Multi-dimensional degenerate operators in L^p – spaces

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Abstract

This paper is concerned with second-order elliptic operators whose diffusion coefficients degenerate at the boundary in first order. In this borderline case, the behavior strongly depends on the size and direction of the drift term. Mildly inward (or outward) pointing and strongly outward pointing drift terms were studied before. Here we treat the intermediate case equipped with Dirichlet boundary conditions, and show generation of an analytic positive C_0-semigroup. The main result is a precise description of the domain of the generator, which is more involved than in the other cases and exhibits reduced regularity compared to them.

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1 Introduction

In the present paper we investigate regularity and generation properties of second-order elliptic operators in L^p whose diffusion coefficients degenerate in first order at the boundary. This type of degeneration is a borderline case in the sense that the drift term in normal direction is of the same 'order' as the diffusion part, roughly speaking. Accordingly, the size and direction of the drift term in normal direction play a crucial role in our results and proofs. In this sense, first-order degeneration is the most interesting situation in this context. We treat the case where all diffusion coefficients behave as the distance to the boundary. In [10] we had studied the case that this is only true for their tangential component. Here the drift term is a small perturbation of the diffusion part, and thus the result in [10] does not depend on size or the direction of the drift term.

Besides the intrinsic PDE motivation, degenerate operators of such type also occur, for instance, in mathematical finance (Heston volatility model), population biology (generalized Kimura diffusion), or in the treatment of nonlinear equations (e.g. porous medium). Also motivated by such applications, they have been studied in (weighted) Sobolev or supnorm spaces or in Hölder spaces for an adapted metric in e.g. [2], [3], [6], [7], [8], [14], [15], [16],

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[17]. These papers deal with different situations as our paper, however, where [17] is closest to us. (See also [10] for references to earlier work.)

We explain the effects of the drift term in the present paper on the level of the model operator

$$A = -y\Delta + \beta D_y + b \cdot \nabla_x \tag{1.1}$$

with constant drift coefficients $b \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$ acting on the strip $S = \{z = (x, y) \in \mathbb{R}^{N+1} \,|\, x \in \mathbb{R}^N, 0 < y < 1\}$. Let $p \in (1, \infty)$. It follows from the paper [9] (co-authored by three of the present authors) that the operator $-A$ with the domain

$$D^0_{reg} = \big\{u \in W^{1,p}_0(S) \cap W^{2,p}_{\mathrm{loc}}(S) \,\big|\, y|D^2 u| \in L^p(S)\big\} \tag{1.2}$$

generates an analytic C_0-semigroup of positive contractions on $L^p(S)$ if $\beta > -1/p$. This condition means that the drift points inward at the boundary or only mildly outward. Correspondingly, one has to impose Dirichlet boundary conditions.

In the more recent contribution [23] the case of a strongly outward pointing drift with $\beta \le -1$ was studied. Here one derives these generation properties for $-A$ on the domain

$$D_{reg} = \big\{u \in W^{1,p}(S) \cap W^{2,p}_{\mathrm{loc}}(S) \,\big|\, y|D^2 u| \in L^p(S)\big\}. \tag{1.3}$$

In both cases one has the full regularity that one can expect reasonably. In the second one there are no (apparent) boundary conditions. Moreover, a one-dimensional example in [9] shows that $-A$ with domain D^0_{reg} is not a generator if $\beta \le -1/p$.

The present paper focuses on the intermediate case $-1 < \beta \le -1/p$. We equip $-A$ with Dirichlet boundary conditions and establish its sectoriality in $L^p(S)$ on the domain

$$D^0_{par} = \big\{u \in L^p(S) \,\big|\, y|D^2_x u|,\, y|D^2_{xy} u|,\, |\nabla_x u|,\, D_y u - \tfrac{\beta+1}{y} u,\, y D^2_y u - \beta D_y u \in L^p(S)\big\} \tag{1.4}$$

exhibiting only partial regularity. The generated semigroup is again positive. Example 3.15 indicates that this domain is optimal. By the same approach we also reprove the above mentioned results from [9] and [23] which were originally shown by quite different methods, see Theorems 4.2, 6.3 and 6.4.

We construct the resolvent and the semigroup for the above operators by approximation. To this aim, we consider A on the strip $S_\varepsilon = \mathbb{R}^N \times (\varepsilon, 1)$ (where it is non-degenerate), equip it with Dirichlet boundary conditions, and let $\varepsilon \to 0^+$. In [11] we studied the one-dimensional case in great detail. In particular, using the Neumann condition $u'(\varepsilon) = 0$ (instead of $u(\varepsilon) = 0$) in the approximation we showed that also the operator $(-A, D_{reg})$ is sectorial if $\beta \in (-1, -1/p)$. The results of [11] heavily rely on explicit formulas for the solution of the equation $Au = f$ on $(0, 1)$. Employing the Neumann approximation, we could establish the sectoriality of $(-A, D_{reg})$ also in the multidimensional case if $p = 2$ in [12]. The methods used there fail for $p \ne 2$. By completely different techniques involving singular integrals, Theorem 4.6.5 of [17] shows maximal regularity for the parabolic problem in L^p under (a variant of) Neumann boundary conditions and also obtaining full regularity as in D_{reg}.

Hence, our present setting in the intermediate range $-1 < \beta \le -1/p$ and with Dirichlet boundary conditions differs considerably from the previous results in so far that one has only reduced regularity in D^0_{par} which involves mixed conditions as $yD^2_y u - \beta D_y u \in L^p(S)$ exhibiting cancellations.

We describe our reasoning in more detail, again focusing on the model operator. The Dirichlet condition on S_ε yields ε-independent variational estimates for $A_\varepsilon = (A, W^{2,p}(S_\varepsilon) \cap W^{1,p}_0(S_\varepsilon))$. Combined with standard elliptic regularity, in the limit $\varepsilon \to 0$ we obtain the resolvent of the generator $-A_p$ of a positive contraction semigroup in $L^p(S)$, where A_p is the realization of the model operator A on a (yet unknown) domain D_p. Moreover, the solutions u_ε of $A_\varepsilon u_\varepsilon = f$ tend to $u = A_p^{-1} f$ in $W^{2,p}(S_\delta)$ for each $\delta \in (0,1)$.

To compute D_p, we first let $b = 0$ on (1.1). (The tangential drift $b \cdot \nabla$ is added later using the Kalton–Weis theorem on operator sums from operator-valued harmonic analysis.) The main observation is that the mapping $u \mapsto v := yu$ transforms Au into

$$Au = \mathcal{A}v := -\Delta_x v - D_y^2 v + (\beta+2)\frac{D_y v}{y} - (\beta+2)\frac{v}{y^2} =: -\Delta_x v + \mathcal{L}v,$$

thus decoupling the variables x and y. Since the inverse transformation is not uniformly bounded on $L^p(S_\varepsilon)$, the operators A and $\mathcal{A}$ are not similar in a reasonable sense, but by the above equation estimates on $\mathcal{A}v$ itself transfer to Au. Singular operators like $\mathcal{L}$ were studied in detail in [19] and [21], and we employ kernel estimates established there. Since Δ_x and $\mathcal{L}$ commute, we can then use the Kalton–Weis theorem to deduce ε-independent apriori estimates for A.

Next, we isolate the derivatives in y and rewrite $A_\varepsilon u_\varepsilon = f$ on S_ε as

$$-yD_y^2 u_\varepsilon + \beta D_y u_\varepsilon = f + y\Delta_x u_\varepsilon.$$

For fixed $x \in \mathbb{R}^N$ we now apply formulas from the one-dimensional case shown in our previous paper [11]. Combined with the apriori estimates mentioned above, by this approach we reprove the domain characterizations in (1.2) and (1.3) for $\beta > -1/p$ and $\beta \le -1$, respectively. In the intermediate range $-1 < \beta \le -1/p$ we obtain (1.4) except for the (a bit unexpected) conditions $|\nabla_x u|$, $y|D_{xy}u| \in L^p(S)$. These can be deduced from very recent results in [20] using again the transformation $u \mapsto v = yu$.

The analyticity of the semigroup then follows from the corresponding result in [9] for $\beta > -1/p$, Stein interpolation and duality. In a final step we use a localization procedure to show corresponding generation theorems on a bounded smooth domain in $\mathbb{R}^{N+1}$. Here we can use ideas from [9] and [10], for instance, but it requires some effort and care to deal with the more complicated domain of the model operator.

The plan of the paper is the following. In Section 2 we give the construction of the semigroup for the model operator. In Section 3 we investigate the domain of the generator when $b = 0$, as indicated above. In Section 4 we use the domain description to deduce analyticity of the semigroup for $b = 0$. The analyticity then allows us to add the drift term $b \cdot \nabla_x$ to our prototype operator on the strip. In Section 5 we extend our results to operators on S with variable coefficients. Finally, in the last section we prove the main result in bounded smooth domains of $\mathbb{R}^{N+1}$. Here we also state the hypotheses for the coefficients in this setting.

2 Construction of the semigroup

We investigate the operator A given by (1.1) on $S = \mathbb{R}^N \times (0,1)$. For every $\varepsilon \in (0,1/2]$ and $p \in (1,\infty)$, set $S_\varepsilon = \mathbb{R}^N \times (\varepsilon,1)$, $D_{p,\varepsilon} = W^{2,p}(S_\varepsilon) \cap W_0^{1,p}(S_\varepsilon)$, and $S_0 = S$. We first prove an accretivity inequality for $A_{p,\varepsilon} = (A, D_{p,\varepsilon})$ with constants independent of ε. This will allow us to construct a generator A_p in the limit $\varepsilon \to 0$. We also show a related inequality implying the positivity of the semigroup.

It is well known that the operator $-A_{p,\varepsilon}$ generates an analytic C_0-semigroup $(T_{p,\varepsilon}(t))_{t\ge 0}$ on $L^p(S_\varepsilon)$ which is consistent; i.e., one has $T_{p,\varepsilon}(t)f = T_{q,\varepsilon}(t)f$ for all $f \in L^p(S_\varepsilon) \cap L^q(S_\varepsilon)$ for $q \in (1,\infty)$ and $t \ge 0$. We thus also write $T_\varepsilon(t)$ instead of $T_{p,\varepsilon}(t)$.

Lemma 2.1. *Let $\beta \in \mathbb{R}$, $b \in \mathbb{R}^N$, $\varepsilon \in (0, 1/2]$, $u \in D_{p,\varepsilon}$, and $u^* = u|u|^{p-2}$. Then we have*

$$\int_{S_\varepsilon} Au\, u^* = (p-1) \int_{S_\varepsilon} y|\nabla u|^2 |u|^{p-2} \chi_{\{u \neq 0\}}, \tag{2.1}$$

$$\int_{S_\varepsilon} Au\, (u^-)^{p-1} = -(p-1) \int_{S_\varepsilon} y|\nabla u|^2 (u^-)^{p-2} \chi_{\{u<0\}}. \tag{2.2}$$

Proof. We first prove (2.1). Let $p \geq 2$ and $u \in D_{p,\varepsilon}$. Recall that $\nabla u^* = (p-1)|u|^{p-2}\nabla u$ and $u^* Du = p^{-1} D|u|^p$. Integrating by parts and using the Dirichlet conditions in $D_{p,\varepsilon}$, we infer

$$\begin{aligned}
\int_{S_\varepsilon} Au\, u^* &= (p-1) \int_\varepsilon^1 y|\nabla u|^2 |u|^{p-2} + (\beta+1) \int_{S_\varepsilon} (D_y u) u |u|^{p-2} + \int_{S_\varepsilon} b \cdot \nabla_x u \, |u|^{p-2} u \\
&= (p-1) \int_{S_\varepsilon} y|\nabla u|^2 |u|^{p-2} + \frac{\beta+1}{p} \int_{S_\varepsilon} D_y |u|^p + \frac{1}{p} \int_{S_\varepsilon} b \cdot \nabla_x |u|^p \\
&= (p-1) \int_{S_\varepsilon} y|\nabla u|^2 |u|^{p-2}.
\end{aligned} \tag{2.3}$$

The case $p \in (1, 2)$ can be handled by a regularization argument as in Lemmas 2.1 and 2.2 of [11]. Assertion (2.2) is shown similarly, using $\nabla u^- = -\nabla u \chi_{\{u<0\}}$. □

The next result will imply that the limit operator A_p is invertible.

Lemma 2.2. *Let $\beta \in \mathbb{R}$, $b \in \mathbb{R}^N$, and $\varepsilon \in (0, 1/2]$. There exists a number $\omega_p > 0$ such that*

$$\omega_p \int_{S_\varepsilon} |u|^p \leq \int_{S_\varepsilon} Au\, u^*, \tag{2.4}$$

$$\omega_p \int_{S_\varepsilon} (u^-)^p \leq -\int_{S_\varepsilon} Au\, (u^-)^{p-1}. \tag{2.5}$$

for each $u \in D_{p,\varepsilon}$.

Proof. As in the previous lemma we focus on (2.4) for $p \geq 2$. Let $u \in D_{p,\varepsilon}$. We compute

$$\begin{aligned}
|u(x,y)|^{\frac{p}{2}} &= -\frac{p}{2} \int_y^1 \Big((D_y u) u |u|^{\frac{p}{2}-2} \Big)(x, \eta)\, d\eta \\
&\leq \frac{p}{2} \left(\int_y^1 \eta \big((D_y u)^2 |u|^{p-2} \big)(x,\eta)\, d\eta \right)^{\frac{1}{2}} \left(\int_y^1 \frac{1}{\eta}\, d\eta \right)^{\frac{1}{2}}
\end{aligned}$$

for $(x, y) \in S_\varepsilon$. Equality (2.1) then implies

$$\int_{S_\varepsilon} |u(x,y)|^p \leq \frac{p^2}{4} \left(\int_0^1 |\log y|\, dy \right) \left(\int_{S_\varepsilon} y (D_y u)^2 |u|^{p-2} \right) \leq \omega_p^{-1} \int_{S_\varepsilon} Au\, u^*,$$

where we set $\omega_p^{-1} = \frac{p^2}{4(p-1)} \int_0^1 |\log y|\, dy$. □

We can now derive the desired uniform accretivity inequality and positivity.

Corollary 2.3. *Let $\beta \in \mathbb{R}$, $b \in \mathbb{R}^N$, $\varepsilon \in (0,1/2]$. For any $\lambda > -\omega_p$ and $u \in D_{p,\varepsilon}$ we have*

$$(\lambda + \omega_p)\|u\|_{L^p(S_\varepsilon)} \le \|(\lambda + A)u\|_{L^p(S_\varepsilon)}.$$

In particular, $(A - \omega_p, D_{p,\varepsilon})$ is accretive and $(A, D_{p,\varepsilon})$ is invertible in $L^p(S_\varepsilon)$. Moreover, u is non-negative if $\lambda u + Au \ge 0$.

Proof. Let $u \in D_{p,\varepsilon}$ and $\lambda > -\omega_p$. Formula (2.4) and Hölder's inequality yield

$$(\lambda + \omega_p)\|u\|^p_{L^p(S_\varepsilon)} \le \int_{S_\varepsilon} (\lambda u + Au)u^* \le \|(\lambda + A)u\|_{L^p(S_\varepsilon)} \|u\|^{p-1}_{L^p(S_\varepsilon)}.$$

Let $f \ge 0$. We multiply the equation $\lambda u + Au = f$ by $(u^-)^{p-1}$ and integrate over S_ε. Employing (2.5), we obtain

$$\int_{S_\varepsilon} f(u^-)^{p-1} = \int_{S_\varepsilon} \big(-\lambda(u^-)^p + Au(u^-)^{p-1}\big) \le -(\lambda + \omega_p)\int_{S_\varepsilon} (u^-)^p$$

and thus a contradiction if $u^- \ne 0$; i.e., $u \ge 0$. □

As in [1] and using the above estimates, we next construct a contraction semigroup generated by the restriction of $-A$ to a suitable domain. To this aim, we set

$$\begin{aligned} D_{max} &= \Big\{u \in L^p(S) \cap \bigcap\nolimits_{\delta\in(0,1)} W^{2,p}(S_\delta) \,\Big|\, Au \in L^p(S),\ u(\cdot,1) = 0\Big\}, \\ D_{en} &= \Big\{u \in D_{max} \,\Big|\, \int_S y|\nabla u|^2|u|^{p-2}\chi_{\{u\ne 0\}} < +\infty\Big\}. \end{aligned} \tag{2.6}$$

Moreover, $P_\varepsilon : L^p(S) \to L^p(S_\varepsilon)$ is the restriction operator and $E_\varepsilon : L^p(S_\varepsilon) \to L^p(S)$ the extension by 0.

Proposition 2.4. *Let $\beta \in \mathbb{R}$ and $b \in \mathbb{R}^N$. There exists a subspace $D_p = D_p(\beta, b) \subseteq D_{en}$ such that $-A_p := (-A, D_p)$ generates a positive strongly continuous semigroup $(T_p(t))_{t\ge 0}$ on $L^p(S)$ with $\|T_p(t)\| \le e^{-\omega_p t}$ for $t \ge 0$. If $1 < q < \infty$, then $T_p(t)f = T_q(t)f$ for all $t \ge 0$ and $f \in L^p(S) \cap L^q(S)$. The operator $T_p(t)$ is the strong limit in $L^p(S)$ of $E_\varepsilon T_{p,\varepsilon}(t)P_\varepsilon$ as $\varepsilon \to 0$, and analogously for the resolvents of the generators.*

The generator A_p is invertible. Moreover, each $u \in D_p$ is the limit in $L^p(S)$ of the (zero extension) of the functions $u_\varepsilon \in D_{p,\varepsilon}$ satisfying $Au_\varepsilon = Au$ on S_ε. The maps u_ε converge to u also in $W^{2,p}(S_\delta)$ for every $\delta \in (0,1)$.

Proof. 1) Fix $f \in L^p(S)$, $f \ge 0$, and $\lambda > -\omega_p$. For every $\varepsilon \in (0,1/2]$, let $u_\varepsilon \ge 0$ be the unique solution in $D_{p,\varepsilon}$ of $\lambda u_\varepsilon + Au_\varepsilon = P_\varepsilon f$ in S_ε given by Corollary 2.3. Since

$$(\lambda + \omega_p)\|u_\varepsilon\|_{L^p(S_\varepsilon)} \le \|f\|_{L^p(S)},$$

we obtain the uniform bound

$$\|Au_\varepsilon\|_{L^p(S_\varepsilon)} \le \|f\|_{L^p(S_\varepsilon)} + |\lambda|\|u_\varepsilon\|_{L^p(S_\varepsilon)} \le \Big(1 + \frac{|\lambda|}{\lambda + \omega_p}\Big)\|f\|_{L^p(S)}.$$

To show that the net (u_ε) increases pointwise as $\varepsilon \to 0$, we proceed as in Theorem 3.1 of [1]. Let $\varepsilon > \delta > 0$. Then $u_\varepsilon, u_\delta \ge 0$ in S_ε. The function $v = u_\delta - u_\varepsilon \in W^{2,p}(S_\varepsilon)$ is non-negative on ∂S_ε and satisfies $\lambda v + Av = 0$ in S_ε. We multiply this identity by $(v^-)^{p-1}$ and integrate by parts as in (2.3), where the boundary terms vanish since v^- has trace 0 on ∂S_ε. Also using an estimate analogous to (2.5), we infer

$$0 = \int_{S_\varepsilon} \left(-\lambda (v^-)^p + Av(v^-)^{p-1}\right) \leq -(\lambda + \omega_p) \int_{S_\varepsilon} (v^-)^p \leq 0.$$

Hence $v^- = 0$, and the monotonicity of (u_ε) follows. Let $u \geq 0$ be the pointwise limit u_ε on S. By the Beppo Levi theorem, the functions u_ε converge to u in $L^p(S)$ and u satisfies

$$(\lambda + \omega_p)\|u\|_p \leq \|f\|_p. \tag{2.7}$$

Standard results on $W^{2,p}$–regularity imply that (u_ε) tends to u in $W^{2,p}(S_\delta)$ for every $\delta \in (0,1)$. (See [13, Theorem 9.11], the proof for balls easily extends to strips.) As a result, the limit u belongs to D_{max} and fulfills $\lambda u + Au = f$ in S. Moreover, u is even contained in D_{en} by (2.1).

2) Recall that $-A_{p,\varepsilon} = (-A, D_{p,\varepsilon})$ generates the C_0–semigroup $(T_{p,\varepsilon}(t))$. Corollary 2.3 implies that $T_{p,\varepsilon}(t)$ is bounded by $e^{-\omega_p t}$. In view of step 1) the operators $E_\varepsilon(\lambda + A_{p,\varepsilon})^{-1}P_\varepsilon$ have the strong limit $R(\lambda)$ in $L^p(S)$ for $\lambda > -\omega_p$, with $R(\lambda) \geq 0$ and $\|R(\lambda)\| \leq (\lambda + \omega_p)^{-1}$. One can check that $\{R(\lambda) \,|\, \lambda > -\omega_p\}$ is a pseudoresolvent, see the proof of Theorem 3.1 of [1]. Our construction yields $f = (\lambda + A)R(\lambda)f$ so that $R(\lambda)$ is injective. By Proposition III.4.6 of [5] there exists a closed operator $-A_p$ with (λ–independent) domain $D_p = R(\lambda)L^p(S)$ in $L^p(S)$ satisfying $R(\lambda) = (\lambda + A_p)^{-1}$ for $\lambda > -\omega_p$.

3) From the previous steps we infer the inclusion $D_p \subseteq D_{en}$ and that A_p is a restriction of A. Since $\|R(\mu - \omega_p)\| \leq 1/\mu$ for $\mu > 0$, Corollary II.3.20 of [5] shows that the operator $\omega_p - A_p$ is densely defined and generates a contraction semigroup $(S_p(t))_{t\geq 0}$. It is positive due to the positivity of the resolvent and Theorem VI.1.8 of [5]. It follows that $-A_p$ generates $(T_p(t))_{t\geq 0} = (e^{-\omega_p t}S_p(t))_{t\geq 0}$. The asserted convergence of $E_\varepsilon T_{p,\varepsilon}(t)P_\varepsilon$ to $T_p(t)$ is a consequence of the corresponding property of the resolvents and of the proof of the Trotter-Kato theorem given in [22, Theorem 3.4.2]. We finally deduce the consistency of the semigroup $(T_p(t))$ from that of $(T_{p,\varepsilon}(t))$. □

Since $T_p(t) = T_q(t)$ on $L^p(S) \cap L^q(S)$, we often write $T(t)$ instead of $T_p(t)$. Before we can show the analyticity of this semigroup, we have to determine the domain of its generator A_p more precisely.

3 Description of the domain if $b = 0$

In order to obtain information on the regularity of the functions of D_p, we thoroughly investigate the properties of the approximating functions u_ε. After some preparations, we show a precise description of D_p for the cases $\beta \leq -1$ and $\beta > -1$ separately. Our reasoning requires that $b = 0$, so that we have to restrict ourselves to this special case at first. The drift term $b \cdot \nabla_x$ will be added in the next section.

At some points we use deep facts from operator-valued harmonic analysis. We refer the reader to [18] for an introduction to the background on R-sectoriality, H^∞–calculus, and operator sums.

3.1 Preliminary regularity results

The starting point of our arguments is the transformation $u \mapsto v = yu$ for a function $u \in W^{2,p}(S_\varepsilon)$ (or $u \in W^{2,p}(S)$). A straightforward calculation yields $Au = \mathcal{A}v$, where

$$\mathcal{A}v := -\Delta_x v - D_y^2 v + (\beta+2)\frac{D_y v}{y} - (\beta+2)\frac{v}{y^2} = -\Delta_x v + \mathcal{L}v, \tag{3.1}$$
$$\mathcal{L}v := -D_y^2 v + (\beta+2)\frac{D_y v}{y} - (\beta+2)\frac{v}{y^2} = -yD_y^2 u + \beta D_y u,$$

so that the variables x and y are separated in $\mathcal{A}v$. We stress that the inverse transformation $v \mapsto u = v/y$ is not uniformly bounded in $L^p(S_\varepsilon)$ as $\varepsilon \to 0$. Nevertheless, in the next lemma formula (3.1) leads to uniform bounds on certain higher-order derivatives of u_ε.

Lemma 3.1. *Let $\beta \in \mathbb{R}$ and $b = 0$. Then there exists a positive constant c such that for every $u_\varepsilon \in D_{p,\varepsilon}$ and $\varepsilon \in (0, 1/2]$ we have*

$$\|y\Delta_x u_\varepsilon\|_{L^p(S_\varepsilon)} + \|yD_y^2 u_\varepsilon - \beta D_y u_\varepsilon\|_{L^p(S_\varepsilon)} \le c\,\|Au_\varepsilon\|_{L^p(S_\varepsilon)}.$$

Moreover, every $u \in D_p$ satisfies

$$\|yD_x^2 u\|_{L^p(S)} + \|yD_y^2 u - \beta D_y u\|_{L^p(S)} \le c\,\|Au\|_{L^p(S)}.$$

Proof. 1) We first establish several properties of the operators in (3.1). In $L^p(S_\varepsilon)$ we endow Δ_x and $\mathcal{L}$ with the domains

$$D_\varepsilon(\Delta_x) = \{v \in L^p(S_\varepsilon)\,|\, v(\cdot, y) \in W^{2,p}(\mathbb{R}^N),\ y \in (\varepsilon, 1),\ \Delta_x v \in L^p(S_\varepsilon)\},$$
$$D_\varepsilon(\mathcal{L}) = \{v \in L^p(S_\varepsilon)\,|\, v(x, \cdot) \in W^{2,p}(\varepsilon, 1) \cap W_0^{1,p}(\varepsilon, 1),\ x \in \mathbb{R}^N,\ \mathcal{L}v \in L^p(S_\varepsilon)\}.$$

The Laplacian is sectorial and has an H^∞–calculus on $L^p(\mathbb{R}^N)$, both of angle 0, see Example 10.2 in [18]. It is then easy to see that Δ_x has the same property on $L^p(S_\varepsilon)$ with uniform bounds.

The scalar version $-\mathcal{L}_\varepsilon$ of $-\mathcal{L}$ generates a positive bounded analytic C_0–semigroup $(V_\varepsilon(t))_{t\ge 0}$ on $L^p(\varepsilon, 1)$, when endowed with Dirichlet boundary conditions. For our purposes it is crucial to bound its heat kernel $p_t^\varepsilon > 0$, independently of ε. To this aim, we also consider $\mathcal{L}$ as an operator $\mathcal{L}_+$ on $\mathbb{R}_+$. Following Section 4 of [19], $\mathcal{L}_+$ is the self-adjoint operator in $L^2(\mathbb{R}_+; r^{-(\beta+2)}\,dr)$ associated with the closure of the form

$$\mathfrak{b}(u, v) := \int_0^\infty \left(u_r \overline{v_r} - (\beta+2)\frac{u\overline{v}}{r^2}\right) r^{-(\beta+2)} dr, \qquad D(\mathfrak{b}) := C_c^\infty(\mathbb{R}_+).$$

On $D(\mathfrak{b})$ this operator is indeed given as in (3.1), see Proposition 4.2 in [19]. Due to Corollary 4.15 of [21] or Proposition 4.14 of [19] the semigroup $(V(t))_{t\ge 0}$ generated by $\mathcal{L}_+$ has a positive kernel p_t which is bounded by

$$0 \le p_t(x, y) \le \frac{\kappa}{\sqrt{t}}\left(\frac{|x|}{t^{1/2}} \wedge 1\right)^{-s_1}\left(\frac{|y|}{t^{1/2}} \wedge 1\right)^{-s_1^*} \exp\left(-\frac{|x-y|^2}{mt}\right) \le \frac{\kappa}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{mt}\right) \tag{3.2}$$

for $t > 0$, $x, y > 0$, and constants $\kappa, m > 0$. We note that in [19] the estimate is given in a slightly different form. The exponents are given by

$$s_1 = -(\beta+2), \quad s_1^* = 0 \qquad \text{if } \beta \ge -1,$$

$$s_1 = -1, \quad s_1^* = \beta + 1 \qquad \text{if } \beta < -1.$$

The last inequality in (3.2) follows by virtue of these identities. The semigroup is strongly continuous also in $L^p(\mathbb{R}_+)$ with respect to the Lebesgue measure and its domain can be described explicitly, however we do not need these facts.

We employ the same construction for $\mathcal{L}_\varepsilon$ and $(V_\varepsilon(t))_{t\geq 0}$, that is we restrict the above form $\mathfrak{b}$ to $H_0^1(\varepsilon, 1)$. Note that this form indeed induces the operator $\mathcal{L}_\varepsilon$ with domain $W^{2,p}(\varepsilon,1) \cap W_0^{1,p}(\varepsilon,1)$ on $L^p(\varepsilon,1)$. Since by [19, Lemma 4.1] the domain of the closure of $\mathfrak{b}$ consists of H^1_{loc} functions, the Beurling-Deny criteria give $0 \leq V_\varepsilon(t)f \leq V(t)f$ for $f \geq 0$ (extending $V_\varepsilon(t)f$ by 0). Hence, $0 \leq p_t^\varepsilon \leq p_t$ so that the kernel $p_t^\varepsilon > 0$ also fulfills (3.2) on $(\varepsilon, 1)$.

2) Take $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg \lambda| \leq \theta$ for some $\theta < \pi/2$. Since the resolvent of $-\mathcal{L}_\varepsilon$ is given by the Laplace transform of $(V_\varepsilon(t))$, it has a kernel k_λ^ε bounded by

$$|k_\lambda^\varepsilon(x,y)| = \Big| \int_0^\infty e^{-\lambda t} p_t^\varepsilon(x,y)\,dt \Big| \leq \kappa \int_0^\infty e^{-\operatorname{Re}\lambda t} \frac{1}{\sqrt{t}} e^{-\frac{|x-y|^2}{mt}}\,dt \leq \kappa \int_0^\infty \frac{e^{-2c|\lambda|t}}{\sqrt{t}} e^{-\frac{|x-y|^2}{mt}}\,dt$$

for $x, y \in (\varepsilon, 1)$ and a constant $c > 0$. Splitting the integral at $\tau = |x-y|/|\lambda|^{1/2}$, we compute

$$\begin{aligned}
|k_\lambda^\varepsilon(x,y)| &\leq \kappa \int_0^\tau \frac{e^{-2c|\lambda|t}}{\sqrt{t}} e^{-m^{-1}|x-y|\,|\lambda|^{1/2}}\,dt + \kappa \int_\tau^\infty \frac{e^{-c|\lambda|t}}{\sqrt{t}} e^{-c|x-y|\,|\lambda|^{1/2}} e^{-\frac{|x-y|^2}{mt}}\,dt \\
&\leq \frac{\kappa e^{-m^{-1}|x-y|\,|\lambda|^{1/2}}}{|\lambda|^{1/2}} \int_0^\infty \frac{e^{-2cs}}{\sqrt{s}}\,ds + \frac{\kappa e^{-c|x-y|\,|\lambda|^{1/2}}}{|\lambda|^{1/2}} \int_0^\infty \frac{e^{-cs}}{\sqrt{s}}\,ds \\
&\leq \frac{\overline{c}\, e^{-c'|x-y|\,|\lambda|^{1/2}}}{|\lambda|^{1/2}},
\end{aligned}$$

where the constants do not depend on ε. This Poisson estimate and Theorem 4.8 of [4] show that $\mathcal{L}_\varepsilon$ is R-sectorial of angle smaller than or equal $\pi - \theta$ in $L^p(\varepsilon, 1)$. The R-sectoriality can be extended to the operator $(\mathcal{L}, D_\varepsilon(\mathcal{L}))$ on $L^p(S_\varepsilon)$ with the same constants.

3) The resolvents of Δ_x and $(\mathcal{L}, D_\varepsilon(\mathcal{L}))$ commute in $L^p(S_\varepsilon)$ since the semigroups commute in view of their kernel representations. The Kalton-Weis Theorem 12.13 in [18] thus yields a constant $C > 0$ independent of $\varepsilon \in (0, 1/2]$ such that

$$\|\Delta_x v + \mathcal{L}v\|_{L^p(S_\varepsilon)} \geq C\big(\|\Delta_x v\|_{L^p(S_\varepsilon)} + \|\mathcal{L}v\|_{L^p(S_\varepsilon)}\big), \tag{3.3}$$

for every $v \in D_\varepsilon(\Delta_x) \cap D_\varepsilon(\mathcal{L})$ in $L^p(S_\varepsilon)$. Let $u_\varepsilon \in D_{p,\varepsilon}$. The function $v_\varepsilon := y u_\varepsilon$ belongs to $D_{p,\varepsilon} \subseteq D_\varepsilon(\Delta_x) \cap D_\varepsilon(\mathcal{L})$. Applying (3.3), we estimate

$$C\big(\|\Delta_x v_\varepsilon\|_{L^p(S_\varepsilon)} + \|\mathcal{L}v_\varepsilon\|_{L^p(S_\varepsilon)}\big) \leq \|\mathcal{A}v_\varepsilon\|_{L^p(S_\varepsilon)} = \|Au_\varepsilon\|_{L^p(S_\varepsilon)}.$$

In particular, as $\Delta_x v_\varepsilon = y\Delta_x u_\varepsilon$ and $\mathcal{L}v_\varepsilon = -yD_y^2 u_\varepsilon + \beta D_y u_\varepsilon$, we infer that

$$\|y\Delta_x u_\varepsilon\|_{L^p(S_\varepsilon)} + \|yD_y^2 u_\varepsilon - \beta D_y u_\varepsilon\|_{L^p(S_\varepsilon)} \leq C^{-1}\|Au_\varepsilon\|_{L^p(S_\varepsilon)}, \tag{3.4}$$

which proves the first inequality of the statement with $c = C^{-1}$.

Next, let $u \in D_p$ and set $f = Au$. Proposition 2.4 yields a function u_ε in $D_{p,\varepsilon}$ such that $Au_\varepsilon = f$ in S_ε, for each $\varepsilon \in (0, 1/2]$. Moreover, the maps u_ε converge to u in $W^{2,p}(S_\delta)$ for every $\delta > 0$. Estimate (3.4) provides the uniform bound

$$\|y\Delta_x u_\varepsilon\|_{L^p(S_\varepsilon)} + \|yD_y^2 u_\varepsilon - \beta D_y u_\varepsilon\|_{L^p(S_\varepsilon)} \leq C^{-1}\|f\|_{L^p(S)}.$$

From the Calderón–Zygmund theorem and (2.7) we deduce

$$\|D_x^2(yu_\varepsilon)\|_{L^p(S_\varepsilon)} \le c(\|\Delta_x(yu_\varepsilon)\|_{L^p(S_\varepsilon)} + \|yu_\varepsilon\|_{L^p(S_\varepsilon)}) \le c\|f\|_{L^p(S)}.$$

(The constant c in the first inequality does not depend on $0 < \varepsilon \le 1/2$ by a simple scaling argument.) Taking the limit as $\varepsilon \to 0$, we obtain the second assertion. $\square$

The regularity gained in the previous lemma allows us to rewrite the equation $Au_\varepsilon = f$ isolating the derivatives with respect to y. In this way we obtain an ordinary differential equation in y (with fixed x) whose solutions can be computed explicitly. This equation and its limit version have the inhomogeneities

$$g_\varepsilon := f + y\Delta_x u_\varepsilon \qquad \text{and} \qquad g := f + y\Delta_x u \tag{3.5}$$

for $f \in L^p(S)$ and the functions $u_\varepsilon \in D_{p,\varepsilon}$ and $u \in D_p$ solving $Aw = f$ on S_ε and S, respectively. In the next lemma we express u_ε by g_ε.

Lemma 3.2. *Let $\beta \in \mathbb{R}$, $b = 0$, $\varepsilon \in (0, 1/2]$, and $f \in L^p(S)$. The solution $u_\varepsilon \in D_{p,\varepsilon}$ to the equation $Au_\varepsilon = f$ given by Proposition 2.4 satisfies*

$$\begin{aligned} u_\varepsilon(x,y) &= \frac{c_\varepsilon(x)}{\beta+1}(y^{\beta+1}-1) - \frac{1}{\beta+1}\int_y^1 g_\varepsilon(x,t)\,dt + \frac{y^{\beta+1}}{\beta+1}\int_y^1 \frac{g_\varepsilon(x,t)}{t^{\beta+1}}\,dt, \\ c_\varepsilon(x) &= \frac{1}{\varepsilon^{\beta+1}-1}\left(\int_\varepsilon^1 g_\varepsilon(x,t)\,dt - \varepsilon^{\beta+1}\int_\varepsilon^1 \frac{g_\varepsilon(x,t)}{t^{\beta+1}}\,dt\right) \end{aligned} \tag{3.6}$$

if $\beta \ne -1$, and

$$\begin{aligned} u_\varepsilon(x,y) &= c_\varepsilon(x)\log y - \int_y^1 g_\varepsilon(x,t)\log\frac{t}{y}\,dt, \\ c_\varepsilon(x) &= -\int_\varepsilon^1 g_\varepsilon(x,t)\,dt + (\log\varepsilon)^{-1}\int_\varepsilon^1 g_\varepsilon(x,t)\log t\,dt \end{aligned} \tag{3.7}$$

if $\beta = -1$. Here $x \in \mathbb{R}^N$, $y \in (\varepsilon, 1)$ and $g_\varepsilon = f + y\Delta_x u_\varepsilon$.

Proof. The proof is based on elementary calculus, as it consists in solving the ordinary differential equation $-yD_y^2 u_\varepsilon + \beta D_y u_\varepsilon = g_\varepsilon$ in y (and for fixed $x \in \mathbb{R}^N$) with 0-boundary conditions at $y = \varepsilon$ and $y = 1$. The details can be found in formula (2.4) and the proof of Proposition 2.8 in [11]. $\square$

In the next step we take the limit as $\varepsilon \to 0$ in the above formulas. We first prove that g_ε from (3.5) tend to g in some sense.

Lemma 3.3. *Let $\beta \in \mathbb{R}$, $b = 0$, $\varepsilon \in (0, 1/2]$, and $u \in D_p$. Let u_ε be the maps given by Proposition 2.4 and represented in (3.6) and (3.7). Define g and g_ε as in* (3.5) *for $f = Au$. We have*

$$\|g\|_{L^p(S)},\ \|g_\varepsilon\|_{L^p(S_\varepsilon)} \le M \tag{3.8}$$

for a suitable constant $M > 0$ independent of ε. Moreover,

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1 \left|g_\varepsilon(\cdot,t) - g(\cdot,t)\right|dt = 0 \quad \text{in } \ L^p(\mathbb{R}^N).$$

Proof. Lemma 3.1 implies (3.8). We recall that the maps u_ε converge to u in $W^{2,p}(S_\delta)$ for every $\delta > 0$ by Proposition 2.4. Hence, the functions $g_\varepsilon = f + y\Delta_x u_\varepsilon$ tend to g in $L^p(S_\delta)$ for every $\delta > 0$. To show the asserted limit, we fix $\delta \in (0, 1/2]$, take $\varepsilon \in (0, \delta)$, and write

$$\int_\varepsilon^1 \big|g_\varepsilon(x,t) - g(x,t)\big|\, dt = \int_\varepsilon^\delta \big|g_\varepsilon(x,t) - g(x,t)\big|\, dt + \int_\delta^1 \big|g_\varepsilon(x,t) - g(x,t)\big|\, dt$$

for $x \in \mathbb{R}^N$. Hölder's inequality and (3.8) yield

$$\int_{\mathbb{R}^N} \bigg(\int_\varepsilon^\delta \big|g_\varepsilon(x,t) - g(x,t)\big|\, dt\bigg)^p dx \le \delta^{p-1} \|g_\varepsilon - g\|^p_{L^p(S_\varepsilon)} \le C\delta^{p-1},$$
$$\int_{\mathbb{R}^N} \bigg(\int_\delta^1 \big|g_\varepsilon(x,t) - g(x,t)\big|\, dt\bigg)^p dx \le \|g_\varepsilon - g\|^p_{L^p(S_\delta)}.$$

Next, fix $\eta > 0$ and choose $\delta > 0$ with $C\delta^{p-1} < \eta$. Since $\|g_\varepsilon - g\|_{L^p(S_\delta)} \to 0$, there exists a number $\varepsilon_0 \in (0, \delta)$ such that $\|g_\varepsilon - g\|^p_{L^p(S_\delta)} < \eta$ for every $\varepsilon < \varepsilon_0$. It follows

$$\int_{\mathbb{R}^N} \bigg(\int_\varepsilon^1 \big|g_\varepsilon(x,t) - g(x,t)\big|\, dt\bigg)^p dx < 2^p \eta$$

for every $\varepsilon < \varepsilon_0$, and the claim is proved. □

In the next two results we compute the limit as $\varepsilon \to 0$ of the functions c_ε in identities (3.6) and (3.7). This leads to an implicit formula for the limit u of u_ε and later to the desired description of the domain D_p. From now on, we have to distinguish between the cases $\beta \le -1$ and $\beta > -1$ since the limits of (c_ε) differ.

Lemma 3.4. *Let $\beta \le -1$, $b = 0$, and $u \in D_p$. Set $f = Au$ and $g = f + y\Delta_x u$. Let u_ε be the functions given by Proposition 2.4 and represented in (3.6) and (3.7). We then have*

$$\lim_{\varepsilon\to 0} c_\varepsilon = -\int_0^1 \frac{g(\cdot,t)}{t^{\beta+1}}\, dt \qquad \text{in } L^p(\mathbb{R}^N).$$

For $(x, y) \in S$, we thus obtain the expressions

$$u(x,y) = \frac{1}{\beta+1}\int_0^1 \frac{g(x,t)}{t^{\beta+1}}\, dt - \frac{y^{\beta+1}}{\beta+1}\int_0^y \frac{g(x,t)}{t^{\beta+1}}\, dt - \frac{1}{\beta+1}\int_y^1 g(x,t)\, dt \tag{3.9}$$

if $\beta < -1$, and

$$u(x,y) = -\log y \int_0^1 g(x,t) dt - \int_y^1 g(x,t) \log\frac{t}{y}\, dt \tag{3.10}$$

if $\beta = -1$.

Proof. Let $x \in \mathbb{R}^N$. Set $\tilde c(x) = -\int_0^1 g(x,t) t^{-(\beta+1)} dt$. We first assume that $\beta < -1$. The definition of c_ε in (3.6) yields

$$\begin{aligned} c_\varepsilon(x) - \tilde c(x) = \frac{1}{\varepsilon^{\beta+1} - 1}\bigg(\int_\varepsilon^1 g_\varepsilon(x,t)\, dt - \int_\varepsilon^1 \frac{g(x,t)}{t^{\beta+1}}\, dt\bigg) + \int_0^\varepsilon \frac{g(x,t)}{t^{\beta+1}}\, dt \\ + \frac{\varepsilon^{\beta+1}}{\varepsilon^{\beta+1} - 1}\int_\varepsilon^1 \frac{g(x,t) - g_\varepsilon(x,t)}{t^{\beta+1}}\, dt. \end{aligned} \tag{3.11}$$

Since $\beta + 1 < 0$, we have

$$\int_\varepsilon^1 \frac{|g(x,t) - g_\varepsilon(x,t)|}{t^{\beta+1}}\,dt \le \int_\varepsilon^1 |g(x,t) - g_\varepsilon(x,t)|\,dt,$$

and the last integral converges to 0 in $L^p(\mathbb{R}^N)$ by Lemma 3.3. Moreover, Hölder's inequality and (3.8) imply the uniform bounds

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left| \int_\varepsilon^1 g_\varepsilon(x,t)\,dt \right|^p dx \le \|g_\varepsilon\|^p_{L^p(S_\varepsilon)} \le M^p, \\
&\int_{\mathbb{R}^N} \left| \int_\varepsilon^1 \frac{g(x,t)}{t^{\beta+1}}\,dt \right|^p dx \le \|g\|^p_{L^p(S)}, \\
&\int_{\mathbb{R}^N} \left| \int_0^\varepsilon g(x,t)\,dt \right|^p dx \le \varepsilon^{p-1}\|g\|^p_{L^p(S)}.
\end{aligned} \tag{3.12}$$

Since $\varepsilon^{\beta+1} \to +\infty$ as $\varepsilon \to 0$, we conclude that all the addends in the right-hand side of (3.11) tend to 0 in $L^p(\mathbb{R}^N)$.

Next, take $\beta = -1$. From formula (3.7) we infer

$$c_\varepsilon(x) - \tilde{c}(x) = \int_\varepsilon^1 \big(g(x,t) - g_\varepsilon(x,t)\big)\,dt + (\log\varepsilon)^{-1}\int_\varepsilon^1 g_\varepsilon(x,t)\log t\,dt + \int_0^\varepsilon g(x,t)\,dt.$$

The first and third addends converge to 0 in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$ thanks to Lemma 3.3 and (3.12), respectively. As regards the second addend, Hölder's inequality and (3.8) lead to the estimate

$$\left| \int_\varepsilon^1 g_\varepsilon(x,t)\log t\,dt \right|^p \le \left(\int_\varepsilon^1 |g_\varepsilon(x,t)|^p dt \right) \left(\int_0^1 |\log t|^{p'} dt \right)^{\frac{p}{p'}} \le C.$$

Therefore the claim is proved also in this case. The representation formulas for u then follow in a similar way. ☐

We next show an analogous result for $\beta > -1$.

Lemma 3.5. *Let $\beta > -1$, $b = 0$, and $u \in D_p$. Set $f = Au$ and $g = f + y\Delta_x u$. Let u_ε be the functions given by Proposition 2.4 and represented in (3.6). We have*

$$\lim_{\varepsilon\to 0} c_\varepsilon = -\int_0^1 g(\cdot,t)\,dt \qquad \text{in } L^p(\mathbb{R}^N). \tag{3.13}$$

For $(x,y) \in S$, we thus obtain the expression

$$u(x,y) = \frac{1}{\beta+1}\int_0^y g(x,t)\,dt - \frac{y^{\beta+1}}{\beta+1}\int_0^1 g(x,t)\,dt + \frac{y^{\beta+1}}{\beta+1}\int_y^1 \frac{g(x,t)}{t^{\beta+1}}\,dt. \tag{3.14}$$

Proof. Let $x \in \mathbb{R}^N$ and set $\hat{c}(x) = -\int_0^1 g(x,t)\,dt$. From equation (3.6) we deduce

$$c_\varepsilon(x) - \hat{c}(x) = \frac{1}{\varepsilon^{\beta+1}-1}\int_\varepsilon^1 g_\varepsilon(x,t)\,dt + \int_0^1 g(x,t)\,dt + \frac{\varepsilon^{\beta+1}}{1-\varepsilon^{\beta+1}}\int_\varepsilon^1 \frac{g_\varepsilon(x,t)}{t^{\beta+1}}\,dt.$$

We first treat the last summand. Hölder's inequality yields

$$\varepsilon^{\beta+1}\int_\varepsilon^1 \frac{|g_\varepsilon(x,t)|}{t^{\beta+1}}\,dt \le \left(\int_\varepsilon^1 |g_\varepsilon(x,t)|^p\,dt\right)^{\frac{1}{p}} \left(\frac{\varepsilon^{(\beta+1)p'}-\varepsilon}{1-(\beta+1)p'}\right)^{\frac{1}{p'}},$$

if $\beta \neq -\frac{1}{p}$ and therefore

$$\int_{\mathbb{R}^N}\left|\varepsilon^{\beta+1}\int_\varepsilon^1 \frac{g_\varepsilon(x,t)}{t^{\beta+1}}\,dt\right|^p dx \le \left(\frac{\varepsilon^{(\beta+1)p'}-\varepsilon}{(\beta+1)p'-1}\right)^{p-1} \|g_\varepsilon\|^p_{L^p(S_\varepsilon)}.$$

For $\beta = -\frac{1}{p}$ we have $(\beta+1)p' = 1$. In the above computation we thus replace the quantity $\frac{\varepsilon^{(\beta+1)p'}-\varepsilon}{1-(\beta+1)p'}$ by $\varepsilon \log \frac{1}{\varepsilon}$. In both cases the bound (3.8) leads to the limit

$$\varepsilon^{\beta+1}\int_\varepsilon^1 \frac{g_\varepsilon(\cdot,t)}{t^{\beta+1}}\,dt \quad \longrightarrow\ 0 \quad \text{in} \quad L^p(\mathbb{R}^N), \quad \text{as} \quad \varepsilon\ \to 0.$$

The other terms in $c_\varepsilon - \hat{c}$ are written as

$$\begin{aligned}\frac{1}{\varepsilon^{\beta+1}-1}\int_\varepsilon^1 g_\varepsilon(x,t)\,dt + \int_0^1 g(x,t)\,dt &= \frac{1}{\varepsilon^{\beta+1}-1}\int_\varepsilon^1 \big(g_\varepsilon(x,t)-g(x,t)\big)\,dt \\ &\quad + \frac{\varepsilon^{\beta+1}}{\varepsilon^{\beta+1}-1}\int_\varepsilon^1 g(x,t)\,dt + \int_0^\varepsilon g(x,t)\,dt\end{aligned} \tag{3.15}$$

The first addend on the right hand side of (3.15) converges to 0 in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$ by Lemma 3.3. Using also (3.12), we conclude that the right-hand side of (3.15) tends to 0 in $L^p(\mathbb{R}^N)$. We have thus proved (3.13). Formula (3.14) is then shown by similar arguments. □

3.2 The domain for $b = 0$ and $\beta \le -1$

We first treat the case $\beta \le -1$. Here, the functions in D_p do not satisfy a boundary condition at $y = 0$, but have 'full' regularity as described by the space

$$D_{reg} := \big\{u \in D_{max} \cap W^{1,p}(S) \,\big|\, y|D^2u| \in L^p(S)\big\}. \tag{3.16}$$

In the next result we deduce the inclusion $D_p \subseteq D_{reg}$ using formulas (3.9) and (3.10) as well as the Calderón–Zygmund and Hardy inequalities applied to $v = yu$. For the other inclusion we show that $I + A$ is injective on D_{reg}. A slight variant of Proposition 3.7 has been proved in [23] using different methods. For the proof, we need the following lemma.

Lemma 3.6. *Let $v \in W^{2,p}(S_\delta)$ for every $0 < \delta < 1$, $v(\cdot,1) = 0$, and $v, \Delta v \in L^p(S)$. Assume that the function $u = v/y$ satisfies $u, D_y u \in L^p(S)$. Then v belongs to $W^{2,p}(S) \cap W_0^{1,p}(S)$.*

Proof. Let $u = v/y$ be given as in the statement. We have

$$u(x,\varepsilon) = \int_\varepsilon^1 D_y u(x,y)\,dy, \qquad \int_{\mathbb{R}^N} |u(x,\varepsilon)|^p\,dx \le \int_S |D_y u|^p\,dx\,dy.$$

Observe that

$$\liminf_{\varepsilon\to 0} \varepsilon^p \int_{\mathbb{R}^N} |D_y u(x,\varepsilon)|^p\, dx = 0,$$

as otherwise $|D_y u|^p$ would not be integrable on S. Since $v = yu$ and $D_y v = y D_y u + u$, the above inequalities yield

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} |D_y v(x,\varepsilon)|\, |v(x,\varepsilon)|^{p-1}\, dx \\ &\qquad \le \liminf_{\varepsilon\to 0} \left(\int_{\mathbb{R}^N} |D_y v(x,\varepsilon)|^p\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |v(x,\varepsilon)|^p\right)^{1-\frac{1}{p}} = 0. \end{aligned} \tag{3.17}$$

Let $w \in W^{2,p}(S) \cap W_0^{1,p}(S)$ be such that $w - \Delta w = v - \Delta v$. Then w satisfies (3.17) and the same holds for $z = v - w$. We multiply $z - \Delta z = 0$ by $z|z|^{p-2}$ and integrate by parts on S_ε, obtaining

$$0 = \int_{S_\varepsilon} (z - \Delta z) z|z|^{p-2} = \int_{S_\varepsilon} |z|^p + (p-1)\int_{S_\varepsilon} |\nabla z|^2 |z|^{p-2} + \int_{\mathbb{R}^N} (D_y z) z|z|^{p-2}(x,\varepsilon)\, dx.$$

Letting $\varepsilon \to 0$ along an appropriate sequence, we deduce $z = 0$ and hence $v = w$. □

Proposition 3.7. *Let $\beta \le -1$ and $b = 0$. Then $D_p = D_{reg}$.*

Proof. 1) We first show the inclusion $D_p \subseteq D_{reg}$. Let $u \in D_p$. Differentiating formulas (3.9) and (3.10) with respect to y, we find

$$D_y u(x,y) = -y^\beta \int_0^y \frac{g(x,t)}{t^{\beta+1}}\, dt = -\int_0^1 \frac{g(x,sy)}{s^{\beta+1}}\, ds$$

for $(x,y) \in S$. Minkowski's (integral) inequality then yields

$$\left(\int_0^1 |D_y u(x,y)|^p dy\right)^{\frac{1}{p}} \le \int_0^1 \frac{1}{s^{\beta+1}} \left(\int_0^1 |g(x,sy)|^p dy\right)^{\frac{1}{p}} ds \le \int_0^1 \frac{ds}{s^{\beta+1+\frac{1}{p}}} \left(\int_0^1 |g(x,z)|^p dz\right)^{\frac{1}{p}}.$$

Raising to the power p and integrating with respect to x, we obtain the bound $\|D_y u\|_{L^p(S)} \le C\|g\|_{L^p(S)}$. Since $yD_x^2 u$ belongs to $L^p(S)$ due to Lemma 3.1 and $Au \in L^p(S)$, by difference the function $yD_y^2 u$ is an element of $L^p(S)$.

2) Setting $v = yu$ again, we infer that $\Delta_x v = y\Delta_x u$ and $D_y^2 v = yD_y^2 u + 2D_y u$ are contained in $L^p(S)$. Moreover, v vanishes at $y = 1$ and $u, D_y u \in L^p(S)$ by point 1). An application of Lemma 3.6 thus yields $v \in W^{2,p}(S) \cap W_0^{1,p}(S)$. Hence $|D_{xy}^2 v|$ belongs to $L^p(S)$, implying that $|yD_{xy}^2 u + \nabla_x u| \in L^p(S)$. On the other hand, as $D_{x_i} v$ vanishes at $y = 0$, Hardy's inequality shows that $(D_{x_i} v)/y = D_{x_i} u$ is contained in $L^p(S)$, so that also $y|D_{xy}^2 u| \in L^p(S)$. Summing up, u is an element of D_{reg}.

3) It remains to establish the injectivity of $(I + A, D_{reg})$ which follows from accretivity. To this aim, let $u \in C_c^\infty(\mathbb{R}^{N+1})$ vanish for $y = 1$. As in Lemma 2.1 we compute

$$\int_S (Au)\, u|u|^{p-2} = (p-1)\int_S y|\nabla u|^2 |u|^{p-2} \chi_{\{u\neq 0\}} - \frac{\beta+1}{p}\int_{\mathbb{R}^N} |u(x,0)|^p dx \ge 0$$

Such functions u are dense in D_{reg} by the proof of [9, Lemma 2.1] where the same result is shown in the infinite strip $\mathbb{R}^N \times [0,\infty)$. Hence, A is accretive on D_{reg}. □

Remark 3.8. Note that point 3) in the above proof gives $D_{reg} \subseteq D_{en}$ for $\beta \le -1$.

3.3 The domains for $b = 0$ and $\beta > -1$

We treat the case $\beta > -1$ in a similar way, now based on the represention formula (3.14). In the range $\beta > -1$, the value $-1/p$ represents a threshold for regularity. For $\beta > -1/p$ the domain D_p has again full regularity and coincides with the space

$$D^0_{reg} := \{u \in D_{reg} \,|\, u(\cdot, 0) = 0\},$$

cf. (3.16). Indeed, since u belongs to $W^{1,p}((0,1), L^p(\mathbb{R}^N))$, the above boundary condition is understood in $C([0,1], L^p(\mathbb{R}^N))$. This result has already been shown in [9] in the half-space $\mathbb{R}^N \times (0, +\infty)$. We give here an independent proof for $b = 0$.

Proposition 3.9. *Let $\beta > -1/p$ and $b = 0$. We then have $D_p = D^0_{reg}$.*

Proof. Let $u \in D_p$. Formula (3.14) gives

$$D_y u(x, y) = -y^\beta \int_0^1 g(x,t)\, dt + y^\beta \int_y^1 \frac{g(x,t)}{t^{\beta+1}}\, dt. \tag{3.18}$$

Proceeding as in Lemma 2.10 (iii) of [11], one can see that $D_y u \in L^p(S)$. The arguments in step 3) of the proof of Proposition 3.7 then imply that u belongs D_{reg}. To show that $u \in D^0_{reg}$, we verify that $\|u(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$ using (3.14); i.e.,

$$u(x, \varepsilon) = \frac{1}{\beta+1} \int_0^\varepsilon g(x,t)\, dt - \frac{\varepsilon^{\beta+1}}{\beta+1} \int_0^1 g(x,t)\, dt + \frac{\varepsilon^{\beta+1}}{\beta+1} \int_\varepsilon^1 \frac{g(x,t)}{t^{\beta+1}}\, dt.$$

The convergence property is clear for the first two addends on the right hand side, since $\beta + 1 > 0$. Concerning the third, say w, Hölder's inequality and $\beta > -1/p$ yield

$$|w(x, \varepsilon)|^p \le C\big(\varepsilon - \varepsilon^{(\beta+1)p'}\big)^{p-1} \int_0^1 |g(x,t)|^p\, dt \le C\varepsilon^{p-1} \int_0^1 |g(x,t)|^p\, dt.$$

Integrating with respect to $x \in \mathbb{R}^N$, the thesis follows. This shows that $D_p \subseteq D^0_{reg}$.

To prove the converse, we take a function $u \in D^0_{reg}$ such that $u + Au = 0$. Employing the boundary conditions, we compute

$$0 = \int_S |u|^p + \int_S (Au)\, u|u|^{p-2} = \int_S |u|^p + (p-1) \int_S y|\nabla u|^2 |u|^{p-2} \chi_{\{u \neq 0\}}.$$

It follows that $u = 0$ and so $D_p = D^0_{reg}$. □

Remark 3.10. Note that the above proof yields $D^0_{reg} \subseteq D_{en}$, also for $\beta > -1/p$.

In the case $-1 < \beta \le -1/p$, we do not have full regularity as shown by Example 3.15. Nevertheless, certain linear combinations of the functions $D_y u$, $y D_y^2 u$ and u/y still belong to D_p in this case.

Lemma 3.11. *Let $-1 < \beta \le -1/p$ and $b = 0$. Then for every $u \in D_p$ the functions $D_y u - (\beta+1)u/y$ and $-y D_y^2 u + \beta D_y u$ belong to $L^p(S)$.*

Proof. Let $u \in D_p$. The second assertion was already shown in Lemma 3.1. Equations (3.18) and (3.14) yield

$$D_y u - (\beta+1)\frac{u}{y} = -\frac{1}{y} \int_0^y g(x,t)\, dt. \tag{3.19}$$

The first statement now follows from Hardy's inequality. □

The first property proven above already implies a Dirichlet and a Neumann-type boundary condition as $y \to 0$.

Lemma 3.12. *Let $-1 < \beta \le -1/p$ and $u \in W^{1,p}(S_\varepsilon)$ for every $0 < \varepsilon \le 1/2$ such that $u(\cdot,1) = 0$ and $D_y u - (\beta+1)u/y \in L^p(S)$. Then $\lim_{\varepsilon\to 0} \|u(\cdot,\varepsilon)\|_{L^p(\mathbb{R}^N)} = 0$ and $\liminf_{\varepsilon\to 0} \varepsilon \|D_y u(\cdot,\varepsilon)\|_{L^p(\mathbb{R}^N)} = 0$.*

Proof. Set $f = D_y u - (\beta+1)u/y$. Due to the boundary condition at $y = 1$, integration by parts and Hölder's inequality yield

$$
\begin{aligned}
u(x,\varepsilon) &= -\varepsilon^{\beta+1} \int_\varepsilon^1 f(x,y) y^{-(\beta+1)}\,dy, \\
|u(x,\varepsilon)|^p &\le \varepsilon^{p(\beta+1)} \int_0^1 |f(x,y)|^p\,dy \left(\int_\varepsilon^1 y^{-(\beta+1)p'}\,dy \right)^{p-1}.
\end{aligned} \tag{3.20}
$$

The last integral is bounded if $\beta < -1/p$ and grows logarithmically when $\beta = -1/p$. In both cases, it follows that $\|u(\cdot,\varepsilon)\|_{L^p(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$.

We also have $yD_y u(x,y) = (\beta+1)u + yf$. Observe that $\liminf_{\varepsilon\to 0} \varepsilon\|f(\cdot,\varepsilon)\|_{L^p(\mathbb{R}^N)} = 0$, since otherwise $|f|^p$ would not be integrable in S. So also the second statement is true. $\square$

We can now give a first description of D_p in the intermediate range $-1 < \beta \le -1/p$. Here we only have partial regularity but Dirichlet boundary conditions at $y = 0$, see Remark 3.14.

Proposition 3.13. *Let $-1 < \beta \le -1/p$ and $b = 0$. We then have the equality*

$$
D_p = \left\{ u \in D_{en} \,\middle|\, y|D_x^2 u|,\ D_y u - (\beta+1)u/y,\ -yD_y^2 u + \beta D_y u \in L^p(S) \right\}.
$$

Proof. Let D be the right-hand side. The inclusion $D_p \subseteq D$ follows from Proposition 2.4 and Lemmas 3.1 and 3.11. It remains to check the injectivity of $(I + A, D)$. Let $u \in D$ satisfy $u + Au = 0$. Let $\varepsilon \in (0, 1/2]$. We compute

$$
\begin{aligned}
0 = \int_{S_\varepsilon} |u|^p + \int_{S_\varepsilon} (Au)\, u|u|^{p-2} &= \int_{S_\varepsilon} |u|^p + (p-1) \int_{S_\varepsilon} y|\nabla u|^2 |u|^{p-2} \\
&\quad - \frac{\beta+1}{p} \int_{\mathbb{R}^N} |u(x,\varepsilon)|^p dx + \int_{\mathbb{R}^N} \varepsilon D_y u(x,\varepsilon) u|u|^{p-2}(x,\varepsilon) dx.
\end{aligned}
$$

The first boundary integral tends to 0 by Lemma 3.12. The last integral is estimated by

$$
\left| \int_{\mathbb{R}^N} \varepsilon D_y u(x,\varepsilon) u|u|^{p-2}(x,\varepsilon) dx \right| \le \left(\int_{\mathbb{R}^N} |\varepsilon D_y u(x,\varepsilon)|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |u(x,\varepsilon)|^p dx \right)^{\frac{1}{p'}}
$$

using Hölder's inequality. It thus tends to 0 on a sequence $\varepsilon_n \to 0$ because of Lemma 3.12 again. Letting $\varepsilon_n \to 0$ in the previous identity, we obtain

$$
\int_S |u|^p + (p-1) \int_S y|\nabla u|^2 |u|^{p-2} \chi_{\{u \ne 0\}} = 0
$$

which yields $u = 0$. $\square$

Remark 3.14. There are boundary conditions hidden in the regularity properties of D_p for $-1 < \beta \le -1/p$. In fact, Lemma 3.12 yields $\|u(\cdot,\varepsilon)\|_{L^p(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$. Concerning $D_y u$, Lemma 3.12 gives only $\liminf_{\varepsilon\to 0} \varepsilon \|D_y(\cdot,\varepsilon)\|_{L^p(\mathbb{R}^N)} = 0$ but we actually have $\lim_{\varepsilon\to 0} \varepsilon \|D_y(\cdot,\varepsilon)\|_{L^p(\mathbb{R}^N)} = 0$. This easily follows from identity (3.19)

$$yD_y u - (\beta+1)u = -\int_0^y g(x,t)\,dt$$

valid for $u \in D_p$, the Dirichlet condition on u, and (3.8).

The next example shows that D_p is not contained in D_{reg} for $-1 < \beta \le -1/p$.

Example 3.15. Let $-1 < \beta \le -1/p$. Take functions $0 \ne \varphi \in C_c^\infty(\mathbb{R}^N)$ and $\psi \in C^\infty([0,1])$ with $\psi = 1$ near $\{y = 0\}$ and $\psi(1) = 0$. We set $u(x,y) = y^{\beta+1}\varphi(x)\psi(y)$ on S. Note that $y^{\beta+1}$ belongs to the kernel of the operator $-yD_y^2 + \beta D_y$. It is easy to check that u is an element of D_p. On the other hand, near $y = 0$ the functions $D_y u$, $yD_y^2 u$ and u/y all behave like $y^\beta\varphi$ and are thus do not belong to $L^p(S)$.

Finally, we establish that $|\nabla_x u|$ and $y|D_{xy}^2 u|$ belong to $L^p(S)$ also if $-1 < \beta \le -1/p$. This a bit unexpected fact relies on the results recently proved in [20]. In order to explain how our operator fits in the setting of [20], we recall from (3.1) the definition of

$$\mathcal{L}_y w = -D_y^2 w + (\beta+2)\frac{D_y w}{y} - (\beta+2)\frac{w}{y^2},$$

on smooth functions $w = w(y)$ and compute $s_1 = -\beta - 2$, $s_2 = -1$, according to the definitions given in Section 4 of [20]. Set

$$D(\mathcal{L}_y) = \big\{ w \in L^p(0,+\infty) \cap W^{2,p}_{loc}(0,+\infty) \,\big|\, \mathcal{L}_y w \in L^p(0,+\infty) \text{ and } y^{-2\theta} w \in L^p(0,+\infty) \\ \text{for every } \theta \in [0,1] \text{ s.t. } -\beta - 2 + 2\theta < \tfrac{1}{p} \big\}$$

(see Proposition 4.2 of [20]). By Theorem 8.8 of [20], the operator $-\mathcal{A} = \Delta_x - \mathcal{L}_y$ endowed with the domain

$$D(\mathcal{A}) = \big\{ v \in L^p(\mathbb{R}^{N+1}_+) \,\big|\, v(\cdot,y) \in W^{2,p}(\mathbb{R}^N) \text{ for a.a. } y > 0, |\nabla_x v|, |D_x^2 v| \in L^p(\mathbb{R}^{N+1}_+), \\ v(x,\cdot) \in W^{2,p}_{loc}(0,+\infty) \text{ for a.a. } x \in \mathbb{R}^N, \mathcal{L}_y v, y^{-2\theta} v \in L^p(\mathbb{R}^{N+1}_+) \\ \text{for every } \theta \in [0,1] \text{ s.t. } -\beta - 2 + 2\theta < \tfrac{1}{p} \big\} \tag{3.21}$$

generates a bounded analytic semigroup in $L^p(\mathbb{R}^{N+1}_+)$. Under more restrictive assumptions on the parameter β, which are fulfilled in our setting, it turns out that

$$D(\mathcal{A}) = \big\{ v \in L^p(E) \,\big|\, v(\cdot,y) \in W^{2,p}(\mathbb{R}^N) \text{ for a.a. } y > 0,\ |\nabla_x v|, |D_x^2 v|, \frac{|\nabla_x v|}{y}, |D_{xy}^2 v| \in L^p(E), \\ v(x,\cdot) \in W^{2,p}_{loc}(0,+\infty) \text{ for a.a. } x \in \mathbb{R}^N,\ \mathcal{L}_y v, y^{-2\theta} v \in L^p(E) \\ \text{for every } \theta \in [0,1] \text{ s.t. } -\beta - 2 + 2\theta < \tfrac{1}{p} \big\}, \tag{3.22}$$

see Theorem 8.9 of [20], where $E = \mathbb{R}^{N+1}_+$.

We can now complete the description of D_p in the case $-1 < \beta \le -1/p$ which cannot be much improved in view of Example 3.15.

Proposition 3.16. *Let $-1 < \beta \leq -1/p$ and $b = 0$. Then the domain D_p is equal to*

$$D^0_{par} := \{u \in D_{en} \,|\, y|D^2_x u|,\ |yD^2_{xy}u|,\ |\nabla_x u|,\ D_y u - (\beta+1)u/y,\ -yD^2_y u + \beta D_y u \in L^p(S)\}.$$

Proof. Because of Proposition 3.13, we only have to show that $D_i u$ and $yD^2_{iy}u$ are contained in $L^p(S)$ for $u \in D_p$, $i \in \{1, \dots, N\}$ and $D_i = D_{x_i}$. Set $v = yu$. We claim that v belongs to the domain of $\mathcal{A}$ given by (3.21). We already know that $v, |\nabla_x v|, |D^2_x v|, |\mathcal{L}_y v| \in L^p(S)$ (after truncating v near to $y = 1$ and then extending it smoothly in the whole halfspace). It remains to show that $y^{-2\theta} v \in L^p(S)$ for every $\theta \in [0, 1]$ such that $-\beta - 2 + 2\theta < \frac{1}{p}$. Recalling estimate (3.20), we have

$$y^{-2\theta p}|v(x,y)|^p = y^{(1-2\theta)p}|u(x,y)|^p \leq y^{p(\beta+2-2\theta)} \int_0^1 |f(x,t)|^p\,dt \left(\int_y^1 t^{-(\beta+1)p'}\,dt\right)^{p-1}.$$

The last integral is bounded if $\beta < -1/p$ and grows logarithmically when $\beta = -1/p$. In both cases, because of since $p(\beta + 2 - 2\theta) > -1$ and the choice of θ, we have the desired assertion. From (3.22) we infer that $\frac{|D_x v|}{y}$ and $|D^2_{xy}v| \in L^p(S)$. Thus, by computing it explicitly, we infer that $|\nabla_x u|$ and $yD^2_{xy}u \in L^p(S)$ and the proof is complete. □

4 Analyticity and the main result on the strip

In a first step we use the domain description to deduce analyticity of the semigroup for $b = 0$ from previous results in [9]. The analyticity then allows us to add the drift term $b \cdot \nabla_x$ to our prototype operator on the strip.

Proposition 4.1. *Let $\beta \in \mathbb{R}$ and $b = 0$. Then the semigroup $(T(t))_{t\geq 0}$ from Proposition 2.4 is analytic.*

Proof. We give the proof distinguishing the cases $\beta > -1$, $\beta < -1$ and $\beta = -1$.

1a) Assume that $\beta > -1$. First, let $\beta > -1/p$. In Theorem 2.10 of [9] it is shown that the operator $-A^\sharp = (-A, D^\sharp_{reg})$ generates a strongly continuous analytic semigroup of positive contractions in $L^p(\mathbb{R}^{N+1}_+)$, where $\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, +\infty)$ and

$$D^\sharp_{reg} = \{u \in L^p(\mathbb{R}^{N+1}_+) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^{N+1}_+) \,|\, |\nabla u|, \sqrt{y}|\nabla u|, y|D^2 u| \in L^p(\mathbb{R}^{N+1}_+), u(\cdot, 0) = 0\}.$$

We transfer this result to the spatial domain S, using the operator $A_{p,\frac{1}{2}} = (A, D_{p,\frac{1}{2}})$ with $\varepsilon = 1/2$ according to our notation. Choose functions $\eta_1, \eta_2 \in C^\infty(\mathbb{R})$ such that $0 \leq \eta_1, \eta_2 \leq 1$, $\operatorname{supp}\eta_1 \subseteq (-1, 1)$, $\operatorname{supp}\eta_2 \subseteq \mathbb{R} \setminus \left(-\frac{1}{3}, \frac{1}{3}\right)$ and $\eta_1^2 + \eta_2^2 = 1$. Fix $f \in L^p(S)$ and take $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. We define

$$\mathcal{R}(\lambda)f = \eta_1(\lambda + A^\sharp)^{-1}(\eta_1 f) + \eta_2(\lambda + A_{p,\frac{1}{2}})^{-1}(\eta_2 f)$$

on S, where $\eta_i = \eta_i(y)$ and the functions are restricted and extended by 0 appropriately.

Since $\mathcal{R}(\lambda)f \in W^{1,p}_0(S)$ and $y|D^2\mathcal{R}(\lambda)f| \in L^p(S)$, the map $\mathcal{R}(\lambda)f$ is an element of D^0_{reg} and thus of D_p by Proposition 3.9. Moreover $(\lambda + A)\mathcal{R}(\lambda)f = f + S_1(\lambda)f + S_2(\lambda)f$, where

$$S_1(\lambda)f = [\lambda + A, \eta_1](\lambda + A^\sharp)^{-1}(\eta_1 f), \qquad S_2(\lambda)f = [\lambda + A, \eta_2](\lambda + A_{p,\frac{1}{2}})^{-1}(\eta_2 f).$$

These functions are supported in $\left(\frac{1}{3}, 1\right)$. Setting $u_1 = (\lambda + A^\sharp)^{-1}(\eta_1 f)$, one computes

$$S_1(\lambda)f = -2y\eta_1' D_y u_1 - y\eta_1'' u_1 + \beta\eta_1' u_1.$$

The estimates for $\|u_1\|_p$ and $\|\sqrt{y}D_y u_1\|_p$ from Lemma 2.13 and Corollary 2.14 of [9] yield positive constants C and r_0 such that

$$\|S_1(\lambda)f\|_p \le \frac{C}{|\lambda|^{\frac12}}\|f\|_p$$

for $|\lambda| \ge r_0$. A similar estimate can be derived for $\|S_2(\lambda)f\|_p$. These facts imply that the operator $S_1(\lambda)+S_2(\lambda)$ has norm less than $\frac12$ and thus $I+S_1(\lambda)+S_2(\lambda)$ is invertible, possibly choosing a larger number r_0. Denoting its inverse by $V(\lambda)$, we infer that $\|V(\lambda)\| \le 2$ for $|\lambda| \ge r_0$ and that the function $u = \mathcal{R}(\lambda)V(\lambda)f$ belongs to $D_p = D^0_{reg}$ and solves $\lambda u + Au = f$. The sectoriality of $A^\sharp$ and $A_{p,\frac12}$ further imply the inequality $\|u\|_p \le C|\lambda|^{-1}\|f\|_p$, so that $-A_p$ generates an analytic semigroup.

1b) Let $-1 < \beta \le -1/p$ and fix $q < p < r$ such that $\beta > -\frac{1}{q}$. By step 1a), the semigroup on $L^q(S)$ generated by $-A_q = (-A, D_q)$ is analytic. On the other hand, $-A_r$ is the generator of a C_0–semigroup on $L^r(S)$ and the semigroups are consistent by Proposition 2.4. So the Stein interpolation theorem shows that $-A_p$ generates an analytic semigroup on $L^p(S)$.

2) Assume $\beta < -1$. For every $\varepsilon \in (0, 1/2]$, the adjoint of $A_{p,\varepsilon}$ is given by $A^* = -y\Delta - (\beta+2)D_y$ endowed with the domain $D_{p',\varepsilon}$. Letting $\varepsilon \to 0$, we infer that the adjoint semigroup of $(T(t))_{t\ge0}$ is the semigroup $(T^*(t))$ generated in $L^{p'}(S)$ by $(-A^*, D_{p'})$, according to Proposition 2.4. Since $-(\beta+2) > -1$, the semigroup $(T^*(t))_{t\ge0}$ is analytic by part 1) and hence also $(T(t))_{t\ge0}$ is analytic, by duality.

3) Finally, assume $\beta = -1$. Here the operators $A_{2,\varepsilon}$ are self-adjoint in $L^2(S_\varepsilon)$ for every $\varepsilon \in (0, 1/2]$. By approximation, it follows that A_2 is self-adjoint in $L^2(S)$. Since $\omega_2 - A_2$ is dissipative, we infer that A_2 is sectorial. Therefore $(T(t))$ is analytic in $L^2(S)$. Again, the Stein interpolation theorem implies that the semigroup $(T(t))$ is analytic in every $L^p(S)$. □

We can now add the gradient term $b \cdot \nabla_x u$ to the operator with $b = 0$ using a theorem by Kalton and Weis on operator sums. This strategy was already used in [12]. Recall the definition of D_{en} in (2.6).

Theorem 4.2. *Let $\beta \in \mathbb{R}$ and $b \in \mathbb{R}^N$. Then the operator $-A_p = (-A, D_p)$ generates an analytic semigroup on $L^p(S)$. Its domain D_p is equal to*

$$\begin{aligned}
D_{reg} &= \{u \in D_{max} \cap W^{1,p}(S) \,|\, y|D^2 u| \in L^p(S)\} \quad \text{if } \beta \le -1,\\
D^0_{par} &= \{u \in D_{en} \,|\, y|D^2_x u|,\, y|D^2_{xy}u|,\, |\nabla_x u|,\, D_y u - \tfrac{\beta+1}{y}u,\, yD^2_y u - \beta D_y u \in L^p(S)\}\\
&\qquad\qquad \text{if } -1 < \beta \le -\tfrac1p,\\
D^0_{reg} &= \{u \in D_{reg} \,|\, u(\cdot, 0) = 0\} \quad \text{if } \beta > -\tfrac1p.
\end{aligned}$$

Proof. 1) For $b = 0$ the result has been shown in Propositions 3.7, 3.9, 3.16 and 4.1.

2) Let $b \ne 0$. We split A defined on $D_p(\beta, 0)$ into $A_0 = -y\Delta + \beta D_y$ and $B = b \cdot \nabla_x$ with domain $D(B) = \{u \in L^p(S) \,|\, Bu \in L^p(S)\}$. By part 1) and Proposition 2.4, we know $D_p(\beta, 0) \subseteq D(B)$ and that $(-A_0, D_p(\beta, 0))$ generates the analytic C_0-semigroup $(T_0(t))_{t\ge0}$ of positive contractions on $L^p(S)$. Combined with Theorem 2.20 of [18], the corollary in Paragraph 4d) of [24] now shows that $(A_0, D_p(\beta, 0))$ is R-sectorial of angle smaller that $\pi/2$. On the other hand, B generates the positive and contractive C_0-group $(S(t))_{t\ge0}$ on $L^p(S)$ given by $(S(t)f)(x, y) = f(x + bt, y)$. It thus has a bounded H^∞–calculus of angle $\pi/2$ by Theorem 10.7 of [18].

The explicit formula for $S(t)$ and part 1) imply that $S(t)D_p(\beta,0) \subseteq D_p(\beta,0)$ and $A_0S(t)u = S(t)A_0u$ for $u \in D_p(\beta,0)$ and $t \geq 0$. Therefore, $(\lambda+A_0)^{-1}S(t) = S(t)(\lambda+A_0)^{-1}$ for $\lambda > 0$ so that $T_0(t)S(t) = S(t)T_0(t)$ because of the resolvent approximation formula for $T_0(t)$ from Corollary III.5.5 in [5]. Paragraph II.2.7 of this monograph also shows that the closure of $(-(A_0 + B), D_p(\beta,0))$ is the generator of the C_0–semigroup given by $U(t) = T_0(t)S(t)$.

Since the semigroups commute, the resolvents of A_0 and B also commute. As the sum of the above angles is less than π, Theorem 12.13 of [18] then shows that $A = A_0 + B$ is closed on $D_p(\beta,0)$. Hence, $(-A, D_p(\beta,0))$ generates $(U(t))_{t\geq 0}$. We still have to prove that $(U(t))_{t\geq 0}$ coincides with the semigroup $(T(t))_{t\geq 0}$ generated by $(-A, D_p(\beta,b))$ according to Proposition 2.4. Let $S_\varepsilon(t)$ be the restriction of $S(t)$ to $L^p(S_\varepsilon)$. As above one deduces the identity $T_\varepsilon(t) = T_{0,\varepsilon}(t)S_\varepsilon(t)$, restricting A, A_0 and B to $D_{p,\varepsilon}$. By Proposition 2.4 the operators $E_\varepsilon T_{0,\varepsilon}(t)P_\varepsilon$ and $E_\varepsilon T_\varepsilon(t)P_\varepsilon$ converge strongly to $T_0(t)$ and $T(t)$ in $L^p(S)$ as $\varepsilon \to 0$, respectively. Therefore the product

$$E_\varepsilon T_\varepsilon(t)P_\varepsilon = E_\varepsilon T_{0,\varepsilon}(t)P_\varepsilon E_\varepsilon S_\varepsilon(t)P_\varepsilon$$

also tends to $T_0(t)S(t) = U(t)$. We conclude $U(t) = T(t)$ and thus $D_p = D_p(\beta,b) = D_p(\beta,0)$ is given as in the statement.

3) It remains to check the analyticity in the case $b \neq 0$. By the open mapping theorem, the graph norms of A and A_0 are equivalent on D_p. Since $(T_0(t))_{t\geq 0}$ is analytic by Proposition 4.1, we can thus estimate

$$\|AT(t)u\|_p \leq c\,(\|A_0T_0(t)S(t)u\|_p + \|T_0(t)S(t)u\|_p) \leq \tfrac{c}{t}\|u\|_p$$

for $t \in (0,1]$ and $u \in L^p(S)$; i.e. $(T(t))_{t\geq 0}$ is analytic. $\square$

In the following corollary we state explicitly some estimates that have been proved along the proofs and will be needed in the next section to deal with operators having variables coefficients.

Corollary 4.3. *Let $-1 < \beta \leq -\frac{1}{p}$ and $b \in \mathbb{R}^N$. Then there is a constant $C > 0$ such that*

$$\|yD_x^2u\|_p + \|yD_{xy}^2u\|_p + \|\nabla_x u\|_p + \|D_yu - (\beta+1)\tfrac{u}{y}\|_p + \|yD_y^2u - \beta D_yu\|_p \leq C\,\|Au\|_p$$

for every $u \in D_p$.

We conclude the section by showing the generation result in a strip with arbitrary width.

Proposition 4.4. *Let $a > 0$ and set $S^a = \mathbb{R}^N \times (0,a)$. Let $\beta \in \mathbb{R}$ and $b \in \mathbb{R}^N$. Then the operator $(-A, D_p^a)$ generates an analytic semigroup on $L^p(S^a)$, where the domain D_p^a is given as in Theorem 4.2 with S replaced by S^a.*

Proof. Let $T : L^p(S^a) \to L^p(S)$ be defined by $Tu(x,y) = u\left(\frac{x}{a}, \frac{y}{a}\right)$. Then $T^{-1}AT$ endowed with domain $T^{-1}D_p$ generates an analytic semigroup in $L^p(S^a)$. By straightforward computations $T^{-1}AT = a^{-1}A$ and $T^{-1}D_p = D_p^a$, so that the statement follows. $\square$

5 Operators with variable coefficients

In this section we extend our results to operators on S with variable coefficients.

5.1 Non-isotropic diffusion coefficients

Our investigation starts by considering operators with constant coefficients

$$\hat{A} = -y \sum_{i,j=1}^{N+1} a_{ij} D_{ij} + b \cdot \nabla_x + \beta D_y,$$

where $a_{ij} = a_{ji} \in \mathbb{R}$ $b \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$. We assume

$$\sum_{i,j=1}^{N+1} a_{ij} \xi_i \xi_j \geq \alpha |\xi|^2$$

for some $\alpha > 0$ and all $\xi \in \mathbb{R}^{N+1}$. Set $M_a = \max |a_{ij}|$. The role of β in the previous section is now played by the coefficient

$$\gamma = \beta \, a_{N+1\, N+1}^{-1}.$$

We are mainly interested in the case where $-1 < \gamma \leq -\frac{1}{p}$, since in the remaining cases the domain of the operator has full regularity and one can argue as in [9], see Theorem 6.4 below. Set

$$D^\gamma = \left\{ u \in D_{en} \,\middle|\, y|D_x^2 u|, y|D_{xy}^2 u|,\ |\nabla_x u|,\ D_y u - (\gamma+1)\tfrac{u}{y},\ yD_y^2 u - \gamma D_y u \in L^p(S) \right\} \quad (5.1)$$

and for any $u \in D^\gamma$

$$\|u\|_{D^\gamma} = \|yD_x^2 u\|_p + \|yD_{xy}^2 u\|_p + \|\nabla_x u\|_p + \|yD_y^2 u - \gamma D_y u\|_p + \|D_y u - (\gamma+1)\tfrac{u}{y}\|_p.$$

The following theorem establishes the sectoriality of $(-\hat{A}, D^\gamma)$.

Proposition 5.1. *Assume $-1 < \gamma \leq -\frac{1}{p}$ and the above hypotheses. Then there are constants $\sigma \in \mathbb{R}$ and $C > 0$, depending on N, p, b, γ, M_a and α, such that for every $\operatorname{Re}\lambda > \sigma$ and $f \in L^p(S)$, there exists a unique solution u in D^γ of $\lambda u + \hat{A}u = f$. It satisfies*

$$\|u\|_p \leq C|\lambda|^{-1} \|f\|_p$$

and

$$\|u\|_{D^\gamma} \leq C\, \|\hat{A}u\|_p \leq C\, \|f\|_p. \quad (5.2)$$

Proof. Let $\varphi \in C^2(\mathbb{R}^{N+1})$. Let Q_1 be a non-singular matrix such that $\sum_{i,j=1}^{N+1} a_{ij} D_{ij}\varphi(z) = \Delta\psi(Q_1 z)$ whenever $\varphi(z) = \psi(Q_1 z)$ with $z = (x, y)$. Since the Laplacian is rotation invariant, we may choose $Q = PQ_1$ with $P^{-1} = P^*$ in such a way that $Q^* e_{N+1} = k e_{N+1}$ for some $k > 0$ and

$$\sum_{i,j=1}^{N+1} a_{ij} D_{ij}\varphi(z) = \Delta\psi(Qz)$$

whenever $\varphi(z) = \psi(Qz)$. This identity then yields $k^2 a_{N+1\, N+1} = 1$. Let $\operatorname{Re}\lambda > 0$ and $f \in L^p(S)$ be fixed. Let $z = (x, y) \in S$ and set $Qz = (\xi, \eta) = \zeta$. Since the last row of Q is $k\, e_{N+1}$, we deduce that $\eta = ky$, therefore $Qz \in \mathbb{R}^N \times (0, k) = S^k$. By a straightforward computation, the equation $\lambda u(z) + \hat{A}u(z) = f(z)$ in S is equivalent to

$$\lambda k v(\zeta) - \eta \Delta v(\zeta) + b_1 \cdot \nabla_\xi v(\zeta) + \gamma D_\eta v(\zeta) = k f(\zeta),$$

in S^k for $u(z) = v(Qz)$ and a suitable $b_1 \in \mathbb{R}^N$. Proposition 4.4 yields a unique solution $v \in D_{en}$ of this equation satisfying

$$\eta|D^2_\xi v|,\ \eta|D^2_{\xi\eta} v|,\ |\nabla_\xi v|,\ D_\eta v - (\gamma+1)\frac{v}{\eta},\ \eta D^2_\eta v - \gamma D_\eta v \in L^p(S^k).$$

From the formula $\nabla u(z) = Q^* \nabla v(Qz)$ we deduce

$$D_{x_i} u(z) = \langle \nabla v(Qz), Qe_i \rangle.$$

This function belongs to $L^p(S)$ since the last component of Qe_i is zero and $|\nabla_\xi v| \in L^p(S^k)$. Analogously, we have

$$D_y u(z) = \langle \nabla v(Qz), Qe_{N+1} \rangle = \langle \nabla_\xi v(Qz), \bar{q} \rangle + k D_\eta v$$

where $\bar{q} \in \mathbb{R}^N$ contains the first N components of Qe_{N+1}. It follows that

$$D_y u(z) - (\gamma+1)\frac{u}{y} = \langle \nabla_\xi v(Qz), \bar{q} \rangle + k\Big(D_\eta v - (\gamma+1)\frac{v}{\eta}\Big)$$

which yields $D_y u(z) - (\gamma+1)\frac{u}{y} \in L^p(S)$. Next we compute the second order derivatives, starting from the general formula $D^2 u(z) = Q^* D^2 v(Qz) Q$. Set $Q = (q_{ij})$. Recalling that $q_{N+1\, i} = 0$ if $i \le N$ and $q_{N+1\, N+1} = k$, we derive

$$\begin{aligned}
yD_{x_i x_j} u &= k^{-1}\eta \sum_{r,s=1}^N q_{ri}\, q_{sj}\, D_{\xi_r \xi_s} v \ \in L^p(S), \\
yD_{x_i y} u &= k^{-1}\eta \sum_{r,s=1}^N q_{ri}\, q_{s\, N+1}\, D_{\xi_r \xi_s} v + k^{-1}\eta \sum_{r=1}^N q_{ri}\, k\, D_{\xi_r \eta} v \ \in L^p(S), \\
yD^2_y u - \gamma D_y u &= k^{-1}\eta \sum_{r,s=1}^N q_{r\, N+1}\, q_{s\, N+1}\, D_{\xi_r \xi_s} v + \eta \sum_{r=1}^N q_{r\, N+1} D_{\xi_r \eta} v \\
&\quad - \gamma \langle \nabla_\xi v(Qz), \bar{q} \rangle + k\big(\eta\, D_{\eta\eta} v - \gamma D_\eta v\big) \ \in L^p(S).
\end{aligned}$$

We have shown that there exists a unique solution u in D^γ of $\lambda u(z) + \hat{A} u(z) = f(z)$. Proposition 4.4 also implies the asserted estimates. □

Corollary 5.2. *Assume $-1 < \gamma \le -\frac{1}{p}$. Then there exist $\sigma', C > 0$ such that for every $u \in D^\gamma$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \sigma'$ we have*

$$\|\sqrt{y}\, \nabla_x u\|_p + \|y\, D_y u\|_p \le C\, |\lambda|^{-1/2} \|\lambda u + \hat{A} u\|_p.$$

Proof. By Lemma 2.7 (iv) of [9] there exists a constant $C > 0$ such that

$$\|\sqrt{y}\, D_{x_i} u\|_{L^p(S)} \le \eta \|y D^2_{x_i} u\|_{L^p(S)} + \frac{C}{\eta} \|u\|_{L^p(S)} \tag{5.3}$$

for every $0 < \eta < 1$. So the estimates of Proposition 5.1 yield

$$\|\sqrt{y}\, D_{x_i} u\|_{L^p(S)} \le C\,\eta \|\hat{A} u\|_p + \frac{C}{\eta}\, \|u\|_p \le C\,\eta \|\lambda u + \hat{A} u\|_p + \frac{C}{|\lambda|} \left(\eta|\lambda| + \frac{1}{\eta}\right) \|\lambda u + \hat{A} u\|_p$$

if $\mathrm{Re}\,\lambda > \sigma$. Choosing $\eta = |\lambda|^{-1/2}$, we get the first of the asserted estimates for any λ with $\mathrm{Re}\,\lambda > \sigma' = \max\{\sigma, 1\}$. For the second estimate, we start from the interpolative inequality

$$\|y\, D_y u\|_{L^p(S)} \le \eta \|y^2 D_y^2 u\|_{L^p(S)} + \frac{C}{\eta}\|u\|_{L^p(S)}$$

which holds for any $0 < \eta < 1$ and can be proved as (5.3). Then

$$\|yD_y u\|_{L^p(S)} \le \eta\|y(y\, D_y^2 u - \gamma D_y u)\|_{L^p(S)} + \eta|\gamma|\|yD_y u\|_{L^p(S)} + \frac{C}{\eta}\|u\|_{L^p(S)}.$$

Since $|\gamma| < 1$, choosing $\eta < 1/2$ we infer

$$\|yD_y u\|_{L^p(S)} \le 2\eta\, \|yD_y^2 u - \gamma D_y u\|_{L^p(S)} + \frac{2C}{\eta}\|u\|_{L^p(S)}. \tag{5.4}$$

We can now argue as before and conclude the proof. □

5.2 Operators with x-dependent coefficients

As second step we consider operators of the form

$$\tilde{A} = -y \sum_{i,j=1}^{N+1} a_{ij}(x) D_{ij} + b(x)\cdot\nabla_x + \beta(x) D_y,$$

whose coefficients depend only on the tangential variables. This step is important to understand the general case. We assume that

(H0) $a_{ij} = a_{ji}$, b_i and β are bounded and uniformly continuous real functions on $\mathbb{R}^N$ and $\sum_{i,j=1}^{N+1} a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2$ for some $\alpha > 0$ and all $\xi \in \mathbb{R}^{N+1}$, $x \in \mathbb{R}^N$;

(H1) there exists $\gamma_0 \in \left]-1, -\frac{1}{p}\right]$ such that $\beta(x) = \gamma_0\, a_{N+1\,N+1}(x)$ for every $x \in \mathbb{R}^N$, i.e., the ratio $\dfrac{\beta(x)}{a_{N+1\,N+1}(x)}$ is constant in $\mathbb{R}^N$.

Set

$$M_{a,b} = \max\{\|a_{ij}\|_\infty, \|b_i\|_\infty\}.$$

Proposition 5.3. *Assume (H0) and (H1). Then there are constants $\sigma_1, C > 0$ (depending on $N, p, \gamma_0, M_{a,b}, \alpha$) such that for every $\mathrm{Re}\,\lambda > \sigma_1$ and $f \in L^p(S)$ there exists a unique solution u in D^{γ_0} of the equation $\lambda u + \tilde{A}u = f$. It satisfies*

$$\|u\|_p \le C|\lambda|^{-1}\|f\|_p \quad \text{and} \quad \|u\|_{D^{\gamma_0}} \le C\|f\|_p.$$

Proof. The basic ideas of the proof are the same as Theorem 3.1 in [9] though we have to take care since the domain of the operator is not a weighted Sobolev space anymore. For the sake of clarity, we give the complete proof.

1) We first solve $\lambda u + \tilde{A}u = f$. Let $\{B(x_n, r)\}$ be a countable family of balls in $\mathbb{R}^N$ which covers $\mathbb{R}^N$ and such that at most c_N among them overlap. To shorten the notation, in the

sequel we simply write B_n instead of $B(x_n,r)$. Let $\{\eta_n^2\}$ be a partition of unity subordinate to such a covering; i.e., every η_n is a C^∞ function with support contained in B_n, $0\le\eta_n\le 1$, $\sum_{n=1}^\infty \eta_n^2(x)=1$ for every $x\in\mathbb{R}^N$ and $\sup_{n\in\mathbb{N}}\|\eta_n\|_{C^2(\mathbb{R}^N)}<+\infty$. Let $\varepsilon>0$ and choose $r>0$ such that

$$|a_{ij}(x)-a_{ij}(x_n)|+|b_i(x)-b_i(x_n)|<\varepsilon \qquad \text{if} \quad |x-x_n|<r, \tag{5.5}$$

for every $n\in\mathbb{N}$. Set

$$\hat{A}_n=-y\sum_{i,j=1}^{N+1}a_{ij}(x_n)D_{ij}+b(x_n)\cdot\nabla_x+\beta(x_n)D_y$$

and notice that $\beta(x_n)\,a^{-1}_{N+1\,N+1}(x_n)=\gamma_0$ with γ_0 defined in (H1). For every $f\in L^p(S)$ and λ with $\operatorname{Re}\lambda>\sigma$, Proposition 5.1 then provides a unique $u_n\in D^{\gamma_0}$ solving $\lambda u_n+\hat{A}_nu_n=\eta_nf$. It satisfies

$$\|u_n\|_p\le\frac{C}{|\lambda|}\|\eta_nf\|_p \quad \text{and} \quad \|u_n\|_{D^{\gamma_0}}\le C\|\hat{A}_nu_n\|_p,$$

for some constant C depending on N, p, γ_0, M and α, but not on n. Set

$$R_n(\lambda)f=\eta_nu_n.$$

Then $v_n=R_n(\lambda)f\in D^{\gamma_0}$ and

$$(\lambda+\tilde{A})R_n(\lambda)f=(\lambda+\hat{A}_n)v_n+(\tilde{A}-\hat{A}_n)v_n=\eta_n^2f+(\tilde{A}-\hat{A}_n)v_n+[\hat{A}_n,\eta_n]u_n.$$

To estimate $(\tilde{A}-\hat{A}_n)v_n$, we add and subtract $\gamma_0(a_{N+1\,N+1}(x)-a_{N+1\,N+1}(x_n))D_yv_n$. Recalling (5.5), we thus obtain the pointwise bound

$$\begin{aligned}|(\tilde{A}-\hat{A}_n)v_n|\le{}& C\,\varepsilon\big(|yD_x^2v_n|+|yD_{xy}^2v_n|+|\nabla_xv_n|+|yD_y^2v_n-\gamma_0D_yv_n|\big)\\ &+\big|-\gamma_0(a_{N+1\,N+1}(x)-a_{N+1\,N+1}(x_n))+(\beta(x)-\beta(x_n))\big||D_yv_n|.\end{aligned} \tag{5.6}$$

By (H1) we have

$$-\gamma_0\big(a_{N+1\,N+1}(x)-a_{N+1\,N+1}(x_n)\big)+\beta(x)-\beta(x_n)=0$$

and therefore

$$\begin{aligned}\|(\tilde{A}-\hat{A}_n)v_n\|_p\le{}& C\varepsilon\big(\|yD_x^2u_n\|_p+\|yD_{xy}^2u_n\|_p+\|\nabla_xu_n\|_p+\|yD_yu_n\|_p+\|yD_y^2u_n-\gamma_0D_yu_n\|_p\big)\\ \le{}& C\varepsilon\big(\|\hat{A}_nu_n\|_p+\|u_n\|_p\big)\le C\varepsilon\big((|\lambda|+1)\|u_n\|_p+\|\eta_nf\|_p\big)\\ \le{}& C\varepsilon\|f\|_{L^p(B_n\times(0,1))}.\end{aligned} \tag{5.7}$$

Notice that $yD_yu_n=y\big(D_yu_n-(\gamma_0+1)\frac{u_n}{y}\big)+(\gamma_0+1)u_n$ and therefore $\|yD_yu_n\|_p$ can be estimated as well. Next, we compute

$$[\hat{A}_n,\eta_n]u_n=-2y\sum_{i,j=1}^{N+1}a_{ij}(x_n)D_i\eta_nD_ju_n-\left(y\sum_{i,j=1}^{N+1}a_{ij}(x_n)D_{ij}\eta_n+b(x_n)\cdot\nabla_x\eta_n\right)u_n\,.$$

By Corollary 5.2 it follows that

$$\|[\hat{A}_n,\eta_n]u_n\|_p\le\frac{C}{|\lambda|^{1/2}}\|f\|_{L^p(B_n\times(0,1))}, \tag{5.8}$$

if $\operatorname{Re}\lambda > \sigma'$. Set

$$R(\lambda)f = \sum_{n=1}^{+\infty} v_n, \qquad S(\lambda)f = \sum_{n=1}^{+\infty} \big((\tilde{A} - \hat{A}_n)v_n + [\hat{A}_n, \eta_n]u_n\big).$$

From above we deduce that

$$(\lambda + \tilde{A})R(\lambda)f = f + S(\lambda)f. \tag{5.9}$$

Estimates (5.7) and (5.8) imply that

$$\|S(\lambda)f\|_p \le \sum_{n=1}^{+\infty} C\varepsilon \|f\|_{L^p((B_n\times(0,1))} + \sum_{n=1}^{+\infty} \frac{C}{|\lambda|^{1/2}} \|f\|_{L^p(B_n\times(0,1))}.$$

Since at most c_N among the balls overlap, we get

$$\|S(\lambda)f\|_p \le c_N C \left(\varepsilon + \frac{1}{|\lambda|^{1/2}}\right) \|f\|_p.$$

We can now choose $\varepsilon > 0$ sufficiently small and $|\lambda|$ large enough to get $\|S(\lambda)\| \le 1/2$. Hence, for some $\omega > 0$ and all $\operatorname{Re}\lambda \ge \omega$ the operator $I + S(\lambda) : L^p(S) \to L^p(S)$ is invertible and its inverse $V(\lambda)$ is bounded by $\|V(\lambda)\| \le 2$. Equation (5.9) with $V(\lambda)f$ instead of f thus shows that $u = R(\lambda)V(\lambda)f \in D^{\gamma_0}$ solves the equation $\lambda u + \tilde{A}u = f$ satisfying $\|u\|_p \le \frac{C}{|\lambda|}\|f\|_p$.

2) We next show the injectivity of $\lambda + \tilde{A}$. According to the notation introduced in the first step, if $u \in D^{\gamma_0}$ and $\operatorname{Re}\lambda > \omega$, we can write

$$R_n(\lambda)(\lambda + \tilde{A})u = \eta_n^2 u + F_n u + G_n u,$$

where

$$F_n u = \eta_n\big((\lambda + \hat{A}_n)^{-1}(\tilde{A} - \hat{A}_n)(\eta_n u)\big), \qquad G_n u = \eta_n\big((\lambda + \hat{A}_n)^{-1}\big([\eta_n, \tilde{A}]u\big)\big).$$

Summing over n, it turns out that

$$\sum_{n=1}^{+\infty} R_n(\lambda)(\lambda + \tilde{A})u = u + \sum_{n=1}^{+\infty}(F_n u + G_n u),$$

for every $u \in D^{\gamma_0}$. Let $u \in D^{\gamma_0}$ satisfy $(\lambda + \tilde{A})u = 0$. The expression above implies that

$$u = -\sum_{n=1}^{+\infty} \big(F_n u + G_n u\big). \tag{5.10}$$

We claim that $u = 0$. To prove this, we need to bound $\|u\|_{D^{\gamma_0}}$ and $\|u\|_p$. It is useful to set

$$\begin{aligned}
\|v\|_{p,n} &= \|v\|_{L^p(B_n\times(0,1))},\\
\|v\|_{\gamma_0,n} &= \|y\, D_x^2 v\|_{p,n} + \|y\, D_{xy}^2 v\|_{p,n} + \|\nabla_x v\|_{p,n} + \|y D_y^2 v - \gamma_0 D_y v\|_{p,n}\\
&\quad + \|D_y v - (\gamma_0 + 1)\tfrac{v}{y}\|_{p,n} + \|v\|_{p,n}.
\end{aligned}$$

We estimate $F_n u$ and $G_n u$ for every $n \geq 1$. To simplify the notation, we define

$$f_n = (\tilde{A} - \hat{A}_n)(\eta_n u), \qquad g_n = [\eta_n, \tilde{A}]u,$$
$$\varphi_n = (\lambda + \hat{A}_n)^{-1} f_n, \qquad \psi_n = (\lambda + \hat{A}_n)^{-1} g_n.$$

As a consequence, we can write $F_n u = \eta_n \varphi_n$ and $G_n u = \eta_n \psi_n$. It is easily seen that

$$\|F_n u\|_{D^{\gamma_0}} \leq \|\varphi_n\|_{\gamma_0,n} + C\big(\|\varphi_n\|_{p,n} + \|\sqrt{y}\,\nabla_x \varphi_n\|_{p,n} + \|y D_y \varphi_n\|_{p,n}\big). \tag{5.11}$$

Estimates (5.2) and (5.6) imply

$$\|\varphi_n\|_{\gamma_0,n} \leq C\|f_n\|_p \leq C\varepsilon \|\eta_n u\|_{\gamma_0,n},$$
$$\|\varphi_n\|_{p,n} \leq \frac{C}{|\lambda|}\|f_n\|_p \leq \frac{C\varepsilon}{|\lambda|}\,\|\eta_n u\|_{\gamma_0,n}.$$

On the other hand, Corollary 5.2 yields

$$\|\sqrt{y}\,\nabla_x \varphi_n\|_{p,n} + \|y D_y \varphi_n\|_{p,n} \leq \frac{C}{|\lambda|^{1/2}}\|f_n\|_p \leq \frac{C\varepsilon}{|\lambda|^{1/2}}\,\|\eta_n u\|_{\gamma_0,n} \tag{5.12}$$

for $|\lambda| \geq 1$. As

$$\|\eta_n u\|_{\gamma_0,n} \leq \|u\|_{\gamma_0,n} + C\big(\|u\|_{p,n} + \|\nabla_x u\|_{p,n} + \|y D_y u\|_{p,n}\big) \leq \|u\|_{\gamma_0,n},$$

we finally obtain

$$\|F_n u\|_{D^{\gamma_0}} \leq \left(C\varepsilon + \frac{C}{|\lambda|^{1/2}}\right)\|\eta_n u\|_{\gamma_0,n} \leq \left(C\varepsilon + \frac{C}{|\lambda|^{1/2}}\right)\|u\|_{\gamma_0,n} \tag{5.13}$$

We also need a better estimate of the L^p norm of $F_n u$, namely,

$$\|F_n u\|_p \leq \frac{C}{|\lambda|}\|f_n\|_p \leq \frac{C}{|\lambda|}\|u\|_{\gamma_0,n}, \tag{5.14}$$

which easily follows from Proposition 5.1. Next, we consider the term $G_n u$. Observe that

$$\|g_n\|_p \leq C(\|u\|_{p,n} + \|\sqrt{y}\,\nabla_x u\|_{p,n} + \|y\,D_y u\|_{p,n}).$$

Replacing φ_n and f_n by ψ_n and g_n, respectively, in (5.11)–(5.12), we then infer

$$\|G_n u\|_{D^{\gamma_0}} \leq C(\|u\|_{p,n} + \|\sqrt{y}\,\nabla_x u\|_{p,n} + \|y\,D_y u\|_{p,n}), \tag{5.15}$$

and

$$\|G_n u\|_p \leq \frac{C}{|\lambda|}\|g_n\|_p \leq \frac{C}{|\lambda|}\|u\|_{\gamma_0,n}. \tag{5.16}$$

Formulas (5.10), (5.13) and (5.15) lead to

$$\begin{aligned}\|u\|_{D^{\gamma_0}} &\leq \sum_{n=1}^{+\infty}\left(C\varepsilon + \frac{C}{|\lambda|^{1/2}}\right)\|u\|_{\gamma_0,n} + \sum_{n=1}^{+\infty} C(\|u\|_{p,n} + \|\sqrt{y}\,\nabla_x u\|_{p,n} + \|y\,D_y u\|_{p,n}) \\ &\leq c_N C\left(C\varepsilon + \frac{C}{|\lambda|^{1/2}}\right)\|u\|_{D^{\gamma_0}} + c_N C(\|u\|_p + \|\sqrt{y}\,\nabla_x u\|_p + \|y\,D_y u\|_p).\end{aligned}$$

We can now fix a sufficiently small $\varepsilon > 0$ and choose sufficiently large $|\lambda|$ to obtain

$$\|u\|_{D^{\gamma_0}} \le C(\|u\|_p + \|\sqrt{y}\,\nabla_x u\|_p + \|y\, D_y u\|_p).$$

The interpolative estimates (5.3) and (5.4) then yield

$$\|u\|_{D^{\gamma_0}} \le C\|u\|_p.$$

Moreover, from (5.10), (5.14) and (5.16) it follows that

$$\|u\|_p \le \frac{C}{|\lambda|}\|u\|_{D^{\gamma_0}}.$$

Combining the last two estimates, we conclude

$$\|u\|_{D^{\gamma_0}} \le \frac{C}{|\lambda|}\|u\|_{D^{\gamma_0}}.$$

Taking large $|\lambda|$, we arrive at $u = 0$. Therefore, $\lambda + A : D^{\gamma_0} \to L^p(S)$ is bijective for every $\lambda \in \mathbb{C}$ with sufficiently large real part. By step 1), the operator A is sectorial which also implies last estimate in the statement. □

As in Corollary 5.2 one obtains mixed estimates.

Corollary 5.4. *Assume (H0) and (H1). Then there are constants $\sigma_1', C > 0$ such that for every $u \in D^{\gamma_0}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \sigma_1'$ we have*

$$\|\sqrt{y}\,\nabla_x u\|_p + \|y\, D_y u\|_p \le C\,|\lambda|^{-1/2}\|\lambda u + \tilde{A}u\|_p.$$

5.3 General operators

Finally we consider general operators

$$A = -y\sum_{i,j=1}^{N+1} a_{ij}(x,y)D_{ij} + b(x,y)\cdot\nabla_x + \beta(x,y)D_y,$$

under the following assumptions on the coefficients.

(Ha) $a_{ij} = a_{ji}$, b_i, β are bounded, uniformly continuous real functions on $\mathbb{R}^N \times [0,1]$ such that $\sum_{i,j=1}^{N+1} a_{ij}(x,y)\xi_i\xi_j \ge \alpha|\xi|^2$ for some $\alpha > 0$ and all $\xi \in \mathbb{R}^{N+1}$, $x \in \mathbb{R}^N$, $y \in [0,1]$;

(Hb) there exists $\gamma_0 \in \left]-1, -\frac{1}{p}\right]$ such that

$$\beta(x,0) = \gamma_0 a_{N+1\,N+1}(x,0)$$

for every $x \in \mathbb{R}^N$, i.e., the ratio $\dfrac{\beta(x,0)}{a_{N+1\,N+1}(x,0)}$ is constant;

(Hc) there exists $L > 0$ such that $\sup_{x\in\mathbb{R}^N}\left|\dfrac{\beta(x,y)}{a_{N+1\,N+1}(x,y)} - \dfrac{\beta(x,0)}{a_{N+1\,N+1}(x,0)}\right| \le L|y|$ for some $\bar{\delta} > 0$ and every $y \in [0,\bar{\delta}]$.

Set

$$M = \max\{\|a_{ij}\|_\infty, \|b_i\|_\infty, \|\beta\|_\infty\}.$$

Recall the definition of D^{γ_0} in (5.1).

Theorem 5.5. *Assume (Ha), (Hb) and (Hc). Then there are constants $\sigma_2, C > 0$ (depending on $N, p, \gamma_0, M, \alpha$) such that for every $\mathrm{Re}\,\lambda > \sigma_2$ and $f \in L^p(S)$, there exists a unique solution u in D^{γ_0} of the equation $\lambda u + Au = f$. It satisfies*

$$\|u\|_p \leq C|\lambda|^{-1}\|f\|_p \quad \textit{and} \quad \|u\|_{D^{\gamma_0}} \leq C\|f\|_p.$$

Proof. We construct the solution by splitting the problem into one for the operator

$$\tilde{A}_0 = -y\sum_{i,j=1}^{N+1} a_{ij}(x,0)D_{ij} + b(x,0)\cdot\nabla_x + \beta(x,0)D_y$$

and one for the realization A_0 of the nondegenerate operator A in $L^p(\mathbb{R}^N \times I_2)$ with Dirichlet boundary conditions. Let $\varepsilon > 0$. By (Ha) there exists $\delta_\varepsilon > 0$ such that

$$|a_{ij}(x,y) - a_{ij}(x,0)| + |b_i(x,y) - b_i(x,0)| \leq \varepsilon,$$

for every $x \in \mathbb{R}^N$ and y with $|y| < \delta_\varepsilon$, for every i, j. Without loss of generality we assume that $\delta_\varepsilon < \bar{\delta}$, where $\bar{\delta}$ is given in (Hc). Set $I_1 = (0, \delta_\varepsilon)$ and $I_2 = (\frac{\delta_\varepsilon}{2}, 1)$. Choose functions $\eta_1, \eta_2 \in C^\infty(\mathbb{R})$ such that $0 \leq \eta_1, \eta_2 \leq 1$, $\mathrm{supp}\,\eta_1 \subseteq (-\delta_\varepsilon, \delta_\varepsilon)$, $\mathrm{supp}\,\eta_2 \subseteq \mathbb{R} \setminus (-\frac{\delta_\varepsilon}{2}, \frac{\delta_\varepsilon}{2})$ and $\eta_1^2 + \eta_2^2 = 1$. Fix $f \in L^p(S)$ and define

$$R(\lambda)f = \eta_1(\lambda + \tilde{A}_0)^{-1}(\eta_1 f) + \eta_2(\lambda + A_0)^{-1}(\eta_2 f)$$

on S, where $\eta_i = \eta_i(y)$ and the functions are restricted and extended by 0 appropriately. Then $R(\lambda)f \in D^{\gamma_0}$ and, setting $u_1 = (\lambda + \tilde{A}_0)^{-1}(\eta_1 f)$ and $u_2 = (\lambda + A_0)^{-1}(\eta_2 f)$, we get

$$(\lambda + A)R(\lambda)f = f + \eta_1(A - \tilde{A}_0)u_1 + [A, \eta_1]u_1 + [A, \eta_2]u_2.$$

The proof then proceeds as that of Proposition 5.3. We explain the only change which regards the estimate of $(A - \tilde{A}_0)u_1$ (see (5.6)) and in particular the coefficient of $D_y u_1$. It is now given by

$$c(x,y) = -\gamma_0 a_{N+1\,N+1}(x,y) + \beta(x,y) = a_{N+1\,N+1}(x,y)\Big(\frac{\beta(x,y)}{a_{N+1\,N+1}(x,y)} - \frac{\beta(x,0)}{a_{N+1\,N+1}(x,0))}\Big).$$

Thanks to assumption (Hc), the estimate

$$|c(x,y)| \leq ML\,|y|, \quad \text{and thus} \quad |c(x,y)D_y u_1| \leq Cy|D_y u_1|,$$

holds true for any $(x,y) \in \mathbb{R}^N \times I_1$. As in (5.7) and using Corollary 5.4, we deduce

$$\|\eta_1(A - \tilde{A}_0)u_1\|_p \leq C(\varepsilon\|f\|_{L^p(\mathbb{R}^N\times I_1)} + \|yD_y u_1\|_{L^p(\mathbb{R}^N\times I_1)}) \leq C\Big(\varepsilon + \frac{1}{|\lambda|^{1/2}}\Big)\|f\|_{L^p(\mathbb{R}^N\times I_1)}.$$

Moreover, as in (5.8) we estimate

$$\|[A, \eta_1]u_1\|_p \leq \frac{C}{|\lambda|^{1/2}}\|f\|_{L^p(\mathbb{R}^N\times I_1)},$$

if $|\lambda|$ is large enough. The term $[A, \eta_2]u_2$ is treated similarly, invoking classical estimates. The remaining part of the proof can be performed adapting the ideas of Proposition 5.3. □

6 The localization procedure

Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ with boundary of class C^2 and let ϱ be a function in $C^2(\mathbb{R}^{N+1})$ such that

$$\Omega = \{\varrho > 0\}, \quad \partial\Omega = \{\varrho = 0\} \text{ and } \nabla\varrho(\xi) = \nu(\xi), \ \xi \in \partial\Omega.$$

Here, $\nu(\xi)$ is the inward unitary normal vector to $\partial\Omega$ at ξ. Such a function ϱ can be constructed by extending the distance function from the boundary of Ω. Let us introduce the operator L defined as

$$L = -\varrho(\xi) \sum_{i,j=1}^{N+1} a_{ij}(\xi) D_{ij} + \sum_{i=1}^{N+1} b_i(\xi) D_i, \qquad \xi \in \overline{\Omega}. \tag{6.1}$$

In the remainder of the section we shall assume the following conditions on the coefficients.

(H1) $a_{ij} = a_{ji}$ are real–valued functions in $C^1(\overline{\Omega})$ and satisfy the ellipticity condition

$$\sum_{i,j=1}^{N+1} a_{ij}(\xi)\zeta_i\zeta_j \geq \mu_0|\zeta|^2$$

for some constant $\mu_0 > 0$ and all $\xi \in \Omega$, $\zeta \in \mathbb{R}^{N+1}$.

(H2) b_i are real–valued functions in $C^1(\overline{\Omega})$.

(H3) There exists $\mathsf{m} \in \,]-1, -\frac{1}{p}]$ such that

$$\mathsf{m} = \frac{\mathbf{b}(\xi) \cdot \nu(\xi)}{\langle a(\xi)\nu(\xi), \nu(\xi)\rangle}$$

for all $\xi \in \partial\Omega$, where $\mathbf{b} = (b_1, \ldots, b_{N+1})$.

Define

$$\mathsf{M} = \max_{1 \leq i,j \leq N+1} \{\|a_{ij}\|_\infty, \|b_i\|_\infty\}.$$

Let $\xi_0 \in \partial\Omega$ be fixed. Following [9] and [10], in a neighborhood $U = U(\xi_0)$ of ξ_0 we consider functions $\theta_1, \ldots, \theta_N \in C^2(U)$ solving the equation

$$\sum_{i=1}^{N+1} \partial_i\varrho(\xi)\, \partial_i\theta(\xi) = 0, \qquad \xi \in U, \tag{6.2}$$

such that $\nabla\theta_1(\xi_0), \ldots, \nabla\theta_N(\xi_0)$ are linearly independent. We then define the transformation

$$J : U \to \mathbb{R}^{N+1}, \quad \xi \mapsto (\boldsymbol{\theta}(\xi), \varrho(\xi))$$

where $\boldsymbol{\theta}(\xi) = (\theta_1(\xi), \ldots, \theta_N(\xi))$. Due to (6.2), the Jacobian matrix of J at ξ_0 is non–singular. Therefore, possibly taking U smaller, we obtain that J is a C^2–diffeormorphism from U onto $J(U)$. It further holds that

$$J(U \cap \Omega) = J(U) \cap \mathbb{R}^{N+1}_+ \qquad J(U \cap \partial\Omega) = J(U) \cap \{y = 0\}.$$

So (U, J) is a local chart. We denote by H the inverse of J. We can cover $\partial\Omega$ by the finite union $V = U_1 \cup \cdots \cup U_m$ of open sets of the above type. Thus, below we may always assume that $U(\xi_0) \subset U_i$ for some of the U_i and that J and H are restrictions of the diffeomorphism on U_i. Hence, all the derivatives of J and H up to the second order may be assumed to be bounded by a constant independent of ξ_0. To fix the notation, we suppose that

$$\|J_k\|_\infty + \|\nabla J_k\|_\infty + \|D^2 J_k\|_\infty \le \mathsf{L},$$
$$\|H_k\|_\infty + \|\nabla H_k\|_\infty + \|D^2 H_k\|_\infty \le \mathsf{L}$$

for any $k = 1, \dots, N+1$. Such local coordinates have the advantage of transforming all the vectors $\nabla\varrho(\xi)$ at points $\xi \in U \cap \Omega$ into the normal direction at $\{y = 0\}$ because of

$$\big(\mathrm{Jac}\, J(\xi)\big)\nabla\varrho(\xi) = \big|\nabla\varrho(\xi)\big|^2 e_{N+1}.$$

It follows that

$$\big(\mathrm{Jac}\, H(z)\big)e_{N+1} = \frac{\nabla\varrho(\xi)}{|\nabla\varrho(\xi)|^2} \tag{6.3}$$

for $z = J(\xi)$. Define $\phi(z) = \varrho(Hz)$, for $z \in J(U) \cap \mathbb{R}^{N+1}_+$. Using Taylor's formula with respect to the last variable, for $z = (x, y)$ we find that

$$\phi(z) = \phi(x, y) = \phi(x, 0) + \partial_y\phi(x, t)\, y$$

for some $t \in (0, y)$. Recalling (6.3), we obtain

$$\partial_y\phi(z) = \big\langle \nabla\varrho(Hz), \big(\mathrm{Jac}\, H(z)\big)e_{N+1}\big\rangle = 1$$

with $\xi = Hz$. Therefore

$$\phi(z) = y \qquad z \in J(U) \cap \mathbb{R}^{N+1}_+.$$

Given a function $u : U \cap \Omega \to \mathbb{R}$, set

$$Tu = u \circ H \quad \text{on } J(U) \cap \mathbb{R}^{N+1}_+.$$

Of course,

$$u \in L^p(U \cap \Omega) \quad \Longleftrightarrow \quad Tu \in L^p(J(U) \cap \mathbb{R}^{N+1}_+).$$

If $u \in W^{2,p}_{\mathrm{loc}}(U \cap \Omega)$, then one can check that $\nabla Tu = (\mathrm{Jac}\, H)^*(\nabla u) \circ H$. Therefore

$$\varrho|\nabla u| \in L^p(U \cap \Omega) \quad \Longleftrightarrow \quad y|\nabla Tu| \in L^p(J(U) \cap \mathbb{R}^{N+1}_+).$$

Moreover, equality (6.3) yields

$$D_y Tu(z) = \langle \nabla Tu(z), e_{N+1}\rangle = \frac{\nabla\varrho(\xi) \cdot \nabla u(\xi)}{|\nabla\varrho(\xi)|^2}$$

for $\xi = Hz$ which implies

$$D_y Tu(z) - (\mathsf{m}+1)\frac{Tu(z)}{y} = \frac{\nabla\varrho(\xi) \cdot \nabla u(\xi)}{|\nabla\varrho(\xi)|^2} - (\mathsf{m}+1)\frac{u(\xi)}{\varrho(\xi)}. \tag{6.4}$$

By the definition of J we infer that

$$\big(\mathrm{Jac}\, J(\xi)\big)^* e_{N+1} = \nabla\varrho(\xi)$$

implying

$$\langle (\operatorname{Jac} H)e_i, \nabla\varrho\rangle = \langle e_i, (\operatorname{Jac} H)^*\nabla\varrho\rangle = \langle e_i, e_{N+1}\rangle = 0$$

for every $i = 1, \dots, N$. As $D_{x_i}(Tu)(z) = \langle \nabla u(\xi), (\operatorname{Jac} H)e_i\rangle$, we have

$$D_{x_i}(Tu) \in L^p(J(U)\cap\mathbb{R}^{N+1}_+) \quad\Longleftrightarrow\quad \langle \nabla u, \tau\rangle \in L^p(U\cap\Omega),$$

for any τ such that $\langle \tau, \nabla\varrho(\xi)\rangle = 0$ for $\xi \in U\cap\Omega$. Concerning second order derivatives, we first compute

$$D^2_{x_ix_j}Tu = \big\langle (D^2u)(\operatorname{Jac} H)e_i, (\operatorname{Jac} H)e_j\big\rangle + \sum_{\ell=1}^{N+1} D_\ell u\, D_{x_ix_j}H_\ell.$$

Let u satisfy $\varrho|\nabla u| \in L^p(U\cap\Omega)$. Then

$$yD^2_x(Tu) \in L^p(J(U)\cap\mathbb{R}^{N+1}_+) \quad\Longleftrightarrow\quad \varrho\langle (D^2u)\tau, \tilde\tau\rangle \in L^p(U\cap\Omega),$$

for any τ and $\tilde\tau$ such that $\langle \tau, \nabla\varrho(\xi)\rangle = 0$ and $\langle \tilde\tau, \nabla\varrho(\xi)\rangle = 0$, respectively, for $\xi \in U\cap\Omega$. Notice that $\varrho|\nabla u| \in L^p(U\cap\Omega)$ follows if one supposes that the right hand side of (6.4) belongs to $L^p(U\cap\Omega)$ as well as $\langle \nabla u, \tau\rangle$. Analogously,

$$yD_{xy}(Tu) \in L^p(J(U)\cap\mathbb{R}^{N+1}_+) \quad\Longleftrightarrow\quad \varrho\langle (D^2u)\tau, \nabla\varrho\rangle \in L^p(U\cap\Omega),$$

for any τ such that $\langle \tau, \nabla\varrho(\xi)\rangle = 0$ for $\xi \in U\cap\Omega$. Finally, by the identity

$$yD_{yy}Tu - \mathsf{m}D_yTu = \frac{\varrho\big\langle (D^2u)\nabla\varrho, \nabla\varrho\big\rangle}{|\nabla\varrho|^4} - \mathsf{m}\frac{\nabla\varrho\cdot\nabla u}{|\nabla\varrho|^2} + \sum_{\ell=1}^{N+1} \varrho D_\ell u\, D_{yy}H_\ell$$

it holds

$$yD_{yy}Tu - \mathsf{m}D_yTu \in L^p(J(U)\cap\mathbb{R}^{N+1}_+) \quad\Longleftrightarrow\quad \frac{\varrho\langle (D^2u)\nabla\varrho, \nabla\varrho\rangle}{|\nabla\varrho|^2} - \mathsf{m}\nabla\varrho\cdot\nabla u \in L^p(U\cap\Omega).$$

Moreover, all the constants involved in these equivalences are independent of ξ_0. We define $D(L)$ as follows.

Definition 6.1. *A function $u \in L^p(\Omega)\cap W^{2,p}_{\mathrm{loc}}(\Omega)$ belongs to $D(L)$ iff*

$$\begin{aligned}
&\langle \nabla u, \tau\rangle \in L^p(\Omega) \quad \text{for any } \tau \text{ s.t. } \langle \tau, \nabla\varrho\rangle = 0,\\
&\frac{\nabla\varrho\cdot\nabla u}{|\nabla\varrho|^2} - (\mathsf{m}+1)\frac{u}{\varrho} \in L^p(\Omega), \qquad \frac{\varrho\langle (D^2u)\nabla\varrho, \nabla\varrho\rangle}{|\nabla\varrho|^2} - \mathsf{m}\nabla\varrho\cdot\nabla u \in L^p(\Omega), \qquad (6.5)\\
&\varrho\langle (D^2u)\tau, \tilde\tau\rangle \in L^p(\Omega), \quad \varrho\langle (D^2u)\tau, \nabla\varrho\rangle \in L^p(\Omega) \quad \text{for any } \tau, \tilde\tau \text{ s.t. } \langle \tau, \nabla\varrho\rangle = 0, \langle \tilde\tau, \nabla\varrho\rangle = 0,
\end{aligned}$$

where m is given in (H3).

Remark 6.2. If $u \in D(L)$ then $\varrho|\nabla u| \in L^p(\Omega)$. This can be seen by the first requirement of (6.5) and the second one after a multiplication by ϱ.

The differential operator L is locally transformed into the operator $\mathcal{L}$ given by

$$\mathcal{L} = -y\sum_{h,k=1}^{N+1} \alpha_{hk}(z)\partial_{hk} + \sum_{k=1}^{N+1} \big(\gamma_k(z) - y\beta_k(z)\big)\partial_k \qquad (6.6)$$

with the coefficients

$$
\begin{aligned}
\alpha_{hk}(z) &= \sum_{i,j=1}^{N+1} a_{ij}(Hz)\partial_{\xi_j} J_h(Hz)\partial_{\xi_i} J_k(Hz),\\
\beta_k(z) &= \sum_{i,j=1}^{N+1} a_{ij}(Hz)\partial_{\xi_i\xi_j} J_k(Hz),\\
\gamma_k(z) &= \sum_{i=1}^{N+1} b_i(Hz)\partial_{\xi_i} J_k(Hz).
\end{aligned}
\tag{6.7}
$$

Notice that the sup–norms of all the coefficients of $\mathcal{L}$ are controlled by constants depending on M, L, $\|D^2\varrho\|_\infty$, but not on ξ_0. In order to deal with the class of operators introduced in §5.3, we shall verify that $\mathcal{L}$ satisfies assumptions (Hb) and (Hc). Since

$$
\begin{aligned}
\gamma_{N+1}(x,0) &= \mathbf{b}(\xi)\cdot\nu(\xi)\\
\alpha_{N+1\,N+1}(x,0) &= \langle a(\xi)\nu(\xi),\nu(\xi)\rangle
\end{aligned}
$$

for $\xi \in \partial\Omega \cap U$, (H3) yields that (Hb) is fulfilled with $\gamma_0 = \mathsf{m}$. Next, consider the ratio

$$
\frac{\gamma_{N+1}(z) - y\beta_{N+1}(z)}{\alpha_{N+1\,N+1}(z)} = \frac{\gamma_{N+1}(z)}{\alpha_{N+1\,N+1}(z)} - y\frac{\beta_{N+1}(z)}{\alpha_{N+1\,N+1}(z)}
$$

and set

$$
\Psi(z) = \frac{\gamma_{N+1}(z)}{\alpha_{N+1\,N+1}(z)} = T\psi,
$$

where $\psi = \frac{\mathbf{b}\cdot\nabla\varrho}{\langle a\nabla\varrho,\nabla\varrho\rangle}$. It follows that

$$
D_y\Psi = \frac{\nabla\varrho(\xi)\cdot\nabla\psi(\xi)}{|\nabla\varrho(\xi)|^2} = \frac{\nabla\varrho(\xi)}{|\nabla\varrho(\xi)|^2}\cdot\nabla\left(\frac{\mathbf{b}\cdot\nabla\varrho}{\langle a\nabla\varrho,\nabla\varrho\rangle}\right)
$$

and this function is bounded in a neighborhood of $\{y = 0\}$ by the boundedness of the coefficients a_{ij}, b_i and their first order derivatives. Therefore the operator $\mathcal{L}$ verifies (Hc). Eventually we are ready to state the main result of the section.

Theorem 6.3. *Assume* **(H1)**, **(H2)** *and* **(H3)**. *The operator* $-L$ *from* (6.1) *endowed with the domain* $D(L)$ *given in Definition 6.1 is sectorial in* $L^p(\Omega)$.

Proof. We only sketch the proof. We cover $\overline{\Omega}$ by the open sets $U_1,\dots,U_m$ constructed before and an open set Ω_0 whose closure is contained in Ω. We construct a solution in $D(L)$ of the resolvent equation $\lambda u + Lu = f$, for a given function $f \in L^p(\Omega)$, by solving the equations $\lambda u + Lu = f\eta_i$ in U_i for $i = 1,\dots,m$ and $\lambda u + Lu = f\eta_0$ in Ω_0, where $\{\eta_i^2\}_{i=0}^m$ is a partition of unity subordinate to the covering $\{\Omega_0, U_1,\dots,U_m\}$. The equation $\lambda u + Lu = f\eta_0$ admits a unique solution $u_0 \in W_0^{1,p}(\Omega_0)\cap W^{2,p}(\Omega_0)$, by classical results for nondegenerate operators. For every $i = 1,\dots,m$ there exists a unique solution $u_i \in D(L)$ of the equation $\lambda u + Lu = f\eta_i$, obtained by using the local chart constructed before and applying the results of subsection 5.3 to the transformed operator. One then has to glue together the functions $u_0,\dots,u_m$ as in the proof of Theorem 5.3. To this aim, we observe that the commutators that turn out from the computations are first order operators involving the first order derivatives multiplied by ϱ. These terms are estimated by interpolative inequalities which are obtained from those of Corollary 5.2 via local charts. The injectivity of $\lambda + L$ on $D(L)$ can be proved by similar arguments. □

For completeness, we also state the variants of the above result if $\beta > -1/p$ or $\beta \le -1$ on the level of the model problem. We omit the proof since it can be carried out as in [9] based on the domain descriptions in Theorem 4.2. As noted, these results have already been proved in [9] and [23] by different methods. We use the function

$$q(\xi) = \frac{\mathbf{b}(\xi) \cdot \nu(\xi)}{\langle a(\xi)\nu(\xi), \nu(\xi)\rangle}, \qquad \xi \in \partial\Omega,$$

and the domains

$$D_{reg}(L) = \{u \in W^{2,p}_{loc}(\Omega) \cap W^{1,p}(\Omega) \mid \varrho D^2 u \in L^p(\Omega)\},$$
$$D^0_{reg}(L) = \{u \in W^{2,p}_{loc}(\Omega) \cap W^{1,p}_0(\Omega) \mid \varrho D^2 u \in L^p(\Omega)\}.$$

Theorem 6.4. *Assume* **(H1)** *and* **(H2)***. If also* $\min_{\partial\Omega} q > -1/p$*, then* $-L$ *from* (6.1) *endowed with the domain* $D^0_{reg}(L)$ *is sectorial in* $L^p(\Omega)$*. If* $\max_{\partial\Omega} q \le -1$*, then* $-L$ *on* $D_{reg}(L)$ *is sectorial in* $L^p(\Omega)$*.*

Example 6.5. Let us consider $\Omega = B_1(0)$ in $\mathbb{R}^{N+1}$ and the operator

$$L = -(1 - |\xi|^2)\Delta + c\,\xi \cdot \nabla$$

for c constant. Then $\varrho = 1 - |\xi|^2$ and $\nabla\varrho = -2\xi$. Therefore

$$\frac{\mathbf{b} \cdot \nabla\varrho}{\langle a\nabla\varrho, \nabla\varrho\rangle} = -\frac{c}{2}$$

The above theorems then show:

- If $\frac{2}{p} \le c < 2$ then $-L$ generates an analytic semigroup in $L^p(\Omega)$ endowed with the domain given in Definition 6.1.
- If $c < \frac{2}{p}$ then $-L$ generates an analytic semigroup endowed with domain $D^0_{reg}(L)$.
- If $c \ge 2$, then $-L$ generates an analytic semigroup endowed with domain $D_{reg}(L)$.

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