

SPECTRAL OPTIMIZATION OF INHOMOGENEOUS PLATES

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ABSTRACT. This article is devoted to the study of spectral optimisation for inhomogeneous plates. In particular, we optimise the first eigenvalue of a vibrating plate with respect to its thickness and/or density. Our result is threefold. First, we prove existence of an optimal thickness, using fine tools hinging on topological properties of rearrangement classes. Second, in the case of a circular plate, we provide a characterisation of this optimal thickness by means of Talenti inequalities. Finally, we prove a stability result when assuming that the thickness and the density of the plate are linearly related. This proof relies on H -convergence tools applied to biharmonic operators.

1. INTRODUCTION

The study of eigenmodes optimisation is central to the theory of inhomogeneous elastic plates and is of great applicative relevance. A vast literature has been devoted to the analysis of spectral optimisation problems for biharmonic operators, modelling plates of varying density and thickness under different settings [3, 4, 6, 8, 9, 10, 11, 14, 18, 19]. In addition, several contributions are devoted to inverse problems arising in the study of such inhomogeneous plates [17, 25, 26]. In the latter context, the main objective is to identify some structural descriptors of the plate under consideration, such as its thickness or its bending stiffness, and the outlook on the problem is mostly computational.

The goal of this article is to provide answers to several theoretical questions that, to the best of our knowledge, have not received a mathematical treatment so far. We focus on the optimization of thickness and/or density with respect to the first eigenvalue. For fixed density, we prove the existence of an optimal thickness. This calls for the implementation of a delicate argument, based on rearrangements. We then investigate the symmetry of the optimal solution in specific geometries, showing analogies with previously studied cases [3, 4]. Eventually, we prove a stability result for the case in which the thickness and the density of the plate are linearly related.

In order to make the discussion more precise, let Ω be a bounded domain in \mathbb{R}^2 with \mathcal{C}^2 boundary, representing the reference mid-surface configuration of a thin plate at rest, and let $D, g \in L^\infty(\Omega)$. The function D describes the varying *thickness* of the plate. Its lower bound is normalized by assuming that $D \geq 1$, where the inequality is meant to hold almost everywhere in Ω . The function g accounts for the *heterogeneity* of the plate. We are hence led to consider the first eigenvalue associated with the natural vibration of the plate. In variational terms, this eigenvalue admits the following Rayleigh-quotient representation

$$\tilde{\Lambda}(D, g) = \inf_{u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), u \neq 0} \frac{\int_{\Omega} D (\Delta u)^2}{\int_{\Omega} g u^2}. \quad (1.1)$$

The associated eigenfunction $v_{D,g}$ satisfies the following elliptic problem

$$\begin{cases} \Delta (D \Delta v_{D,g}) = \tilde{\Lambda}(D, g) g v_{D,g} & \text{in } \Omega, \\ v_{D,g} = \Delta v_{D,g} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

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The most general formulation of the optimisation problem under consideration, covering questions from [3, 4, 6, 14, 18, 19], is the study of the qualitative properties of solutions to the minimisation problem

$$\inf_{D,g} \tilde{\Lambda}(D, g). \quad (1.3)$$

From the modeling viewpoint, the reference application consists in reinforcing the plate locally by adding a layer of material, hence increasing the thickness, or by combining two materials, hence increasing the density. These cases translate in the choice

$$D = 1 + \beta_0 \mathbf{1}_\omega, \quad g = 1 + \delta_0 \mathbf{1}_{\omega'} \quad (1.4)$$

for two measurable subsets ω, ω' on which we can act, where $\mathbf{1}$ is the corresponding characteristic function. This in turn leads to considering L^∞ and L^1 constraints on D and g . We hence introduce the following admissible classes for thickness and heterogeneity, where $\beta_0, \delta_0, D_0, g_0$ are fixed positive parameters:

$$\mathcal{N}(\Omega) := \left\{ D \in L^\infty(\Omega) : 1 \leq D \leq 1 + \beta_0, \int_\Omega D = D_0 \right\}, \quad (1.5)$$

$$\mathcal{N}'(\Omega) := \left\{ g \in L^\infty(\Omega) : 1 \leq g \leq 1 + \delta_0, \int_\Omega g = g_0 \right\}. \quad (1.6)$$

The main minimization problem (1.3) is then specified as follows

$$\inf_{D \in \mathcal{N}(\Omega), g \in \mathcal{N}'(\Omega)} \tilde{\Lambda}(D, g). \quad (1.7)$$

Let us start by removing a difficulty related to the definition of the eigenvalue, which is that the potential term $\tilde{\Lambda}(D, g)gv_{D,g}$ in (1.2) appears in a multiplicative form. As it is customary in eigenvalue optimisation, arguing as in [13, Theorem 13] we reformulate the problem by referring to the *density (excess)* ρ of the plate instead of its heterogeneity. In particular, we introduce the class of admissible densities

$$\mathcal{M}(\Omega) := \left\{ \rho \in L^\infty(\Omega) : 0 \leq \rho \leq 1, \int_\Omega \rho = \rho_0 \right\} \quad (1.8)$$

and, for $D \in \mathcal{N}(\Omega)$, we define the first eigenvalue

$$\Lambda(D, \rho) := \inf_{u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), u \neq 0} \frac{\int_\Omega D (\Delta u)^2 - \int_\Omega \rho u^2}{\int_\Omega u^2}. \quad (1.9)$$

Up to a scaling factor, proceeding along the lines of [13, Theorem 13], solving (1.7) is equivalent to finding solutions to

$$\inf_{D \in \mathcal{N}(\Omega), \rho \in \mathcal{M}(\Omega)} \Lambda(D, \rho). \quad (1.10)$$

We prefer to work with formulation (1.10), for the normalization term $\int_\Omega u^2 = 1$ in the denominator in (1.9) is independent of ρ (compare with (1.1)).

Most contributions on the minimization problem (1.10) focus on the case of fixed thickness $D \equiv 1$ and the optimisation is carried out with respect to ρ only, either under Navier boundary conditions, or under clamped boundary conditions, see for instance [3, 4, 18]. In these contributions, rearrangements arguments and Talenti inequalities are used in order to derive Faber-Krahn-like inequalities, delivering information on the geometry of minimizers. On the other hand, the optimisation of the thickness D is mostly treated numerically [6, 17, 25, 26] and the existence of a minimizer D^* is usually not ascertained, to the best of our knowledge. Let us stress that existence in this setting can be quite delicate to obtain. As a matter of comparison, let us recall that in the somehow related case of optimisation of the first eigenvalue of two-phases operators $-\nabla \cdot (D\nabla)$ under the constraint $D \in \mathcal{N}(\Omega)$, it is well-known [29, 12] that no solution exists if Ω is not a ball.

The first main result of the paper is hence an existence proof for an optimal thickness D^* for (1.10), under fixed $\rho \equiv 0$. In particular, setting $\mu(D) := \Lambda(D, 0)$, we investigate the minimisation problem $\inf_{D \in \mathcal{N}(\Omega)} \mu(D)$. We prove in Theorem 2.1 that, in any domain Ω , a minimiser exists. Note that it is in sharp contrast with several other models involving heterogeneity in the leading term of the underlying

elliptic operator (such as classical two-phases operators), where existence strongly depends on the choice of the ambient space Ω . The proof of Theorem 2.1 relies on delicate topological properties of constraint classes defined through rearrangements and we will make use of some related results from [2, 15].

Our second main result, Theorem 2.2, focuses on the case when Ω is a ball. In this case, we are able to characterise the optimal thickness D^* as being piecewise constant and radially symmetric. The argument is in the spirit of [4, 3]. In particular, we use Talenti inequalities in combination with rearrangement arguments.

In our last main result, Theorem 2.3, we investigate the case of coupled thickness and density. For simplicity, we focus on the case of a linear relation between these two quantities, namely, $D = 1 + \alpha\rho$ for a small parameter $\alpha > 0$. Albeit linear, this case already proves very challenging. By defining $\lambda_\alpha(\rho) := \Lambda(1 + \alpha\rho, \rho)$, we give a fine stability analysis in the case where Ω is a ball, α is small, and all the functions involved are assumed to be radial. In particular we obtain a stationary result: the minimisers ρ^* in the case $\alpha = 0$, which were already studied in [3, 4, 18], remain optimal for $\alpha > 0$ small enough. This proof relies on H -convergence-like tools, generalising to biharmonic operators a strategy developed in [27].

The paper is organised as follows. In Section 2 we specify the precise assumptions for our analysis and state our three main results. In Section 3 we collect some preliminary technical results. Sections 4–6 are devoted to the proofs of Theorems 2.1–2.3. Eventually, Section 7 contains a summary of our findings.

2. MATHEMATICAL SETTING AND RESULTS

Throughout the paper, inequalities will always be meant in the sense of L^1 functions, namely almost everywhere in the corresponding set where the different quantities are defined.

2.1. Optimisation with respect to the thickness. We first investigate optimisation with respect to the thickness of the plate. Given two positive parameters β_0, D_0 , the admissible class of thicknesses $\mathcal{N}(\Omega)$ is defined in (1.5), where nonetheless we assume that $D_0 > \text{Vol}(\Omega)$ in order to ensure that this class is not empty or reduced to a single element. For any $D \in \mathcal{N}(\Omega)$ we define the first eigenvalue $\mu(D)$ given by the Rayleigh quotient

$$\mu(D) = \inf_{u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega), u \neq 0} \frac{\int_\Omega D (\Delta u)^2}{\int_\Omega u^2}, \quad (2.1)$$

which is associated with the following eigenequation (where we have chosen a L^2 normalisation):

$$\begin{cases} \Delta(D\Delta u_D) = \mu(D)u_D & \text{in } \Omega, \\ u_D = \Delta u_D = 0 & \text{on } \partial\Omega, \\ \int_\Omega u_D^2 = 1. \end{cases} \quad (2.2)$$

We emphasise once again that this corresponds to problem (1.7) with $g \equiv 1$. The first optimisation problem we consider is

$$\inf_{D \in \mathcal{N}(\Omega)} \mu(D). \quad (2.3)$$

Our first result establishes the existence of a solution:

Theorem 2.1 (Existence of minimisers). *For any bounded domain $\Omega \subset \mathbb{R}^2$ with \mathcal{C}^2 boundary there exists $D^* \in \mathcal{N}(\Omega)$ such that*

$$\inf_{D \in \mathcal{N}(\Omega)} \mu(D) = \mu(D^*). \quad (2.4)$$

Furthermore, there exists a measurable set $\omega^ \subset \Omega$ such that $D^* = 1 + \beta_0 \mathbb{1}_{\omega^*}$.*

The proof of this theorem relies on rather fine topological arguments which yield compactness of sequences of minimisers. Let us note that, as is classical in this class of problems, one can not use the direct method in the calculus of variations: indeed, the best convergence one could get on a minimising sequence $\{D_k\}_{k \in \mathbb{N}}$ (and on the associated sequence of eigenfunctions $\{u_k\}_{k \in \mathbb{N}}$) is the weak-* convergence in L^∞ of $\{D_k\}_{k \in \mathbb{N}}$ and weak convergence of $\{u_k\}_{k \in \mathbb{N}}$ in $W^{2,2}(\Omega)$, thus forbidding to pass to the limit

in the Rayleigh-quotient formulation (2.1) of $\{\mu(D_k)\}_{k \in \mathbb{N}}$. This is a known conundrum in the study of two-phases operators [29], impairing the proof of the existence of optimisers for general Ω . We present here a way to circumvent this difficulty in the case of biharmonic operators.

In general domains, it is hopeless to give an explicit characterisation of the optimal thickness D^* . In the case of a ball, however, using Talenti inequalities, we obtain an inequality of Faber-Krahn type. Consider the case in which our plate coincides with the ball of radius $R > 0$ centered in the origin, i.e. $\Omega = \mathbb{B}(0; R)$. Define the function $\bar{D}_\#$ as follows:

$$\bar{D}_\# = (1 + \beta_0)\mathbf{1}_\mathbb{A} + \mathbf{1}_{\mathbb{A}^c} \quad (2.5)$$

where, in radial coordinates, $\mathbb{A} = \{r_0 < r < R\}$ and $\text{Vol}(\mathbb{A}) = (D_0 - \text{Vol}(\Omega))/\beta_0$. Note that the set \mathbb{A} is uniquely defined and the volume constraint ensures that $\bar{D}_\# \in \mathcal{N}(\Omega)$. Our second result reads as follows.

Theorem 2.2 (The case of the ball). *Let $\Omega = \mathbb{B}(0; R)$ for some $R > 0$. Then, $\bar{D}_\#$ minimises μ in $\mathcal{N}(\Omega)$, namely,*

$$\mu(\bar{D}_\#) \leq \mu(D) \quad \forall D \in \mathcal{N}(\Omega). \quad (2.6)$$

It should be noted that this is the exact opposite result with respect to the optimisation on the density ρ (i.e. keeping $D \equiv 1$). In fact, by minimizing w.r.t. ρ it is shown in [3, 4] that the unique optimal material density ρ^* when $\Omega = \mathbb{B}(0; R)$ corresponds to having a maximal density in the center, and a minimal density close to the boundary: $\rho^* = \mathbf{1}_{\mathbb{B}(0; r^*)}$ with r^* chosen so as to satisfy the volume constraint. This observation motivates our interest in investigating optimality with respect to density *and* thickness. We tackle this topic in the next subsection, by assuming a linear relation between ρ and D .

2.2. Density-dependent thickness. In this subsection, we consider another version of (1.7)-(1.10), by assuming a linear dependency of the thickness D of the plate with respect to the density of the material. In other words, we consider a real parameter $\alpha \geq 0$, and assume that the thickness D depends on the density of the material via the relation

$$D = 1 + \alpha\rho. \quad (2.7)$$

Keeping in mind that ρ corresponds to the repartition of some material inside the elastic plate Ω , we recall the admissible class $\mathcal{M}(\Omega)$ of densities from (1.8) and, for any $\rho \in \mathcal{M}(\Omega)$, we consider the first eigenvalue $\lambda_\alpha(\rho)$ of $u \mapsto \Delta((1 + \alpha\rho)\Delta u) - \rho u$. In its Rayleigh-quotient formulation, this is given by

$$\lambda_\alpha(\rho) := \inf_{u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), u \neq 0} \frac{\int_\Omega (1 + \alpha\rho)(\Delta u)^2 - \int_\Omega \rho u^2}{\int_\Omega u^2}. \quad (2.8)$$

Up to a L^2 normalisation, the associated eigenfunction $u_{\alpha,\rho}$ satisfies

$$\begin{cases} \Delta((1 + \alpha\rho)\Delta u_{\alpha,\rho}) = \lambda_\alpha(\rho)u_{\alpha,\rho} + \rho u_{\alpha,\rho} & \text{in } \Omega, \\ u_{\alpha,\rho} = \Delta u_{\alpha,\rho} = 0 & \text{on } \partial\Omega, \\ \int_\Omega u_{\alpha,\rho}^2 = 1. \end{cases} \quad (2.9)$$

We prove in Lemma 3.2 that $\lambda_\alpha(\rho)$ is a simple eigenvalue and that the associated first eigenfunction has a constant sign.

For a fixed parameter $\alpha \geq 0$, we consider the optimisation problem

$$\inf_{\rho \in \mathcal{M}(\Omega)} \lambda_\alpha(\rho). \quad (2.10)$$

We assume that $\Omega = \mathbb{B}(0; R)$ for some $R > 0$ and focus on the geometry of minimizers for $\alpha > 0$ small. Indeed, an explicit characterisation of the minimisers for $\alpha = 0$ was given in [4]: if \mathbb{B}^* is the unique ball centered in the origin, contained in $\Omega = \mathbb{B}(0; R)$ with $\text{Vol}(\mathbb{B}^*) = \rho_0$, then the unique minimiser of λ_0 in $\mathcal{M}(\Omega)$ is

$$\rho^* = \mathbf{1}_{\mathbb{B}^*}. \quad (2.11)$$

On the other hand, Theorem 2.2 seems to indicate that, for $\alpha \rightarrow \infty$, the optimal ρ should behave as $\mathbf{1}_\mathbb{A}$, where $\mathbb{A} = \{r_0 < r < R\}$ is the only annulus of volume ρ_0 .

Theorem 2.3 (Stability for small α in the ball for radially symmetric distributions). *Let $\Omega = \mathbb{B}(0; R)$ for some $R > 0$, and define $\rho^* := \mathbb{1}_{\mathbb{B}^*}$. Then, there exists $\bar{\alpha} > 0$ such that, for any $0 \leq \alpha \leq \bar{\alpha}$,*

$$\lambda_\alpha(\rho^*) \leq \lambda_\alpha(\rho) \quad \forall \rho \in \mathcal{M}(\mathbb{B}), \rho \text{ radially symmetric.} \quad (2.12)$$

The proof of this theorem relies on fine arguments inspired from H -convergence theory [1, 29], and can be linked to some stationarity results in two-phases problems [23, 27]. In the proof, the radial symmetry assumption of competitors is crucial.

3. PRELIMINARY TECHNICAL RESULTS

We first gather in this section several preliminary results that are used throughout the rest of the paper.

Let us begin by presenting a straightforward application of the maximum principle.

Lemma 3.1 (Positivity principle). *Let $\rho \in \mathcal{M}(\Omega)$. Assume that $u \in W_0^{2,2}(\Omega)$ satisfies, for some $f \in L^2(\Omega)$,*

$$\begin{cases} \Delta((1 + \alpha\rho)\Delta u) = f \geq 0 \text{ in } \Omega, \\ u = \Delta u = 0 \text{ on } \partial\Omega. \end{cases} \quad (3.1)$$

Then

$$u \geq 0 \text{ and } (1 + \alpha\rho)\Delta u \leq 0 \text{ in } \Omega. \quad (3.2)$$

Proof of Lemma 3.1. Let ρ, u be as in the statement of the lemma. First of all, by elliptic regularity,

$$(1 + \alpha\rho)\Delta u \in W^{2,2}(\Omega).$$

Let us introduce $z = -(1 + \alpha\rho)\Delta u$. Then $z \in W_0^{1,2}(\Omega)$ satisfies

$$\begin{cases} -\Delta z = f \geq 0 \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega. \end{cases}$$

As a consequence of the maximum principle for the Laplacian we obtain $z \geq 0$ in Ω . Hence, $\Delta u \leq 0$. Since $u \in W^{2,2}(\Omega)$, $\Delta u \in L^2(\Omega)$. We can then apply the maximum principle to the inequality $-\Delta u \geq 0$ in Ω to conclude that $u \geq 0$ in Ω . \square

In the next lemma we collect some basic facts about the underlying spectral and optimisation problems.

Lemma 3.2. (1) *For any $D \in \mathcal{N}(\Omega)$, $\alpha \geq 0$, and $\rho \in \mathcal{M}(\Omega)$, the eigenfunctions u_D and $u_{\alpha,\rho}$ can be assumed to have constant sign. Hence, the first eigenvalue is the only one whose eigenfunction is constant in sign.*

(2) *There exists $M > 0$ such that, for any $D \in \mathcal{N}(\Omega)$,*

$$|\mu(D)|, \|u_D\|_{W^{2,2}(\Omega)} \leq M. \quad (3.3)$$

(3) *For any $\bar{\alpha} > 0$, there exists $M(\bar{\alpha})$ such that, for any $\rho \in \mathcal{M}(\Omega)$ and any $\alpha \in [0; \bar{\alpha}]$,*

$$|\lambda_\alpha(\rho)|, \|u_{\alpha,\rho}\|_{W^{2,2}(\Omega)} \leq M(\bar{\alpha}). \quad (3.4)$$

Proof of Lemma 3.2. To prove point 1 of the Lemma, we adapt [7, Lemma 16]. We detail this argument for $\lambda_\alpha(\rho)$ only, for an analogous proof yields the conclusion for $\mu(D)$, as well. In order to prove point 1, it suffices to establish the following fact: for any $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and any $\rho \in \mathcal{M}(\Omega)$ there exists $w \in W^{2,2}(\Omega) \cap W^{1,2}(\Omega)$ such that

$$w \geq 0, \int_\Omega (1 + \alpha\rho)(\Delta w)^2 - \rho w^2 \leq \int_\Omega (1 + \alpha\rho)(\Delta u)^2 - \rho u^2, \text{ and } \int_\Omega w^2 \geq \int_\Omega u^2. \quad (3.5)$$

Indeed, since $u \in W^{2,2}(\Omega)$ does not imply $|u| \in W^{2,2}(\Omega)$, it is not possible to simply replace u by its absolute value. Let us hence consider $\rho \in \mathcal{M}(\Omega)$, $\alpha \geq 0$, $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and define w as the unique solution of

$$\begin{cases} -\Delta w = |\Delta u| & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.6)$$

We first observe that $\int_{\Omega}(1 + \alpha\rho)(\Delta w)^2 = \int_{\Omega}(1 + \alpha\rho)(\Delta u)^2$. Besides, by the maximum principle, $w \geq 0$ in Ω . Furthermore, from the definition of w , we get that $-\Delta w \geq -\Delta u$, $-\Delta w \geq \Delta u$, whence $w \geq u$, and $w \geq -u$ in Ω . As a consequence, $w \geq |u|$ in Ω . Thus, $\int_{\Omega} w^2 \geq \int_{\Omega} u^2$. Since $\rho \geq 0$, we have that

$$\int_{\Omega} \rho u^2 \leq \int_{\Omega} \rho w^2$$

which yields the conclusion. It should be noted that this construction proves that any eigenfunction associated with the first eigenvalue has a constant sign, whence the simplicity of the first eigenvalues $\mu(D)$ and $\lambda_{\alpha}(\rho)$.

We now proceed with the proof of point 2. Point 3 follows from the exact same arguments. Let us consider $D \in \mathcal{N}(\Omega)$. From the Rayleigh-quotient formulation (2.1) of $\mu(D)$, we get that $\mu(D) \geq 0$ (for $\lambda_{\alpha}(\rho)$, we would get $\lambda_{\alpha}(\rho) \geq -1$). Let us consider the first eigenvalue $\eta_1(\Omega)$ of the biharmonic operator in Ω defined as

$$\eta_1(\Omega) := \inf_{u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \int_{\Omega} u^2 = 1} \int_{\Omega} (\Delta u)^2. \quad (3.7)$$

Let w_1 be an associated eigenfunction. Then

$$\mu(D) \leq \int_{\Omega} D (\Delta w_1)^2 \leq (1 + \beta_0) \int_{\Omega} (\Delta w_1)^2 = (1 + \beta_0) \eta_1(\Omega), \quad (3.8)$$

which yields the required uniform bound on the eigenvalue. Next, by multiplying the eigenequation (2.2) by u_D and integrating by parts we obtain

$$\int_{\Omega} (\Delta u_D)^2 \leq \int_{\Omega} D (\Delta u_D)^2 = \mu(D) \leq (1 + \beta_0) \eta_1(\Omega).$$

Since, by elliptic regularity, for any $u \in W_0^{1,2}(\Omega)$,

$$\|u_D\|_{W^{2,2}(\Omega)} \leq C \|\Delta u_D\|_{L^2(\Omega)} \quad (3.9)$$

we obtain the required bound. \square

Henceforth, with no loss of generality we assume u_D and $u_{\alpha,\rho}$ to be nonnegative, up to multiplying them by -1 . Our next step is hence to establish the concavity of the eigenvalue maps.

Lemma 3.3. *Let $\alpha \geq 0$ be fixed. The two maps*

$$\mathcal{N}(\Omega) \ni D \mapsto \mu(D), \quad \mathcal{M}(\Omega) \ni \rho \mapsto \lambda_{\alpha}(\rho) \quad (3.10)$$

are concave.

Proof of Lemma 3.3. Each of these two maps is defined as an infimum of linear functionals in their respective variables, so that they are concave. \square

This concavity property enables one to write the seemingly naive but in fact crucial reformulation of the eigenvalue problems in terms of bang-bang functions, which we now define.

Definition 3.4. A function $D \in \mathcal{N}(\Omega)$ is called bang-bang if $D = 1 + \beta_0 \mathbf{1}_{\omega}$ for some measurable subset ω of Ω . Such functions are the extremal points of $\mathcal{N}(\Omega)$ and are denoted $\text{Ext}(\mathcal{N}(\Omega))$.

A function $\rho \in \mathcal{M}(\Omega)$ is called bang-bang if $\rho = \mathbf{1}_{\omega'}$ for some measurable subset ω' of Ω . Such functions are the extremal points of $\mathcal{M}(\Omega)$ and are denoted $\text{Ext}(\mathcal{M}(\Omega))$.

The definition of bang-bang functions in terms of extremal points is classical [16, Proposition 7.2.17]. As an immediate consequence of Lemma 3.3 and of the convexity of the admissible sets $\mathcal{M}(\Omega)$ and $\mathcal{N}(\Omega)$, we obtain the following lemma:

Lemma 3.5. *We have that*

$$\begin{aligned} \inf_{D \in \mathcal{N}(\Omega)} \mu(D) &= \inf_{D \in \text{Ext}(\mathcal{N}(\Omega))} \mu(D), \\ \inf_{\rho \in \mathcal{M}(\Omega)} \lambda_\alpha(\rho) &= \inf_{\rho \in \text{Ext}(\mathcal{M}(\Omega))} \lambda_\alpha(\rho). \end{aligned}$$

4. PROOF OF THEOREM 2.1

The proof relies on several preliminary results that we recall in Subsection 4.1. The proof is then presented in Subsection 4.2.

4.1. Preliminary material about rearrangements. Let us briefly recall the key concepts of the Schwarz rearrangement. For a comprehensive introduction to rearrangements, we refer to [5, 21, 22]. For a \mathcal{C}^2 domain of \mathbb{R}^2 , let $\Omega^\# = \mathbb{B}(0; R^\#)$ be the centered ball with the same volume as Ω . For any function $\varphi \in L^2(\Omega)$, $\varphi \geq 0$, the Schwarz rearrangement of φ is the unique non-increasing function $\varphi^\# : \Omega^\# \rightarrow \mathbb{R}_+$ such that, for any $t \geq 0$,

$$\text{Vol}(\{\varphi > t\}) = \text{Vol}(\{\varphi^\# > t\}). \quad (4.1)$$

Of particular importance are the following properties of this rearrangement:

- (1) Equimeasurability: for any $\varphi \in L^2(\Omega)$, $\varphi \geq 0$,

$$\|\varphi\|_{L^2(\Omega)} = \|\varphi^\#\|_{L^2(\Omega^\#)}.$$

- (2) Hardy-Littlewood inequality: for any non-negative functions $\varphi_0, \varphi_1 \in L^2(\Omega)$,

$$\int_{\Omega} \varphi_0 \varphi_1 \leq \int_{\Omega^\#} \varphi_0^\# \varphi_1^\#.$$

Another key tool is the Talenti inequality [31] which reads as follows.

Proposition 4.1 (Talenti inequality, [31, Theorem 1]). *Let Ω be a Lipschitz bounded domain, and let \mathbb{B} be the ball centered in the origin and such that $\text{Vol}(\Omega) = \text{Vol}(\mathbb{B})$. Let $\psi \in L^2(\Omega)$, $\psi \geq 0$. Let $\phi \in W^{1,2}(\Omega)$ be the solution of*

$$\begin{cases} -\Delta \phi = \psi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.2)$$

and $\tilde{\phi}$ be the solution of

$$\begin{cases} -\Delta \tilde{\phi} = \psi^\# & \text{in } \mathbb{B}, \\ \tilde{\phi} = 0 & \text{on } \partial\mathbb{B}. \end{cases} \quad (4.3)$$

Then the inequality

$$\phi^\# \leq \tilde{\phi} \quad (4.4)$$

holds pointwise in \mathbb{B} .

The proof of Theorem 2.1 relies on some results of Alvino, Lions, and Trombetti [2]. These results have been used to show existence properties for two-phases optimisation problems in the case of balls by Conca, Mahadevan, and Sanz [15]. The strategy from [2] reads as follows: using a suitable rearrangement one checks that, when $\Omega = \mathbb{B}(0; R)$ for a suitable $R > 0$, one can restrict to minimising sequences of radially symmetric functions. Such symmetry then enables to use a powerful compactness result to obtain existence of a minimiser. What is notable in our approach is that the structure of the biharmonic operator makes it so that we do not require any symmetry property of the domain, nor of the elements of the minimising sequence.

Let us introduce a comparison relation: for any two functions $f, g \in L^2(\Omega)$, $f, g \geq 0$, we write

$$f \prec g \quad (4.5)$$

if, for any $r \in [0; R^\#]$,

$$\int_{\mathbb{B}(0;r)} f^\# \leq \int_{\mathbb{B}(0;r)} g^\# \quad (4.6)$$

and if

$$\int_{\mathbb{B}(0,R^\#)} f^\# = \int_{\mathbb{B}(0,R^\#)} g^\#. \quad (4.7)$$

Remark 4.2. *It should be noted that if g is L^∞ , and if $f \prec g$, then f is L^∞ as well and $\|f\|_{L^\infty} \leq \|g\|_{L^\infty}$.*

Let $\Omega^\# = \mathbb{B}(0; R^\#)$, and let $\mathbb{B}^* := \mathbb{B}(0; r^*)$ be the only ball centered in the origin of volume $(D_0 - \text{Vol}(\Omega))/\beta_0$. We define $\bar{D}^\#$ as

$$\bar{D}^\# = 1 + \beta_0 \mathbf{1}_{\mathbb{B}^*}. \quad (4.8)$$

First of all let us notice that for any $D \in \mathcal{N}(\Omega)$ we have

$$D^\# \prec \bar{D}^\# \quad \text{and} \quad \int_{\Omega} D = \int_{\Omega^\#} \bar{D}^\#. \quad (4.9)$$

We define the class

$$\mathcal{C}(\bar{D}^\#) := \left\{ f \in L^2(\Omega) : f \geq 0, f^\# = \bar{D}^\# \right\}, \quad (4.10)$$

which exactly corresponds to the set of bang-bang functions:

$$\text{Ext}(\mathcal{N}(\Omega)) = \mathcal{C}(\bar{D}^\#). \quad (4.11)$$

This class $\mathcal{C}(\psi)$ is not closed under weak-* L^∞ convergence. Its weak-* L^∞ compactification has been proved in [2] to be

$$\mathcal{K}(\bar{D}^\#) := \left\{ f \in L^2(\Omega) : f \geq 0, f \prec \bar{D}^\# \right\}. \quad (4.12)$$

From [2, Theorem 2.2], $\mathcal{K}(\bar{D}^\#)$ is closed and weakly-* compact for the L^∞ -topology (this result is a generalisation of a result by Migliaccio [28]). Furthermore, from [2, Theorem 2.2] we have

$$\text{Ext}(\mathcal{K}(\bar{D}^\#)) = \mathcal{C}(\bar{D}^\#). \quad (4.13)$$

As a consequence of the general result [16, Proposition 2.1] or directly from weak-* convergence to extreme points of convex sets, if a sequence $\{f_k\}_{k \in \mathbb{N}} \in \mathcal{K}(\bar{D}^\#)$ weakly-* converges to $f \in \mathcal{C}(\bar{D}^\#)$, then the convergence is strong in L^p , $p \in [1; +\infty)$, see [32].

4.2. Proof of Theorem 2.1. What should be noted is that, here, the weak-* L^∞ convergence of a sequence $\{D_k\}_{k \in \mathbb{N}} \in \mathcal{N}(\Omega)^\mathbb{N}$ does not imply the convergence of the associated sequence of eigenvalues $\{\mu(D_k)\}_{k \in \mathbb{N}}$. As is clear from the eigenequation

$$\begin{cases} \Delta(D\Delta u_\rho) = \mu(D)u_D \text{ in } \Omega, \\ u_D = \Delta u_D = 0 \text{ on } \partial\Omega, \end{cases} \quad (4.14)$$

the correct convergence that would imply lower-semi continuity of the eigenfunction is the convergence of the sequence $\left\{ \frac{1}{D_k} \right\}_{k \in \mathbb{N}}$.

Lemma 4.3. *Let δ and M_1 be two positive constants. Let $\{D_k\}_{k \in \mathbb{N}} \in L^\infty(\Omega)^\mathbb{N}$, $\inf_{k,\Omega} D_k \geq \delta > 0$, and $\sup_k \|D_k\|_{L^\infty(\Omega)} \leq M_1$. Assume there exists $C_\infty \in L^\infty(\Omega)$ such that*

$$\frac{1}{D_k} \xrightarrow[k \rightarrow +\infty]{} C_\infty \text{ weakly-* in } L^\infty. \quad (4.15)$$

Then, up to a subsequence,

$$\mu(D_k) \xrightarrow[k \rightarrow +\infty]{} \mu\left(\frac{1}{C_\infty}\right). \quad (4.16)$$

Proof of Lemma 4.3. To lighten notations, for any $k \in \mathbb{N}$ we denote by u_k the eigenfunction associated with $\mu(D_k)$ and we define

$$z_k = -D_k \Delta u_k. \quad (4.17)$$

From Lemma 3.1, we have $z_k = 0$ on $\partial\Omega$ and $z_k \geq 0$ in Ω .

Furthermore from Lemma 3.2 the sequence $\{\mu(D_k)\}_{k \in \mathbb{N}}$ is bounded. We can thus choose $\mu_\infty \in \mathbb{R}$ such that $\mu(D_k) \xrightarrow[k \rightarrow \infty]{} \mu_\infty$ for some not relabelled subsequence.

By assumption, we know that $\frac{1}{D_k} \xrightarrow[k \rightarrow \infty]{} C_\infty$ weakly-* in L^∞ . Since $\frac{1}{1+\beta_0} \leq \frac{1}{D_k} \leq 1$, the same L^∞ bounds hold for $\frac{1}{C_\infty}$. By Lemma 3.2, we have a uniform $W^{2,2}(\Omega)$ bound on the family $\{u_k\}_{k \in \mathbb{N}}$. Since z_k solves the equation

$$\begin{cases} -\Delta z_k = \mu(D_k) u_k & \text{in } \Omega, \\ z_k = 0 & \text{on } \partial\Omega \end{cases} \quad (4.18)$$

we obtain a uniform $W_0^{1,2}(\Omega)$ bound on $\{z_k\}_{k \in \mathbb{N}}$, namely, there exists M such that

$$\forall k \in \mathbb{N}, \|z_k\|_{W_0^{1,2}(\Omega)} \leq M. \quad (4.19)$$

As a consequence, there exists $z_\infty \in L^2(\Omega)$ such that

$$z_k \xrightarrow[k \rightarrow \infty]{} z_\infty \text{ weakly in } W_0^{1,2}(\Omega) \text{ and strongly in } L^2(\Omega) \quad (4.20)$$

for some not relabelled subsequence. There also exists $u_\infty \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ such that

$$u_k \xrightarrow[k \rightarrow \infty]{} u_\infty \text{ weakly in } W^{2,2}(\Omega) \text{ and strongly in } W_0^{1,2}(\Omega) \quad (4.21)$$

for some not relabelled subsequence. Passing to the limit in the weak formulation of (4.17), the triple $(u_\infty, C_\infty, z_\infty)$ solves

$$\begin{cases} -\Delta u_\infty = C_\infty z_\infty & \text{in } \Omega, \\ z_\infty = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.22)$$

and since, for any k , $u_k \geq 0$ and $\int_\Omega u_k^2 = 1$, we have

$$u_\infty \geq 0 \quad \text{and} \quad \int_\Omega u_\infty^2 = 1.$$

Passing to the limit in the weak formulation (4.18) we obtain that $(z_\infty, \mu_\infty, u_\infty)$ solves

$$\begin{cases} -\Delta z_\infty = \mu_\infty u_\infty & \text{in } \Omega, \\ z_\infty = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.23)$$

As a consequence, (C_∞, u_∞) solves

$$\begin{cases} \Delta \left(\frac{1}{C_\infty} \Delta u_\infty \right) = \mu_\infty u_\infty & \text{in } \Omega, \\ u_\infty = \Delta u_\infty = 0 & \text{on } \partial\Omega, \\ u_\infty \geq 0, \int_\Omega u_\infty^2 = 1. \end{cases} \quad (4.24)$$

However, the first eigenvalue being the only having a constant sign eigenfunction, we conclude that (u_∞, μ_∞) is the first eigencouple associated to $\frac{1}{C_\infty}$ or, in other words, that $\mu_\infty = \mu\left(\frac{1}{C_\infty}\right)$. Thus, the sequence $\{\mu(D_k)\}_{k \in \mathbb{N}}$ has a unique closure point, and hence the entire sequence converges, so that

$$\lim_{k \rightarrow \infty} \mu(D_k) = \mu\left(\frac{1}{C_\infty}\right). \quad (4.25)$$

□

We now treat the optimisation problem (2.3) in a slightly different way. For any $D \in L^\infty(\Omega)$ with $\inf D > 0$ we set

$$\eta(D) := \mu\left(\frac{1}{D}\right). \quad (4.26)$$

We recall that from Lemma 3.5 and Subsection 4.1 we have

$$\inf_{D \in \mathcal{N}(\Omega)} \mu(D) = \inf_{\{D \in \mathcal{N}(\Omega) : D^\# = \overline{D}^\#\}} \mu(D).$$

Since $\mathcal{C}\left(\overline{D}^\#\right) = \{D \in \mathcal{N}(\Omega) : D^\# = \overline{D}^\#\}$ this can be equivalently rewritten as

$$\inf_{D \in \mathcal{N}(\Omega)} \mu(D) = \inf_{D \in \mathcal{C}\left(\overline{D}^\#\right)} \mu(D). \quad (4.27)$$

Eventually, as $\overline{D}^\#$ is bang-bang, it follows that $D \in \mathcal{C}\left(\overline{D}^\#\right)$ if and only if $\frac{1}{D} \in \mathcal{C}\left(\left(\frac{1}{\overline{D}^\#}\right)^\#\right)$.

Given the definition of η , problem (2.3) is equivalent to

$$\inf_{\left\{E \in \mathcal{C}\left(\left(\frac{1}{\overline{D}^\#}\right)^\#\right)\right\}} \eta(E), \quad (4.28)$$

in the sense that, if E solves (4.28) then $\frac{1}{E}$ solves (2.3).

The key lemma is thus the following:

Lemma 4.4. (1) *The variational problem*

$$\inf_{E \in \mathcal{X}\left(\left(\frac{1}{\overline{D}^\#}\right)^\#\right)} \eta(E) \quad (4.29)$$

has a solution E^ .*

(2) *The solutions of the variational problem (4.29) belong to $\mathcal{C}\left(\left(\frac{1}{\overline{D}^\#}\right)^\#\right)$.*

Proof of Lemma 4.4. Point 1. The existence of a minimiser for problem (4.29) follows from the weak- $*$ L^∞ compactness of the set $\mathcal{X}\left(\left(\frac{1}{\overline{D}^\#}\right)^\#\right)$. Let $\{E_k\}_{k \in \mathbb{N}} \in \mathcal{X}\left(\left(\frac{1}{\overline{D}^\#}\right)^\#\right)^\mathbb{N}$ be a minimising sequence, and let $E_\infty \in \mathcal{X}\left(\left(\frac{1}{\overline{D}^\#}\right)^\#\right)$ be one of its weak closure points. From Lemma 4.3,

$$\eta(E_k) = \mu\left(\frac{1}{E_k}\right) \xrightarrow{k \rightarrow \infty} \mu\left(\frac{1}{E_\infty}\right) = \eta(E_\infty). \quad (4.30)$$

Hence, E_∞ is a solution of (4.29).

Point 2. To prove the second point of the lemma, it suffices to prove that no interior point $E \in \mathcal{X}\left(\left(\frac{1}{\overline{D}^\#}\right)^\#\right)$ satisfies local first order optimality conditions. By standard theorems [20] the simplicity of $\eta(E)$, obtained as in Lemma 3.2, enables us to differentiate it with respect to E . Let $E \in \mathcal{X}\left(\left(\frac{1}{\overline{D}^\#}\right)^\#\right)$ and h be an admissible perturbation at E (i.e. $E + th \in \mathcal{X}\left(\left(\frac{1}{\overline{D}^\#}\right)^\#\right)$ for $t > 0$ small enough). For the sake of notational simplicity, let u_E be the eigenfunction associated with $\eta(E)$. Let $\dot{\eta}$ and \dot{u} be the derivative of $\eta(E + th)$ and its associated eigenfunction with respect to t evaluated in the origin, respectively. Then, $(\dot{u}, \dot{\eta})$ solves

$$\begin{cases} \Delta\left(\frac{1}{E}\Delta\dot{u}\right) - \Delta\left(\frac{h}{E^2}\Delta u_E\right) = \dot{\eta}u_E + \eta(E)\dot{u}, \\ \dot{u} = \Delta\dot{u} = 0 \text{ on } \partial\Omega, \\ \int_\Omega u_E \dot{u} = 0. \end{cases} \quad (4.31)$$

As a consequence, testing (4.31) against u_E , using the eigenequation for u_E , and integrating by parts, from the fact that $\int_{\Omega} u_E^2 = 1$, we find

$$\dot{\eta} = \int_{\Omega} \frac{h}{E^2} (\Delta u_E)^2. \quad (4.32)$$

Thus, if E is not a bang-bang function, that is, if $\omega_0 := \left\{ \frac{1}{1+\beta_0} < E < 1 \right\}$ is a set of positive measure, there exists a constant C such that $\frac{(\Delta u_E)^2}{E^2} = C$ in ω_0 , see for instance [30, Theorem 1, Remark 1]. Plugging this in the eigenequation $\Delta \left(\frac{1}{E} \Delta u_E \right) = \eta(E) u_E$ we obtain

$$u_E = 0 \text{ in } \omega_0.$$

This contradicts the positivity of u_E inside Ω , which is a consequence of the strong maximum principle and of Lemma 3.2. \square

Relying on Lemma 4.4, we can eventually prove Theorem 2.1 by computing

$$\begin{aligned} \inf_{D \in \mathcal{C}} \mu(D) &= \inf_{D \in \mathcal{C}} \eta \left(\frac{1}{D} \right) = \min_{E \in \mathcal{X}} \eta(E) \\ &= \min_{E \in \mathcal{C}} \eta(E) = \eta(E^*) = \mu \left(\frac{1}{E^*} \right). \end{aligned}$$

Since $E^* \in \mathcal{C} \left(\frac{1}{D^{\#}} \right)^{\#}$, we have that $\frac{1}{E^*} \in \mathcal{C} \left(\overline{D^{\#}} \right)$. This entails the existence of a minimizer, hence Theorem 2.1 holds.

5. PROOF OF THEOREM 2.2

Recall that here $\Omega = \mathbb{B}(0, R)$ for some $R > 0$. The core idea of the proof is to use the Talenti inequality, as was done in [4] to solve (2.3). Let us briefly recall this inequality:

Let $D \in \mathcal{N}(\Omega)$ and u_D be the associated eigenfunction solving (2.2). Let z_D be defined as

$$-\Delta u_D = z_D \text{ in } \Omega. \quad (5.1)$$

Since $u_D \in H^2(\Omega)$ we have that $z_D \in L^2(\Omega)$.

From $\Delta u_D = 0$ on $\partial\Omega$ we obtain $z_D = 0$ on $\partial\Omega$. Furthermore, from Lemma 3.1, there holds $z_D \geq 0$ in Ω . Let us consider its Schwarz rearrangement $z_D^{\#}$. Since Ω is a ball centered in the origin, clearly $\Omega^{\#} = \Omega$. Let \tilde{u}_D be the solution of

$$-\Delta \tilde{u}_D = z_D^{\#} \text{ in } \Omega, \tilde{u}_D = 0 \text{ on } \partial\Omega. \quad (5.2)$$

From the Talenti inequality, Proposition 4.1, we have

$$0 \leq u_D^{\#} \leq \tilde{u}_D \text{ in } \Omega. \quad (5.3)$$

This inequality holds pointwise and hence guarantees

$$1 = \int_{\Omega} u_D^2 = \int_{\Omega} (u_D^{\#})^2 \leq \int_{\Omega} \tilde{u}_D^2. \quad (5.4)$$

Furthermore, since $z_D^{\#}$ is a rearrangement of z_D , for any $V \in [0; \text{Vol}(\Omega)]$ we have that

$$\inf_{F \subset \Omega, \text{Vol}(F)=V} \int_F z_D^2 = \inf_{G \subset \mathbb{B}, \text{Vol}(G)=V} \int_G (z_D^{\#})^2. \quad (5.5)$$

Take now $V = (D_0 - \text{Vol}\Omega)/\beta_0$. The function $z_D^{\#}$ being non-increasing, the so-called *bathtub* principle [24, Theorem 1.14] ensures that

$$\inf_{G \subset \mathbb{B}, \text{Vol}(G)=V} \int_{\Omega} (1 + \beta_0 \mathbb{1}_G) (z_D^{\#})^2 = \int_{\mathbb{B}} \overline{D}_{\#} (z_D^*)^2. \quad (5.6)$$

On the one hand, the Schwarz rearrangement is measure preserving, hence

$$\int_{\Omega} (\Delta u_D)^2 = \int_{\Omega} z_D^2 = \int_{\mathbb{B}} (z_D^{\#})^2 = \int_{\mathbb{B}} (\Delta \tilde{u}_D)^2. \quad (5.7)$$

On the other hand, from (5.5)-(5.6) we get

$$\int_{\Omega} D(\Delta u_D)^2 \geq \int_{\mathbb{B}} \bar{D}_{\#} (\Delta \tilde{u}_D)^2. \quad (5.8)$$

Combining (5.8) with (5.4) and plugging these estimates in the Rayleigh-quotient formulation of the eigenvalues we obtain

$$\mu(D) = \frac{\int_{\Omega} D(\Delta u_D)^2}{\int_{\Omega} u_D^2} \geq \frac{\int_{\mathbb{B}} \bar{D}_{\#} (\Delta \tilde{u}_D)^2}{\int_{\Omega} \tilde{u}_D^2} \geq \mu(\bar{D}_{\#}), \quad (5.9)$$

and the assertion follows.

6. PROOF OF THEOREM 2.3

We are now working under the assumption that $\Omega = \mathbb{B}(0, R)$ for some $R > 0$. Recall that $\rho^* = \mathbb{1}_{\mathbb{B}^*}$ is the characteristic function of a ball centered in the origin and of volume V_0 . By the same arguments as in [4, Theorem 3.3], ρ^* is the unique minimiser of λ_0 in $\mathcal{M}(\Omega)$ and u_{0,ρ^*} is radially symmetric non-increasing.

Furthermore, u_{0,ρ^*} is strictly decreasing and there holds

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0, \forall r \in (\varepsilon; R] : \left| \frac{\partial u_{0,\rho^*}}{\partial r} \right| \geq \delta(\varepsilon). \quad (6.1)$$

Indeed, this follows from the following fact: replacing u_{0,ρ^*} with the solution w to

$$\begin{cases} -\Delta w = |\Delta u|^* & \text{in } \mathbb{B}(0; R), \\ w \in W_0^{1,2}(\Omega), \end{cases}$$

we obtain, combining the arguments of Lemma 3.2 and the Talenti inequality, that

$$\frac{\int_{\Omega} (\Delta w)^2 - \int_{\Omega} \rho^* w^2}{\int_{\Omega} w^2} \leq \frac{\int_{\Omega} (\Delta u_{0,\rho^*})^2 - \int_{\Omega} \rho^* u_{0,\rho^*}^2}{\int_{\Omega} u_{0,\rho^*}^2}.$$

Thus, w is also an eigenfunction. By simplicity of $\lambda_0(\rho^*)$, u_{0,ρ^*} and w are linearly dependent. As a consequence, $u_{0,\rho^*} = cw$ for some constant $c > 0$; this sign condition comes from the fact that both u_{0,ρ^*} and w are non-negative. Thus, it follows that $-\Delta u_{0,\rho^*} = |\Delta u_{0,\rho^*}|^*$. Since $\Delta u_{0,\rho^*} \neq 0$, $|\Delta u_{0,\rho^*}|^*(0) > 0$. Setting $z = \Delta u_{0,\rho^*} = -|\Delta u_{0,\rho^*}|^*$ we have, in radial coordinates

$$r \frac{\partial u_{0,\rho^*}}{\partial r}(r) = \int_0^r \tau z(\tau) d\tau < 0,$$

which concludes the proof.

We prove Theorem 2.3 by contradiction and assume that, for any $\alpha > 0$, there exists a radially symmetric $\rho_{\alpha} \in \mathcal{M}(\Omega)$ such that

$$\lambda_{\alpha}(\rho_{\alpha}) \leq \lambda_{\alpha}(\rho^*), \quad \rho_{\alpha} \neq \rho^*. \quad (6.2)$$

Let us prepare a preliminary lemma.

Lemma 6.1. *We can assume that ρ_{α} is a bang-bang function. Furthermore, we have $\rho_{\alpha} \xrightarrow{\alpha \rightarrow 0} \rho^*$ strongly in L^1 .*

Proof of Lemma 6.1. The first point follows from the concavity of the functional. For the second point, we first observe that, for any weak-* L^{∞} closure point ρ_0 of $\{\rho_{\alpha}\}_{\alpha \rightarrow 0}$, there holds $\lambda_0(\rho_0) \leq \liminf_{\alpha \rightarrow 0} \lambda_{\alpha}(\rho_{\alpha})$: setting, for notational convenience, $u_{\alpha} := u_{\alpha,\rho_{\alpha}}$, we have, from Lemma 3.2, a uniform $W^{2,2}(\Omega)$ bound on

this sequence. This allows to pick a weak $W^{2,2}(\Omega)$ and strong $L^2(\Omega)$ closure point u_0 of a not relabelled subsequence. By weak lower-semicontinuity of convex functions

$$\int_{\Omega} (\Delta u_0)^2 \leq \liminf_{\alpha \rightarrow 0} \int_{\Omega} (1 + \alpha \rho_{\alpha}) (\Delta u_{\alpha})^2,$$

while

$$\int_{\Omega} \rho_0 u_0^2 = \lim_{\alpha \rightarrow 0} \int_{\Omega} \rho_{\alpha} u_{\alpha}^2, \quad \int_{\Omega} u_0^2 = 1.$$

From the variational formulation (2.8) of $\lambda_0(\rho_0)$, we obtain the conclusion. Let us then observe that for any fixed $\rho \in \mathcal{M}(\Omega)$ (in particular for $\rho = \rho^*$), there holds $\lambda_0(\rho) = \lim_{\alpha \rightarrow 0} \lambda_{\alpha}(\rho)$. Passing to the limit in the inequality $\lambda_{\alpha}(\rho_{\alpha}) \leq \lambda_{\alpha}(\rho^*)$ we obtain $\lambda_0(\rho_0) \leq \lambda_0(\rho^*)$. Since ρ^* is the unique minimiser of λ_0 we have $\rho_0 = \rho^*$. As ρ^* is an extreme point of $\mathcal{M}(\Omega)$, from [16, Proposition 2.1] this convergence is strong in L^1 . \square

Henceforth, we can hence assume that the sequence $\{\rho_{\alpha}\}_{\alpha \rightarrow 0}$ fulfilling (6.2) consists of bang-bang functions. We use this information to proceed with the proof, which rests upon fine properties of the switch function. We need to use one of the core idea of H -convergence to make sure this function is regular enough. Let us explain why some concepts from homogenisation are needed: if we consider the map $D \mapsto \lambda_{\alpha}(D)$ and if we define $u_{\alpha,\rho}$ as the eigenfunction associated with $\lambda_{\alpha}(\rho)$, the simplicity of the eigenvalue (Lemma 3.2) ensures that $\rho \mapsto (\lambda_{\alpha}(\rho), u_{\alpha,\rho})$ is Gâteaux-differentiable. Furthermore, for any $\rho \in \mathcal{M}(\mathbb{B}(0, R))$ and any admissible perturbation h at ρ (i.e a function h such that, for every $\varepsilon > 0$ small enough $\rho + \varepsilon h \in \mathcal{M}(\mathbb{B}(0, R))$), the Gâteaux-derivatives $\dot{u}_{\alpha,\rho}$ and $\dot{\lambda}_{\alpha}(\rho)$ (we omit the dependency on h for notational convenience) solve

$$\begin{cases} \Delta((1 + \alpha\rho)\Delta\dot{u}_{\alpha,\rho}) + \alpha\Delta(h\Delta u_{\alpha,\rho}) = (\lambda_{\alpha,\rho} + \rho)\dot{u}_{\alpha,\rho} + (\dot{\lambda}_{\alpha,\rho} + h)u_{\alpha,\rho} & \text{in } \mathbb{B}(0, R), \\ \dot{u}_{\alpha,\rho} = \Delta\dot{u}_{\alpha,\rho} = 0 & \text{on } \partial\mathbb{B}(0, R), \\ \int_{\mathbb{B}(0, R)} u_{\alpha,\rho}\dot{u}_{\alpha,\rho} = 0. \end{cases} \quad (6.3)$$

Multiplying the equation by $u_{\alpha,\rho}$, integrating by parts, and using the equation (2.9) on $u_{\alpha,\rho}$ we obtain the following expression for $\dot{\lambda}_{\alpha}(\rho)$:

$$\dot{\lambda}_{\alpha}(\rho) = \int_{\mathbb{B}(0, R)} h \left\{ \alpha (\Delta u_{\alpha,\rho})^2 - u_{\alpha,\rho}^2 \right\}. \quad (6.4)$$

This leads to defining the switch function associated with the problem as

$$U_{\alpha,\rho} := \alpha (\Delta u_{\alpha,\rho})^2 - u_{\alpha,\rho}^2. \quad (6.5)$$

In other words, with this approach, we have $\dot{\lambda}_{\alpha}(\rho) = \int_{\mathbb{B}(0, R)} U_{\alpha,\rho} h$. Ideally, we would use Lemma 6.1 to approximate $U_{\alpha,\rho}$ by U_{0,ρ^*} in the \mathcal{C}^1 norm. However, since $\Delta u_{\alpha,\rho}$ is merely L^{∞} , $U_{\alpha,\rho}$ is not regular enough. To overcome this problem, we rely on some general ideas borrowed from H -convergence and homogenisation theory [1, 29]. We introduce, for any $\rho \in \mathcal{M}(\mathbb{B}(0, R))$, the harmonic mean $\mathcal{J}_-(\rho)$ of $1 + \alpha\rho$, defined as

$$\mathcal{J}_-(\rho) := \frac{1 + \alpha}{1 + \alpha(1 - \rho)}. \quad (6.6)$$

We define an auxiliary eigenvalue $\Lambda_{\alpha}(\rho)$ as follows:

$$\Lambda_{\alpha}(\rho) := \min_{u \in W^{2,2}(\mathbb{B}(0, R)) \cap W_0^{1,2}(\mathbb{B}(0, R)) \text{ } u \neq 0} \frac{\int_{\mathbb{B}(0, R)} \mathcal{J}_-(\rho) (\Delta u)^2 - \int_{\mathbb{B}(0, R)} \rho u^2}{\int_{\mathbb{B}(0, R)} u^2}. \quad (6.7)$$

From this variational formulation, since $\rho \mapsto \mathcal{J}_-(\rho)$ is concave, we have that $\rho \mapsto \Lambda_{\alpha}(\rho)$ is concave too. If ρ is a bang-bang function, that is, if $\rho = \mathbf{1}_E$ for some measurable subset E , then $\mathcal{J}_-(\rho) = 1 + \alpha\rho$ so that

$$\text{For any bang-bang } \rho \text{ one has that } \lambda_{\alpha}(\rho) = \Lambda_{\alpha}(\rho). \quad (6.8)$$

Hence for all $\alpha > 0$, we have that

$$\lambda_{\alpha}(\rho_{\alpha}) = \Lambda_{\alpha}(\rho_{\alpha}) \text{ and } \lambda_{\alpha}(\rho^*) = \Lambda_{\alpha}(\rho^*).$$

To see why this allows to overcome the aforementioned regularity issues, let us compute the Gâteaux-derivative of the map $\rho \mapsto \Lambda_\alpha(\rho)$. Let us define $v_{\alpha,\rho}$ to be the eigenfunction associated with $\Lambda_\alpha(\rho)$. This can be chosen positive and normalized in L^2 . In particular, $v_{\alpha,\rho}$ solves

$$\begin{cases} \Delta(\mathcal{J}_-(\rho)\Delta v_{\alpha,\rho}) = \Lambda_\alpha(\rho)v_{\alpha,\rho} + \rho v_{\alpha,\rho} & \text{in } \mathbb{B}(0, R), \\ v_{\alpha,\rho} = \Delta v_{\alpha,\rho} = 0 & \text{on } \partial\mathbb{B}(0, R), \\ \int_{\mathbb{B}(0, R)} v_{\alpha,\rho}^2 = 1, v_{\alpha,\rho} \geq 0. \end{cases} \quad (6.9)$$

From the same arguments as in Lemma 3.2, $\Lambda_\alpha(\rho)$ is a simple eigenvalue, and so the map $\rho \mapsto (\Lambda_\alpha(\rho), v_{\alpha,\rho})$ is Gâteaux-differentiable and, for $\rho \in \mathcal{M}(\mathbb{B}(0, R))$ and an admissible perturbation h at ρ , if we denote with a dot the Gâteaux-differentiated quantities, the couple $(\dot{\Lambda}_\alpha(\rho), \dot{v}_{\alpha,\rho})$ solves

$$\begin{cases} \Delta(\mathcal{J}_-(\rho)\Delta \dot{v}_{\alpha,\rho}) + \frac{\alpha}{1+\alpha}\Delta(h\mathcal{J}_-(\rho)^2\Delta v_{\alpha,\rho}) = (\Lambda_\alpha(\rho) + \rho)\dot{v}_{\alpha,\rho} \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \dot{\Lambda}_\alpha(\rho)v_{\alpha,\rho} + hv_{\alpha,\rho} & \text{in } \mathbb{B}(0, R), \\ \dot{v}_{\alpha,\rho} = \Delta \dot{v}_{\alpha,\rho} = 0 & \text{on } \partial\mathbb{B}(0, R), \\ \int_{\mathbb{B}(0, R)} \dot{v}_{\alpha,\rho}v_{\alpha,\rho} = 0. \end{cases} \quad (6.10)$$

This equation has a unique solution by the Fredholm alternative. Multiplying the first equation in (6.10) by $v_{\alpha,\rho}$, integrating by part and using (6.9) yields

$$\dot{\Lambda}_\alpha(\rho) = \int_{\mathbb{B}(0, R)} h \left\{ \frac{\alpha}{1+\alpha} \mathcal{J}_-(\rho)^2 (\Delta v_{\alpha,\rho})^2 - v_{\alpha,\rho}^2 \right\}. \quad (6.11)$$

The new switch function

$$\psi_{\alpha,\rho} := \frac{\alpha}{1+\alpha} \mathcal{J}_-(\rho)^2 (\Delta v_{\alpha,\rho})^2 - v_{\alpha,\rho}^2 \quad (6.12)$$

is now more regular, since the function $\mathcal{J}_-(\rho)\Delta v_{\alpha,\rho}$ is itself the solution of an elliptic problem. Let us now consider the two bang-bang-densities $\rho_\alpha, \rho^* \in \mathcal{M}(\mathbb{B}(0, R))$. Instead of considering the path $t \mapsto \lambda_\alpha(\rho_\alpha + t(\rho^* - \rho_\alpha))$, which would lead to the irregular switch function (6.5), we set $\rho_t := \rho_\alpha + t(\rho^* - \rho_\alpha)$ and we consider the path

$$f_\alpha : t \mapsto \Lambda_\alpha(\rho_t). \quad (6.13)$$

For $t \in [0, 1]$, let us define v_t to be the eigenfunction associated with $\Lambda_\alpha(\rho_t)$ and

$$\Psi_t := \frac{\alpha}{1+\alpha} \mathcal{J}_-(\rho_t)^2 (\Delta v_t)^2 - v_t^2 \quad (6.14)$$

By Lemma 6.1 and by the mean value Theorem, we write

$$\lambda_\alpha(\rho^*) - \lambda_\alpha(\rho_\alpha) = \Lambda_\alpha(\rho^*) - \Lambda_\alpha(\rho_\alpha) = \int_{\mathbb{B}(0, R)} \Psi_t(\rho^* - \rho_\alpha) \quad (6.15)$$

for some $t = t(\alpha) \in [0, 1]$. From Lemma 6.1, we know that $\rho_{t(\alpha)} \xrightarrow{\alpha \rightarrow 0} \rho^*$ strongly in $L^1(\mathbb{B}(0, R))$. From standard elliptic regularity, there exists a constant $M > 0$ such that $\|\mathcal{J}_-(\rho_{t(\alpha)})\Delta v_{t(\alpha)}\|_{C^1(\mathbb{B}(0, R))} \leq M$. Again from elliptic regularity, we also have that $v_{t(\alpha)} \xrightarrow{\alpha \rightarrow 0} u_{0,\rho^*}$ in \mathcal{C}^1 . Hence, $\Psi_{t(\alpha)} \xrightarrow{\alpha \rightarrow 0} -u_{0,\rho^*}^2$ in \mathcal{C}^1 . Since $\Psi_{t(\alpha)}$ is radial, the strict monotonicity (6.1) implies that ρ^* is the unique solution of

$$\inf_{\rho \in \mathcal{M}(\mathbb{B}(0, R))} \int_{\mathbb{B}(0, R)} \Psi_{t(\alpha)} \rho \quad (6.16)$$

for $\alpha > 0$ small enough. Indeed, from (6.1) and the \mathcal{C}^1 convergence of $\{\Psi_{t(\alpha)}\}_{\alpha \rightarrow 0}$ to $-u_{0,\rho^*}^2$, for $\alpha > 0$ small enough, \mathbb{B}^* is the unique level set of $\Psi_{t(\alpha)}$ of volume ρ_0 .

Hence, $\int_{\mathbb{B}(0, R)} \Psi_{t(\alpha)}(\rho_\alpha - \rho^*) \geq 0$, which in turn implies that $\Lambda_\alpha(\rho^*) - \Lambda_\alpha(\rho_\alpha) \leq 0$. By Lemma 6.1, this leads to contradicting (6.2) and concludes the proof of the theorem.

7. CONCLUSION

In this article, we have studied several theoretical aspects related with the spectral optimisation of inhomogeneous plates. It is worth underlining that the existence result, Theorem 2.1, is in sharp contrast with other results in the context of the optimisation of two-phase problems.

Note moreover that the stationarity of minimisers of λ_α , as $\alpha \rightarrow 0^+$ is proved with respect to radial competitors only. We believe that the case of not radially symmetric competitors is presently out of reach, given the available rearrangement tools. In fact, Theorem 2.2 indicates that the correct rearrangement when handling thickness optimisation is expected to be the *increasing* rearrangement, whereas previous results [4] point to the fact that optimisation with respect to the density should rather involve the *decreasing* rearrangement.

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