σ . The energy consists of the sum of three terms. The first is ferromagnetic as it favors aligned nearest-neighboring spins, whereas the second and the third one are antiferromagnetic as they favor antipodal second-neighboring and third-neighboring spins, respectively.

In the case where $J_2 = J_3 = 0$ the energy above describes the so-called XY model, a ferromagnetic model which can be considered a lattice version of the Ginzburg-Landau model for type II superconductors. The latter is an energy functional which has drawn the attention of the mathematical community since several decades (see, *e.g.*, [12, 52] and the references therein) and which shares with the XY functional many similarities as pointed out in [3]. The variational analysis of the XY model has been carried out in [2] also in connection to the theory of dislocations [48, 4]. We also mention the more recent results in [16] on a variant of the XY model on a non-flat lattice and the results in [20, 18, 19] regarding its connections with the N -clock model.

In the case $J_2 = 0$ and $J_3 > 0$, E becomes the energy of the J_1 - J_3 model considered in [17]. In that paper, it has been shown that the ferromagnetic and the antiferromagnetic terms in E compete and give rise to ground states in the form of helices of possibly different chiralities (for recent experimental evidences on helical ground states of the J_1 - J_3 and of the J_1 - J_2 - J_3 models see, *e.g.*, [53, 55]). Referring the energy E to that of such helimagnetic ground states, one can then investigate the energetic behavior of low-energy spin configurations in a bounded domain as the lattice spacing vanishes. In terms of Γ -convergence, one can prove the existence of a specific energy scaling at which chirality transitions take place and describe the energetic behavior of the system in terms of an effective macroscopic energy which gives the cost of such chirality transitions. The goal of this paper is to follow a similar approach in the complete J_1 - J_2 - J_3 model. We explain this approach below in more details.

To study the asymptotic variational limit of the energy E as the number of particles diverges, we consider the sequence of energies E_n obtained as follows: We fix a bounded open set $\Omega \subset \mathbb{R}^2$ and we scale the lattice spacing by a small parameter $\lambda_n > 0$. Given $u: \lambda_n \mathbb{Z}^2 \cap \Omega \rightarrow \mathbb{S}^1$, writing $\sigma \in \mathbb{Z}^2$ in components as (i, j) , and letting $u^{i,j}$ denote the value of u at $(\lambda_n i, \lambda_n j)$, the energy per particle in Ω reduces to the sequence of energies

$$\begin{aligned} E_n(u) := & -\alpha \lambda_n^2 \sum_{(i,j)} \left(u^{i,j} \cdot u^{i+1,j} + u^{i,j} \cdot u^{i,j+1} \right) + \beta \lambda_n^2 \sum_{(i,j)} \left(u^{i,j} \cdot u^{i+1,j+1} + u^{i,j} \cdot u^{i-1,j+1} \right) \\ & + \lambda_n^2 \sum_{(i,j)} \left(u^{i,j} \cdot u^{i+2,j} + u^{i,j} \cdot u^{i,j+2} \right), \end{aligned} \tag{1.1}$$

where $\alpha = J_1/J_3$, $\beta = J_2/J_3$, and the sums are taken over all those $(i, j) \in \mathbb{Z}^2$ for which all evaluations of u above are defined.

We are interested in the case where the parameters α and β depend on the lattice spacing λ_n , hence we write $\alpha = \alpha_n$ and $\beta = \beta_n$. We focus on the range $0 \leq \beta_n \leq 2$ and we note that, depending on the parameter α_n , the ground states of the system are either ferromagnetic or helimagnetic as depicted in the phase diagram reported in Figure 2 (*cf.* also [49, Figure 2]). To explain the emergence of the different types of ground states, it is convenient to rewrite the energy $E_n(u)$ (up to an additive constant and neglecting error

terms at the boundary of Ω) as

$$\begin{aligned}
 F_n(u) := & \frac{\beta_n}{4} \lambda_n^2 \sum_{(i,j)} \left| u^{i+1,j} - \frac{\alpha_n}{\beta_n + 2} u^{i,j} + u^{i-1,j} + u^{i,j+1} - \frac{\alpha_n}{\beta_n + 2} u^{i,j} + u^{i,j-1} \right|^2 \\
 & + \frac{2 - \beta_n}{4} \lambda_n^2 \sum_{(i,j)} \left| u^{i+1,j} - \frac{\alpha_n}{\beta_n + 2} u^{i,j} + u^{i-1,j} \right|^2 + \left| u^{i,j+1} - \frac{\alpha_n}{\beta_n + 2} u^{i,j} + u^{i,j-1} \right|^2.
 \end{aligned} \tag{1.2}$$

We refer to Subsection 2.5 for the details. If the ferromagnetic nearest-neighbor interaction parameter α_n is large enough, one expects the ferromagnetic order to dominate, leading to ground states made of parallel spins $u \equiv \text{const.} \in \mathbb{S}^1$. The range of all α_n leading to this behavior is characterized by the inequality $\frac{\alpha_n}{\beta_n + 2} \geq 2$, which can be explained by the following simple heuristic argument. One starts by observing that, for $\frac{\alpha_n}{\beta_n + 2} = 2$, ferromagnetic states are the only spin configurations which make F_n zero. As a consequence, since larger values of $\frac{\alpha_n}{\beta_n + 2}$ increase the weight of the ferromagnetic interactions versus the antiferromagnetic interactions even more, ferromagnetic ground states should appear also for $\frac{\alpha_n}{\beta_n + 2} > 2$. A rigorous proof of this argument is based on a simple comparison argument already used in the one-dimensional case investigated in [21] and that can be repeated in the present case verbatim. If instead $\frac{\alpha_n}{\beta_n + 2} < 2$, the ground states have a different geometry. If $\beta_n < 2$, they are completely characterized by the requirement that all the squares in (1.2) are zero. This can be achieved only by choosing a helical spin field $u: \lambda_n \mathbb{Z}^2 \cap \Omega \rightarrow \mathbb{S}^1$ such that

$$u^{i,j} = (\cos(\theta_0 + i\theta^{\text{hor}} + j\theta^{\text{ver}}), \sin(\theta_0 + i\theta^{\text{hor}} + j\theta^{\text{ver}})), \tag{1.3}$$

where $\theta^{\text{hor}}, \theta^{\text{ver}} \in \{\pm \arccos(\frac{\alpha_n}{2(\beta_n + 2)})\}$ and $\theta_0 \in [0, 2\pi)$. Indeed, for such a spin field we have that

$$u^{i+1,j} + u^{i-1,j} = u^{i,j+1} + u^{i,j-1} = \frac{\alpha_n}{\beta_n + 2} u^{i,j}.$$

The four possible families of ground states obtained by choosing the signs of θ^{hor} and θ^{ver} correspond to left-handed or right-handed helices directed along the lattice rows or columns, respectively. A concise description of this discrete ground state degeneracy is made possible by introducing the notion of chirality vector χ . Roughly speaking, χ represents the direction along which the helical configuration is rotating most and is given by

$$\chi \simeq \frac{1}{\sqrt{2} \arccos(\frac{\alpha_n}{2(\beta_n + 2)})} (\theta^{\text{hor}}, \theta^{\text{ver}}), \tag{1.4}$$

i.e., by normalizing the vector $(\theta^{\text{hor}}, \theta^{\text{ver}})$ of the angles between horizontally and vertically adjacent spins¹. According to this definition, the four families of ground states in the regime $\frac{\alpha_n}{\beta_n + 2} < 2$, $\beta_n < 2$, correspond to χ taking one of the four values

$$\frac{1}{\sqrt{2}}(+1, +1), \frac{1}{\sqrt{2}}(+1, -1), \frac{1}{\sqrt{2}}(-1, +1), \frac{1}{\sqrt{2}}(-1, -1) \tag{1.5}$$

(see, *e.g.*, the second picture in Figure 1 for an illustration of the value $\frac{1}{\sqrt{2}}(-1, +1)$). When $\beta_n = 2$ and $\frac{\alpha_n}{\beta_n + 2} < 2$, ground states only need to satisfy the weaker condition

$$u^{i+1,j} + u^{i-1,j} + u^{i,j+1} + u^{i,j-1} = \frac{2\alpha_n}{\beta_n + 2} u^{i,j} = \frac{\alpha_n}{2} u^{i,j}.$$

This can be achieved by helical fields as in (1.3) with $\theta^{\text{hor}}, \theta^{\text{ver}}$ satisfying the relation $\cos(\theta^{\text{hor}}) + \cos(\theta^{\text{ver}}) = \frac{\alpha_n}{4}$. The latter condition is equivalent to requiring the chirality vector to have unitary length, namely $\chi \in \mathbb{S}^1$. Figure 1 shows the helical ground state u corresponding to different choices of $\chi \in \mathbb{S}^1$.

¹Notice that in the sequel it will be convenient to use a non-linear variant of (1.4) to define the chirality vector χ , *cf.* (2.8).

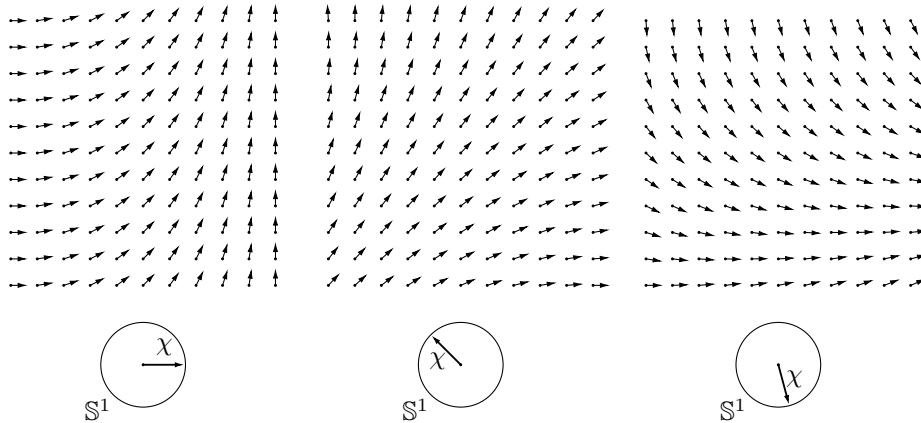


Figure 1. Three examples of ground states of the J_1 - J_2 - J_3 model. The three ground states are distinguished by different chirality vectors that set the speed of rotation of the spin in the horizontal and vertical direction. The chirality vector can be any direction in \mathbb{S}^1 .

In this paper we investigate the chirality properties of spin fields with low J_1 - J_2 - J_3 energy for a choice of parameters corresponding to spin configurations close to the helimagnet-ferromagnet transition point. This is equivalent to assuming that $0 \leq \beta_n \leq 2$ and that $2(\beta_n + 2) - \alpha_n \searrow 0$. Within this range of parameters, the asymptotic behavior of (an appropriate scaling of) F_n is established by rewriting the energy in terms of a microscopic notion of chirality that we associate to any admissible spin configuration. Such a chirality (still denoted by) χ will then be a discrete vector field defined on $\lambda_n \mathbb{Z}^2 \cap \Omega$, the order parameter of the system.

In the case $\beta_n \equiv 0$, this program has already been carried out in [17]. In that paper, it has been proved that transitions in the chirality parameter χ cost an energy of order $(4 - \alpha_n)^{3/2} \lambda_n$. Moreover, expressed in terms of $\chi = (\chi_1, \chi_2)$, the accordingly scaled energies $((4 - \alpha_n)^{-3/2} \lambda_n^{-1}) F_n$ behave like a functional of the form

$$\frac{1}{2} \int \frac{1}{\varepsilon_n} \left(\left| \frac{1}{2} - |\chi_1|^2 \right|^2 + \left| \frac{1}{2} - |\chi_2|^2 \right|^2 \right) + \varepsilon_n (|\partial_1 \chi_1|^2 + |\partial_2 \chi_2|^2) dx,$$

where $\varepsilon_n \simeq (4 - \alpha_n)^{-\frac{1}{2}} \lambda_n \rightarrow 0$. In addition, the crucial observation that χ is forced to be approximately a curl-free vector field, say $\chi \simeq \nabla \varphi$, has made possible to recognize the functional above as a Modica-Mortola type functional written in the gradient variable $\nabla \varphi$. This functional features a four-well potential, whose zeros correspond to the four possible chiralities of the ground states mentioned in (1.5). Exploiting these observations it has been proved that the Γ -limit of $((4 - \alpha_n)^{-3/2} \lambda_n^{-1}) F_n$ is finite on $BV(\Omega; \{ \pm \frac{1}{\sqrt{2}} \}^2)$ chiralities with vanishing curl and takes the form of an interfacial energy between regions with different constant chiralities.

It can be observed that the full J_1 - J_2 - J_3 model shares similarities with the J_1 - J_3 model mentioned above, if $\sup_n \beta_n < 2$, see Remark 4.6 below. (This is related to the fact that, as in the J_1 - J_3 model, ground states of the J_1 - J_2 - J_3 energy can only have one of the four possible chiralities in (1.5) for all $\beta_n < 2$.) If, instead, $\beta_n \rightarrow 2$, the behavior of the J_1 - J_2 - J_3 system can be substantially different. To single out the new features of the model, in this paper we consider the extreme case $\beta_n \equiv 2$. In Remark 4.6 we explain how to obtain a satisfactory description of the model in more general cases by combining the analysis of the case $\beta_n \equiv 0$ examined in [17] with the results in the case $\beta_n \equiv 2$. With this particular choice of β_n , the helimagnet-ferromagnet transition point we are interested in corresponds to $\alpha_n \nearrow 8$.

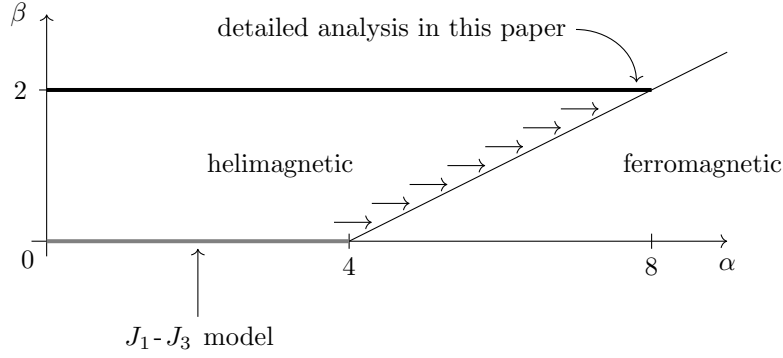


Figure 2. A schematic representation of the case studied in this paper. For $\beta \in [0, 2]$, the line $\beta = \frac{\alpha-4}{2}$ separates the cases where the ground states are helimagnetic and ferromagnetic. We are interested in helimagnet/ferromagnet transitions, *i.e.*, in the case where the values (α, β) approach the aforementioned line. The boundary case $\beta \equiv 0$ corresponds to the so-called J_1 - J_3 model, whose variational analysis at the helimagnet/ferromagnet transition point $\alpha \rightarrow 4$ has been carried out in [17]. In this paper, we examine in detail the opposite boundary case $\beta \equiv 2$ when α approaches the value 8. The main features of the in-between cases $\beta \in (0, 2)$ can be obtained by combining the behaviors in the two extreme cases, see Remark 4.6.

Our analysis of the case $\beta_n \equiv 2$ is made possible by the key observation that, written in terms of χ , suitable rescalings of F_n resemble a discrete version of the Aviles-Giga functional. In the following we present a heuristic computation which motivates such an analogy, referring to Subsection 2.6 for a more rigorous derivation. Let us introduce the small parameter $\delta_n := 4 - \frac{\alpha_n}{2}$ which we will also use throughout the paper. Roughly speaking, an angular lifting ψ such that $u = (\cos \psi, \sin \psi)$ is related to the angles θ^{hor} and θ^{ver} between horizontally and vertically neighboring spins *via* $(\theta^{\text{hor}}, \theta^{\text{ver}}) \simeq \lambda_n \nabla \psi$. According to that, in view of (1.4) (for $\beta_n = 2$), we can write

$$\chi \simeq \frac{\lambda_n}{\sqrt{2} \arccos(1 - \frac{\delta_n}{4})} \nabla \psi \simeq \frac{\lambda_n}{\sqrt{2} \sqrt{\frac{\delta_n}{2}}} \nabla \psi = \nabla \varphi,$$

where we have set $\varphi := \frac{\lambda_n}{\sqrt{\delta_n}} \psi$. To rewrite F_n in terms of χ , for λ_n small enough, we may write $(u^{i+1,j} - 2u^{i,j} + u^{i-1,j})/\lambda_n^2 \simeq \partial_{11} u$ and $(u^{i,j+1} - 2u^{i,j} + u^{i,j-1})/\lambda_n^2 \simeq \partial_{22} u$. Therefore,

$$\begin{aligned} F_n(u) &= \frac{1}{2} \lambda_n^2 \sum_{(i,j)} \left| u^{i+1,j} + u^{i-1,j} + u^{i,j+1} + u^{i,j-1} - \frac{\alpha_n}{2} u^{i,j} \right|^2 \\ &= \frac{1}{2} \lambda_n^2 \sum_{(i,j)} \left| \delta_n u^{i,j} + \lambda_n^2 \frac{u^{i+1,j} - 2u^{i,j} + u^{i-1,j}}{\lambda_n^2} + \lambda_n^2 \frac{u^{i,j+1} - 2u^{i,j} + u^{i,j-1}}{\lambda_n^2} \right|^2 \\ &\simeq \frac{1}{2} \int \delta_n^2 + 2\lambda_n^2 \delta_n u \cdot (\partial_{11} u + \partial_{22} u) + \lambda_n^4 |\partial_{11} u + \partial_{22} u|^2 dx \\ &= \frac{1}{2} \int \delta_n^2 + 2\lambda_n^2 \delta_n u \cdot \Delta u + \lambda_n^4 |\Delta u|^2 dx. \end{aligned}$$

We observe that $u \cdot \Delta u = -|\nabla \psi|^2$ and $|\Delta u|^2 = |\nabla \psi|^4 + |\Delta \psi|^2$. As a consequence, the above integral reads

$$\frac{1}{2} \int \delta_n^2 - 2\lambda_n^2 \delta_n |\nabla \psi|^2 + \lambda_n^4 |\nabla \psi|^4 + \lambda_n^4 |\Delta \psi|^2 dx = \frac{1}{2} \int |\delta_n - \lambda_n^2 |\nabla \psi|^2|^2 + \lambda_n^4 |\Delta \psi|^2 dx.$$

Thus,

$$F_n(u) \simeq \frac{1}{2} \int \delta_n^2 |1 - |\nabla \varphi|^2|^2 + \lambda_n^2 \delta_n |\Delta \varphi|^2 dx = \delta_n^{3/2} \lambda_n \frac{1}{2} \int \frac{1}{\varepsilon_n} |1 - |\nabla \varphi|^2|^2 + \varepsilon_n |\Delta \varphi|^2 dx,$$

where we have set $\varepsilon_n = \frac{\lambda_n}{\sqrt{\delta_n}}$. To make these computations rigorous, in Subsection 2.6 we introduce the functionals $H_n(\chi, \Omega) \simeq \frac{1}{\delta_n^{3/2} \lambda_n} F_n(u)$. These resemble a discretization of the

functionals

$$AG_{\varepsilon_n}^{\Delta}(\varphi, \Omega) := \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} |1 - |\nabla\varphi|^2|^2 + \varepsilon_n |\Delta\varphi|^2 dx, \quad (1.6)$$

where $\chi \simeq \nabla\varphi$. The latter are variants of the classical Aviles-Giga functionals

$$AG_{\varepsilon_n}(\varphi, \Omega) := \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} |1 - |\nabla\varphi|^2|^2 + \varepsilon_n |\nabla^2\varphi|^2 dx \quad (1.7)$$

and share with them most of their properties related to their Γ -convergence as $\varepsilon_n \rightarrow 0$. We will study the asymptotic properties of the functionals H_n for $\lambda_n \ll \sqrt{\delta_n}$, the regime which corresponds to $\varepsilon_n \rightarrow 0$.

The sequence of Aviles-Giga functionals has been introduced by Aviles and Giga [7] and Gioia and Ortiz [45] to study smectic liquid crystals and blistering in thin films. Although similar in form to the sequence of Ginzburg-Landau functionals, its asymptotic behavior as $\varepsilon \rightarrow 0$ is completely different due to the curl-free constraint on the vector field $\nabla\varphi$. In [7] it has been conjectured that the Γ -limit as $\varepsilon \rightarrow 0$ of AG_{ε} is a functional finite on functions $\varphi \in W^{1,\infty}(\Omega)$ solving the eikonal equation

$$|\nabla\varphi| = 1 \text{ a.e. in } \Omega \quad (1.8)$$

and charges jumps of the gradient field $\nabla\varphi$. The analysis of one-dimensional transition profiles suggests that the Γ -limit behaves as the defect energy

$$\frac{1}{6} \int_{J_{\nabla\varphi}} |[\nabla\varphi]|^3 d\mathcal{H}^1, \quad (1.9)$$

where $J_{\nabla\varphi}$ is the jump set of $\nabla\varphi$, $[\nabla\varphi](x)$ is the jump of $\nabla\varphi$ at $x \in J_{\nabla\varphi}$, and \mathcal{H}^1 is the one-dimensional Hausdorff measure.

If one assumes that φ belongs to the set of functions solving (1.8) and such that $\nabla\varphi \in BV(\Omega)$, then it has been proved (cf. [34, 8, 5, 22, 46]) that AG_{ε} Γ -converge with respect to the $W^{1,1}(\Omega)$ topology at φ to (1.9). However, in [5, 22] it is observed that this set is only strictly contained in the domain of the Γ -limit of AG_{ε} . To identify the asymptotic admissible set, one can exploit the conservation law structure of the eikonal equation (1.8). In particular, suitable notions of entropies (see Remark 3.4 for a short overview) have been exploited to prove compactness properties of the functionals AG_{ε} (cf. [5, 26], see also [33] for an approach *via* the kinetic formulation). Entropies have also been used to define an asymptotic lower bound on the family of functionals $AG_{\varepsilon}(\cdot, \Omega)$, cf. Remark 3.5. In Section 3 we introduce the functional H , defined in (3.5), which is obtained by taking the supremum of entropy productions over a suitable class of entropies given in Definition 3.1 subject to a normalization constraint. The functional H satisfies the lower bound $H(\nabla\varphi, \Omega) \leq \liminf_{\varepsilon} AG_{\varepsilon}(\varphi_{\varepsilon}, \Omega)$ for $\varphi_{\varepsilon} \rightarrow \varphi$ in $W^{1,1}(\Omega)$, see (3.12). Moreover, $H(\nabla\varphi, \Omega)$ is given by (1.9) if $\nabla\varphi \in BV$ (cf. Corollary 3.8). As a side note, we mention that the behavior of the sequence of Aviles-Giga functionals is related that of the micromagnetic energies investigated in [50, 51, 43], for which the notion of entropy plays a fundamental role as well.

By carefully adapting to our setting some of the strategies recently exploited to investigate the Aviles-Giga functionals, we can describe the asymptotic behavior of the rescaled J_1 - J_2 - J_3 energies $H_n \simeq \frac{1}{\delta_n^{3/2} \lambda_n} F_n$. In the main theorem of this paper we prove a compactness and Γ -convergence result for the functionals H_n that we briefly outline below.

In Theorem 4.1-*i*) we prove that every sequence $(\chi_n)_n \in L_{\text{loc}}^1(\mathbb{R}^2; \mathbb{R}^2)$ such that

$$\sup_n H_n(\chi_n, \Omega) < +\infty,$$

is precompact in $L_{\text{loc}}^p(\Omega)$ for every $p \in [1, 6)$. Moreover, the limit χ satisfies $H(\chi, \Omega) < +\infty$ and, in particular, it solves the eikonal equation in the sense that

$$|\chi| = 1 \text{ a.e. in } \Omega, \quad \text{curl}(\chi) = 0 \text{ in } \mathcal{D}'(\Omega).$$

In Theorem 4.1-*ii*) we show that the following liminf inequality holds for H_n : If $(\chi_n)_n, \chi \in L_{\text{loc}}^1(\mathbb{R}^2; \mathbb{R}^2)$ are such that $\chi_n \rightarrow \chi$ in $L_{\text{loc}}^1(\Omega; \mathbb{R}^2)$, then

$$H(\chi, \Omega) \leq \liminf_n H_n(\chi_n, \Omega).$$

Finally, assuming the additional scaling assumption $\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0$ as $n \rightarrow \infty$, in Theorem 4.1-*iii*) we prove the following limsup inequality: If $\chi \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) \cap BV(\Omega; \mathbb{R}^2)$, then there exists a sequence $(\chi_n)_n \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ such that $\chi_n \rightarrow \chi$ in $L^1(\Omega; \mathbb{R}^2)$ and

$$\limsup_n H_n(\chi_n, \Omega) \leq H(\chi, \Omega).$$

It is by now well-understood that the variational analysis of discrete-to-continuum problems often does not reduce to the comparison with an analogue continuum model by merely estimating discretization errors. In this sense, compared to the Aviles-Giga functionals, the J_1 - J_2 - J_3 model features new difficulties, some of which can be recognized by the presence of perturbations of the terms in the energy H_n with respect to those of the Aviles-Giga, see (2.16). In the following we highlight some of the major difficulties in proving our main result. For technical reasons, throughout the paper we will use several different variants of the chirality order parameter, all asymptotically equivalent. Although for the rest of the paper the energy H_n will be defined in terms of the variant denoted by χ , to describe some of the arising difficulties in this introduction, we rewrite it in terms of the parameter $\bar{\chi} = (\bar{\chi}_1, \bar{\chi}_2)$ defined in (2.19) with a slight abuse of notation as follows:

$$H_n = \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W^{\text{d}} \left(\frac{\frac{2}{\sqrt{\delta_n}} \sin(\frac{\sqrt{\delta_n}}{2} \bar{\chi}_1)}{\frac{2}{\sqrt{\delta_n}} \sin(\frac{\sqrt{\delta_n}}{2} \bar{\chi}_2)} \right) + \varepsilon_n \left| A^{\text{d}} \left(\frac{\frac{2}{\sqrt{\delta_n}} \sin(\frac{\sqrt{\delta_n}}{2} \bar{\chi}_1)}{\frac{2}{\sqrt{\delta_n}} \sin(\frac{\sqrt{\delta_n}}{2} \bar{\chi}_2)} \right) \right|^2 dx. \quad (1.10)$$

In the formula above, W^{d} is a discrete approximation of the potential $W(\xi) = (1 - |\xi|^2)^2$ of the Aviles-Giga functionals, and A^{d} is an approximation of the divergence operator. More precisely, it is a discrete approximation of the composition of the divergence operator with a δ_n -dependent non-linear perturbation of the identity.

To prove the compactness result Theorem 4.1-*i*), as a first key step we need to prove a bound on an Aviles-Giga-like energy with unperturbed potential and derivative terms, which we achieve in Proposition 2.6. The crucial step therein is to obtain from the bound on the derivative term featuring A^{d} in (1.10) a bound on (a discrete analogue of) the full derivative $D\bar{\chi}$. This is achieved by recognizing that the derivative term in (1.10) is a non-linear elliptic operator and by employing suitable regularity estimates. Subsequently, in Section 5 we will adapt to our setting the main arguments used in [26] to prove compactness properties of the Aviles-Giga functionals in (1.7).

We prove the liminf inequality in Theorem 4.1-*ii*) in Section 6. This is achieved by carefully estimating entropy productions in terms of the Aviles-Giga energy as outlined in Remark 6.1, making use of a key observation in [26] that allows us to conveniently rewrite entropy productions. Additionally, in the proof of both the compactness result and the liminf inequality, we have to take care of the fact that $\bar{\chi}$ has possibly non-zero curl, due to the possible formation of vortices in the discrete spin field u . In Lemma 2.3 we prove that the number of such vortex cells can be controlled in terms of the energy. This leads to a rate of convergence of $\text{curl}(\bar{\chi})$ to zero in L^1 which we need to use as a replacement of the curl-free condition. The situation we are dealing with here, where the curl concentrates on a controlled number of cells of a certain size, is only natural in the discrete. Nevertheless, the question for alternatives to the vanishing curl condition on $\nabla\varphi_\varepsilon$ in the Aviles-Giga functionals $AG_\varepsilon(\varphi_\varepsilon, \Omega)$ that still lead to the same Γ -limiting behavior as $\varepsilon \rightarrow 0$ can be asked and may be of interest also in the continuum.

The proof of the limsup inequality in Theorem 4.1-*iii*) is contained in Section 7. We resort to a technique which has originally been introduced in [46] to prove upper bounds for the Aviles-Giga functionals in (1.7), and has then been generalized to more general singular perturbation functionals in [47]. The latter applies in particular to the energies AG_ε^Δ in (1.6). This method has already been successfully applied in [17] to the discrete-to-continuum Γ -convergence analysis of the simpler J_1 - J_3 model already mentioned in this introduction.

In adapting to our setting the arguments used for the proofs of both the liminf and the limsup inequality a major additional difficulty needs to be overcome. This is due to the fact that in (1.10) the potential term featuring W^{d} is, in terms of $\bar{\chi}$, a δ_n -dependent

perturbation of the Aviles-Giga potential W with moving wells, *i.e.*, its set of zeros is δ_n -dependent. We stress that in the Γ -convergence analysis of the Aviles-Giga functionals, dealing with such scale-dependent potentials poses some difficulties even in the continuum case. Due to this issue, we require the additional scaling assumption $\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0$ for the proof of the limsup inequality. In contrast, we succeed in proving the liminf inequality without additional assumptions by introducing a class of approximate entropies (*cf.* Lemma 6.3).

As a final remark, we would like to mention that any rigorous numerical approximation of the Aviles-Giga functionals requires the proof of a Γ -convergence result of (unperturbed) discretizations of the Aviles-Giga energies, such as the functionals AG_n^d defined in (2.23), as both the discretization parameter λ_n and the singular perturbation parameter ε_n vanish. In the case that $\lambda_n \ll \varepsilon_n$ as $n \rightarrow \infty$, such a result follows as a byproduct of our analysis, *cf.* Remark 4.5. In fact, for that analysis many of the steps of our proofs can be simplified since several of the aforementioned difficulties due to the non-vanishing curl, the presence of a scale-dependent potential, and the non-linear elliptic derivative term do not take place.

2. PRELIMINARIES AND THE J_1 - J_2 - J_3 MODEL

2.1. Basic notation. Given two vectors $a, b \in \mathbb{R}^m$ we let $a \cdot b$ denote their scalar product. If $a, b \in \mathbb{R}^2$, their cross product is the scalar given by $a \times b = a_1 b_2 - a_2 b_1$. As usual, we let $|a| = \sqrt{a \cdot a}$ denote the norm of a . We use the notation \mathbb{S}^1 for the unit circle in \mathbb{R}^2 . Given $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^M$, their tensor product is the matrix $a \otimes b = (a_i b_j)_{i=1, \dots, N, j=1, \dots, M}^i \in \mathbb{R}^{N \times M}$. Given a vector $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we use the notation $\xi^\perp := (-\xi_2, \xi_1)$ for the vector obtained by rotating ξ by 90 degrees counterclockwise around the origin.

Given an open set $\Omega \subset \mathbb{R}^d$, we let $\mathcal{M}_b(\Omega; \mathbb{R}^\ell)$ denote the space of \mathbb{R}^ℓ -valued Radon measures on Ω with finite total variation. If $\ell = 1$, *i.e.*, for the space of finite signed Radon measures, we instead use the notation $\mathcal{M}_b(\Omega)$. We define the supremum $\bigvee_{t \in \mathcal{T}} \mu_t$ of a family of non-negative measures $(\mu_t)_{t \in \mathcal{T}} \in \mathcal{M}_b(\Omega)$ (with \mathcal{T} not necessarily countable) by

$$\bigvee_{t \in \mathcal{T}} \mu_t(B) := \sup \left\{ \sum_{t' \in \mathcal{T}'} \mu_{t'}(B_{t'}) : \mathcal{T}' \subset \mathcal{T} \text{ finite, } B_{t'} \subset B \text{ disjoint Borel sets} \right\}.$$

Then $\bigvee_{t \in \mathcal{T}} \mu_t$ is a Borel measure (not necessarily a Radon measure). We recall that if $\mu_t = f_t \mu$ for a non-negative measure $\mu \in \mathcal{M}_b(\Omega)$ and $f_t \geq 0$ Borel, then $\bigvee_{t \in \mathcal{T}} \mu_t = (\sup_{t \in \mathcal{T}} f_t) \mu$.

Unless specified otherwise, we always let C denote a positive and finite constant that may change at each of its occurrences.

2.2. BV functions. In the following we recall some basic facts about *BV* functions, referring to the book [6] for a comprehensive treatment on the subject. Moreover, we recall the notion of *BVG* function introduced in [46].

Let $\Omega \subset \mathbb{R}^d$ be an open set. A function $v \in L^1(\Omega; \mathbb{R}^m)$ is a function of bounded variation if its distributional derivative Dv is a finite matrix-valued Radon measure, *i.e.*, $Dv \in \mathcal{M}_b(\Omega; \mathbb{R}^{m \times d})$.

The distributional derivative $Dv \in \mathcal{M}_b(\Omega; \mathbb{R}^{m \times d})$ of a function $v \in BV(\Omega; \mathbb{R}^m)$ can be decomposed in the sum of three mutually singular matrix-valued measures

$$Dv = D^a v + D^c v + D^j v = \nabla v \mathcal{L}^d + D^c v + [v] \otimes \nu_v \mathcal{H}^{d-1} \llcorner J_v, \quad (2.1)$$

where \mathcal{L}^d is the Lebesgue measure and \mathcal{H}^{d-1} is the $(d-1)$ -dimensional Hausdorff measure; $\nabla v \in L^1(\Omega; \mathbb{R}^{m \times d})$ is the approximate gradient of v ; $D^c v$ is the so-called Cantor part of the derivative satisfying $D^c v(B) = 0$ for every Borel set B with $\mathcal{H}^{n-1}(B) < \infty$; J_v denotes the jump set of v , ν_v denotes the direction of the jump, $[v] = (v^+ - v^-)$, and v^+ and v^- denote the one-sided approximate limits of v on J_v . These are defined for a general $w \in L^1_{\text{loc}}(\Omega; \mathbb{R}^m)$ as follows (*cf.* for example [6, Definition 3.67]): J_w is the set of points $x \in \Omega$ such that there exist $a, b \in \mathbb{R}^m$, $a \neq b$, and $\nu \in \mathbb{S}^1$ such that

$$\lim_{r \rightarrow 0} \frac{1}{r^2} \int_{B_r^+(x, \nu)} |w(y) - a| dy = 0, \quad \lim_{r \rightarrow 0} \frac{1}{r^2} \int_{B_r^-(x, \nu)} |w(y) - b| dy = 0 \quad (2.2)$$

with $B_r^\pm(x, \nu) = \{y \in B_r(x) : \pm(y - x) \cdot \nu > 0\}$. The triple (a, b, ν) is unique up to the change to $(b, a, -\nu)$ and referred to as $(w^+(x), w^-(x), \nu_w(x))$. We let $[w](x) := w^+(x) - w^-(x)$.

We recall that every function $v \in BV(\Omega; \mathbb{R}^m)$ is approximately continuous at \mathcal{H}^{d-1} -a.e. point $x \in \Omega \setminus J_v$, in the sense that

$$\lim_{r \rightarrow 0} \frac{1}{r^d} \int_{B_r(x)} |v(y) - \xi| \, dy = 0$$

for some $\xi \in \mathbb{R}^m$. The point ξ is called the approximate limit of v at x and coincides with $v(x)$ for \mathcal{L}^d -a.e. x .

Let us furthermore recall the Vol'pert chain rule: Let $v \in BV(\Omega; \mathbb{R}^m)$ and let $\Phi \in C^1(\mathbb{R}^m; \mathbb{R}^\ell)$ be Lipschitz. If $\mathcal{L}^d(\Omega) = +\infty$, assume moreover that $\Phi(0) = 0$. Then, $\Phi \circ v \in BV(\Omega; \mathbb{R}^\ell)$ and

$$D(\Phi \circ v) = D\Phi(v)(D^a v + D^c v) + (\Phi(v^+) - \Phi(v^-)) \otimes \nu_v \mathcal{H}^{d-1} \llcorner J_v.$$

Note carefully that here the term $D\Phi(v)$ has to be understood as the function defined up to an \mathcal{H}^{d-1} -null set on $\Omega \setminus J_v$ by $D\Phi(v)(x) := D\Phi(\xi)$, where ξ is the approximate limit of v at x .

Finally, we recall the space $BVG(\Omega)$ introduced in [46]. It is defined by

$$BVG(\Omega) := \{\varphi \in W^{1,\infty}(\Omega) : \nabla \varphi \in BV(\Omega; \mathbb{R}^2)\}.$$

In [46], the author proves a convenient extension result for functions in $BVG(\Omega)$ under suitable conditions on the regularity of the set Ω . A bounded, open set $\Omega \subset \mathbb{R}^d$ is called a BVG domain if Ω can be described locally at its boundary as the epigraph of a BVG function $\mathbb{R}^{d-1} \rightarrow \mathbb{R}$ with respect to a suitable choice of the axes, *i.e.*, if every $x \in \partial\Omega$ has a neighborhood $U_x \subset \mathbb{R}^d$ such that there exists a function $\psi_x \in BVG(\mathbb{R}^{d-1})$ and a rigid motion $R_x: \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying

$$R_x(\Omega \cap U_x) = \{y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{d-1} : y_1 > \psi_x(y')\} \cap R_x(U_x).$$

Every BVG domain is an extension domain for BVG functions in the following sense.

Proposition 2.1 (Proposition 4.1 in [46]). *Let Ω be a BVG domain. Then for every $\varphi \in BVG(\Omega)$ there exists $\bar{\varphi} \in BVG(\mathbb{R}^d)$ such that $\bar{\varphi} = \varphi$ in Ω and $|D\nabla \bar{\varphi}|(\partial\Omega) = 0$.*

2.3. Jumps of functions with vanishing curl. We recall here how the curl-free constraint of a vector field enforces a relation between the geometry of its jump set and its one-sided approximate limits on both sides of the jump. For simplicity, we restrict to vector fields in dimension $d = 2$. In the following, Ω is an open subset of \mathbb{R}^2 .

Given a vector field $v \in L^1_{\text{loc}}(\Omega; \mathbb{R}^2)$, we define its (distributional) curl by $\text{curl}(v) := \partial_1 v_2 - \partial_2 v_1$, the partial derivatives being taken in the distributional sense.

If $v \in BV(\Omega; \mathbb{R}^2)$, it is clear from (2.1) that $\text{curl}(v) = 0$ implies that

$$0 = \text{curl}(v) \llcorner J_v = [v] \cdot \nu_v^\perp \mathcal{H}^1 \llcorner J_v$$

and, as a consequence, $[v]$ is parallel to ν_v at \mathcal{H}^1 -a.e. point in J_v .

If $v \in L^1_{\text{loc}}(\Omega; \mathbb{R}^2)$ satisfies $\text{curl}(v) = 0$, it can be observed that still $[v]$ is parallel to ν_v , and in fact this holds *everywhere* on J_v . Indeed, being $\text{curl}(v) = 0$, the same is true for the rescaled functions $v^{x,r}(y) := v(x + ry)$ for $x \in \Omega$ and $r > 0$. Taking $x \in J_v$ and letting $r \rightarrow 0$, by (2.2) we get that $v^{x,r}$ converge in $L^1(B_1(0))$ to the pure jump function

$$j_{v^+(x), v^-(x)}^{\nu_v(x)} : y \mapsto \begin{cases} v^+(x) & \text{if } y \cdot \nu_v(x) > 0, \\ v^-(x) & \text{if } y \cdot \nu_v(x) < 0. \end{cases}$$

As a consequence we get that $\text{curl}(j_{v^+(x), v^-(x)}^{\nu_v(x)}) = 0$. Since $j_{v^+(x), v^-(x)}^{\nu_v(x)}$ is a BV vector field, this yields that $[v](x)$ is parallel to $\nu_v(x)$.

2.4. Discrete functions. We introduce here the notation used for functions defined on a square lattice in \mathbb{R}^2 . For the whole paper, λ_n denotes a sequence of positive lattice spacings that converges to zero. Given $i, j \in \mathbb{Z}$, we define the half-open square $Q_{\lambda_n}(i, j)$ with left-bottom corner in $(\lambda_n i, \lambda_n j)$ by $Q_{\lambda_n}(i, j) := (\lambda_n i, \lambda_n j) + [0, \lambda_n)^2$. We refer to $Q_{\lambda_n}(i, j)$ as a cell of the lattice $\lambda_n \mathbb{Z}^2$. For a given set S , we introduce the class of functions with values in S which are piecewise constant on the cells of the lattice $\lambda_n \mathbb{Z}^2$:

$$\mathcal{PC}_{\lambda_n}(S) := \{v: \mathbb{R}^2 \rightarrow S : v(x) = v(\lambda_n i, \lambda_n j) \text{ for } x \in Q_{\lambda_n}(i, j)\}.$$

With a slight abuse of notation, we will always identify a function $v \in \mathcal{PC}_{\lambda_n}(S)$ with the function defined on the points of the lattice \mathbb{Z}^2 given by $(i, j) \mapsto v^{i,j} := v(\lambda_n i, \lambda_n j)$ for $(i, j) \in \mathbb{Z}^2$. Conversely, given values $v^{i,j} \in S$ for $(i, j) \in \mathbb{Z}^2$, we define $v \in \mathcal{PC}_{\lambda_n}(S)$ by $v(x) := v^{i,j}$ for $x \in Q_{\lambda_n}(i, j)$. Given a sequence $v_n \in \mathcal{PC}_{\lambda_n}(\mathbb{R}^m)$, we use the notation $v_n^{i,j}$ to refer to the k -th component of v_n .

Given $v \in \mathcal{PC}_{\lambda_n}(\mathbb{R}^m)$, we define its discrete partial derivatives $\partial_1^d v, \partial_2^d v \in \mathcal{PC}_{\lambda_n}(\mathbb{R}^m)$ by $\partial_1^d v^{i,j} := \frac{1}{\lambda_n}(v^{i+1,j} - v^{i,j})$ and $\partial_2^d v^{i,j} := \frac{1}{\lambda_n}(v^{i,j+1} - v^{i,j})$. Using these discrete derivatives, we have analogues of any differential operator in the discrete. In particular, we define $D^d v \in \mathcal{PC}_{\lambda_n}(\mathbb{R}^{m \times 2})$ to be the matrix whose k -th column is given by $\partial_k^d v$. If $m = 1$, we will often interpret $D^d v^{i,j}$ instead as a vector in \mathbb{R}^2 . Moreover, if $m = 2$, we define $\text{div}^d(v) \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$ and $\text{curl}^d(v) \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$ by

$$\text{div}^d(v)^{i,j} := \text{tr}(D^d v^{i,j}) = \partial_1^d v_1^{i,j} + \partial_2^d v_2^{i,j} \quad \text{and} \quad \text{curl}^d(v)^{i,j} := \partial_1^d v_2^{i,j} - \partial_2^d v_1^{i,j}$$

and call them the discrete divergence and the discrete curl of v , respectively. It is to be noted that in some contexts the proper discrete analogue of the Laplacian Δ of a field $v \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$ is given by

$$\Delta_s^d v^{i,j} := \partial_{11}^d v^{i-1,j} + \partial_{22}^d v^{i,j-1}, \quad (2.3)$$

i.e., suitable shifts in the lattice points are needed. To reflect this fact we add to our notation the subscript s which stands for “shifted”.

Next, we mention here a specific type of interpolation, which we shall use several times throughout the paper, mainly to relate the discrete divergence of a discrete vector field to its distributional divergence. For any $v \in \mathcal{PC}_{\lambda_n}(\mathbb{R}^2)$ we define $\mathcal{I}v: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as follows: Given any cell $Q_{\lambda_n}(i, j)$ of the lattice $\lambda_n \mathbb{Z}^2$ and any $x \in Q_{\lambda_n}(i, j)$, we write $x = \lambda_n(i, j) + \lambda_n y$, where $y = (y_1, y_2) \in [0, 1)^2$. We set

$$\mathcal{I}v(x) := \begin{pmatrix} (1 - y_1)v_1^{i,j} + y_1 v_1^{i+1,j} \\ (1 - y_2)v_2^{i,j} + y_2 v_2^{i,j+1} \end{pmatrix}. \quad (2.4)$$

We observe that $\text{div}(\mathcal{I}v) = \text{div}^d(v)$ in the sense of distributions. In particular, $\text{curl}^d(v) = -\text{div}(\mathcal{I}(v^\perp))$. Moreover, we note that

$$|\mathcal{I}v(x) - v(x)| = \left| \begin{pmatrix} y_1 \lambda_n \partial_1^d v_1^{i,j} \\ y_2 \lambda_n \partial_2^d v_2^{i,j} \end{pmatrix} \right| \leq C \lambda_n |D^d v(x)|. \quad (2.5)$$

for $x = \lambda_n(i, j) + \lambda_n y \in Q_{\lambda_n}(i, j)$.

The energy of the model (*cf.* Subsection 2.5 below) is defined on spin fields $u \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$. To every such u we associate the oriented angles $\theta^{\text{hor}}(u), \theta^{\text{ver}}(u) \in \mathcal{PC}_{\lambda_n}([-\pi, \pi])$ between adjacent spins by

$$\begin{aligned} (\theta^{\text{hor}}(u))^{i,j} &:= \text{sign}(u^{i,j} \times u^{i+1,j}) \arccos(u^{i,j} \cdot u^{i+1,j}), \\ (\theta^{\text{ver}}(u))^{i,j} &:= \text{sign}(u^{i,j} \times u^{i,j+1}) \arccos(u^{i,j} \cdot u^{i,j+1}), \end{aligned} \quad (2.6)$$

where we used the convention $\text{sign}(0) = -1$. We shall often drop the dependence on u as it will be clear from the context and for shortness we adopt the notation θ_n^{hor} and θ_n^{ver} for the angles associated to u_n .

2.5. Derivation of the energy model. The main subject of our study will be the sequence of functionals H_n which we define in Subsection 2.6 below. We show here how these are derived from the energies E_n in (1.1).

We start by showing how the energy E_n in (1.1) can be written in terms of the energy F_n in (1.2). In the following, we let the sums run over indices (i, j) such that $(\lambda_n i, \lambda_n j)$ belongs to a fixed set Ω . We shall specify later the precise assumptions on Ω , as now we present a formal computation. We split the terms in the sum involving α_n as follows

$$\begin{aligned} E_n(u) &= -\alpha_n \lambda_n^2 \sum_{(i,j)} \left(u^{i,j} \cdot u^{i+1,j} + u^{i,j} \cdot u^{i,j+1} \right) \\ &\quad + \beta_n \lambda_n^2 \sum_{(i,j)} \left(u^{i,j} \cdot u^{i+1,j+1} + u^{i,j} \cdot u^{i-1,j+1} \right) + \lambda_n^2 \sum_{(i,j)} \left(u^{i,j} \cdot u^{i+2,j} + u^{i,j} \cdot u^{i,j+2} \right) \\ &= \lambda_n^2 \sum_{(i,j)} \left(\left(-\frac{\beta_n \alpha_n}{\beta_n + 2} - \frac{2\alpha_n}{\beta_n + 2} \right) u^{i,j} \cdot u^{i+1,j} + \left(-\frac{\beta_n \alpha_n}{\beta_n + 2} - \frac{2\alpha_n}{\beta_n + 2} \right) u^{i,j} \cdot u^{i,j+1} \right) \\ &\quad + \beta_n \lambda_n^2 \sum_{(i,j)} \left(u^{i,j} \cdot u^{i+1,j+1} + u^{i,j} \cdot u^{i-1,j+1} \right) + \lambda_n^2 \sum_{(i,j)} \left(u^{i,j} \cdot u^{i+2,j} + u^{i,j} \cdot u^{i,j+2} \right). \end{aligned}$$

Then we shift coordinates: in $u^{i,j} \cdot u^{i+1,j}$ to get $\frac{1}{2}u^{i,j} \cdot u^{i+1,j}$ and $\frac{1}{2}u^{i-1,j} \cdot u^{i,j}$; in $u^{i,j} \cdot u^{i,j+1}$ to get $\frac{1}{2}u^{i,j} \cdot u^{i,j+1}$ and $\frac{1}{2}u^{i,j-1} \cdot u^{i,j}$; in $u^{i,j} \cdot u^{i-1,j+1}$ to get $\frac{1}{2}u^{i+1,j} \cdot u^{i,j+1}$ and $\frac{1}{2}u^{i-1,j} \cdot u^{i,j-1}$; in $u^{i,j} \cdot u^{i+1,j+1}$ to get $\frac{1}{2}u^{i+1,j} \cdot u^{i,j-1}$ and $\frac{1}{2}u^{i-1,j} \cdot u^{i,j+1}$; in $u^{i,j} \cdot u^{i+2,j}$ to get $u^{i-1,j} \cdot u^{i+1,j}$; in $u^{i,j} \cdot u^{i,j+2}$ to get $u^{i,j-1} \cdot u^{i,j+1}$.

The shifting procedure above may produce energy errors when applied to points $(\lambda_n i, \lambda_n j)$ close to the boundary of Ω . For instance a pair (i, j) such that $(\lambda_n i, \lambda_n j) \in \Omega$ could be transformed into a new shifted pair (i', j') such that $(\lambda_n i', \lambda_n j') \notin \Omega$ and, as such, it could no more be an element of the sum. Letting B_n denote these errors, we obtain

$$\begin{aligned} E_n(u) &= \lambda_n^2 \sum_{(i,j)} \left(-\frac{\alpha_n}{\beta_n + 2} u^{i,j} \cdot u^{i+1,j} + u^{i-1,j} \cdot u^{i+1,j} - \frac{\alpha_n}{\beta_n + 2} u^{i-1,j} \cdot u^{i,j} \right. \\ &\quad \left. - \frac{\alpha_n}{\beta_n + 2} u^{i,j} \cdot u^{i,j+1} + u^{i,j-1} \cdot u^{i,j+1} - \frac{\alpha_n}{\beta_n + 2} u^{i,j-1} \cdot u^{i,j} \right) \\ &\quad + \frac{\beta_n}{2} \lambda_n^2 \sum_{(i,j)} \left(u^{i+1,j} \cdot u^{i,j+1} - \frac{\alpha_n}{\beta_n + 2} u^{i,j} \cdot u^{i+1,j} + u^{i+1,j} \cdot u^{i,j-1} \right. \\ &\quad \left. - \frac{\alpha_n}{\beta_n + 2} u^{i,j} \cdot u^{i,j+1} - \frac{\alpha_n}{\beta_n + 2} u^{i,j-1} \cdot u^{i,j} \right. \\ &\quad \left. + u^{i-1,j} \cdot u^{i,j+1} - \frac{\alpha_n}{\beta_n + 2} u^{i-1,j} \cdot u^{i,j} + u^{i-1,j} \cdot u^{i,j-1} \right) + B_n. \end{aligned}$$

By reorganizing the terms we get that

$$\begin{aligned} E_n(u) &= \frac{1}{2} \lambda_n^2 \sum_{(i,j)} \left| u^{i+1,j} - \frac{\alpha_n}{\beta_n + 2} u^{i,j} + u^{i-1,j} \right|^2 + \left| u^{i,j+1} - \frac{\alpha_n}{\beta_n + 2} u^{i,j} + u^{i,j-1} \right|^2 \\ &\quad + \frac{\beta_n}{2} \lambda_n^2 \sum_{(i,j)} \left(u^{i+1,j} - \frac{\alpha_n}{\beta_n + 2} u^{i,j} + u^{i-1,j} \right) \cdot \left(u^{i,j+1} - \frac{\alpha_n}{\beta_n + 2} u^{i,j} + u^{i,j-1} \right) \\ &\quad - \lambda_n^2 \sum_{(i,j)} 2 + \frac{2\alpha_n^2}{2(\beta_n + 2)^2} + \frac{\alpha_n^2 \beta_n}{2(\beta_n + 2)^2} + B_n \\ &= F_n(u) - \lambda_n^2 \sum_{(i,j)} \left(\frac{\alpha_n^2}{2(\beta_n + 2)} + 2 \right) + B_n, \end{aligned}$$

where F_n is given by (1.2). As here we are not interested in the energy due to boundary layers, we shall neglect the error term B_n . Removing from E_n the bulk energy $-\lambda_n^2 \sum_{(i,j)} \left(\frac{\alpha_n^2}{2(\beta_n + 2)} + 2 \right)$ corresponding to the energy of the ground states, we are led to study the energy F_n .

As explained in the introduction, the main results in this paper concern the case where $\alpha_n < 8$, $\alpha_n \rightarrow 8$, and $\beta_n \equiv 2$. In that case the energy F_n reads

$$F_n(u) = \frac{1}{2} \lambda_n^2 \sum_{(i,j)} \left| u^{i+1,j} + u^{i-1,j} + u^{i,j+1} + u^{i,j-1} - \frac{\alpha_n}{2} u^{i,j} \right|^2. \quad (2.7)$$

We find it convenient to parametrize the convergence $\alpha_n \rightarrow 8$ by introducing the positive sequence $\delta_n := 4 - \frac{\alpha_n}{2}$ such that $\delta_n \rightarrow 0$.

Next, we introduce an order parameter $\chi(u)$ representing the chirality of the spin field u and we express the above energy in terms of this parameter. A rescaling will then lead to the energies H_n . Due to technical reasons we need to work with several variants of the chirality parameter. Specifically, we define

$$\begin{aligned} \chi(u)^{i,j} &:= (\chi_1(u)^{i,j}, \chi_2(u)^{i,j}), & \tilde{\chi}(u)^{i,j} &:= (\tilde{\chi}_1(u)^{i,j}, \tilde{\chi}_2(u)^{i,j}), \text{ where} \\ \chi_1(u)^{i,j} &:= \frac{2}{\sqrt{\delta_n}} \sin\left(\frac{(\theta^{\text{hor}})^{i,j}}{2}\right), & \tilde{\chi}_1(u)^{i,j} &:= \frac{1}{\sqrt{\delta_n}} \sin((\theta^{\text{hor}})^{i,j}), \\ \chi_2(u)^{i,j} &:= \frac{2}{\sqrt{\delta_n}} \sin\left(\frac{(\theta^{\text{ver}})^{i,j}}{2}\right), & \tilde{\chi}_2(u)^{i,j} &:= \frac{1}{\sqrt{\delta_n}} \sin((\theta^{\text{ver}})^{i,j}), \end{aligned} \quad (2.8)$$

where $\theta^{\text{hor}} = \theta^{\text{hor}}(u)$ and $\theta^{\text{ver}} = \theta^{\text{ver}}(u)$ are given by (2.6). A third variant $\bar{\chi}(u)$ will be introduced in (2.19) below. In our notation we shall often drop the dependence on u as it will be clear from the context. In addition, given a sequence of spin fields u_n , we will write $\chi_n, \tilde{\chi}_n$ in place of $\chi(u_n), \tilde{\chi}(u_n)$, respectively. Note that $\tilde{\chi}$ can be written as a function of χ , *e.g.*, $\tilde{\chi}_1^{i,j} = \frac{1}{\sqrt{\delta_n}} \sin(2 \arcsin(\frac{\sqrt{\delta_n}}{2} \chi_1^{i,j}))$. Since $\delta_n \rightarrow 0$, the reader can formally assume that $\chi \simeq \tilde{\chi}$ as $n \rightarrow \infty$ to ease the reading of the statements.

Given (i, j) , we rewrite the corresponding contribution to the energy in (2.7) in terms of $\chi(u)$. We observe that the \mathbb{S}^1 -symmetry of the energy allows us to assume, without loss of generality, that $u^{i,j} = \exp(i\theta) = (1, 0)$. Here and in the following, we interpret vectors in \mathbb{S}^1 as complex numbers *via* the relation $(\cos \theta, \sin \theta) = e^{i\theta}$, where i is the imaginary unit. As a consequence, the spins appearing in (2.7) can be rewritten in terms of the relative angles as

$$\begin{aligned} u^{i+1,j} &= e^{i(\theta^{\text{hor}})^{i,j}}, & u^{i-1,j} &= e^{-i(\theta^{\text{hor}})^{i-1,j}}, \\ u^{i,j+1} &= e^{i(\theta^{\text{ver}})^{i,j}}, & u^{i,j-1} &= e^{-i(\theta^{\text{ver}})^{i,j-1}}. \end{aligned}$$

We rewrite the energy $F_n(u)$ in terms of θ^{hor} and θ^{ver} as follows:

$$\begin{aligned} F_n(u) &= \frac{1}{2} \lambda_n^2 \sum_{(i,j)} \left| e^{i(\theta^{\text{hor}})^{i,j}} + e^{-i(\theta^{\text{hor}})^{i-1,j}} + e^{i(\theta^{\text{ver}})^{i,j}} + e^{-i(\theta^{\text{ver}})^{i,j-1}} - \frac{\alpha_n}{2} (1, 0) \right|^2 \\ &= \frac{1}{2} \lambda_n^2 \sum_{(i,j)} \left(\cos(\theta^{\text{hor}})^{i,j} + \cos(\theta^{\text{hor}})^{i-1,j} + \cos(\theta^{\text{ver}})^{i,j} + \cos(\theta^{\text{ver}})^{i,j-1} - \frac{\alpha_n}{2} \right)^2 \\ &\quad + \frac{1}{2} \lambda_n^2 \sum_{(i,j)} \left(\sin(\theta^{\text{hor}})^{i,j} - \sin(\theta^{\text{hor}})^{i-1,j} + \sin(\theta^{\text{ver}})^{i,j} - \sin(\theta^{\text{ver}})^{i,j-1} \right)^2. \end{aligned} \quad (2.9)$$

Using the trigonometric identity $\cos(\theta) = 1 - 2 \sin^2(\frac{\theta}{2})$ and recalling that $\delta_n = 4 - \frac{\alpha_n}{2}$, we get that

$$\begin{aligned} F_n(u) &= \frac{1}{2} \lambda_n^2 \sum_{(i,j)} \left(\delta_n - 2 \sin^2\left(\frac{(\theta^{\text{hor}})^{i,j}}{2}\right) - 2 \sin^2\left(\frac{(\theta^{\text{hor}})^{i-1,j}}{2}\right) \right. \\ &\quad \left. - 2 \sin^2\left(\frac{(\theta^{\text{ver}})^{i,j}}{2}\right) - 2 \sin^2\left(\frac{(\theta^{\text{ver}})^{i,j-1}}{2}\right) \right)^2 \\ &\quad + \frac{1}{2} \lambda_n^2 \sum_{(i,j)} \left(\lambda_n \partial_1^d \sin((\theta^{\text{hor}})^{i-1,j}) + \lambda_n \partial_2^d \sin((\theta^{\text{ver}})^{i,j-1}) \right)^2. \end{aligned} \quad (2.10)$$

Finally, using the definition of χ , we obtain that

$$\begin{aligned}
 F_n(u) &= \frac{1}{2} \lambda_n^2 \sum_{(i,j)} \frac{\delta_n^2}{4} \left(2 - |\chi_1^{i,j}|^2 - |\chi_1^{i-1,j}|^2 - |\chi_2^{i,j}|^2 - |\chi_2^{i,j-1}|^2 \right)^2 \\
 &\quad + \delta_n \lambda_n^2 \left| \partial_1^d \tilde{\chi}_1^{i-1,j} + \partial_2^d \tilde{\chi}_2^{i,j-1} \right|^2 \\
 &= \frac{\delta_n^{3/2} \lambda_n}{2} \lambda_n^2 \sum_{(i,j)} \frac{\sqrt{\delta_n}}{\lambda_n} W^d(\chi)^{i,j} + \frac{\lambda_n}{\sqrt{\delta_n}} |A^d(\chi)^{i,j}|^2 \\
 &= \frac{\delta_n^{3/2} \lambda_n}{2} \int_{\Omega_{\lambda_n}} \frac{1}{\varepsilon_n} W^d(\chi) + \varepsilon_n |A^d(\chi)|^2 dx.
 \end{aligned} \tag{2.11}$$

In the above formula, we let Ω_{λ_n} denote the union of cells of the lattice appearing in the sum and we define $\varepsilon_n := \frac{\lambda_n}{\sqrt{\delta_n}}$. Moreover, we have associated to χ the piecewise constant functions $W^d(\chi)$, $A^d(\chi) \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$ defined by

$$W^d(\chi)^{i,j} := \frac{1}{4} \left(2 - |\chi_1^{i,j}|^2 - |\chi_1^{i-1,j}|^2 - |\chi_2^{i,j}|^2 - |\chi_2^{i,j-1}|^2 \right)^2, \quad A^d(\chi)^{i,j} := \partial_1^d \tilde{\chi}_1^{i-1,j} + \partial_2^d \tilde{\chi}_2^{i,j-1}, \tag{2.12}$$

with χ , $\tilde{\chi}$ given by the relations (2.8) and recalling that $\tilde{\chi}$ can be written as a function of χ . The integral in the right-hand side of (2.11) defines the functional we are interested in in this paper. However, working with the integral on Ω_{λ_n} instead of Ω gives rise to minor technical issues, which are only tedious to fix. For this reason, in this paper we study directly the integral functional on Ω , which we define precisely in Subsection 2.6 below.

2.6. Assumptions on the model, the energies H_n , and the Aviles-Giga functionals. Throughout the paper we assume that λ_n, δ_n are two sequences of positive real numbers that converge to zero such that

$$\varepsilon_n := \frac{\lambda_n}{\sqrt{\delta_n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{2.13}$$

In particular, we have that $\lambda_n \ll \varepsilon_n$ as $n \rightarrow \infty$.

Our main result is valid whenever the domain Ω belongs to the class of admissible domains defined by

$$\mathcal{A}_0 := \{ \Omega \subset \mathbb{R}^2 : \Omega \text{ is an open, bounded, simply connected, } BVG \text{ domain} \}. \tag{2.14}$$

We recall that simply connected sets are by definition connected. Since parts of our results remain true under more general assumptions on Ω (*cf.* Remark 4.3), let us also introduce

$$\mathcal{A} := \{ \Omega \subset \mathbb{R}^2 : \Omega \text{ is an open and bounded set} \}. \tag{2.15}$$

In the rest of this section, Ω is always a domain in \mathcal{A} . The theorems in this paper will be stated for the functionals $H_n: L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) \times \mathcal{A} \rightarrow [0, +\infty]$ defined by

$$H_n(\chi, \Omega) := \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W^d(\chi) + \varepsilon_n |A^d(\chi)|^2 dx, \tag{2.16}$$

if $\chi = \chi(u)$ as in (2.8) for some $u \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$, and H_n extended to $+\infty$ otherwise. As a conclusion of subsection 2.5, we have established in which sense

$$\frac{1}{\delta_n^{3/2} \lambda_n} (E_n(u) - \min E_n) \sim H_n(\chi, \Omega). \tag{2.17}$$

As remarked in the introduction, the functionals H_n are related to the Aviles-Giga functionals. Indeed, let us note that W^d is a discrete approximation of the potential

$$W: \mathbb{R}^2 \rightarrow [0, +\infty), \quad W(\xi) := (1 - |\xi|^2)^2 \tag{2.18}$$

with suitable shifts in the discrete variable. Moreover, let us note that in a similar way, we have that $A^d(\chi) \simeq \text{div}^d(\tilde{\chi})$. Since $\tilde{\chi} \simeq \chi$ for large n , the functionals H_n resemble a discretization of the functionals $\chi \mapsto \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W(\chi) + \varepsilon_n |\text{div}(\chi)|^2 dx$. In Remark 2.5 below we show how $\text{curl}(\chi) \simeq 0$ which fully establishes the relation of H_n to the Aviles-Giga-like

functional $AG_{\varepsilon_n}^\Delta$ in (1.6), which in turn is related to the classical Aviles-Giga functional AG_{ε_n} in (1.7).

In the following we will explore this relation in more detail and to this end, as well as for later use, prove several *a priori* estimates on χ that can be obtained from the energy bound $H_n(\chi, \Omega) \leq C$.

Remark 2.2. The potential part in H_n provides L^4 bounds on the variable χ . More precisely, if $\sup_n H_n(\chi_n, \Omega) < +\infty$, then $\sup_n \|\chi_n\|_{L^4(\Omega)} < +\infty$. Indeed, using Young's inequality, we have that $(2-a)^2 \geq \frac{1}{2}a^2 - 4$ for $a = |\chi_{1,n}^{i,j}|^2 + |\chi_{1,n}^{i-1,j}|^2 + |\chi_{2,n}^{i,j}|^2 + |\chi_{2,n}^{i,j-1}|^2$. As a consequence,

$$C \geq \frac{1}{4\varepsilon_n} \int_{\Omega} \frac{1}{2} (|\chi_{1,n}^{i,j}|^2 + |\chi_{1,n}^{i-1,j}|^2 + |\chi_{2,n}^{i,j}|^2 + |\chi_{2,n}^{i,j-1}|^2)^2 - 4 \, dx \geq \frac{1}{4\varepsilon_n} \int_{\Omega} \frac{1}{2} |\chi_n|^4 - 4 \, dx.$$

This bound can be improved by additionally exploiting the derivative part in H_n as explained in detail below in Proposition 2.7.

Using the potential part of H_n , in the following lemma we count the number of cells where the angles between adjacent spins defined in (2.6) are far from 0. This counting argument will be often put to use throughout the paper.

Lemma 2.3. *Assume that $\sup_n H_n(\chi_n, \Omega) < +\infty$. Then for every $t \in (0, +\infty)$ there exists $C(t) \in (0, +\infty)$ such that*

$$\#\{(i, j) \in \mathbb{Z}^2 : Q_{\lambda_n}(i, j) \subset \Omega, |(\theta_n^{\text{hor}})^{i,j}| > t \text{ or } |(\theta_n^{\text{ver}})^{i,j}| > t\} \leq C(t) \frac{\delta_n^{3/2}}{\lambda_n}.$$

Proof. We may assume that $t < \pi$ since otherwise the statement is trivial. Then, if $|(\theta_n^{\text{hor}})^{i,j}| > t$ or $|(\theta_n^{\text{ver}})^{i,j}| > t$, we get that $\max\{|\chi_{1,n}^{i,j}|^2, |\chi_{2,n}^{i,j}|^2\} \geq \frac{4}{\delta_n} \sin^2\left(\frac{t}{2}\right) \geq C \frac{t^2}{\delta_n}$. Hence, for δ_n sufficiently small, this implies that $W^d(\chi_n)^{i,j} \geq C \frac{t^4}{\delta_n^2}$. Thus we get that

$$\begin{aligned} C &\geq \int_{\Omega} \frac{1}{\varepsilon_n} W^d(\chi_n) \, dx \\ &\geq \lambda_n^2 \#\{(i, j) \in \mathbb{Z}^2 : Q_{\lambda_n}(i, j) \subset \Omega, |(\theta_n^{\text{hor}})^{i,j}| > t \text{ or } |(\theta_n^{\text{ver}})^{i,j}| > t\} C \frac{1}{\varepsilon_n} \frac{t^4}{\delta_n^2}. \end{aligned}$$

Since $\varepsilon_n = \frac{\lambda_n}{\sqrt{\delta_n}}$, this implies the claim. \square

A first consequence of the counting argument in Lemma 2.3 is the following estimate on the discrete curl of sequences χ_n with equibounded energies. For the precise statement, it is convenient to introduce the auxiliary variable $\bar{\chi}_n$ defined by

$$\bar{\chi}_n^{i,j} := (\bar{\chi}_{1,n}^{i,j}, \bar{\chi}_{2,n}^{i,j}), \quad \bar{\chi}_{1,n}^{i,j} := \frac{1}{\sqrt{\delta_n}} (\theta_n^{\text{hor}})^{i,j}, \quad \bar{\chi}_{2,n}^{i,j} := \frac{1}{\sqrt{\delta_n}} (\theta_n^{\text{ver}})^{i,j}. \quad (2.19)$$

This is the linearized version of the order parameter χ_n , cf. its definition in (2.8).

Lemma 2.4. *Assume that $\sup_n H_n(\chi_n, \Omega) < +\infty$. Then for every $\Omega' \subset\subset \Omega$ there exists $C \in (0, +\infty)$ such that*

$$\|\text{curl}^d(\bar{\chi}_n)\|_{L^1(\Omega')} \leq C \delta_n.$$

Proof. Let u_n be such that $\chi_n = \chi(u_n)$ as in (2.8). We start by observing that

$$\lambda_n \sqrt{\delta_n} \text{curl}^d(\bar{\chi}_n)^{i,j} = (\theta_n^{\text{hor}})^{i,j} + (\theta_n^{\text{ver}})^{i+1,j} - (\theta_n^{\text{ver}})^{i,j} - (\theta_n^{\text{hor}})^{i,j+1} \in 2\pi\mathbb{Z}$$

since $(\theta_n^{\text{hor}})^{i,j} + (\theta_n^{\text{ver}})^{i+1,j}$ and $(\theta_n^{\text{ver}})^{i,j} + (\theta_n^{\text{hor}})^{i,j+1}$ both represent an oriented angle between the spins $u_n^{i,j}$ and $u_n^{i+1,j+1}$ and thus must be equal modulo 2π . Moreover, since $(\theta_n^{\text{hor}})^{i,j}, (\theta_n^{\text{ver}})^{i+1,j}, (\theta_n^{\text{ver}})^{i,j}, (\theta_n^{\text{hor}})^{i,j+1} \in [-\pi, \pi)$, we actually get that $\lambda_n \sqrt{\delta_n} \text{curl}^d(\bar{\chi}_n)^{i,j} \in \{-2\pi, 0, 2\pi\}$. If moreover

$$|(\theta_n^{\text{hor}})^{i,j}|, |(\theta_n^{\text{ver}})^{i+1,j}|, |(\theta_n^{\text{ver}})^{i,j}|, |(\theta_n^{\text{hor}})^{i,j+1}| < \frac{\pi}{2}, \quad (2.20)$$

then we even have that $\lambda_n \sqrt{\delta_n} \text{curl}^d(\bar{\chi}_n)^{i,j} = 0$. For n large enough all cells $Q_{\lambda_n}(i, j)$ that intersect Ω' as well as all their neighboring cells are contained in Ω . As a consequence, by

Lemma 2.3 we have that (2.20) only fails on a subset of Ω' of measure less than $\lambda_n^2 C \frac{\delta_n^{3/2}}{\lambda_n}$. Hence we conclude that

$$\|\operatorname{curl}^d(\bar{\chi}_n)\|_{L^1(\Omega')} \leq \frac{1}{\lambda_n \sqrt{\delta_n}} \cdot 2\pi \lambda_n^2 C \frac{\delta_n^{3/2}}{\lambda_n} = C\delta_n.$$

□

Remark 2.5. Lemma 2.4 implies, in particular, that

$$\operatorname{curl}^d(\chi_n) \rightharpoonup 0 \quad \text{in the sense of distributions.} \quad (2.21)$$

Indeed, using the inequality $|2 \sin(\frac{s}{2}) - s| \leq \frac{1}{24}|s|^3$ and writing χ_n in terms of $\bar{\chi}_n$, we get $|\chi_n^{i,j} - \bar{\chi}_n^{i,j}|^2 \leq C\delta_n^2 |\bar{\chi}_n^{i,j}|^6 \leq C\delta_n |\chi_n^{i,j}|^4$, where we have used that $|\bar{\chi}_n^{i,j}| \leq C|\chi_n^{i,j}| \leq \frac{C}{\sqrt{\delta_n}}$. Thus, the L^4 -bounds on χ_n obtained in Remark 2.2 yield $\|\chi_n - \bar{\chi}_n\|_{L^2(\Omega)} \leq C\delta_n$. A discrete integration by parts shows that $\operatorname{curl}^d(\chi_n - \bar{\chi}_n) \rightarrow 0$ in $\mathcal{D}'(\Omega)$ and together with Lemma 2.4 we obtain the claim.

As an alternative to the discrete integration by parts, we can observe that $\operatorname{curl}^d(\chi_n - \bar{\chi}_n) = -\operatorname{div}(\mathcal{I}(\chi_n^\perp - \bar{\chi}_n^\perp))$, where \mathcal{I} is defined by (2.4). Since for every open $\Omega' \subset\subset \Omega$ we have that $\|\mathcal{I}(\chi_n^\perp - \bar{\chi}_n^\perp)\|_{L^2(\Omega')} \leq 2\|\chi_n - \bar{\chi}_n\|_{L^2(\Omega)}$ for n large enough, this allows us to assert that in fact $\operatorname{curl}^d(\chi_n - \bar{\chi}_n) \rightarrow 0$ strongly in H^{-1} locally in Ω . We will later make use of this observation (*cf.* Proposition 5.2, Step 2).

As we have observed previously, Remark 2.5 suggests that the functionals H_n share similarities with the Aviles-Giga functionals. To give a rigorous statement, we introduce the auxiliary functionals H_n^* defined as follows: for $\chi \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, we set

$$H_n^*(\chi, \Omega) := \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W(\chi) + \varepsilon_n |\mathbb{D}^d \chi|^2 dx \quad (2.22)$$

if $\chi = \chi(u)$ as in (2.8) for some $u \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$, and H_n^* extended to $+\infty$ otherwise, where W is defined by (2.18). Up to replacing the condition $\operatorname{curl}^d(\chi_n) \rightharpoonup 0$ (*cf.* Remark 2.5) with the condition $\operatorname{curl}^d(\chi_n) \equiv 0$, the functionals H_n^* are the discrete Aviles-Giga energies $AG_n^d: L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}) \times \mathcal{A} \rightarrow [0, +\infty]$ defined by

$$AG_n^d(\varphi, \Omega) := \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W(\mathbb{D}^d \varphi) + \varepsilon_n |\mathbb{D}^d \mathbb{D}^d \varphi|^2 dx \quad (2.23)$$

if $\varphi \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$, and AG_n^d extended to $+\infty$ otherwise.² In the next proposition we prove that the energy bound $H_n(\chi, \Omega) \leq C$ implies a local bound on the energies H_n^* . Note that the functionals H_n^* feature the full discrete derivative matrix of χ , and not just the discrete divergence-type term $A^d(\chi)$ as the functionals H_n . Nonetheless, for sequences χ_n with equibounded energies H_n , the full discrete derivative matrix can be controlled by exploiting the vanishing curl condition obtained in Lemma 2.4. Our proof of this fact is inspired by the well-known technique used to prove H^2 -regularity for weak solutions of elliptic second order PDE.

Proposition 2.6. *Let $(\chi_n)_n \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$. We have that*

$$\sup_n H_n(\chi_n, \Omega) < +\infty \quad \implies \quad \sup_n H_n^*(\chi_n, \Omega') < +\infty \quad \text{for every } \Omega' \subset\subset \Omega.$$

Proof. Step 1. (Bound on the derivative term in H_n^* .) We claim that

$$\sup_n \int_{\Omega'} \varepsilon_n |\mathbb{D}^d \bar{\chi}_n|^2 dx < +\infty \quad \text{for every } \Omega' \subset\subset \Omega. \quad (2.24)$$

Note that the 1-Lipschitz continuity of the map $s \mapsto \frac{2}{\sqrt{\delta_n}} \sin(\frac{\sqrt{\delta_n}}{2}s)$ and the definition of χ in (2.8) and of $\bar{\chi}$ in (2.19) imply that $|\mathbb{D}^d \chi_n| \leq |\mathbb{D}^d \bar{\chi}_n|$ and thus

$$\sup_n \int_{\Omega'} \varepsilon_n |\mathbb{D}^d \chi_n|^2 dx < +\infty \quad \text{for every } \Omega' \subset\subset \Omega, \quad (2.25)$$

²Notice that every $\chi \in \mathcal{PC}_{\lambda_n}(\mathbb{R}^2)$ with $\operatorname{curl}^d(\chi) = 0$ in Ω admits, at least locally in Ω , a discrete potential φ such that $\chi = \mathbb{D}^d \varphi$.

providing the first bound needed for H_n^* . For later use let us note that using (2.8), (2.19), and the 1-Lipschitz continuity of the map $s \mapsto \frac{1}{\sqrt{\delta_n}} \sin(\sqrt{\delta_n} s)$ we also get that $|\mathbb{D}^d \tilde{\chi}_n| \leq |\mathbb{D}^d \bar{\chi}_n|$ and, as a consequence,

$$\sup_n \int_{\Omega'} \varepsilon_n |\mathbb{D}^d \tilde{\chi}_n|^2 dx < +\infty \quad \text{for every } \Omega' \subset\subset \Omega. \quad (2.26)$$

To prove (2.24) let us start by considering an additional open set Ω'' with $\Omega' \subset\subset \Omega'' \subset\subset \bar{\Omega}$ and a smooth cut-off function $\zeta \in C_c^\infty(\Omega''; [0, 1])$ with $\zeta \equiv 1$ on a neighborhood of $\bar{\Omega}'$. Although not necessary, it will be convenient for our computations to introduce the discretizations $\zeta_n \in \mathcal{PC}_{\lambda_n}([0, 1])$ by $\zeta_n^{i,j} := \zeta(\lambda_n(i, j))$. Next, let us observe that by (2.8) and (2.19) we have that

$$A^d(\chi_n)^{i,j} = \partial_1^d \left(\frac{1}{\sqrt{\delta_n}} \sin(\sqrt{\delta_n} \bar{\chi}_{1,n}) \right)^{i-1,j} + \partial_2^d \left(\frac{1}{\sqrt{\delta_n}} \sin(\sqrt{\delta_n} \bar{\chi}_{2,n}) \right)^{i,j-1}.$$

Therefore, using twice a discrete integration by parts, we get that

$$\begin{aligned} I_n &:= \int_{\mathbb{R}^2} A^d(\chi_n) \partial_1^d (|\zeta_n|^2 \bar{\chi}_{1,n})^{\bullet-e_1} dx \\ &= \int_{\mathbb{R}^2} \partial_1^d \left(\frac{1}{\sqrt{\delta_n}} \sin(\sqrt{\delta_n} \bar{\chi}_{1,n}) \right) \partial_1^d (|\zeta_n|^2 \bar{\chi}_{1,n}) + \partial_1^d \left(\frac{1}{\sqrt{\delta_n}} \sin(\sqrt{\delta_n} \bar{\chi}_{2,n}) \right) \partial_2^d (|\zeta_n|^2 \bar{\chi}_{1,n}) dx. \end{aligned} \quad (2.27)$$

In the following we show how (2.27) can be used to deduce the bound $\int_{\Omega'} \varepsilon_n |\mathbb{D}^d \bar{\chi}_{1,n}|^2 dx \leq C$. The remaining bound $\int_{\Omega'} \varepsilon_n |\mathbb{D}^d \bar{\chi}_{2,n}|^2 dx \leq C$ can be proved analogously starting instead from the equation

$$\begin{aligned} &\int_{\mathbb{R}^2} A^d(\chi_n) \partial_2^d (|\zeta_n|^2 \bar{\chi}_{2,n})^{\bullet-e_2} dx \\ &= \int_{\mathbb{R}^2} \partial_2^d \left(\frac{1}{\sqrt{\delta_n}} \sin(\sqrt{\delta_n} \bar{\chi}_{1,n}) \right) \partial_1^d (|\zeta_n|^2 \bar{\chi}_{2,n}) + \partial_2^d \left(\frac{1}{\sqrt{\delta_n}} \sin(\sqrt{\delta_n} \bar{\chi}_{2,n}) \right) \partial_2^d (|\zeta_n|^2 \bar{\chi}_{2,n}) dx. \end{aligned}$$

We rewrite the right-hand side of (2.27) by using a discrete chain rule and a particular version of a discrete product rule which takes the form $\partial_k^d(vw) = \frac{1}{2}(w + w^{\bullet+e_k})\partial_k^d v + \frac{1}{2}(v + v^{\bullet+e_k})\partial_k^d w$ for $v, w \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$. We obtain that

$$\begin{aligned} I_n &= \frac{1}{2} \int_{\mathbb{R}^2} (|\zeta_n|^2 + |\zeta_n^{\bullet+e_1}|^2) \cos(\sqrt{\delta_n} X_{1,n}) |\partial_1^d \bar{\chi}_{1,n}|^2 \\ &\quad + (|\zeta_n|^2 + |\zeta_n^{\bullet+e_2}|^2) \cos(\sqrt{\delta_n} X_{2,n}) |\partial_2^d \bar{\chi}_{1,n}|^2 dx + R_n, \end{aligned} \quad (2.28)$$

where

$$\begin{aligned} R_n &= \frac{1}{2} \int_{\mathbb{R}^2} \cos(\sqrt{\delta_n} X_{1,n}) \partial_1^d \bar{\chi}_{1,n} (\bar{\chi}_{1,n} + \bar{\chi}_{1,n}^{\bullet+e_1}) \partial_1^d (|\zeta_n|^2) \\ &\quad + \cos(\sqrt{\delta_n} X_{2,n}) \partial_1^d \bar{\chi}_{2,n} (\bar{\chi}_{1,n} + \bar{\chi}_{1,n}^{\bullet+e_2}) \partial_2^d (|\zeta_n|^2) \\ &\quad + (|\zeta_n|^2 + |\zeta_n^{\bullet+e_2}|^2) \cos(\sqrt{\delta_n} X_{2,n}) \partial_2^d \bar{\chi}_{1,n} (\partial_1^d \bar{\chi}_{2,n} - \partial_2^d \bar{\chi}_{1,n}) dx \end{aligned}$$

and where $X_{1,n}^{i,j}$ is an intermediate point between $\bar{\chi}_{1,n}^{i,j}$ and $\bar{\chi}_{1,n}^{i+1,j}$ and $X_{2,n}^{i,j}$ lies between $\bar{\chi}_{2,n}^{i,j}$ and $\bar{\chi}_{2,n}^{i+1,j}$. In the following, we may restrict all integrations to Ω'' with the understanding that the resulting estimates hold for n large enough. To estimate R_n let us recall that by Lemma 2.4 we have that $\|\partial_1^d \bar{\chi}_{2,n} - \partial_2^d \bar{\chi}_{1,n}\|_{L^1(\Omega'')} = \|\text{curl}^d(\bar{\chi}_n)\|_{L^1(\Omega'')} \leq C\delta_n$. Moreover, $|\partial_2^d \bar{\chi}_{1,n}| \leq \frac{C}{\lambda_n \sqrt{\delta_n}}$ and, as a consequence,

$$\int_{\mathbb{R}^2} (|\zeta_n|^2 + |\zeta_n^{\bullet+e_2}|^2) \cos(\sqrt{\delta_n} X_{2,n}) \partial_2^d \bar{\chi}_{1,n} (\partial_1^d \bar{\chi}_{2,n} - \partial_2^d \bar{\chi}_{1,n}) dx \leq C \frac{\sqrt{\delta_n}}{\lambda_n} = \frac{C}{\varepsilon_n}. \quad (2.29)$$

Furthermore, we have that $\partial_k^d(|\zeta_n|^2) = \frac{1}{\lambda_n}|\zeta_n^{*+e_k} - \zeta_n||\zeta_n^{*+e_k} + \zeta_n| \leq C(|\zeta_n^{*+e_k} + \zeta_n|)$ because $D^d\zeta_n$ are bounded in $L^\infty(\mathbb{R}^2)$. Using Young's inequality we get that

$$\begin{aligned} & \int_{\mathbb{R}^2} |\cos(\sqrt{\delta_n}X_{1,n})\partial_1^d\bar{\chi}_{1,n}(\bar{\chi}_{1,n} + \bar{\chi}_{1,n}^{*+e_1})\partial_1^d(|\zeta_n|^2)| dx \\ & \leq C \int_{\Omega''} \frac{1}{M} |\partial_1^d\bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{*+e_1}|^2) + M|\bar{\chi}_{1,n}|^2 dx, \end{aligned} \quad (2.30)$$

where M is an arbitrary positive number. Similarly, using first the triangle inequality, we also get the estimate

$$\begin{aligned} & \int_{\mathbb{R}^2} |\cos(\sqrt{\delta_n}X_{2,n})\partial_1^d\bar{\chi}_{2,n}(\bar{\chi}_{1,n} + \bar{\chi}_{1,n}^{*+e_2})\partial_2^d(|\zeta_n|^2)| dx \\ & \leq \int_{\mathbb{R}^2} |\partial_2^d\bar{\chi}_{1,n}(\bar{\chi}_{1,n} + \bar{\chi}_{1,n}^{*+e_2})\partial_2^d(|\zeta_n|^2)| + |\text{curl}^d(\bar{\chi}_n)(\bar{\chi}_{1,n} + \bar{\chi}_{1,n}^{*+e_2})\partial_2^d(|\zeta_n|^2)| dx \\ & \leq C \int_{\Omega''} \frac{1}{M} |\partial_2^d\bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{*+e_2}|^2) + M|\bar{\chi}_{1,n}|^2 dx + C\sqrt{\delta_n}, \end{aligned} \quad (2.31)$$

where we have used Lemma 2.4 and the fact that $|\bar{\chi}_{1,n}| \leq \frac{C}{\sqrt{\delta_n}}$. By (2.29)–(2.31) we get that

$$|R_n| \leq \frac{C}{M} \int_{\Omega''} |\partial_1^d\bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{*+e_1}|^2) + |\partial_2^d\bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{*+e_2}|^2) dx + CM + \frac{C}{\varepsilon_n},$$

where we have used that $\sqrt{\delta_n} \leq \frac{1}{\varepsilon_n}$ for n large enough (by (2.13)) and the fact that $\bar{\chi}_{1,n}$ are bounded in $L^2(\Omega'')$. The latter bound is due to Remark 2.2 and the fact that $|\bar{\chi}_n| \leq \frac{\pi}{2}|\chi_n|$. With the bound on R_n in place, we now return to (2.28) and estimate I_n from below as follows:

$$\begin{aligned} I_n & \geq \left(\frac{1}{4} - \frac{C}{M}\right) \int_{\Omega''} |\partial_1^d\bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{*+e_1}|^2) + |\partial_2^d\bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{*+e_2}|^2) dx \\ & \quad + \frac{1}{2} \int_{\Omega''} |\partial_1^d\bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{*+e_1}|^2) (\cos(\sqrt{\delta_n}X_{1,n}) - \frac{1}{2}) \\ & \quad \quad + |\partial_2^d\bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{*+e_2}|^2) (\cos(\sqrt{\delta_n}X_{2,n}) - \frac{1}{2}) dx \\ & \quad - CM - \frac{C}{\varepsilon_n}. \end{aligned} \quad (2.32)$$

For all indices $(i, j) \in \mathbb{Z}^2$ such that

$$|\bar{\chi}_{1,n}^{i,j}|, |\bar{\chi}_{1,n}^{i+1,j}|, |\bar{\chi}_{2,n}^{i,j}|, |\bar{\chi}_{2,n}^{i+1,j}| \leq \frac{\arccos \frac{1}{2}}{\sqrt{\delta_n}}, \quad (2.33)$$

we have that $\cos(\sqrt{\delta_n}X_{1,n}), \cos(\sqrt{\delta_n}X_{2,n}) \geq \frac{1}{2}$ on the cell $Q_{\lambda_n}(i, j)$. On the other hand, for n large enough all cells $Q_{\lambda_n}(i, j)$ that intersect Ω'' as well as all their neighboring cells are contained in Ω and thus in view of (2.19), Lemma 2.3 implies that

$$\#\{(i, j) \in \mathbb{Z}^2 : Q_{\lambda_n}(i, j) \cap \Omega'' \neq \emptyset \text{ and (2.33) fails}\} \leq C \frac{\delta_n^{3/2}}{\lambda_n}.$$

This allows us to estimate the second integral in (2.32) from below by splitting it into the integral on the cells where (2.33) holds and the integral on the cells where it fails: On the former, the integrand is non-negative. On the latter cells, we use that $|D^d\bar{\chi}_{1,n}|^2 \leq \frac{C}{\lambda_n^2\delta_n}$ and consequently obtain that the second integral in (2.32) is bounded from below by $-C\lambda_n^2 \frac{\delta_n^{3/2}}{\lambda_n} \frac{1}{\lambda_n^2\delta_n} = -\frac{C}{\varepsilon_n}$. Thus,

$$I_n \geq \left(\frac{1}{4} - \frac{C}{M}\right) \int_{\Omega''} |\partial_1^d\bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{*+e_1}|^2) + |\partial_2^d\bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{*+e_2}|^2) dx - CM - \frac{C}{\varepsilon_n}. \quad (2.34)$$

To find the desired L^2 estimate on $D^d \bar{\chi}_{1,n}$, we combine this lower bound with an upper bound on the left-hand side of (2.27). Using Young's inequality, a discrete product rule, and the bound on the energy H_n we get that

$$\begin{aligned} I_n &\leq \frac{1}{2} \int_{\Omega''} MA^d(\chi_n)^2 + \frac{1}{M} |\partial_1^d (|\zeta_n|^2 \bar{\chi}_{1,n})|^2 dx \\ &\leq \frac{CM}{\varepsilon_n} + \frac{1}{M} \int_{\Omega''} |\partial_1^d (|\zeta_n|^2)|^2 |\bar{\chi}_{1,n}^{+e_1}|^2 + |\zeta_n|^4 |\partial_1^d \bar{\chi}_{1,n}|^2 dx. \end{aligned}$$

Finally, as already observed in this proof, we use that $\bar{\chi}_{1,n}$ are bounded in $L^2(\Omega)$, $D^d(|\zeta_n|^2)$ are bounded in L^∞ , and $|\zeta_n|^4 \leq |\zeta_n|^2 \leq |\zeta_n|^2 + |\zeta_n^{+e_1}|^2$ to obtain that

$$I_n \leq \frac{CM}{\varepsilon_n} + \frac{C}{M} + \frac{1}{M} \int_{\Omega''} (|\zeta_n|^2 + |\zeta_n^{+e_1}|^2) |\partial_1^d \bar{\chi}_{1,n}|^2 dx.$$

Together with (2.34) this implies that

$$\left(\frac{1}{4} - \frac{C}{M}\right) \int_{\Omega''} |\partial_1^d \bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{+e_1}|^2) + |\partial_2^d \bar{\chi}_{1,n}|^2 (|\zeta_n|^2 + |\zeta_n^{+e_2}|^2) dx \leq \frac{C}{\varepsilon_n} (1+M) + \frac{C}{M}.$$

As none of the constants C depend on M , choosing M sufficiently large and if n is large enough, the left-hand side provides an upper bound on $\frac{1}{8} \int_{\Omega'} |D^d \bar{\chi}_{1,n}|^2 dx$. Thus we get that $\int_{\Omega'} \varepsilon_n |D^d \bar{\chi}_{1,n}|^2 dx \leq C$ as desired.

Step 2. (Bound on the potential term in H_n^* .) We claim that

$$\sup_n \int_{\Omega'} \frac{1}{\varepsilon_n} W(\chi_n) dx < +\infty \quad \text{for every } \Omega' \subset\subset \Omega. \quad (2.35)$$

By the reverse triangle inequality we have that

$$\begin{aligned} &|\sqrt{W^d}(\chi_n^{i,j}) - \sqrt{W}(\chi_n^{i,j})| \\ &\leq \frac{1}{2} \left| 2 - |\chi_{1,n}^{i,j}|^2 - |\chi_{1,n}^{i-1,j}|^2 - |\chi_{2,n}^{i,j}|^2 - |\chi_{2,n}^{i,j-1}|^2 - (2 - 2|\chi_{1,n}^{i,j}|^2 - 2|\chi_{2,n}^{i,j}|^2) \right| \\ &= \frac{1}{2} \left| (\chi_{1,n}^{i,j} + \chi_{1,n}^{i-1,j}) \lambda_n \partial_1^d \chi_{1,n}^{i-1,j} + (\chi_{2,n}^{i,j} + \chi_{2,n}^{i,j-1}) \lambda_n \partial_2^d \chi_{2,n}^{i,j-1} \right|. \end{aligned} \quad (2.36)$$

Let Ω'' be another open set with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Using (2.25), the fact that $|\chi_{1,n}|, |\chi_{2,n}| \leq \frac{C}{\sqrt{\delta_n}}$, and (2.13), we obtain for n large enough that

$$\frac{1}{\sqrt{\varepsilon_n}} \|\sqrt{W^d}(\chi_n^{i,j}) - \sqrt{W}(\chi_n^{i,j})\|_{L^2(\Omega')} \leq C \frac{\lambda_n}{\sqrt{\varepsilon_n} \sqrt{\delta_n}} \|D^d \chi\|_{L^2(\Omega'')} \leq C \frac{\lambda_n}{\varepsilon_n \sqrt{\delta_n}} = C.$$

Writing $W^d - W = (2\sqrt{W^d} - (\sqrt{W^d} + \sqrt{W}))(\sqrt{W^d} - \sqrt{W})$, we infer that

$$\int_{\Omega'} \frac{1}{\varepsilon_n} |W^d(\chi_n) - W(\chi_n)| dx \leq \left(\frac{2}{\sqrt{\varepsilon_n}} \|\sqrt{W^d}(\chi_n)\|_{L^2(\Omega')} + C \right) \cdot C \leq C,$$

where we have used that $H_n(\chi_n, \Omega) \leq C$ implies that $\|\sqrt{W^d}(\chi_n)\|_{L^2(\Omega')} \leq C\sqrt{\varepsilon_n}$.

This concludes the proof. \square

We conclude the section by investigating a first consequence of Proposition 2.6. For the classical Aviles-Giga functionals in dimension two it is known that a uniform bound on the energies $AG_\varepsilon(\varphi_\varepsilon, \Omega)$ implies a bound on $\nabla \varphi_\varepsilon$ not only in $L^4(\Omega)$ but even in $L^6(\Omega)$ (cf. [5, Theorem 6.1]). Using Proposition 2.6 and exploiting the analogy between H_n^* and the classical Aviles-Giga, in the following proposition we improve the L^4 bound obtained in Remark 2.2.

Proposition 2.7. *Let $(\chi_n)_n \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ and assume that $\sup_n H_n(\chi_n, \Omega) < +\infty$. Then, for every $\Omega' \subset\subset \Omega$, $(\chi_n)_n$ is bounded in $L^6(\Omega')$.*

Proof. We let $\Omega' \subset \subset \Omega$ be fixed. We start by introducing a piecewise affine interpolation $\widehat{\chi}_n$ of the discrete functions $|\chi_n|$. To this end, let $T_{\lambda_n}^-(i, j)$ and $T_{\lambda_n}^+(i, j)$ be the two triangles partitioning the cell $Q_{\lambda_n}(i, j)$ defined by

$$\begin{aligned} T_{\lambda_n}^- &:= \{\lambda_n(i, j) + \lambda_n y \in Q_{\lambda_n}(i, j) : y_1 \in [0, 1], y_2 \in [0, 1 - y_1]\}, \\ T_{\lambda_n}^+ &:= \{\lambda_n(i, j) + \lambda_n y \in Q_{\lambda_n}(i, j) : y_1 \in (0, 1), y_2 \in (1 - y_1, 1)\}. \end{aligned}$$

We define the function $\widehat{\chi}_n$ on $T_{\lambda_n}^-(i, j)$ by interpolating the values on the three vertices of $T_{\lambda_n}^-(i, j)$, *i.e.*,

$$\widehat{\chi}_n(\lambda_n(i, j) + \lambda_n y) := (1 - y_1 - y_2)|\chi_n^{i,j}| + y_1|\chi_n^{i+1,j}| + y_2|\chi_n^{i,j+1}|.$$

Analogously, for $\lambda_n(i, j) + \lambda_n y \in T_{\lambda_n}^+(i, j)$,

$$\widehat{\chi}_n(\lambda_n(i, j) + \lambda_n y) := (1 - y_1)|\chi_n^{i,j+1}| + (1 - y_2)|\chi_n^{i+1,j}| + (y_1 + y_2 - 1)|\chi_n^{i+1,j+1}|.$$

Below, we will exploit Sobolev embeddings to show that $(\widehat{\chi}_n)_n$ is bounded in $L^6(\Omega'')$ for some open set Ω'' with $\Omega' \subset \subset \Omega'' \subset \subset \Omega$. This will conclude the proof since we can control the L^6 norm of χ_n by that of $\widehat{\chi}_n$ as follows: Given any $(i, j) \in \mathbb{Z}^2$, on the sub-triangle

$$T_{\lambda_n}^{1/2}(i, j) := \{\lambda_n(i, j) + \lambda_n y \in Q_{\lambda_n}(i, j) : y_1 \in [0, \frac{1}{2}], y_2 \in [0, \frac{1}{2} - y_1]\} \subset T_{\lambda_n}^-(i, j)$$

we have that $|\widehat{\chi}_n| \geq \frac{1}{2}|\chi_n^{i,j}|$. Therefore,

$$\|\widehat{\chi}_n\|_{L^6(Q_{\lambda_n}(i,j))}^6 \geq C \mathcal{L}^2(T_{\lambda_n}^{1/2}(i, j)) |\chi_n^{i,j}|^6 = C \|\chi_n\|_{L^6(Q_{\lambda_n}(i,j))}^6,$$

where we have used that $\mathcal{L}^2(T_{\lambda_n}^{1/2}(i, j)) = C \mathcal{L}^2(Q_{\lambda_n}(i, j))$ with C independent of n, i, j . For all n large enough, every cell $Q_{\lambda_n}(i, j)$ that intersects Ω' is contained in Ω'' and thus we conclude that

$$\|\chi_n\|_{L^6(\Omega')} \leq C \|\widehat{\chi}_n\|_{L^6(\Omega'')} \quad (2.37)$$

for all n large enough.

To estimate $\widehat{\chi}_n$ in L^6 , let us fix an open and smooth set Ω'' and an additional open set Ω''' satisfying $\Omega' \subset \subset \Omega'' \subset \subset \Omega''' \subset \subset \Omega$. We observe that $\widehat{\chi}_n$ belongs to $W_{\text{loc}}^{1,\infty}(\mathbb{R}^2; \mathbb{R})$ with a Sobolev gradient that is constant on $T_{\lambda_n}^\pm(i, j)$ and given by

$$\begin{aligned} \nabla \widehat{\chi}_n &= (\partial_1^d |\chi_n|^{i,j}, \partial_2^d |\chi_n|^{i,j}) \quad \text{in } T_{\lambda_n}^-(i, j), \\ \nabla \widehat{\chi}_n &= (\partial_1^d |\chi_n|^{i,j+1}, \partial_2^d |\chi_n|^{i+1,j}) \quad \text{in } T_{\lambda_n}^+(i, j). \end{aligned}$$

This entails the estimate $\|\nabla \widehat{\chi}_n\|_{L^2(\Omega'')} \leq \|D^d |\chi_n|\|_{L^2(\Omega''')}$ for n large enough. By use of the reverse triangle inequality, $D^d |\chi_n|$ is bounded by $D^d \chi_n$ and thus, by Proposition 2.6, we get that

$$\varepsilon_n \|\nabla \widehat{\chi}_n\|_{L^2(\Omega'')} \leq C. \quad (2.38)$$

Next, we introduce the convex function

$$V: \mathbb{R} \rightarrow \mathbb{R}, \quad V(s) := \begin{cases} 0 & \text{if } -1 < s < 1, \\ s^2 - 1 & \text{if } |s| \geq 1, \end{cases}$$

and set $V^2(s) := |V(s)|^2$. V^2 is the convex envelope of the double-well potential $s \mapsto (1 - s^2)^2$. By convexity of V^2 and by the definition of $\widehat{\chi}_n$ we have that

$$V^2(\widehat{\chi}_n(\lambda_n(i, j) + \lambda_n y)) \leq (1 - y_1 - y_2)V^2(|\chi_n^{i,j}|) + y_1 V^2(|\chi_n^{i+1,j}|) + y_2 V^2(|\chi_n^{i,j+1}|)$$

for $\lambda_n(i, j) + \lambda_n y \in T_{\lambda_n}^-(i, j)$ and

$$V^2(\widehat{\chi}_n(\lambda_n(i, j) + \lambda_n y)) \leq (1 - y_1)V^2(|\chi_n^{i,j+1}|) + (1 - y_2)V^2(|\chi_n^{i+1,j}|) + (y_1 + y_2 - 1)V^2(|\chi_n^{i+1,j+1}|)$$

for $\lambda_n(i, j) + \lambda_n y \in T_{\lambda_n}^+(i, j)$. Since $V^2(|\chi_n|) \leq (1 - |\chi_n|^2)^2 = W(\chi_n)$, Proposition 2.6 gives us that

$$\int_{\Omega''} \frac{1}{\varepsilon_n} V^2(\widehat{\chi}_n) \, dx \leq \int_{\Omega'''} \frac{1}{\varepsilon_n} W(\chi_n) \, dx \leq C$$

for n large enough. Combining this with (2.38), we have obtained the following energy bound on $\widehat{\chi}_n$:

$$\int_{\Omega''} \frac{1}{\varepsilon_n} V^2(\widehat{\chi}_n) + \varepsilon_n |\nabla \widehat{\chi}_n|^2 dx \leq C. \quad (2.39)$$

Next we introduce a primitive P of the function V , namely the C^1 function

$$P: \mathbb{R} \rightarrow \mathbb{R}, \quad P(s) := \begin{cases} \frac{1}{3}s^3 - s - \frac{2}{3} & \text{if } s \geq 1, \\ 0 & \text{if } -1 < s < 1, \\ \frac{1}{3}s^3 - s + \frac{2}{3} & \text{if } s \leq -1. \end{cases}$$

The function P has cubic growth, *i.e.*,

$$c_1 |s|^3 + c_2 \leq |P(s)| \leq C_1 |s|^3 + C_2 \quad (2.40)$$

with constants $c_1, C_1 > 0$ and $c_2, C_2 \in \mathbb{R}$. In particular, $P \circ \widehat{\chi}_n$ are bounded in $L^1(\Omega'')$ since $\widehat{\chi}_n$ are bounded in $L^4(\Omega'')$, being the piecewise affine interpolations of $|\chi_n|$, which are bounded in $L^4(\Omega)$ by Remark 2.2. Moreover, since P is C^1 and locally Lipschitz and $\widehat{\chi}_n$ belong to $W^{1,\infty}(\Omega''; \mathbb{R})$, by the chain rule $P \circ \widehat{\chi}_n$ are Sobolev functions as well and $\nabla(P \circ \widehat{\chi}_n) = (V \circ \widehat{\chi}_n) \nabla \widehat{\chi}_n$. Using Young's inequality, we get that

$$\|\nabla(P \circ \widehat{\chi}_n)\|_{L^1(\Omega'')} \leq \frac{1}{2} \int_{\Omega''} \frac{1}{\varepsilon_n} |V(\widehat{\chi}_n)|^2 + \varepsilon_n |\nabla \widehat{\chi}_n|^2 dx \leq C$$

by (2.39). Thus, $P \circ \widehat{\chi}_n$ are bounded in $W^{1,1}(\Omega'')$ and recalling that we have chosen Ω'' to be a smooth domain, Poincaré's inequality leads to a bound on $P \circ \widehat{\chi}_n$ in $L^2(\Omega'')$. Finally, (2.40) yields $\|\widehat{\chi}_n\|_{L^6(\Omega'')}^6 \leq \|c_1^{-1}(P \circ \widehat{\chi}_n - c_2)\|_{L^2(\Omega'')}^2 \leq C$ and thereby the desired L^6 bound. Then by (2.37) we conclude the proof. \square

3. ENTROPIES AND THE LIMIT FUNCTIONAL

In this section we define the notion of entropy that we will use in this paper and define the limit functional H for our energies H_n .

Definition 3.1. We say that a map $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an *entropy* if $\Phi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$ and it satisfies

$$\xi \cdot (D\Phi(\xi)\xi^\perp) = 0 \quad \text{for all } \xi \in \mathbb{R}^2. \quad (3.1)$$

We define the space $\text{Ent} := \{\Phi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2), \Phi \text{ is an entropy}\}$.

This notion of entropy strongly resembles the one used in [26]. (There it is not required that Φ is zero in a neighborhood of zero.) As in [26, Lemma 2.2] we associate to every $\Phi \in \text{Ent}$ a pair of functions (Ψ, α) defined by

$$\alpha(\xi) := \frac{\xi^\perp \cdot (D\Phi(\xi)\xi^\perp)}{|\xi|^2}, \quad (3.2)$$

$$\Psi(\xi) := -\frac{1}{2|\xi|^2} (D\Phi(\xi) - \alpha(\xi)\text{Id})\xi. \quad (3.3)$$

Note that $\text{supp}(\Psi), \text{supp}(\alpha) \subset \text{supp}(\Phi) \subset \mathbb{R}^2 \setminus \{0\}$ and $\Psi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$ and $\alpha \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$, since $\Phi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$. This will be useful for technical reasons in the proofs.

Using property (3.1) and the identity $\text{Id} = \frac{1}{|\xi|^2} \xi \otimes \xi + \frac{1}{|\xi|^2} \xi^\perp \otimes \xi^\perp$, one sees that the pair (Ψ, α) satisfies (and in fact is characterized uniquely by) the relation

$$D\Phi(\xi) + 2\Psi(\xi) \otimes \xi = \alpha(\xi)\text{Id}. \quad (3.4)$$

Definition 3.2. Given $\Phi \in \text{Ent}$, we define

$$\|\Phi\|_{\text{Ent}} := \text{Lip}(\Psi),$$

where $\text{Lip}(\Psi)$ is the Lipschitz constant of the function Ψ given by (3.3).

We remark that $\|\cdot\|_{\text{Ent}}$ is a norm on Ent . Indeed, Ψ and α are linear in Φ , see (3.3) and (3.2). Moreover, recalling that Φ , Ψ , and α have compact support, if $\text{Lip}(\Psi) = 0$, then $\Psi \equiv 0$ and (3.4) yields $D\Phi = \alpha \text{Id}$. Since the row-wise $\text{curl}(\alpha \text{Id})$ equals $\nabla^\perp \alpha$, we get $\alpha \equiv 0$ and thus $\Phi \equiv 0$.

Let \mathcal{A} be the class of open and bounded subsets of \mathbb{R}^2 as in (2.15). To state our main result, we introduce the functional $H: L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) \times \mathcal{A} \rightarrow [0, +\infty]$ defined by

$$H(\chi, \Omega) := \bigvee_{\substack{\Phi \in \text{Ent} \\ \|\Phi\|_{\text{Ent}} \leq 1}} |\text{div}(\Phi \circ \chi^\perp)|(\Omega), \quad (3.5)$$

if χ satisfies

$$|\chi| = 1 \text{ a.e. in } \Omega, \quad \text{curl}(\chi) = 0 \text{ in } \mathcal{D}'(\Omega), \quad \text{div}(\Phi \circ \chi^\perp) \in \mathcal{M}_b(\Omega) \text{ for all } \Phi \in \text{Ent}, \quad (3.6)$$

and H extended to $+\infty$ otherwise in $L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$. For a discussion on the role played by the functional H in the analysis of the classical Aviles-Giga functionals, we refer to Remark 3.5 below.

Using compactly supported instead of non-compactly supported entropies in the definition of H is not restrictive, as we show in Proposition 3.3 below. In particular, taking the supremum in (3.5) over the entropies introduced in [26] does not affect the values of the functional H .

Proposition 3.3. *Let $\Phi \in C^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$ be a function satisfying (3.1) for $\xi \neq 0$. Notice that for such Φ , (3.2), (3.3) define functions $\alpha \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and $\Psi \in C^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$. Assume that $\text{Lip}(\Psi) \leq 1$. Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set and let $\chi \in L^\infty(\Omega; \mathbb{S}^1)$ satisfy $\text{curl}(\chi) = 0$ in $\mathcal{D}'(\Omega)$. Let $\Omega' \subset \Omega$ be an open set. Then we have that*

$$|\text{div}(\Phi \circ \chi^\perp)|(\Omega') \leq H(\chi, \Omega').$$

Proof. We start by showing that the singularities of Φ, Ψ, α at 0 can be removed. To this end we note that for Φ, Ψ, α (3.4) holds true in $\mathbb{R}^2 \setminus \{0\}$. Computing the row-wise curl of both sides of this identity, and using that the curl of the identity $\xi \mapsto \xi$ vanishes, we get that

$$\nabla^\perp \alpha(\xi) = -2D\Psi(\xi) \cdot \xi^\perp.$$

Since $\text{Lip}(\Psi) \leq 1$, we obtain that $|\nabla \alpha(\xi)| \leq 2|\xi|$. Note that this implies that α is Lipschitz in $B_1(0) \setminus \{0\}$ and thus admits a unique continuous extension to the whole \mathbb{R}^2 . In the same way, Ψ admits a unique continuous extension to \mathbb{R}^2 which still satisfies $\text{Lip}(\Psi) \leq 1$. By (3.4) we then infer that also $D\Phi$ extends continuously to \mathbb{R}^2 , and, as a consequence Φ can be extended to a C^1 function on the whole \mathbb{R}^2 .

Next, we reduce the claim to the ‘‘effective entropy’’ Φ^{eff} defined on \mathbb{R}^2 by

$$\Phi^{\text{eff}}(\xi) := \Phi(\xi) - \Phi(0) - \alpha(0)\xi + |\xi|^2\Psi(0).$$

Observe that Φ^{eff} is C^1 on \mathbb{R}^2 , smooth on $\mathbb{R}^2 \setminus \{0\}$ and satisfies (3.1). Since $|\chi| = 1$ and $\text{curl}(\chi) = 0$, we have that

$$\text{div}(\Phi^{\text{eff}} \circ \chi^\perp) = \text{div}(\Phi \circ \chi^\perp) - \text{div}(\Phi(0)) - \alpha(0)\text{div}(\chi^\perp) + \text{div}(|\chi|^2\Psi(0)) = \text{div}(\Phi \circ \chi^\perp). \quad (3.7)$$

Moreover, the functions α^{eff} and Ψ^{eff} associated to Φ^{eff} are given by

$$\alpha^{\text{eff}}(\xi) = \alpha(\xi) - \alpha(0) \quad \text{and} \quad \Psi^{\text{eff}}(\xi) = \Psi(\xi) - \Psi(0)$$

and by our previous bound on $\nabla \alpha$ we infer that $|\alpha^{\text{eff}}(\xi)| \leq C|\xi|^2$. We furthermore obtain the bounds

$$|\Psi^{\text{eff}}(\xi)| \leq |\xi|, \quad |D\Phi^{\text{eff}}(\xi)| \leq C|\xi|^2, \quad |\Phi^{\text{eff}}(\xi)| \leq C|\xi|^3 \quad (3.8)$$

by recalling that $\text{Lip}(\Psi) \leq 1$ and then using that (3.4) holds for $\Phi^{\text{eff}}, \Psi^{\text{eff}}, \alpha^{\text{eff}}$ and that $\Phi^{\text{eff}}(0) = 0$. Note moreover that $\text{Lip}(\Psi^{\text{eff}}) \leq 1$.

Let us now approximate Φ^{eff} by entropies $\Phi_k \in \text{Ent}$. To this end we consider a sequence of functions $\zeta_k \in C^\infty((0, \infty))$ with $\zeta_k(1) = 1$ and such that

$$0 \leq \zeta_k(s) \leq 1, \quad |\zeta'_k(s)| \leq \frac{C}{ks}, \quad |\zeta''_k(s)| \leq \frac{C}{ks^2} \quad (3.9)$$

for all $s > 0$, where the constant C is independent of k and s . To find the functions ζ_k , we first construct a sequence of functions $\rho_k \in W^{2,\infty}((0, \infty))$ with compact supports, satisfying the bounds in (3.9) and such that $\rho_k = 1$ in a neighborhood of 1. This can be achieved following the scheme shown in Figure 3. The desired functions ζ_k are then obtained by mollifying ρ_k on a sufficiently small scale.

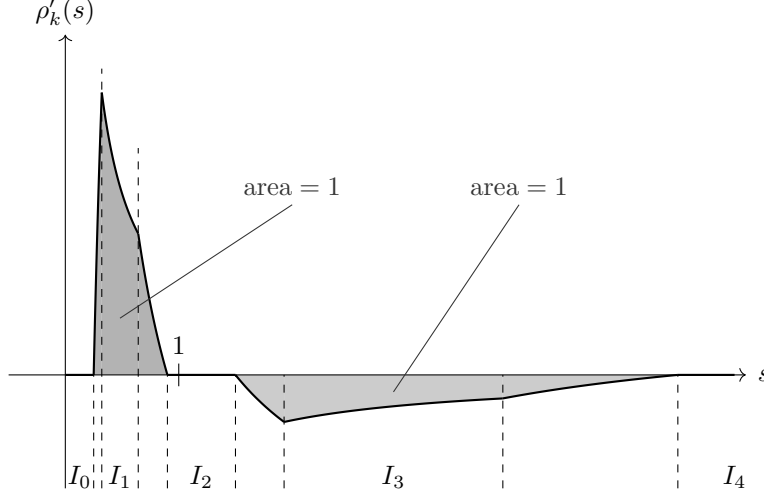


Figure 3. The figure shows the construction of ρ'_k . It is $\rho'_k = 0$ in $I_0 \cup I_2 \cup I_4$, I_2 being a neighborhood of 1. In I_1 and I_3 , ρ'_k takes the form of a hyperbolic arc and is given by $\pm \frac{1}{ks}$. In the four intervals in between, the pieces are joined together with hyperbolic arcs of the form $\pm \frac{2}{ks} + c$, where the constant c is chosen suitably for each individual interval. Notice that by positioning I_1 close enough to $s = 0$ and by letting I_3 extend far enough to the right, it is possible to achieve that both gray areas each have an area of 1. This is due to the fact that the integral of $\frac{1}{ks}$ is infinite both close to 0 and close to ∞ . The primitive ρ_k of ρ'_k with $\rho_k(0) = 0$ has the desired properties.

Let us define the approximations $\Phi_k \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$ by

$$\Phi_k(\xi) := \zeta_k(|\xi|)\Phi^{\text{eff}}(\xi)$$

and observe that they indeed satisfy (3.1). Let us estimate $\|\Phi_k\|_{\text{Ent}}$. The function Ψ_k associated to Φ_k through (3.2), (3.3) is given by

$$\Psi_k(\xi) = \zeta_k(|\xi|)\Psi^{\text{eff}}(\xi) - \frac{1}{2} \frac{\zeta'_k(|\xi|)}{|\xi|} \Phi^{\text{eff}}(\xi).$$

Moreover,

$$D\Psi_k(\xi) = \zeta_k(|\xi|)D\Psi^{\text{eff}}(\xi) + R_k(\xi),$$

where

$$R_k(\xi) = \zeta'_k(|\xi|)\Psi^{\text{eff}}(\xi) \otimes \frac{\xi}{|\xi|} - \frac{1}{2} \frac{\zeta'_k(|\xi|)}{|\xi|} D\Phi^{\text{eff}}(\xi) + \frac{1}{2} \frac{\zeta'_k(|\xi|)}{|\xi|^2} \Phi^{\text{eff}}(\xi) \otimes \frac{\xi}{|\xi|} - \frac{1}{2} \frac{\zeta''_k(|\xi|)}{|\xi|} \Phi^{\text{eff}}(\xi) \otimes \frac{\xi}{|\xi|}.$$

By virtue of (3.8) and (3.9) we have that $R_k \rightarrow 0$ uniformly as $k \rightarrow \infty$. Since $|\zeta_k(|\xi|)| \leq 1$ this implies that $\|\Phi_k\|_{\text{Ent}} = \text{Lip}(\Psi_k) \leq \text{Lip}(\Psi^{\text{eff}}) + o_k(1) \leq 1 + o_k(1)$.

Finally, we show that

$$|\text{div}(\Phi^{\text{eff}} \circ \chi^\perp)|(\Omega') \leq (1+t)H(\chi, \Omega')$$

for all $t > 0$. Indeed, choose k such that $\|\Phi_k\|_{\text{Ent}} \leq 1+t$. The case $\Phi_k = 0$ being trivial, we may assume that $\|\Phi_k\|_{\text{Ent}} > 0$. Then, since $\Phi^{\text{eff}} = \Phi_k$ on \mathbb{S}^1 we get that

$$|\text{div}(\Phi^{\text{eff}} \circ \chi^\perp)|(\Omega') \leq \|\Phi_k\|_{\text{Ent}} \left| \text{div} \left(\frac{\Phi_k}{\|\Phi_k\|_{\text{Ent}}} \circ \chi^\perp \right) \right|(\Omega') \leq (1+t)H(\chi, \Omega').$$

In view of (3.7) this concludes the proof. \square

Remark 3.4 (Notions of entropy and the domain of the Γ -limit). Entropies are a central tool in the analysis of the Aviles-Giga functionals AG_ε in (1.7). In this remark we give an overview of some notions of entropy in the context of Aviles-Giga functionals available in the literature.

As explained above, our definition of entropies is inspired by that given in [26], where entropies are used to prove compactness properties of sequences with equibounded Aviles-Giga energies.

With the aim of better understanding the fine properties of solutions of the eikonal equation selected by the Aviles-Giga functionals, another definition of entropy has been given in [25]. There the authors explain that the asymptotic admissible set of the Aviles-Giga functionals is contained in the space $A(\Omega)$ of solutions to the eikonal equation $|\nabla\varphi| = 1$ satisfying

$$\operatorname{div}(\Phi \circ \nabla^\perp \varphi) \in \mathcal{M}_b(\Omega)$$

for all smooth $\Phi: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ (the entropies in [25]) with the property that

$$\text{if } U \subset \mathbb{R}^2 \text{ is open, } m: U \rightarrow \mathbb{S}^1 \text{ is smooth, and } \operatorname{div}(m) = 0, \quad \text{then } \operatorname{div}(\Phi \circ m) = 0. \quad (3.10)$$

This notion of entropy (also used in other variants in [32, 24, 28, 35, 42]) and the one in Definition 3.1 (or in [26]) are basically equivalent. Specifically, every entropy Φ of the type (3.10) admits an extension to a smooth function on \mathbb{R}^2 that is an entropy in the sense of Definition 3.1. Conversely, for every entropy in the sense of Definition 3.1, its restriction to \mathbb{S}^1 satisfies (3.10). In particular, condition (3.6) for $\chi = \nabla\varphi$ is equivalent to requiring that $\varphi \in A(\Omega)$.

A smaller class of entropies has been considered in [34, 8, 5]. They are of the form

$$\Sigma_{\nu, \nu^\perp}(\xi) := \frac{2}{3}((\xi \cdot \nu^\perp)^3 \nu + (\xi \cdot \nu)^3 \nu^\perp), \quad \nu \in \mathbb{S}^1. \quad (3.11)$$

In [5] they are used to prove compactness of sequences with equibounded Aviles-Giga energy and to formulate an asymptotic lower bound (*cf.* Remark 3.5 below). In particular, it is shown that the asymptotic admissible set of the Aviles-Giga functionals is contained in the space $AG(\Omega)$ of solutions to the eikonal equation $|\nabla\varphi| = 1$ satisfying

$$\operatorname{div}(\Sigma_{\nu, \nu^\perp} \circ \nabla^\perp \varphi) \in \mathcal{M}_b(\Omega)$$

for all $\nu \in \mathbb{S}^1$ (in fact, it is equivalent to require this only for $\nu_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\nu_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$). As Σ_{ν, ν^\perp} satisfy (3.10), the inclusion $A(\Omega) \subset AG(\Omega)$ holds true.

To the best of our knowledge, it is not known whether $A(\Omega) = AG(\Omega)$, *i.e.*, whether all entropy productions $\operatorname{div}(\Phi \circ \nabla^\perp \varphi)$ can be controlled by only the entropy productions $\operatorname{div}(\Sigma_{\nu, \nu^\perp} \circ \nabla^\perp \varphi)$ if φ solves $|\nabla\varphi| = 1$. This problem has been intensively studied in the recent years and several partial results have been obtained. As a first evidence, in [37, 35] it has been proved that if $\operatorname{div}(\Sigma_{\nu, \nu^\perp} \circ \nabla^\perp \varphi) = 0$ for $\nu = \nu_1, \nu_2$, then all entropy productions $\operatorname{div}(\Phi \circ \nabla^\perp \varphi)$ vanish. In [37] this follows from the result that, under the previous assumption, $\nabla\varphi$ satisfies rigidity, *i.e.*, $\nabla\varphi$ is locally Lipschitz outside a locally finite set of vortex-like singularities. In [32, 24] it is shown that also suitable fractional Sobolev regularity of $\nabla\varphi$ triggers the same rigidity. A further step towards understanding the threshold regularity for rigidity has been achieved in [28]. There it is shown that requiring that all entropy productions $\operatorname{div}(\Phi \circ \nabla^\perp \varphi)$ are finite measures is locally equivalent to the Besov regularity $\nabla\varphi \in B_{3, \infty}^{1/3}$. Already the stronger regularity $\nabla\varphi \in B_{3, q}^{1/3}$ for $q < \infty$ yields rigidity. In [38], the authors raise the question whether $B_{3p, \infty}^{1/3}$ regularity, $p > 1$, triggers this rigidity, too. As a partial result, they prove that this regularity implies that the entropy productions $\operatorname{div}(\Phi \circ \nabla^\perp \varphi)$ belong to L^p , which they conjecture to be enough to deduce rigidity. Furthermore, the authors obtain further evidence that $\operatorname{div}(\Phi \circ \nabla^\perp \varphi)$ can be controlled by $\operatorname{div}(\Sigma_{\nu, \nu^\perp} \circ \nabla^\perp \varphi)$ for $\nu = \nu_1, \nu_2$. More precisely, it is shown that if $p \geq \frac{4}{3}$, then $\operatorname{div}(\Sigma_{\nu, \nu^\perp} \circ \nabla^\perp \varphi) \in L^p$ implies that $\operatorname{div}(\Phi \circ \nabla^\perp \varphi) \in L^p$ for all entropies Φ . Moreover, according to [41], a preliminary result on the question whether this can be extended to the case of measures is available as a consequence of recent developments, specifically, the eikonal equation's kinetic formulation established in [28], a Lagrangian representation method [13, 39, 43, 40, 42],

and ideas used in [38]. The precise statement requires the introduction of a subclass of parametrized entropies $\{\Phi_f : f: \mathbb{S}^1 \rightarrow \mathbb{R}\}$ (cf. [28, Subsection 3.1]), which is rich enough to establish the kinetic formulation. The results in [28] imply that if $\text{div}(\Phi_f \circ \nabla^\perp \varphi) \in \mathcal{M}_b$ for all f , then $\text{div}(\Phi \circ \nabla^\perp \varphi) \in \mathcal{M}_b$ for all entropies Φ . By [41], if it is assumed *a priori* that the entropy productions $\text{div}(\Phi_f \circ \nabla^\perp \varphi)$ are finite measures for all f , then the precise structure of the kinetic defect measure obtained in [42, Proposition 1.7] allows one to control $\text{div}(\Phi_f \circ \nabla^\perp \varphi)$, for all f , in terms of $\text{div}(\Sigma_{\nu, \nu^\perp} \circ \nabla^\perp \varphi)$, $\nu = \nu_1, \nu_2$, up to a multiplicative constant depending only on Φ_f .

Remark 3.5. We introduce the functional H in (3.5) for the Γ -convergence analysis of the functionals H_n . In fact, $\varphi \mapsto H(\nabla \varphi, \Omega)$ is also a candidate for the Γ -limit of the classical Aviles-Giga functionals $AG_\varepsilon(\cdot, \Omega)$ defined (1.7). In particular, it can be shown that the liminf inequality holds true, *i.e.*, that $\varphi_\varepsilon \rightarrow \varphi$ in $W_{\text{loc}}^{1,1}(\Omega)$ implies that

$$\liminf_{\varepsilon \rightarrow 0} AG_\varepsilon(\varphi_\varepsilon, \Omega) \geq H(\nabla \varphi, \Omega). \quad (3.12)$$

We remark that all arguments required for the proof of this liminf inequality are contained in Section 6; we refer to Remark 6.1 for an outline of the proof.³

We remark that in the analysis of the Aviles-Giga functionals AG_ε the candidate Γ -limit most often used in the literature is given by $\varphi \mapsto H^0(\nabla \varphi, \Omega)$, where H^0 slightly differs from (3.5). Specifically,

$$H^0(\chi, \Omega) := \bigvee_{\nu \in \mathbb{S}^1} |\text{div}(\Sigma_{\nu, \nu^\perp} \circ \chi^\perp)|(\Omega) = \left| \begin{pmatrix} \text{div}(\Sigma_{\nu_1, \nu_1^\perp} \circ \chi^\perp) \\ \text{div}(\Sigma_{\nu_2, \nu_2^\perp} \circ \chi^\perp) \end{pmatrix} \right|(\Omega). \quad (3.13)$$

Here, $\nu_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\nu_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and Σ_{ν, ν^\perp} are the entropies defined by (3.11). The functional $H^0(\chi, \Omega)$ is defined by (3.13) if χ satisfies

$$|\chi| = 1 \text{ a.e. in } \Omega, \quad \text{curl}(\chi) = 0 \text{ in } \mathcal{D}'(\Omega), \quad \text{div}(\Sigma_{\nu, \nu^\perp} \circ \chi^\perp) \in \mathcal{M}_b(\Omega) \text{ for all } \nu \in \mathbb{S}^1, \quad (3.14)$$

and extended to $+\infty$ otherwise. The functional H^0 has first been considered in [8, 5], where it has been shown that $\varphi \mapsto H^0(\nabla \varphi, \Omega)$ provides a lower bound on the Γ -liminf of the Aviles-Giga functionals $AG_\varepsilon(\cdot, \Omega)$. As it is still not known whether the domain of H^0 is contained in $A(\Omega)$ (cf. Remark 3.4), it is natural to look for a limit functional that takes into account all entropy productions, such as H in (3.5).

Let us discuss next why the lower bound (3.12) is coherent with the already known results on the Γ -limiting behavior of the Aviles-Giga functionals. In Corollary 3.6 below, we show that $H \geq H^0$. For a discussion about whether $H = H^0$, see Remark 3.4 above. Since $H \geq H^0$, H provides a lower bound of the Γ -liminf AG_ε that is possibly sharper than H^0 . Moreover, in Corollary 3.8 below we show that

$$\chi \in BV(\Omega; \mathbb{S}^1) \text{ and } \text{curl}(\chi) = 0 \quad \implies \quad H(\chi, \Omega) = H^0(\chi, \Omega) = \frac{1}{6} \int_{J_\chi} |\chi|^3 d\mathcal{H}^1. \quad (3.15)$$

In particular, the lower bound obtained from H is optimal on φ if $\nabla \varphi \in BV(\Omega; \mathbb{S}^1)$, as for such φ the limsup inequality corresponding to H^0 has been proved in [22, 46].

As we show in Proposition 3.7 below, the theory established in [25] allows us to prove that, even if χ is not BV , the restriction of $H(\chi, \cdot)$ to the jump set J_χ is still given by $\frac{1}{6} \int_{J_\chi} |\chi|^3 d\mathcal{H}^1$. It is however not known whether H is concentrated on J_χ . This is related to a conjecture raised in [25, Conjecture 1], which would imply that the identity $H(\chi, \Omega) = \frac{1}{6} \int_{J_\chi} |\chi|^3 d\mathcal{H}^1$ holds for all χ satisfying (3.6). We remark that concentration results of this kind have been proved for related models in [43, 40].

The following result is a consequence of Proposition 3.3.

Corollary 3.6. *Let H be the functional in (3.5) and let H^0 be defined by (3.13). We have that $H \geq H^0$.*

³The last step in that proof is not required for the proof of (3.12), but only needed to prove the same liminf inequality for the variants AG_ε^Δ in (1.6).

Proof. For $\nu \in \mathbb{S}^1$, we compute the derivative of the function Σ_{ν, ν^\perp} defined by (3.11) to be

$$D\Sigma_{\nu, \nu^\perp}(\xi) = 2((\xi \cdot \nu^\perp)^2 \nu \otimes \nu^\perp + (\xi \cdot \nu)^2 \nu^\perp \otimes \nu).$$

Using the elementary identities $\xi^\perp \cdot \nu^\perp = \xi \cdot \nu$ and $\xi^\perp \cdot \nu = -\xi \cdot \nu^\perp$ we obtain that Σ_{ν, ν^\perp} satisfies (3.1). Computing the functions α and Ψ associated to Σ_{ν, ν^\perp} through (3.2), (3.3), we obtain

$$\begin{aligned} \alpha(\xi) &= -2(\xi \cdot \nu^\perp)(\xi \cdot \nu), \\ \Psi(\xi) &= -(\xi \cdot \nu^\perp)\nu - (\xi \cdot \nu)\nu^\perp = -(\nu^\perp \otimes \nu + \nu \otimes \nu^\perp)\xi, \end{aligned}$$

where we have used the identities $|\xi|^2 = (\xi \cdot \nu)^2 + (\xi \cdot \nu^\perp)^2$ and $\xi = (\xi \cdot \nu)\nu + (\xi \cdot \nu^\perp)\nu^\perp$. Since the matrix $\nu^\perp \otimes \nu + \nu \otimes \nu^\perp$ is orthogonal, we find that $\text{Lip}(\Psi) = 1$. As a consequence, applying Proposition 3.3 to Σ_{ν, ν^\perp} , we get for every χ satisfying (3.14) that

$$|\text{div}(\Sigma_{\nu, \nu^\perp} \circ \chi^\perp)|(\Omega') \leq H(\chi, \Omega')$$

for every open $\Omega' \subset \Omega$. By considering partitions of Ω to pass to the supremum, we then infer that $H^0(\chi, \Omega') \leq H(\chi, \Omega)$ as desired. \square

For the next result we recall that the jump set J_ν is defined for every $\nu \in L^1_{\text{loc}}(\Omega; \mathbb{R}^2)$ according to Subsection 2.3.

Proposition 3.7. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set and let $\chi \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ satisfy (3.6). Let J_χ be the jump set of $\chi|_\Omega$. Then we have that*

$$\bigvee_{\substack{\Phi \in \text{Ent} \\ \|\Phi\|_{\text{Ent}} \leq 1}} |\text{div}(\Phi \circ \chi^\perp)|(\Omega) = \frac{1}{6} \int_{J_\chi} |\chi|^3 d\mathcal{H}^1.$$

Proof. Due to the relation between entropies in Ent and functions Φ satisfying (3.10) as explained in Remark 3.4, the theory in [25] and specifically [25, Theorem 1] applies to χ . (More precisely, as the authors in [25] work with divergence-free fields instead of curl-free fields, we apply their results to χ^\perp .) According to this theory, there exists a set $J \subset \Omega$, coinciding with J_χ up to a \mathcal{H}^1 -null set, such that

$$\begin{aligned} \text{div}(\Phi \circ \chi^\perp) \llcorner J &= \left(\Phi((\chi^\perp)^+) - \Phi((\chi^\perp)^-) \right) \cdot \nu_\chi \mathcal{H}^1 \llcorner J, \\ \text{div}(\Phi \circ \chi^\perp) \llcorner K &= 0 \quad \text{for all } K \subset \Omega \setminus J \text{ with } \mathcal{H}^1(K) < +\infty \end{aligned}$$

for all $\Phi: \mathbb{S}^1 \mapsto \mathbb{R}^2$ satisfying (3.10). As a consequence,

$$\text{div}(\Phi \circ \chi^\perp) \llcorner J_\chi = \left(\Phi((\chi^\perp)^+) - \Phi((\chi^\perp)^-) \right) \cdot \nu_\chi \mathcal{H}^1 \llcorner J_\chi.$$

Since the restriction to \mathbb{S}^1 of any $\Phi \in \text{Ent}$ satisfies (3.10), the above equation is also true for every $\Phi \in \text{Ent}$. The same applies to $\Phi = \Sigma_{\nu, \nu^\perp}$ for any $\nu \in \mathbb{S}^1$ as well. As a consequence we have that

$$\mu(J_\chi) = \int_{J_\chi} \sup_{\substack{\Phi \in \text{Ent} \\ \|\Phi\|_{\text{Ent}} \leq 1}} \left| \left(\Phi((\chi^\perp)^+) - \Phi((\chi^\perp)^-) \right) \cdot \nu_\chi \right| d\mathcal{H}^1 \quad (3.16)$$

and

$$\mu^0(J_\chi) = \int_{J_\chi} \sup_{\nu \in \mathbb{S}^1} \left| \left(\Sigma_{\nu, \nu^\perp}((\chi^\perp)^+) - \Sigma_{\nu, \nu^\perp}((\chi^\perp)^-) \right) \cdot \nu_\chi \right| d\mathcal{H}^1, \quad (3.17)$$

where we have set

$$\mu := \bigvee_{\substack{\Phi \in \text{Ent} \\ \|\Phi\|_{\text{Ent}} \leq 1}} |\text{div}(\Phi \circ \chi^\perp)| \quad \text{and} \quad \mu^0 := \bigvee_{\nu \in \mathbb{S}^1} |\text{div}(\Sigma_{\nu, \nu^\perp} \circ \chi^\perp)|.$$

Let us note that from Corollary 3.6 it follows that $\mu \geq \mu^0$. Let us also note that from $|\chi| = 1$ a.e. it follows that $\chi^+(x), \chi^-(x) \in \mathbb{S}^1$ for every $x \in J_\chi$. Let us fix $x \in J_\chi$. We recall from Subsection 2.3 that there exists a $d \in \mathbb{R}$ such that $\chi^+(x) - \chi^-(x) = d\nu_\chi(x)$.

We now claim that for all $a, b \in \mathbb{S}^1$ and $\nu \in \mathbb{S}^1$ with the properties that $a \neq b$ and $(a - b) = d\nu$ for some $d \in \mathbb{R}$, we have that

$$|(\Phi(a^\perp) - \Phi(b^\perp)) \cdot \nu| \leq \frac{1}{6}|a - b|^3 \quad \text{for all } \Phi \in \text{Ent with } \|\Phi\|_{\text{Ent}} \leq 1 \quad (3.18)$$

and

$$|(\Sigma_{\nu, \nu^\perp}(a^\perp) - \Sigma_{\nu, \nu^\perp}(b^\perp)) \cdot \nu| = \frac{1}{6}|a - b|^3. \quad (3.19)$$

As a consequence, the supremum in (3.17) at x is attained for $\nu = \nu_\chi(x)$, takes the value $\frac{1}{6}|\chi(x)|^3$, and coincides with the supremum in (3.16) at x . This concludes the proof.

To prove (3.18), let us note that the conditions on a, b, ν, d imply that $a \cdot \nu^\perp = b \cdot \nu^\perp \in \{\pm\sqrt{1 - |d|^2/4}\}$ and $a \cdot \nu = -b \cdot \nu = \frac{d}{2}$. For $\Phi \in \text{Ent}$ with $\|\Phi\|_{\text{Ent}} \leq 1$ we get that

$$\begin{aligned} (\Phi(a^\perp) - \Phi(b^\perp)) \cdot \nu &= \int_0^{\frac{d}{2}} \nu \cdot \frac{d}{ds} \left(\Phi(- (a \cdot \nu^\perp)\nu + s\nu^\perp) - \Phi(- (a \cdot \nu^\perp)\nu - s\nu^\perp) \right) ds \\ &= -2 \int_0^{\frac{d}{2}} \nu \cdot \left(\Psi(- (a \cdot \nu^\perp)\nu + s\nu^\perp) - \Psi(- (a \cdot \nu^\perp)\nu - s\nu^\perp) \right) s ds \end{aligned}$$

where we have used that (3.4) yields $\nu \cdot (D\Phi(\xi)\nu^\perp) = -2(\nu \cdot \Psi(\xi))(\xi \cdot \nu^\perp)$, Ψ being the function associated to Φ through (3.3). By Definition 3.2 we have that $\text{Lip}(\Psi) \leq 1$ and thus we infer that

$$|(\Phi(a^\perp) - \Phi(b^\perp)) \cdot \nu| \leq 2 \int_0^{\frac{|d|}{2}} 2s^2 ds = \frac{4}{3} \frac{|d|^3}{8} = \frac{1}{6}|a - b|^3$$

as desired.

To prove (3.19), from the definition of Σ_{ν, ν^\perp} in (3.11) we compute that

$$|(\Sigma_{\nu, \nu^\perp}(a^\perp) - \Sigma_{\nu, \nu^\perp}(b^\perp)) \cdot \nu| = \frac{2}{3} |(a^\perp \cdot \nu^\perp)^3 - (b^\perp \cdot \nu^\perp)^3| = \frac{2}{3} \frac{|d|^3}{4} = \frac{1}{6}|a - b|^3,$$

where we have used that $a \cdot \nu = -b \cdot \nu = \frac{d}{2}$. \square

Corollary 3.8. *Let Ω , χ , and J_χ be as in Proposition 3.7. If additionally $\chi \in BV(\Omega; \mathbb{S}^1)$, then we have that*

$$H(\chi, \Omega) = \frac{1}{6} \int_{J_\chi} |\chi|^3 d\mathcal{H}^1.$$

Proof. By Proposition 3.7 and the definition (3.5), it remains only to prove that $|\text{div}(\Phi \circ \chi^\perp)|(\Omega \setminus J_\chi) = 0$ for every $\Phi \in \text{Ent}$. Fix $\Phi \in \text{Ent}$, let Ψ be defined by (3.3), and let us set $\tilde{\Phi}(\xi) := \Phi(\xi) - (1 - |\xi|^2)\Psi(\xi)$. We observe that $\Phi \circ \chi^\perp = \tilde{\Phi} \circ \chi^\perp$ a.e. in Ω . Moreover, $\tilde{\Phi} \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$ and therefore, by the Vol'pert chain rule (cf. Subsection 2.2), we have that

$$|\text{div}(\Phi \circ \chi^\perp)|(\Omega \setminus J_\chi) = |\text{div}(\tilde{\Phi} \circ \chi^\perp)|(\Omega \setminus J_\chi) = |\text{tr}(D\tilde{\Phi}(\chi^\perp)(D^a \chi^\perp + D^c \chi^\perp))|(\Omega).$$

Recall that in the above formula, $D\tilde{\Phi}$ is evaluated at the approximate limits of χ^\perp . Since $\chi^\perp \in \mathbb{S}^1$ a.e. in Ω , its approximate limit lies in \mathbb{S}^1 at every point where it is defined. Next, observe that $D\tilde{\Phi}(\xi) = \alpha(\xi)\text{Id} - (1 - |\xi|^2)D\Psi(\xi)$ by (3.4). As a consequence,

$$\text{tr}(D\tilde{\Phi}(\chi^\perp)(D^a \chi^\perp + D^c \chi^\perp)) = \alpha(\chi^\perp)\text{tr}(D^a \chi^\perp + D^c \chi^\perp) = 0,$$

since $\text{curl}(\chi) = 0$ implies that the absolutely continuous and Cantor parts of $\text{div}(\chi^\perp)$ vanish. This concludes the proof. \square

4. STATEMENT OF THE MAIN RESULTS

4.1. List of variables, parameters, and symbols. For the reader's convenience we summarize in the following list the main variables and parameters used in the paper:

- λ_n is the lattice spacing. We assume that $\lambda_n \rightarrow 0$.
- α_n is the parameter in the energy (1.1) depending on λ_n . We assume that $\alpha_n \rightarrow 0$. Moreover, $\beta_n \equiv 2$.
- $\delta_n := 4 - \frac{\alpha_n}{2}$ is set to get the identities (2.10)–(2.11). We have that $\delta_n \rightarrow 0$;
- $\varepsilon_n := \frac{\lambda_n}{\sqrt{\delta_n}}$ is the parameter corresponding to the parameter ε in the analogy between the energies H_n and the Aviles-Giga functionals AG_ε in (1.7). We assume that $\varepsilon_n \rightarrow 0$.
- We let $u \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$ denote spin fields, interpreted as \mathbb{S}^1 -valued piecewise constant functions.
- θ^{hor} and θ^{ver} are the oriented angles between adjacent spins of the spin field u as defined in (2.6).
- χ is the relevant variable for the main result in the paper. It is defined in terms of θ^{hor} and θ^{ver} in (2.8) and represents the direction along which the helical configuration is rotating most, see Figure 1.
- $\tilde{\chi}$ is a variant of χ defined in (2.8).
- $\bar{\chi}$ is the linearized variant of χ defined in (2.19). As $n \rightarrow \infty$ we heuristically have that $\chi \simeq \tilde{\chi} \simeq \bar{\chi}$.
- \mathcal{A}_0 is the class of admissible domains Ω in our problem defined by (2.14).
- H_n are the discrete functionals studied in this paper and defined by (2.16).
- W^{d} and A^{d} are discrete operators used to define H_n . They are defined in (2.12).
- H_n^* are the auxiliary Aviles-Giga-like discrete functionals defined by (2.22), which help in providing bounds on χ through Proposition 2.6.
- W is the potential in the classical Aviles-Giga functionals, $W(\xi) = (1 - |\xi|^2)^2$.
- H is the candidate discrete-to-continuum Γ -limit of the energies H_n . It is defined in (3.5).
- Ent is the space of entropies defined in Definition 3.1 and $\|\cdot\|_{\text{Ent}}$ is a norm on Ent defined by Definition 3.2.

4.2. The main result. We state here the main result in the paper.

Theorem 4.1. *Let $\Omega \in \mathcal{A}_0$. The following results hold true:*

i) (Compactness) *Let $(\chi_n)_n \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ be a sequence such that*

$$\sup_n H_n(\chi_n, \Omega) < +\infty.$$

Then there exists $\chi \in L^\infty(\mathbb{R}^2; \mathbb{S}^1)$ solving

$$|\chi| = 1 \text{ a.e. in } \Omega, \quad \text{curl}(\chi) = 0 \text{ in } \mathcal{D}'(\Omega), \quad (4.1)$$

such that, up to a subsequence, $\chi_n \rightarrow \chi$ in $L^p_{\text{loc}}(\Omega; \mathbb{R}^2)$ for every $p \in [1, 6)$.

ii) (liminf inequality) *Let $(\chi_n)_n, \chi \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ be such that $\chi_n \rightarrow \chi$ in $L^1_{\text{loc}}(\Omega; \mathbb{R}^2)$. Then*

$$H(\chi, \Omega) \leq \liminf_n H_n(\chi_n, \Omega). \quad (4.2)$$

iii) (limsup inequality) *Assume that $\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0$ as $n \rightarrow \infty$. Let $\chi \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$. Assume additionally that $\chi \in BV(\Omega; \mathbb{R}^2)$. Then there exists a sequence $(\chi_n)_n \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ such that $\chi_n \rightarrow \chi$ in $L^1(\Omega; \mathbb{R}^2)$ and*

$$\limsup_n H_n(\chi_n, \Omega) \leq H(\chi, \Omega).$$

More precisely, if $H(\chi, \Omega) < +\infty$, then $\chi \in L^\infty(\Omega; \mathbb{S}^1)$ and the recovery sequence $(\chi_n)_n$ is bounded in $L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ and satisfies $\chi_n \rightarrow \chi$ in $L^p(\Omega; \mathbb{R}^2)$ for every $p \in [1, \infty)$.

Remark 4.2. Note that, if $\sup_n H_n(\chi_n) < +\infty$, then Theorem 4.1-*i*) implies that there is a subsequence (not relabeled) such that $\chi_n \rightarrow \chi$ in $L^p_{\text{loc}}(\Omega; \mathbb{R}^2)$ for every $p \in [1, 6)$, and χ satisfies the eikonal equation (4.1). Additionally, by Theorem 4.1-*ii*) we deduce that $H(\chi) < +\infty$, namely χ satisfies

$$\operatorname{div}(\Phi \circ \chi^\perp) \in \mathcal{M}_b(\Omega) \text{ for all } \Phi \in \text{Ent},$$

and $\bigvee\{|\operatorname{div}(\Phi \circ \chi^\perp)| : \Phi \in \text{Ent}, \|\Phi\|_{\text{Ent}} \leq 1\}$ is a finite measure. Hence, χ is a (strong) finite entropy production solution of the eikonal equation (*cf.* [28, Definition 2.3] for a similar definition).

Remark 4.3. The proof of the compactness Theorem 4.1-*i*) as well as that of the liminf inequality Theorem 4.1-*ii*) do not require the simple connectedness of Ω and the regularity of its boundary.

Remark 4.4. Our Γ -convergence result is partial in that the limsup inequality requires that χ is BV and the additional scaling assumption $\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0$. The former assumption reflects the fact that the limsup inequality for the classical Aviles-Giga functionals is only known for BV fields, *cf.* [22, 46]. Improving Theorem 4.1-*iii*) by only requiring that χ is such that $H(\chi, \Omega) < +\infty$ is out of the scope of this paper and it requires new developments in the analysis of the Aviles-Giga functionals.

The scaling assumption $\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0$ is technical. It is due to the fact that the variable χ_n , which enters the potential W^{d} in our energy H_n , is not equal to the curl-free variable $\bar{\chi}_n$ that we use in our construction of the recovery sequence. In the energy we therefore commit a bulk error (that is, away from the jump set J_χ , where all of the asymptotic energy H concentrates). The scaling assumption is needed to control this bulk error.

We remark that we do not require an additional scaling assumption in our liminf inequality, as we are able to solve the mentioned problem in this case, through the introduction of approximate entropies (*cf.* (6.5)–(6.8) and Lemma 6.3).

We finally remark that, in terms of λ_n and ε_n the scaling assumption $\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0$ can be read as an additional assumption on the asymptotic relation $\lambda_n \ll \varepsilon_n$. Indeed, the scaling assumption is satisfied whenever $\frac{\lambda_n^4}{\varepsilon_n^4} \rightarrow 0$, *e.g.*, if $\lambda_n = \varepsilon_n^p$ with $p > \frac{5}{4}$.

Remark 4.5. The Γ -convergence analysis carried out for the functionals H_n to prove Theorem 4.1 can be applied with minor modifications also to the discrete Aviles-Giga functionals AG_n^{d} defined by (2.23) in the regime $\frac{\lambda_n}{\varepsilon_n} \rightarrow 0$ as $n \rightarrow \infty$. Hence, the analogous results as in Theorem 4.1 can be proved for the functionals AG_n^{d} , too. Moreover, in many cases our arguments can be simplified as we explain in Remarks 5.3, 6.2, and 7.4. In particular, we stress that the analogue of the limsup inequality in Theorem 4.1-*iii*) holds true for the functionals AG_n^{d} without the additional scaling assumption $\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0$ (where $\delta_n = \frac{\lambda_n}{\varepsilon_n}$).

Remark 4.6. We recall that the functionals H_n represent the behavior of the J_1 - J_2 - J_3 energies F_n close to the helimagnet/ferromagnet transition point ($\alpha_n - (4 + 2\beta_n) \nearrow 0$) if the next-to-nearest neighbors interaction parameter β_n is chosen as $\beta_n \equiv 2$. We collect here some remarks about the cases where $0 \leq \beta_n < 2$. Setting $\delta_n := 4 - \frac{2\alpha_n}{2+\beta_n}$ and rescaling (1.2), a computation similar to (2.9)–(2.11) shows that the rescaled energy $\frac{1}{\delta_n^{3/2}\lambda_n} F_n$ is given by the convex combination $\frac{\beta_n}{2} H_n^{(2)} + (1 - \frac{\beta_n}{2}) H_n^{(0)}$. Here, $H_n^{(2)}$ is given by the same expression as H_n in (2.16) (with ε_n adapted using $\delta_n = 4 - \frac{2\alpha_n}{2+\beta_n}$) and shares the same compactness properties. Moreover, $H_n^{(0)}$ corresponds to the J_1 - J_3 energy studied in [17]. The observations therein show that $H_n^{(0)}$ takes the form

$$H_n^{(0)}(\chi) = \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W_{(0)}^{\text{d}}(\chi) + \varepsilon_n |A_{(0)}^{\text{d}}(\chi)|^2 \, dx,$$

where $W_{(0)}^{\text{d}}(\chi) \simeq (\frac{1}{2} - |\chi_1|^2)^2 + (\frac{1}{2} - |\chi_2|^2)^2$ and $A_{(0)}^{\text{d}}(\chi) \simeq (|\partial_1^{\text{d}} \chi_1|^2 + |\partial_2^{\text{d}} \chi_2|^2)^{1/2}$ as $n \rightarrow \infty$. Although similar in form to $H_n^{(2)}$, the behavior of this energy is very different from that of

$H_n^{(2)}$. Indeed, its compactness is substantially stronger as it allows the values of the limit χ only to lie in four isolated points and, moreover, $\chi \in BV$, *cf.* [17, Theorem 2.1-*i*]).

The analysis in [17] together with the analysis carried out in this paper, allows us to understand the compactness properties of the rescaled F_n for general $\beta_n \in [0, 2]$ as a combination of the compactness properties of $H_n^{(0)}$ and $H_n^{(2)}$.

In the case that $\sup_n \beta_n < 2$, a bound on the energies $\frac{1}{\delta_n^{3/2} \lambda_n} F_n$ implies a bound on $H_n^{(0)}$. Since moreover $H_n^{(2)}$ can be controlled by $H_n^{(0)}$ up to a multiplicative constant, in this case the compactness of the rescaled F_n is the same as in the J_1 - J_3 model, *cf.* also [17, Remark 2.3].

If instead $\beta_n \rightarrow 2$, a bound on $\frac{1}{\delta_n^{3/2} \lambda_n} F_n$ implies only a bound on $H_n^{(2)}$ and on $(1 - \frac{\beta_n}{2})H_n^{(0)}$. The question whether the latter term improves the compactness of the energy $H_n^{(2)}$ as in Theorem 4.1-*i*), *ii*) depends on the relative speed of the convergences $\beta_n \rightarrow 2$ and $\varepsilon_n \rightarrow 0$. In the case that $\frac{2-\beta_n}{\varepsilon_n} \leq C$, no improved compactness can be expected. Indeed, it can be observed that $|A_{(0)}^d(\chi)|^2 \leq C|D^d \bar{\chi}|^2$ and that

$$\sup_n \int_{\Omega} W(\chi_n) dx < +\infty \implies \sup_n \int_{\Omega} W_{(0)}^d(\chi_n) dx < +\infty$$

for all $\chi_n = \chi(u_n)$, $u_n \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$. As a consequence, it can be seen that a uniform bound on $H_n^{(2)}(\chi_n, \Omega)$ already implies (locally in Ω) a uniform bound on $(1 - \frac{\beta_n}{2})H_n^{(0)}(\chi)$ through Proposition 2.6 and (2.24).

However, if $\frac{2-\beta_n}{\varepsilon_n} \rightarrow +\infty$, then the bound $C \geq (1 - \frac{\beta_n}{2})H_n^{(0)}(\chi_n) \geq \frac{2-\beta_n}{2\varepsilon_n} \int_{\Omega} W_{(0)}^d(\chi_n) dx$ implies that the limit χ (obtained from the compactness of $H_n^{(2)}$) satisfies $\chi(x) \in \{\pm \frac{1}{\sqrt{2}}\}^2$ for a.e. x . In particular, it attains only finitely many values and by Proposition 4.7 below we obtain $\chi \in BV(\Omega; \{\pm \frac{1}{\sqrt{2}}\}^2)$. Thus, *a posteriori* the stronger compactness of the J_1 - J_3 model is recovered.

Proposition 4.7. *Let $\chi \in L^\infty(\Omega; \mathbb{S}^1)$ be such that $H(\chi, \Omega) < +\infty$. If χ attains values in a finite set a.e., then $\chi \in BV(\Omega; \mathbb{S}^1)$.*

Proof. We recall that thanks to [28, Theorem 2.6], χ being a finite entropy production solution implies that $\chi \in B_{3,\infty}^{1/3}(\Omega')$ for all open sets $\Omega' \subset\subset \Omega$. (As in [28] the authors work with divergence-free fields, we apply their results to χ^\perp .) Accordingly, (*cf.* also [36, Definition 14.1])

$$\sup_{t>0} \sup_{|z|\leq t} \int_{\Omega' \cap (\Omega' - z)} \frac{|\chi(x+z) - \chi(x)|^3}{t} dx < +\infty.$$

Since χ takes only finitely many values, we find a constant C such that $|\chi(x+z) - \chi(x)| \leq C|\chi(x+z) - \chi(x)|^3$ for a.e. $x \in \Omega$. As a consequence $\sup_{z \neq 0} \int_{\Omega' \cap (\Omega' - z)} \frac{|\chi(x+z) - \chi(x)|}{|z|} dx < +\infty$, which implies that $\chi \in BV_{\text{loc}}(\Omega'; \mathbb{S}^1)$ (*cf.* [36, Theorem 13.48]). Applying now Corollary 3.8 locally in Ω , we obtain that

$$\sup_{\Omega' \subset\subset \Omega} \int_{J_\chi \cap \Omega'} |\chi| d\mathcal{H}^1 \leq C \int_{J_\chi} |\chi|^3 d\mathcal{H}^1 = CH(\chi, \Omega) < +\infty.$$

In conclusion, $D\chi = D^j \chi \in \mathcal{M}_b(\Omega)$ and this concludes the proof. \square

5. PROOF OF COMPACTNESS

In this section we prove a series of results which lead to the compactness statement in Theorem 4.1-*i*). Some of the steps are inspired by the proof of compactness in the continuum setting in [26].

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set. Let $(\chi_n)_n \in L_{\text{loc}}^1(\mathbb{R}^2; \mathbb{R}^2)$ and $\chi \in L_{\text{loc}}^1(\mathbb{R}^2; \mathbb{R}^2)$ be such that $\chi_n \rightarrow \chi$ in $L_{\text{loc}}^1(\Omega; \mathbb{R}^2)$ and*

$$\sup_n H_n(\chi_n, \Omega) < +\infty.$$

Then χ solves

$$|\chi| = 1 \text{ a.e. in } \Omega, \quad \operatorname{curl}(\chi) = 0 \text{ in } \mathcal{D}'(\Omega). \quad (5.1)$$

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set. Let $(\chi_n)_n \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ be such that*

$$\sup_n H_n(\chi_n, \Omega) < +\infty.$$

Then there exists $\chi \in L^\infty(\mathbb{R}^2; \mathbb{S}^1)$ such that, up to a subsequence, $\chi_n \rightarrow \chi$ in $L^p_{\text{loc}}(\Omega; \mathbb{R}^2)$ for every $p \in [1, 6)$.

Propositions 5.1 and 5.2 yield Theorem 4.1-*i*).

Proof of Proposition 5.1. By Proposition 2.6 we have that $\int_{\Omega'} W(\chi_n) dx \leq C\varepsilon_n$ for every $\Omega' \subset\subset \Omega$ and as a consequence $|\chi_n|^2 \rightarrow 1$ in $L^2_{\text{loc}}(\Omega)$. Thus, we find a (non-reabeled) subsequence with $|\chi_n| \rightarrow 1$ and $\chi_n \rightarrow \chi$ a.e. in Ω . In particular, $|\chi| = 1$ a.e. in Ω . To show that $\operatorname{curl}(\chi) = 0$ in the distributional sense, let us recall that by Remark 2.5, $\operatorname{curl}^{\text{d}}(\chi_n) \rightharpoonup 0$ in the sense of distributions. Thus it is sufficient to show that $\operatorname{curl}(\chi_n) - \operatorname{curl}^{\text{d}}(\chi_n) \rightharpoonup 0$ in the sense of distributions. Using the interpolation \mathcal{I} defined in (2.4), we have that $\operatorname{curl}(\chi_n) - \operatorname{curl}^{\text{d}}(\chi_n) = -\operatorname{div}(\chi_n^\perp - \mathcal{I}(\chi_n^\perp))$ as distributions. Moreover, using (2.5) and Proposition 2.6, we obtain that

$$\|\chi_n^\perp - \mathcal{I}(\chi_n^\perp)\|_{L^2(\Omega')} \leq C\lambda_n \|\mathbf{D}^{\text{d}}\chi_n\|_{L^2(\Omega')} \leq C \frac{\lambda_n}{\sqrt{\varepsilon_n}} = C\lambda_n^{1/2}\delta_n^{1/4} \rightarrow 0$$

for every $\Omega' \subset\subset \Omega$, and the desired distributional convergence $\operatorname{curl}(\chi_n) - \operatorname{curl}^{\text{d}}(\chi_n) \rightharpoonup 0$ follows.⁴ \square

Proof of Proposition 5.2. Step 1. (Recasting the discrete entropy productions.) Let $\Phi \in \text{Ent}$ and let α and Ψ be as in (3.2) and (3.3). We show that there are discrete functions $r_n^{(1)}, r_n^{(2)} \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$ such that

$$\operatorname{div}^{\text{d}}(\Phi \circ \chi_n^\perp) = (\Psi \circ \chi_n^\perp) \cdot \mathbf{D}^{\text{d}}(1 - |\chi_n|^2) + r_n^{(1)} + r_n^{(2)},$$

where $r_n^{(1)}$ and $r_n^{(2)}$ are estimated below in Step 2. By a discrete chain rule we get that

$$\operatorname{div}^{\text{d}}(\Phi \circ \chi_n^\perp) = \nabla\Phi_1(X_n) \cdot \partial_1^{\text{d}}\chi_n^\perp + \nabla\Phi_2(Y_n) \cdot \partial_2^{\text{d}}\chi_n^\perp \quad \text{in } \mathcal{PC}_{\lambda_n}(\mathbb{R}),$$

where $(X_n)^{i,j}$ is a vector on the segment connecting $(\chi_n^\perp)^{i,j}$ and $(\chi_n^\perp)^{i+1,j}$, and $(Y_n)^{i,j}$ lies on the segment connecting $(\chi_n^\perp)^{i,j}$ and $(\chi_n^\perp)^{i,j+1}$. By (3.4) we get that

$$\begin{aligned} \operatorname{div}^{\text{d}}(\Phi \circ \chi_n^\perp) &= (\nabla\Phi_1(X_n) - \nabla\Phi_1(\chi_n^\perp)) \cdot \partial_1^{\text{d}}\chi_n^\perp + (\nabla\Phi_2(Y_n) - \nabla\Phi_2(\chi_n^\perp)) \cdot \partial_2^{\text{d}}\chi_n^\perp \\ &\quad - 2\chi_n^\perp \cdot (\Psi_1(\chi_n^\perp)\partial_1^{\text{d}}\chi_n^\perp + \Psi_2(\chi_n^\perp)\partial_2^{\text{d}}\chi_n^\perp) \\ &\quad + \alpha(\chi_n^\perp)(\partial_1^{\text{d}}\chi_{1,n}^\perp + \partial_2^{\text{d}}\chi_{2,n}^\perp). \end{aligned}$$

By a discrete chain rule we also have

$$\partial_1^{\text{d}}(1 - |\chi_n|^2) = \partial_1^{\text{d}}(1 - |\chi_n^\perp|^2) = -2\tilde{X}_n \cdot \partial_1^{\text{d}}\chi_n^\perp \quad \text{and} \quad \partial_2^{\text{d}}(1 - |\chi_n|^2) = -2\tilde{Y}_n \cdot \partial_2^{\text{d}}\chi_n^\perp,$$

where $(\tilde{X}_n)^{i,j}$ lies between $(\chi_n^\perp)^{i,j}$ and $(\chi_n^\perp)^{i+1,j}$, and $(\tilde{Y}_n)^{i,j}$ lies between $(\chi_n^\perp)^{i,j}$ and $(\chi_n^\perp)^{i,j+1}$. Therefore we get

$$\operatorname{div}^{\text{d}}(\Phi \circ \chi_n^\perp) = (\Psi \circ \chi_n^\perp) \cdot \mathbf{D}^{\text{d}}(1 - |\chi_n|^2) + r_n^{(1)} + r_n^{(2)},$$

where

$$\begin{aligned} r_n^{(1)} &= (\nabla\Phi_1(X_n) - \nabla\Phi_1(\chi_n^\perp)) \cdot \partial_1^{\text{d}}\chi_n^\perp + (\nabla\Phi_2(Y_n) - \nabla\Phi_2(\chi_n^\perp)) \cdot \partial_2^{\text{d}}\chi_n^\perp \\ &\quad - 2\Psi_1(\chi_n^\perp)(\chi_n^\perp - \tilde{X}_n) \cdot \partial_1^{\text{d}}\chi_n^\perp - 2\Psi_2(\chi_n^\perp)(\chi_n^\perp - \tilde{Y}_n) \cdot \partial_2^{\text{d}}\chi_n^\perp + \alpha(\chi_n^\perp) \operatorname{div}^{\text{d}}(\bar{\chi}_n^\perp) \end{aligned} \quad (5.2)$$

and

$$r_n^{(2)} = \alpha(\chi_n^\perp) \operatorname{div}^{\text{d}}(\chi_n^\perp - \bar{\chi}_n^\perp). \quad (5.3)$$

⁴Instead of using the interpolation \mathcal{I} , one can prove that $\operatorname{curl}(\chi_n) - \operatorname{curl}^{\text{d}}(\chi_n) \rightharpoonup 0$ in $\mathcal{D}'(\Omega)$ through a discrete integration by parts. This argument only requires boundedness of χ_n locally in L^1 and no bound on $\mathbf{D}^{\text{d}}\chi_n$.

Here we recall that $\bar{\chi}_n$ is the linearized version of the order parameter χ_n defined by (2.19).

Step 2. (Estimates for the remainders $r_n^{(1)}$ and $r_n^{(2)}$.) By the Lipschitz continuity of $D\Phi$ and the fact that $|(X_n)^{i,j} - (\chi_n^\perp)^{i,j}| \leq |(\chi_n^\perp)^{i+1,j} - (\chi_n^\perp)^{i,j}|$ we have that

$$|\nabla\Phi_1(X_n) - \nabla\Phi_1(\chi_n^\perp)| \leq C|X_n - \chi_n^\perp| \leq C\lambda_n|\partial_1^d\chi_n^\perp|, \quad (5.4)$$

a similar estimate being true for $|\nabla\Phi_2(Y_n) - \nabla\Phi_2(\chi_n^\perp)|$. Similarly, by the boundedness of Ψ , we get that

$$|\Psi_1(\chi_n^\perp)(\chi_n^\perp - \tilde{X}_n)| \leq C\lambda_n|\partial_1^d\chi_n^\perp| \quad \text{and} \quad |\Psi_2(\chi_n^\perp)(\chi_n^\perp - \tilde{Y}_n)| \leq C\lambda_n|\partial_2^d\chi_n^\perp|. \quad (5.5)$$

Using (5.4) and (5.5) in (5.2), we get that

$$|r_n^{(1)}| \leq C\lambda_n|D^d\chi_n|^2 + C|\text{curl}^d(\bar{\chi}_n)|,$$

where we have also used the boundedness of α and the identity $|\text{div}^d(\bar{\chi}_n^\perp)| = |\text{curl}^d(\bar{\chi}_n)|$. For every $\Omega' \subset\subset \Omega$ we have by Proposition 2.6 and (2.13) that $\|\lambda_n|D^d\chi_n|^2\|_{L^1(\Omega')} \leq C\frac{\lambda_n}{\varepsilon_n} = C\sqrt{\delta_n}$ and by Lemma 2.4 that $\|\text{curl}^d(\bar{\chi}_n)\|_{L^1(\Omega')} \leq C\delta_n$. Therefore,

$$\|r_n^{(1)}\|_{L^1(\Omega')} = \mathcal{O}(\sqrt{\delta_n}).$$

Let us prove that

$$r_n^{(2)} \rightarrow 0 \quad \text{in } H^{-1}(\Omega') \text{ for every } \Omega' \subset\subset \Omega.$$

We first use boundedness of α to infer that $|r_n^{(2)}| \leq C|\text{div}^d(\chi_n^\perp - \bar{\chi}_n^\perp)|$. As observed in Remark 2.5, the fact that $\chi_n^\perp - \bar{\chi}_n^\perp \rightarrow 0$ in $L^2(\Omega)$ implies that $\text{div}^d(\chi_n^\perp - \bar{\chi}_n^\perp) \rightarrow 0$ in $H^{-1}(\Omega')$ for every open $\Omega' \subset\subset \Omega$ through the use of the interpolation \mathcal{I} defined by (2.4). As a consequence, $r_n^{(2)} \rightarrow 0$ in $H^{-1}(\Omega')$ for every $\Omega' \subset\subset \Omega$ as desired.

Step 3. (Compactness in H^{-1} of the discrete entropy productions.) Let us prove that the sequence $(\text{div}^d(\Phi \circ \chi_n^\perp))_n$ is compact in $H^{-1}(\Omega')$, for every $\Omega' \subset\subset \Omega$. To this end we apply Lemma 5.4 below. Let us first show how to write $\text{div}^d(\Phi \circ \chi_n^\perp)$ as the distributional divergences of L^2 vector fields whose squares are uniformly integrable on Ω' , where $\Omega' \subset\subset \Omega$ is a fixed open set. Using again the interpolation \mathcal{I} defined by (2.4), we get that $\text{div}^d(\Phi \circ \chi_n^\perp) = \text{div}(\mathcal{I}(\Phi \circ \chi_n^\perp))$. Moreover, we observe that $(\mathcal{I}(\Phi \circ \chi_n^\perp))_n$ is bounded in L^∞ since Φ is a bounded function. As a consequence, $|\mathcal{I}(\Phi \circ \chi_n^\perp)|^2$ is uniformly integrable on Ω' .

To apply Lemma 5.4, let us now use a discrete product rule to write

$$\text{div}^d((\Psi \circ \chi_n^\perp)(1 - |\chi_n|^2)) = (\Psi \circ \chi_n^\perp) \cdot D^d(1 - |\chi_n|^2) + R_n \quad \text{in } \mathcal{PC}_{\lambda_n}(\mathbb{R}),$$

where

$$R_n^{i,j} = \partial_1^d(\Psi_1 \circ \chi_n^\perp)^{i,j}(1 - |\chi_n|^2)^{i+1,j} + \partial_2^d(\Psi_2 \circ \chi_n^\perp)^{i,j}(1 - |\chi_n|^2)^{i,j+1}.$$

In view of Step 1 this leads to

$$\text{div}^d(\Phi \circ \chi_n^\perp) = \text{div}^d((\Psi \circ \chi_n^\perp)(1 - |\chi_n|^2)) - R_n + r_n^{(1)} + r_n^{(2)}$$

and we will show that

$$\begin{aligned} (a) \quad & \text{div}^d((\Psi \circ \chi_n^\perp)(1 - |\chi_n|^2)) + r_n^{(2)} \rightarrow 0 \text{ in } H^{-1}(\Omega') \text{ and} \\ (b) \quad & -R_n + r_n^{(1)} \in L^2(\Omega'), \quad \sup_n \| -R_n + r_n^{(1)} \|_{L^1(\Omega')} < +\infty. \end{aligned} \quad (5.6)$$

By Step 2, to prove (a) in (5.6) it remains to show that $\text{div}^d((\Psi \circ \chi_n^\perp)(1 - |\chi_n|^2)) \rightarrow 0$ in $H^{-1}(\Omega')$. Since Ψ is a bounded function and $1 - |\chi_n|^2 \rightarrow 0$ in $L^2(\Omega)$ in view of Proposition 2.6, we can proceed as in the estimate of $r_n^{(2)}$ in Step 2: For the interpolated fields $\mathcal{I}((\Psi \circ \chi_n^\perp)(1 - |\chi_n|^2))$ defined by (2.4) we get that $\mathcal{I}((\Psi \circ \chi_n^\perp)(1 - |\chi_n|^2)) \rightarrow 0$ in $L^2(\Omega')$ and $\text{div}^d((\Psi \circ \chi_n^\perp)(1 - |\chi_n|^2)) = \text{div}(\mathcal{I}((\Psi \circ \chi_n^\perp)(1 - |\chi_n|^2)))$. Thereby, the desired convergence to 0 in $H^{-1}(\Omega')$ follows.

To prove (b) in (5.6), we first observe that for every fixed n , $r_n^{(1)}$ and R_n belong to $L^\infty(\Omega')$ since they only attain finitely many values on Ω' . In view of Step 2 it remains only

to show that $(R_n)_n$ is bounded in $L^1(\Omega')$. We observe that $|\mathbb{D}^d(\Psi \circ \chi_n^\perp)| \leq C|\mathbb{D}^d\chi_n|$ since Ψ is a Lipschitz function. By Young's inequality we get that

$$|R_n^{i,j}| \leq C \left(\varepsilon_n |\mathbb{D}^d\chi_n^{i,j}|^2 + \frac{1}{\varepsilon_n} \left((1 - |\chi_n^{i+1,j}|^2)^2 + (1 - |\chi_n^{i,j+1}|^2)^2 \right) \right)$$

and we obtain boundedness in $L^1(\Omega')$ from Proposition 2.6.

Step 4. (Compactness in H^{-1} of the distributional entropy productions.) Let us prove that the sequence $(\operatorname{div}(\Phi \circ \chi_n^\perp))_n$ is compact in $H^{-1}(\Omega')$, for every $\Omega' \subset\subset \Omega$.

We again use the interpolation defined by (2.4): For every $\Omega' \subset\subset \Omega$, $\operatorname{div}(\mathcal{I}(\Phi \circ \chi_n^\perp)) = \operatorname{div}^d(\Phi \circ \chi_n^\perp)$ is compact in $H^{-1}(\Omega')$ by Step 3 and, as a consequence, it is enough to show that $\mathcal{I}(\Phi \circ \chi_n^\perp) - (\Phi \circ \chi_n^\perp) \rightarrow 0$ in $L^2(\Omega')$. Using the Lipschitz continuity of Φ we have that $|\mathcal{I}(\Phi \circ \chi_n^\perp) - (\Phi \circ \chi_n^\perp)| \leq C\lambda_n |\mathbb{D}^d(\Phi \circ \chi_n^\perp)| \leq C\lambda_n |\mathbb{D}^d\chi_n|$ and in view of Proposition 2.6 and (2.13) this yields that $\|\mathcal{I}(\Phi \circ \chi_n^\perp) - (\Phi \circ \chi_n^\perp)\|_{L^2(\Omega')} \leq C \frac{\lambda_n}{\sqrt{\varepsilon_n}} \rightarrow 0$.

Step 5. (Bounds in L^6 for χ_n .) By Proposition 2.7 the sequence $(\chi_n)_n$ is bounded in $L^6(\Omega')$ for every $\Omega' \subset\subset \Omega$.

Step 6. (Compactness in L^p_{loc} , $p \in [1, 6)$, for χ_n .) We fix again $\Omega' \subset\subset \Omega$. We will show that there exists a $\chi \in L^\infty(\Omega'; \mathbb{S}^1)$ and a (non-relabeled) subsequence $\chi_n \rightarrow \chi$ in $L^p(\Omega'; \mathbb{R}^2)$ for all $p \in [1, 6)$. The claim of Proposition 5.2 then finally follows by exhausting Ω with a sequence of compactly contained subsets and using a diagonal argument. To prove the compactness in $L^p(\Omega')$, $p < 6$, we make use of the theory of Young measures. There exists a (non-relabeled) subsequence of $(\chi_n^\perp)_n$ and a Young measure $\nu = (\nu_x)_{x \in \Omega'}$ such that for every $g \in C_0(\mathbb{R}^2)$ we have that

$$g \circ \chi_n^\perp \xrightarrow{*} \bar{g} \text{ weakly* in } L^\infty(\Omega'), \text{ where } \bar{g}(x) = \int_{\mathbb{R}^2} g \, d\nu_x. \quad (5.7)$$

For later use, let us record several additional properties of the Young measure ν : By Proposition 2.6 we have that $\int_{\Omega'} (1 - |\chi_n^\perp|^2) \, dx \leq C\varepsilon_n \rightarrow 0$ and, as a consequence, ν_x is supported on \mathbb{S}^1 for a.e. $x \in \Omega'$.⁵ Since $(\chi_n^\perp)_n$ is bounded in $L^6(\Omega')$ by Step 5, we moreover have that ν_x is a probability measure for a.e. $x \in \Omega'$ ⁶ and that

$$g \circ \chi_n^\perp \rightharpoonup \bar{g} \text{ weakly in } L^{6/p}(\Omega'), \text{ where } \bar{g}(x) = \int_{\mathbb{R}^2} g \, d\nu_x \quad (5.8)$$

for every $p < 6$ and every function $g \in C(\mathbb{R}^2)$ with $|g(\xi)| \leq C(1 + |\xi|^p)$.⁷ In particular, taking as g the components of the identity on \mathbb{R}^2 we get that $(\chi_n^\perp)_n$ itself converges weakly in $L^6(\Omega')$.

To improve this to strong convergence, we will now show that for a.e. $x \in \Omega'$, ν_x is a Dirac measure. For the moment, let us fix two entropies $\Phi_{(1)}, \Phi_{(2)} \in \text{Ent}$. Applying (5.7) to the components of $\Phi_{(1)}$ and $\Phi_{(2)}$ we get that

$$\Phi_{(k)} \circ \chi_n^\perp \xrightarrow{*} \bar{\Phi}_{(k)} \text{ weakly* in } L^\infty(\Omega'; \mathbb{R}^2), \quad \bar{\Phi}_{(k)}(x) = \int_{\mathbb{R}^2} \Phi_{(k)} \, d\nu_x$$

for $k = 1, 2$. Now we recall that by Step 4, $(\operatorname{div}(\Phi_{(1)} \circ \chi_n^\perp))_n$ and $(\operatorname{curl}(\Phi_{(2)}^\perp \circ \chi_n^\perp))_n = (\operatorname{div}(\Phi_{(2)} \circ \chi_n^\perp))_n$ are compact in $H^{-1}(\Omega')$. Therefore, the div-curl lemma (cf. [44, 54]) yields that

$$(\Phi_{(1)} \circ \chi_n^\perp) \cdot (\Phi_{(2)}^\perp \circ \chi_n^\perp) \rightharpoonup \bar{\Phi}_{(1)} \cdot \bar{\Phi}_{(2)}^\perp \quad \text{in the sense of distributions on } \Omega'.$$

⁵In fact, the sole assumption that $\chi_n^\perp \rightarrow K$ in measure for some closed set $K \subset \mathbb{R}^2$ implies that $\operatorname{supp} \nu_x \subset K$ for a.e. x , cf. [11]

⁶In fact, for the Young measures to have mass 1 it is already sufficient to satisfy the much weaker condition $\int_{\Omega'} \phi(|\chi_n^\perp(x)|) \, dx \leq C$ for some increasing and continuous function ϕ with $\lim_{s \rightarrow \infty} \phi(s) = +\infty$, cf. [11]

⁷For a continuous function g to satisfy $g \circ \chi_n^\perp \rightharpoonup \bar{g}$ weakly in L^1 it is enough that $(g \circ \chi_n^\perp)_n$ is a weakly compact sequence in L^1 , cf. [11]. If $|g(\xi)| \leq C(1 + |\xi|^p)$ for $p < 6$, then $(g \circ \chi_n^\perp)_n$ is bounded in $L^{6/p}$ and thus weakly compact even in $L^{6/p}$, improving the weak L^1 convergence to weak $L^{6/p}$ convergence.

On the other hand, (5.7) applied to $\Phi_{(1)} \cdot \Phi_{(2)}^\perp$ leads to

$$(\Phi_{(1)} \circ \chi_n^\perp) \cdot (\Phi_{(2)}^\perp \circ \chi_n^\perp) \xrightarrow{*} \overline{\Phi_{(1)} \cdot \Phi_{(2)}^\perp} \text{ weakly* in } L^\infty(\Omega'; \mathbb{R}^2).$$

In conclusion,

$$\left(\int_{\mathbb{R}^2} \Phi_{(1)} d\nu_x \right) \cdot \left(\int_{\mathbb{R}^2} \Phi_{(2)}^\perp d\nu_x \right) = \left(\int_{\mathbb{R}^2} \Phi_{(1)} \cdot \Phi_{(2)}^\perp d\nu_x \right) \quad \text{for a.e. } x \in \Omega'.$$

The exceptional null set depends on $\Phi_{(1)}, \Phi_{(2)} \in \text{Ent}$. Nonetheless, we can get rid of this dependence since both sides of the above equation are continuous under uniform convergence of $\Phi_{(1)}, \Phi_{(2)}$ and since the space Ent is separable with respect to the L^∞ norm, being a subspace of the separable metric space $C_0(\mathbb{R}^2; \mathbb{R}^2)$. This allows us to apply [26, Lemma 2.6] to obtain that ν_x is a Dirac measure for a.e. $x \in \Omega'$.⁸ To this end let us recall that we have already shown that ν_x is supported on \mathbb{S}^1 for a.e. x .

Defining

$$\chi(x) := \int_{\mathbb{R}^2} -\xi^\perp d\nu_x(\xi), \quad x \in \Omega',$$

we now have that $\chi \in L^\infty(\mathbb{R}^2; \mathbb{S}^1)$ and for a.e. $x \in \Omega'$, ν_x is the Dirac measure in the point $\chi^\perp(x)$. Applying (5.8) with $p = 1$ to the components of $\xi \mapsto -\xi^\perp$ we moreover obtain that $\chi_n \rightharpoonup \chi$ weakly in $L^6(\Omega')$. Now let us fix $p \in [1, 6)$ and show that the convergence is in fact strong in $L^p(\Omega')$. Applying (5.8) to $g(\xi) = |\xi|^p$ we get that $|\chi_n|^p \rightharpoonup |\chi|^p$ weakly in $L^{6/p}(\Omega')$ because ν_x is the Dirac measure in the point $\chi^\perp(x)$. Testing this weak convergence with the characteristic function of Ω' we get that $\|\chi_n\|_{L^p(\Omega')}^p \rightarrow \|\chi\|_{L^p(\Omega')}^p$. Since convergence of the norms improves weak convergence to strong convergence in L^q for $q > 1$, we conclude that $\chi_n \rightarrow \chi$ strongly in $L^p(\Omega')$. This concludes the proof of Proposition 5.2. \square

Remark 5.3. The same strategy can be used to prove the following compactness result for the discrete Aviles-Giga functionals AG_n^d defined by (2.23): If $AG_n^d(\varphi_n, \Omega) \leq C$, then, up to a subsequence, $D^d \varphi_n$ converges in $L_{\text{loc}}^p(\Omega)$ for every $p < 6$ and the limit is curl-free and valued in \mathbb{S}^1 a.e.

In fact, several of the steps in the proofs of Propositions 5.1 and 5.2 simplify due to the fact that when $\chi_n = D^d \varphi_n$, we have that $\text{curl}^d(\chi_n) \equiv 0$ in place of only $\text{curl}^d(\chi_n) \simeq 0$. In particular, the term $\alpha(\chi_n^\perp) \text{div}^d(\overline{\chi_n^\perp})$ in (5.2) as well as the remainder $r_n^{(2)}$ in (5.3) are not present. Then, all later steps in the proof of Proposition 5.2 apply with only few obvious modifications, noting that the bounds obtained applying Proposition 2.6 follow in this case directly from the energy bound $AG_n^d(\varphi_n, \Omega) \leq C$.

We conclude this section by stating and proving a technical result used in the proof of Proposition 5.2. It is a slightly modified version of [26, Lemma 3.1]. Nevertheless, we provide the proof for completeness.

Lemma 5.4. *Let $U \subset \mathbb{R}^d$ be an open bounded set. Let $(f_n)_n$ be a sequence in $L^2(U; \mathbb{R}^d)$ such that $(|f_n|^2)_n$ is uniformly integrable. If $\text{div}(f_n) = a_n + b_n$, where $(a_n)_n$ is compact in $H^{-1}(U)$ and $(b_n)_n$ is a sequence in $L^2(U)$ with $\sup_n \|b_n\|_{L^1(U)} < +\infty$, then $(\text{div}(f_n))_n$ is compact in $H^{-1}(U)$.*

Proof. Let us fix a sequence $(\varphi_n)_n$ in $H_0^1(U)$ such that $\varphi_n \rightharpoonup 0$ weakly in $H_0^1(U)$. We will prove that $\langle \text{div}(f_n), \varphi_n \rangle_{H^{-1}(U), H_0^1(U)} \rightarrow 0$.⁹ For such a sequence (φ_n) we have that $\varphi_n \rightarrow 0$ strongly in $L^2(U)$ and in particular

$$\mathcal{L}^d(U \cap \{|\varphi_n| > \delta\}) \rightarrow 0 \quad \text{for every } \delta > 0. \quad (5.9)$$

⁸Our notion of an entropy is slightly more restrictive than in [26] since we don't allow 0 to lie in the support of any entropy. Nevertheless, since the approximation in [26, Lemma 2.5] can be achieved with entropies whose supports don't contain 0, [26, Lemma 2.6] remains true under our more restrictive notion of entropy.

⁹We recall that for any separable and reflexive Banach space X , strong compactness of $(v_n^*)_n \subset X^*$ is equivalent to $\langle v_n^*, v_n \rangle \rightarrow 0$ for every sequence $(v_n)_n \subset X$ with $v_n \rightharpoonup 0$ weakly in X .

We fix $\delta > 0$, define the truncated functions

$$\varphi_n^{(1)} := \begin{cases} -\delta & \text{on } \{\varphi_n < -\delta\}, \\ \varphi_n & \text{on } \{|\varphi_n| \leq \delta\}, \\ \delta & \text{on } \{\varphi_n > \delta\}, \end{cases}$$

and have $\varphi_n^{(1)} \in H_0^1(U)$ with $\nabla \varphi_n^{(1)} = \nabla \varphi_n \cdot \mathbf{1}_{\{|\varphi_n| \leq \delta\}}$. We moreover set $\varphi_n^{(2)} := \varphi_n - \varphi_n^{(1)}$. We claim that $\varphi_n^{(2)} \rightharpoonup 0$ in $H_0^1(U)$ and therefore also $\varphi_n^{(1)} \rightharpoonup 0$ in $H_0^1(U)$. To prove this claim, let $\psi^* \in H^{-1}(U)$. Let $\psi \in H_0^1(U)$ solve $-\Delta \psi = \psi^*$. Then,

$$|\langle \psi^*, \varphi_n^{(2)} \rangle| = \left| \int_{\{|\varphi_n| > \delta\}} \nabla \psi \cdot \nabla \varphi_n \, dx \right| \leq \|\nabla \psi\|_{L^2(\{|\varphi_n| > \delta\})} \|\nabla \varphi_n\|_{L^2(U)}.$$

By (5.9) and since weak convergence of φ_n in $H_0^1(U)$ implies that $\|\nabla \varphi_n\|_{L^2(U)}$ is bounded, we infer that $\langle \psi^*, \varphi_n^{(2)} \rangle \rightarrow 0$, which proves our claim.

Now we write $\langle \operatorname{div}(f_n), \varphi_n \rangle = \langle a_n, \varphi_n^{(1)} \rangle + \langle b_n, \varphi_n^{(1)} \rangle + \langle \operatorname{div}(f_n), \varphi_n^{(2)} \rangle$. Since $(a_n)_n$ is compact in $H^{-1}(U)$, we have that $\langle a_n, \varphi_n^{(1)} \rangle \rightarrow 0$. Moreover, since b_n are functions in $L^2(U)$, the $(H^{-1}(U), H_0^1(U))$ -pairing between b_n and $\varphi_n^{(1)}$ is given by $\int_U b_n \varphi_n^{(1)} \, dx$ and thus we have that

$$|\langle b_n, \varphi_n^{(1)} \rangle| = \left| \int_U b_n \varphi_n^{(1)} \, dx \right| \leq \delta \sup_n \|b_n\|_{L^1(U)}.$$

Finally,

$$|\langle \operatorname{div}(f_n), \varphi_n^{(2)} \rangle| = \left| \int_U f_n \cdot \nabla \varphi_n^{(2)} \, dx \right| \leq \|f_n\|_{L^2(\{|\varphi_n| > \delta\})} \|\varphi_n\|_{L^2(U)},$$

which goes to zero by boundedness of $(\varphi_n)_n$ in $L^2(U)$, by (5.9), and by the uniform integrability of $(|f_n|^2)_n$. In conclusion we obtain that

$$\limsup_{n \rightarrow \infty} |\langle \operatorname{div}(f_n), \varphi_n \rangle| \leq \delta \sup_n \|b_n\|_{L^1(U)}.$$

Since $\delta > 0$ is arbitrary and $(b_n)_n$ is bounded in $L^1(U)$, this concludes the proof. \square

6. PROOF OF THE LIMINF INEQUALITY

In this section we prove Theorem 4.1-ii). We assume for the whole section that $\Omega \subset \mathbb{R}^2$ is an open and bounded set. Let us fix $(\chi_n)_n$ and $\chi \in L_{\text{loc}}^1(\mathbb{R}^2; \mathbb{R}^2)$ such that $\chi_n \rightarrow \chi$ in $L_{\text{loc}}^1(\Omega; \mathbb{R}^2)$. Let us assume, without loss of generality, that $\liminf_n H_n(\chi_n, \Omega) = \lim_n H_n(\chi_n, \Omega) < +\infty$. By Proposition 5.1 we get that χ satisfies (5.1), *i.e.*, the first two conditions in (3.6). In the following we prove (4.2), which yields, in particular, the third condition in (3.6).

Let us fix $\Phi \in \text{Ent}$ with $\|\Phi\|_{\text{Ent}} \leq 1$. We let Ψ and α denote the functions given by (3.2), (3.3). We start by noticing that the condition $|\chi| = 1$ a.e. in Ω yields

$$\Phi \circ \chi^\perp = \tilde{\Phi} \circ \chi^\perp \quad \text{a.e. in } \Omega,$$

where $\tilde{\Phi}(\xi) := \Phi(\xi) - (1 - |\xi|^2)\Psi(\xi)$. Hence, it suffices to estimate the total variation of $\operatorname{div}(\tilde{\Phi} \circ \chi^\perp)$.

Remark 6.1 (Heuristic argument in a continuum setting). We estimate the total variation of $\operatorname{div}(\tilde{\Phi} \circ \chi^\perp)$ below in several steps. To outline the proof, we first illustrate the argument in a continuum setting. Assume that $\omega_n \in H^1(\Omega; \mathbb{R}^2)$, $\operatorname{curl}(\omega_n) = 0$, $\omega_n \rightarrow \chi$ in $L_{\text{loc}}^1(\mathbb{R}^2; \mathbb{R}^2)$ and $\sup_n \frac{1}{2} \int_\Omega \frac{1}{\varepsilon_n} W(\omega_n) + \varepsilon_n |\operatorname{div}(\omega_n)|^2 \, dx < \infty$. In the following we sketch how to show that

$$|\operatorname{div}(\tilde{\Phi} \circ \chi^\perp)|(\Omega) \leq \liminf_n \frac{1}{2} \int_\Omega \frac{1}{\varepsilon_n} W(\omega_n) + \varepsilon_n |\operatorname{div}(\omega_n)|^2 \, dx. \quad (6.1)$$

Note that the energies on the right-hand side of (6.1) are continuum analogues of our energies H_n . In view of Proposition 2.6, let us assume moreover that

$$\sup_n \frac{1}{2} \int_\Omega \frac{1}{\varepsilon_n} (1 - |\omega_n|^2)^2 + \varepsilon_n |\operatorname{D}\omega_n|^2 \, dx < +\infty. \quad (6.2)$$

Step (Passing to the limit.) Given $\zeta \in C_c^\infty(\Omega)$ we have that

$$\langle \operatorname{div}(\tilde{\Phi} \circ \chi^\perp), \zeta \rangle = \lim_n \int_\Omega \zeta \operatorname{div}(\tilde{\Phi} \circ \omega_n^\perp) \, dx.$$

Step (Expanding the divergence using (3.4).) The relation (3.4) yields that

$$\operatorname{div}(\tilde{\Phi} \circ \omega_n^\perp) = \alpha(\omega_n^\perp) \operatorname{div}(\omega_n^\perp) - q(\omega_n) \operatorname{div}(\Psi \circ \omega_n^\perp) = -q(\omega_n) \operatorname{div}(\Psi \circ \omega_n^\perp), \quad (6.3)$$

where we have put $q(\xi) := (1 - |\xi|^2)$ and used that $\operatorname{curl}(\omega_n) = 0$.

Step (Young's inequality.) By Young's inequality we have that

$$\begin{aligned} - \int_\Omega \zeta q(\omega_n) \operatorname{div}(\Psi \circ \omega_n^\perp) \, dx &\leq \frac{1}{2} \int_\Omega \frac{1}{\varepsilon_n} q(\omega_n)^2 + \varepsilon_n |\zeta|^2 |\operatorname{div}(\Psi \circ \omega_n^\perp)|^2 \, dx \\ &= \frac{1}{2} \int_\Omega \frac{1}{\varepsilon_n} W(\omega_n) + \varepsilon_n |\zeta|^2 |\operatorname{div}(\Psi \circ \omega_n^\perp)|^2 \, dx, \end{aligned}$$

where we have used that $q(\xi)^2 = W(\xi)$.

Step (From divergence to full derivative matrix.) We have that

$$\begin{aligned} \varepsilon_n \int_\Omega |\zeta|^2 |\operatorname{D}(\Psi \circ \omega_n^\perp)|^2 \, dx &= \varepsilon_n \int_\Omega |\zeta|^2 \left(|\operatorname{div}(\Psi \circ \omega_n^\perp)|^2 + |\operatorname{curl}(\Psi \circ \omega_n^\perp)|^2 - 2 \det \operatorname{D}(\Psi \circ \omega_n^\perp) \right) \, dx \\ &\geq \varepsilon_n \int_\Omega |\zeta|^2 |\operatorname{div}(\Psi \circ \omega_n^\perp)|^2 \, dx + o_n(1), \end{aligned} \quad (6.4)$$

where we have used that $\det \operatorname{D}(\Psi \circ \omega_n^\perp) = \operatorname{curl}((\Psi_1 \circ \omega_n^\perp) \nabla(\Psi_2 \circ \omega_n^\perp))$ and thus, integrating by parts,

$$\begin{aligned} \left| \int_\Omega |\zeta|^2 \det \operatorname{D}(\Psi \circ \omega_n^\perp) \, dx \right| &= \left| \int_\Omega \nabla^\perp(|\zeta|^2) \cdot ((\Psi_1 \circ \omega_n^\perp) \nabla(\Psi_2 \circ \omega_n^\perp)) \, dx \right| \\ &\leq C \|\operatorname{D}(\Psi \circ \omega_n^\perp)\|_{L^2} \leq C \|\operatorname{D}\omega_n\|_{L^2} \leq \frac{C}{\sqrt{\varepsilon_n}}. \end{aligned}$$

Here we have used (6.2) and the fact that $\operatorname{Lip}(\Psi) = \|\Phi\|_{\text{Ent}} \leq 1$ implies that $|\operatorname{D}(\Psi \circ \omega_n^\perp)| \leq |\operatorname{D}\omega_n|$. Using the latter in (6.4) we now obtain that

$$\varepsilon_n \int_\Omega |\zeta|^2 |\operatorname{div}(\Psi \circ \omega_n^\perp)|^2 \, dx \leq \varepsilon_n \int_\Omega |\zeta|^2 |\operatorname{D}\omega_n|^2 \, dx + o_n(1).$$

Step (From full derivative matrix to divergence.) Similarly to the previous step we get that

$$\varepsilon_n \int_\Omega |\zeta|^2 |\operatorname{D}\omega_n|^2 \, dx = \varepsilon_n \int_\Omega |\zeta|^2 |\operatorname{div}(\omega_n)|^2 \, dx + o_n(1),$$

where we have used that $\operatorname{curl}(\omega_n) = 0$ and

$$\left| \int_\Omega |\zeta|^2 \det \operatorname{D}\omega_n \, dx \right| = \left| \int_\Omega \nabla^\perp(|\zeta|^2) \cdot (\omega_{1,n} \nabla \omega_{2,n}) \, dx \right| \leq C \|\omega_n\|_{L^2} \|\operatorname{D}\omega_n\|_{L^2} \leq \frac{C}{\sqrt{\varepsilon_n}}.$$

By all the previous steps we now get that

$$\langle \operatorname{div}(\tilde{\Phi} \circ \chi^\perp), \zeta \rangle \leq \liminf_n \frac{1}{2} \int_\Omega \frac{1}{\varepsilon_n} W(\omega_n) + \varepsilon_n |\operatorname{div}(\omega_n)|^2 \, dx$$

and taking the supremum over ζ we obtain (6.1).

To follow the previous steps in the discrete setting, we first need to introduce functions q_n such that $W(\chi_n) = q_n(\bar{\chi}_n)^2$, namely

$$q_n(\xi) := 1 - \frac{4}{\delta_n} \sin^2\left(\frac{\sqrt{\delta_n}}{2} \xi_1\right) - \frac{4}{\delta_n} \sin^2\left(\frac{\sqrt{\delta_n}}{2} \xi_2\right).$$

Here we recall that $\bar{\chi}_n$ is the linearized version of the order parameter χ_n defined by (2.19). The functions q_n are approximations of the function q . In fact, as we observe in the proof of Lemma 6.3 below (*cf.* (6.36)), they converge locally in $C^k(\mathbb{R}^2)$ for every k . Moreover,

we introduce suitable approximations $\tilde{\Phi}_n \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$ of $\tilde{\Phi}$ and functions $\Psi_n \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$ and $\alpha_n \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ with the following properties:

$$\tilde{\Phi}_n \rightarrow \tilde{\Phi}, \Psi_n \rightarrow \Psi, \alpha_n \rightarrow \alpha \quad \text{in } C^2, \quad (6.5)$$

$$D\tilde{\Phi}_n = \alpha_n \text{Id} - q_n D\Psi_n \quad \text{in } \mathbb{R}^2, \quad (6.6)$$

$$\text{Lip}(\Psi_n) \rightarrow \text{Lip}(\Psi) = \|\Phi\|_{\text{Ent}}, \quad (6.7)$$

$$\text{supp}(\tilde{\Phi}_n), \text{supp}(\Psi_n), \text{supp}(\alpha_n) \subset\subset (-M, M)^2 \quad (6.8)$$

for some $M > 1$ independent of n . The existence of the latter approximations is proved in Lemma 6.3 below.

The reason to make use of these approximations is that, by using (6.6), they allow us to prove a relation similar to (6.3), namely (in a formal fashion)

$$\text{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) \simeq -q_n(\bar{\chi}_n) \text{div}^d(\Psi_n \circ \bar{\chi}_n^\perp).$$

The precise relation is obtained in (6.21) below. As can be seen below in Step 5, the fact that q_n appears in place of q in the above formula allows us to recover the potential term in the energy H_n .

In the next steps, let us fix an open set $\Omega' \subset \Omega$ and $\zeta \in C_c^\infty(\Omega')$ with $\|\zeta\|_{L^\infty(\Omega')} \leq 1$ and let us prove that

$$\langle \text{div}(\tilde{\Phi} \circ \chi^\perp), \zeta \rangle \leq \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega'} \frac{1}{\varepsilon_n} W^d(\chi_n) + \varepsilon_n |A^d(\chi_n)|^2 dx. \quad (6.9)$$

Replacing Ω' by a sufficiently small neighborhood of $\text{supp}(\zeta)$ if necessary, we may assume, without loss of generality, that $\Omega' \subset\subset \Omega$.

Step 1. (Passing to the limit.) We prove that

$$\langle \text{div}(\tilde{\Phi} \circ \chi^\perp), \zeta \rangle = \lim_n \int_{\Omega'} \zeta \text{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) dx. \quad (6.10)$$

This follows from the fact that $\text{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) \rightharpoonup \text{div}^d(\tilde{\Phi} \circ \chi^\perp)$ in the sense of distributions. Indeed, we have that $\tilde{\Phi}_n \circ \bar{\chi}_n^\perp \rightarrow \tilde{\Phi} \circ \chi^\perp$ in $L_{\text{loc}}^1(\Omega; \mathbb{R}^2)$ and $\text{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) - \text{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) \rightharpoonup 0$ in $\mathcal{D}'(\Omega)$. The former is a consequence of (6.5), our assumption that $\chi_n \rightarrow \chi$ in $L_{\text{loc}}^1(\Omega; \mathbb{R}^2)$, and the fact that $\chi_n - \bar{\chi}_n \rightarrow 0$ in $L^2(\Omega; \mathbb{R}^2)$ (cf. Remark 2.5). On the other hand, the latter is proved by observing that $\text{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) - \text{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) = \text{div}(\mathcal{I}(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) - (\tilde{\Phi}_n \circ \bar{\chi}_n^\perp))$, where \mathcal{I} is defined by (2.4), and that

$$|\mathcal{I}(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) - (\tilde{\Phi}_n \circ \bar{\chi}_n^\perp)| \leq C\lambda_n |D^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp)| \leq C\lambda_n |D^d \bar{\chi}_n| \rightarrow 0 \quad \text{in } L_{\text{loc}}^2(\Omega).$$

Here we have used the fact that $\tilde{\Phi}_n$ are equi-Lipschitz and (2.24) in the proof of Proposition 2.6.¹⁰

Step 2. (Removing cells where $\bar{\chi}_n^\perp$ lies outside of the support of $\tilde{\Phi}_n$.) In the integral in (6.10) we remove the cells where $\bar{\chi}_n$ is far from zero by exploiting that $\tilde{\Phi}_n$ have compact support.¹¹ More precisely, we fix $M > 1$ such that $\text{supp}(\tilde{\Phi}_n) \subset\subset (-M, M)^2$ for all n (cf. (6.8)) and we introduce the collection of cells

$$\mathcal{Q}_n^{\text{supp}} := \{Q_{\lambda_n}(i, j) : (i, j) \in \mathbb{Z}^2, Q_{\lambda_n}(i, j) \cap \Omega' \neq \emptyset, |\bar{\chi}_n| \leq M \text{ on } Q_{\lambda_n}(i, j) \text{ and on all of its adjacent cells}\}. \quad (6.11)$$

By ‘‘adjacent cells’’ we mean that they share a side. We claim that

$$\int_{\Omega'} \zeta \text{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) dx = \int_{\Omega_n^{\text{supp}}} \zeta \text{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) dx + o_n(1), \quad (6.12)$$

¹⁰Similarly as in the proof of Proposition 5.1 we here could also use a discrete integration by parts instead of the interpolation \mathcal{I} . This argument would not require any bound on $D^d \chi_n$.

¹¹We need this technical step to obtain a bound on $\bar{\chi}_n$ in L^∞ . Notice that, in general, $\|\bar{\chi}_n\|_{L^\infty}$ can be an unbounded sequence. The L^∞ bound will help us in the later steps of the proof to estimate several of the error terms which emerge due to the discrete setting.

where $\Omega_n^{\text{supp}} := \Omega' \cap \bigcup_{Q \in \mathcal{Q}_n^{\text{supp}}} Q$. We start by observing that a discrete chain rule yields

$$\operatorname{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) = \nabla \tilde{\Phi}_{1,n}(X_n) \cdot \partial_1^d \bar{\chi}_n^\perp + \nabla \tilde{\Phi}_{2,n}(Y_n) \cdot \partial_2^d \bar{\chi}_n^\perp \quad \text{in } \mathcal{PC}_{\lambda_n}(\mathbb{R}), \quad (6.13)$$

where $(X_n)^{i,j}$ are vectors on the segment connecting $(\bar{\chi}_n^\perp)^{i,j}$ and $(\bar{\chi}_n^\perp)^{i+1,j}$ and the vectors $(Y_n)^{i,j}$ belong to the segment connecting $(\bar{\chi}_n^\perp)^{i,j}$ and $(\bar{\chi}_n^\perp)^{i,j+1}$. Suppose that $x \in \Omega' \setminus \Omega_n^{\text{supp}}$ and $x \in Q_{\lambda_n}(i, j)$. Let $(i', j') \in \mathbb{Z}^2$ be such that $|(i', j') - (i, j)| \leq 1$ (possibly $(i', j') = (i, j)$) and $|\bar{\chi}_n^\perp| > M$ on $Q_{\lambda_n}(i', j')$ (thus $(\bar{\chi}_n^\perp)^{i', j'} \notin \operatorname{supp}(\tilde{\Phi}_n)$). Then we have that

$$\begin{aligned} |(X_n)^{i,j} - (\bar{\chi}_n^\perp)^{i', j'}| &\leq |(X_n)^{i,j} - (\bar{\chi}_n^\perp)^{i,j}| + |(\bar{\chi}_n^\perp)^{i,j} - (\bar{\chi}_n^\perp)^{i', j'}| \\ &\leq C\lambda_n (|D^d \bar{\chi}_n^{i,j}| + |D^d \bar{\chi}_n^{i-1,j}| + |D^d \bar{\chi}_n^{i,j-1}|), \end{aligned}$$

a similar estimate being true for Y_n . Using that $D\tilde{\Phi}_n$ are equi-Lipschitz by (6.5) and that $D\tilde{\Phi}_n((\bar{\chi}_n^\perp)^{i', j'}) = 0$, we get from (6.13) that

$$|\operatorname{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp)(x)| \leq C\lambda_n (|D^d \bar{\chi}_n^{i,j}|^2 + |D^d \bar{\chi}_n^{i-1,j}|^2 + |D^d \bar{\chi}_n^{i,j-1}|^2).$$

Fixing an open set Ω'' with $\Omega' \subset \subset \Omega'' \subset \subset \Omega$ we conclude that for all n large enough

$$\left| \int_{\Omega' \setminus \Omega_n^{\text{supp}}} \zeta \operatorname{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) dx \right| \leq C\lambda_n \|D^d \bar{\chi}_n\|_{L^2(\Omega'')}^2 \leq C \frac{\lambda_n}{\varepsilon_n} \rightarrow 0,$$

where we have used (2.24) and (2.13). This implies (6.12).

Step 3. (Expanding the divergence using (6.6).) Let us observe first that after expanding $\operatorname{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp)$ on Ω_n^{supp} by (6.13), we can employ similar arguments as in Step 2 to replace the points X_n and Y_n in this formula by $\bar{\chi}_n^\perp$. Indeed, we have that $|X_n - \bar{\chi}_n^\perp|, |Y_n - \bar{\chi}_n^\perp| \leq C\lambda_n |D^d \bar{\chi}_n|$ and using that $D\tilde{\Phi}_n$ are equi-Lipschitz and (2.24) we get that

$$\int_{\Omega_n^{\text{supp}}} \zeta \operatorname{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) dx = \int_{\Omega_n^{\text{supp}}} \zeta (\nabla \tilde{\Phi}_{1,n}(\bar{\chi}_n^\perp) \cdot \partial_1^d \bar{\chi}_n^\perp + \nabla \tilde{\Phi}_{2,n}(\bar{\chi}_n^\perp) \cdot \partial_2^d \bar{\chi}_n^\perp) dx + o_n(1). \quad (6.14)$$

By (6.6) we get that

$$\begin{aligned} &\nabla \tilde{\Phi}_{1,n}(\bar{\chi}_n^\perp) \cdot \partial_1^d \bar{\chi}_n^\perp + \nabla \tilde{\Phi}_{2,n}(\bar{\chi}_n^\perp) \cdot \partial_2^d \bar{\chi}_n^\perp \\ &= \alpha_n(\bar{\chi}_n^\perp) (\partial_1^d \bar{\chi}_{1,n}^\perp + \partial_2^d \bar{\chi}_{2,n}^\perp) - q_n(\bar{\chi}_n) (\nabla \Psi_{1,n}(\bar{\chi}_n^\perp) \cdot \partial_1^d \bar{\chi}_n^\perp + \nabla \Psi_{2,n}(\bar{\chi}_n^\perp) \cdot \partial_2^d \bar{\chi}_n^\perp). \end{aligned} \quad (6.15)$$

We now exploit the fact that $\operatorname{curl}^d(\bar{\chi}_n)$ is approximately zero to obtain that

$$\int_{\Omega'} |\alpha_n(\bar{\chi}_n^\perp) (\partial_1^d \bar{\chi}_{1,n}^\perp + \partial_2^d \bar{\chi}_{2,n}^\perp)| dx = o_n(1) \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (6.16)$$

Indeed, since $|\alpha_n(\bar{\chi}_n^\perp) (\partial_1^d \bar{\chi}_{1,n}^\perp + \partial_2^d \bar{\chi}_{2,n}^\perp)| = |\alpha_n(\bar{\chi}_n^\perp)| |\operatorname{curl}^d(\bar{\chi}_n)|$, this is a consequence of Lemma 2.4 and the fact that α_n are equibounded by (6.5). Combining (6.12), (6.14), (6.15), and (6.16), we have shown that

$$\begin{aligned} &\int_{\Omega'} \zeta \operatorname{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) dx \\ &= - \int_{\Omega_n^{\text{supp}}} \zeta q_n(\bar{\chi}_n) \left(\nabla \Psi_{1,n}(\bar{\chi}_n^\perp) \cdot \partial_1^d \bar{\chi}_n^\perp + \nabla \Psi_{2,n}(\bar{\chi}_n^\perp) \cdot \partial_2^d \bar{\chi}_n^\perp \right) dx + o_n(1). \end{aligned} \quad (6.17)$$

Next, we replace $\nabla \Psi_{1,n}(\bar{\chi}_n^\perp) \cdot \partial_1^d \bar{\chi}_n^\perp + \nabla \Psi_{2,n}(\bar{\chi}_n^\perp) \cdot \partial_2^d \bar{\chi}_n^\perp$ with $\operatorname{div}^d(\Psi_n \circ \tilde{\chi}_n^\perp)$ up to a small error, thus recovering the analogue of (6.3) in the discrete, cf. (6.21).¹² We start by using a discrete chain rule to get that

$$\operatorname{div}^d(\Psi_n \circ \tilde{\chi}_n^\perp) = \nabla \Psi_{1,n}(\tilde{X}_n) \cdot \partial_1^d \tilde{\chi}_n^\perp + \nabla \Psi_{2,n}(\tilde{Y}_n) \cdot \partial_2^d \tilde{\chi}_n^\perp$$

where $(\tilde{X}_n)^{i,j}$ belongs to the segment connecting $(\tilde{\chi}_n^\perp)^{i,j}$ and $(\tilde{\chi}_n^\perp)^{i+1,j}$, and $(\tilde{Y}_n)^{i,j}$ to the segment connecting $(\tilde{\chi}_n^\perp)^{i,j}$ and $(\tilde{\chi}_n^\perp)^{i,j+1}$. Using that $|\bar{\chi}_n^\perp| \leq M$ on Ω_n^{supp} , that q_n

¹²Instead, we could also replace this term with $\operatorname{div}^d(\Psi_n \circ \bar{\chi}_n^\perp)$. However, since the derivative part $A^d(\chi)$ of our energy H_n features discrete derivatives of the parameter $\bar{\chi}$, it is useful to us to replace $\bar{\chi}$ by $\tilde{\chi}$ in this term.

are locally equibounded, that $|\tilde{X}_n - \tilde{\chi}_n^\perp|, |\tilde{Y}_n - \tilde{\chi}_n^\perp| \leq C\lambda_n |D^d \tilde{\chi}_n|$, and that $D\Psi_n$ are equi-Lipschitz by (6.5), we get that

$$q_n(\bar{\chi}_n) \left| \operatorname{div}^d(\Psi_n \circ \tilde{\chi}_n^\perp) - \left(\nabla \Psi_{1,n}(\tilde{\chi}_n^\perp) \cdot \partial_1^d \tilde{\chi}_n^\perp + \nabla \Psi_{2,n}(\tilde{\chi}_n^\perp) \cdot \partial_2^d \tilde{\chi}_n^\perp \right) \right| \leq C\lambda_n |D^d \tilde{\chi}_n|^2 \text{ on } \Omega_n^{\text{supp}}.$$

By (2.26) and (2.13) we obtain that

$$\begin{aligned} & - \int_{\Omega_n^{\text{supp}}} \zeta q_n(\bar{\chi}_n) \operatorname{div}^d(\Psi_n \circ \tilde{\chi}_n^\perp) dx \\ & = - \int_{\Omega_n^{\text{supp}}} \zeta q_n(\bar{\chi}_n) \left(\nabla \Psi_{1,n}(\tilde{\chi}_n^\perp) \cdot \partial_1^d \tilde{\chi}_n^\perp + \nabla \Psi_{2,n}(\tilde{\chi}_n^\perp) \cdot \partial_2^d \tilde{\chi}_n^\perp \right) dx + o_n(1). \end{aligned} \quad (6.18)$$

Next, for $k = 1, 2$ we estimate

$$\left| \nabla \Psi_{k,n}(\bar{\chi}_n^\perp) \cdot \partial_k^d \bar{\chi}_n^\perp - \nabla \Psi_{k,n}(\tilde{\chi}_n^\perp) \cdot \partial_k^d \tilde{\chi}_n^\perp \right| \leq C|\bar{\chi}_n^\perp - \tilde{\chi}_n^\perp| |\partial_k^d \bar{\chi}_n^\perp| + C|\partial_k^d \bar{\chi}_n^\perp - \partial_k^d \tilde{\chi}_n^\perp|, \quad (6.19)$$

where we have used again that $D\Psi_n$ are equi-Lipschitz and equibounded. The term $|\bar{\chi}_n^\perp - \tilde{\chi}_n^\perp|$ can be estimated as the difference $|\chi_n - \bar{\chi}_n|$ in Remark 2.5. Indeed, writing $\tilde{\chi}_{h,n} = \frac{1}{\sqrt{\delta_n}} \sin(\sqrt{\delta_n} \bar{\chi}_{h,n})$ for $h = 1, 2$ and using that $|s - \sin(s)| \leq C|s|^3$ we get $|\bar{\chi}_n^\perp - \tilde{\chi}_n^\perp| \leq C\delta_n |\bar{\chi}_n|$. Using a discrete chain rule in the above representation of $\tilde{\chi}_{h,n}$ we also get that

$$|\partial_k^d \bar{\chi}_{h,n} - \partial_k^d \tilde{\chi}_{h,n}| = |1 - \cos(\sqrt{\delta_n} X_{h,k,n})| |\partial_k^d \bar{\chi}_{h,n}| \leq C\delta_n (|\bar{\chi}_{h,n}|^2 + |\bar{\chi}_{h,n}^{+e_k}|^2) |\partial_k^d \bar{\chi}_{h,n}|, \quad (6.20)$$

for $k, h = 1, 2$, where $X_{h,k,n}^{i,j}$ belongs to the segment connecting $\bar{\chi}_{h,n}^{(i,j)}$ and $\bar{\chi}_{h,n}^{(i,j)+e_k}$ and we have used that $|1 - \cos(s)| \leq C|s|^2$. Returning to (6.19) we now infer that

$$\begin{aligned} \left| \nabla \Psi_{k,n}(\bar{\chi}_n^\perp) \cdot \partial_k^d \bar{\chi}_n^\perp - \nabla \Psi_{k,n}(\tilde{\chi}_n^\perp) \cdot \partial_k^d \tilde{\chi}_n^\perp \right| & \leq C\delta_n |\partial_k^d \bar{\chi}_n| (|\bar{\chi}_n|^3 + |\bar{\chi}_n|^2 + |\bar{\chi}_n^{+e_k}|^2) \\ & \leq CM^3 \delta_n |D^d \bar{\chi}_n| \quad \text{on } \Omega_n^{\text{supp}}, \end{aligned}$$

cf. (6.11). Thus, by (6.17) and (6.18) we infer that

$$\begin{aligned} & \left| \int_{\Omega'} \zeta \operatorname{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) dx + \int_{\Omega_n^{\text{supp}}} \zeta q_n(\bar{\chi}_n) \operatorname{div}^d(\Psi_n \circ \tilde{\chi}_n^\perp) dx \right| \\ & \leq CM^3 \delta_n \|q_n(\bar{\chi}_n)\|_{L^2(\Omega_n^{\text{supp}})} \|D^d \bar{\chi}_n\|_{L^2(\Omega_n^{\text{supp}})} + o_n(1) \leq CM^3 \delta_n \frac{\sqrt{\varepsilon_n}}{\sqrt{\varepsilon_n}} + o_n(1) \rightarrow 0, \end{aligned}$$

where we have used the fact that $q_n(\bar{\chi}_n)^2 = W(\chi_n)$, Proposition 2.6, and (2.24) in the proof thereof. In conclusion, we have proved that

$$\int_{\Omega'} \zeta \operatorname{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) dx = - \int_{\Omega_n^{\text{supp}}} \zeta q_n(\bar{\chi}_n) \operatorname{div}^d(\Psi_n \circ \tilde{\chi}_n^\perp) dx + o_n(1). \quad (6.21)$$

Step 4. (Young's inequality.) Applying Young's inequality to (6.21) we get that

$$\int_{\Omega'} \zeta \operatorname{div}^d(\tilde{\Phi}_n \circ \bar{\chi}_n^\perp) dx \leq \frac{1}{2} \int_{\Omega_n^{\text{supp}}} \frac{1}{\varepsilon_n} q_n(\bar{\chi}_n)^2 dx + \frac{1}{2} \int_{\Omega'} \varepsilon_n |\zeta|^2 |\operatorname{div}^d(\Psi_n \circ \tilde{\chi}_n^\perp)|^2 dx + o_n(1). \quad (6.22)$$

Step 5. (Recovering the potential term.) We prove that

$$\frac{1}{\varepsilon_n} \int_{\Omega_n^{\text{supp}}} q_n(\bar{\chi}_n)^2 dx \leq \frac{1}{\varepsilon_n} \int_{\Omega'} W^d(\chi_n) dx + o_n(1). \quad (6.23)$$

We proceed similarly as in Step 2 of the proof of Proposition 2.6: By (2.36) we have that

$$\left| \sqrt{W^d}(\chi_n^{i,j}) - \sqrt{W}(\chi_n^{i,j}) \right| \leq CM\lambda_n (|D^d \chi_n^{i-1,j}| + |D^d \chi_n^{i,j-1}|) \quad \text{on } \Omega_n^{\text{supp}}$$

according to (6.11), where we have used that $|\chi_n| \leq |\bar{\chi}_n|$. (This is seen by using in (2.8) the fact that $|\sin(x)| \leq |x|$.) Let Ω'' again be an open set with $\Omega' \subset \subset \Omega'' \subset \subset \Omega$. By the bound (2.25) and by (2.13) we obtain for n large enough that

$$\frac{1}{\sqrt{\varepsilon_n}} \left\| \sqrt{W^d}(\chi_n) - \sqrt{W}(\chi_n) \right\|_{L^2(\Omega_n^{\text{supp}})} \leq CM \frac{\lambda_n}{\sqrt{\varepsilon_n}} \|D^d \chi_n\|_{L^2(\Omega'')} \leq CM \frac{\lambda_n}{\varepsilon_n} \rightarrow 0.$$

Since $W^d - W = (2\sqrt{W^d} - (\sqrt{W^d} + \sqrt{W}))(\sqrt{W^d} - \sqrt{W})$ and $q_n(\bar{\chi}_n)^2 = W(\chi_n)$ we get that

$$\frac{1}{\varepsilon_n} \int_{\Omega_n^{\text{supp}}} |W^d(\chi_n) - q_n(\bar{\chi}_n)^2| dx \leq \left(\frac{2}{\sqrt{\varepsilon_n}} \|\sqrt{W^d}(\chi_n)\|_{L^2(\Omega_n^{\text{supp}})} + o_n(1) \right) \cdot o_n(1) \rightarrow 0,$$

where we have used that $\|W^d(\chi_n)\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon_n}$. Since $\Omega_n^{\text{supp}} \subset \Omega'$ and $W^d \geq 0$, we conclude the proof of (6.23).

Step 6. (From discrete divergence to full discrete derivative matrix.) In the next steps we recover the derivative term $|A^d(\chi_n)|^2$. We start by claiming that

$$\varepsilon_n \int_{\Omega'} |\zeta|^2 |\text{div}^d(\Psi_n \circ \tilde{\chi}_n^\perp)|^2 dx \leq \varepsilon_n \int_{\Omega'} |\zeta|^2 |\mathbf{D}^d(\Psi_n \circ \tilde{\chi}_n^\perp)|^2 dx + o_n(1), \quad (6.24)$$

see (6.4) for the analogous inequality in the continuum. Let us use the short-hand notation $V_n := \Psi_n \circ \tilde{\chi}_n^\perp$. We prove the claim first with a perturbed version of $\mathbf{D}^d(\Psi_n \circ \tilde{\chi}_n^\perp)$, where we add certain shifts in the lattice point. Specifically, let us observe that

$$\begin{aligned} & |\partial_1^d V_{1,n}|^2 + |\partial_1^d V_{2,n}^{*+e_2}|^2 + |\partial_2^d V_{1,n}^{*+e_1}|^2 + |\partial_2^d V_{2,n}|^2 \\ &= |\text{div}^d(V_n)|^2 + |\partial_1^d V_{2,n}^{*+e_2} - \partial_2^d V_{1,n}^{*+e_1}|^2 - 2\partial_1^d(V_{1,n} \partial_2^d V_{2,n}) + 2\partial_2^d(V_{1,n}^{*+e_1} \partial_1^d V_{2,n}), \end{aligned} \quad (6.25)$$

because $-\partial_1^d(V_{1,n} \partial_2^d V_{2,n}) + \partial_2^d(V_{1,n}^{*+e_1} \partial_1^d V_{2,n}) = -\partial_1^d V_{1,n} \partial_2^d V_{2,n} + \partial_2^d V_{1,n}^{*+e_1} \partial_1^d V_{2,n}^{*+e_2}$ by the discrete product rule. Since ζ is compactly supported in Ω , a discrete integration by parts allows us to conclude that

$$\begin{aligned} \varepsilon_n \int_{\Omega'} |\zeta|^2 |\text{div}^d(V_n)|^2 dx &\leq \varepsilon_n \int_{\Omega'} |\zeta|^2 (|\partial_1^d V_{1,n}|^2 + |\partial_1^d V_{2,n}^{*+e_2}|^2 + |\partial_2^d V_{1,n}^{*+e_1}|^2 + |\partial_2^d V_{2,n}|^2) dx \\ &\quad - 2\varepsilon_n \int_{\Omega'} \partial_1^d(|\zeta|^2) (V_{1,n} \partial_2^d V_{2,n})^{*+e_1} - \partial_2^d(|\zeta|^2) (V_{1,n}^{*+e_1} \partial_1^d V_{2,n})^{*+e_2} dx \end{aligned} \quad (6.26)$$

for all n large enough. Notice that, even although $|\zeta|^2$ is not a discrete function, it is still possible to use a discrete integration by parts when we extend the notion of discrete derivatives to non-discrete functions by making use of difference quotients. Specifically, for any function f on \mathbb{R}^2 we set $\partial_k^d f(x) := \frac{1}{\lambda_n} (f(x + \lambda_n e_k) - f(x))$ for $k = 1, 2$, where it will always be clear from the context which lattice spacing λ_n is to be considered. Since Ψ_n are equibounded, we have that $|V_n| \leq C$ with C independent of n . Since moreover $|\partial_k^d(|\zeta|^2)| \leq \|\nabla(|\zeta|^2)\|_{L^\infty}$, we can estimate the modulus of the last integral above by $C\varepsilon_n \|\mathbf{D}^d V_n\|_{L^1(\Omega')}$ for n large enough. Since Ψ_n are equi-Lipschitz, this can be further estimated by $C\varepsilon_n \|\mathbf{D}^d \tilde{\chi}_n\|_{L^2(\Omega')}$ which goes to zero, since $\|\mathbf{D}^d \tilde{\chi}_n\|_{L^2(\Omega')} \leq \frac{C}{\sqrt{\varepsilon_n}}$ by (2.26). Using a shift of variables we now obtain that

$$\begin{aligned} \varepsilon_n \int_{\Omega'} |\zeta|^2 |\text{div}^d(V_n)|^2 dx &\leq \varepsilon_n \int_{\Omega'} |\zeta|^2 |\mathbf{D}^d V_n|^2 dx + o_n(1) \\ &\quad + \varepsilon_n \int_{\Omega'} (|\zeta(x - \lambda_n e_2)|^2 - |\zeta(x)|^2) |\partial_1^d V_{2,n}|^2 \\ &\quad + (|\zeta(x - \lambda_n e_1)|^2 - |\zeta(x)|^2) |\partial_2^d V_{1,n}|^2 dx. \end{aligned}$$

Using that $|\zeta|^2$ is Lipschitz, that Ψ_n are equi-Lipschitz, and (2.24), we can estimate the last integral above by $C\lambda_n = o_n(1)$. Thus we obtain (6.24).

Step 7. (Identifying “bad” cells.) Next, we want to use the inequality

$$|\mathbf{D}^d(\Psi_n \circ \tilde{\chi}_n^\perp)|^2 \leq \text{Lip}(\Psi_n)^2 |\mathbf{D}^d \tilde{\chi}_n|^2$$

and afterwards the fact that $\text{curl}^d(\tilde{\chi}_n)$ is approximately zero to later recover the term $|A^d(\chi_n)|^2$ (a discrete divergence of $\tilde{\chi}_n$ with shifts in the lattice points) from $|\mathbf{D}^d \tilde{\chi}_n|^2$. However, before dismissing the approximations Ψ_n , we need to exploit their uniform boundedness in cells where $\text{curl}^d(\tilde{\chi}_n)$ is large. (This step can be avoided under the additional scaling assumption $\frac{\delta_n^{3/2}}{\lambda_n} \rightarrow 0$, cf. Footnote 13.)

For a small parameter $t > 0$ (we take $t < \frac{\pi}{2}$, see below), we introduce the collection of bad cells

$$\mathcal{Q}_{n,t}^{\text{bad}} := \{Q_{\lambda_n}(i, j) : (i, j) \in \mathbb{Z}^2, Q_{\lambda_n}(i, j) \cap \Omega' \neq \emptyset, \\ |(\bar{\chi}_n)^{i', j'}| > \frac{t}{\sqrt{\delta_n}} \text{ for some vertex } \lambda_n(i', j') \text{ of } Q_{\lambda_n}(i, j)\}$$

and the set

$$\Omega_{n,t}^{\text{bad}} := \bigcup_{Q \in \mathcal{Q}_{n,t}^{\text{bad}}} Q.$$

We note that we have an L^2 control on $\text{curl}^{\text{d}}(\tilde{\chi}_n)$ on the remaining set $\Omega' \setminus \Omega_{n,t}^{\text{bad}}$. To see this, let us recall from the proof of Lemma 2.4 that under the assumption (2.20) we have that $\text{curl}^{\text{d}}(\bar{\chi}_n)^{i,j} = 0$. As a consequence, recalling the definition (2.19) we have that $\text{curl}^{\text{d}}(\bar{\chi}_n) \equiv 0$ on $\Omega' \setminus \Omega_{n,t}^{\text{bad}}$ if $t < \frac{\pi}{2}$. Then, as in (6.20) we get that

$|\text{curl}^{\text{d}}(\tilde{\chi}_n)| = |\text{curl}^{\text{d}}(\tilde{\chi}_n) - \text{curl}^{\text{d}}(\bar{\chi}_n)| \leq C\delta_n(|\bar{\chi}_n|^2 + |\bar{\chi}_n^{+e_1}|^2 + |\bar{\chi}_n^{+e_2}|^2)|\text{D}^{\text{d}}\bar{\chi}_n| \leq Ct^2|\text{D}^{\text{d}}\bar{\chi}_n|$ on $\Omega' \setminus \Omega_{n,t}^{\text{bad}}$. Using the estimate (2.24) from the proof of Proposition 2.6 we conclude that

$$\varepsilon_n \int_{\Omega' \setminus \Omega_{n,t}^{\text{bad}}} |\text{curl}^{\text{d}}(\tilde{\chi}_n)|^2 dx \leq Ct^4. \quad (6.27)$$

We also note that we have an estimate on the number of bad cells: In view of Lemma 2.3, the number of vertices $\lambda_n(i', j')$ with $|(\bar{\chi}_n)^{i', j'}| > \frac{t}{\sqrt{\delta_n}}$ and such that $Q_{\lambda_n}(i', j') \subset \Omega$ is at most $C(t)\frac{\delta_n^{3/2}}{\lambda_n}$. Since $\Omega' \subset \subset \Omega$, for large enough n , $\#\mathcal{Q}_{n,t}^{\text{bad}}$ is at most four times larger. Thus, $\#\mathcal{Q}_{n,t}^{\text{bad}} \leq C(t)\frac{\delta_n^{3/2}}{\lambda_n}$.¹³

Step 8. (Removing a neighborhood of the “bad” cells.) We introduce a neighborhood of the “bad” cells

$$N_{n,t} := \Omega_{n,t}^{\text{bad}} + B_{r_n},$$

where B_{r_n} is the ball centered at 0 with radius r_n . If $r_n \ll \varepsilon_n$ (e.g., $r_n := \delta_n^{1/4}\varepsilon_n$) we claim that

$$\varepsilon_n \int_{\Omega'} |\zeta|^2 |\text{D}^{\text{d}}(\Psi_n \circ \tilde{\chi}_n^\perp)|^2 dx \leq \text{Lip}(\Psi_n)^2 \varepsilon_n \int_{\Omega' \setminus N_{n,t}} |\zeta|^2 |\text{D}^{\text{d}}\tilde{\chi}_n|^2 dx + o_n(1). \quad (6.28)$$

Indeed, we have that $\varepsilon_n \int_{N_{n,t}} |\zeta|^2 |\text{D}^{\text{d}}(\Psi_n \circ \tilde{\chi}_n^\perp)|^2 dx \rightarrow 0$ and $|\text{D}^{\text{d}}(\Psi_n \circ \tilde{\chi}_n^\perp)| \leq \text{Lip}(\Psi_n)|\text{D}^{\text{d}}\tilde{\chi}_n|$. To prove the former, let us note that for every $Q \in \mathcal{Q}_{n,t}^{\text{bad}}$, $Q + B_{r_n}$ is contained in a ball of radius $r_n + \frac{\lambda_n}{\sqrt{2}}$. Thus, $\mathcal{L}^2(N_{n,t}) \leq \#\mathcal{Q}_{n,t}^{\text{bad}} \pi(r_n + \frac{\lambda_n}{\sqrt{2}})^2 \leq C(t)\frac{\delta_n^{3/2}}{\lambda_n}(r_n^2 + \lambda_n^2)$, where we have used the estimate on $\#\mathcal{Q}_{n,t}^{\text{bad}}$ derived in Step 7. Moreover, we have that $|\text{D}^{\text{d}}(\Psi_n \circ \tilde{\chi}_n^\perp)| \leq \frac{C}{\lambda_n}$ since Ψ_n are bounded in L^∞ . In conclusion, using (2.13),

$$\varepsilon_n \int_{N_{n,t}} |\zeta|^2 |\text{D}^{\text{d}}(\Psi_n \circ \tilde{\chi}_n^\perp)|^2 dx \leq C(t)\varepsilon_n \frac{\delta_n^{3/2}}{\lambda_n}(r_n^2 + \lambda_n^2) \frac{1}{\lambda_n^2} = C(t) \frac{r_n^2 + \lambda_n^2}{\varepsilon_n^2} \rightarrow 0.$$

Step 9. (From full discrete derivative matrix to A^{d} , outside “bad” cells.) To ease the integration by parts, we introduce cut-off functions $\rho_{n,t} \in C^\infty(\mathbb{R}^2; [0, 1])$ such that $\rho_{n,t} = 0$ in $\Omega_{n,t}^{\text{bad}}$, $\rho_{n,t} = 1$ in $\Omega' \setminus \bar{N}_{n,t}$, and $|\nabla \rho_{n,t}| \leq \frac{C}{r_n}$. We set $\eta_{n,t} := \rho_{n,t}|\zeta|^2$. By (6.28) we have that

$$\varepsilon_n \int_{\Omega'} |\zeta|^2 |\text{D}^{\text{d}}(\Psi_n \circ \tilde{\chi}_n^\perp)|^2 dx \leq \text{Lip}(\Psi_n)^2 \varepsilon_n \int_{\Omega'} \eta_{n,t} |\text{D}^{\text{d}}\tilde{\chi}_n|^2 dx + o_n(1). \quad (6.29)$$

Let us observe that, similarly to (6.25),

$$|\partial_1^{\text{d}} \tilde{\chi}_{1,n}^{-e_1}|^2 + |\partial_1^{\text{d}} \tilde{\chi}_{2,n}|^2 + |\partial_2^{\text{d}} \tilde{\chi}_{1,n}|^2 + |\partial_2^{\text{d}} \tilde{\chi}_{2,n}^{-e_2}|^2 \\ = |\partial_1^{\text{d}} \tilde{\chi}_{1,n}^{-e_1} + \partial_2^{\text{d}} \tilde{\chi}_{2,n}^{-e_2}|^2 + |\text{curl}^{\text{d}}(\tilde{\chi}_n)|^2 - 2\partial_1^{\text{d}}(\tilde{\chi}_{1,n}^{-e_1} \partial_2^{\text{d}} \tilde{\chi}_{2,n}^{-e_2}) + 2\partial_2^{\text{d}}(\tilde{\chi}_{1,n} \partial_1^{\text{d}} \tilde{\chi}_{2,n}^{-e_2}),$$

¹³Under the scaling assumption that $\frac{\delta_n^{3/2}}{\lambda_n} \rightarrow 0$ we have that $\mathcal{Q}_{n,t}^{\text{bad}} = \emptyset$ for large enough n . As a consequence, the following technical steps can be simplified substantially. Without this additional scaling assumption however, to our knowledge these technicalities cannot be avoided.

and note that $|\partial_1^d \tilde{\chi}_{1,n}^{-e_1} + \partial_2^d \tilde{\chi}_{2,n}^{-e_2}|^2 = |A^d(\chi_n)|^2$. Thus, by shifting variables and using a discrete integration by parts, we get that for n large enough

$$\begin{aligned}
 & \varepsilon_n \int_{\Omega'} \eta_{n,t} |D^d \tilde{\chi}_n|^2 dx \\
 &= \varepsilon_n \int_{\Omega'} \eta_{n,t} (|\partial_1^d \tilde{\chi}_{1,n}^{-e_1}|^2 + |\partial_1^d \tilde{\chi}_{2,n}|^2 + |\partial_2^d \tilde{\chi}_{1,n}|^2 + |\partial_2^d \tilde{\chi}_{2,n}^{-e_2}|^2) dx \\
 & \quad + \varepsilon_n \int_{\Omega'} |\partial_1^d \tilde{\chi}_{1,n}|^2 (\eta_{n,t}(x) - \eta_{n,t}(x + \lambda_n e_1)) + |\partial_2^d \tilde{\chi}_{2,n}|^2 (\eta_{n,t}(x) - \eta_{n,t}(x + \lambda_n e_2)) dx \\
 &= \varepsilon_n \int_{\Omega'} \eta_{n,t} |A^d(\chi_n)|^2 dx + \varepsilon_n \int_{\Omega'} \eta_{n,t} |\operatorname{curl}^d(\tilde{\chi}_n)|^2 dx \\
 & \quad + 2\varepsilon_n \int_{\Omega'} \partial_1^d \eta_{n,t} (\tilde{\chi}_{1,n}^{-e_1} \partial_2^d \tilde{\chi}_{2,n}^{-e_2})^{\cdot+e_1} - \partial_2^d \eta_{n,t} (\tilde{\chi}_{1,n} \partial_2^d \tilde{\chi}_{2,n}^{-e_2})^{\cdot+e_2} dx \\
 & \quad - \varepsilon_n \int_{\Omega'} |\partial_1^d \tilde{\chi}_{1,n}|^2 \lambda_n \partial_1^d \eta_{n,t} + |\partial_2^d \tilde{\chi}_{2,n}|^2 \lambda_n \partial_2^d \eta_{n,t} dx,
 \end{aligned} \tag{6.30}$$

where as in (6.26) we let $\partial_1^d \eta_{n,t}$, $\partial_2^d \eta_{n,t}$ denote difference quotients of the function $\eta_{n,t}$. By (6.27) we have that

$$\varepsilon_n \int_{\Omega'} \eta_{n,t} |\operatorname{curl}^d(\tilde{\chi}_n)|^2 dx \leq Ct^4. \tag{6.31}$$

Moreover, since $\varepsilon_n \int_{\Omega'} |D^d \tilde{\chi}_n|^2 dx \leq C$ (by (2.26)) and $|\partial_k^d \eta_{n,t}| \leq \|\nabla \eta_{n,t}\|_{L^\infty} \leq \frac{C}{r_n}$, we obtain that

$$\left| \varepsilon_n \int_{\Omega'} |\partial_1^d \tilde{\chi}_{1,n}|^2 \lambda_n \partial_1^d \eta_{n,t} + |\partial_2^d \tilde{\chi}_{2,n}|^2 \lambda_n \partial_2^d \eta_{n,t} dx \right| \leq C \frac{\lambda_n}{r_n} \rightarrow 0, \tag{6.32}$$

provided we choose r_n such that $\lambda_n \ll r_n$, e.g., $r_n = \delta_n^{1/4} \varepsilon_n$ as proposed in Step 8. Finally, we show that

$$2\varepsilon_n \int_{\Omega'} \partial_1^d \eta_{n,t} (\tilde{\chi}_{1,n}^{-e_1} \partial_2^d \tilde{\chi}_{2,n}^{-e_2})^{\cdot+e_1} - \partial_2^d \eta_{n,t} (\tilde{\chi}_{1,n} \partial_2^d \tilde{\chi}_{2,n}^{-e_2})^{\cdot+e_2} dx = C(t) o_n(1). \tag{6.33}$$

To this end, let us use that $|\partial_k^d \eta_{n,t}(x)| \leq \|\nabla \eta_{n,t}\|_{L^\infty(B_{\lambda_n}(x))}$ and that $\nabla \eta_{n,t} = \rho_{n,t} \nabla(|\zeta|^2) + |\zeta|^2 \nabla \rho_{n,t}$ is bounded by C on $\Omega' \setminus N_{n,t}$ and by $\frac{C}{r_n}$ on $N_{n,t}$. As a consequence, for n large enough,

$$\begin{aligned}
 & \left| 2\varepsilon_n \int_{\Omega'} \partial_1^d \eta_{n,t} (\tilde{\chi}_{1,n}^{-e_1} \partial_2^d \tilde{\chi}_{2,n}^{-e_2})^{\cdot+e_1} - \partial_2^d \eta_{n,t} (\tilde{\chi}_{1,n} \partial_2^d \tilde{\chi}_{2,n}^{-e_2})^{\cdot+e_2} dx \right| \\
 & \leq C \frac{\varepsilon_n}{r_n} \int_{N_{n,t} + B_{\lambda_n}} |\tilde{\chi}_{1,n}| |\partial_2^d \tilde{\chi}_{2,n}^{\cdot+e_1-e_2}| + |\tilde{\chi}_{1,n}^{\cdot+e_2}| |\partial_1^d \tilde{\chi}_{2,n}| dx \\
 & \quad + C \varepsilon_n \|\tilde{\chi}_{1,n}\|_{L^2(\Omega')} \|D^d \tilde{\chi}_{2,n}\|_{L^2(\Omega')}.
 \end{aligned}$$

To further estimate this expression, let us observe that the function $s \mapsto |s| - \sqrt{(|s|^2 - 1)^+}$ belongs to $C_0(\mathbb{R}; \mathbb{R})$ and, as a consequence, is bounded. Since $|\tilde{\chi}_{1,n}| \leq |\chi_{1,n}|$ by their definition (2.8), we get that $|\tilde{\chi}_{1,n}| \leq \sqrt{(|\chi_{1,n}|^2 - 1)^+} + C$. Using Hölder's inequality, this allows us to infer that

$$\begin{aligned}
 & \left| 2\varepsilon_n \int_{\Omega'} \partial_1^d \eta_{n,t} (\tilde{\chi}_{1,n}^{-e_1} \partial_2^d \tilde{\chi}_{2,n}^{-e_2})^{\cdot+e_1} - \partial_2^d \eta_{n,t} (\tilde{\chi}_{1,n} \partial_2^d \tilde{\chi}_{2,n}^{-e_2})^{\cdot+e_2} dx \right| \\
 & \leq C \varepsilon_n \|D^d \tilde{\chi}_{2,n}\|_{L^2(\Omega')} \left(\frac{1}{r_n} (\mathcal{L}^2(N_{n,t} + B_{3\lambda_n}))^{1/4} \|\sqrt{(|\chi_{1,n}|^2 - 1)^+}\|_{L^4(\Omega')} \right. \\
 & \quad \left. + \frac{1}{r_n} (\mathcal{L}^2(N_{n,t} + B_{3\lambda_n}))^{1/2} + \|\tilde{\chi}_{1,n}\|_{L^2(\Omega')} \right)
 \end{aligned}$$

for n large enough. We recall that $\|D^d \tilde{\chi}_{2,n}\|_{L^2(\Omega')} \leq \frac{C}{\sqrt{\varepsilon_n}}$ by (2.26). We observe moreover that $\|\sqrt{(|\chi_{1,n}|^2 - 1)^+}\|_{L^4(\Omega')} \leq \|W(\chi_n)\|_{L^1(\Omega')}^{1/4} \leq C \varepsilon_n^{1/4}$ by Proposition 2.6. Furthermore, we have that $\|\tilde{\chi}_{1,n}\|_{L^2(\Omega')} \leq C$ by Remark 2.2. Finally, since $\lambda_n \ll r_n$ (according to our

choice of r_n) and in view of the bound on $\#\mathcal{Q}_{n,t}^{\text{bad}}$ from Step 7 we have that $\mathcal{L}^2(N_{n,t} + B_{3\lambda_n}) \leq \#\mathcal{Q}_{n,t}^{\text{bad}} Cr_n^2 \leq C(t) \frac{\delta_n^{3/2} r_n^2}{\lambda_n}$. Therefore, using (2.13) we obtain that

$$\begin{aligned} & \left| 2\varepsilon_n \int_{\Omega'} \partial_1^d \eta_{n,t} (\tilde{\chi}_{1,n}^{-e_1} \partial_2^d \tilde{\chi}_{2,n}^{-e_2})^{+e_1} - \partial_2^d \eta_{n,t} (\tilde{\chi}_{1,n} \partial_2^d \tilde{\chi}_{2,n}^{-e_2})^{+e_2} dx \right| \\ & \leq C(t) \sqrt{\varepsilon_n} \left(\frac{\delta_n^{3/8} \varepsilon_n^{1/4}}{r_n^{1/2} \lambda_n^{1/4}} + \frac{\delta_n^{3/4}}{\lambda_n^{1/2}} + 1 \right) = C(t) \left(\frac{\lambda_n^{1/2}}{r_n^{1/2}} + \delta_n^{1/2} + \varepsilon_n^{1/2} \right) = C(t) o_n(1). \end{aligned}$$

Thus we have shown (6.33). Now, using (6.31)–(6.33) in (6.30) and returning to (6.29), we get that

$$\varepsilon_n \int_{\Omega'} |\zeta|^2 |D^d(\Psi_n \circ \tilde{\chi}_n^\perp)|^2 dx \leq \text{Lip}(\Psi_n)^2 \varepsilon_n \int_{\Omega'} \eta_{n,t} |A^d(\chi_n)|^2 dx + Ct^4 + C(t) o_n(1).$$

By (6.7) and our assumption that $\|\Phi\|_{\text{Ent}} \leq 1$ we have that $\limsup_n \text{Lip}(\Psi_n) \leq 1$. Thus, letting $n \rightarrow \infty$ and then $t \rightarrow 0$ we infer that

$$\limsup_{n \rightarrow \infty} \varepsilon_n \int_{\Omega'} |\zeta|^2 |D^d(\Psi_n \circ \tilde{\chi}_n^\perp)|^2 dx \leq \liminf_{n \rightarrow \infty} \varepsilon_n \int_{\Omega'} |A^d(\chi_n)|^2 dx, \quad (6.34)$$

where we have used that $|\eta_{n,t}| \leq 1$.

Step 10. (Conclusion.) By (6.10), (6.22), (6.23), (6.24), and (6.34) we conclude that

$$\langle \text{div}(\tilde{\Phi} \circ \chi^\perp), \zeta \rangle \leq \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega'} \frac{1}{\varepsilon_n} W^d(\chi_n) + \varepsilon_n |A^d(\chi_n)|^2 dx,$$

i.e., (6.9) holds true as desired for all open $\Omega' \subset \Omega$ and $\zeta \in C_c^\infty(\Omega')$ with $\|\zeta\|_{L^\infty(\Omega')} \leq 1$. Passing to the supremum in ζ , the left-hand side of the inequality becomes the total variation $|\text{div}(\tilde{\Phi} \circ \chi^\perp)|(\Omega') = |\text{div}(\Phi \circ \chi^\perp)|(\Omega')$. Then, considering partitions of Ω to pass to the supremum in Φ in the sense of measures, we get that

$$\bigvee_{\substack{\Phi \in \text{Ent} \\ \|\Phi\|_{\text{Ent}} \leq 1}} |\text{div}(\Phi \circ \chi^\perp)|(\Omega) \leq \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W^d(\chi_n) + \varepsilon_n |A^d(\chi_n)|^2 dx.$$

This is the claim (4.2) and concludes the proof of Theorem 4.1-ii).

Remark 6.2. To prove an analogous liminf inequality for the functionals AG_n^d defined by (2.23) in place of H_n , a similar proof can be used and several steps can be simplified substantially. In particular, the introduction of the approximations $\tilde{\Phi}_n$ is not required. Moreover, there is no necessity to work with three different order parameters $\chi, \tilde{\chi}, \bar{\chi}$ and to estimate errors that are created when replacing one with another.

For the functionals AG_n^d , instead of (6.10), we have that

$$\langle \text{div}(\tilde{\Phi} \circ \chi^\perp), \zeta \rangle = \lim_n \int_{\Omega'} \zeta \text{div}^d(\tilde{\Phi} \circ D^d \varphi_n^\perp) dx,$$

where we assume that $D^d \varphi_n \rightarrow \chi$ in L_{loc}^1 . Using that $\text{curl}^d(D^d \varphi_n) \equiv 0$, with the obvious simplifications Steps 2–4 yield that

$$\int_{\Omega'} \zeta \text{div}^d(\tilde{\Phi} \circ D^d \varphi_n^\perp) dx \leq \frac{1}{2} \int_{\Omega_n^{\text{supp}}} \frac{1}{\varepsilon_n} q(D^d \varphi_n)^2 dx + \frac{1}{2} \int_{\Omega'} \varepsilon_n |\zeta|^2 |\text{div}^d(\Psi \circ D^d \varphi_n^\perp)|^2 dx.$$

In place of Step 5, we get that $\int_{\Omega_n^{\text{supp}}} q(D^d \varphi_n)^2 dx \leq \int_{\Omega'} W(D^d \varphi_n) dx$, which is immediate, since $q(\xi)^2 = W(\xi)$. Then, performing a discrete integration by parts as in Step 6 we get that

$$\int_{\Omega'} \varepsilon_n |\zeta|^2 |\text{div}^d(\Psi \circ D^d \varphi_n^\perp)|^2 dx \leq \int_{\Omega'} \varepsilon_n |\zeta|^2 |D^d(\Psi \circ D^d \varphi_n^\perp)|^2 dx + o_n(1)$$

and using that $|D^d(\Psi \circ D^d \varphi_n^\perp)|^2 \leq |D^d D^d \varphi_n|^2$ leads to the conclusion. The technical Steps 7–9 are not required.

In the next lemma we provide details about the approximate entropies $\tilde{\Phi}_n$ that we have used above.

Lemma 6.3. *Let $\Phi \in \text{Ent}$, let (Ψ, α) be as in (3.2), (3.3), and let $\tilde{\Phi}(\xi) = \Phi(\xi) - (1 - |\xi|^2)\Psi(\xi)$. Then, for n large enough, there exist functions $\tilde{\Phi}_n \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$, $\Psi_n \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$, and $\alpha_n \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ satisfying (6.5)–(6.8) above.*

Proof. Following [26, Lemma 2.4, Formula (2.7)], let us define the scalar function $\phi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ by $\phi(\xi) := \frac{1}{|\xi|^2} \Phi(\xi) \cdot \xi$. Using (3.1), it can be checked that Φ is characterized by ϕ through

$$\Phi(\xi) = \phi(\xi)\xi + (\nabla\phi(\xi) \cdot \xi^\perp)\xi^\perp. \quad (6.35)$$

Before defining $\tilde{\Phi}_n$, we first introduce an approximation Φ_n of Φ by using an approximate version of the above formula. We set $h_n(\xi) := -\frac{1}{2}q_n(\xi)$. Then, as $n \rightarrow \infty$ we have that $\nabla h_n(\xi) \rightarrow \xi$ and $\nabla^2 h_n(\xi) \rightarrow \text{Id}$ locally uniformly in $\xi \in \mathbb{R}^2$. In fact, we even have¹⁵

$$h_n(\xi) \rightarrow -\frac{1}{2}q(\xi) = \frac{1}{2}(|\xi|^2 - 1) \quad \text{locally in } C^k \text{ for every } k \in \mathbb{N} \quad (6.36)$$

as $n \rightarrow \infty$, since the functions $s \mapsto \frac{2}{\sqrt{\delta_n}} \sin\left(\frac{\sqrt{\delta_n}}{2}s\right)$ converge to the identity $s \mapsto s$ locally in C^k for $k \in \mathbb{N}$. We define

$$\Phi_n := \phi \nabla h_n + \frac{|\nabla h_n|^2 (\nabla\phi \cdot \nabla^\perp h_n) + \phi \nabla h_n \cdot (\nabla^2 h_n \nabla^\perp h_n)}{\nabla^\perp h_n \cdot (\nabla^2 h_n \nabla^\perp h_n)} \nabla^\perp h_n. \quad (6.37)$$

For large enough n , this defines a function $\Phi_n \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$. Indeed, for large n , $\nabla^2 h_n$ is positive definite and thus the denominator in the formula above can only be zero if $\nabla h_n = 0$. For large n this can only occur in a small neighborhood of the origin on which $\phi = 0$. From (6.35)–(6.37) we also get that $\Phi_n \rightarrow \Phi$ in C^2 . The function Φ_n is defined in such a way that it satisfies an approximate version of condition (3.1), namely

$$\nabla h_n(\xi) \cdot (\text{D}\Phi_n(\xi) \nabla^\perp h_n(\xi)) = 0 \quad \text{for all } \xi \in \mathbb{R}^2. \quad (6.38)$$

To prove this, let us use the short-hand notation

$$f_n := \frac{|\nabla h_n|^2 (\nabla\phi \cdot \nabla^\perp h_n) + \phi \nabla h_n \cdot (\nabla^2 h_n \cdot \nabla^\perp h_n)}{\nabla^\perp h_n \cdot (\nabla^2 h_n \cdot \nabla^\perp h_n)}.$$

We have that

$$\text{D}\Phi_n = \nabla h_n \otimes \nabla\phi + \phi \nabla^2 h_n + \nabla^\perp h_n \otimes \nabla f_n + f_n \text{D}\nabla^\perp h_n.$$

As a consequence,

$$\begin{aligned} \nabla h_n \cdot (\text{D}\Phi_n \nabla^\perp h_n) &= |\nabla h_n|^2 (\nabla\phi \cdot \nabla^\perp h_n) + \phi \nabla h_n \cdot (\nabla^2 h_n \nabla^\perp h_n) \\ &\quad + f_n \nabla h_n \cdot (\text{D}\nabla^\perp h_n \nabla^\perp h_n). \end{aligned}$$

Let for the moment $R \in SO(2)$ denote the rotation $x \mapsto x^\perp$. We observe that its inverse $R^{-1} = R^T$ is given by $-R$. Using that

$$\begin{aligned} (\nabla h_n)^T \text{D}\nabla^\perp h_n &= (\nabla h_n)^T \text{D}(R\nabla h_n) = (\nabla h_n)^T R \nabla^2 h_n = (R^T \nabla h_n)^T \nabla^2 h_n \\ &= -(\nabla^\perp h_n)^T \nabla^2 h_n, \end{aligned}$$

we get that

$$\begin{aligned} \nabla h_n \cdot (\text{D}\Phi_n \nabla^\perp h_n) &= |\nabla h_n|^2 (\nabla\phi \cdot \nabla^\perp h_n) + \phi \nabla h_n \cdot (\nabla^2 h_n \nabla^\perp h_n) \\ &\quad - f_n \nabla^\perp h_n \cdot (\nabla^2 h_n \nabla^\perp h_n) \\ &= 0 \end{aligned}$$

by the definition of f_n . This is (6.38).

Next we observe that (6.38) implies that

$$\text{D}\Phi_n \nabla^\perp h_n = \frac{\nabla^\perp h_n \cdot (\text{D}\Phi_n \nabla^\perp h_n)}{|\nabla h_n|^2} \nabla^\perp h_n.$$

¹⁴Indeed, we have that $\Phi(\xi) \cdot \xi^\perp$ is the derivative of $\xi \mapsto \Phi(\xi) \cdot \xi$ in direction ξ^\perp , which in turn is given by $|\xi|^2 (\nabla\phi(\xi) \cdot \xi^\perp)$.

¹⁵For our purposes, it is sufficient to have local convergence in C^4 .

Using twice that $\text{Id} = \frac{1}{|\nabla h_n|^2}(\nabla h_n \otimes \nabla h_n + \nabla^\perp h_n \otimes \nabla^\perp h_n)$, except in a small neighborhood of 0 in which $\Phi_n = 0$, the previous formula yields that

$$\begin{aligned} D\Phi_n &= \frac{1}{|\nabla h_n|^2} (D\Phi_n \nabla h_n \otimes \nabla h_n + D\Phi_n \nabla^\perp h_n \otimes \nabla^\perp h_n) \\ &= \frac{\nabla^\perp h_n \cdot (D\Phi_n \nabla^\perp h_n)}{|\nabla h_n|^2} \text{Id} + \frac{1}{|\nabla h_n|^2} \left(D\Phi_n \nabla h_n - \frac{\nabla^\perp h_n \cdot (D\Phi_n \nabla^\perp h_n)}{|\nabla h_n|^2} \nabla h_n \right) \otimes \nabla h_n. \end{aligned}$$

Thus, we get an approximate version of (3.4), namely $D\Phi_n + 2\Psi_n \otimes \nabla h_n = \alpha_n \text{Id}$, where we have set

$$\alpha_n := \frac{\nabla^\perp h_n \cdot (D\Phi_n \nabla^\perp h_n)}{|\nabla h_n|^2} \quad \text{and} \quad \Psi_n := -\frac{1}{2|\nabla h_n|^2} (D\Phi_n - \alpha_n \text{Id}) \nabla h_n.$$

Since $\Phi_n \rightarrow \Phi$ in C^2 , by (6.36) and in view of (3.2), (3.3) we have that $\alpha_n \rightarrow \alpha$ and $\Psi_n \rightarrow \Psi$ in C^2 . Now we define $\tilde{\Phi}_n := \Phi_n - q_n \Psi_n$ and have that $\tilde{\Phi}_n \rightarrow \tilde{\Phi}$ in C^2 as claimed. This proves (6.5). Moreover, we have that

$$D\tilde{\Phi}_n = \alpha_n \text{Id} - 2\Psi_n \otimes h_n - \Psi_n \otimes \nabla q_n - q_n D\Psi_n = \alpha_n \text{Id} - q_n D\Psi_n,$$

which proves (6.6). Recalling Definition 3.2, (6.7) follows from (6.5). Finally, to obtain (6.8), let us observe that by the definition of ϕ , Φ_n , α_n , Ψ_n , and $\tilde{\Phi}_n$, we have that

$$\text{supp}(\tilde{\Phi}_n), \text{supp}(\Psi_n), \text{supp}(\alpha_n) \subset \text{supp}(\Phi_n) \subset \text{supp}(\phi) \subset \text{supp}(\Phi),$$

which is a compact set. \square

7. PROOF OF THE LIMSUP INEQUALITY

In this section we prove Theorem 4.1-iii). We recall that for the proof we need the additional assumption

$$\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We fix $\Omega \in \mathcal{A}_0$ and $\chi \in BV(\Omega; \mathbb{R}^2)$ and we will prove that there exists a sequence $(\chi_n)_n \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ with $\chi_n \rightarrow \chi$ in $L^1(\Omega; \mathbb{R}^2)$ and

$$\limsup_{n \rightarrow \infty} H_n(\chi_n, \Omega) \leq H(\chi, \Omega). \quad (7.1)$$

If $H(\chi, \Omega) = +\infty$ the statement is trivial. Hence, in what follows we will assume that $H(\chi, \Omega) < +\infty$ and in particular that $\chi \in L^\infty(\Omega; \mathbb{S}^1)$ and $\text{curl}(\chi) = 0$ in $\mathcal{D}'(\Omega)$. We recall that under our assumptions on $\Omega \in \mathcal{A}_0$, such a field χ admits a potential $\varphi \in BVG(\Omega)$ such that $\nabla \varphi = \chi$ (cf. [17, Lemma 3.4]). The potential φ will be used in the construction of the recovery sequence below.

For a function χ with the properties listed above we will moreover show that there exists a sequence $(\chi_n)_n \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ satisfying (7.1), such that additionally

$$\sup_n \|\chi_n\|_{L^\infty(\mathbb{R}^2)} < +\infty, \quad \chi_n \rightarrow \chi \text{ in } L^p(\Omega; \mathbb{R}^2) \quad \text{for every } p < \infty. \quad (7.2)$$

Relying on the idea that the functionals H_n resemble a discrete version of Aviles-Giga functionals, we resort to the technique used in [46] to prove the limsup inequality for the classical Aviles-Giga functional. This technique has later been generalized by the same author in [47] to prove upper bounds for generic singular perturbation problems of the form

$$\frac{1}{\varepsilon_n} \int_{\Omega} F(\varepsilon_n \nabla^2 \varphi(x), \nabla \varphi(x)) \, dx.$$

Led by the observation that the functionals H_n resemble more closely the Aviles-Giga like energies $AG_{\varepsilon_n}^\Delta$ in (1.6), we will apply [47, Theorems 6.1, 6.2] to the sequence of functionals

$$\frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W(\nabla \varphi) + \varepsilon_n |\Delta \varphi|^2 \, dx, \quad (7.3)$$

i.e., to the case

$$F(A, b) = \frac{1}{2}(W(b) + |\operatorname{tr}(A)|^2) \quad \text{for } A \in \mathbb{R}^{2 \times 2}, b \in \mathbb{R}. \quad (7.4)$$

Before proving (7.1), we recall that the technique proposed in [47] uses a sequence of mollifications of φ to obtain a candidate for the recovery sequence. This leads to an asymptotic upper bound for the functionals in (7.3) which depends on the choice of the mollifier. Subsequently, the limsup inequality is obtained by optimizing the upper bound over all admissible mollifiers.

To define a mollification of φ on Ω we first extend it to the whole \mathbb{R}^2 . Since Ω is a BVG domain, by Proposition 2.1 we can find a compactly supported function $\bar{\varphi} \in BVG(\mathbb{R}^2)$ such that $\bar{\varphi} = \varphi$ a.e. in Ω and $|\operatorname{D}\bar{\varphi}|(\partial\Omega) = 0$.

We define a sequence φ^ε by convolving $\bar{\varphi}$ with suitable kernels. Following [47], we introduce the class $\mathcal{V}(\Omega)$ consisting of mollifiers $\eta \in C_c^3(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R})$ satisfying

$$\int_{\mathbb{R}^2} \eta(z, x) dz = 1 \quad \text{for all } x \in \Omega. \quad (7.5)$$

Remark 7.1. In [47] the author only requires C^2 regularity for the mollifiers. We remark that the proofs of [47, Theorem 6.1, Theorem 6.2] also work under this stronger regularity assumption on the convolution kernels.

Let us fix an arbitrary mollifier $\eta \in \mathcal{V}(\Omega)$ and let us define

$$\varphi^\varepsilon(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \eta\left(\frac{y-x}{\varepsilon}, x\right) \bar{\varphi}(y) dy = \int_{\mathbb{R}^2} \eta(z, x) \bar{\varphi}(x + \varepsilon z) dz \quad \text{for } x \in \mathbb{R}^2. \quad (7.6)$$

Evaluating the sequence of functionals in (7.3) on the functions φ^{ε_n} , we obtain a first asymptotic upper bound. More precisely, by [47, Theorem 6.1] we have that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{1}{\varepsilon_n} W(\nabla \varphi^{\varepsilon_n}) + \varepsilon_n |\Delta \varphi^{\varepsilon_n}|^2 dx = Y[\eta](\varphi), \quad (7.7)$$

where an explicit formula for $Y[\eta](\varphi)$ is given in [47, Formula (6.4)]. The precise expression of $Y[\eta](\varphi)$ is not relevant for our purposes. It is however important to derive the expression obtained when we optimize $Y[\eta](\varphi)$ with respect to $\eta \in \mathcal{V}(\Omega)$.

Proposition 7.2. *The following equality holds true:*

$$\inf_{\eta \in \mathcal{V}(\Omega)} Y[\eta](\varphi) = \frac{1}{6} \int_{J_\chi} |\chi|^3 d\mathcal{H}^1.$$

Proof. We recall that [47, Theorem 6.2] gives

$$\inf_{\eta \in \mathcal{V}(\Omega)} Y[\eta](\varphi) = \int_{J_{\nabla\varphi}} \sigma(\nabla\varphi^+(x), \nabla\varphi^-(x), \nu_{\nabla\varphi}(x)) d\mathcal{H}^1(x),$$

where the surface density σ is obtained by optimizing the energy for a transition from $\nabla\varphi^-(x)$ to $\nabla\varphi^+(x)$ over one-dimensional profiles and is given by

$$\sigma(a, b, \nu) := \inf_{\gamma} \left\{ \int_{-\infty}^{+\infty} F(-\gamma'(t)\nu \otimes \nu, \gamma(t)\nu + b) dt : \gamma \in C^1(\mathbb{R}), \text{ there exists } L > 0 \right. \\ \left. \text{s.t. for } t \geq L \text{ we have } \gamma(-t) = d \text{ and } \gamma(t) = 0 \right\}$$

for every $a, b \in \mathbb{R}^2$ and $\nu \in \mathbb{S}^1$ such that $(a - b) = d\nu$ for some $d \in \mathbb{R}$. This exhaustively defines the energy for the triple $(\nabla\varphi^+(x), \nabla\varphi^-(x), \nu_{\nabla\varphi}(x))$ for every $x \in J_{\nabla\varphi}$, cf. Subsection 2.3.

We claim that for all $a, b \in \mathbb{S}^1$, $a \neq b$, and $\nu \in \mathbb{S}^1$ with $(a - b) = d\nu$, $d \in \mathbb{R}$, we have that

$$\sigma(a, b, \nu) = \frac{1}{6} |a - b|^3. \quad (7.8)$$

In particular, since $\nabla\varphi^\pm = \chi^\pm \in \mathbb{S}^1$ a.e., we obtain that $\sigma(\nabla\varphi^+, \nabla\varphi^-, \nu_{\nabla\varphi}) = \frac{1}{6} |\chi|^3$ \mathcal{H}^1 -a.e. on $J_{\nabla\varphi} = J_\chi$. This will conclude the proof.

To prove (7.8), let us consider any admissible profile γ in the infimum problem defining $\sigma(a, b, \nu)$. Using the definition of F in (7.4) together with $|\operatorname{tr}(\nu \otimes \nu)| = |\nu|^2 = 1$ and writing $\gamma(t)\nu + b = (b \cdot \nu^\perp)\nu^\perp + (\gamma(t) + b \cdot \nu)\nu$, we get that

$$\int_{-\infty}^{+\infty} F(-\gamma'(t)\nu \otimes \nu, \gamma(t)\nu + b) dt = \frac{1}{2} \int_{-\infty}^{+\infty} (1 - |b \cdot \nu^\perp|^2 - |\gamma(t) + b \cdot \nu|^2)^2 + |\gamma'(t)|^2 dt.$$

Next, note that our assumptions $a, b, \nu \in \mathbb{S}^1$, $a \neq b$, and $(a - b) = d\nu$ imply that $a \cdot \nu = -b \cdot \nu = \frac{d}{2}$ and $1 - |b \cdot \nu^\perp|^2 = \frac{|d|^2}{4}$. In conclusion we obtain that

$$\begin{aligned} \int_{-\infty}^{+\infty} F(-\gamma'(t)\nu \otimes \nu, \gamma(t)\nu + b) dt &= \frac{1}{2} \int_{-\infty}^{+\infty} \left(\frac{|d|^2}{4} - |\gamma(t) - \frac{d}{2}|^2 \right)^2 + |\gamma'(t)|^2 dt \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{|d|^4}{16} \left(1 - |\tilde{\gamma}(\frac{|d|}{2}t)|^2 \right)^2 + \frac{|d|^4}{16} |\tilde{\gamma}'(\frac{|d|}{2}t)|^2 dt, \end{aligned}$$

where we have put $\tilde{\gamma}(t) := \frac{2}{d}(\gamma(\frac{2}{|d|}t) - \frac{d}{2})$. Using the change of variables $s = \frac{|d|}{2}t$ we infer that

$$\int_{-\infty}^{+\infty} F(-\gamma'(t)\nu \otimes \nu, \gamma(t)\nu + b) dt = \frac{|d|^3}{16} \int_{-\infty}^{+\infty} (1 - |\tilde{\gamma}(s)|^2)^2 + |\tilde{\gamma}'(s)|^2 ds.$$

Note that $\tilde{\gamma}(s) = -1$ for s large enough and $\tilde{\gamma}(s) = 1$ for s small enough. Thus, up to the multiplicative factor $\frac{|d|^3}{16}$, the infimum problem that defines $\sigma(a, b, \nu)$ coincides with the infimum problem for the optimal profile of the Modica-Mortola functional, *cf.* for example [14, Chapter 6]. In conclusion, $\sigma(a, b, \nu) = \frac{|d|^3}{16} 2 \left| \int_{-1}^1 (1 - s^2) ds \right| = \frac{|d|^3}{6}$. Since $|a - b| = |d|$, we conclude (7.8). \square

As a consequence, to prove Theorem 4.1-iii) we now only need to construct a sequence of spin fields $u_n \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$ such that their associated chirality variables χ_n satisfy (7.2) and

$$\limsup_n H_n(\chi_n, \Omega) \leq Y[\eta](\chi). \quad (7.9)$$

Indeed, in view of Proposition 7.2 and Corollary 3.8, the existence of a recovery sequence satisfying (7.1), (7.2) is then obtained by a diagonal argument (*cf.* also [47, Section 5]). To find such a sequence $(u_n)_n$, we first discretize the functions φ^{ε_n} defined by (7.6) on the lattice $\lambda_n \mathbb{Z}^2$. Specifically, we define $\varphi_n \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$ by

$$\varphi_n^{i,j} := \varphi^{\varepsilon_n}(\lambda_n i, \lambda_n j).$$

In the next proposition we prove that the Aviles-Giga-like functionals in (7.7) are the same as their discrete counterparts evaluated on φ_n , up to an error that vanishes when $n \rightarrow \infty$.

Proposition 7.3. *We have that*

$$\frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W(\nabla \varphi^{\varepsilon_n}) + \varepsilon_n |\Delta \varphi^{\varepsilon_n}|^2 dx = \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W(D^d \varphi_n) + \varepsilon_n |\Delta_s^d \varphi_n|^2 dx + o_n(1),$$

where $\Delta_s^d \varphi_n$ is defined by (2.3).

Proof. Step 1. (L^∞ -bounds on derivatives of φ^{ε_n} .) We claim that there exists a constant $C > 0$ such that

$$|\nabla \varphi^{\varepsilon_n}(x)| \leq C, \quad |D^d \varphi_n(x)| \leq C, \quad (7.10)$$

$$|\nabla^2 \varphi^{\varepsilon_n}(x)| \leq \frac{C}{\varepsilon_n}, \quad |D^d D^d \varphi_n(x)| \leq \frac{C}{\varepsilon_n}, \quad (7.11)$$

$$|\nabla^3 \varphi^{\varepsilon_n}(x)| \leq \frac{C}{\varepsilon_n^2}, \quad (7.12)$$

for every $x \in \mathbb{R}^2$ and every n . From the very definition of φ^{ε_n} in (7.6) and by integrating by parts we get

$$\partial_k \varphi^{\varepsilon_n}(x) = \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_k \bar{\varphi}(y) dy + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \partial_{x_k} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \bar{\varphi}(y) dy, \quad (7.13)$$

$$\begin{aligned} \partial_{hk}\varphi^{\varepsilon_n}(x) &= -\frac{1}{\varepsilon_n^3} \int_{\mathbb{R}^2} \partial_{z_h} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_k \bar{\varphi}(y) \, dy + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \partial_{x_h x_k} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \bar{\varphi}(y) \, dy \\ &\quad + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \partial_{x_h} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_k \bar{\varphi}(y) \, dy + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \partial_{x_k} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_h \bar{\varphi}(y) \, dy, \end{aligned} \quad (7.14)$$

and

$$\begin{aligned} \partial_{hkl}\varphi^{\varepsilon_n}(x) &= \frac{1}{\varepsilon_n^4} \int_{\mathbb{R}^2} \partial_{z_h z_\ell} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_k \bar{\varphi}(y) \, dy - \frac{1}{\varepsilon_n^3} \int_{\mathbb{R}^2} \partial_{z_h x_\ell} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_k \bar{\varphi}(y) \, dy \\ &\quad - \frac{1}{\varepsilon_n^3} \int_{\mathbb{R}^2} \partial_{z_\ell x_h x_k} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \bar{\varphi}(y) \, dy + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \partial_{x_h x_k x_\ell} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \bar{\varphi}(y) \, dy \\ &\quad - \frac{1}{\varepsilon_n^3} \int_{\mathbb{R}^2} \partial_{z_\ell x_h} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_k \bar{\varphi}(y) \, dy + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \partial_{x_h x_\ell} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_k \bar{\varphi}(y) \, dy \\ &\quad - \frac{1}{\varepsilon_n^3} \int_{\mathbb{R}^2} \partial_{z_\ell x_k} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_h \bar{\varphi}(y) \, dy + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \partial_{x_k x_\ell} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_h \bar{\varphi}(y) \, dy, \end{aligned} \quad (7.15)$$

where $\partial_{z_h} \eta(z, x)$ and $\partial_{x_h} \eta(z, x)$ denote the derivative with respect to the h -th variable in the first and second group of variables of $\eta(z, x)$ respectively.

By the assumptions on η , the function $y \mapsto \eta\left(\frac{y-x}{\varepsilon_n}, x\right)$ is supported on a ball $B_{R\varepsilon_n}(x)$ for a suitable $R > 0$ (independent of n and x). Together with the condition $\bar{\varphi} \in W^{1,\infty}(\mathbb{R}^2)$, (7.13)–(7.15) yield the first inequalities in (7.10)–(7.12).

Next, we observe that, as a consequence,

$$|\partial_1^d \varphi_n^{i,j}| = \left| \frac{\varphi^{\varepsilon_n}(\lambda_n(i+1), \lambda_n j) - \varphi^{\varepsilon_n}(\lambda_n i, \lambda_n j)}{\lambda_n} \right| \leq \int_0^1 |\partial_1 \varphi^{\varepsilon_n}(\lambda_n(i+t), \lambda_n j)| \, dt \leq C.$$

With analogous computations for $|\partial_2^d \varphi_n^{i,j}|$ we conclude the second inequality in (7.10).

In a similar way we also get that

$$|\partial_{kh}^d \varphi_n^{i,j}| \leq \int_0^1 \int_0^1 |\partial_{kh} \varphi^{\varepsilon_n}(\lambda_n(i, j) + \lambda_n t e_k + \lambda_n s e_h)| \, ds \, dt \leq \frac{C}{\varepsilon_n} \quad (7.16)$$

and thereby the second inequality in (7.11).

Step 2. (L^1 -bounds on derivatives of order 2.) We prove that

$$\int_{\mathbb{R}^2} \sup_{B_{\sqrt{5}\lambda_n}(x)} |\nabla^2 \varphi^{\varepsilon_n}| \, dx \leq C, \quad (7.17)$$

$$\|\mathbb{D}^d \mathbb{D}^d \varphi_n\|_{L^1(\mathbb{R}^2)} \leq C. \quad (7.18)$$

Recalling that $\nabla \bar{\varphi} \in BV(\mathbb{R}^2; \mathbb{R}^2)$, we can integrate by parts in (7.14) and obtain that

$$\begin{aligned} \partial_{hk}\varphi^{\varepsilon_n}(x) &= \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \mathbb{D}_h \partial_k \bar{\varphi}(y) \, dy + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \partial_{x_h x_k} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \bar{\varphi}(y) \, dy \\ &\quad + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \partial_{x_h} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_k \bar{\varphi}(y) \, dy + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \partial_{x_k} \eta\left(\frac{y-x}{\varepsilon_n}, x\right) \partial_h \bar{\varphi}(y) \, dy, \end{aligned}$$

where we let $\mathbb{D}_h \partial_k \bar{\varphi}$ denote the h -th component of the distributional derivative of $\partial_k \bar{\varphi}$. Since the function $y \mapsto \eta\left(\frac{y-x}{\varepsilon_n}, x\right)$ is supported on a ball $B_{R\varepsilon_n}(x)$, we observe that for every $x \in \mathbb{R}^2$

$$|\nabla^2 \varphi^{\varepsilon_n}(x)| \leq \|\eta\|_\infty \frac{1}{\varepsilon_n^2} |\mathbb{D} \nabla \bar{\varphi}|(B_{R\varepsilon_n}(x)) + C$$

and therefore

$$\sup_{B_{\sqrt{5}\lambda_n}(x)} |\nabla^2 \varphi^{\varepsilon_n}| \leq C \left(1 + \frac{1}{\varepsilon_n^2} |\mathbb{D} \nabla \bar{\varphi}|(B_{\sqrt{5}\lambda_n + R\varepsilon_n}(x)) \right).$$

Since $\bar{\varphi}$ is compactly supported in \mathbb{R}^2 , all φ^{ε_n} are supported in a common bounded set K . As a consequence, we get that

$$\begin{aligned} \int_{\mathbb{R}^2} \sup_{B_{\sqrt{5}\lambda_n}(x)} |\nabla^2 \varphi^{\varepsilon_n}| \, dx &\leq \int_{K+B_{\sqrt{5}\lambda_n}} C \left(1 + \frac{1}{\varepsilon_n^2} |\mathbb{D} \nabla \bar{\varphi}|(B_{\sqrt{5}\lambda_n + R\varepsilon_n}(x)) \right) \, dx \\ &\leq C + \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} |\mathbb{D} \nabla \bar{\varphi}|(B_{\sqrt{5}\lambda_n + R\varepsilon_n}(x)) \, dx. \end{aligned}$$

By Fubini we have that

$$\frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} |\mathrm{D}\nabla\bar{\varphi}|(B_{\sqrt{5}\lambda_n+R\varepsilon_n}(x)) \, dx = \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \mathcal{L}^2(B_{\sqrt{5}\lambda_n+R\varepsilon_n}(x')) \, d|\mathrm{D}\nabla\bar{\varphi}|(x') \leq C|\mathrm{D}\nabla\bar{\varphi}|(\mathbb{R}^2),$$

where we have used that $\lambda_n \ll \varepsilon_n$ as $n \rightarrow \infty$ by (2.13). This concludes the proof of (7.17). To prove (7.18) it only remains to observe that with the estimate (7.16) we get that, for $x \in Q_{\lambda_n}(i, j)$

$$|\partial_{kh}^d \varphi_n(x)| \leq \int_0^1 \int_0^1 |\partial_{kh} \varphi^{\varepsilon_n}(\lambda_n(i, j) + \lambda_n t e_k + \lambda_n s e_h)| \, ds \, dt \leq \sup_{B_{\sqrt{5}\lambda_n}(x)} |\nabla^2 \varphi^{\varepsilon_n}|.$$

Step 3. (Estimates on the error in the potential part.) We show that

$$\frac{1}{\varepsilon_n} \int_{\Omega} |W(\nabla \varphi^{\varepsilon_n}(x)) - W(\mathrm{D}^d \varphi_n(x))| \, dx \rightarrow 0.$$

We start by observing that for every $x \in Q_{\lambda_n}(i, j)$

$$\begin{aligned} |\partial_1 \varphi^{\varepsilon_n}(x) - \partial_1^d \varphi_n(x)| &= |\partial_1 \varphi^{\varepsilon_n}(x) - \partial_1^d \varphi_n^{i,j}(x)| \leq \int_0^1 |\partial_1 \varphi^{\varepsilon_n}(x) - \partial_1 \varphi^{\varepsilon_n}(\lambda_n(i+t), \lambda_n j)| \, dt \\ &\leq \sup_{B_{\sqrt{2}\lambda_n}(x)} |\nabla^2 \varphi^{\varepsilon_n}| \sqrt{2} \lambda_n, \end{aligned} \tag{7.19}$$

a similar computation being true for the discrete partial derivatives in the direction of e_2 . By (7.10) and since W is locally Lipschitz, there exists a constant L independent of n and x , such that

$$|W(\nabla \varphi^{\varepsilon_n}(x)) - W(\mathrm{D}^d \varphi_n(x))| \leq L |\nabla \varphi^{\varepsilon_n}(x) - \mathrm{D}^d \varphi_n(x)| \leq L \sup_{B_{\sqrt{2}\lambda_n}(x)} |\nabla^2 \varphi^{\varepsilon_n}| \sqrt{2} \lambda_n.$$

By (7.17) and using (2.13) we get that

$$\frac{1}{\varepsilon_n} \int_{\Omega} |W(\nabla \varphi^{\varepsilon_n}(x)) - W(\mathrm{D}^d \varphi_n(x))| \, dx \leq \sqrt{2} L C \frac{\lambda_n}{\varepsilon_n} = C \sqrt{\delta_n} \rightarrow 0.$$

Step 4. (Estimates on the error in the derivative part.) We show that

$$\varepsilon_n \int_{\Omega} \left| |\Delta \varphi^{\varepsilon_n}|^2 - |\Delta_s^d \varphi_n|^2 \right| \, dx \rightarrow 0.$$

To this end, we observe again that for $x \in Q_{\lambda_n}(i, j)$

$$\partial_{11}^d \varphi_n^{i-1,j} - \partial_{11} \varphi^{\varepsilon_n}(x) = \int_0^1 \int_0^1 \partial_{11} \varphi^{\varepsilon_n}(\lambda_n(i-1+s+t), \lambda_n j) - \partial_{11} \varphi^{\varepsilon_n}(x) \, ds \, dt$$

and thus, noting that $|x - (\lambda_n(i-1+s+t), \lambda_n j)| \leq \sqrt{5}\lambda_n$, we conclude that

$$|\partial_{11}^d \varphi_n^{i-1,j} - \partial_{11} \varphi^{\varepsilon_n}(x)| \leq \sqrt{5} \lambda_n \|\nabla^3 \varphi^{\varepsilon_n}\|_{L^\infty(\mathbb{R}^2)} \leq C \frac{\lambda_n}{\varepsilon_n^2},$$

where we have used (7.12). Since the same estimate holds true for $|\partial_{22}^d \varphi_n^{i,j-1} - \partial_{22} \varphi^{\varepsilon_n}(x)|$, we infer that

$$\begin{aligned} \varepsilon_n \int_{\mathbb{R}^2} \left| |\Delta \varphi^{\varepsilon_n}|^2 - |\Delta_s^d \varphi_n|^2 \right| \, dx &= \varepsilon_n \int_{\mathbb{R}^2} |\Delta \varphi^{\varepsilon_n} - \Delta_s^d \varphi_n| |\Delta \varphi^{\varepsilon_n} + \Delta_s^d \varphi_n| \, dx \\ &\leq C \frac{\lambda_n}{\varepsilon_n} \left(\|\nabla^2 \varphi^{\varepsilon_n}\|_{L^1(\mathbb{R}^2)} + \|\mathrm{D}^d \mathrm{D}^d \varphi_n\|_{L^1(\mathbb{R}^2)} \right) \leq C \sqrt{\delta_n} \rightarrow 0, \end{aligned}$$

where we have used (7.17)–(7.18) and (2.13). This concludes the proof. \square

Remark 7.4. In view of (7.7), Proposition 7.3 yields

$$\lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W(\mathrm{D}^d \varphi_n) + \varepsilon_n |\Delta_s^d \varphi_n|^2 \, dx = Y[\eta](\varphi). \tag{7.20}$$

Together with $\varphi_n \rightarrow \varphi$ in $W^{1,1}(\Omega)$ (see the proof of Proposition 7.5 below) this allows us to prove the limsup inequality on the space of $\varphi \in BVG(\Omega)$ such that $|\nabla \varphi| = 1$ a.e. for the

discrete functionals in (7.20). In a similar fashion it is possible to prove the same limsup inequality for the discrete Aviles-Giga functionals defined in (2.23). Note that both results hold without assuming the additional scaling assumption $\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0$ and instead require merely that $\frac{\lambda_n}{\varepsilon_n} \rightarrow 0$ as $n \rightarrow \infty$.

Using the discrete functions φ_n , we can now define the sequence χ_n . To this end it is convenient to introduce the spin fields $u_n \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$ by

$$u_n^{i,j} := \left(\cos\left(\frac{\sqrt{\delta_n}}{\lambda_n} \varphi_n^{i,j}\right), \sin\left(\frac{\sqrt{\delta_n}}{\lambda_n} \varphi_n^{i,j}\right) \right).$$

We then define $\chi_n := \chi(u_n)$ through (2.8) as the chirality variable associated to u_n . Moreover, let us again use the notation $\tilde{\chi}_n := \tilde{\chi}(u_n)$ for the order parameters defined as well in (2.8), and $\bar{\chi}_n$ for the auxiliary variables defined by (2.19).

Note that the construction of u_n is done in such a way that

$$\bar{\chi}_n = D^d \varphi_n \quad \text{for all } n \text{ large enough.} \quad (7.21)$$

Indeed, by (7.10) we have that $\sqrt{\delta_n} |\partial_1^d \varphi_n| < \pi$ for all n large enough. Thus, evaluating (2.6) and using standard trigonometric identities we get that

$$(\theta^{\text{hor}}(u_n))^{i,j} = \text{sign}(\sin(\sqrt{\delta_n} \partial_1^d \varphi_n^{i,j})) \arccos(\cos(\sqrt{\delta_n} \partial_1^d \varphi_n^{i,j})) = \sqrt{\delta_n} \partial_1^d \varphi_n^{i,j}$$

for all i, j . Analogously, $(\theta^{\text{ver}}(u_n))^{i,j} = \sqrt{\delta_n} \partial_2^d \varphi_n^{i,j}$. Then, in view of (2.19) we obtain (7.21).

Let us prove that the sequence $(\chi_n)_n$ satisfies the conditions in (7.2).

Proposition 7.5. *There exists a constant $C > 0$ such that*

$$\|\chi_n\|_{L^\infty(\mathbb{R}^2)} \leq C \quad \text{and} \quad \|\bar{\chi}_n\|_{L^\infty(\mathbb{R}^2)} \leq C \quad (7.22)$$

for all n . Moreover, $\chi_n \rightarrow \chi$ in $L^p(\Omega; \mathbb{R}^2)$ for all $p < \infty$.

Proof. From (7.10) and (7.21) we immediately get boundedness of $(\bar{\chi}_n)_n$ in L^∞ . Writing χ_n in terms of $\bar{\chi}_n$ and using that $|\sin(s)| \leq |s|$ we have that $|\chi_n| \leq |\bar{\chi}_n|$, which concludes the proof of (7.22).

To show that $\chi_n \rightarrow \chi$ in $L^p(\Omega; \mathbb{R}^2)$ for all $p < \infty$ observe that due to the L^∞ bound on the sequence $(\chi_n)_n$ it is enough to show the convergence only in $L^1(\Omega; \mathbb{R}^2)$.

We start by showing that $\bar{\chi}_n \rightarrow \chi$ in $L^1(\Omega; \mathbb{R}^2)$. Let us recall that $\chi = \nabla \varphi$. By (7.21) we get that

$$\|\bar{\chi}_n - \chi\|_{L^1(\Omega)} \leq \|D^d \varphi_n - \nabla \varphi^{\varepsilon_n}\|_{L^1(\Omega)} + \|\nabla \varphi^{\varepsilon_n} - \nabla \varphi\|_{L^1(\Omega)}$$

for n large enough. Using the bounds (7.17) and (7.19) (together with its analogue for discrete partial derivatives in the direction of e_2) already proven in Proposition 7.3, we obtain for the first term

$$\|D^d \varphi_n - \nabla \varphi^{\varepsilon_n}\|_{L^1(\Omega)} \leq C \lambda_n \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Moreover, from (7.5) we deduce that

$$\nabla \varphi(x) = \int_{\mathbb{R}^2} \nabla_x \eta(z, x) \bar{\varphi}(x) + \eta(z, x) \nabla \bar{\varphi}(x) \, dz, \quad \text{for } x \in \Omega,$$

where $\nabla_x \eta(z, x)$ denotes the gradient of η with respect to the second group of variables. Together with (7.6), this yields

$$\begin{aligned} \int_{\Omega} |\nabla \varphi^{\varepsilon_n}(x) - \nabla \varphi(x)| \, dx &\leq \int_{\Omega} \int_{\mathbb{R}^2} |\nabla_x \eta(z, x)| |\bar{\varphi}(x + \varepsilon_n z) - \bar{\varphi}(x)| \, dz \, dx \\ &\quad + \int_{\Omega} \int_{\mathbb{R}^2} |\eta(z, x)| |\nabla \bar{\varphi}(x + \varepsilon_n z) - \nabla \bar{\varphi}(x)| \, dz \, dx \\ &\leq \|\nabla \eta\|_{L^\infty} \int_{B_R} \|\bar{\varphi}(\cdot + \varepsilon_n z) - \bar{\varphi}\|_{L^1(\Omega)} \, dz \\ &\quad + \|\eta\|_{L^\infty} \int_{B_R} \|\nabla \bar{\varphi}(\cdot + \varepsilon_n z) - \nabla \bar{\varphi}\|_{L^1(\Omega)} \, dz \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, where $R > 0$ is a radius (independent of n and x) such that $z \mapsto \eta(z, x)$ is supported in B_R and we have used the continuity of translations of L^1 functions. This concludes the proof that $\bar{\chi}_n \rightarrow \chi$ in $L^1(\Omega; \mathbb{R}^2)$.

Hence, it remains to show that $\|\chi_n - \bar{\chi}_n\|_{L^1(\Omega)} \rightarrow 0$. Similarly as in Remark 2.5 we have that $|\chi_n - \bar{\chi}_n| \leq C\delta_n |\bar{\chi}_n|^3$. Thus, by (7.22) we even have that $\chi_n - \bar{\chi}_n \rightarrow 0$ in $L^\infty(\Omega)$. \square

It remains to show that $\limsup_n H_n(\chi_n, \Omega) \leq Y[\eta](\chi)$. We will achieve this by comparing the energies $H_n(\chi_n, \Omega)$ to the discrete Aviles-Giga-like energies from Proposition 7.3. This is the only part of the proof in which we require the scaling assumption $\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0$.

Proposition 7.6. *Assume that $\frac{\delta_n^{5/2}}{\lambda_n} \rightarrow 0$. Then,*

$$H_n(\chi_n, \Omega) = \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W(D^d \varphi_n) + \varepsilon_n |\Delta_S^d \varphi_n|^2 dx + o_n(1).$$

Proof. Step 1. (Estimate of $|W(\chi_n) - W(D^d \varphi_n)|$.) We prove that

$$\frac{1}{\varepsilon_n} \int_{\Omega} |W(\chi_n) - W(D^d \varphi_n)| dx \rightarrow 0.$$

First, by (7.21) we have that

$$\begin{aligned} |W(\chi_n) - W(D^d \varphi_n)| &= |1 - |\chi_n|^2 + 1 - |\bar{\chi}_n|^2| | |\bar{\chi}_n|^2 - |\chi_n|^2 | \\ &\leq \left(2|1 - |\bar{\chi}_n|^2| + ||\bar{\chi}_n|^2 - |\chi_n|^2|\right) ||\bar{\chi}_n|^2 - |\chi_n|^2|. \end{aligned}$$

Next, as in Remark 2.5 we obtain that $|\chi_n - \bar{\chi}_n| \leq C\delta_n |\bar{\chi}_n|^3$. In view of (7.22) and (2.13) we get

$$\frac{1}{\sqrt{\varepsilon_n}} \||\bar{\chi}_n|^2 - |\chi_n|^2\|_{L^2(\Omega)} = \frac{1}{\sqrt{\varepsilon_n}} \||\bar{\chi}_n + \chi_n||\bar{\chi}_n - \chi_n|\|_{L^2(\Omega)} \leq C \frac{\delta_n}{\sqrt{\varepsilon_n}} = C \left(\frac{\delta_n^{5/2}}{\lambda_n}\right)^{1/2} \rightarrow 0.$$

Moreover, by (7.7) and Proposition 7.3,

$$\frac{1}{\varepsilon_n} \||1 - |\bar{\chi}_n|^2\|_{L^2(\Omega)}^2 = \frac{1}{\varepsilon_n} \int_{\Omega} W(D^d \varphi_n) dx \leq C.$$

In conclusion, by Hölder's inequality,

$$\frac{1}{\varepsilon_n} \int_{\Omega} |W(\chi_n) - W(D^d \varphi_n)| dx \leq \left(\frac{2}{\sqrt{\varepsilon_n}} \||1 - |\bar{\chi}_n|^2\|_{L^2(\Omega)} + o_n(1)\right) \cdot o_n(1) \rightarrow 0.$$

Step 2. (Estimate of $|W^d(\chi_n) - W(\chi_n)|$.) We prove that

$$\frac{1}{\varepsilon_n} \int_{\Omega} |W^d(\chi_n) - W(\chi_n)| dx \rightarrow 0.$$

As in (2.36) we have that

$$\begin{aligned} |\sqrt{W^d}(\chi_n^{i,j}) - \sqrt{W}(\chi_n^{i,j})| &\leq \frac{1}{2} |(\chi_{1,n}^{i,j} + \chi_{1,n}^{i-1,j})\lambda_n \partial_1^d \chi_{1,n}^{i-1,j} + (\chi_{2,n}^{i,j} + \chi_{2,n}^{i,j-1})\lambda_n \partial_2^d \chi_{2,n}^{i,j-1}| \\ &\leq C\lambda_n (|D^d \chi_n^{i-1,j}| + |D^d \chi_n^{i,j-1}|), \end{aligned}$$

where we have used (7.22). By writing χ_n in terms of $\bar{\chi}_n$ and using the 1-Lipschitz continuity of the map $s \mapsto \frac{2}{\sqrt{\delta_n}} \sin(\frac{\sqrt{\delta_n}}{2}s)$ we get that $|D^d \chi_n| \leq |D^d \bar{\chi}_n| = |D^d D^d \varphi_n|$. Let us observe that by the bounds (7.11) and (7.18) we have that $\|D^d D^d \varphi_n\|_{L^2(\mathbb{R}^2)} \leq \frac{C}{\sqrt{\varepsilon_n}}$ and, as a consequence, by (2.13),

$$\frac{1}{\sqrt{\varepsilon_n}} \|\sqrt{W^d}(\chi_n) - \sqrt{W}(\chi_n)\|_{L^2(\Omega)} \leq C \frac{\lambda_n}{\varepsilon_n} \rightarrow 0.$$

Writing $W^d - W = (2\sqrt{W} + (\sqrt{W^d} - \sqrt{W}))(\sqrt{W^d} - \sqrt{W})$ we infer that

$$\frac{1}{\varepsilon_n} \int_{\Omega} |W^d(\chi_n) - W(\chi_n)| dx \leq \left(\frac{2}{\sqrt{\varepsilon_n}} \|\sqrt{W}(\chi_n)\|_{L^2(\Omega)} + o_n(1)\right) \cdot o_n(1) \rightarrow 0,$$

where we have used that that $\frac{1}{\sqrt{\varepsilon_n}} \|\sqrt{W(\chi_n)}\|_{L^2(\Omega)} \leq C$ by (7.7), Proposition 7.3, and Step 1.

Step 3. (Estimate of $||A^d(\chi_n)|^2 - |\Delta_s^d \varphi_n|^2|$) We prove that

$$\varepsilon_n \int_{\Omega} ||A^d(\chi_n)|^2 - |\Delta_s^d \varphi_n|^2| dx \rightarrow 0.$$

To show this we observe that

$$||A^d(\chi_n)|^2 - |\Delta_s^d \varphi_n|^2| = |A^d(\chi_n) + \Delta_s^d \varphi_n| |A^d(\chi_n) - \Delta_s^d \varphi_n|,$$

where, by (7.21),

$$|A^d(\chi_n)^{i,j} + \Delta_s^d \varphi_n^{i,j}| \leq |D^d \tilde{\chi}_n^{i-1,j}| + |D^d \tilde{\chi}_n^{i,j-1}| + |D^d \bar{\chi}_n^{i-1,j}| + |D^d \bar{\chi}_n^{i,j-1}| \quad (7.23)$$

and

$$|A^d(\chi_n) - \Delta_s^d \varphi_n| \leq |D^d \tilde{\chi}_n^{i-1,j} - D^d \bar{\chi}_n^{i-1,j}| + |D^d \tilde{\chi}_n^{i,j-1} - D^d \bar{\chi}_n^{i,j-1}|. \quad (7.24)$$

To estimate the right-hand side in (7.23) we use the 1-Lipschitz continuity of the map $s \mapsto \frac{1}{\sqrt{\delta_n}} \sin(\sqrt{\delta_n} s)$ to obtain that $|D^d \tilde{\chi}_n| \leq |D^d \bar{\chi}_n| = |D^d D^d \varphi_n|$. As in Step 2 we have that $\|D^d D^d \varphi_n\|_{L^2(\mathbb{R}^2)} \leq \frac{C}{\sqrt{\varepsilon_n}}$ by the bounds (7.11) and (7.18) in the proof of Proposition 7.3. As a consequence,

$$\|A^d(\chi_n) + \Delta_s^d \varphi_n\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\varepsilon_n}}.$$

To estimate the right-hand side in (7.24) we proceed similarly as in Step 3 in Section 6. Specifically, as in (6.20) we have that

$$|\partial_k^d \tilde{\chi}_{h,n} - \partial_k^d \bar{\chi}_{h,n}| \leq C \delta_n (|\bar{\chi}_{h,n}|^2 + |\bar{\chi}_{h,n}^{+e_k}|^2) |\partial_k^d \bar{\chi}_{h,n}| \leq C \delta_n |\partial_{kh}^d \varphi_n|$$

for $k, h = 1, 2$, where the last inequality is due to (7.21) and (7.22). Using again that $\|D^d D^d \varphi_n\|_{L^2(\mathbb{R}^2)} \leq \frac{C}{\sqrt{\varepsilon_n}}$, we obtain that $\|\partial_k^d \tilde{\chi}_{h,n} - \partial_k^d \bar{\chi}_{h,n}\|_{L^2(\Omega)} \leq C \frac{\delta_n}{\sqrt{\varepsilon_n}}$. This yields

$$\|A^d(\chi_n) - \Delta_s^d \varphi_n\|_{L^2(\Omega)} \leq C \frac{\delta_n}{\sqrt{\varepsilon_n}}.$$

Finally, our estimates lead to

$$\varepsilon_n \int_{\Omega} ||A^d(\chi_n)|^2 - |\Delta_s^d \varphi_n|^2| dx \leq C \delta_n \rightarrow 0.$$

Recalling that

$$H_n(\chi_n, \Omega) = \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon_n} W^d(\chi_n) + \varepsilon_n |A^d(\chi_n)|^2 dx,$$

Steps 1–3 yield the claim of the proposition. \square

Thanks to (7.7) and Propositions 7.3 and 7.6, we have proved (7.9). Since by Proposition 7.5 the sequence $(\chi_n)_n$ moreover satisfies (7.2), this concludes the proof of Theorem 4.1-iii).

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