SHARP GRADIENT STABILITY FOR THE SOBOLEV INEQUALITY

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Abstract. We prove a sharp quantitative version of the p-Sobolev inequality for any $1 < p < n$, with a control on the strongest possible distance from the class of optimal functions. Surprisingly, the sharp exponent is constant for $p \leq 2$, while it depends on p for $p > 2$.

1. Introduction

Motivated by important applications to problems in the calculus of variations and evolution PDEs, recently, there has been a growing interest in understanding quantitative stability for functional and geometric inequalities. For instance, there have been several works investigating the stability of Sobolev and Sobolev-type inequalities [3, 2, 9, 29, 10, 31, 8, 25, 37, 6, 26, 35, 36], isoperimetric inequalities [30, 24, 11, 22, 27], and the Brunn-Minkowski inequality [23, 18, 19, 32, 33], with a variety of applications [20, 7, 12, 14, 21, 15]. The interested reader may also look at [17, 28, 4] for further bibliography, also related to the stability of other inequalities.

Following this line of research, in this paper we shall investigate the stability of minimizers to the classical Sobolev inequality.

1.1. **The Sobolev inequality.**

The question of quantitative stability for the Sobolev inequality was first raised by Brezis and Lieb [5]. Before describing the problem and the state of the art, we first introduce some useful definitions.

Given $n \geq 2$ and $1 < p < n$, denote by $\dot{W}^{1,p}(\mathbb{R}^n)$ the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\dot{W}^{1,p}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |Du|^p \, dx\right)^{\frac{1}{p}}.$$

The Sobolev inequality guarantees the existence of a positive constant $S = S(n,p)$ such that

$$\|Du\|_{L^p(\mathbb{R}^n)} \geq S\|u\|_{L^{p^*}(\mathbb{R}^n)},$$

where $p^* = \frac{np}{n-p}$. We call the largest constant S satisfying this property the *optimal Sobolev constant.*

Let $\mathcal{M}$ be the $(n+2)$-dimensional manifold of all functions of the form

$$v_{a,b,x_0}(x) := \frac{a}{\left(1 + b|x - x_0|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}}, \qquad a \in \mathbb{R} \setminus \{0\},\ b > 0,\ x_0 \in \mathbb{R}^n.$$

As shown in [1, 39, 13], $\mathcal{M}$ coincides with the space of all weak solutions to the equation

$$-\Delta_p v = S^p \|v\|_{L^{p^*}(\mathbb{R}^n)}^{p-p^*} |v|^{p^*-2} v \qquad \text{in } \mathbb{R}^n \tag{1.1}$$

that do not change sign. Here, S is the optimal Sobolev constant and

$$-\Delta_p v = \operatorname{div}(|Dv|^{p-2} Dv).$$

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It is also proven that $\mathcal{M}$ coincides with the set of all the extremal functions in the Sobolev inequality; in particular,

$$\|Dv\|_{L^p(\mathbb{R}^n)} = S\|v\|_{L^{p^*}(\mathbb{R}^n)} \qquad \forall\, v \in \mathcal{M}.$$

1.2. **The stability question: the generalized Brezis-Lieb's problem.** To formulate our stability problem, we introduce the notion of *p-Sobolev deficit*:

$$\delta(u) := \frac{\|Du\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}} - S \qquad \forall\, u \in \dot{W}^{1,p}(\mathbb{R}^n). \tag{1.2}$$

Note that $\delta \geq 0$, and it vanishes only on $\mathcal{M}$.

In [5], Brezis and Lieb asked whether, for $p = 2$, the deficit can be estimated from below by some appropriate distance between u and $\mathcal{M}$, together with a suitable decay. This problem was settled few years later by Bianchi and Egnell [3]: they showed the existence of a constant $c = c(n) > 0$ such that

$$\delta(u) \geq c \inf_{v\in\mathcal{M}} \left(\frac{\|Du - Dv\|_{L^2(\mathbb{R}^n)}}{\|Du\|_{L^2(\mathbb{R}^n)}} \right)^2 \qquad \forall\, u \in \dot{W}^{1,2}(\mathbb{R}^n).$$

This estimate is optimal both in terms of the strength of the distance from $\mathcal{M}$, and in terms of the exponent 2 appearing in the right hand side.

After this work, it became immediately of interest understanding whether Brezis-Lieb's question could be solved also for general values of p. Unfortunately, Bianchi-Egnell's method heavily depended on the Hilbert structure of $\dot{W}^{1,2}(\mathbb{R}^n)$, so new ideas and techniques were needed.

Almost 20 years later, in [10], Cianchi, Fusco, Maggi, and Pratelli proved a stability version for every $p \in (1, n)$ with distance given by

$$\inf_{v\in\mathcal{M}} \left(\frac{\|u - v\|_{L^{p^*}(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}} \right)^\alpha \qquad \forall\, u \in \dot{W}^{1,p}(\mathbb{R}^n), \tag{1.3}$$

together with the explicit decay exponent $\alpha = \alpha_{CFMP} := \left[p^*\left(3 + 4p - \frac{3p+1}{n}\right)\right]^2$. Although most likely the result was not sharp, this was the first stability result valid for the full range of p. In addition, their proof introduced in this problem a beautiful combination of techniques coming from symmetrization theory and optimal transport.

These technique were further developed by Figalli, Maggi, and Pratelli in [25] to provide a sharp stability result —both in terms of the notion of distance and of the decay exponent— in the special case $p = 1$ (for this case, see also the earlier results [9, 28, 29]).

Still, until few years ago, it remained a major open problem whether the p-Sobolev deficit could control the closeness to $\mathcal{M}$ at the level of the gradients (i.e., the strongest distance that one may hope to control with $\delta(u)$), as in the case of Bianchi and Egnell.

A first answer to this question was given by Figalli and Neumayer in [26] in the case $p > 2$, where they developed in $\dot{W}^{1,p}(\mathbb{R}^n)$ a suitable analogue of the strategy in [3] to prove the existence of a constant $c = c(n, p) > 0$ such that

$$\delta(u) \geq c \inf_{v\in\mathcal{M}} \left(\frac{\|D(u - v)\|_{L^p(\mathbb{R}^n)}}{\|Du\|_{L^p(\mathbb{R}^n)}} \right)^\alpha \qquad \forall\, u \in \dot{W}^{1,p}(\mathbb{R}^n), \tag{1.4}$$

where $\alpha = p\alpha_{CFMP}$ with α_{CFMP} as above. The appearance of the exponent α_{CFMP} comes from the fact that, in one of the steps in the proof, the authors need to rely on the result in [10].

Very recently, in [36], Neumayer extended (1.4) to the full range $1 < p < n$. While her proof is much simpler than the one in [26], it relies heavily on the result in [10] and her strategy cannot give the sharp exponent in (1.4), even if one could prove (1.3) with a sharp exponent. In particular, her approach provides the same exponent as the one in [26] when $p > 2$, while it gives (1.4) with $\alpha = \frac{p}{p-1}\alpha_{CFMP}$ when $p \in (1, 2)$.

Despite all these developments, the stability exponent appearing in all these previous results was far from optimal. The aim of this paper is to give a final answer to this problem by proving (1.4) for all $1 < p < n$ with a sharp exponent.

Here is our theorem:

Theorem 1.1. *Let $1 < p < n$, and define $\delta(\,\cdot\,)$ as in (1.2). There exists a constant $c = c(n, p) > 0$ such that (1.4) holds with $\alpha = \max\{2,\, p\}$.*

Remark 1.2. The decay exponent $\alpha = \max\{2,\, p\}$ is sharp, as we now explain.
Fix $v = v_{1,1,0} \in \mathcal{M}$ and consider first $u_i := v(A_i x)$, where $A_i \in \mathbb{R}^{n\times n}$ denotes the diagonal matrix

$$A_i = \operatorname{diag}\left(1, \dots, 1, 1 + \frac{1}{i}\right).$$

It is not difficult to check that $\delta(u_i)$ behaves as i^{-2}, while the right hand side of (1.4) behaves as $i^{-\alpha}$, hence (1.4) cannot hold with $\alpha < 2$.
On the other hand, fix $\varphi \in C_c^\infty(B_1)$ a nontrivial function, and consider now $\tilde{u}_i := v + \varphi(x_i + \cdot)$, where $x_i \in \mathbb{R}^n$ is a sequence of points converging towards ∞. One can check that

$$\|D\tilde{u}_i\|^p_{L^p(\mathbb{R}^n)} = \|Dv\|^p_{L^p(\mathbb{R}^n)} + \|D\varphi\|^p_{L^p(\mathbb{R}^n)} + r_{i,1}$$

and

$$\|\tilde{u}_i\|^{p^*}_{L^{p^*}(\mathbb{R}^n)} = \|v\|^{p^*}_{L^{p^*}(\mathbb{R}^n)} + \|\varphi\|^{p^*}_{L^{p^*}(\mathbb{R}^n)} + r_{i,2},$$

with $|r_{i,1}| + |r_{i,2}| \le C\big(v(x_i) + |Dv(x_i)|\big) \le Cv(x_i) \to 0$ as $i \to \infty$. Hence, choosing a sequence $\epsilon_i \to 0$ such that $v(x_i) \ll \epsilon_i \ll 1$, the functions $\hat{u}_i := v + \epsilon_i \varphi(x_i + \cdot)$ satisfy

$$\|D\hat{u}_i\|^p_{L^p(\mathbb{R}^n)} = \|Dv\|^p_{L^p(\mathbb{R}^n)} + \epsilon_i^p \|D\varphi\|^p_{L^p(\mathbb{R}^n)} + o(\epsilon_i^p)$$

and

$$\|\hat{u}_i\|^{p^*}_{L^{p^*}(\mathbb{R}^n)} = \|v\|^{p^*}_{L^{p^*}(\mathbb{R}^n)} + \epsilon_i^{p^*} \|\varphi\|^{p^*}_{L^{p^*}(\mathbb{R}^n)} + o(\epsilon_i^{p^*}).$$

Thanks to these facts, one easily deduces that $\delta(\hat{u}_i)$ behaves as ϵ_i^p, while the right hand side of (1.4) behaves as ϵ_i^α. Thus (1.4) cannot hold with $\alpha < p$.

1.3. **Strategy of the proof.** As in [26], the beginning of the proof follows [3].

More precisely, given u close to $\mathcal{M}$, one chooses $v \in \mathcal{M}$ close to u and set $\varphi := \frac{u-v}{\|\nabla u - \nabla v\|_{L^p(\mathbb{R}^n)}}$ and $\epsilon := \|\nabla u - \nabla v\|_{L^p(\mathbb{R}^n)}$, so that u can be written as $v + \epsilon\varphi$. Then one expands $\delta(u)$ in ϵ, and aims to use this expansion to control $\nabla\varphi$ in L^p.

When $p = 2$, as shown in [3], the expansion of $\delta(u)$ gives

$$\delta(v + \epsilon\varphi) = \epsilon^2 Q_v[\varphi] + o\big(\epsilon^2 \|D\varphi\|^2_{L^2(\mathbb{R}^n)}\big),$$

where $Q_v[\,\cdot\,]$ is a quadratic form depending on v. In addition, if φ is orthogonal to $T_v\mathcal{M}$ in the weighted space $L^2(\mathbb{R}^n; v^{2^*-2})$, spectral analysis shows that $Q_v[\varphi]$ controls $\|D\varphi\|^2_{L^2}$ from above, thus

$$\delta(v + \epsilon\varphi) \ge c\epsilon^2 \|D\varphi\|^2_{L^2(\mathbb{R}^n)} + o\big(\epsilon^2 \|D\varphi\|^2_{L^2(\mathbb{R}^n)}\big).$$

Hence the result follows for $\epsilon \ll 1$, provided orthogonality can be ensured. In the case $p = 2$, this can be easily guaranteed by choosing v which minimizes

$$\mathcal{M} \ni v \mapsto \|\nabla u - \nabla v\|_{L^2(\mathbb{R}^n)},$$

completing the proof.

For $p > 2$, in [26] the authors tried to mimic the strategy of [3]. More precisely, the expansion of $\delta(u)$ gives

$$\delta(v + \epsilon\varphi) = \epsilon^2 Q_v[\varphi] + o\big(\epsilon^2 \|D\varphi\|^2_{L^p(\mathbb{R}^n)}\big),$$

where $Q_v[\,\cdot\,]$ is a quadratic form depending on v and p. Again, if φ is orthogonal to $T_v\mathcal{M}$ in the weighted space $L^2(\mathbb{R}^n; v^{p^*-2})$, spectral analysis shows that $Q_v[\varphi]$ controls the weighted norm $\|D\varphi\|^2_{L^2(\mathbb{R}^n;|Dv|^{p-2})}$ from above, thus

$$\delta(v+\epsilon\varphi) \geq c\epsilon^2\|D\varphi\|^2_{L^2(\mathbb{R}^n;|Dv|^{p-2})} + o\big(\epsilon^2\|D\varphi\|^2_{L^p(\mathbb{R}^n)}\big).$$

Unfortunately, now this argument is not sufficient, since for $p > 2$ the L^p norm of $D\varphi$ may not be controllable by its weighted L^2 norm. Furthermore, finding the correct orthogonality condition in this non-Hilbertian context requires new ideas. All this creates a series of challenges that were overcome in [26] by relying also on the L^{p^*} stability result of [10], as explained in detail in [26, Section 2].

In this paper, to handle the general case $1 < p < n$ and prove a stability estimate with sharp exponent, we need to face several new difficulties. The idea is again to expand the deficit $\delta(v+\epsilon\varphi)$. However, the argument in [26] shows that, for $p \neq 2$, a standard Taylor expansion creates error terms that cannot be controlled. Even worse, a second order expansion of the deficit naturally leads to a quadratic form consisting of a weighted $\dot{W}^{1,2}$ and a weighted L^2 norm. However, when $p < 2$, the $\dot{W}^{1,p}$ norm is weaker than any weighted $\dot{W}^{1,2}$ norm, so we cannot expand the deficit at order 2 (this was the main reason why [26] could only deal with the case $p \geq 2$). In addition, when $p \leq \frac{2n}{n+2}$ (equivalently $p^* \leq 2$), the L^{p^*} norm is not sufficient to control any weighted L^2 norms, and this creates even further challenges. For all these reasons, our arguments are different in the three regimes $p \in (1, \frac{2n}{n+2}]$, $p \in (\frac{2n}{n+2}, 2)$, and $p \in [2, n)$.

To briefly explain the main ideas in the proof, let us focus on the case $p \in (1, \frac{2n}{n+2}]$ (note that this set is nonempty only for $n \geq 3$). As mentioned above, a first problem consists in understanding how to expand the deficit. With no loss of generality, we can assume that $v > 0$.

Our first new tool is provided by the following inequalities: for any $\kappa > 0$ there exists $C_1 > 0$ such that, for ϵ sufficiently small,

$$\begin{aligned}\|Dv+\epsilon D\varphi\|^p_{L^p(\mathbb{R}^n)} \geq &\int_{\mathbb{R}^n}|Dv|^p\,dx + \epsilon p\int_{\mathbb{R}^n}|Dv|^{p-2}Dv\cdot D\varphi\,dx\\ &+ \frac{\epsilon^2 p(1-\kappa)}{2}\left(\int_{\mathbb{R}^n}|Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2}\left(\frac{|Du|-|Dv|}{\epsilon}\right)^2 dx\right)\end{aligned}$$

and

$$\begin{aligned}\|v+\epsilon\varphi\|^p_{L^{p^*}(\mathbb{R}^n)} \leq &\|v\|^p_{L^{p^*}(\mathbb{R}^n)}\\ &+ \|v\|^{p-p^*}_{L^{p^*}(\mathbb{R}^n)}\left(\epsilon p\int_{\mathbb{R}^n}v^{p^*-1}\varphi\,dx + \epsilon^2\left(\frac{p(p^*-1)}{2}+\frac{p\kappa}{p^*}\right)\int_{\mathbb{R}^n}\frac{(v+C_1|\epsilon\varphi|)^{p^*}}{v^2+|\epsilon\varphi|^2}|\varphi|^2\,dx\right),\end{aligned}$$

where $w = w(Dv, Du)$ is obtained by taking a suitable combination of Dv and Du (depending on their respective sizes) as in Lemma 2.1.

Combining these inequalities and using (1.1), one gets

$$\begin{aligned}C(n,p)\delta(u) \geq &\frac{\epsilon^2 p(1-\kappa)}{2}\left(\int_{\mathbb{R}^n}|Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2}\left(\frac{|Du|-|Dv|}{\epsilon}\right)^2 dx\right)\\ &- \epsilon^2\|v\|^{p-p^*}_{L^{p^*}(\mathbb{R}^n)}S^p\left(\frac{p(p^*-1)}{2}+\frac{p\kappa}{p^*}\right)\int_{\mathbb{R}^n}\frac{(v+C_1|\epsilon\varphi|)^{p^*}}{v^2+|\epsilon\varphi|^2}|\varphi|^2\,dx, \quad (1.5)\end{aligned}$$

so the result would be proved if we could show that, under some suitable orthogonality relation between v and φ, the right hand side above controls $\|\epsilon D\varphi\|^{\max\{2,p\}}_{L^p(\mathbb{R}^n)}$ for $\epsilon \ll 1$. Unfortunately this is false for $p < 2$, since

$$\epsilon^2|Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2}\big(|Du|-|Dv|\big)^2 \sim \epsilon|Dv|^{p-1}|D\varphi| \qquad \text{for } |Dv| \leq \epsilon|D\varphi|$$

(cp. (2.2)), and in general this weighted $W^{1,1}$ norm of φ is not enough to control the last term in (1.5).

Hence, our second goal consists in showing that we can improve the expansion of $\|Dv+\epsilon D\varphi\|^p_{L^p(\mathbb{R}^n)}$ (see Lemma 2.1), so that we can add the extra term

$$c_0\int_{\mathbb{R}^n}\min\left\{\epsilon^p|D\varphi|^p,\,\epsilon^2|Dv|^{p-2}|D\varphi|^2\right\}dx \tag{1.6}$$

to the right hand side of (1.5). With this extra term at our disposal, we now want to use the right hand side of (1.5) to control $\|\epsilon D\varphi\|^{\max\{2,p\}}_{L^p(\mathbb{R}^n)}$.

The main idea behind the proof of this fact consists of two steps:
(a) show that the result is true if one replaces the two integrands in the right hand side of (1.5) by their limit as $\epsilon\to 0$;
(b) prove that the result holds also for ϵ sufficiently small.

The argument for Step (a) is relatively standard (although delicate), and it boils down to proving a compact embedding and performing a Sturm-Liouville analysis with singular weights, see Propositions 3.2 and 3.6 and Appendix B.

On the other hand, Step (b) turns out to be highly challenging. A key difficulty comes from the fact that the integrand appearing in the last term of (1.5) behaves like $v^{p^*-2}|\varphi|^2$ when $|\varphi|\ll\frac{v}{\epsilon}$, and like $\epsilon^{p^*-2}|\varphi|^{p^*}$ otherwise. Analogously, the integrand inside the extra term (1.6) behaves like $\epsilon^2|Dv|^{p-2}|D\varphi|^2$ when $|D\varphi|\le\frac{|Dv|}{\epsilon}$, and like $\epsilon^p|D\varphi|^p$ otherwise. These substantial changes of behavior, and the fact that a change in size of the gradients does not necessarily correspond to a change in size of the functions, make the proofs of several estimates (in particular the ones in Lemma 3.4 and Proposition 3.8) very involved.

Finally, once all these difficulties have been solved, in Section 4 we introduce a new minimization principle to select v so to guarantee orthogonality and conclude the proof.

1.4. **Structure of the paper.** In Section 2, we prove a series of new vectorial inequalities that play a crucial role in the expansion of the deficit. In Section 3, we prove the compactness and spectral gaps estimates required for the proof of Theorem 1.1, which is then postponed to Section 4. Finally, we collect some technical estimates in two appendices.

Notation. In our estimates we often write positive constants as $C(\cdot)$ and $c(\cdot)$, with the parentheses including all the parameters on which the constant depends. Usually we use C to denote a constant larger than 1, and c for a constant less than 1. We simply write C or c if the constant is absolute. The constant $C(\cdot)$ may vary between appearances, even within a chain of inequalities. The notation $a\sim b$ indicates that both inequalities $a\le Cb$ and $b\le Ca$ hold. We denote the closure of a set $A\subset\mathbb{R}^n$ by $\overline{A}$. Finally, the Euclidean ball centered at x with radius r is denoted by $B(x,r)$.

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2. Sharp vector inequalities in Euclidean spaces

We start with the following sharp inequalities on vectors, which improve the ones in [26, Section 3.2]. The basic idea behind these inequalities is the following: to apply the strategy described in Section 1.3, for fixed $x\in\mathbb{R}^n$ we would like to find a non-negative quadratic expression in y that controls $|x+y|^p-|x|^p+p|x|^{p-2}x\cdot y$ from below, and that for $|y|\ll 1$ behaves like the Hessian of $z\mapsto|z|^p$ at x (this is needed in order to exploit later Proposition 3.6). Unfortunately this is impossible, so we introduce a weight $|w|=|w(x,x+y)|$ that depends on the sizes of $|x|$ and $|x+y|$ and modulates the quadratic-type expression appearing in the right hand side of our estimates. Analogously, in Lemma 2.4(i) we need to consider a weighted expression in front of $|b|^2$ in order to obtain a sufficiently precise expansion. We note that, as explained in Section 1.3, the extra term (the one multiplied by c_0) appearing in Lemma 2.1 will be crucial to prove our main theorem.

Lemma 2.1. *Let $x, y \in \mathbb{R}^n$. Then, for any $\kappa > 0$, there exists a constant $c_0 = c_0(p, \kappa) > 0$ such that the following holds:*
(i) For $1 < p < 2$,

$$|x+y|^p \geq |x|^p + p|x|^{p-2}x \cdot y + \frac{1-\kappa}{2}\left(p|x|^{p-2}|y|^2 + p(p-2)|w|^{p-2}\big(|x|-|x+y|\big)^2\right) + c_0 \min\left\{|y|^p,\ |x|^{p-2}|y|^2\right\},$$

where

$$w = w(x, x+y) := \begin{cases} \left(\frac{|x+y|}{(2-p)|x+y|+(p-1)|x|}\right)^{\frac{1}{p-2}} x & \text{if } |x| < |x+y| \\ x & \text{if } |x+y| \leq |x| \end{cases}.$$

(ii) For $p \geq 2$,[1]

$$|x+y|^p \geq |x|^p + p|x|^{p-2}x \cdot y + \frac{1-\kappa}{2}\left(p|x|^{p-2}|y|^2 + p(p-2)|w|^{p-2}\big(|x|-|x+y|\big)^2\right) + c_0|y|^p,$$

where

$$w = w(x, x+y) := \begin{cases} x & \text{if } |x| \leq |x+y| \\ \left(\frac{|x+y|}{|x|}\right)^{\frac{1}{p-2}} (x+y) & \text{if } |x+y| \leq |x| \end{cases}.$$

Remark 2.2. Note that the constant c_0 appearing in the statement above is said to depend on p and κ, but not on the dimension n. The reason is that, to prove the inequality, one can always restrict to the 2-dimensional plane generated by x and y, therefore the dimension n of the ambient space plays no role.

Remark 2.3. One may be tempted to define directly the weight $\tilde{w} := |w|^{p-2}$ with w as above, and then use $\tilde{w}$ in place of $|w|^{p-2}$ everywhere. However our notation has the advantage that $w \to x$ as $y \to 0$. Not only this emphasizes better the similarities with a Taylor expansion, but it will also be convenient in the proof of Proposition 3.8.

Proof. We split the proof in several steps.
• *Proof of (i): the case $1 < p < 2$.* By approximation we can assume that $|x| \neq 0$.
- *Step (i)-1: we show that*

$$|x+y|^p \geq \left(1 - \frac{1}{2}p\right)|x|^p + \frac{1}{2}p|x|^{p-2}|x+y|^2 + \frac{1}{2}p(p-2)|w|^{p-2}\big(|x|-|x+y|\big)^2. \tag{2.1}$$

To prove this, we set $z = x + y$ and distinguish two cases.

In the case $|z| < |x|$ we set $t := \frac{|z|}{|x|}$. Then (2.1) is equivalent to proving that

$$h(t) := t^p - \left(1 - \frac{1}{2}p\right) - \frac{1}{2}pt^2 - \frac{1}{2}p(p-2)(1-t)^2 \geq 0, \qquad \forall\, 0 < t < 1.$$

For this, it suffices to notice that $h(1) = h'(1) = 0$, and that

$$h''(t) = p\left((p-1)t^{p-2} - 1 + (2-p)\right) \geq 0 \qquad \forall\, 0 < t < 1$$

as $1 < p < 2$. So (2.1) holds for $|z| < |x|$.

[1] Since for $p = 2$ the coefficient $p(p-2)$ vanishes, the exact definition of w is irrelevant in this case.

On the other hand, in the case $|z| \geq |x|$ we set $t := \frac{|x|}{|z|}$ and we claim that

$$h(t) := 1 - \left(1 - \frac{1}{2}p\right) t^p - \frac{1}{2}pt^{p-2} - \frac{1}{2}p(p-2)\frac{1}{(2-p)+(p-1)t}t^{p-2}(t-1)^2 \geq 0, \qquad \forall\, 0 < t \leq 1.$$

Since $h(1) = 0$ and

$$\begin{aligned} h'(t) =&\frac{1}{2}p(p-2)\left[t^{p-1} - t^{p-3} + 2(1-t)t^{p-2}[(2-p)+(p-1)t]^{-1}\right.\\ &\left.+(2-p)t^{p-3}(t-1)^2[(2-p)+(p-1)t]^{-1} + (p-1)t^{p-2}(t-1)^2[(2-p)+(p-1)t]^{-2}\right]\\ =&\frac{1}{2}p(p-2)(t-1)t^{p-3}\left[t+1-\frac{2t}{(2-p)+(p-1)t}+\frac{(p-2)(1-t)}{(2-p)+(p-1)t}-\frac{(p-1)t(1-t)}{((2-p)+(p-1)t)^2}\right]\\ =&-\frac{1}{2}p(2-p)\frac{t^{p-2}}{(2-p)+(p-1)t}(p-1)(t-1)^2\left[1+\frac{1}{(2-p)+(p-1)t}\right] \leq 0 \qquad \forall\, 0 \leq t \leq 1, \end{aligned}$$

we deduce that $h(t) \geq h(1) = 0$, concluding the proof of (2.1).

- *Step (i)-2: we prove that, for any $x \neq 0$, the function*

$$G(x,y) := p|x|^{p-2}|y|^2 + p(p-2)|w|^{p-2}\big(|x| - |x+y|\big)^2$$

satisfies the lower bound

$$G(x,y) \geq c(p)\frac{|x|}{|x|+|y|}|x|^{p-2}|y|^2, \qquad \textit{for some } c(p) > 0. \tag{2.2}$$

Indeed, when $|x+y| < |x|$, by the triangle inequality and the fact that $1 < p < 2$ we get

$$G(x,y) = p|x|^{p-2}\Big(|y|^2 - (2-p)\big(|x|-|x+y|\big)^2\Big) \geq p|x|^{p-2}\Big(|y|^2 - (2-p)|y|^2\Big) = p(p-1)|x|^{p-2}|y|^2,$$

which implies (2.2). On the other hand, when $|x+y| \geq |x| > 0$ we have

$$|w|^{p-2} = \frac{|x+y|}{(2-p)|x+y| + (p-1)|x|}|x|^{p-2}.$$

Therefore, using again the triangle inequality we obtain

$$\begin{aligned} G(x,y) &\geq p\left(|x|^{p-2}|y|^2 + (p-2)|w|^{p-2}|y|^2\right)\\ &= p|x|^{p-2}|y|^2\frac{(p-1)|x|}{(2-p)|x+y|+(p-1)|x|} \geq p|x|^{p-2}|y|^2\frac{(p-1)|x|}{(2-p)|y|+|x|}, \end{aligned}$$

and (2.2) follows.

- *Step (i)-3: conclusion.* As a consequence of (2.2), we know that $G(x,y) \geq 0$ and it vanishes only if $y = 0$ (by assumption $x \neq 0$). Thanks to this fact and recalling (2.1), we get the following: for any $\kappa > 0$ and $x \neq 0$, the inequality

$$|x+y|^p \geq |x|^p + p|x|^{p-2}x\cdot y + \frac{1-\kappa}{2}\Big(p|x|^{p-2}|y|^2 + p(p-2)|w|^{p-2}\big(|x|-|x+y|\big)^2\Big)$$

holds, and equality is attained if and only if $y = 0$.

We now prove the inequality in the statement of the lemma by contradiction: If the inequality is false, there exist sequences x_i and y_i such that

$$\begin{aligned} |x_i+y_i|^p \leq |x_i|^p + p|x_i|^{p-2}x_i\cdot y_i + \frac{1-\kappa}{2}\Big(p|x_i|^{p-2}|y_i|^2 + p(p-2)|w_i|^{p-2}\big(|x_i|-|x_i+y_i|\big)^2\Big)&\\ +\frac{1}{i}\min\left\{|y_i|^p,\, |x_i|^{p-2}|y_i|^2\right\},& \end{aligned} \tag{2.3}$$

where w_i corresponds to x_i and $x_i + y_i$. By homogeneity (rescaling both x_i and y_i by the same factor $\frac{1}{|x_i|}$) we may assume that $|x_i| = 1$, and up to passing to a subsequence we can assume that $x_i \to \bar{x}$ as $i \to \infty$.

Note that, when $|y_i|$ is large enough, the left hand side in (2.3) behaves like $|y_i|^p$ while the right hand side is bounded by $C(p)|y_i| + \frac{1}{i}|y_i|^p$. This implies that the sequence y_i is uniformly bounded, and up to a subsequence y_i converges to $\bar{y}$. Hence, taking the limit in (2.3) we deduce that

$$|\bar{x}+\bar{y}|^p \le |\bar{x}|^p + p|\bar{x}|^{p-2}\bar{x}\cdot\bar{y} + \frac{1-\kappa}{2}\left(p|\bar{x}|^{p-2}|\bar{y}|^2 + p(p-2)|\bar{w}|^{p-2}\big(|\bar{x}| - |\bar{x}+\bar{y}|\big)^2\right),$$

which is possible only if $\bar{y} = 0$. This means that $y_i \to 0$. However, for $|x| = 1$ and $|y| \ll 1$, it follows from a Taylor expansion that

$$|x+y|^p - \left[|x|^p + p|x|^{p-2}x\cdot y + \frac{1-\kappa}{2}\left(p|x|^{p-2}|y|^2 + p(p-2)|w|^{p-2}\big(|x| - |x+y|\big)^2\right)\right] \ge \frac{\kappa}{3}|y|^2,$$

which is incompatible with (2.3) when $i \gg 1$. This leads to a contradiction and proves the lemma for $1 < p < 2$.

• *Proof of (ii): the case $p \ge 2$.* By approximation we can assume that $|x+y| \ne 0$ and $|x| \ne 0$.

- *Step (ii)-1: we show that*

$$|x+y|^p \ge |x|^p + p|x|^{p-2}x\cdot y + \frac{1}{2}\left(p|x|^{p-2}|y|^2 + p(p-2)|w|^{p-2}\big(|x| - |x+y|\big)^2\right). \tag{2.4}$$

Setting $z = x + y$, this is equivalent to proving that

$$|z|^p \ge \left(1 - \frac{1}{2}p\right)|x|^p + \frac{1}{2}p|x|^{p-2}|z|^2 + \frac{1}{2}p(p-2)|w|^{p-2}\big(|x| - |z|\big)^2.$$

Set $f(z) := |z|^p$ and

$$g(z) := \left(1 - \frac{1}{2}p\right)|x|^p + \frac{1}{2}p|x|^{p-2}|z|^2 + \frac{1}{2}p(p-2)|w|^{p-2}\big(|x| - |z|\big)^2.$$

In the case $|z| \ge |x|$ we note that $f = g$ and $Df = Dg$ on $\partial B(0, |x|)$. Also,

$$D^2 f(z)\frac{z}{|z|}\cdot\frac{z}{|z|} = p(p-1)|z|^{p-2} \ge p(p-1)|x|^{p-2} = D^2 g(z)\frac{z}{|z|}\cdot\frac{z}{|z|} \qquad \forall\, |z| \ge |x|.$$

Hence, integrating the Hessian of $f - g$ along the segment $\left[\frac{|x|}{|z|}z, z\right]$, we obtain that $f(z) \ge g(z)$ for $|z| \ge |x|$.

On the other hand, in the case $|z| < |x|$, our aim is to prove that

$$|z|^p \ge \left(1 - \frac{1}{2}p\right)|x|^p + \frac{1}{2}p|x|^{p-2}|z|^2 + \frac{1}{2}p(p-2)\frac{|z|}{|x|}|z|^{p-2}\big(|x| - |z|\big)^2.$$

Setting $t := \frac{|x|}{|z|}$, this is equivalent to saying that

$$h(t) := 1 - \left(1 - \frac{1}{2}p\right)t^p - \frac{1}{2}pt^{p-2} - \frac{1}{2}p(p-2)\frac{(t-1)^2}{t} \ge 0 \qquad \forall\, t \ge 1.$$

Since $p \ge 2$, a direct computation shows that, for $t \ge 1$,

$$\begin{aligned} h'(t) &= -\left(1 - \frac{1}{2}p\right)pt^{p-1} - \frac{1}{2}p(p-2)t^{p-3} - p(p-2)t^{-1}(t-1) + \frac{1}{2}p(p-2)t^{-2}(t-1)^2 \\ &= \frac{1}{2}p(p-2)\left[t^{p-1} - t^{p-3} - 2t^{-1}(t-1) + t^{-2}(t-1)^2\right] \\ &= \frac{1}{2}p(p-2)\frac{t-1}{t^2}\left[t^{p-1}(t+1) - 2t + (t-1)\right] = \frac{1}{2}p(p-2)\frac{t-1}{t^2}(t^{p-1} - 1)(t+1) \ge 0. \end{aligned}$$

This implies that $h(t) \geq h(1) = 0$ for $t \geq 1$, concluding the proof of (2.4).

- *Step (ii)-2: conclusion.* Thanks to Step (ii)-1 we deduce that, for any $\kappa > 0$ and $x \neq 0$, the inequality

$$|x+y|^p \geq |x|^p + p|x|^{p-2}x \cdot y + \frac{1-\kappa}{2}\Big(p|x|^{p-2}|y|^2 + p(p-2)|w|^{p-2}\big(|x| - |x+y|\big)^2\Big)$$

becomes an equality if and only if $y = 0$ (note that, since $p \geq 2$, the last term above is trivially positive for $y \neq 0$). So, if the statement of the lemma does not hold, we can find sequences x_i and y_i such that

$$|x_i+y_i|^p \leq |x_i|^p + p|x_i|^{p-2}x_i \cdot y_i + \frac{1-\kappa}{2}\Big(p|x_i|^{p-2}|y_i|^2 + p(p-2)|w_i|^{p-2}\big(|x_i| - |x_i+y_i|\big)^2\Big) + \frac{1}{i}|y_i|^p,$$

where w_i corresponds to x_i and $x_i + y_i$. As before, by homogeneity we may assume that $|x_i| = 1$, and that $x_i \to \bar{x}$ as $i \to \infty$. Also, for $|y_i| \gg 1$, the left hand side above behaves like $|y_i|^p$ while the right hand side is bounded by $(1-\kappa)\frac{p(p-1)}{2}|y_i|^2 + \frac{1}{i}|y_i|^p$. Hence, since $\kappa > 0$ and $p \geq 2$, we deduce that y_i cannot go to ∞. This implies that y_i are uniformly bounded, and as in the previous case we deduce that the only possibility is that $y_i \to 0$. However, since

$$|x+y|^p - \left[|x|^p + p|x|^{p-2}x \cdot y + \frac{1-\kappa}{2}\Big(p|x|^{p-2}|y|^2 + p(p-2)|w|^{p-2}\big(|x| - |x+y|\big)^2\Big)\right] \geq \frac{\kappa}{3}|y|^2,$$

for $|x| = 1$ and $|y| \ll 1$, this leads to a contradiction when i is sufficiently large. □

We end this section with the following simple lemma.

Lemma 2.4. *(i) Let $1 < p \leq \frac{2n}{n+2}$. For any $\kappa > 0$ there exists $C_1 = C_1(p^*, \kappa) > 0$ such that, for every $a, b \in \mathbb{R}$ with $a \neq 0$, we have*

$$|a+b|^{p^*} \leq |a|^{p^*} + p^*|a|^{p^*-2}ab + \left(\frac{p^*(p^*-1)}{2} + \kappa\right)\frac{(|a| + C_1|b|)^{p^*}}{|a|^2 + |b|^2}|b|^2.$$

(ii) Let $\frac{2n}{n+2} < p < n$. For any $\kappa > 0$ there exists $C_1 = C_1(p^*, \kappa) > 0$ such that, for every $a, b \in \mathbb{R}$ with $a \neq 0$, we have

$$|a+b|^{p^*} \leq |a|^{p^*} + p^*|a|^{p^*-2}ab + \left(\frac{p^*(p^*-1)}{2} + \kappa\right)|a|^{p^*-2}|b|^2 + C_1|b|^{p^*}.$$

Proof. Note that (ii) follows from [26, Lemma 3.2], so we only need to show (i). Observe that in this case $p^* \leq 2$.

Setting $t := \frac{b}{a}$, our statement is equivalent to proving that

$$|1+t|^{p^*} - 1 - p^*t - \left(\frac{p^*(p^*-1)}{2} + \kappa\right)\frac{(1 + C_1|t|)^{p^*}}{1 + |t|^2}|t|^2 \leq 0 \tag{2.5}$$

for any $t \in \mathbb{R}$ and some $C_1 > 0$.

First of all, by a Taylor expansion,

$$|1+t|^{p^*} = 1 + p^*t + \frac{p^*(p^*-1)}{2}|t|^2 + o(|t|^2) \qquad \forall\, |t| \ll 1.$$

Also, by the concavity of $t \mapsto t^{\frac{1}{p^*}}$ we have

$$1 + \frac{1}{p^*}|t|^2 \geq (1 + |t|^2)^{\frac{1}{p^*}} \qquad \forall\, |t| \leq 1.$$

Therefore there exists $t_0 = t_0(p^*) > 0$ small such that, for any $C_1 \geq \frac{1}{p^*}$,

$$|1+t|^{p^*} - 1 - p^*t \leq \left(\frac{p^*(p^*-1)}{2} + \kappa\right)\frac{(1 + C_1|t|)^{p^*}}{1 + |t|^2}|t|^2 \qquad \forall\, t \in [-t_0, t_0].$$

On the other hand, for $|t| > t_0$ we can rewrite (2.5) as

$$\left[\left((1+|t|^2)\frac{|1+t|^{p^*}-1-p^*t}{\left(\frac{p^*(p^*-1)}{2}+\kappa\right)|t|^2}\right)^{\frac{1}{p^*}}-1\right]|t|^{-1} \le C_1. \tag{2.6}$$

Since the left-hand side of (2.6) is bounded as $|t| \to +\infty$, the existence of a constant $C_1 < +\infty$ such that (2.6) holds on $\mathbb{R} \setminus (-t_0, t_0)$ follows by compactness. □

3. Spectral gaps

Let $v = v_{a,b,x_0} \in \mathcal{M}$. The goal of this section is to study some embedding/compacteness theorems and spectral gaps for weighted Sobolev/Orlicz-type spaces, where the weights depend on v. Throughout this section we assume that $a_0 > 0$, $b = 1$, and $x_0 = 0$, that is

$$v(x) = \frac{a_0}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}}.$$

Also, we assume that $a_0 > 0$ is such that $\frac{1}{2} \le \|v\|_{L^{p^*}} \le 2$.

Given $\Omega \subset \mathbb{R}^n$, $q \ge 1$, and a non-negative locally integrable function $g_0 : \mathbb{R}^n \to \mathbb{R}$, we define the Banach space $L^q(\Omega; g_0)$ as the space of measurable functions $\varphi : \Omega \to \mathbb{R}$ whose norm

$$\|\varphi\|_{L^q(\Omega;\, g_0)} := \left(\int_\Omega |\varphi|^q\, g_0(x)\, dx\right)^{\frac{1}{q}}$$

is finite. Also, given $g_1 \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ non-negative, we denote by $C^1_{c,0}(\mathbb{R}^n)$ the space of compactly supported functions of class C^1 that are constant in a neighborhood of the origin, and we define $\dot W^{1,q}(\mathbb{R}^n; g_1)$ as the closure of $C^1_{c,0}(\mathbb{R}^n)$ with respect to the norm

$$\|\varphi\|_{\dot W^{1,q}(\mathbb{R}^n;\, g_1)} := \left(\int_{\mathbb{R}^n} |D\varphi|^q\, g_1(x)\, dx\right)^{\frac{1}{q}}.$$

Remark 3.1. It is important for us to consider weights that are not necessarily integrable at the origin, since $|Dv|^{p-2} \sim |x|^{\frac{p-2}{p-1}} \notin L^1(B_1)$ for $p \le \frac{n+2}{n+1}$. This is why, when defining weighted Sobolev spaces, we consider the space $C^1_{c,0}(\mathbb{R}^n)$, so that gradients vanish near 0. Of course, replacing $C^1_c(\mathbb{R}^n)$ by $C^1_{c,0}(\mathbb{R}^n)$ plays no role in the case $p > \frac{n+2}{n+1}$.

3.1. Compact embedding.

The following embedding theorem generalizes [26, Corollary 6.2].

Proposition 3.2. *Let $1 < p < n$. The space $\dot W^{1,2}(\mathbb{R}^n; |Dv|^{p-2})$ compactly embeds into $L^2(\mathbb{R}^n; v^{p^*-2})$.*

To prove this result, we first show an intermediate estimate that will be useful also later.

Lemma 3.3. *Let $1 < p < n$, and $\varphi \in \dot W^{1,2}(\mathbb{R}^n; |Dv|^{p-2}) \cap L^2(\mathbb{R}^n; v^{p^*-2})$. Then*

$$\int_{\mathbb{R}^n} v^{p^*-2}|\varphi|^2\, dx \le C(n,p)\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2\, dx. \tag{3.1}$$

Also, there exists $\vartheta = \vartheta(n,p) > 0$ such that, for any $\rho \in (0,1)$, we have

$$\int_{B(0,\rho)} v^{p^*-2}|\varphi|^2\, dx \le C(n,p)\rho^\vartheta \int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2\, dx \tag{3.2}$$

and

$$\int_{\mathbb{R}^n\setminus B(0,\rho^{-1})} v^{p^*-2}|\varphi|^2\, dx \le \frac{C(n,p)}{|\log\rho|^2}\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2\, dx. \tag{3.3}$$

Proof. To prove (3.1), we can assume by approximation that $\varphi \in C^1_{c,0}(\mathbb{R}^n)$ (see Remark 3.1).

We note that, thanks to Fubini's theorem and using polar coordinates,

$$\begin{aligned}\int_{\mathbb{R}^n} v^{p^*-2}|\varphi|^2\,dx &\le C(n,p)\int_{\mathbb{S}^{n-1}}\int_0^\infty r^{n-1}(1+r^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}|\varphi(r\theta)|^2\,dr\,d\theta\\ &\le C(n,p)\int_{\mathbb{S}^{n-1}}\int_0^\infty r^{n-1}(1+r^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}\int_r^\infty |\varphi(t\theta)||D\varphi(t\theta)|\,dt\,dr\,d\theta\\ &\le C(n,p)\int_{\mathbb{S}^{n-1}}\int_0^\infty\int_0^t |\varphi(t\theta)||D\varphi(t\theta)|r^{n-1}(1+r^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}\,dr\,dt\,d\theta\\ &\le C(n,p)\int_{\mathbb{S}^{n-1}}\int_0^\infty |\varphi(t\theta)||D\varphi(t\theta)|t^n(1+t^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}\,dt\,d\theta.\end{aligned}$$

Thus, by Cauchy-Schwarz inequality we get

$$\begin{aligned}\int_{\mathbb{R}^n} v^{p^*-2}|\varphi|^2\,dx \le C(n,p)&\Big(\int_{\mathbb{S}^{n-1}}\int_0^\infty |D\varphi(t\theta)|^2t^{n+1}(1+t^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}\,dt\,d\theta\Big)^{1/2}\cdot\\ &\cdot\Big(\int_{\mathbb{S}^{n-1}}\int_0^\infty t^{n-1}(1+t^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}|\varphi(t\theta)|^2\,dt\,d\theta\Big)^{1/2},\end{aligned}$$

and since the last term in the right hand side coincides with $\|\varphi\|_{L^2(\mathbb{R}^n;v^{p^*-2})}$ (up to a multiplicative constant), we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} v^{p^*-2}|\varphi|^2\,dx &\le C(n,p)\int_{\mathbb{S}^{n-1}}\int_0^\infty |D\varphi(t\theta)|^2t^{n+1}(1+t^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}\,dt\,d\theta\\ &\le C(n,p)\int_{\mathbb{R}^n}|D\varphi(x)|^2|x|^2(1+|x|^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}\,dx.\end{aligned}\tag{3.4}$$

We now observe that

$$|x|^2\sim|x|^{1+\frac{1}{p-1}}|Dv|^{p-2}\le C(n,p)\,|Dv|^{p-2}\qquad\text{when } |x|\in(0,1],$$

and

$$|x|^2(1+|x|^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}\sim|x|^{-\frac{p}{p-1}}|Dv|^{p-2}\le C(n,p)\,|Dv|^{p-2}\qquad\text{when } |x|\in(1,\infty),$$

so (3.1) follows from (3.4).

To prove (3.2), we apply (3.1) and the Sobolev inequality with radial weights (see e.g. [34, Section 2.1]). More precisely, since $|Dv|^{p-2}\ge c(n,p)|x|$ inside $B(0,1)$,

$$\begin{aligned}\int_{\mathbb{R}^n}|Dv|^{p-2}|D\varphi|^2\,dx &\ge c(n,p)\int_{\mathbb{R}^n}\big(v^{p^*-2}|\varphi|^2+|Dv|^{p-2}|D\varphi|^2\big)\,dx\\ &\ge c(n,p)\int_{B(0,1)}\big(|\varphi|^2+|x|\,|D\varphi|^2\big)\,dx\ge\Big(\int_{B(0,1)}|\varphi|^q\,dx\Big)^{\frac{2}{q}},\end{aligned}$$

where $q=q(n)>2$. Thus, by Hölder inequality, for any $\rho\in(0,1)$ we get

$$\begin{aligned}\int_{B(0,\rho)} v^{p^*-2}|\varphi|^2\,dx &\le C(n,p)\int_{B(0,\rho)}|\varphi|^2\,dx\\ &\le C(n,p)\rho^{n\left(1-\frac{2}{q}\right)}\Big(\int_{B(0,\rho)}|\varphi|^q\,dx\Big)^{\frac{2}{q}}\le C(n,p)\rho^{n\left(1-\frac{2}{q}\right)}\int_{\mathbb{R}^n}|Dv|^{p-2}|D\varphi|^2\,dx,\end{aligned}$$

as desired.

To prove (3.3), we define

$$\chi_\rho(x) := \begin{cases} 0 & \text{for } |x| < \rho^{-1/2} \\ \frac{2\log|x| - |\log\rho|}{|\log\rho|} & \text{for } \rho^{-1/2} \le |x| \le \rho^{-1} \\ 1 & \text{for } \rho^{-1} \le |x| \end{cases}$$

and we apply (3.4) to the function $\phi_\rho := \chi_\rho \varphi$:

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B(0,\rho^{-1})} v^{p^*-2}|\varphi|^2\,dx &\le \int_{\mathbb{R}^n} v^{p^*-2}|\phi_\rho|^2\,dx \le C(n,p)\int_{\mathbb{R}^n} |x|^2(1+|x|^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}|D\phi_\rho|^2\,dx \\
&\le C(n,p)\int_{\mathbb{R}^n} |x|^2(1+|x|^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}\chi_\rho^2\,|D\varphi|^2\,dx \\
&\quad + C(n,p)\int_{\mathbb{R}^n} |x|^2(1+|x|^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}|D\chi_\rho|^2\varphi^2\,dx \\
&\le C(n,p)\int_{\mathbb{R}^n\setminus B(0,\rho^{-1/2})} |x|^2(1+|x|^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}|D\varphi|^2\,dx \\
&\quad + C(n,p)|\log\rho|^{-2}\int_{B(0,\rho^{-1})\setminus B(0,\rho^{-1/2})} (1+|x|^{\frac{p}{p-1}})^{-\frac{n(p-2)}{p}-2}\varphi^2\,dx \\
&\le C(n,p)\rho^{\frac{p}{p-1}}\int_{\mathbb{R}^n\setminus B(0,\rho^{-1})} |Dv|^{p-2}|D\varphi|^2\,dx \\
&\quad + C(n,p)|\log\rho|^{-2}\int_{B(0,\rho^{-1})\setminus B(0,\rho^{-1/2})} |v|^{p^*-2}\varphi^2\,dx \\
&\le C(n,p)|\log\rho|^{-2}\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2\,dx,
\end{aligned}$$

where the last inequality follows from (3.1). □

Proof of Proposition 3.2. Let φ_i be a sequence of functions in $\dot W^{1,2}(\mathbb{R}^n;|Dv|^{p-2})$ with uniformly bounded norm. It follows by (3.1) that their $L^2(\mathbb{R}^n;v^{p^*-2})$ norm is uniformly bounded as well.

Since both $|Dv|^{p-2}$ and v^{p^*-2} are locally bounded away from zero and infinity in $\mathbb{R}^n\setminus\{0\}$, by Rellich-Kondrachov Theorem and a diagonal argument we deduce that, up to a subsequence, φ_i converges to some function φ both weakly in $\dot W^{1,2}(\mathbb{R}^n;|Dv|^{p-2})\cap L^2(\mathbb{R}^n;v^{p^*-2})$ and strongly in $L^2_{\rm loc}(\mathbb{R}^n\setminus\{0\};v^{p^*-2})$.

Also, (3.2) and (3.3) imply that, for any $\rho\in(0,1)$,

$$\int_{\mathbb{R}^n\setminus B(0,\rho)} v^{p^*-2}|\varphi_i|^2\,dx \le C(n,p)\rho^\vartheta, \qquad \int_{\mathbb{R}^n\setminus B(0,\rho^{-1})} v^{p^*-2}|\varphi_i|^2\,dx \le \frac{C(n,p)}{|\log\rho|^2}.$$

We conclude the proof considering the compact set $K_\rho := \overline{B(0,\rho^{-1})}\setminus B(0,\rho)$ and applying the strong convergence of φ_i inside K_ρ, together with the arbitrariness of ρ (that can be chosen arbitrarily small). □

As we shall see, the previous result allows us to deal with the case $p > \frac{2n}{n+2}$. However, when $1 < p \le \frac{2n}{n+2}$, we will need a much more delicate compactness result that we now present.

Lemma 3.4. *Let $1 < p \le \frac{2n}{n+2}$, and let ϕ_i be a sequence of functions in $\dot W^{1,p}(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} \big(|Dv| + \epsilon_i|D\phi_i|\big)^{p-2}|D\phi_i|^2\,dx \le 1, \tag{3.5}$$

where $\epsilon_i \in (0,1)$ is a sequence of positive numbers converging to 0. Then, up to a subsequence, ϕ_i converges weakly in $\dot W^{1,p}(\mathbb{R}^n)$ to some function $\phi \in \dot W^{1,p}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n;v^{p^*-2})$. Also, given any constant

$C_1 \geq 0$ *it holds*[2]

$$\int_{\mathbb{R}^n} \frac{(v + C_1\epsilon_i\phi_i)^{p^*}}{v^2 + |\epsilon_i\phi_i|^2}|\phi_i|^2\,dx \to \int_{\mathbb{R}^n} v^{p^*-2}|\phi|^2\,dx \qquad \text{as } i \to \infty. \tag{3.6}$$

Proof. Up to replacing ϕ_i by $|\phi_i|$, we can assume that $\phi_i \geq 0$. Note that $p < p^* \leq 2$ under our assumption.

Observe that, by Hölder inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} |D\phi_i|^p\,dx &\leq \left(\int_{\mathbb{R}^n} \big(|Dv| + \epsilon_i|D\phi_i|\big)^{p-2}|D\phi_i|^2\,dx\right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^n} \big(|Dv| + \epsilon_i|D\phi_i|\big)^{p}\,dx\right)^{1-\frac{p}{2}} \\ &\leq C(n,p)\left(\int_{\mathbb{R}^n} \big(|Dv| + \epsilon_i|D\phi_i|\big)^{p-2}|D\phi_i|^2\,dx\right)^{\frac{p}{2}} \left(1 + \epsilon_i^p\int_{\mathbb{R}^n} |D\phi_i|^p\,dx\right)^{1-\frac{p}{2}},\end{aligned}$$

that combined with (3.5) gives

$$\left(\int_{\mathbb{R}^n} |D\phi_i|^p\,dx\right)^{\frac{2}{p}} \leq C(n,p)\int_{\mathbb{R}^n} \big(|Dv| + \epsilon_i|D\phi_i|\big)^{p-2}|D\phi_i|^2\,dx \leq C(n,p). \tag{3.7}$$

Thus, up to a subsequence, ϕ_i converges weakly in $\dot{W}^{1,p}(\mathbb{R}^n)$ and also a.e. to some function $\phi \in \dot{W}^{1,p}(\mathbb{R}^n)$. Hence, to conclude the proof, we need to show that $\phi \in L^2(\mathbb{R}^n; v^{p^*-2})$ and the validity of (3.6).

We first prove these facts under the assumption that $\epsilon_i\phi_i \leq \zeta v$, with $\zeta = \zeta(n,p,C_1) \in (0,1)$ a small constant to be determined. Later, we will remove this assumption.

• *Step 1: proof of* (3.6) *when* $\epsilon_i\phi_i \leq \zeta v$. Since $\epsilon_i\phi_i$ is bounded by $\zeta v \leq v$, we have that $\left(1 + \frac{\epsilon_i\phi_i}{v}\right) \leq 2$, thus

$$\begin{aligned}\int_{\mathbb{R}^n} (v + \epsilon_i\phi_i)^{p^*-2}|\phi_i|^2\,dx &\leq \int_{\mathbb{R}^n} v^{p^*-2}\left(1 + \frac{\epsilon_i\phi_i}{v}\right)^{p^*-2}|\phi_i|^2\,dx \\ &\leq 2^{p^*-p}\int_{\mathbb{R}^n} v^{p^*-2}\left(1 + \frac{\epsilon_i\phi_i}{v}\right)^{p-2}|\phi_i|^2\,dx.\end{aligned}$$

Recall that

$$v \sim (1 + |x|^{\frac{p}{p-1}})^{1-\frac{n}{p}} \qquad \text{and} \qquad |Dv| \sim (1 + |x|^{\frac{p}{p-1}})^{-\frac{n}{p}}|x|^{\frac{1}{p-1}}, \tag{3.8}$$

where the constants depend only on p and n. Moreover, the following Hardy-Poincaré inequality holds [38][3]: For any $p > 1$ and $\gamma \geq 1$, and any compactly supported function $\xi \in W^{1,p}(\mathbb{R}^n)$, one has

$$\int_{\mathbb{R}^n} |\xi|^p\left[\left(1 + |x|^{\frac{p}{p-1}}\right)^{p-1}\right]^{\gamma-1} dx \leq C(n,p,\gamma)\int_{\mathbb{R}^n} |D\xi|^p\left[\left(1 + |x|^{\frac{p}{p-1}}\right)^{p-1}\right]^{\gamma} dx.$$

By approximation, we can apply this inequality with

$$\gamma = 1 + \frac{(2-p^*)\left(\frac{n}{p} - 1\right)}{p-1} \qquad \text{and} \qquad \xi = \left(1 + \frac{\epsilon_i\phi_i}{v}\right)^{\frac{p-2}{p}}|\phi_i|^{\frac{2}{p}}.$$

[2]As already noticed in the introduction, the expression appearing in the left hand side of (3.6) behaves like $v^{p^*-2}|\phi_i|^2$ when $|\phi_i| \ll \frac{v}{\epsilon_i}$, and like $\epsilon_i^{p^*-2}|\phi_i|^{p^*}$ otherwise. Analogously, the expression in (3.5) behaves like $|Dv|^{p-2}|D\phi_i|^2$ when $|D\phi_i| \ll \frac{|Dv|}{\epsilon_i}$, and like $\epsilon_i^{p-2}|D\phi_i|^p$ otherwise. These substantial changes of behavior, and the fact that the change in size of the gradients does not necessarily correspond to a change in size of the functions, make the proof particularly delicate.

[3]More precisely, the case $\gamma > 1$ is stated in [38, Theorem 3.1], while the case $\gamma = 1$ follows from the classical Hardy inequality (see for instace [38, Theorem 4.1]).

Thus, since $v^{p^*-2} \sim \left[\left(1+|x|^{\frac{p}{p-1}}\right)^{p-1}\right]^{\gamma-1}$, we get

$$\begin{aligned}\int_{\mathbb{R}^n} (v+\epsilon_i\phi_i)^{p^*-2}|\phi_i|^2\,dx &\le C(n,p)\int_{\mathbb{R}^n} v^{p^*-2}\left(1+\frac{\epsilon_i\phi_i}{v}\right)^{p-2}|\phi_i|^2\,dx\\ &\le C(n,p)\left\|\left(1+\frac{\epsilon_i\phi_i}{v}\right)^{\frac{p-2}{p}}|\phi_i|^{\frac{2}{p}}\right\|^p_{\dot W^{1,p}\left(\mathbb{R}^n;v^{p^*-2}\left(1+|x|^{\frac{p}{p-1}}\right)^{p-1}\right)}\\ &\le C(n,p)\int_{\mathbb{R}^n} v^{p^*-2}\left(1+|x|^{\frac{p}{p-1}}\right)^{p-1}\cdot \\ &\quad\cdot\left[\left(1+\frac{\epsilon_i\phi_i}{v}\right)^{-2}|\phi_i|^2\left(\frac{\epsilon_i\phi_i|Dv|}{v^2}+\frac{\epsilon_i|D\phi_i|}{v}\right)^p+\left(1+\frac{\epsilon_i\phi_i}{v}\right)^{p-2}|\phi_i|^{2-p}|D\phi_i|^p\right]dx\\ &\le C(n,p)\int_{\mathbb{R}^n} v^{p^*-2}\left(1+|x|^{\frac{p}{p-1}}\right)^{p-1}\left[|\phi_i|^2\left(\frac{\zeta|Dv|}{v}+\frac{\epsilon_i|D\phi_i|}{v}\right)^p+|\phi_i|^{2-p}|D\phi_i|^p\right]dx,\end{aligned} \tag{3.9}$$

where, in the last inequality, we used that $0\le\frac{\epsilon_i\phi_i}{v}\le\zeta<1$.

We now apply (C.2) to the last integrand in (3.9) with $\epsilon=\epsilon_i, r=|x|, a=|\phi_i|, b=|D\phi_i|$. In this way, thanks to (3.9) and since $v+\epsilon_i\phi_i\le 2v$, we deduce that for any $\epsilon_0>0$ there exists $\zeta=\zeta(\epsilon_0)\in(0,1)$ such that

$$\begin{aligned}\int_{\mathbb{R}^n} v^{p^*-2}|\phi_i|^2\,dx &\le 2^{2-p^*}\int_{\mathbb{R}^n}(v+\epsilon_i\phi_i)^{p^*-2}|\phi_i|^2\,dx\\ &\le C(n,p)\left\|\left(1+\frac{\epsilon_i\phi_i}{v}\right)^{\frac{p-2}{p}}|\phi_i|^{\frac{2}{p}}\right\|^p_{\dot W^{1,p}\left(\mathbb{R}^n;v^{p^*-2}\left(1+|x|^{\frac{p}{p-1}}\right)^{p-1}\right)}\\ &\le C(n,p)\epsilon_0\int_{\mathbb{R}^n} v^{p^*-2}|\phi_i|^2\,dx+C(\epsilon_0,n,p)\int_{\mathbb{R}^n}\left(|Dv|+\epsilon_i|D\phi_i|\right)^{p-2}|D\phi_i|^2\,dx.\end{aligned}$$

Thus, fixing ϵ_0 small enough so that $C(n,p)\epsilon_0\le\frac12$, it follows from (3.5) and the inequality above that

$$\begin{aligned}\int_{\mathbb{R}^n} v^{p^*-2}|\phi_i|^2\,dx+\left\|\left(1+\frac{\epsilon_i\phi_i}{v}\right)^{\frac{p-2}{p}}|\phi_i|^{\frac{2}{p}}\right\|^p_{\dot W^{1,p}\left(\mathbb{R}^n;v^{p^*-2}\left(1+|x|^{\frac{p}{p-1}}\right)^{p-1}\right)}&\\ \le C(n,p)\int_{\mathbb{R}^n}\left(|Dv|+\epsilon_i|D\phi_i|\right)^{p-2}|D\phi_i|^2\,dx\le C(n,p).&\end{aligned} \tag{3.10}$$

In particular, the sequence $\left(1+\frac{\epsilon_i\phi_i}{v}\right)^{\frac{p-2}{p}}|\phi_i|^{\frac{2}{p}}$ is uniformly bounded in $\dot W^{1,p}_{\rm loc}(\mathbb{R}^n)\subset L^{p^*}_{\rm loc}(\mathbb{R}^n)$. Since $\left(1+\frac{\epsilon_i\phi_i}{v}\right)\sim 1$, this implies that $|\phi_i|^{\frac{2}{p}}\in L^{p^*}_{\rm loc}(\mathbb{R}^n)$. Combining this higher integrability estimate with the a.e. convergence of ϕ_i to ϕ, by dominated convergence we deduce that, for any $R>1$,

$$\int_{B(0,R)}\frac{(v+C_1\epsilon_i\phi_i)^{p^*}}{v^2+|\epsilon_i\phi_i|^2}|\phi_i|^2\,dx\to\int_{B(0,R)} v^{p^*-2}|\phi|^2\,dx \qquad \text{as } i\to\infty \tag{3.11}$$

(recall that $\epsilon_i\to 0$).

Also, since $1<p\le\frac{2n}{n+2}$ it follows that $n\ge 3$, and therefore

$$\frac{-np+2n-2p}{p-1}+n=\frac{n-2p}{p-1}>0.$$

This allows us to apply Lemma A.1 to ϕ_i with

$$\alpha=\frac{np-2n+2p}{p-1},$$

and similarly to (3.9) we obtain (recall (3.8))

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus B(0,R)} \frac{(v+C_1\epsilon_i\phi_i)^{p^*}}{v^2+|\epsilon_i\phi_i|^2}|\phi_i|^2\,dx\\
&\quad\le C(n,p,C_1)\int_{\mathbb{R}^n\setminus B(0,R)} v^{p^*-2}\left(1+\frac{\epsilon_i\phi_i}{v}\right)^{p-2}|\phi_i|^2\,dx\\
&\quad\le C(n,p,C_1)\int_{\mathbb{R}^n\setminus B(0,R)} |x|^{\frac{-np+2n-2p}{p-1}+p}\,\cdot\\
&\qquad\cdot\left[\left(1+\frac{\epsilon_i\phi_i}{v}\right)^{-2}|\phi_i|^2\left(\frac{\epsilon_i\phi_i|Dv|}{v^2}+\frac{\epsilon_i|D\phi_i|}{v}\right)^p+\left(1+\frac{\epsilon_i\phi_i}{v}\right)^{p-2}|\phi_i|^{2-p}|D\phi_i|^p\right]dx\\
&\quad\le C(n,p,C_1)\int_{\mathbb{R}^n\setminus B(0,R)} |x|^{\frac{-np+2n-2p}{p-1}+p}\left[|\phi_i|^2\left(\frac{\zeta|Dv|}{v}+\frac{\epsilon_i|D\phi_i|}{v}\right)^p+|\phi_i|^{2-p}|D\phi_i|^p\right]dx.
\end{aligned}
$$

Then, applying (C.1) to the last term above with $\epsilon=\epsilon_i, r=|x|, a=|\phi_i|, b=|D\phi_i|$, we obtain that for any $\epsilon_0'>0$ there exists $\zeta=\zeta(\epsilon_0')\in(0,1)$ such that

$$
\begin{aligned}
\int_{\mathbb{R}^n\setminus B(0,R)} \frac{(v+C_1\epsilon_i\phi_i)^{p^*}}{v^2+|\epsilon_i\phi_i|^2}|\phi_i|^2\,dx &\le C(n,p,C_1)\epsilon_0'\int_{\mathbb{R}^n\setminus B(0,R)} v^{p^*-2}|\phi_i|^2\,dx\\
&\quad+C(\epsilon_0',n,p,C_1)R^{-\frac{p}{p-1}}\int_{\mathbb{R}^n\setminus B(0,R)}\big(|Dv|+\epsilon_i|D\phi_i|\big)^{p-2}|D\phi_i|^2\,dx\\
&\le C(n,p,C_1)\epsilon_0'\int_{\mathbb{R}^n\setminus B(0,R)} \frac{(v+C_1\epsilon_i\phi_i)^{p^*}}{v^2+|\epsilon_i\phi_i|^2}|\phi_i|^2\,dx\\
&\quad+C(\epsilon_0',n,p,C_1)R^{-\frac{p}{p-1}}\int_{\mathbb{R}^n\setminus B(0,R)}\big(|Dv|+\epsilon_i|D\phi_i|\big)^{p-2}|D\phi_i|^2\,dx.
\end{aligned}
$$

Thus, by fixing ϵ_0' so that $C(n,p,C_1)\epsilon_0'\le\frac{1}{2}$, it follows that

$$
\int_{\mathbb{R}^n\setminus B(0,R)} \frac{(v+C_1\epsilon_i\phi_i)^{p^*}}{v^2+|\epsilon_i\phi_i|^2}|\phi_i|^2\,dx\le C(n,p,C_1)R^{-\frac{p}{p-1}}\int_{\mathbb{R}^n}\big(|Dv|+\epsilon_i|D\phi_i|\big)^{p-2}|D\phi_i|^2\,dx\le C(n,p)R^{-\frac{p}{p-1}}.
$$

Combining this bound with (3.10) and (3.11), by the arbitrariness of R we conclude that $\phi\in L^2(\mathbb{R}^n;v^{p^*-2})$ and that (3.6) holds. This concludes the proof under the assumption that $\epsilon_i\phi_i\le\zeta v$ with $\zeta=\zeta(n,p,C_1)>0$ sufficiently small.

• *Step 2: proof of* (3.6) *in the general case.* Throughout this part, we assume that $\zeta=\zeta(n,p,C_1)>0$ is a small constant so that Step 1 applies.

Observe that, by (1.1), ζv is a supersolution for the operator

$$
L_v[\psi]:=-\mathrm{div}\left(\big(|Dv|+|D\psi|\big)^{p-2}D\psi+(p-2)\big(|Dv|+|D\psi|\big)^{p-3}|D\psi|D\psi\right),
$$

namely $L_v[\zeta v]\ge0$. Therefore, multiplying $L_v[\zeta v]\ge0$ by $(\epsilon_i\phi_i-\zeta v)_+$ and integrating by parts, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\big(|Dv|+\zeta|Dv|\big)^{p-2}\zeta Dv\cdot D(\epsilon_i\phi_i-\zeta v)_+\,dx\\
&\qquad+(p-2)\int_{\mathbb{R}^n}\big(|Dv|+\zeta|Dv|\big)^{p-3}\zeta^2|Dv|Dv\cdot D(\epsilon_i\phi_i-\zeta v)_+\,dx\ge0. \qquad (3.12)
\end{aligned}
$$

Also, by the convexity of

$$
\mathbb{R}^n\ni z\mapsto F_t(z):=(t+|z|)^{p-2}|z|^2,\qquad t\ge0,
$$

we have

$$F_t(z) + DF_t(z)\cdot(z'-z) \le F_t(z') \qquad \forall\, z,z'\in\mathbb{R}^n,\ t\ge 0.$$

Hence, applying this inequality with $t=|Dv|$, $z=\zeta Dv$, and $z'=\epsilon_i D\phi_i$, it follows by (3.12) that

$$\begin{aligned}
c(n,p)\epsilon_i^{-2}\int_{\{\epsilon_i\phi_i>\zeta v\}}|Dv|^p\,dx &\le \epsilon_i^{-2}\int_{\{\epsilon_i\phi_i>\zeta v\}}\big(|Dv|+\zeta|Dv|\big)^{p-2}\zeta^2|Dv|^2\,dx\\
&\le \epsilon_i^{-2}\int_{\{\epsilon_i\phi_i>\zeta v\}}\big(|Dv|+\zeta|Dv|\big)^{p-2}\zeta^2|Dv|^2\,dx\\
&\quad+\epsilon_i^{-2}\int_{\{\epsilon_i\phi_i>\zeta v\}}\big(|Dv|+\zeta|Dv|\big)^{p-2}\zeta Dv\cdot D(\epsilon_i\phi_i-\zeta v)_+\,dx\\
&\quad+\epsilon_i^{-2}(p-2)\int_{\{\epsilon_i\phi_i>\zeta v\}}\big(|Dv|+\zeta|Dv|\big)^{p-3}\zeta^2|Dv|Dv\cdot D(\epsilon_i\phi_i-\zeta v)_+\,dx\\
&\le \int_{\{\epsilon_i\phi_i>\zeta v\}}\big(|Dv|+\epsilon_i|D\phi_i|\big)^{p-2}|D\phi_i|^2\,dx.
\end{aligned}\tag{3.13}$$

We now write $\phi_i=\phi_{i,1}+\phi_{i,2}$, where

$$\phi_{i,1}:=\min\left\{\phi_i,\ \frac{\zeta v}{\epsilon_i}\right\},\qquad \phi_{i,2}:=\phi_i-\phi_{i,1}.$$

Note that, as a consequence of (3.5) and (3.13),

$$\begin{aligned}
\int_{\mathbb{R}^n}\big(|Dv|+\epsilon_i|D\phi_{i,1}|\big)^{p-2}|D\phi_{i,1}|^2\,dx+\int_{\mathbb{R}^n}\big(|Dv|+\epsilon_i|D\phi_{i,2}|\big)^{p-2}|D\phi_{i,2}|^2\,dx\\
\le C(n,p)\int_{\mathbb{R}^n}\big(|Dv|+\epsilon_i|D\phi_i|\big)^{p-2}|D\phi_i|^2\,dx\le C(n,p).
\end{aligned}\tag{3.14}$$

Hence, it follows by the analogue of (3.7) that

$$\int_{\mathbb{R}^n}|D\phi_{i,1}|^p\,dx+\int_{\mathbb{R}^n}|D\phi_{i,2}|^p\,dx\le C(n,p).\tag{3.15}$$

In particular we deduce that $\phi_{i,2}\rightharpoonup 0$ in $\dot W^{1,p}(\mathbb{R}^n)$ (as $|\{\epsilon_i\phi_i>\zeta v\}\cap B(0,R)|\to 0$ for any $R>1$) and that, up to a subsequence, both ϕ_i and $\phi_{i,1}$ converge weakly in $\dot W^{1,p}(\mathbb{R}^n)$ and also a.e. to the same function $\phi\in\dot W^{1,p}(\mathbb{R}^n)$.

Let $\eta=\eta(n,p)>0$ be a small exponent to be fixed. We analyze two cases.

- *Case 1.* If

$$\int_{\{\epsilon_i\phi_i>\zeta v\}}|\phi_{i,1}|^{p^*}\,dx>\epsilon_i^{-\eta}\int_{\{\epsilon_i\phi_i>\zeta v\}}\left(\phi_i-\frac{\zeta v}{\epsilon_i}\right)_+^{p^*}dx=\epsilon_i^{-\eta}\int_{\{\epsilon_i\phi_i>\zeta v\}}|\phi_{i,2}|^{p^*}\,dx,$$

since ϕ is also the limit of $\phi_{i,1}$, we can apply Step 1 to $\phi_{i,1}$ to deduce that $\phi\in L^2(\mathbb{R}^n;v^{p^*-2})$ and

$$\int_{\mathbb{R}^n}\frac{(v+C_1\epsilon_i\phi_i)^{p^*}}{v^2+|\epsilon_i\phi_i|^2}|\phi_i|^2\,dx=\big(1+O(\epsilon_i^\eta)\big)\int_{\mathbb{R}^n}\frac{(v+C_1\epsilon_i\phi_{i,1})^{p^*}}{v^2+|\epsilon_i\phi_{i,1}|^2}|\phi_{i,1}|^2\,dx\to\int_{\mathbb{R}^n}v^{p^*-2}|\phi|^2\,dx,$$

which proves (3.6).

- *Case 2.* Assume now that

$$\int_{\{\epsilon_i\phi_i>\zeta v\}}|\phi_{i,1}|^{p^*}\,dx\le\epsilon_i^{-\eta}\int_{\{\epsilon_i\phi_i>\zeta v\}}|\phi_{i,2}|^{p^*}\,dx.\tag{3.16}$$

We claim that

$$\epsilon_i^{p^*-2}\int_{\mathbb{R}^n}|\phi_{i,2}|^{p^*}\,dx=O(\epsilon_i^\eta).\tag{3.17}$$

To prove this, denote $A_i := \{\epsilon_i\phi_i > \zeta v\}$ and define

$$E_i := \left\{|D\phi_{i,2}| \le \frac{|Dv|}{\epsilon_i}\right\} \cap A_i, \qquad F_i := \left\{|D\phi_{i,2}| > \frac{|Dv|}{\epsilon_i}\right\} \cap A_i.$$

Then, since $|Dv| + \epsilon_i|D\phi_{i,2}| \le 2|Dv|$ inside E_i, it follows by Hölder inequality that

$$\begin{aligned}
\int_{\mathbb{R}^n} |D\phi_{i,2}|^p\,dx &= \int_{E_i} |D\phi_{i,2}|^p\,dx + \int_{F_i} |D\phi_{i,2}|^p\,dx \\
&\le \left(\int_{E_i} |Dv|^{p-2}|D\phi_{i,2}|^2\,dx\right)^{\frac{p}{2}} \left(\int_{E_i} |Dv|^p\,dx\right)^{1-\frac{p}{2}} + \int_{F_i} |D\phi_{i,2}|^p\,dx \\
&\le \left(2^{2-p}\int_{E_i} \big(|Dv| + \epsilon_i|D\phi_{i,2}|\big)^{p-2}|D\phi_{i,2}|^2\,dx\right)^{\frac{p}{2}} \left(\int_{E_i} |Dv|^p\,dx\right)^{1-\frac{p}{2}} + \int_{F_i} |D\phi_{i,2}|^p\,dx \\
&\le C(n,p)\left(\int_{E_i} \big(|Dv| + \epsilon_i|D\phi_{i,2}|\big)^{p-2}|D\phi_{i,2}|^2\,dx\right)^{\frac{p}{2}} \left(\int_{E_i} |Dv|^p\,dx\right)^{1-\frac{p}{2}} + \int_{F_i} |D\phi_{i,2}|^p\,dx.
\end{aligned} \tag{3.18}$$

Also, using (3.8) and (3.16) together with Hölder inequality (note that, since $1 < p \le \frac{2n}{n+2}$, we have $n \ge 3$) we get

$$\begin{aligned}
\int_{E_i} |Dv|^p\,dx &\le C(n,p)\int_{E_i} (1+|x|^{\frac{p}{p-1}})^{-n}|x|^{\frac{p}{p-1}}\,dx \\
&\le C(n,p)\left(\int_{E_i} \left((1+|x|^{\frac{p}{p-1}})^{-n+1}|x|^{\frac{p}{p-1}}\right)^{\frac{n}{n-2}}dx\right)^{1-\frac{2}{n}} \left(\int_{\mathbb{R}^n} (1+|x|^{\frac{p}{p-1}})^{-\frac{n}{2}}\,dx\right)^{\frac{2}{n}} \\
&\le C(n,p)\left(\int_{A_i} \left(\frac{\epsilon_i\phi_i}{\zeta v}\right)^{p^*} \left((1+|x|^{\frac{p}{p-1}})^{-n+1}|x|^{\frac{p}{p-1}}\right)^{\frac{n}{n-2}}dx\right)^{1-\frac{2}{n}} \\
&\le C(n,p)\left(\epsilon_i^{p^*}\int_{A_i} |\phi_i|^{p^*}\,dx\right)^{1-\frac{2}{n}} \le C(n,p)\left(\epsilon_i^{p^*-\eta}\int_{A_i} |\phi_{i,2}|^{p^*}\,dx\right)^{1-\frac{2}{n}},
\end{aligned} \tag{3.19}$$

where we used that $\frac{np}{2(p-1)} > n$ (since $p \le \frac{2n}{n+2} < 2$) and that

$$v^{-p^*}\left((1+|x|^{\frac{p}{p-1}})^{-n+1}|x|^{\frac{p}{p-1}}\right)^{\frac{n}{n-2}} \le C(n,p).$$

Therefore, introducing the notation

$$N_{i,2} := \int_{E_i} \big(|Dv| + \epsilon_i|D\phi_{i,2}|\big)^{p-2}|D\phi_{i,2}|^2\,dx,$$

by Sobolev inequality, (3.18), and (3.19), we deduce that

$$\begin{aligned}
\epsilon_i^{p^*-2}\int_{\mathbb{R}^n} |\phi_{i,2}|^{p^*}\,dx &\le C(n,p)\epsilon_i^{p^*-2}\left(\int_{\mathbb{R}^n} |D\phi_{i,2}|^p\,dx\right)^{\frac{p^*}{p}} \\
&\le C(n,p)\epsilon_i^{p^*-2}\left[N_{i,2}^{\frac{p^*}{2}}\left(\int_{E_i} |Dv|^p\,dx\right)^{\frac{(2-p)p^*}{2p}} + \left(\int_{F_i} |D\phi_{i,2}|^p\,dx\right)^{\frac{p^*}{p}}\right] \\
&\le C(n,p)\epsilon_i^{p^*-2}\left[N_{i,2}^{\frac{p^*}{2}}\left(\epsilon_i^{p^*-\eta}\int_{A_i} |\phi_{i,2}|^{p^*}\,dx\right)^{\frac{(2-p)(n-2)}{2(n-p)}} + \int_{F_i} |D\phi_{i,2}|^p\,dx\right],
\end{aligned} \tag{3.20}$$

where in the last inequality we used (3.15) and the fact that $\frac{p^*}{p} \ge 1$.

Suppose first that

$$\int_{F_i} |D\phi_{i,2}|^p\,dx \geq N_{i,2}^{\frac{p^*}{2}} \left(\epsilon_i^{p^*-\eta}\int_{A_i} |\phi_{i,2}|^{p^*}\,dx\right)^{\frac{(2-p)(n-2)}{2(n-p)}}.$$

Then, since $|Dv| \leq \epsilon_i|D\phi_{i,2}| \sim \epsilon_i|D\phi_i|$ inside F_i (recall that $\zeta < 1$), (3.5) and (3.20) yield

$$\begin{aligned}\epsilon_i^{p^*-2}\int_{\mathbb{R}^n} |\phi_{i,2}|^{p^*}\,dx &\leq C(n,p)\epsilon_i^{p^*-2}\int_{F_i} |D\phi_{i,2}|^p\,dx\\ &= C(n,p)\epsilon_i^{p^*-p}\int_{F_i}\big(\epsilon_i|D\phi_{i,2}|\big)^{p-2}|D\phi_{i,2}|^2\,dx\\ &\leq C(n,p)\epsilon_i^{p^*-p}\int_{F_i}\big(|Dv|+\epsilon_i|D\phi_{i,2}|\big)^{p-2}|D\phi_{i,2}|^2\,dx,\end{aligned} \tag{3.21}$$

which proves (3.17) choosing $\eta \leq p^* - p$ (recall (3.14)).

Consider instead the case

$$\int_{F_i} |D\phi_{i,2}|^p\,dx < N_{i,2}^{\frac{p^*}{2}} \left(\epsilon_i^{p^*-\eta}\int_{A_i} |\phi_{i,2}|^{p^*}\,dx\right)^{\frac{(2-p)(n-2)}{2(n-p)}},$$

and set $\theta := \frac{(2-p)(n-2)}{2(n-p)}$, so that (3.20) yields

$$\begin{aligned}\epsilon_i^{p^*-2}\int_{\mathbb{R}^n} |\phi_{i,2}|^{p^*}\,dx &\leq C(n,p)\epsilon_i^{p^*-2}N_{i,2}^{\frac{p^*}{2}}\left(\epsilon_i^{p^*-\eta}\int_{A_i}|\phi_{i,2}|^{p^*}\,dx\right)^{\theta}\\ &= C(n,p)\epsilon_i^{p^*-2+(2-\eta)\theta}N_{i,2}^{\frac{p^*}{2}}\left(\epsilon_i^{p^*-2}\int_{A_i}|\phi_{i,2}|^{p^*}\,dx\right)^{\theta}.\end{aligned}$$

Since $\theta < 1$, recalling the definition of $N_{i,2}$ and (3.14), this gives

$$\begin{aligned}\epsilon_i^{p^*-2}\int_{\mathbb{R}^n} |\phi_{i,2}|^{p^*}\,dx &\leq C(n,p)\epsilon_i^{\frac{p^*-2+(2-\eta)\theta}{1-\theta}}\left(\int_{E_i}\big(|Dv|+\epsilon_i|D\phi_{i,2}|\big)^{p-2}|D\phi_{i,2}|^2\,dx\right)^{\frac{p^*}{2(1-\theta)}}\\ &\leq C(n,p)\epsilon_i^{\eta}\int_{E_i}\big(|Dv|+\epsilon_i|D\phi_{i,2}|\big)^{p-2}|D\phi_{i,2}|^2\,dx,\end{aligned} \tag{3.22}$$

where the last inequality follows by choosing $\eta > 0$ sufficiently small (notice that $p^* - 2 + 2\theta > 0$ and $\frac{p^*}{2(1-\theta)} > 1$). This proves (3.17) also in this case.

Now, combining (3.16) and (3.17), we finally get

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\frac{(v+C_1\epsilon_i\phi_i)^{p^*}}{v^2+|\epsilon_i\phi_i|^2}|\phi_i|^2\,dx - \int_{\mathbb{R}^n}\frac{(v+C_1\epsilon_i\phi_{i,1})^{p^*}}{v^2+|\epsilon_i\phi_{i,1}|^2}|\phi_{i,1}|^2\,dx\right|\\ &\qquad\leq C(n,p,C_1)\left(\epsilon_i^{p^*-2}\int_{A_i}|\phi_{i,2}|^{p^*}\,dx + \epsilon_i^2\int_{A_i}\frac{(v+C_1\zeta v)^{p^*}}{v^2+|\zeta v|^2}|\zeta v|^2\,dx\right) = o(1).\end{aligned}$$

Thanks to this estimate, and since ϕ is also the limit of $\phi_{i,1}$, applying Step 1 to $\phi_{i,1}$ we conclude the proof of the lemma. □

An important consequence of the proof of Lemma 3.4 is the following Orlicz-type Poincaré inequality:

Corollary 3.5. *Let $1 < p \leq \frac{2n}{n+2}$. There exists $\epsilon_0 = \epsilon_0(n,p) > 0$ small such that the following holds: For any $\epsilon \in (0,\epsilon_0)$ and any $\phi \in \dot W^{1,p}(\mathbb{R}^n) \cap \dot W^{1,2}(\mathbb{R}^n;\, |Dv|^{p-2})$ with*

$$\int_{\mathbb{R}^n}\big(|Dv|+\epsilon|D\phi|\big)^{p-2}|D\phi|^2\,dx \leq 1,$$

we have

$$\int_{\mathbb{R}^n} (v+\epsilon\phi)^{p^*-2}|\phi|^2\,dx \le C(n,p)\int_{\mathbb{R}^n} \big(|Dv|+\epsilon|D\phi|\big)^{p-2}|D\phi|^2\,dx. \tag{3.23}$$

Proof. As in the proof of Lemma 3.4, it suffices to consider the case $\phi \ge 0$. Also, let $\zeta \in (0,1)$ be the small constant provided in the proof of Lemma 3.4 with $C_1 = 1$.

Write $\phi = \phi_1 + \phi_2$, where

$$\phi_1 := \min\left\{\phi,\, \frac{\zeta v}{\epsilon}\right\}, \qquad \phi_2 := \phi - \phi_1.$$

Since $\epsilon\phi_1 \le \zeta v$ we have $v \sim v + \epsilon\phi_1$, so (3.23) for ϕ_1 follows from the analogue of (3.10).

For ϕ_2 we discuss two cases.

If

$$\int_{\{\epsilon\phi>\zeta v\}} |\phi_1|^{p^*}\,dx > \int_{\{\epsilon\phi>\zeta v\}} \left(\phi - \frac{\zeta v}{\epsilon}\right)_+^{p^*} dx = \int_{\{\epsilon\phi>\zeta v\}} |\phi_2|^{p^*}\,dx,$$

then

$$\int_{\mathbb{R}^n} \epsilon^{p^*-2}|\phi_2|^{p^*}\,dx \le C(n,p)\int_{\mathbb{R}^n} v^{p^*-2}|\phi_1|^2\,dx.$$

Thus, applying (3.23) to ϕ_1, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} (v+\epsilon\phi)^{p^*-2}|\phi|^2\,dx &\le C(n,p)\int_{\mathbb{R}^n} v^{p^*-2}|\phi_1|^2\,dx\\ &\le C(n,p)\int_{\mathbb{R}^n} \big(|Dv|+\epsilon|D\phi_1|\big)^{p-2}|D\phi_1|^2\,dx\\ &\le C(n,p)\int_{\mathbb{R}^n} \big(|Dv|+\epsilon|D\phi|\big)^{p-2}|D\phi|^2\,dx,\end{aligned}$$

where the last step follows from the analogue of (3.14).

On the other hand, when

$$\int_{\{\epsilon\phi>\zeta v\}} |\phi_1|^{p^*}\,dx \le \int_{\{\epsilon\phi>\zeta v\}} |\phi_2|^{p^*}\,dx,$$

we can repeat the proofs of (3.21) and (3.22) with $\eta = 0$ to deduce the validity of (3.23) for ϕ_2.

Thus, by (3.14) for ϕ, and (3.23) for ϕ_1 and ϕ_2, we eventually obtain

$$\begin{aligned}\int_{\mathbb{R}^n} (v+\epsilon\phi)^{p^*-2}|\phi|^2\,dx &\le C(n,p)\int_{\mathbb{R}^n} v^{p^*-2}|\phi_1|^2\,dx + C(n,p)\int_{\mathbb{R}^n} \epsilon^{p^*-2}|\phi_2|^{p^*}\,dx\\ &\le C(n,p)\int_{\mathbb{R}^n} \big(|Dv|+\epsilon|D\phi_1|\big)^{p-2}|D\phi_1|^2\,dx + C(n,p)\int_{\mathbb{R}^n} \big(|Dv|+\epsilon|D\phi_2|\big)^{p-2}|D\phi_2|^2\,dx\\ &\le C(n,p)\int_{\mathbb{R}^n} \big(|Dv|+\epsilon|D\phi|\big)^{p-2}|D\phi|^2\,dx,\end{aligned}$$

which concludes the proof of the corollary. □

3.2. **Spectral gap.** Let $v = v_{a_0,1,0}$ be as in the previous section, and recall that

$$T_v\mathcal{M} := \operatorname{span}\{v,\, \partial_b v,\, \partial_{x_1} v,\, \ldots,\, \partial_{x_n} v\},$$

which is a subspace of $L^2(\mathbb{R}^n; v^{p^*-2})$.

Consider the linearized p-Laplacian operator

$$\mathcal{L}_v[\varphi] := -\operatorname{div}\big(|Dv|^{p-2}D\varphi + (p-2)|Dv|^{p-4}(Dv\cdot D\varphi)Dv\big)$$

on the space $L^2(\mathbb{R}^n; v^{p^*-2})$, and observe that this operator has a discrete spectrum for any $1 < p < n$, thanks to Proposition 3.2.

In [26, Proposition 3.1] it is proved that, for $p > 2$, $T_v\mathcal{M}$ generates the first and the second eigenspaces corresponding to $\mathcal{L}_v$. As shown in Appendix B, thanks to Proposition 3.2 and a modification of the arguments in [26, Section 6.2], this fact holds in the full range $1 < p < n$.

As a consequence, functions orthogonal to $T_v\mathcal{M}$ enjoy a quantitative improvement in the Poincaré inequality induced by $\mathcal{L}_v$. More precisely, the following holds:

Proposition 3.6. *Given $1 < p < n$, and any function $\varphi \in L^2(\mathbb{R}^n; v^{p^*-2})$ orthogonal to $T_v\mathcal{M}$, there exists a constant $\lambda = \lambda(n,p) > 0$ so that*

$$\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2 + (p-2)|Dv|^{p-4}|Dv\cdot D\varphi|^2\,dx \geq \big((p^*-1)S^p + 2\lambda\big)\|v\|^{p-p^*}_{L^{p^*}(\mathbb{R}^n)} \int_{\mathbb{R}^n} v^{p^*-2}|\varphi|^2\,dx,$$

where $S = S(n,p)$ is the optimal Sobolev constant.

In our proof, it will be important to give a meaning to the notion of "orthogonality to $T_v\mathcal{M}$" for functions which are not necessarily in $L^2(\mathbb{R}^n; v^{p^*-2})$.

Definition 3.7. *Observe that, for any $\xi \in T_v\mathcal{M}$, it holds $v^{p^*-2}\xi \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^n) = \big(L^{p^*}(\mathbb{R}^n)\big)'$. Hence, by abuse of notation, for any function $\psi \in L^{p^*}(\mathbb{R}^n)$ we say that ψ is orthogonal to $T_v\mathcal{M}$ in $L^2(\mathbb{R}^n; v^{p^*-2})$ if*

$$\int_{\mathbb{R}^n} v^{p^*-2}\xi\,\psi\,dx = 0 \qquad \forall\,\xi \in T_v\mathcal{M}.$$

Note that, by Hölder inequality, $L^{p^*}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n; v^{p^*-2})$ if $p^* \geq 2$. Hence, the notion of orthogonality introduced above is relevant only when $p^* < 2$ (equivalently, $p < \frac{2n}{n+2}$). We also observe that, by Sobolev embedding, the previous remark gives a meaning to the orthogonality to $T_v\mathcal{M}$ for functions in $\dot{W}^{1,p}(\mathbb{R}^n)$.

The main result of this section is the following spectral gap-type estimate.

Proposition 3.8. *Let $S = S(n,p)$ be the optimal Sobolev constant, and let $\lambda = \lambda(n,p) > 0$ be as in Proposition 3.6. For any $\gamma_0 > 0$ and $C_1 > 0$ there exists $\bar{\delta} = \bar{\delta}(n,p,\gamma_0,C_1) > 0$ such that the following holds:*

Let $\varphi \in \dot{W}^{1,p}(\mathbb{R}^n)$ be orthogonal to $T_v\mathcal{M}$ in $L^2(\mathbb{R}^n; v^{p^*-2})$, with

$$\|\varphi\|_{\dot{W}^{1,p}(\mathbb{R}^n)} \leq \bar{\delta}.$$

Then:
(i) when $1 < p \leq \frac{2n}{n+2}$, we have

$$\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2}\big(|D(v+\varphi)| - |Dv|\big)^2 + \gamma_0 \min\big\{|D\varphi|^p,\, |Dv|^{p-2}|D\varphi|^2\big\}\,dx$$
$$\geq \big((p^*-1)S^p + \lambda\big)\|v\|^{p-p^*}_{L^{p^*}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{(v + C_1|\varphi|)^{p^*}}{v^2 + |\varphi|^2}|\varphi|^2\,dx,$$

where $w : \mathbb{R}^n \to \mathbb{R}^n$ is defined in analogy to Lemma 2.1:

$$w = \begin{cases} \left(\frac{|D(v+\varphi)|}{(2-p)|D(v+\varphi)| + (p-1)|Dv|}\right)^{\frac{1}{p-2}} Dv & \text{if } |Dv| < |D(v+\varphi)| \\ Dv & \text{if } |D(v+\varphi)| \leq |Dv| \end{cases};$$

(ii) when $\frac{2n}{n+2} < p < 2$, we have

$$\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2}\big(|D(v+\varphi)| - |Dv|\big)^2 + \gamma_0 \min\big\{|D\varphi|^p,\, |Dv|^{p-2}|D\varphi|^2\big\}\,dx$$
$$\geq \big((p^*-1)S^p + \lambda\big)\|v\|^{p-p^*}_{L^{p^*}(\mathbb{R}^n)} \int_{\mathbb{R}^n} v^{p^*-2}|\varphi|^2\,dx,$$

where $w : \mathbb{R}^n \to \mathbb{R}^n$ is defined in analogy to Lemma 2.1:

$$w = \begin{cases} \left(\frac{|D(v+\varphi)|}{(2-p)|D(v+\varphi)|+(p-1)|Dv|}\right)^{\frac{1}{p-2}} Dv & \text{if } |Dv| < |D(v+\varphi)| \\ Dv & \text{if } |D(v+\varphi)| \le |Dv| \end{cases};$$

(iii) when $p \ge 2$, we have

$$\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2}\big(|D(v+\varphi)| - |Dv|\big)^2\,dx \ge \big((p^*-1)S^p + \lambda\big)\|v\|^{p-p^*}_{L^{p^*}(\mathbb{R}^n)} \int_{\mathbb{R}^n} v^{p^*-2}|\varphi|^2\,dx,$$

where $w : \mathbb{R}^n \to \mathbb{R}^n$ is defined in analogy to Lemma 2.1:

$$w = \begin{cases} Dv & \text{if } |Dv| < |D(v+\varphi)| \\ \left(\frac{|D(v+\varphi)|}{|Dv|}\right)^{\frac{1}{p-2}} D(v+\varphi) & \text{if } |D(v+\varphi)| \le |Dv| \end{cases}.$$

Proof. We can assume that $\|v\|_{L^{p^*}(\mathbb{R}^n)} = 1$, as the general case follows by a scaling. Also, as in the proof of Lemma 3.4, it suffices to consider the case $\varphi \ge 0$.

We argue by contradiction in all three cases.

• *The case* $1 < p \le \frac{2n}{n+2}$. Suppose the inequality does not hold. Then there exists a sequence $0 \not\equiv \varphi_i \to 0$ in $\dot W^{1,p}(\mathbb{R}^n)$, with φ_i orthogonal to $T_v\mathcal{M}$, such that

$$\begin{aligned}\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi_i|^2 + (p-2)|w_i|^{p-2}\big(|D(v+\varphi_i)| - |Dv|\big)^2 + \gamma_0 \min\big\{|D\varphi_i|^p,\, |Dv|^{p-2}|D\varphi_i|^2\big\}\,dx \\ < \big((p^*-1)S^p + \lambda\big)\int_{\mathbb{R}^n} \frac{(v + C_1\varphi_i)^{p^*}}{v^2 + |\varphi_i|^2}|\varphi_i|^2\,dx, \quad (3.24)\end{aligned}$$

where w_i corresponds to φ_i as in the statement.

Let

$$\epsilon_i := \left(\int_{\mathbb{R}^n} \big(|Dv| + |D\varphi_i|\big)^{p-2}|D\varphi_i|^2\,dx\right)^{\frac{1}{2}},$$

and set $\hat\varphi_i := \frac{\varphi_i}{\epsilon_i}$. Since $p < 2$ it holds

$$\int_{\mathbb{R}^n} \big(|Dv| + |D\varphi_i|\big)^{p-2}|D\varphi_i|^2\,dx \le \int_{\mathbb{R}^n} |D\varphi_i|^{p-2}|D\varphi_i|^2\,dx = \int_{\mathbb{R}^n} |D\varphi_i|^p\,dx \to 0,$$

and hence $\epsilon_i \to 0$.

For any $R > 1$, set

$$\mathcal{R}_i := \{2|Dv| \ge |D\varphi_i|\}, \qquad \mathcal{S}_i := \{2|Dv| < |D\varphi_i|\},$$

$$\mathcal{R}_{i,R} := \big(B(0,R) \setminus B(0,1/R)\big) \cap \mathcal{R}_i, \qquad \mathcal{S}_{i,R} := \big(B(0,R) \setminus B(0,1/R)\big) \cap \mathcal{S}_i.$$

Since the integrand in the left hand side of (3.24) is nonnegative (see (2.2)), we deduce that

$$\begin{aligned}\int_{B(0,R)\setminus B(0,1/R)} |Dv|^{p-2}|D\hat\varphi_i|^2 + (p-2)|w_i|^{p-2}\left(\frac{|Dv + D\varphi_i| - |Dv|}{\epsilon_i}\right)^2 \\ + \gamma_0 \min\big\{\epsilon_i^{p-2}|D\hat\varphi_i|^p,\, |Dv|^{p-2}|D\hat\varphi_i|^2\big\}\,dx \le \big((p^*-1)S^p + \lambda\big)\int_{\mathbb{R}^n} \frac{(v + C_1\varphi_i)^{p^*}}{v^2 + |\varphi_i|^2}|\hat\varphi_i|^2\,dx \quad (3.25)\end{aligned}$$

for any $R > 1$. Also, by (2.2),

$$|Dv|^{p-2}|D\hat{\varphi}_i|^2 + (p-2)|w_i|^{p-2}\left(\frac{|Dv + D\varphi_i| - |Dv|}{\epsilon_i}\right)^2 \geq c(p)\frac{|Dv|}{|Dv| + |D\varphi_i|}|Dv|^{p-2}|D\hat{\varphi}_i|^2 \geq c(p)|Dv|^{p-2}|D\hat{\varphi}_i|^2 \quad \text{on } \mathcal{R}_{i,R}.$$

Thus, combining this bound with (3.25), we get

$$\begin{aligned} c(p)\int_{\mathcal{R}_{i,R}} |Dv|^{p-2}|D\hat{\varphi}_i|^2\,dx + \gamma_0\int_{\mathcal{S}_{i,R}} \epsilon_i^{p-2}|D\hat{\varphi}_i|^p\,dx \\ \leq \int_{B(0,R)\setminus B(0,1/R)} |Dv|^{p-2}|D\hat{\varphi}_i|^2 + (p-2)|w_i|^{p-2}\left(\frac{|Dv + D\varphi_i| - |Dv|}{\epsilon_i}\right)^2 \\ + \gamma_0 \min\{\epsilon_i^{p-2}|D\hat{\varphi}_i|^p,\, |Dv|^{p-2}|D\hat{\varphi}_i|^2\}\,dx \\ \leq \big((p^*-1)S^p + \lambda\big)\int_{\mathbb{R}^n} \frac{(v + C_1\varphi_i)^{p^*}}{v^2 + |\varphi_i|^2}|\hat{\varphi}_i|^2\,dx. \end{aligned} \tag{3.26}$$

In particular, this implies that

$$\begin{aligned} 1 = \epsilon_i^{-2}\int_{\mathbb{R}^n} \big(|Dv| + |D\varphi_i|\big)^{p-2}|D\varphi_i|^2\,dx \\ \leq C(p)\left[\int_{\mathcal{R}_i} |Dv|^{p-2}|D\hat{\varphi}_i|^2\,dx + \int_{\mathcal{S}_i} \epsilon_i^{p-2}|D\hat{\varphi}_i|^p\,dx\right] \\ \leq C(n,p,\gamma_0)\big((p^*-1)S^p + \lambda\big)\int_{\mathbb{R}^n} \frac{(v + C_1\varphi_i)^{p^*}}{v^2 + |\varphi_i|^2}|\hat{\varphi}_i|^2\,dx. \end{aligned} \tag{3.27}$$

Furthermore, thanks to Corollary 3.5, for i large enough (so that $\epsilon_i \leq \epsilon_0$) we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{(v + C_1\varphi_i)^{p^*}}{v^2 + |\varphi_i|^2}|\hat{\varphi}_i|^2\,dx \leq C(n,p,C_1)\int_{\mathbb{R}^n} (v + |\varphi_i|)^{p^*-2}|\hat{\varphi}_i|^2\,dx \\ \leq C(n,p,C_1)\int_{\mathbb{R}^n} \big(|Dv| + |D\varphi_i|\big)^{p-2}|D\hat{\varphi}_i|^2\,dx \leq C(n,p,C_1). \end{aligned} \tag{3.28}$$

Hence, combining (3.26) with (3.28), by the definition of $\mathcal{S}_{i,R}$ we get

$$\epsilon_i^{-2}\int_{\mathcal{S}_{i,R}} |Dv|^p\,dx \leq \epsilon_i^{p-2}\int_{\mathcal{S}_{i,R}} |D\hat{\varphi}_i|^p\,dx \leq C(n,p,C_1),$$

and since $|Dv|$ is uniformly bounded away from zero inside $B(0,R)\setminus B(0,1/R)$, we conclude that

$$|\mathcal{S}_{i,R}| \to 0 \qquad \text{as } i \to \infty, \quad \forall\, R > 1. \tag{3.29}$$

Now, according to Lemma 3.4, we have that $\hat{\varphi}_i$ converges weakly in $\dot{W}^{1,p}(\mathbb{R}^n)$ to some function $\hat{\varphi} \in \dot{W}^{1,p}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, v^{p^*-2})$, and

$$\int_{\mathbb{R}^n} \frac{(v + C_1\varphi_i)^{p^*}}{v^2 + |\varphi_i|^2}|\hat{\varphi}_i|^2\,dx \to \int_{\mathbb{R}^n} v^{p^*-2}|\hat{\varphi}|^2. \tag{3.30}$$

Also, using again (3.26) and (3.28),

$$\int_{\mathcal{R}_{i,R}} |Dv|^{p-2}|D\hat{\varphi}_i|^2\,dx \leq C(n,p,C_1),$$

therefore (3.29) and the weak convergence of $\hat{\varphi}_i$ to $\hat{\varphi}$ in $\dot{W}^{1,p}(\mathbb{R}^n)$ imply that, up to a subsequence,

$$D\hat{\varphi}_i\chi_{\mathcal{R}_{i,R}} \rightharpoonup D\hat{\varphi}\chi_{B(0,R)\setminus B(0,1/R)} \quad \text{in } L^2(\mathbb{R}^n,\mathbb{R}^n), \qquad \forall\, R > 1.$$

In particular, $\hat\varphi \in \dot W^{1,2}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$. In addition, letting $i \to \infty$ in (3.27) and (3.28), and using (3.30), we deduce that

$$0 < c(n,p,\gamma_0) \le \|\hat\varphi\|_{L^2(\mathbb{R}^n; v^{p^*-2})} \le C(n,p,C_1). \tag{3.31}$$

Let us write

$$\hat\varphi_i = \hat\varphi + \psi_i, \qquad \text{with } \psi_i := \hat\varphi_i - \hat\varphi,$$

so that

$$\psi_i \rightharpoonup 0 \ \text{ in } \dot W^{1,p}(\mathbb{R}^n) \qquad \text{and} \qquad D\psi_i\chi_{\mathcal{R}_i} \rightharpoonup 0 \ \text{ in } L^2_{\rm loc}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n).$$

We now look at the left hand side of (3.25).

The strong $\dot W^{1,p}(\mathbb{R}^n)$ convergence of φ_i to 0 implies that, up to a subsequence, $|w_i| \to |Dv|$ a.e. Also, we can rewrite

$$\begin{aligned}\left(\frac{|Dv + D\varphi_i| - |Dv|}{\epsilon_i}\right)^2 &= \left(\left[\int_0^1 \frac{Dv + tD\varphi_i}{|Dv + tD\varphi_i|}\,dt\right]\cdot D\hat\varphi_i\right)^2\\ &= \left(\left[\int_0^1 \frac{Dv + tD\varphi_i}{|Dv + tD\varphi_i|}\,dt\right]\cdot \big(D\hat\varphi + D\psi_i\big)\right)^2.\end{aligned}$$

Hence, if we set

$$f_{i,1} := \left[\int_0^1 \frac{Dv + tD\varphi_i}{|Dv + tD\varphi_i|}\,dt\right]\cdot D\hat\varphi, \qquad f_{i,2} := \left[\int_0^1 \frac{Dv + tD\varphi_i}{|Dv + tD\varphi_i|}\,dt\right]\cdot D\psi_i,$$

since $\frac{Dv+tD\varphi_i}{|Dv+tD\varphi_i|} \to \frac{Dv}{|Dv|}$ a.e., it follows from Lebesgue's dominated convergence theorem that

$$f_{i,1} \to \frac{Dv}{|Dv|}\cdot D\hat\varphi \text{ strongly in } L^2_{\rm loc}(\mathbb{R}^n \setminus \{0\}), \qquad f_{i,2}\chi_{\mathcal{R}_i} \rightharpoonup 0 \text{ weakly in } L^2_{\rm loc}(\mathbb{R}^n \setminus \{0\}).$$

Thus, we can control the first two terms of the left hand side of (3.25) from below as follows:

$$\begin{aligned}&\int_{\mathcal{R}_{i,R}} |Dv|^{p-2}|D\hat\varphi_i|^2 + (p-2)|w_i|^{p-2}\left(\frac{|Dv + D\varphi_i| - |Dv|}{\epsilon_i}\right)^2\\ &\quad= \int_{\mathcal{R}_{i,R}} |Dv|^{p-2}\Big(|D\hat\varphi|^2 + 2D\psi_i\cdot D\hat\varphi\Big) + (p-2)|w_i|^{p-2}\left(f_{i,1}^2 + 2f_{i,1}f_{i,2}\right)\\ &\qquad + \int_{\mathcal{R}_{i,R}} |Dv|^{p-2}|D\psi_i|^2 + (p-2)|w_i|^{p-2}f_{i,2}^2\\ &\quad\ge \int_{\mathcal{R}_{i,R}} |Dv|^{p-2}\Big(|D\hat\varphi|^2 + 2D\psi_i\cdot D\hat\varphi\Big) + (p-2)|w_i|^{p-2}\left(f_{i,1}^2 + 2f_{i,1}f_{i,2}\right),\end{aligned} \tag{3.32}$$

where the last inequality follows from the nonnegativity of $|Dv|^{p-2}|D\psi_i|^2 + (p-2)|w_i|^{p-2}f_{i,2}^2$ (thanks to (2.2) and the fact that $f_{i,2}^2 \le |D\psi_i|^2$).

Then, combining the convergences

$$D\psi_i\chi_{\mathcal{R}_i} \rightharpoonup 0, \quad f_{i,1} \to \frac{Dv}{|Dv|}\cdot D\hat\varphi, \quad f_{i,2}\chi_{\mathcal{R}_i} \rightharpoonup 0 \qquad \text{in } L^2_{\rm loc}(\mathbb{R}^n \setminus \{0\}),$$

$$|w_i| \to |Dv| \text{ a.e.}, \qquad |\left(B(0,R) \setminus B(0,1/R)\right) \setminus \mathcal{R}_{i,R}| \to 0,$$

with the fact that

$$|w_i|^{p-2} \le C(p)|Dv|^{p-2},$$

by Lebesgue's dominated convergence theorem we deduce that the last term in (3.32) converges to

$$\int_{B(0,R)\setminus B(0,1/R)} |Dv|^{p-2}|D\hat\varphi|^2 + (p-2)|Dv|^{p-2}\left(\frac{Dv}{|Dv|}\cdot D\hat\varphi\right)^2 dx.$$

Recalling (3.25) and (3.30), since $R > 1$ is arbitrary and the integrand is nonnegative, this proves that

$$\int_{\mathbb{R}^n} |Dv|^{p-2}|D\hat{\varphi}|^2 + (p-2)|Dv|^{p-2}\left(\frac{Dv}{|Dv|}\cdot D\hat{\varphi}\right)^2 dx \le \big((p^*-1)S^p+\lambda\big)\int_{\mathbb{R}^n} v^{p^*-2}|\hat{\varphi}|^2\,dx. \tag{3.33}$$

On the other hand, $\hat{\varphi}$ being the weak limit of $\hat{\varphi}_i$ in $\dot{W}^{1,p}(\mathbb{R}^n)$, it follows that $\hat{\varphi}_i \rightharpoonup \hat{\varphi}$ in $L^{p^*}(\mathbb{R}^n)$. Hence, thanks to Definition 3.7, the orthogonality of φ_i (and so of $\hat{\varphi}_i$) implies that also $\hat{\varphi}$ is orthogonal to $T_v\mathcal{M}$. Since $\hat{\varphi}\in L^2(\mathbb{R}^n; v^{p^*-2})$, (3.31) and (3.33) contradict Proposition 3.6, concluding the proof.

• *The case* $\frac{2n}{n+2} < p < 2$. The proof is very similar to the previous case, except for some small changes and a couple of different estimates.

If the statement fails, then there exists a sequence $0 \not\equiv \varphi_i \to 0$ in $\dot{W}^{1,p}(\mathbb{R}^n)$, with φ_i orthogonal to $T_v\mathcal{M}$, such that

$$\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi_i|^2 + (p-2)|w_i|^{p-2}\big(|D(v+\varphi_i)|-|Dv|\big)^2 + \gamma_0\min\big\{|D\varphi_i|^p,\, |Dv|^{p-2}|D\varphi_i|^2\big\}\,dx$$
$$< \big((p^*-1)S^p+\lambda\big)\int_{\mathbb{R}^n} v^{p^*-2}|\varphi_i|^2\,dx, \tag{3.34}$$

where w_i corresponds to φ_i as in the statement.

As in the case $p \le \frac{2n}{n+2}$, we define

$$\epsilon_i := \left(\int_{\mathbb{R}^n}\big(|Dv|+|D\varphi_i|\big)^{p-2}|D\varphi_i|^2\,dx\right)^{\frac{1}{2}}, \qquad \hat{\varphi}_i = \frac{\varphi_i}{\epsilon_i},$$

and we split $B(0,R)\setminus B(0,1/R) = \mathcal{R}_{i,R}\cup\mathcal{S}_{i,R}$.

Then, the analogues of (3.26) and (3.27) hold also in this case, with the only difference that the last term in both equations now becomes $\big((p^*-1)S^p+\lambda\big)\int_{\mathbb{R}^n} v^{p^*-2}|\hat{\varphi}_i|^2\,dx$.

We now observe that, thanks to Hölder inequality, we have

$$\int_{\mathbb{R}^n}|D\hat{\varphi}_i|^p\,dx \le \left(\int_{\mathbb{R}^n}\big(|Dv|+|D\varphi_i|\big)^{p-2}|D\hat{\varphi}_i|^2\,dx\right)^{\frac{p}{2}}\left(\int_{\mathbb{R}^n}\big(|Dv|+|D\varphi_i|\big)^p dx\right)^{1-\frac{p}{2}}$$
$$= \left(\int_{\mathbb{R}^n}\big(|Dv|+|D\varphi_i|\big)^p dx\right)^{1-\frac{p}{2}} \le C(p)\left[\left(\int_{\mathbb{R}^n}|Dv|^p dx\right)^{1-\frac{p}{2}} + \epsilon_i^{\frac{p(2-p)}{2}}\left(\int_{\mathbb{R}^n}|D\hat{\varphi}_i|^p dx\right)^{1-\frac{p}{2}}\right],$$

from which it follows that

$$\int_{\mathbb{R}^n}|D\hat{\varphi}_i|^p\,dx \le C(n,p). \tag{3.35}$$

Thus, up to a subsequence, $\hat{\varphi}_i \to \hat{\varphi}$ weakly in $\dot{W}^{1,p}(\mathbb{R}^n)$ and strongly in $L^2_{\rm loc}(\mathbb{R}^n)$ (note that now $p^* > 2$). In addition, Hölder and Sobolev inequalities, together with (3.35), yield

$$\int_{\mathbb{R}^n\setminus B(0,\rho)} v^{p^*-2}|\hat{\varphi}_i|^2\,dx \le \left(\int_{\mathbb{R}^n\setminus B(0,\rho)} v^{p^*}\,dx\right)^{1-\frac{2}{p^*}}\left(\int_{\mathbb{R}^n\setminus B(0,\rho)}|\hat{\varphi}_i|^{p^*}\,dx\right)^{\frac{2}{p^*}}$$
$$\le \left(\int_{\mathbb{R}^n\setminus B(0,\rho)} v^{p^*}\,dx\right)^{1-\frac{2}{p^*}}\left(\int_{\mathbb{R}^n}|D\hat{\varphi}_i|^p\,dx\right)^{\frac{2}{p}} \qquad \forall\,\rho\ge 0.$$

Combining this bound with (3.35) and the strong convergence of $\hat{\varphi}_i$ to $\hat{\varphi}$ in $L^2_{\rm loc}(\mathbb{R}^n)$, we conclude that $\hat{\varphi}_i \to \hat{\varphi}$ strongly in $L^2(\mathbb{R}^n;\, v^{p^*-2})$.

In particular, letting $i\to\infty$ in the analogue of (3.27), we obtain

$$0 < c(n,p) \le \|\hat{\varphi}\|_{L^2(\mathbb{R}^n; v^{p^*-2})}.$$

Similarly, the analogue of (3.26) implies that

$$|\mathcal{S}_{i,R}| \to 0 \quad \text{and} \quad \int_{\mathcal{R}_{i,R}} |Dv|^{p-2}|D\hat{\varphi}_i|^2\,dx \le C(n,p) \qquad \forall\, R>1.$$

So, it follows from the weak convergence of $\hat{\varphi}_i$ to $\hat{\varphi}$ in $\dot{W}^{1,p}(\mathbb{R}^n)$ that, up to a subsequence,

$$D\hat{\varphi}_i\chi_{\mathcal{R}_{i,R}} \rightharpoonup D\hat{\varphi}\chi_{B(0,R)\setminus B(0,1/R)} \quad \text{in } L^2(\mathbb{R}^n,\mathbb{R}^n), \qquad \forall\, R>1.$$

Thanks to this bound, we can split

$$\hat{\varphi}_i = \hat{\varphi} + \psi_i, \qquad \text{with } \psi_i := \hat{\varphi}_i - \hat{\varphi},$$

and the very same argument as in the case $p \le \frac{2n}{n+2}$ allows us to deduce that

$$\liminf_{i\to\infty} \int_{\mathcal{R}_{i,R}} |Dv|^{p-2}|D\hat{\varphi}_i|^2 + (p-2)|w_i|^{p-2}\left(\frac{|Dv+D\varphi_i|-|Dv|}{\epsilon_i}\right)^2$$
$$\ge \int_{B(0,R)\setminus B(0,1/R)} |Dv|^{p-2}|D\hat{\varphi}|^2 + (p-2)|Dv|^{p-2}\left(\frac{Dv}{|Dv|}\cdot D\hat{\varphi}\right)^2 dx.$$

Recalling (3.34), since $R>1$ is arbitrary and the integrands above are nonnegative, this proves that (3.33) holds, a contradiction to Proposition 3.6 since $\hat{\varphi}$ is orthogonal to $T_v\mathcal{M}$ (being the strong $L^2(\mathbb{R}^n; v^{p^*-2})$-limit of $\hat{\varphi}_i$).

- *The case* $p \ge 2$. The argument is similar to the case $1<p<2$, but simpler.

If the statement of the lemma fails, then there exists a sequence $0 \not\equiv \varphi_i \to 0$ in $\dot{W}^{1,p}(\mathbb{R}^n)$, with φ_i orthogonal to $T_v\mathcal{M}$, such that

$$\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi_i|^2 + (p-2)|w_i|^{p-2}\big(|D(v+\varphi_i)|-|Dv|\big)^2\,dx < \big((p^*-1)S^p+\lambda\big)\int_{\mathbb{R}^n} v^{p^*-2}|\varphi_i|^2\,dx, \quad (3.36)$$

where w_i corresponds to φ_i as in the statement.

Let

$$\epsilon_i := \|\varphi_i\|_{\dot{W}^{1,2}(\mathbb{R}^n;|Dv|^{p-2})}, \qquad \hat{\varphi}_i = \frac{\varphi_i}{\epsilon_i}.$$

Note that, since $p\ge 2$, it follows by Hölder inequality that

$$\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi_i|^2\,dx \le \left(\int_{\mathbb{R}^n}|Dv|^p\,dx\right)^{1-\frac{p}{2}}\left(\int_{\mathbb{R}^n}|D\varphi_i|^p\,dx\right)^{\frac{p}{2}} \to 0,$$

hence $\epsilon_i \to 0$.

Since $1 = \|\hat{\varphi}_i\|_{\dot{W}^{1,2}(\mathbb{R}^n;|Dv|^{p-2})}$, Proposition 3.2 implies that, up to a subsequence, $\hat{\varphi}_i \to \hat{\varphi}$ weakly in $\dot{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^n;\,|Dv|^{p-2})$ and strongly in $L^2(\mathbb{R}^n; v^{p^*-2})$. Also, since $p\ge 2$, it follows from (3.36) that

$$1 = \int_{\mathbb{R}^n} |Dv|^{p-2}|D\hat{\varphi}_i|^2 \le \big((p^*-1)S^p+\lambda\big)\int_{\mathbb{R}^n} v^{p^*-2}|\hat{\varphi}_i|^2\,dx,$$

so we deduce that

$$\|\hat{\varphi}\|_{L^2(\mathbb{R}^n;v^{p^*-2})} \ge c(n,p) > 0.$$

In addition, since the integrand in the left hand side of (3.36) is nonnegative, we get

$$\int_{B(0,R)\setminus B(0,1/R)} |Dv|^{p-2}|D\hat{\varphi}_i|^2 + (p-2)|w_i|^{p-2}\left(\frac{|Dv+D\varphi_i|-|Dv|}{\epsilon_i}\right)^2 dx$$
$$\le \big((p^*-1)S^p+\lambda\big)\int_{\mathbb{R}^n} v^{p^*-2}|\hat{\varphi}_i|^2\,dx \quad (3.37)$$

for any $R>1$.

Note now that, because

$$0 < c(R) \le |Dv| \le C(R) \qquad \text{inside } B(0,R) \setminus B(0,1/R) \quad \forall\, R > 1,$$

writing

$$\hat\varphi_i = \hat\varphi + \psi_i, \qquad \text{with } \psi_i := \hat\varphi_i - \hat\varphi$$

we have

$$\psi_i \rightharpoonup 0 \quad \text{ in } \dot W^{1,2}_{\rm loc}(\mathbb{R}^n \setminus \{0\}).$$

Then we look at the left hand side of (3.37), and exactly as in the case $\frac{2n}{n+2} < p < 2$ we deduce that

$$\liminf_{i\to\infty} \int_{B(0,R)\setminus B(0,1/R)} |Dv|^{p-2}|D\hat\varphi_i|^2 + (p-2)|w_i|^{p-2}\left(\frac{|Dv + D\varphi_i| - |Dv|}{\epsilon_i}\right)^2$$
$$\ge \int_{B(0,R)\setminus B(0,1/R)} |Dv|^{p-2}|D\hat\varphi|^2 + (p-2)|Dv|^{p-2}\left(\frac{Dv}{|Dv|}\cdot D\hat\varphi\right)^2 dx. \quad (3.38)$$

Recalling (3.37) and since $R > 1$ is arbitrary, this proves that (3.33) holds, which contradicts Proposition 3.6 due to the orthogonality of $\hat\varphi$ to $T_v\mathcal{M}$. □

4. Proof of Theorem 1.1

Thanks to the preliminary estimates performed in the previous sections, we can now follow the compactness strategy of [3, 26].

By scaling, we can assume $\|u\|_{L^{p^*}(\mathbb{R}^n)} = 1$. Also, since the right hand side of (1.4) is trivially bounded by 2, it suffices to prove the result for $\delta(u) \ll 1$.

It follows by concentration-compactess that for any $\hat\epsilon > 0$ there exists a constant $\hat\delta = \hat\delta(n,p,\hat\epsilon)$ such that the following holds: if

$$\|Du\|_{L^p(\mathbb{R}^n)} - S \le \hat\delta,$$

then there exists $\hat v \in \mathcal{M}$ which minimizes the right-hand side of (1.4), $\hat v$ satisfies $\frac34 \le \|\hat v\|_{L^{p^*}} \le \frac43$, and $\|Du - D\hat v\|_{L^p(\mathbb{R}^n)} \le \hat\epsilon$. Also, up to a translation and a rescaling that preserve the L^{p^*}-norm, we can assume that $\hat v = v_{a,1,0}$ with $a > 0$.

As explained in the introduction, the basic idea would be to expand u around $\hat v$. Unfortunately, with our choice of $\hat v$ we do not have the desired orthogonality needed to use the spectral properties proved in the previous section. Hence, we need the following result (recall Definition 3.7 for the notion of orthogonality when a function is in $L^{p^*}(\mathbb{R}^n)$):

Lemma 4.1. *Let $\|u\|_{L^{p^*}(\mathbb{R}^n)} = 1$, and assume that $\|Du - D\hat v\|_{L^p(\mathbb{R}^n)} \le \hat\epsilon$ with $\hat v = v_{a,1,0} \in \mathcal{M}$. There exist $\epsilon' = \epsilon'(n,p) > 0$ and a modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following holds: If $\hat\epsilon \le \epsilon'$ then there exists $v \in \mathcal{M}$ such that $u - v$ is orthogonal to $T_v\mathcal{M}$ and $\|Du - Dv\|_{L^p(\mathbb{R}^n)} \le \omega(\hat\epsilon)$.*

Proof. Given u as in the statement, we consider the minimization of the functional

$$\mathcal{M} \ni v \mapsto \mathcal{F}_u[v] := \frac{1}{p^*}\int_{\mathbb{R}^n} |v|^{p^*}\,dx - \frac{1}{p^*-1}\int_{\mathbb{R}^n} |v|^{p^*-2} v\, u\, dx. \quad (4.1)$$

Assume first that $u = \hat v \in \mathcal{M}$. We claim that the minimizer of (4.1) is unique and coincides with u.

To prove this we note that, by Hölder inequality,

$$\mathcal{F}_u[v] \ge \frac{1}{p^*}\int_{\mathbb{R}^n} |v|^{p^*}\,dx - \frac{1}{p^*-1}\left(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\right)^{\frac{1}{p^*}}\left(\int_{\mathbb{R}^n} |v|^{p^*}\,dx\right)^{\frac{p^*-1}{p^*}} \ge -\frac{1}{p^*(p^*-1)}\int_{\mathbb{R}^n} u^{p^*}\,dx, \quad (4.2)$$

where the second inequality follows from the fact that the function

$$(0,+\infty) \ni s \mapsto \frac{1}{p^*}s^{p^*} - \frac{1}{p^*-1}A\, s^{p^*-1}$$

is uniquely minimized at $s = A$. Noticing that the last term in (4.2) coincides with $\mathcal{F}_u[u]$, and that equality holds in both inequalities of (4.2) if and only if $v = u$, the claim follows.

Now, if u is close to $\hat{v} = v_{a,1,0}$ in $\dot{W}^{1,p}(\mathbb{R}^n)$-norm, it follows by compactness that the minimum of the function

$$\mathbb{R} \times (0, +\infty) \times \mathbb{R}^n \ni (a, b, x_0) \mapsto \mathcal{F}_u[v_{a,b,x_0}]$$

is attained at some values (a', b', x_0') close to $(a, 1, 0)$, hence $\|Dv_{a',b',x_0'} - D\hat{v}\|_{L^p(\mathbb{R}^n)} \ll 1$. Thus, since by assumption u and $\hat{v}$ are $\dot{W}^{1,p}(\mathbb{R}^n)$-close, we deduce that

$$\|Du - Dv_{a',b',x_0'}\|_{L^p(\mathbb{R}^n)} \to 0 \qquad \text{as } \|Du - D\hat{v}\|_{L^p(\mathbb{R}^n)} \to 0,$$

which proves the existence of a modulus of continuity ω as in the statement.

Finally, it is immediate to check that if $v \in \mathcal{M}$ is close to $v_{a,1,0}$ and minimizes $\mathcal{F}_u$, then

$$0 = \frac{d}{dt}\Big|_{t=0} \mathcal{F}_u[v + t\xi] = \int_{\mathbb{R}^n} v^{p^*-2} \xi \, (v - u) \, dx \qquad \forall \, \xi \in T_v\mathcal{M}.$$

This concludes the proof. □

Thanks to Lemma 4.1, given u as at the beginning of the section with $\delta(u)$ sufficiently small, we can find $v \in \mathcal{M}$ close to u such that $u - v$ is orthogonal to $T_v\mathcal{M}$. More precisely, u can be written as $u = v + \epsilon\varphi$, where $\epsilon \le \omega(\hat{\epsilon})$ with $\hat{\epsilon} \le \epsilon'$, $\|D\varphi\|_{L^p(\mathbb{R}^n)} = 1$, and φ is orthogonal to $T_v\mathcal{M}$ (see Definition 3.7). Furthermore, up to a further small translation and rescaling, we can assume that $v = v_{a_0,1,0}$ with $\frac{1}{2} \le \|v\|_{L^{p^*}} \le 2$ so that all the statements in Section 3 hold.

Observe that, for $\delta(u)$ small,

$$\delta(u) = \|Du\|_{L^p(\mathbb{R}^n)} - S \ge c(n,p)\Big(\|Du\|^p_{L^p(\mathbb{R}^n)} - S^p\Big). \tag{4.3}$$

In the following argument several parameters will appear, and these parameters depend on each other. To simplify the notation we shall not explicit their dependence on n and p, but we emphasize how the parameters depend on each other, at least until they have been fixed.

- *The case* $1 < p \le \frac{2n}{n+2}$. Let $\kappa > 0$ be a small constant to be fixed later. By Lemma 2.1 we have

$$\begin{aligned}
\|Du\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |Dv + \epsilon D\varphi|^p \, dx \\
&\ge \int_{\mathbb{R}^n} |Dv|^p \, dx + \epsilon p \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot D\varphi \, dx \\
&\qquad + \frac{\epsilon^2 p(1-\kappa)}{2} \bigg(\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2} \Big(\frac{|Du| - |Dv|}{\epsilon}\Big)^2 dx \bigg) \\
&\qquad\qquad + c_0(\kappa) \int_{\mathbb{R}^n} \min\big\{\epsilon^p |D\varphi|^p, \, \epsilon^2 |Dv|^{p-2}|D\varphi|^2\big\} \, dx,
\end{aligned}$$

where w corresponds to u and v as in Lemma 2.1. On the other hand, by Lemma 2.4 and the concavity of $t \mapsto t^{\frac{p}{p^*}}$,

$$\begin{aligned}
1 = \|u\|^p_{L^{p^*}(\mathbb{R}^n)} &= \bigg(\int_{\mathbb{R}^n} |v + \epsilon\varphi|^{p^*} \, dx\bigg)^{\frac{p}{p^*}} \\
&\le \|v\|^p_{L^{p^*}(\mathbb{R}^n)} + \|v\|^{p-p^*}_{L^{p^*}(\mathbb{R}^n)} \bigg(\epsilon p \int_{\mathbb{R}^n} v^{p^*-1}\varphi \, dx + \epsilon^2 \Big(\frac{p(p^*-1)}{2} + \frac{p\kappa}{p^*}\Big) \int_{\mathbb{R}^n} \frac{(v + C_1(\kappa)|\epsilon\varphi|)^{p^*}}{v^2 + |\epsilon\varphi|^2} |\varphi|^2 \, dx\bigg).
\end{aligned}$$

Since, by (1.1),

$$\epsilon p \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot D\varphi \, dx = \|v\|^{p-p^*}_{L^{p^*}(\mathbb{R}^n)} S^p \epsilon p \int_{\mathbb{R}^n} v^{p^*-1}\varphi \, dx,$$

and $\|Dv\|_{L^p(\mathbb{R}^n)} = S\|v\|_{L^{p^*}(\mathbb{R}^n)}$, we then immediately conclude that

$$\begin{aligned}
C(n,p)\delta(u) &\geq \|Du\|^p_{L^p(\mathbb{R}^n)} - S^p\|u\|^p_{L^{p^*}(\mathbb{R}^n)} \\
&\geq \frac{\epsilon^2 p(1-\kappa)}{2}\bigg(\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2}\Big(\frac{|Du|-|Dv|}{\epsilon}\Big)^2\,dx\bigg) \\
&\qquad + c_0(\kappa)\int_{\mathbb{R}^n} \min\big\{\epsilon^p|D\varphi|^p,\, \epsilon^2|Dv|^{p-2}|D\varphi|^2\big\}\,dx \\
&\qquad\quad - \epsilon^2\|v\|^{p-p^*}_{L^{p^*}(\mathbb{R}^n)}S^p\left(\frac{p(p^*-1)}{2} + \frac{p\kappa}{p^*}\right)\int_{\mathbb{R}^n}\frac{(v+C_1(\kappa)|\epsilon\varphi|)^{p^*}}{v^2+|\epsilon\varphi|^2}|\varphi|^2\,dx.
\end{aligned}$$

Now, for $\delta(u) \leq \delta' = \delta'(\epsilon,\kappa,\gamma_0)$ small enough, Proposition 3.8 allows us to reabsorb the last term above: more precisely, we have

$$\begin{aligned}
&C(n,p)\delta(u) \\
&\quad\geq p\epsilon^2\bigg(\frac{(1-\kappa)}{2} - \frac{(p^*-1)+\frac{2}{p^*}\kappa}{2(p^*-1)+2\lambda S^{-p}}\bigg)\bigg(\int_{\mathbb{R}^n}|Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2}\Big(\frac{|Du|-|Dv|}{\epsilon}\Big)^2\,dx\bigg) \\
&\quad+\bigg(c_0(\kappa) - \gamma_0\frac{p\big[(p^*-1)+\frac{2}{p^*}\kappa\big]}{2(p^*-1)+2\lambda S^{-p}}\bigg)\int_{\mathbb{R}^n}\min\big\{\epsilon^p|D\varphi|^p,\, \epsilon^2|Dv|^{p-2}|D\varphi|^2\big\}\,dx,
\end{aligned}$$

and choosing first $\kappa = \kappa(n,p) > 0$ small enough so that

$$\frac{(1-\kappa)}{2} - \frac{(p^*-1)+\frac{2}{p^*}\kappa}{2(p^*-1)+2\lambda S^{-p}} \geq 0,$$

and then $\gamma_0 = \gamma_0(n,p) > 0$ small enough so that

$$\frac{c_0}{2} \geq \gamma_0\frac{p\big[(p^*-1)+\frac{2}{p^*}\kappa\big]}{2(p^*-1)+2\lambda S^{-p}},$$

we eventually arrive at

$$C(n,p)\delta(u) \geq \frac{c_0}{2}\int_{\mathbb{R}^n}\min\big\{\epsilon^p|D\varphi|^p,\, \epsilon^2|Dv|^{p-2}|D\varphi|^2\big\}\,dx. \tag{4.4}$$

Observe that, since $p<2$, it follows by Hölder inequality that

$$\begin{aligned}
\bigg(\int_{\{\epsilon|D\varphi|<|Dv|\}}|D\varphi|^p\,dx\bigg)^{\frac{2}{p}} &\leq \bigg(\int_{\{\epsilon|D\varphi|<|Dv|\}}|Dv|^p\,dx\bigg)^{\frac{2}{p}-1}\int_{\{\epsilon|D\varphi|<|Dv|\}}|Dv|^{p-2}|D\varphi|^2\,dx \\
&\leq C(n,p)\int_{\{\epsilon|D\varphi|<|Dv|\}}|Dv|^{p-2}|D\varphi|^2\,dx.
\end{aligned}$$

Hence, since $\|D\varphi\|_{L^p(\mathbb{R}^n)} = 1$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\min\big\{\epsilon^p|D\varphi|^p,\, \epsilon^2|Dv|^{p-2}|D\varphi|^2\big\}\,dx \\
&\qquad= \int_{\{\epsilon|D\varphi|\geq|Dv|\}}\epsilon^p|D\varphi|^p\,dx + \int_{\{\epsilon|D\varphi|<|Dv|\}}\epsilon^2|Dv|^{p-2}|D\varphi|^2\,dx \\
&\qquad\geq \int_{\{\epsilon|D\varphi|\geq|Dv|\}}\epsilon^p|D\varphi|^p\,dx + c\bigg(\int_{\{\epsilon|D\varphi|<|Dv|\}}\epsilon^p|D\varphi|^p\,dx\bigg)^{\frac{2}{p}} \geq c\left(\int_{\mathbb{R}^n}\epsilon^p|D\varphi|^p\,dx\right)^{\frac{2}{p}},
\end{aligned} \tag{4.5}$$

where $c = c(n,p) > 0$.

Combining (4.4) and (4.5), we conclude the proof of (1.4) with $\alpha = 2$.

• *The case* $\frac{2n}{n+2} < p < 2$. The proof is very similar to the previous case, with very small changes. By Lemma 2.1 we have

$$\int_{\mathbb{R}^n} |Du|^p\,dx - \int_{\mathbb{R}^n} |Dv|^p\,dx - \epsilon p \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot D\varphi\,dx$$
$$\geq \frac{\epsilon^2 p(1-\kappa)}{2}\left(\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2}\left(\frac{|Du|-|Dv|}{\epsilon}\right)^2 dx\right)$$
$$+ c_0(\kappa)\int_{\mathbb{R}^n} \min\big\{\epsilon^p|D\varphi|^p,\, \epsilon^2|Dv|^{p-2}|D\varphi|^2\big\}\,dx,$$

where w corresponds to u and v as in Lemma 2.1, while by Lemma 2.4

$$\int_{\mathbb{R}^n} |u|^{p^*}\,dx \leq 1 + \epsilon p^* \int_{\mathbb{R}^n} v^{p^*-1}\varphi\,dx + \epsilon^2\left(\frac{p^*(p^*-1)}{2} + \kappa\right)\int_{\mathbb{R}^n} v^{p^*-2}|\varphi|^2\,dx + \epsilon^{p^*} C_1(\kappa)\int_{\mathbb{R}^n} |\varphi|^{p^*}\,dx.$$

Hence, arguing as in the case $1 < p \leq \frac{2n}{n+2}$, it follows from (1.1), Proposition 3.8, and (4.5) that, by choosing first $\kappa > 0$ and then $\gamma_0 > 0$ small enough, for $\delta(u)$ sufficiently small we have

$$\int_{\mathbb{R}^n} |Du|^p\,dx - \int_{\mathbb{R}^n} |Dv|^p\,dx \geq c\left(\int_{\mathbb{R}^n} \epsilon^p|D\varphi|^p\,dx\right)^{\frac{2}{p}} - \epsilon^{p^*}\frac{C_1 p}{p^*}\int_{\mathbb{R}^n} |\varphi|^{p^*}\,dx.$$

Since $p^* > 2$ and $1 = \|D\varphi\|_{L^p(\mathbb{R}^n)} \geq S\|\varphi\|_{L^{p^*}(\mathbb{R}^n)}$, the result follows by the Sobolev inequality, provided ϵ is sufficiently small.

• *The case* $p \geq 2$. By Lemma 2.1 we have

$$\int_{\mathbb{R}^n} |Du|^p\,dx - \int_{\mathbb{R}^n} |Dv|^p\,dx - \epsilon p \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot D\varphi\,dx$$
$$\geq \frac{\epsilon^2 p(1-\kappa)}{2}\left(\int_{\mathbb{R}^n} |Dv|^{p-2}|D\varphi|^2 + (p-2)|w|^{p-2}\left(\frac{|Du|-|Dv|}{\epsilon}\right)^2 dx\right) + \epsilon^p c_0(\kappa)\int_{\mathbb{R}^n} |D\varphi|^p\,dx,$$

where w corresponds to u and v as in Lemma 2.1, while by Lemma 2.4

$$\int_{\mathbb{R}^n} |u|^{p^*}\,dx \leq 1 + \epsilon p^* \int_{\mathbb{R}^n} v^{p^*-1}\varphi\,dx + \epsilon^2\left(\frac{p^*(p^*-1)}{2} + \kappa\right)\int_{\mathbb{R}^n} v^{p^*-2}|\varphi|^2\,dx + \epsilon^{p^*} C_1(\kappa)\int_{\mathbb{R}^n} |\varphi|^{p^*}\,dx.$$

Hence, arguing again as in the case $p \leq \frac{2n}{n+2}$, it follows from (1.1) and Proposition 3.8 that, by choosing $\kappa > 0$ small enough,

$$\int_{\mathbb{R}^n} |Du|^p\,dx - \int_{\mathbb{R}^n} |Dv|^p\,dx \geq \epsilon^p c_0 \int_{\mathbb{R}^n} |D\varphi|^p\,dx - \epsilon^{p^*}\frac{C_1 p}{p^*}\int_{\mathbb{R}^n} |\varphi|^{p^*}\,dx.$$

Since $1 = \|D\varphi\|_{L^p(\mathbb{R}^n)} \geq S\|\varphi\|_{L^{p^*}(\mathbb{R}^n)}$, this implies (1.4) with $\alpha = p$ when ϵ is sufficiently small, concluding the proof of Theorem 1.1. □

Appendix A. A Hardy-Poincare inequality

Lemma A.1. *Let* $\alpha < n$ *and let* $u \in \dot W^{1,p}(\mathbb{R}^n; |x|^{-\alpha})$. *Then, for any* $R > 1$, *we have*

$$\int_{\mathbb{R}^n\setminus B(0,R)} |u|^p|x|^{-\alpha}\,dx \leq C(n,p,\alpha)\int_{\mathbb{R}^n\setminus B(0,R)} |Du|^p|x|^{-\alpha+p}\,dx.$$

Proof. Since $R \geq 1$ and $\alpha < n$, thanks to Fubini's Theorem and using polar coordinates we get

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B(0,R)} |u|^p|x|^{-\alpha}\,dx \\
&\quad\leq C(n,p)\int_{\mathbb{S}^{n-1}}\int_R^\infty |u(r\theta)|^p r^{-\alpha+n-1}\,dr\,d\theta \\
&\quad\leq C(n,p)\int_{\mathbb{S}^{n-1}}\int_R^\infty\int_r^\infty |u(t\theta)|^{p-1}|Du|(t\theta)r^{-\alpha+n-1}\,dt\,dr\,d\theta \\
&\quad\leq C(n,p)\int_{\mathbb{S}^{n-1}}\int_R^\infty\int_1^t |u(t\theta)|^{p-1}|Du|(t\theta)r^{-\alpha+n-1}\,dr\,dt\,d\theta \\
&\quad\leq C(n,p,\alpha)\int_{\mathbb{S}^{n-1}}\int_R^\infty |u(t\theta)|^{p-1}|Du|(t\theta)t^{-\alpha+n}\,dt\,d\theta \\
&\quad\leq C(n,p,\alpha)\left(\int_{\mathbb{S}^{n-1}}\int_R^\infty |u(t\theta)|^p t^{-\alpha+n-1}\,dt\,d\theta\right)^{\frac{p-1}{p}}\cdot \\
&\qquad\qquad\cdot\left(\int_{\mathbb{S}^{n-1}}\int_R^\infty |Du|^p(t\theta)t^{-\alpha+n-1+p}\,dt\,d\theta\right)^{\frac{1}{p}} \\
&\quad\leq C(n,p,\alpha)\left(\int_{\mathbb{R}^n\setminus B(0,R)} |u(x)|^p|x|^{-\alpha}\,dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n\setminus B(0,R)} |Du|^p(x)|x|^{-\alpha+p}\,dx\right)^{\frac{1}{p}}
\end{aligned}$$

where we applied Hölder inequality in the penultimate step. This implies the lemma. □

Appendix B. Spectral analysis

In this appendix we discuss the spectral properties of the operator

$$\mathcal{L}_v[\varphi] := -\mathrm{div}\left(|Dv|^{p-2}D\varphi + (p-2)|Dv|^{p-4}(Dv\cdot D\varphi)Dv\right)$$

on the space $L^2(\mathbb{R}^n; v^{p^*-2})$. As shown in this Proposition 3.2, this operator has a discrete spectrum for any $1 < p < n$.

Note that eigenfunctions belong to the closure of $C^1_{c,0}(\mathbb{R}^n)$ with respect to the $\dot W^{1,2}(\mathbb{R}^n; |Dv|^{p-2})$ norm (see the beginning of Section 3, in particular Remark 3.1), and eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(\mathbb{R}^n; v^{p^*-2})$.

One easily verifies that v is an eigenfunction of $\mathcal{L}_v$ with eigenvalue $(p-1)S^p$, and that $\partial_b v$ and $\partial_{x_i} v$ are eigenfunctions with eigenvalue $(p^*-1)S^p$. Furthermore, since $v > 0$, it follows that $(p-1)S^p$ is the first eigenvalue, which is simple.

To prove that $T_v\mathcal{M}$ generates the first and the second eigenspaces corresponding to $\mathcal{L}_v$, we must show that $(p^*-1)S^p$ is the second eigenvalue and verify that there are no other eigenfunctions corresponding to this eigenvalue. As in the proof of [26, Proposition 3.1], both of these facts follow from separation of variables and Sturm-Liouville theory.

Indeed, given an eigenfunction of the form $\varphi(x) = Y(\theta)f(r)$, where $Y : \mathbb{S}^{n-1} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, the eigenvalue problem corresponds to the following system:

$$0 = \Delta_{\mathbb{S}^{n-1}}Y(\theta) + \mu Y(\theta) \qquad \text{on } \mathbb{S}^{n-1}, \tag{B.1}$$

$$\begin{aligned}
0 = (p-1)|v'|^{p-2}f'' + \frac{(p-1)(n-1)}{r}|v'|^{p-2}f' - \frac{\mu}{r^2}|v'|^{p-2}f& \\
+(p-1)(p-2)|v'|^{p-4}v'v''f' + \alpha v^{p^*-2}f&
\end{aligned} \qquad \text{on } [0,\infty). \tag{B.2}$$

The eigenvalues and eigenfunctions of (B.1) are the spherical harmonics, and the eigenvalues are well-known and nonnegative. In particular, as noted in the proof of [26, Proposition 3.1], what matters for us is that $\mu_1 = 0$ and $\mu_2 = n-1$.

Multiplying by the integrating factor r^{n-1}, the ordinary differential equation (B.2) takes the form of the Sturm-Liouville eigenvalue problem

$$Lf + \alpha f = 0 \quad \text{on} \quad [0, \infty), \tag{B.3}$$

where

$$Lf = \frac{1}{w}[(Pf')' - Qf]$$

with

$$P(r) = (p-1)|v'|^{p-2} r^{n-1}, \qquad Q(r) = \mu r^{n-3}|v'|^{p-2}, \qquad w(r) = v^{p^*-2} r^{n-1}, \tag{B.4}$$

and the eigenfunctions belong to the closure of $C^1_{c,0}([\mathbb{R}^n])$Hilbert space

$$\mathcal{H} := \{g : [0, \infty) \to \mathbb{R} \,:\, g \in L^2([0, \infty); w),\, g' \in L^2[0, \infty); P)\}. \tag{B.5}$$

Remark B.1. It is clear that eigenfunctions of L are smooth on $(0, \infty)$. It is interesting to observe that they are actually continuous up to the origin.
Indeed, since $|Dv|^{p-2} \sim |x|^{\frac{p-2}{p-1}} \in L^1(B_1)$ for $p > \frac{n+2}{n+1}$, as in [26] one can easily check that the operator $\mathcal{L}_v$ is degenerate elliptic with ellipticity matrix that defines an A_2-Muckenhoupt weight. This implies that its eigenfunctions are locally Hölder continuous [16], hence eigenfunctions of L are Hölder continuous near the origin.
On the other hand, we note that $P(r) \sim r^{\frac{p-2}{p-1}+n-1} \geq 1$ on $[0, 1]$ for $p \leq \frac{n+1}{n}$. In particular, since eigenfunctions of L belong to $\mathcal{H}$ (see (B.5)), it follows by Sobolev's embedding that

$$\infty > \int_0^1 P|f'|^2\, dr \geq c \int_0^1 |f'|^2\, dr \qquad \Rightarrow \qquad f \in C^{0,1/2}([0, 1]),$$

that is, eigenfunctions of L are Hölder continuous on $[0, 1]$ for $p \leq \frac{n+1}{n}$.
Since $\frac{n+1}{n} > \frac{n+2}{n+1}$, this shows that eigenfunction of L are continuous on $[0, \infty)$ for any $1 < p < n$.

Remark B.2. As noted above, eigenfunctions of $\mathcal{L}_v$ corresponding to different eigenvalues are orthogonal $L^2(\mathbb{R}^n; v^{p^*-2})$. This implies that if f_1 and f_2 are two eigenfunctions of L corresponding to different eigenvalues, then

$$\int_0^\infty w f_1 f_2\, dr = 0.$$

As shown in the proof of [26, Proposition 3.1], to conclude the argument it suffices to prove the validity of the following:

Lemma B.3. *Consider the Sturm-Liouville problem* (B.3)*.*

(1) If f_1 and f_2 are two eigenfunctions corresponding to the same eigenvalue α, then $f_1 = cf_2$ for some $c \in \mathbb{R}$.
(2) The i-th eigenfunction of L has $i - 1$ interior zeros.

Proof. We begin by noticing that [26, Lemma 6.6] applies verbatim in our situation, and it shows that eigenfunctions of L satisfy the following decay estimates:

$$|f(r)| \leq Cr^{-\beta}, \qquad |f'(r)| \leq Cr^{-\beta-1} \qquad \text{for } r \gg 1, \quad \text{for any } 0 < \beta < \frac{n-p}{p-1}. \tag{B.6}$$

We can now prove the two desired properties.

• *Proof of (1).* Given f_1, f_2 two eigenfunctions for the same eigenvalue α, as shown in the proof of [26, Lemma 6.4] it holds

$$(PW)' = 0 \quad \text{on } (0, \infty), \qquad \text{with } W := f_1 f_2' - f_2 f_1'. \tag{B.7}$$

We claim that $PW \equiv 0$. Indeed, since $P(r) \sim r^{\frac{n-1}{p-1}}$ for $r \gg 1$, thanks to (B.6) with $\beta = \frac{3(n-p)}{4(p-1)}$ it holds

$$|Pf_1'f_2| + |Pf_2'f_1| \le Cr^{\frac{n-1}{p-1}-2\beta-1} = Cr^{-\frac{n-p}{2(p-1)}} \to 0 \qquad \text{as } r \to \infty. \tag{B.8}$$

This implies that $(PW)(r) \to 0$ as $r \to \infty$, so the claim follows from (B.7).

Once the claim is proven, the proof of (1) follows as in [26, Lemma 6.4].

• *Proof of (2).* Suppose that f_1 and f_2 are eigenfunctions of L corresponding to eigenvalues α_1 and α_2 respectively, with $\alpha_1 < \alpha_2$.

Assume first that f_1 has two consecutive zero at r_1 and r_2, with $r_2 \in (r_1, \infty]$. Thanks to (B.6) we can apply the argument in the proof [26, Lemma 6.4] to show that f_2 must have a zero inside (r_1, r_2).

Assume now that r_1 is the first interior zero of f_1, and suppose by contradiction that f_2 has no zero in $(0, r_1)$. Assuming with no loss of generality that both f_1 and f_2 are non-negative on $[0, r_1]$, thanks to Remark B.2, [26, Equation (6.16)], and (B.8), we get

$$0 > (\alpha_1 - \alpha_2) \int_0^{r_1} w f_1 f_2 \, dr = -(\alpha_1 - \alpha_2) \int_{r_1}^{\infty} w f_1 f_2 \, dr = PW(r_1) - \lim_{r \to \infty} (PW)(r) = PW(r_1).$$

Thus $0 > (PW)(r_1) = P(r_1) f_1'(r_1) f_2(r_1)$, a contradiction since $P \ge 0$, $f_1'(r_1) \le 0$, and $f_2(r_1) \ge 0$.

We can now conclude the proof of (2) as in [26, Lemma 6.4]. □

Appendix C. A numerical inequality

Lemma C.1. *Let $1 < p \le \frac{2n}{n+2}$. Given $\epsilon_0 > 0$, there exists $\zeta = \zeta(\epsilon_0)$ small enough so that the following inequality holds for any nonnegative numbers ϵ, r, a, b satisfying $\epsilon \in (0, 1)$ and $\epsilon a \le \zeta\big(1 + r^{\frac{p}{p-1}}\big)^{1-\frac{n}{p}}$:*

$$\begin{aligned}
&\big(1 + r^{\frac{p}{p-1}}\big)^{\left(1-\frac{n}{p}\right)(p^*-2)+p-1} \left[a^2\zeta^p r^{\frac{p}{p-1}} \big(1 + r^{\frac{p}{p-1}}\big)^{-p} + a^2\epsilon^p b^p \big(1 + r^{\frac{p}{p-1}}\big)^{n-p} + a^{2-p}b^p\right] \\
&\quad \le \epsilon_0 \big(1 + r^{\frac{p}{p-1}}\big)^{\left(1-\frac{n}{p}\right)(p^*-2)} a^2 + C(\epsilon_0, n, p)(1 + r)^{-\frac{p}{p-1}} \left(\big(1 + r^{\frac{p}{p-1}}\big)^{-\frac{n}{p}} r^{\frac{1}{p-1}} + \epsilon b\right)^{p-2} b^2 \qquad \text{(C.1)} \\
&\quad \le \epsilon_0 \big(1 + r^{\frac{p}{p-1}}\big)^{\left(1-\frac{n}{p}\right)(p^*-2)} a^2 + C(\epsilon_0, n, p) \left(\big(1 + r^{\frac{p}{p-1}}\big)^{-\frac{n}{p}} r^{\frac{1}{p-1}} + \epsilon b\right)^{p-2} b^2. \qquad \text{(C.2)}
\end{aligned}$$

Proof. Note that (C.2) immediately follows from (C.1), so it suffices to prove (C.1). We distinguish several cases.

• *Case 1:* $0 \le r \le 1$. In this case, up to changing the values of ϵ_0 and ζ by a universal constant, (C.1) is equivalent to

$$a^2\zeta^p r^{\frac{p}{p-1}} + a^2\epsilon^p b^p + a^{2-p}b^p \le \epsilon_0 a^2 + C(\epsilon_0, n, p)\big(r^{\frac{1}{p-1}} + \epsilon b\big)^{p-2} b^2. \tag{C.3}$$

Note that:

- if $\epsilon b \le \left(\frac{\epsilon_0}{3}\right)^{\frac{1}{p}} r^{\frac{1}{p-1}}$ then $a^2\epsilon^p b^p \le \frac{\epsilon_0}{3} a^2$;
- if $\epsilon b > \left(\frac{\epsilon_0}{3}\right)^{\frac{1}{p}} r^{\frac{1}{p-1}}$ then, since $\epsilon a \le \zeta\big(1 + r^{\frac{p}{p-1}}\big)^{1-\frac{n}{p}} \le 2\zeta$,

$$a^2\epsilon^p b^p \le 4\zeta^2\epsilon^{p-2} b^p \le C(\epsilon_0, n, p)\big(r^{\frac{1}{p-1}} + \epsilon b\big)^{p-2} b^2.$$

Similarly:

- if $b \le \left(\frac{\epsilon_0}{3}\right)^{\frac{1}{p}} a$ then $a^{2-p}b^p \le \frac{\epsilon_0}{3} a^2$;
- if $\left(\frac{\epsilon_0}{3}\right)^{\frac{1}{p}} a < b < \epsilon^{-1} r^{\frac{1}{p-1}}$ then

$$a^{2-p}b^p \le C(\epsilon_0, n, p) b^2 \le C(\epsilon_0, n, p) r^{\frac{p-2}{p-1}} b^2 \le C(\epsilon_0, n, p)\big(r^{\frac{1}{p-1}} + \epsilon b\big)^{p-2} b^2;$$

- if $b \ge \epsilon^{-1} r^{\frac{1}{p-1}}$ then, since $\epsilon a \le \zeta\big(1 + r^{\frac{p}{p-1}}\big)^{1-\frac{n}{p}} \le 2\zeta$,

$$a^{2-p}b^p \le 4^{2-p}\zeta^{2-p}\epsilon^{p-2} b^p \le C(\epsilon_0, n, p)\big(r^{\frac{1}{p-1}} + \epsilon b\big)^{p-2} b^2.$$

Thus, choosing $\zeta^p \le \frac{\epsilon_0}{3}$, (C.3) holds in all cases.

• *Case 2:* $r > 1$. In this case, (C.1) is equivalent to

$$r^{\frac{p-n}{p-1}(p^*-2)}a^2\zeta^p + r^{\frac{p-n}{p-1}(p^*-2-p)+p}a^p\epsilon^p b^p + a^{2-p}b^p r^{\frac{p-n}{p-1}(p^*-2)+p} \le \epsilon_0 r^{\frac{p-n}{p-1}(p^*-2)}a^2 + C(\epsilon_0,n,p)r^{-\frac{p}{p-1}}\big(r^{\frac{1-n}{p-1}}+\epsilon b\big)^{p-2}b^2. \quad \text{(C.4)}$$

Again:

- if $b \le \left(\frac{\epsilon_0}{3}\right)^{\frac{1}{p}} r^{\frac{1-n}{p-1}}\epsilon^{-1}$ then

$$r^{\frac{p-n}{p-1}(p^*-2-p)+p}a^2\epsilon^p b^p \le \frac{\epsilon_0}{3}r^{\frac{p-n}{p-1}(p^*-2)}a^2;$$

-if $b > \left(\frac{\epsilon_0}{3}\right)^{\frac{1}{p}} r^{\frac{1-n}{p-1}}\epsilon^{-1}$, we apply the inequality $\epsilon a \le \zeta\big(1+r^{\frac{p}{p-1}}\big)^{1-\frac{n}{p}} \le 2\zeta r^{\frac{p-n}{p-1}}$ to conclude

$$r^{\frac{p-n}{p-1}(p^*-2-p)+p}a^2\epsilon^p b^p \le 4r^{-\frac{p}{p-1}}\zeta^2\epsilon^{p-2}b^p \le C(\epsilon_0,n,p)r^{-\frac{p}{p-1}}\big(r^{\frac{1-n}{p-1}}+\epsilon b\big)^{p-2}b^2.$$

On the other hand:

- if $b \le \left(\frac{\epsilon_0}{3}\right)^{\frac{1}{p}} ar^{-1}$ then

$$a^{2-p}b^p r^{\frac{p-n}{p-1}(p^*-2)+p} \le \frac{\epsilon_0}{3}r^{\frac{p-n}{p-1}(p^*-2)}a^2;$$

- if $\left(\frac{\epsilon_0}{3}\right)^{\frac{1}{p}} ar^{-1} < b < \epsilon^{-1}r^{\frac{1-n}{p-1}}$ then

$$\begin{aligned} a^{2-p}b^p r^{\frac{p-n}{p-1}(p^*-2)+p} &\le C(\epsilon_0,n,p)b^2 r^{\frac{p-n}{p-1}(p^*-2)+2} \\ &= C(\epsilon_0,n,p)r^{-\frac{p}{p-1}}r^{\frac{1-n}{p-1}(p-2)}b^2 \le C(\epsilon_0,n,p)r^{-\frac{p}{p-1}}\big(r^{\frac{1-n}{p-1}}+\epsilon b\big)^{p-2}b^2; \end{aligned}$$

- if $b \ge \epsilon^{-1}r^{\frac{1-n}{p-1}}$ then we apply the inequality $\epsilon a \le \zeta\big(1+r^{\frac{p}{p-1}}\big)^{1-\frac{n}{p}} \le 2\zeta r^{\frac{p-n}{p-1}}$ to get

$$a^{2-p}b^p r^{\frac{p-n}{p-1}(p^*-2)+p} \le 2^{2-p}r^{-\frac{p}{p-1}}\zeta^{2-p}\epsilon^{p-2}b^p \le C(\epsilon_0,n,p)r^{-\frac{p}{p-1}}\big(r^{\frac{1-n}{p-1}}+\epsilon b\big)^{p-2}b^2.$$

This proves (C.4) whenever $\zeta^p \le \frac{\epsilon_0}{3}$, concluding the proof of (C.1). □

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