

EPSILON-REGULARITY FOR THE SOLUTIONS OF A FREE BOUNDARY SYSTEM

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ABSTRACT. This paper is dedicated to a free boundary system arising in the study of a class of shape optimization problems. The problem involves three variables: two functions u and v , and a domain Ω ; with u and v being both positive in Ω , vanishing simultaneously on $\partial\Omega$ and satisfying an overdetermined boundary value problem involving the product of their normal derivatives on $\partial\Omega$. Precisely, we consider solutions $u, v \in C(B_1)$ of

$$-\Delta u = f \quad \text{and} \quad -\Delta v = g \quad \text{in} \quad \Omega = \{u > 0\} = \{v > 0\}, \quad \frac{\partial u}{\partial n} \frac{\partial v}{\partial n} = Q \quad \text{on} \quad \partial\Omega \cap B_1.$$

Our main result is an epsilon-regularity theorem for viscosity solutions of this free boundary system. We prove a partial Harnack inequality near flat points for the couple of auxiliary functions \sqrt{uv} and $\frac{1}{2}(u+v)$. Then, we use the gained space near the free boundary to transfer the improved flatness to the original solutions. Finally, using the partial Harnack inequality, we obtain an improvement-of-flatness result, which allows to conclude that flatness implies $C^{1,\alpha}$ regularity.

1. INTRODUCTION

Let $u, v \in C(B_1)$ be two continuous non-negative functions on the unit ball in \mathbb{R}^d such that

$$\Omega := \{u > 0\} = \{v > 0\}.$$

Suppose that u and v are also solutions of the free boundary problem

$$-\Delta u = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$-\Delta v = 0 \quad \text{in} \quad \Omega, \tag{2}$$

$$\frac{\partial u}{\partial n} \frac{\partial v}{\partial n} = 1 \quad \text{on} \quad \partial\Omega \cap B_1, \tag{3}$$

where the two equations (1) and (2) hold in the classical sense in the open set Ω . On the other hand, since we will not assume that Ω is regular, the boundary condition (3) is to be intended in a generalized sense. Following the classical approach of Caffarelli [8, 9], for simplicity, in the introduction and in [Theorem 1.3](#), we will assume that (3) holds in the sense of [Definition 1.1](#) below. Our main ε -regularity result applies to an even more general notion of solution, but in order to avoid technicalities in the introduction, we postpone this discussion to [Section 2](#).

Definition 1.1 (Definition of solutions). We say that (3) holds, if at any point $x_0 \in \partial\Omega \cap B_1$ at which $\partial\Omega$ admits a one-sided tangent ball we have that the functions u and v can be expanded as

$$u(x) = \alpha((x - x_0) \cdot \nu)_+ + o(|x - x_0|), \tag{4}$$

$$v(x) = \beta((x - x_0) \cdot \nu)_+ + o(|x - x_0|), \tag{5}$$

where ν is a unit vector and α and β are positive real numbers such that $\alpha\beta = 1$.

Our main result is a regularity theorem which applies to solutions, which are sufficiently flat in the sense of the following definition.

Definition 1.2 (Definition of flatness). We say that u and v are ε -flat in B_1 , if there is a unit vector $\nu \in \partial B_1$ and positive constants α and β such that $\alpha\beta = 1$ and

$$\alpha(x \cdot \nu - \varepsilon)_+ \leq u(x) \leq \alpha(x \cdot \nu + \varepsilon)_+ \quad \text{for every} \quad x \in B_1, \tag{6}$$

$$\beta(x \cdot \nu - \varepsilon)_+ \leq v(x) \leq \beta(x \cdot \nu + \varepsilon)_+ \quad \text{for every} \quad x \in B_1. \tag{7}$$

We will also say that u and v are ε -flat in the direction ν .

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We prove the following theorem.

Theorem 1.3. *There is a constant $\varepsilon_0 > 0$ such that the following holds.*

Let u and v be two non-negative continuous functions on B_1 and let $\Omega := \{u > 0\} = \{v > 0\}$.

If u and v are solutions of (1)-(2)-(3) and are ε -flat in B_1 , for some $\varepsilon \in (0, \varepsilon_0]$, then $\partial\Omega$ is $C^{1,\alpha}$ in $B_{1/2}$.

Theorem 1.3 follows directly from Theorem 3.1 (Section 3), in which we prove the same result for a more general notion of solution, which we define in Section 2 in terms of the blow-ups of u and v . The proof of Theorem 3.1 will be given in Section 4 and Section 5.

The rest of the introduction is organized as follows. In Section 1.1 we briefly discuss the relation of the system (1)-(2)-(3) to the well-known one-phase, two-phase and vectorial Bernoulli problems, with which it shares several key features. In Section 1.2 we briefly explain the overall strategy and the novelties of the proof. Finally, in Section 1.3, we discuss the applications of our result to the theory of Shape Optimization.

1.1. The classical one-phase, two-phase and vectorial problems. The free boundary problem (1)-(2)-(3) is a vectorial analogue of the following classical one-phase problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega := \{u > 0\}, \\ |\nabla u| = 1 & \text{on } \partial\Omega \cap B_1, \end{cases} \quad (8)$$

which was introduced by Alt and Caffarelli in [1] in the early 80s. Later, in a series of papers (see [8, 9] and the book [10]), Caffarelli studied the following two-phase problem, in which the solution is given by a single function $u : B_1 \rightarrow \mathbb{R}$ that changes sign:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_+ := \{u > 0\} \text{ and } \Omega_- := \{u < 0\}, \\ (\partial_n^+ u)^2 - (\partial_n^- u)^2 = 1 & \text{on } \partial\Omega_+ \cap \partial\Omega_- \cap B_1, \end{cases} \quad (9)$$

and where the transmission condition on the boundary $\partial\Omega_+ \cap \partial\Omega_-$ is defined in terms of Taylor expansions at points with one-sided tangent ball (contained in Ω_+ or Ω_-), exactly as in Definition 1.1; here n denotes the normal to $\partial\Omega_+ \cap \partial\Omega_-$ at such points. More recently, De Silva [13] gave a different proof to the one-phase ε -regularity theorem from [1]; the method found application to several generalizations of (9) (see [14, 15, 16]) and opened the way to the original two-phase problem of Alt-Caffarelli-Friedman [2], for which the $C^{1,\alpha}$ regularity of the free boundary in every dimension was proved only recently in [12] by a similar argument.

Inspired by a problem arising in the theory of shape optimization, a cooperative vectorial version of the one-phase problem was introduced in [11], [22] and [24]. In this case, the solutions are vector-valued functions

$$U = (u_1, \dots, u_k) : B_1 \rightarrow \mathbb{R}^k$$

satisfying

$$\Delta U = 0 \text{ in } \Omega := \{|U| > 0\}, \quad \sum_{j=1}^k |\nabla u_j|^2 = 1 \text{ on } \partial\Omega \cap B_1. \quad (10)$$

The regularity of the vectorial free boundaries turned out to be quite challenging, especially when it comes to viscosity solutions. This is mainly due to the fact that the regularity techniques, based on the maximum principle and on the comparison of the solutions with suitable test functions (see for instance [13] and [12]), are in general hard to implement in the case of systems. Epsilon-regularity theorems for the vectorial problem (10) were proved in [11, 22, 23, 24, 25, 28] and more recently, in [17], where the regularity of the flat free boundaries was obtained directly for viscosity solutions.

1.2. Outline of the paper and sketch of the proof. The free boundary problem (1)-(2)-(3) is also a vectorial problem and arises in the study of a whole class of shape optimization problems (which we will discuss in Section 1.3). On the other hand, unlike the one-phase ([1]), the two-phase ([2]) and the vectorial problem [11, 24] it doesn't have an underlying variational structure in terms of the functions u and v , thus purely variational arguments as the epiperimetric inequality (see [28]) cannot be applied.

Thus, we prove an improvement-of-flatness result for solutions of (1)-(2)-(3) from which Theorem 1.3 follows by a standard argument (for more details we refer for instance to [29]). Precisely, the aim of the paper is to prove the following theorem (see also the more general Theorem 5.1).

Theorem 1.4 (Improvement of flatness). *There are dimensional constants $\varepsilon_0 > 0$ and $C > 0$ such that the following holds. Let $u : B_1 \rightarrow \mathbb{R}$ and $v : B_1 \rightarrow \mathbb{R}$ be two continuous non-negative functions, which are also solutions to (1)-(2)-(3). Let $\Omega := \{u > 0\} = \{v > 0\}$ and $0 \in \partial\Omega$. If u and v satisfy*

$$(x_d - \varepsilon)_+ \leq u(x) \leq (x_d + \varepsilon)_+ \quad \text{and} \quad (x_d - \varepsilon)_+ \leq v(x) \leq (x_d + \varepsilon)_+ \quad \text{in } B_1,$$

for some $\varepsilon < \varepsilon_0$, then there are a unit vector $\nu \in \mathbb{R}^d$ with $|\nu - e_d| \leq C\varepsilon$ and a radius $\rho \in (0, 1)$ such that

$$\tilde{\alpha}(x \cdot \nu - \varepsilon/2)_+ \leq \frac{u(\rho x)}{\rho} \leq \tilde{\alpha}(x \cdot \nu + \varepsilon/2)_+ \quad \text{and} \quad \tilde{\beta}(x \cdot \nu - \varepsilon/2)_+ \leq \frac{v(\rho x)}{\rho} \leq \tilde{\beta}(x \cdot \nu + \varepsilon/2)_+$$

for all $x \in B_1$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are such that $\tilde{\alpha}\tilde{\beta} = 1$, $|1 - \tilde{\alpha}| \leq C\varepsilon$ and $|1 - \tilde{\beta}| \leq C\varepsilon$.

In order to prove Theorem 1.4, we use the general strategy of De Silva developed in [13] for viscosity solutions of the one-phase problem, which reduces the proof of Theorem 1.4 to two key ingredients (partial Harnack inequality and analysis of the linearized problem) at which are concentrated the whole difficulty of the proof and the insight on the specific problem. The idea is the following. Arguing by contradiction, one considers a sequence of solutions (u_n, v_n) , which are ε_n -flat with $\varepsilon_n \rightarrow 0$, and then produces the linearizing sequence

$$\tilde{u}_n(x) = \frac{u_n(x) - x_d}{\varepsilon_n} \quad \text{and} \quad \tilde{v}_n(x) = \frac{v_n(x) - x_d}{\varepsilon_n}.$$

The argument now can be divided in two main steps.

The first step is to show that \tilde{u}_n and \tilde{v}_n converge (see Section 4) to some functions u_∞ and v_∞ ; this is done by proving a *partial Harnack inequality*, which is the hardest part of the proof. Roughly speaking, the partial Harnack inequality in our case (see Lemma 4.3 for the precise statement) states that if u and v is any couple of ε -flat solutions, for some $\varepsilon < \varepsilon_0$, then there is a constant $c \in (0, 1)$ such that

$$(x_d - (1 - c)\varepsilon)_+ \leq u(x) \leq (x_d + \varepsilon)_+ \quad \text{and} \quad (x_d - (1 - c)\varepsilon)_+ \leq v(x) \leq (x_d + \varepsilon)_+ \quad \text{in } B_{1/2}, \quad (11)$$

or

$$(x_d - \varepsilon)_+ \leq u(x) \leq (x_d + (1 - c)\varepsilon)_+ \quad \text{and} \quad (x_d - \varepsilon)_+ \leq v(x) \leq (x_d + (1 - c)\varepsilon)_+ \quad \text{in } B_{1/2}, \quad (12)$$

that is, the flatness is improved from above or from below, in the same direction e_d , but without the scaling factor that would allow to iterate the statement without going above the threshold ε_0 .

The second step is to show that the u_∞ and v_∞ are solutions to a PDE problem (the so-called *linearized problem* or *limit problem*), from which one can obtain an oscillation decay for u_∞ and v_∞ that can then be transferred back to \tilde{u}_n and \tilde{v}_n , for some n large enough. In our case, the linearized problem is the following system of PDEs on the half-ball $B_1 \cap \{x_d > 0\}$ (see Lemma 5.2, Section 5):

$$\begin{cases} \Delta u_\infty = \Delta v_\infty = 0 & \text{in } B_1 \cap \{x_d > 0\}, \\ u_\infty = v_\infty \quad \text{and} \quad \partial_{x_d} u_\infty + \partial_{x_d} v_\infty = 0 & \text{on } B_1 \cap \{x_d = 0\}. \end{cases} \quad (13)$$

In our case, the most challenging part of the proof is the partial Harnack inequality. In fact, in the one-phase and the two-phase problems (see [13], [14, 15, 16], and [12]) the validity of (11)-(12) is obtained by constructing explicit competitors, which are essentially variations of the constructions in [13]. In our case though, the functions u and v , considered separately, are not solutions (not even sub- or supersolutions) to any free boundary problem. Thus, a key observation in our case, which is inspired by [17] and turns out to be crucial in both the partial Harnack inequality and the proof of (13), is that if u and v are solutions to (1)-(2)-(3), then \sqrt{uv} and $\frac{1}{2}(u + v)$ are respectively a viscosity subsolution and viscosity supersolution of the one-phase problem (8) (see Lemma 2.9 and Remark 4.1¹). Moreover, it is easy to check that both \sqrt{uv} and $\frac{1}{2}(u + v)$ hereditate the flatness of u and v . Thus, by using the competitors from [13] on these functions, we obtain the following dichotomy in $B_{1/2}$:

the flatness $\frac{1}{2}(u + v)$ is improved from above or the flatness of \sqrt{uv} is improved from below.

Now, notice that we cannot transfer this information back to u and v just be an algebraic manipulation; for instance, a bound from below on \sqrt{uv} does not a priori imply a bound from below on both u and v . On the other hand, one can easily notice that the improved flatness of \sqrt{uv} or $\frac{1}{2}(u + v)$, in particular, implies that in $B_{1/2}$ the boundary $\partial\Omega$ is trapped between two nearby translations of a half-space, which are distant at most $(2 - c)\varepsilon$. Using this geometric information and a comparison argument based on the boundary Harnack principle, in Lemma 4.2, we obtain that also the flatness of u and v improves in $B_{1/2}$.

¹This situation is similar to the one of the vectorial problem (10) in which each of the components of U is a viscosity supersolution, while, as it was shown in [24], the modulus $|U|$ satisfies $|\nabla|U|| = 1$ on $\partial\{|U| > 0\}$ and is a viscosity subsolution of (8); this information was used in [17] to prove a partial Harnack inequality and an ε -regularity theorem for the solutions of (10).

1.3. On the boundary condition $\frac{\partial u}{\partial n} \frac{\partial v}{\partial n} = 1$ and its relation to a shape optimization problem. Our result applies to a whole class of shape optimization problems, that is, variational problems of the form

$$\min \left\{ J(\Omega) : \Omega \in \mathcal{A} \right\},$$

where \mathcal{A} is an admissible class of subsets of \mathbb{R}^d and J is a given function on \mathcal{A} . Typically, the admissible set \mathcal{A} is a family of sets of fixed measure, contained in a given bounded open set $D \subset \mathbb{R}^d$, while the functional J is monotone with respect to the set inclusion and depends on the resolvent of an elliptic operator with Dirichlet boundary conditions on $\partial\Omega$. The shape functionals are usually related to models in Engineering, Mechanics and Material Sciences and in most of the cases fall in one of the following main classes: *spectral functionals* and *integral functionals* (for more details we refer to the books [3], [18], [19], [20] and the survey paper [6]).

The spectral functionals are functionals of the form $J(\Omega) = F(\lambda_1(\Omega), \dots, \lambda_k(\Omega))$, where $F : \mathbb{R}^k \rightarrow \mathbb{R}$ is a monotone (in each variable) function and $\lambda_1(\Omega), \dots, \lambda_k(\Omega)$ are the eigenvalues of the Dirichlet Laplacian on Ω . The regularity and the local structure of these optimal sets were studied in [4], [22, 23], [24] (see also [5] and [26] for the special cases $J(\Omega) = \lambda_1(\Omega)$ and $J(\Omega) = \lambda_2(\Omega)$) and are related to the vectorial Bernoulli problem from [11], [24] and [25]. An ε -regularity theorem for general spectral functionals was obtained in [23].

The integral functionals can be written in the general form

$$J(\Omega) = \int_D j(u_\Omega, x) dx, \quad (14)$$

where $j : \mathbb{R} \times D \rightarrow \mathbb{R}$ is a given function and the state function u_Ω is the unique solution of

$$-\Delta u = f \quad \text{in } \Omega, \quad u \in H_0^1(\Omega),$$

the right-hand side $f : D \rightarrow \mathbb{R}$ being a fixed measurable function.

As it was observed already in [7], a free boundary system of the form (1)-(2)-(3) naturally arises in the computation of the first variation of J . This is easy to see if one computes formally the first variation of J for smooth sets. Indeed, if we suppose that Ω is an optimal set and smooth, and that $\xi \in C^\infty(D; \mathbb{R}^d)$ is a compactly supported vector field, then we can define the family of sets $\Omega_t := (Id + t\xi)(\Omega)$ and the family of state functions $u_t := u_{\Omega_t}$. Then, the first variation of J is given by

$$\begin{aligned} \delta J(\Omega)[\xi] &:= \left. \frac{d}{dt} \right|_{t=0} J(\Omega_t) = \left. \frac{d}{dt} \right|_{t=0} \int_{\Omega_t} j(u_t, x) dx \\ &= \left. \frac{d}{dt} \right|_{t=0} \left[\int_D (j(u_t, x) - j(0, x)) dx + \int_{\Omega_t} j(0, x) dx \right] \\ &= \int_\Omega u' \frac{\partial j}{\partial u}(u_\Omega, x) dx + \int_{\partial\Omega} j(0, x) \xi \cdot n_\Omega d\mathcal{H}^{d-1}, \end{aligned}$$

where $n_\Omega(x)$ to be the exterior normal at $x \in \partial\Omega$ and the formal derivative u' (of u_t at $t = 0$) is the solution of the boundary value problem

$$\Delta u' = 0 \quad \text{in } \Omega, \quad u' = -\xi \cdot \nabla u_\Omega \quad \text{on } \partial\Omega,$$

in which the condition on $\partial\Omega$ is a consequence of the fact that, given $x \in \partial\Omega$, we have

$$u_t(x + t\xi(x)) = 0 \quad \text{for every } t \in \mathbb{R}.$$

We next define the function

$$g(x) := -\frac{\partial j}{\partial u}(u_\Omega(x), x),$$

and the solution v_Ω of the problem

$$-\Delta v = g \quad \text{in } \Omega, \quad v \in H_0^1(\Omega).$$

Remark 1.5. Before we continue with the computation of the first variation, we notice that, in order to have the monotonicity of J , it is natural to assume that $f \geq 0$ and $\frac{\partial j}{\partial u} \leq 0$, which of course implies that $g \geq 0$ and that both u_Ω and v_Ω are non-negative. On the other hand, if f and $\frac{\partial j}{\partial u}$ change sign, then in general an optimal set might not exist (see for instance [7]).

In order to complete the computation of $\delta J(\Omega)[\xi]$, we integrate by parts in Ω , obtaining

$$-\int_\Omega u' g(x) dx = \int_\Omega u' \Delta v_\Omega dx = -\int_\Omega \nabla u' \cdot \nabla v_\Omega dx + \int_{\partial\Omega} u' \frac{\partial u_\Omega}{\partial n} = \int_{\partial\Omega} u' \frac{\partial u_\Omega}{\partial n}.$$

But now since ∇u_Ω is parallel to n_Ω at the boundary, we have that $u' = -\xi \cdot \nabla u_\Omega = -(\xi \cdot n_\Omega)(n_\Omega \cdot \nabla u_\Omega)$. Thus, the first variation of J is

$$\delta J(\Omega)[\xi] = \int_{\partial\Omega} \left(-\frac{\partial u_\Omega}{\partial n} \frac{\partial v_\Omega}{\partial n} + j(0, x) \right) n_\Omega \cdot \xi.$$

Since the vector field ξ is arbitrary and since Ω is a minimizer among the sets of prescribed measure, we get that in a neighborhood $B_r(x_0)$ of a point x_0 of the free boundary $\partial\Omega \cap D$, u_Ω and v_Ω are solutions of the system

$$\begin{cases} -\Delta u = f & \text{in } \Omega \cap B_r(x_0), \\ -\Delta v = g & \text{in } \Omega \cap B_r(x_0), \\ \frac{\partial u}{\partial n} \frac{\partial v}{\partial n} = c + j(0, x) & \text{on } \partial\Omega \cap B_r(x_0), \end{cases}$$

where c is a positive constant.

Our definition [Definition 1.1](#) is a generalization of the notion of solution and was proposed by Caffarelli in [\[8, 9\]](#) in the context of a two-phase free boundary problem. In a subsequent paper we will use [Theorem 1.3](#) to obtain a regularity result for optimal sets for functionals of the form [\(14\)](#). We notice that in [Theorem 1.3](#) we do not assume that the functions u and v are minimizers of a functional or solutions of a shape optimization problem of any kind, so this result is of independent interest and can be seen as a one-phase vectorial version of the classical results of Caffarelli [\[8, 9\]](#).

2. ON THE VISCOSITY FORMULATIONS OF SOLUTION

In this section we briefly discuss the boundary condition [\(3\)](#). In particular, in [Definition 2.4](#) we give a more general notion of solution, which we will use throughout the paper. We start by recalling the following definition.

Definition 2.1 (One sided tangent balls). Let $\Omega \subset \mathbb{R}^d$ be an open set and let $x_0 \in \partial\Omega$. We say that Ω admits a one-sided tangent ball at x_0 if one of the following conditions hold:

- (i) there are $r > 0$ and $y_0 \in \Omega$ such that $B_r(y_0) \subset \Omega$ and $\partial B_r(y_0) \cap \partial\Omega = \{x_0\}$,
- (ii) there are $r > 0$ and $y_0 \in \mathbb{R}^d \setminus \bar{\Omega}$ such that $B_r(y_0) \subset \mathbb{R}^d \setminus \bar{\Omega}$ and $\partial B_r(y_0) \cap \partial\Omega = \{x_0\}$.

Moreover, we will use the notation

$$\nu_{x_0, y_0} := \frac{y_0 - x_0}{|y_0 - x_0|} \quad \text{in the case (i),} \quad \text{and} \quad \nu_{x_0, y_0} := -\frac{y_0 - x_0}{|y_0 - x_0|} \quad \text{in the case (ii).} \quad (15)$$

We notice that when Ω is regular, the vector ν_{x_0, y_0} is the inner normal to $\partial\Omega$ at x_0 , while for non-smooth domains it may depend on the ball $B_r(y_0)$.

Let $Q : B_1 \rightarrow \mathbb{R}$ be a $C^{0, \alpha}$ -regular function (for some $\alpha > 0$) and suppose that:

$$\text{There is } C_Q \geq 1 \text{ such that } C_Q^{-1} \leq Q(x) \leq C_Q \text{ for all } x \in B_1. \quad (16)$$

Then, [Definition 1.1](#) can be generalized as follows.

Definition 2.2 (Definition of solutions I). Given two continuous non-negative functions $u, v : B_1 \rightarrow \mathbb{R}$ with the same support $\Omega = \{u > 0\} = \{v > 0\}$, we say that

$$\frac{\partial u}{\partial n} \frac{\partial v}{\partial n} = Q \quad \text{on } \partial\Omega \cap B_1,$$

if at any point $x_0 \in \partial\Omega \cap B_1$, for which $\partial\Omega$ admits a one-sided tangent ball at x_0 , we have that the functions u and v can be expanded as

$$u(x) = \alpha ((x - x_0) \cdot \nu)_+ + o(|x - x_0|), \quad (17)$$

$$v(x) = \beta ((x - x_0) \cdot \nu)_+ + o(|x - x_0|), \quad (18)$$

where ν is the unit vector given by [\(15\)](#) and α and β are positive real numbers such that:

$$\alpha\beta = Q(x_0).$$

In particular, this definition implies that if $\partial\Omega$ admits a one-sided tangent ball at $x_0 \in \partial\Omega$, then this tangent ball is unique, which of course excludes a priori domains with angles and cusps. Moreover, it implies that at such points the blow-ups of u and v are unique. Now, since in many situations this is not a priori known, we will work with couples which are solutions in the more general sense described in the next subsection.

2.1. A more general notion of solution. For every $x_0 \in \partial\Omega \cap B_1$ and every $r > 0$ small enough, we define

$$u_{r,x_0}(x) = \frac{1}{r}u(x_0 + rx) \quad \text{and} \quad v_{r,x_0}(x) = \frac{1}{r}v(x_0 + rx).$$

Throughout the paper we will also adopt the notation $u_r := u_{r,0}$ and $v_r := v_{r,0}$. Therefore, [Definition 2.2](#) can be expressed in terms of the rescaling u_{r,x_0} and v_{r,x_0} in the following way.

Remark 2.3. Let $u, v : B_1 \rightarrow \mathbb{R}$ be two non-negative continuous functions with the same support

$$\Omega := \{u > 0\} = \{v > 0\}.$$

Then, the following are equivalent

- (1) $\frac{\partial u}{\partial n} \frac{\partial v}{\partial n} = Q$ on $\partial\Omega \cap B_1$, in the sense of [Definition 2.2](#).
- (2) At any point $x_0 \in \partial\Omega \cap B_1$, for which one of the conditions (i) and (ii) of [Definition 2.1](#) hold, we have that u_{r,x_0} and v_{r,x_0} converge uniformly in B_1 as $r \rightarrow 0$ respectively to the functions

$$x \mapsto \alpha((x - x_0) \cdot \nu)_+ \quad \text{and} \quad x \mapsto \beta((x - x_0) \cdot \nu)_+ \quad (19)$$

where α and β are positive constants such that $\alpha\beta = Q(x_0)$ and $\nu \in \mathbb{R}^d$ is the unit vector given by [\(15\)](#).

In particular, [Remark 2.3](#) implies that [Definition 2.2](#) can be generalized as follows.

Definition 2.4 (Definition of solutions II). Given two continuous non-negative functions $u, v : B_1 \rightarrow \mathbb{R}$ with the same support $\Omega = \{u > 0\} = \{v > 0\}$, we say that

$$\frac{\partial u}{\partial n} \frac{\partial v}{\partial n} = Q \quad \text{on} \quad \partial\Omega \cap B_1,$$

if at any point $x_0 \in \partial\Omega \cap B_1$, for which one of the conditions (i) and (ii) of [Definition 2.1](#) hold, there exist:

- a decreasing sequence $r_n \rightarrow 0$;
- two positive constants $\alpha > 0$ and $\beta > 0$ such that $\alpha\beta = Q(x_0)$;
- a unit vector $\nu \in \mathbb{R}^d$;

such that the sequences u_{r_n,x_0} and v_{r_n,x_0} converge uniformly in B_1 respectively to the functions

$$u_0(x) := \alpha(x \cdot \nu)_+ \quad \text{and} \quad v_0(x) := \beta(x \cdot \nu)_+ . \quad (20)$$

Remark 2.5. We will say that u_0 and v_0 are blow-up limits of u and v at x_0 . We notice that the blow-up limits at a point may not be unique as they a priori depend on the sequence $r_n \rightarrow 0$.

Remark 2.6. [Theorem 1.3](#) holds also for solutions u and v of (1)-(2)-(3) in the sense of [Definition 2.4](#). In fact, the entire proof of this theorem will be given for solutions in the sense of [Definition 2.4](#).

Remark 2.7. The sequence $r_n \rightarrow 0$ from [Definition 2.4](#) may depend on the tangent ball $B_r(y_0)$ at x_0 (which in turn may not be unique). Thus, in [Definition 2.4](#) we do not assume that the blow-ups of u and v at x_0 , as well as the the tangent ball $B_r(y_0)$, are unique.

2.2. Optimality conditions in viscosity sense. In view of the last notion of solutions, we can finally state the viscosity formulation of the free boundary condition introduced in [Definition 2.4](#).

Definition 2.8. Let $u : B_1 \rightarrow \mathbb{R}$ be a continuous non-negative function, $\varphi \in C^\infty(\mathbb{R}^d)$ be given and

$$\varphi_+(x) := \max\{\varphi(x), 0\}.$$

- We say that φ_+ touches u from below at a point $x_0 \in \partial\{u > 0\} \cap B_1$ if $u(x_0) = \varphi(x_0) = 0$ and
$$\varphi_+(x) \leq u(x) \quad \text{for every } x \text{ in a neighborhood of } x_0.$$
- We say that φ_+ touches u from above at a point $x_0 \in \partial\{u > 0\} \cap B_1$ if $u(x_0) = \varphi(x_0) = 0$ and
$$\varphi_+(x) \geq u(x) \quad \text{for every } x \text{ in a neighborhood of } x_0.$$

Lemma 2.9. *Suppose that u and v satisfy*

$$\frac{\partial u}{\partial n} \frac{\partial v}{\partial n} = 1 \quad \text{on} \quad \partial\Omega \cap B_1$$

in the sense of [Definition 2.4](#), where $\Omega = \{u > 0\} = \{v > 0\}$. Then, the following holds:

- (a) *If φ_+ touches \sqrt{uv} from below at a point $x_0 \in B_1 \cap \partial\Omega$, then $|\nabla\varphi(x_0)| \leq 1$.*
- (b) *If φ_+ touches \sqrt{uv} from above at a point $x_0 \in B_1 \cap \partial\Omega$, then $|\nabla\varphi(x_0)| \geq 1$.*

(c) If a and b are constants such that

$$a > 0, \quad b > 0 \quad \text{and} \quad ab = 1,$$

and if φ_+ touches the function $w_{ab} := \frac{1}{2}(au + bv)$ from above at $x_0 \in B_1 \cap \partial\Omega$, then $|\nabla\varphi(x_0)| \geq 1$.

Proof. We start by proving (c). Suppose that the function φ_+ touches w_{ab} from above at $x_0 \in \partial\Omega$. Then, there is a ball touching $\partial\Omega$ at x_0 from outside (in the sense of [Definition 2.1](#) (ii)). But then, by [Definition 2.4](#), there are blow-up limits of u and v given respectively by

$$u_0(x) = \alpha(x \cdot \nu)_+ \quad \text{and} \quad v_0(x) = \beta(x \cdot \nu)_+. \quad (21)$$

Moreover, since φ is smooth, the blow-up of φ_+ is given by

$$\varphi_0(x) := (x \cdot \nabla\varphi(x_0))_+. \quad (22)$$

By hypothesis, we have that φ_0 touches from above (at zero) the function

$$x \mapsto \frac{1}{2}(au_0(x) + bv_0(x)) = \frac{a\alpha + b\beta}{2}(x \cdot \nu)_+.$$

Now, since $\frac{a\alpha + b\beta}{2} \geq \sqrt{\alpha\alpha\beta\beta} = 1$, we have that φ_0 touches from above (again in zero) also the function

$$x \mapsto (x \cdot \nu)_+.$$

Thus, $\nabla\varphi(x_0) = \nu$ and in particular $|\nabla\varphi(x_0)| \geq 1$.

We next prove (a) and (b). If φ_+ touches \sqrt{uv} from below (resp. above) at $x_0 \in \partial\Omega$, then $\partial\Omega$ has an interior (resp. exterior) tangent ball at x_0 . In particular, by [Definition 2.4](#), u and v have blow-ups u_0 and v_0 given by [\(21\)](#). But then the function

$$\sqrt{u_0v_0} = (x \cdot \nu)_+$$

is a blow-up limit of \sqrt{uv} . Now, using again that the blow-up limit of φ_+ is given by [\(22\)](#), we get the claim. \square

Remark 2.10. We will say that u_0 and v_0 are blow-up limits of u and v at x_0 . We notice that the blow-up limits at a point may not be unique as they a priori depend on the sequence $r_n \rightarrow 0$.

3. STATEMENT OF THE MAIN THEOREM

We now give the statement of our main theorem, which is a generalization of [Theorem 1.3](#).

Theorem 3.1. *Given $f, g \in L^\infty(B_1)$ non-negative and $Q \in C^{0,\alpha}(B_1)$, consider $u, v \in C(B_1)$ non-negative function which have the same support in B_1 and set $\Omega := \{u > 0\} = \{v > 0\}$.*

Suppose, moreover, that u and v are solutions of the system

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ -\Delta v = g & \text{in } \Omega, \\ \frac{\partial u}{\partial n} \frac{\partial v}{\partial n} = Q & \text{on } \partial\Omega \cap B_1, \end{cases} \quad (23)$$

where the free boundary condition holds in the sense of [Definition 2.4](#).

Then, there is $\varepsilon > 0$ such that if u and v are ε -flat in B_1 and

$$\|f\|_{L^\infty(B_1)} + \|g\|_{L^\infty(B_1)} \leq \varepsilon^2 \quad \text{and} \quad \|Q(x) - 1\|_{L^\infty(B_1)} \leq \varepsilon,$$

then $\partial\Omega$ is $C^{1,\alpha}$ in $B_{1/2}$.

Remark 3.2. We notice that the presence of f , g and Q entails only some minor technical adjustments of the proof. The main difference with respect to [Theorem 1.3](#) is in the fact that the boundary condition is given by [Definition 2.4](#) instead of [Definition 2.2](#). In fact, what we will use in the proof is not even [Definition 2.4](#), but the comparison properties listed in [Lemma 2.9](#).

Remark 3.3. The positivity assumption on f and g is technical and is only required for the estimate on the Laplacian of \sqrt{uv} (see [Remark 4.1](#) below). Without this assumption one should know that the functions u and v are comparable on Ω , i.e. that u/v is bounded away from zero and infinity.

4. A PARTIAL HARNACK INEQUALITY

In this Section we prove a Harnack type inequality for solutions to (23) in the spirit of [13]. In our case the boundary condition does not allow us to work separately with the solutions u and v . Our strategy consists in tracking the improvement of the auxiliary functions $\frac{1}{2}(u+v)$ and \sqrt{uv} , in order to trap the boundary $\partial\Omega$ between nearby translations of a half-space. Only at this point, by exploiting the gained space, we are able to improve the estimates on the solution (u, v) .

Remark 4.1. Our approach relies on the fact that the two auxiliary functions are respectively subsolution and supersolution for the scalar one phase problem (8). Indeed, if u and v are harmonic in Ω and satisfy Lemma 2.9 (c), then $w := \frac{1}{2}(u+v)$ is a subsolution since

$$\Delta w = 0 \quad \text{in } \Omega, \quad |\nabla w| \geq 1 \quad \text{on } \partial\Omega.$$

On the other hand, the function $z := \sqrt{uv}$ is a supersolution:

$$\Delta z \leq 0 \quad \text{in } \Omega, \quad |\nabla z| = 1 \quad \text{on } \partial\Omega.$$

The boundary condition follows again from Lemma 2.9, while the superharmonicity in Ω follows from the fact that if $u, v : \Omega \rightarrow \mathbb{R}$ are two positive and superharmonic functions on an open set Ω , then

$$\Delta(\sqrt{uv}) = \operatorname{div} \left(\frac{u\nabla v + v\nabla u}{2\sqrt{uv}} \right) = \frac{(u\Delta v + v\Delta u)}{2\sqrt{uv}} - \frac{|u\nabla v - v\nabla u|^2}{4(uv)^{3/2}} \leq \frac{(u\Delta v + v\Delta u)}{2\sqrt{uv}} \leq 0.$$

We first show that if one is able trap the set Ω between two nearby translations of a half-space, then one can also improve the flatness estimates on u and v . This general principle is formulated in the following lemma.

Lemma 4.2. *Let $\varepsilon > 0$ and $\phi \in C(B_1)$ be a non-negative solution of*

$$-\Delta\phi = f \quad \text{in } B_1 \cap \{\phi > 0\},$$

with $f \in L^\infty(B_1)$. Assume that

$$\gamma(x_d + a)_+ \leq \phi(x) \leq \gamma(x_d + a + \varepsilon)_+ \quad \text{for all } x \in B_1, \quad (24)$$

with $|a| \leq 1$ and either

$$\Omega \supset \{x_d + a + C\varepsilon > 0\} \cap B_{1/4} \quad \text{or} \quad \Omega \subset \{x_d + a + (1-C)\varepsilon > 0\} \cap B_{1/4}, \quad (25)$$

for some universal $C \in (0, 1)$. Then, there exists $\delta \in (0, 1), \rho \in (0, 1)$ dimensional constants, such that either

$$\phi(x) \geq \gamma(x_d + a + \delta\varepsilon)_+ \quad \text{or} \quad \phi(x) \leq \gamma(x_d + a + (1-\delta)\varepsilon)_+,$$

for every $x \in B_\rho$.

Proof. Let us proceed by dividing the proof in two cases.

Case 1. $\Omega \supset \{x_d + a + C\varepsilon > 0\} \cap B_{1/4}$. Set $D = B_{1/4} \setminus \{x_d \leq -a - C\varepsilon\}$ and consider the function

$$\begin{cases} \Delta\varphi = 0 & \text{in } D \\ \varphi = 0 & \text{in } B_{1/4} \setminus D, \\ \varphi = w & \text{on } \partial B_{1/4}. \end{cases} \quad \text{with } w = \phi - \frac{1}{2}(x_d + a + C\varepsilon)_+^2 \|f\|_{L^\infty(B_1)}.$$

Since $\phi > 0$ in D we have $-\Delta\phi \geq -\|f\|_{L^\infty(B_1)}$ in D and consequently $-\Delta w \geq 0$ in D . Then, by applying the maximum principle in D , we get $\varphi \leq w \leq \phi$ in D . On the other hand, since $\phi \geq 0 = \varphi$ in $B_{1/4} \setminus D$, it follows that $\varphi \leq \phi$ in $B_{1/4}$. Therefore, we claim that

$$\varphi \geq \gamma(x_d + a + \delta\varepsilon)_+ \quad \text{for all } x \in B_{1/32}, \quad (26)$$

for some universal $\delta \in (0, 1)$, from which the desired inequality follows immediately. By (24) we know that

$$\gamma(x_d + a + C\varepsilon)_+ \leq \gamma \left(x_d + a + \frac{1+C}{2}\varepsilon \right)_+ \leq \phi + \frac{1+C}{2}\gamma\varepsilon \quad \text{in } \overline{B}_{1/4},$$

and since $\varphi = \phi$ on $\partial B_{1/4}$, we get

$$\gamma(x_d + a + C\varepsilon)_+ \leq \varphi + \frac{1+C}{2}\gamma\varepsilon \quad \text{on } \partial B_{1/4}.$$

Therefore, by applying the maximum principle in D , we get

$$\gamma(x_d + a + C\varepsilon)_+ - \varphi \leq \frac{1+C}{2}\gamma\varepsilon \quad \text{in } B_{1/4}.$$

Consider now the function

$$\begin{cases} \Delta h = 0 & \text{in } D \\ h = 0 & \text{in } B_{3/16} \setminus D \\ h = \frac{1+C}{2}\gamma\varepsilon & \text{on } \partial B_{3/16}. \end{cases}$$

Clearly, $0 \leq h \leq \frac{1+C}{2}\gamma\varepsilon$ and by the maximum principle

$$\gamma(x_d + a + C\varepsilon)_+ - \varphi \leq h \quad \text{in } B_{3/16}. \quad (27)$$

Now, by the boundary Harnack inequality, if we set $\bar{x} = 1/8e_d$ we get

$$h(x) \leq C_1 \frac{h(\bar{x})}{(1/8 + a + C\varepsilon)_+} (x_d + a + C\varepsilon)_+ \leq C_2 \gamma\varepsilon (x_d + a + C\varepsilon)_+ \quad \text{in } B_{1/8},$$

for some universal constants $C_1, C_2 > 0$. This last inequality, together with (27), leads to

$$(1 - C_2\varepsilon)\gamma(x_d + a + C\varepsilon)_+ \leq \varphi \quad \text{in } B_{1/8}.$$

On the other hand, since there exists $\delta \in (0, 1)$, $\delta < C$ and $\rho < 1/8$ such that

$$(x_d + a + \delta\varepsilon)_+ \leq (1 - C_2\varepsilon)(x_d + a + C\varepsilon)_+ \quad \text{for all } x \in B_\rho,$$

we obtain the desired claim (26).

Case 2. $\Omega \subset \{x_d + a + (1 - C)\varepsilon > 0\} \cap B_{1/4}$. Set $D = B_{1/4} \setminus \{x_d \leq -a - (1 - C)\varepsilon\}$ and consider the function

$$\begin{cases} \Delta\varphi = 0 & \text{in } D \\ \varphi = 0 & \text{in } B_{1/4} \setminus D, \\ \varphi = w & \text{on } \partial B_{1/4} \end{cases} \quad \text{with } w = \phi + \frac{1}{2}(x_d + a + (1 - C)\varepsilon)_+^2 \|f\|_{L^\infty(B_1)}.$$

Notice that $w > 0$ and $-\Delta w \leq 0$ in D . Therefore, by maximum principle, since $w = 0$ in $B_{1/4} \setminus D$ we get that $\varphi \geq w$ in $B_{1/4}$. Therefore, since $w \geq \phi$ in $B_{1/4}$, we have $\varphi \geq \phi$ in $B_{1/4}$. Let us claim that

$$\varphi \leq \gamma(x_d + a + (1 - \delta)\varepsilon)_+ \quad \text{for all } x \in B_{1/32},$$

for some universal $\delta \in (0, 1)$, from which the desired inequality follows immediately. By (24) we know that

$$\phi - \frac{1+C}{2}\gamma\varepsilon \leq \gamma \left(x_d + a + \frac{1-C}{2}\varepsilon \right)_+ \leq \gamma(x_d + a + (1 - C)\varepsilon)_+ \quad \text{in } \bar{B}_{1/4},$$

and since $\varphi = \phi$ on $\partial B_{1/4}$, we get

$$\phi - \gamma(x_d + a + (1 - C)\varepsilon)_+ \leq \frac{1+C}{2}\gamma\varepsilon \quad \text{on } \partial B_{1/4}.$$

Therefore, by applying the maximum principle in D , we get

$$\phi - \gamma(x_d + a + (1 - C)\varepsilon)_+ \leq \frac{1+C}{2}\gamma\varepsilon \quad \text{in } B_{1/4}.$$

Consider now the function

$$\begin{cases} \Delta h = 0 & \text{in } D \\ h = 0 & \text{in } B_{3/16} \setminus D \\ h = \frac{1+C}{2}\gamma\varepsilon & \text{on } \partial B_{3/16}. \end{cases}$$

Clearly, $0 \leq h \leq \frac{1+C}{2}\gamma\varepsilon$ and by the maximum principle

$$\phi - \gamma(x_d + a + (1 - C)\varepsilon)_+ \leq h \quad \text{in } B_{3/16}. \quad (28)$$

Now, by the boundary Harnack inequality, if we set $\bar{x} = 1/8e_d$ we get

$$h(x) \leq C_1 \frac{h(\bar{x})}{(1/8 + a + (1 - C)\varepsilon)_+} (x_d + a + (1 - C)\varepsilon)_+ \leq C_2 \gamma\varepsilon (x_d + a + (1 - C)\varepsilon)_+ \quad \text{in } B_{1/8},$$

for some universal constants $C_1, C_2 > 0$. This last inequality, together with (28), leads to

$$\phi \leq (1 + C_2)\gamma(x_d + a + (1 - C)\varepsilon)_+ \quad \text{in } B_{1/8}.$$

On the other hand, since there exists $\delta \in (0, 1)$, $\delta < C$ such that

$$(1 + C_2)(x_d + a + (1 - C)\varepsilon)_+ \leq (x_d + a + (1 - \delta)\varepsilon)_+ \quad \text{for all } x \in B_R,$$

we conclude the proof. \square

Finally, we can prove a partial Harnack inequality for solutions to (23). We notice that this result will be applied to the rescalings u_{r,x_0} and v_{r,x_0} of the solutions u and v at some point $x_0 \in \bar{\Omega}$, where as usual we set $\Omega = \{u > 0\} = \{v > 0\}$.

Lemma 4.3 (Partial Harnack). *Given a constant $K > 0$, there exists $\varepsilon_0, \rho > 0$ such that the following holds. If u and v are solutions of (23) in the sense of Definition 2.4 and are such that $0 \in \bar{\Omega}$ and*

$$\begin{aligned} \alpha(x_d + a)_+ \leq u(x) \leq \alpha(x_d + b)_+ & \quad \text{for all } x \in B_1, \\ \beta(x_d + a)_+ \leq v(x) \leq \beta(x_d + b)_+ & \quad \text{for all } x \in B_1, \end{aligned}$$

for some α and β satisfying

$$0 < \alpha \leq K, \quad 0 < \beta \leq K \quad \text{and} \quad \alpha\beta = 1,$$

and for some a and b such that

$$|a| < \frac{1}{10}, \quad |b| < \frac{1}{10} \quad \text{and} \quad b - a \leq \varepsilon_0,$$

then there are \tilde{a}, \tilde{b} satisfying $\tilde{b} - \tilde{a} \leq (1 - \delta)(b - a)$ for some $\delta > 0$ such that

$$\begin{aligned} \alpha(x_d + \tilde{a})_+ \leq u(x) \leq \alpha(x_d + \tilde{b})_+ \\ \beta(x_d + \tilde{a})_+ \leq v(x) \leq \beta(x_d + \tilde{b})_+ \end{aligned}'$$

hold for all $x \in B_\rho$, with $\rho < 1/8$ a dimensional constant.

Proof. As in [13], we fix a point $\bar{x} := e_d/5$ and we consider the function $w : \mathbb{R}^d \rightarrow \mathbb{R}$ defined as

$$w(x) := \begin{cases} 1 & \text{if } x \in B_{1/20}(\bar{x}), \\ 0 & \text{if } x \notin B_{3/4}(\bar{x}), \\ \bar{c}(|x - \bar{x}|^{-d} - (3/4)^{-d}) & \text{if } x \in B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x}), \end{cases}$$

where $\bar{c} := 20^d - (4/3)^d$. Notice that the function w is nonzero exactly on $B_{3/4}(\bar{x})$ and it satisfies the following properties on $B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x})$:

- (w-i) the function is subharmonic since $\Delta w(x) = 2d\bar{c}|x - \bar{x}|^{-d-2} \geq 2d\bar{c}(3/4)^{-d-2} > 0$;
- (w-ii) $\partial_{x_d} w$ is strictly positive on the half-space $\{x_d < 1/10\}$.

Let us proceed by dividing the proof in three steps.

Step 1. Invariant transformation and flatness estimates. Now, we consider two functions u and v satisfying the hypotheses of the lemma. First we notice that

$$\frac{1}{10K} \leq u(\bar{x}) \leq 10K \quad \text{and} \quad \frac{1}{10K} \leq v(\bar{x}) \leq 10K.$$

Thus, there is a constant

$$\frac{1}{100K} \leq c \leq 100K$$

such that the couple $\tilde{u} = cu$, $\tilde{v} = c^{-1}v$ satisfies

$$\tilde{u}(\bar{x}) = \tilde{v}(\bar{x}),$$

is also a solution to (1)-(2)-(3) and

$$\begin{cases} \tilde{\alpha}(x_d + a)_+ \leq \tilde{u}(x) \leq \tilde{\alpha}(x_d + b)_+ & \text{for all } x \in B_1, \\ \tilde{\beta}(x_d + a)_+ \leq \tilde{v}(x) \leq \tilde{\beta}(x_d + b)_+ & \text{for all } x \in B_1, \end{cases} \quad (29)$$

where

$$\tilde{\alpha} := c\alpha, \quad \tilde{\beta} := c^{-1}\beta \quad \text{and} \quad \tilde{\alpha}\tilde{\beta} = 1.$$

We define the positive constants δ and ε as follows. We set

$$\varepsilon := b - a < \varepsilon_0,$$

and, without loss of generality, we assume

$$\tilde{\alpha} := 1 + \delta \geq 1 \geq \frac{1}{1 + \delta} = \tilde{\beta}.$$

Next, we notice that since $\tilde{u}(\bar{x}) = \tilde{v}(\bar{x})$, by (29), we have

$$\tilde{\alpha}\left(\frac{1}{5} + a\right) \leq \tilde{\beta}\left(\frac{1}{5} + b\right),$$

and thus,

$$1 + \delta = \tilde{\alpha} \leq \frac{1/5 + b}{1/5 + a} \tilde{\beta} \leq \frac{1/5 + b}{1/5 + a} = 1 + \frac{b - a}{1/5 + a} \leq 1 + 10\varepsilon.$$

In particular, this implies that $\delta \leq 10\varepsilon$ and

$$1 \geq \tilde{\beta} = \frac{1}{1 + \delta} \geq 1 - \delta \geq 1 - 10\varepsilon.$$

Finally, we obtain

$$1 \leq \frac{\tilde{\alpha} + \tilde{\beta}}{2} = \frac{1}{2} \left(1 + \delta + \frac{1}{1 + \delta} \right) \leq 1 + \delta^2 \leq 1 + 100\varepsilon^2. \quad (30)$$

This, together with (29), implies that for all $x \in B_1$ we have

$$\begin{cases} (x_d + a)_+ \leq \sqrt{\tilde{u}(x)\tilde{v}(x)} \leq (x_d + b)_+ \\ (x_d + a)_+ \leq \frac{1}{2}(\tilde{u}(x) + \tilde{v}(x)) \leq (1 + 100\varepsilon^2)(x_d + b)_+, \end{cases} \quad (31)$$

with

$$\tilde{u}(\bar{x}) = \tilde{v}(\bar{x}) = \sqrt{\tilde{u}(\bar{x})\tilde{v}(\bar{x})} = \frac{\tilde{u}(\bar{x}) + \tilde{v}(\bar{x})}{2}.$$

Now, using again (29) and choosing ε_0 such that $\delta \leq 10\varepsilon \leq 10\varepsilon_0 \leq \frac{1}{2}$, we have

$$|\tilde{u} - \tilde{v}| \leq 2\varepsilon \quad \text{on } B_1.$$

Moreover, since $a, b \leq 1/10$,

$$1 \geq \tilde{u} \geq \frac{1}{40} \quad \text{on } B_{1/20}(\bar{x}).$$

This implies that we can choose ε_0 small enough such that on $B_{1/20}(\bar{x})$ we have

$$0 \leq \frac{\tilde{u} + \tilde{v}}{2} - \sqrt{\tilde{u}\tilde{v}} = \tilde{u} \left(1 + \frac{1}{2} \frac{\tilde{v} - \tilde{u}}{\tilde{u}} \right) - \tilde{u} \sqrt{1 + \frac{\tilde{v} - \tilde{u}}{\tilde{u}}} \leq C\varepsilon^2, \quad (32)$$

where C is a numerical constant.

Step 2. Gaining space for the domain Ω . We now reason as in [13] and [17] by considering two cases:

Case 1. $\frac{\tilde{u}(\bar{x}) + \tilde{v}(\bar{x})}{2} \geq \frac{\varepsilon}{2} + (\bar{x}_d + a)_+$. Since $|a| < 1/10$, we have $B_{1/10}(\bar{x}) \subset \{x_d + a > 0\}$ and so the function

$$h(x) := \frac{\tilde{u}(x) + \tilde{v}(x)}{2} - (\bar{x}_d + a)_+$$

is non-negative and solves a uniformly elliptic equation in $B_{1/10}(\bar{x})$ with right-hand side bounded from above and below by ε^2 . Therefore, since $h(\bar{x}) \geq \frac{\varepsilon}{2}$, the Harnack inequality gives that

$$h \geq C_{\mathcal{H}}\varepsilon \quad \text{in } B_{1/20}(\bar{x}),$$

where $C_{\mathcal{H}}$ is a (small) positive constant depending only on the dimension d . Now, using (32) and choosing ε_0 small enough (depending on the dimension), we get that

$$\sqrt{\tilde{u}\tilde{v}} - (x_d + a) \geq \frac{1}{2}C_{\mathcal{H}}\varepsilon \quad \text{in } B_{1/20}(\bar{x}).$$

Now consider the family of functions

$$\psi_t(x) := x_d + a + \frac{1}{2}C_{\mathcal{H}}\varepsilon(w(x) - 1) + \frac{1}{2}C_{\mathcal{H}}\varepsilon t,$$

defined for $t \geq 0$ and $x \in B_1$. So far we proved that

$$\sqrt{\tilde{u}(x)\tilde{v}(x)} > (\psi_t(x))_+ \quad \text{for every } x \in B_{1/20}(\bar{x}) \quad \text{and every } t < 1. \quad (33)$$

We will show that the same inequality holds for every $x \in B_1$. Notice that the family of functions ψ_t satisfies, as a consequence of (w-i) and (w-ii) respectively, the following properties:

- (ψ -i) $\Delta\psi_t \geq C\varepsilon > 0$ on $B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x})$;
- (ψ -ii) $|\nabla\psi_t|(x) > 1$ on $(B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x})) \cap \{x_d < 1/10\}$.

We argue by contradiction. Suppose that for some $t < 1$ there exists $y \in B_1$ such that ψ_t touches $\sqrt{\tilde{u}\tilde{v}}$ from below at y . By Remark 4.1 and (ψ -i), we have that $y \notin \Omega \cap (B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x}))$. On the other hand, by (ψ -ii) and Lemma 2.9 we have that $y \notin \partial\Omega \cap (B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x}))$, which is a contradiction. Thus,

$$\sqrt{\tilde{u}(x)\tilde{v}(x)} > \left(x_d + a + \frac{1}{2}C_{\mathcal{H}}\varepsilon w(x)\right)_+ \quad \text{for every } x \in B_1, \quad (34)$$

and in particular,

$$\Omega \supset \{x \in B_1 : x_d + a + \frac{1}{2}C_{\mathcal{H}}\varepsilon w(x) > 0\}. \quad (35)$$

Case 2. $\frac{\tilde{u}(\bar{x}) + \tilde{v}(\bar{x})}{2} \leq \frac{\varepsilon}{2} + (\bar{x}_d + a)_+$, which is equivalent to

$$\frac{\varepsilon}{2} \leq \bar{x}_d + b - \frac{\tilde{u}(\bar{x}) + \tilde{v}(\bar{x})}{2}.$$

Using the above estimate, (31) and the Harnack inequality in $B_{1/10}(\bar{x})$ we get that

$$(1 + 100\varepsilon^2)(x_d + b) - \frac{\tilde{u} + \tilde{v}}{2} \geq C_{\mathcal{H}}\varepsilon \quad \text{in } B_{1/20}(\bar{x}).$$

Now consider the family of functions

$$\eta_t(x) := (1 + 100\varepsilon^2)(x_d + b) - C_{\mathcal{H}}\varepsilon(w(x) - 1) - C_{\mathcal{H}}\varepsilon t,$$

defined for $t \geq 0$ and $x \in B_1$. Then

$$\frac{\tilde{u}(x) + \tilde{v}(x)}{2} > (\eta_t(x))_+ \quad \text{for every } x \in B_{1/20}(\bar{x}) \quad \text{and every } t < 1. \quad (36)$$

Let us prove that the same inequality holds for every $x \in B_1$. Notice that, for every $t > 0$, we have:

- (η -i) $\Delta\eta_t < -C\varepsilon$ on $B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x})$;
- (η -ii) $|\nabla\eta_t|(x) < 1$ on $(B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x})) \cap \{x_d < 1/10\}$.

As in the previous case, we argue by contradiction. Suppose that, for some $t < 1$, there exists $z \in B_1$ such that η_t touches from above $\frac{1}{2}(\tilde{u} + \tilde{v})$ at z . By (η -i), we have that $z \notin \Omega \cap (B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x}))$. On the other hand, by (η -ii) and Lemma 2.9 we have that $z \notin \partial\Omega \cap (B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x}))$, which is a contradiction. Thus,

$$\frac{\tilde{u}(x) + \tilde{v}(x)}{2} \leq \left((1 + 100\varepsilon^2)(x_d + b) - C_{\mathcal{H}}\varepsilon w(x)\right)_+ \quad \text{for every } x \in B_1, \quad (37)$$

and in particular,

$$\Omega \subset \{x \in B_1 : (1 + 100\varepsilon^2)(x_d + b) - C_{\mathcal{H}}\varepsilon w(x) > 0\}.$$

Finally, choosing ε_0 small enough and using that w is bounded away from zero in $B_{1/4}$, we get

$$\Omega \subset \left\{x \in B_{1/4} : x_d + b - \frac{1}{2}C_{\mathcal{H}}\varepsilon w(x) > 0\right\}. \quad (38)$$

Step 3. Conclusion. As a consequence of Step 2, we have one of the two inclusions (35) and (38). More precisely, there is a constant $C > 0$ such that either

$$\Omega \supset \{x_d + a + C\varepsilon > 0\} \cap B_{1/4} \quad \text{or} \quad \Omega \subset \{x_d + b - C\varepsilon > 0\} \cap B_{1/4}. \quad (39)$$

Then, by applying Lemma 4.2 to both u and v (by replacing γ respectively with α and β), there exists a universal constant $\delta \in (0, 1)$ such that either

$$\begin{cases} u(x) \leq \alpha(x_d + b - \delta(b - a))_+ \\ v(x) \leq \beta(x_d + b - \delta(b - a))_+ \end{cases} \quad \text{or} \quad \begin{cases} u(x) \geq \alpha(x_d + a + \delta(b - a))_+ \\ v(x) \geq \beta(x_d + a + \delta(b - a))_+ \end{cases},$$

for every $x \in B_\rho$, with $\rho < 1/8$ a dimensional constant. \square

The following corollary is a consequence of the result above.

Corollary 4.4. *Let (u_n, v_n) be a sequence of solutions to (23) in the sense of Definition 2.4. Let*

$$\Omega_n := \{u_n > 0\} = \{v_n > 0\}$$

and suppose that $0 \in \partial\Omega_n$, for every $n \in \mathbb{N}$. Suppose that there is a sequence $\varepsilon_n \rightarrow 0$ such that for every $x \in B_1$

$$(x_d - \varepsilon_n)_+ \leq u_n(x) \leq (x_d + \varepsilon_n)_+ \quad \text{and} \quad (x_d - \varepsilon_n)_+ \leq v_n(x) \leq (x_d + \varepsilon_n)_+.$$

Then, there are continuous functions

$$\tilde{u}_\infty : \overline{B_1^+} \rightarrow \mathbb{R} \quad \text{and} \quad \tilde{v}_\infty : \overline{B_1^+} \rightarrow \mathbb{R},$$

with $B_1^+ = B_1 \cap \{x_d > 0\}$, such that the following holds:

(a) The graphs over $\overline{\Omega_n}$ of

$$\tilde{u}_n = \frac{u_n - x_d}{\varepsilon_n} \quad \text{and} \quad \tilde{v}_n = \frac{v_n - x_d}{\varepsilon_n},$$

Hausdorff converge respectively to the graphs of u_∞ and v_∞ over $\overline{B_1^+}$.

(b) The graph over $\overline{\Omega_n}$ of

$$\tilde{w}_n = \frac{\sqrt{u_n v_n} - x_d}{\varepsilon_n},$$

Hausdorff converges to the graph of $\frac{1}{2}(u_\infty + v_\infty)$ over $\overline{B_1^+}$.

Proof. The proof of claim (a) follows from Lemma 4.3 precisely as in [13]. In order to prove (b), we first notice that for any fixed $\delta > 0$, the sequences \tilde{u}_n and \tilde{v}_n converge uniformly on $B_1 \cap \{x_d > \delta\}$ respectively to the functions \tilde{u}_∞ and \tilde{v}_∞ . In particular, this implies that

$$\frac{\sqrt{u_n v_n} - x_d}{\varepsilon_n} = \frac{\sqrt{(x_d + \varepsilon_n \tilde{u}_n)(x_d + \varepsilon_n \tilde{v}_n)} - x_d}{\varepsilon_n} = \frac{1}{2}(\tilde{u}_n + \tilde{v}_n) + o(\varepsilon_n),$$

which proves the claim on every $B_1 \cap \{x_d > \delta\}$. Now, in order to have the convergence of the graphs over the whole B_1^+ , we notice that by (31) the oscillation of $\sqrt{u_n v_n} - x_d$ decays when passing from B_1 to a smaller ball B_ρ . Using again the argument from [13], we get that the graphs of

$$\tilde{w}_n = \frac{\sqrt{u_n v_n} - x_d}{\varepsilon_n},$$

Hausdorff converge to the graph of a Hölder continuous function

$$\tilde{w}_\infty : \overline{B_1^+} \rightarrow \mathbb{R}.$$

Now, since $\tilde{w}_\infty = \frac{1}{2}(\tilde{u}_\infty + \tilde{v}_\infty)$ on each set $B_1 \cap \{x_d > \delta\}$, we get that $\tilde{w}_\infty = \frac{1}{2}(\tilde{u}_\infty + \tilde{v}_\infty)$ on $\overline{B_1^+}$. \square

5. IMPROVEMENT OF FLATNESS

In this section we prove our main Theorem, from which the $C^{1,\alpha}$ regularity of a flat free boundary follows by standard arguments (see for example [17] for the vectorial case). In view of the invariance of (23) under suitable multiplication (see Step 1 of the proof of Lemma 4.3) the flatness conditions of Definition 1.2 can be expressed with $\alpha = \beta = 1$.

Theorem 5.1. *Let (u, v) be solutions to (23) in the sense of Definition 2.4. Let*

$$\Omega := \{u > 0\} = \{v > 0\}$$

and suppose that $0 \in \partial\Omega$. There are constants $\varepsilon_0 > 0$ and $C > 0$ such that the following holds. If (u, v) is a couple of solutions satisfying

$$(x_d - \varepsilon)_+ \leq u(x) \leq (x_d + \varepsilon)_+ \quad \text{and} \quad (x_d - \varepsilon)_+ \leq v(x) \leq (x_d + \varepsilon)_+ \quad \text{in } B_1,$$

for some $\varepsilon < \varepsilon_0$, then there are a unit vector $\nu \in \mathbb{R}^d$ with $|\nu - e_d| \leq C\varepsilon$ and a radius $\rho \in (0, 1)$ such that

$$\tilde{\alpha}(x \cdot \nu - \varepsilon/2)_+ \leq u_\rho(x) \leq \tilde{\alpha}(x \cdot \nu + \varepsilon/2)_+ \quad \text{and} \quad \tilde{\beta}(x \cdot \nu - \varepsilon/2)_+ \leq v_\rho(x) \leq \tilde{\beta}(x \cdot \nu + \varepsilon/2)_+$$

for all $x \in B_1$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are positive constants that satisfy

$$\tilde{\alpha}\tilde{\beta} = 1, \quad |1 - \tilde{\alpha}| \leq C\varepsilon \quad \text{and} \quad |1 - \tilde{\beta}| \leq C\varepsilon.$$

We postpone the construction of the limiting problem arising from the linearization to Lemma 5.2 and Lemma 5.4 and we directly prove the improvement of flatness result.

Proof of Theorem 5.1. We argue by contradiction. Let (u_n, v_n) be a sequence of solutions such that

$$(x_d - \varepsilon_n)_+ \leq u_n(x) \leq (x_d + \varepsilon_n)_+ \quad \text{and} \quad (x_d - \varepsilon_n)_+ \leq v_n(x) \leq (x_d + \varepsilon_n)_+,$$

where ε_n is an infinitesimal sequence. Let $\Omega_n := \{u_n > 0\} = \{v_n > 0\}$ and consider

$$\tilde{u}_n = \frac{u_n - x_d}{\varepsilon_n} \quad \text{and} \quad \tilde{v}_n = \frac{v_n - x_d}{\varepsilon_n} \quad \text{on } \overline{\Omega_n}. \quad (40)$$

By the compactness result of [Corollary 4.4](#), we get that $(\tilde{u}_n, \tilde{v}_n)$ converge, up to a subsequence, to a couple of continuous functions

$$\tilde{u}_\infty : B_1 \cap \{x_d \geq 0\} \rightarrow \mathbb{R} \quad \text{and} \quad \tilde{v}_\infty : B_1 \cap \{x_d \geq 0\} \rightarrow \mathbb{R}. \quad (41)$$

By [Lemma 5.2](#), we have that

$$M := \frac{1}{2}(\tilde{u}_\infty + \tilde{v}_\infty) \quad \text{and} \quad D := \frac{1}{2}(\tilde{u}_\infty - \tilde{v}_\infty)$$

are classic solutions to

$$\begin{cases} \Delta M = 0 & \text{in } B_1 \cap \{x_d > 0\}, \\ \partial_{x_d} M = 0 & \text{on } B_1 \cap \{x_d = 0\}. \end{cases} \quad \text{and} \quad \begin{cases} \Delta D = 0 & \text{in } B_1 \cap \{x_d > 0\}, \\ D = 0 & \text{on } B_1 \cap \{x_d = 0\}. \end{cases}$$

Therefore, by the regularity result [Lemma 5.4](#), we get

$$|\tilde{u}_\infty(x) - x \cdot \nabla \tilde{u}_\infty(0)| \leq C_d \rho^2 \quad \text{and} \quad |\tilde{v}_\infty(x) - x \cdot \nabla \tilde{v}_\infty(0)| \leq C_d \rho^2,$$

for every $x \in B_\rho \cap \{x_d \geq 0\}$. Now, we can write this as

$$\begin{cases} x \cdot \nabla \tilde{u}_\infty(0) - C_d \rho^2 \leq \tilde{u}_\infty(x) \leq x \cdot \nabla \tilde{u}_\infty(0) + C_d \rho^2 \\ x \cdot \nabla \tilde{v}_\infty(0) - C_d \rho^2 \leq \tilde{v}_\infty(x) \leq x \cdot \nabla \tilde{v}_\infty(0) + C_d \rho^2 \end{cases} \quad \text{for every } x \in B_\rho \cap \{x_d \geq 0\}.$$

Now, this implies that for n large enough

$$\begin{cases} x \cdot \nabla \tilde{u}_\infty(0) - 2C_d \rho^2 \leq \tilde{u}_n(x) \leq x \cdot \nabla \tilde{u}_\infty(0) + 2C_d \rho^2 \\ x \cdot \nabla \tilde{v}_\infty(0) - 2C_d \rho^2 \leq \tilde{v}_n(x) \leq x \cdot \nabla \tilde{v}_\infty(0) + 2C_d \rho^2 \end{cases} \quad \text{for every } x \in B_\rho \cap \bar{\Omega}_n,$$

which by the definition of \tilde{u}_n and \tilde{v}_n can be written as

$$x \cdot \left(e_d + \varepsilon_n \nabla \tilde{u}_\infty(0) \right) - \varepsilon_n 2C_d \rho \leq (u_n)_\rho(x) \leq x \cdot \left(e_d + \varepsilon_n \nabla \tilde{u}_\infty(0) \right) + \varepsilon_n 2C_d \rho,$$

$$x \cdot \left(e_d + \varepsilon_n \nabla \tilde{v}_\infty(0) \right) - \varepsilon_n 2C_d \rho \leq (v_n)_\rho(x) \leq x \cdot \left(e_d + \varepsilon_n \nabla \tilde{v}_\infty(0) \right) + \varepsilon_n 2C_d \rho,$$

for every $x \in B_1 \cap \left(\frac{1}{\rho} \Omega_n \right)$. We next set

$$V := \nabla M(0) \quad \text{and} \quad c := \frac{\partial D}{\partial x_d}(0).$$

Thus,

$$x \cdot \left(e_d + \varepsilon_n V + c \varepsilon_n e_d \right) - \varepsilon_n 2C_d \rho \leq (u_n)_\rho(x) \leq x \cdot \left(e_d + \varepsilon_n V + c \varepsilon_n e_d \right) + \varepsilon_n 2C_d \rho,$$

$$x \cdot \left(e_d + \varepsilon_n V - c \varepsilon_n e_d \right) - \varepsilon_n 2C_d \rho \leq (v_n)_\rho(x) \leq x \cdot \left(e_d + \varepsilon_n V - c \varepsilon_n e_d \right) + \varepsilon_n 2C_d \rho.$$

Now, since by [Lemma 5.2](#) V and e_d are orthogonal, we can compute

$$|e_d(1 \pm c \varepsilon_n) + \varepsilon_n V| = \sqrt{1 \pm 2c \varepsilon_n + \varepsilon_n^2(c^2 + |V|^2)} = 1 \pm c \varepsilon_n + O(\varepsilon_n^2).$$

Then, fixing $\rho > 0$ small enough and taking ε_n sufficiently small with respect to ρ , we get

$$x \cdot \frac{e_d + \varepsilon_n V}{|e_d + \varepsilon_n V|} - \frac{1}{2} \varepsilon_n \leq \frac{1}{1 + c \varepsilon_n} (u_n)_\rho(x) \leq x \cdot \frac{e_d + \varepsilon_n V}{|e_d + \varepsilon_n V|} + \frac{1}{2} \varepsilon_n,$$

$$x \cdot \frac{e_d + \varepsilon_n V}{|e_d + \varepsilon_n V|} - \frac{1}{2} \varepsilon_n \leq (1 + c \varepsilon_n) (v_n)_\rho(x) \leq x \cdot \frac{e_d + \varepsilon_n V}{|e_d + \varepsilon_n V|} + \frac{1}{2} \varepsilon_n,$$

for every $x \in B_1 \cap \left(\frac{1}{\rho} \Omega_n \right)$. Finally, the contradiction follows by taking

$$\nu = \frac{e_d + \varepsilon_n V}{|e_d + \varepsilon_n V|}, \quad \tilde{\alpha} = 1 + c \varepsilon_n \quad \text{and} \quad \tilde{\beta} = \tilde{\alpha}^{-1}.$$

□

Under the notations of the proof of [Theorem 5.1](#), we introduce the limiting problem arising from the linearization near flat free boundary points.

Lemma 5.2 (The linearized problem). *Let \tilde{u}_∞ and \tilde{v}_∞ be as in (41) and set*

$$M := \frac{1}{2}(\tilde{u}_\infty + \tilde{v}_\infty) \quad \text{and} \quad D := \frac{1}{2}(\tilde{u}_\infty - \tilde{v}_\infty).$$

Then, M and D are classical solutions of

$$\begin{cases} \Delta M = 0 & \text{in } B_1 \cap \{x_d > 0\}, \\ \partial_{x_d} M = 0 & \text{on } B_1 \cap \{x_d = 0\}. \end{cases} \quad \text{and} \quad \begin{cases} \Delta D = 0 & \text{in } B_1 \cap \{x_d > 0\}, \\ D = 0 & \text{on } B_1 \cap \{x_d = 0\}. \end{cases} \quad (42)$$

Proof. We divide the proof in several steps.

Step 1. Equations in $\{x_d > 0\}$. First we notice that the equation

$$\Delta M = \Delta D = 0 \quad \text{in } B_1 \cap \{x_d > 0\}$$

follows from the fact that on every compact subset of $B_1 \cap \{x_d > 0\}$, the functions \tilde{u}_n and \tilde{v}_n given by (40), are harmonic and converge uniformly to \tilde{u}_∞ and \tilde{v}_∞ .

Step 2. Boundary condition for D . In order to prove the boundary condition

$$D = 0 \quad \text{on } B_1 \cap \{x_d = 0\},$$

we notice that the graphs of \tilde{u}_n and \tilde{v}_n over $\partial\Omega_n$ are both given by the graph of the function $-\frac{1}{\varepsilon_n}x_d$. Thus, by the Hausdorff convergence of the graphs, we get that

$$u_\infty = v_\infty \quad \text{on } B_1 \cap \{x_d = 0\}.$$

Finally, it remains to prove that

$$\frac{\partial M}{\partial x_d} = 0 \quad \text{on } B_1 \cap \{x_d = 0\}$$

is satisfied in the viscosity sense.

Notice that, the fact that M is a classical solution of (42) follows by [13, Lemma 2.6]).

Step 3. The boundary condition for M from below. Suppose that a quadratic polynomial P touches M strictly from below at a point $x_0 \in \{x_d = 0\}$. We will show that $\partial_d P(x_0) \leq 0$. Therefore, suppose by contradiction that

$$\frac{\partial P}{\partial x_d}(x_0) > 0, \quad (43)$$

and notice that we can assume that $\Delta P > 0$ in a neighborhood of x_0 . Let now

$$\tilde{w}_n := \frac{\sqrt{u_n v_n} - x_d}{\varepsilon_n} : \bar{\Omega}_n \rightarrow \mathbb{R}.$$

By Corollary 4.4, we have that the sequence of graphs of \tilde{w}_n over $\bar{\Omega}_n$ converge in the Hausdorff sense to the graph of M over $B_1 \cap \{x_d \geq 0\}$. In particular, this means that the graph of P touches from below the graph of \tilde{w}_n at some point $x_n \in \bar{\Omega}_n$. Since \tilde{w}_n is superharmonic in Ω_n (see Remark 4.1), we have that necessary $x_n \in \partial\Omega_n$. But then, we have

$$P(x) \leq \frac{\sqrt{u_n v_n} - x_d}{\varepsilon_n} \quad \text{for } x \in \bar{\Omega}_n,$$

with an equality when $x = x_n$, which can be written as

$$\varepsilon_n P(x) + x_d \leq \sqrt{u_n(x)v_n(x)} \quad \text{for } x \in \bar{\Omega}_n.$$

Setting

$$\varphi(x) := \varepsilon_n P(x) + x_d, \quad (44)$$

we get that φ_+ touches $\sqrt{u_n v_n}$ from below at x_n . On the other hand, by Lemma 2.9 we know that

$$1 + \varepsilon \frac{\partial P}{\partial x_d}(x_n) = \partial_{x_d} \varphi(x_n) \leq |\nabla \varphi(x_n)| \leq 1,$$

which is a contradiction with (43), as we claimed.

Step 4. The boundary condition for M from above. Suppose that a quadratic polynomial P touches M strictly from above at a point $y_0 \in \{x_d = 0\}$. We will prove that $\partial_d P(y_0) \geq 0$. Conversely, suppose that

$$\frac{\partial P}{\partial x_d}(y_0) < 0, \quad (45)$$

and notice that we can assume that $\Delta P < 0$ in a neighborhood of y_0 . By the Hausdorff convergence of the graphs, the graph of P touches from above the graph of $\frac{1}{2}(\tilde{u}_n + \tilde{v}_n)$ at a point $y_n \in \bar{\Omega}_n$. Since \tilde{u}_n and \tilde{v}_n are harmonic in Ω_n we have that $y_n \in \partial\Omega_n$. But then, we have

$$P(x) \geq \frac{1}{2} \left(\frac{u_n - x_d}{\varepsilon_n} + \frac{v_n - x_d}{\varepsilon_n} \right) \quad \text{for } x \in \bar{\Omega}_n,$$

with an equality when $x = y_n$, which can be written as

$$x_d + \varepsilon_n P(x) \geq \frac{u_n(x) + v_n(x)}{2} \quad \text{for } x \in \bar{\Omega}_n.$$

Let now

$$\psi(x) := x_d + \varepsilon_n P(x), \tag{46}$$

Then, ψ_+ touches $\frac{1}{2}(u_n(x) + v_n(x))$ from above at y_n . Thus, by [Lemma 2.9](#),

$$1 \leq |\nabla\psi(y_n)|^2 = 1 + 2\varepsilon_n \frac{\partial P}{\partial x_d}(y_n) + \varepsilon_n^2 |\nabla P(y_n)|^2,$$

which can be written as

$$\frac{\partial P}{\partial x_d}(y_n) + \frac{\varepsilon_n}{2} |\nabla P(y_n)|^2 \geq 0,$$

Passing to the limit as $n \rightarrow \infty$, we get $\frac{\partial P}{\partial x_d}(y_0) \geq 0$, which is a contradiction. \square

Remark 5.3. For the vectorial Bernoulli problem, in [\[22, 17\]](#) the authors proved that the linearized problem arising by the improvement of flatness technique is a system of decoupled equations in which the first component satisfies a Neumann problem, while the others have Dirichlet boundary conditions. On the contrary, in our case, the nonlinear formulation of the free boundary condition [\(3\)](#) requires to consider suitable linear combinations of the solutions u, v in order to detect the problem solved by the limits u_∞, v_∞ .

For the sake of completeness, we briefly sketch the proof of the decay for the solutions of the linearized problem [\(42\)](#), which we used in the proof of [Theorem 5.1](#).

Lemma 5.4 (First and second order estimates for \tilde{u}_∞ and \tilde{v}_∞). *Let \tilde{u}_∞ and \tilde{v}_∞ be as in [Lemma 5.2](#). Then \tilde{u}_∞ and \tilde{v}_∞ are C^∞ in $B_1 \cap \{x_d \geq 0\}$ and we have the estimates*

$$\|\nabla\tilde{u}_\infty\|_{L^\infty(B_{1/2} \cap \{x_d \geq 0\})} + \|\nabla\tilde{v}_\infty\|_{L^\infty(B_{1/2} \cap \{x_d \geq 0\})} \leq C_d, \tag{47}$$

and

$$\begin{cases} |\tilde{u}_\infty(x) - x \cdot \nabla\tilde{u}_\infty(0)| \leq C_d |x|^2 \\ |\tilde{v}_\infty(x) - x \cdot \nabla\tilde{v}_\infty(0)| \leq C_d |x|^2 \end{cases} \quad \text{for every } x \in B_{1/2} \cap \{x_d \geq 0\}. \tag{48}$$

Moreover, we have

$$\nabla\tilde{u}_\infty(0) = \nabla\tilde{v}_\infty(0) + e_d \frac{\partial(\tilde{u}_\infty - \tilde{v}_\infty)}{\partial x_d}(0). \tag{49}$$

Proof. Let M and D be as in [Lemma 5.2](#). Then, both can be extended to (resp. an even and an odd) harmonic functions in the ball B_1 . We can use the classical gradient and second order estimates (see for instance [\[29, Lemma 7.17\]](#)) for a harmonic function $h : B_R \rightarrow \mathbb{R}$, that is,

$$\|\nabla h\|_{L^\infty(B_{R/2})} \leq \frac{C_d}{R} \|h\|_{L^\infty(B_R)},$$

and

$$|h(x) - h(0) - x \cdot \nabla h(0)| \leq \frac{C_d}{R^2} |x|^2 \|h\|_{L^\infty(B_R)} \quad \text{for every } x \in B_{R/2},$$

where C_d is a dimensional constant. Now since $\tilde{u}_\infty(0) = \tilde{v}_\infty(0) = 0$ and

$$|\tilde{u}_\infty| \leq 1 \quad \text{and} \quad |\tilde{v}_\infty| \leq 1 \quad \text{in} \quad B_1 \cap \{x_d \geq 0\},$$

and since $M = \tilde{u}_\infty + \tilde{v}_\infty$ and $D = \tilde{u}_\infty - \tilde{v}_\infty$, we get that

$$\begin{cases} |D(x) - x \cdot \nabla D(0)| \leq C_d |x|^2 \\ |M(x) - x \cdot \nabla M(0)| \leq C_d |x|^2 \end{cases} \quad \text{for every } x \in B_{1/2} \cap \{x_d \geq 0\},$$

which gives [\(48\)](#). Finally, [\(49\)](#) follows from the fact that $D \equiv 0$ on $\{x_d = 0\}$. \square

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