Local Hölder regularity of minimizers for nonlocal denoising problems

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Abstract

We study the regularity of solutions to a nonlocal version of the image denoising model and we show that, in two dimensions, minimizers have the same Hölder regularity as the original image. More precisely, if the datum is (locally) β-Hölder continuous for some $\beta \in (1-s,\, 1]$, where $s \in (0,1)$ is a parameter related to the nonlocal operator, we prove that the solution is also β-Hölder continuous.

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1 Introduction

Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a given function and $f : \mathbb{R}^n \to \mathbb{R}$ be a given datum. We study the minimization problem

$$\min\left\{\mathcal{F}_{K,f}(u) \mid u \in BV_K(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right\} \tag{1.1}$$

where the functional $\mathcal{F}_{K,f}$ is defined as

$$\mathcal{F}_{K,f}(u) \coloneqq \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\,|u(x)-u(y)|\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^n}(u(x)-f(x))^2\,dx \tag{1.2}$$

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for any measurable function $u : \mathbb{R}^n \to \mathbb{R}$, and the space $BV_K(\mathbb{R}^n)$ is the set of all functions such that the first term of (1.2) is finite (see Section 2 for the detail). The function K is a kernel singular at the origin, and a typical example is the function $x \mapsto |x|^{-(n+s)}$ with $s \in (0, 1)$. If K is non-negative and we understand that $\mathcal{F}_{K,f}(u) = +\infty$ when $u \in L^2(\mathbb{R}^n) \setminus BV_K(\mathbb{R}^n)$, then we observe that the functional $\mathcal{F}_{K,f}$ is strictly convex, lower semi-continuous, and coercive in $L^2(\mathbb{R}^n)$. Hence, from the general theory of functional analysis (see, for instance, [8]), we obtain existence and uniqueness of solutions to (1.1).

In this paper we focus on the specific kernel $K(x) = |x|^{-(n+s)}$, with $s \in (0, 1)$, and we study the regularity of the minimizers of $\mathcal{F}_{K,f}$, under some suitable conditions on the datum f. Our minimization problem is motivated by the classical variational problem

$$\min \left\{ \mathcal{F}_f(u) \mid u \in BV(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right\} \tag{1.3}$$

where $\mathcal{F}_f(u)$ is defined as

$$\mathcal{F}_f(u) := \int_{\mathbb{R}^n} |\nabla u| \, dx + \frac{1}{2} \int_{\mathbb{R}^n} |u - f|^2 \, dx. \tag{1.4}$$

Th minimization problem (1.3) has been studied by many authors since the celebrated work by Rudin, Osher, and Fatemi [33], and plays an important role in image denoising and restoration (see for instance [14, 9]). From the perspective of image processing, the datum f in the functional $\mathcal{F}_f$ indicates an observed image and, when the given image has poor quality, then the minimizers of $\mathcal{F}_f$ or solutions to the Euler-Lagrange equation associated with $\mathcal{F}_f$ correspond to regularized images. It is easy to show that the minimizer of (1.4) exists and is unique, as a result of strict convexity, lower semi-continuity and coercivity of the functional. Moreover, the minimizer turns out to be the solution, in a suitable sense, of the Euler-Lagrange equation

$$-\mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right) + u - f = 0 \quad \text{ in } \mathbb{R}^n. \tag{1.5}$$

The regularity of minimizers of $\mathcal{F}_f$ have been studied by several authors. In particular, the global and local regularity was investigated in a series of papers by Caselles, Chambolle and Novaga (see [12, 13, 14]), who proved that the solution of (1.5) inherits the local Hölder or Lipschitz regularity of the datum f, when the space-dimension n is less than or equal to 7. In addition, if f is globally Hölder or Lipschitz in a convex domain $\Omega \subset \mathbb{R}^n$, the global regularity also holds for the solution of (1.5) with homogeneous Neumann boundary condition. In the recent papers [30, 32], some of these results were extended to general dimensions. In [30], Mercier has proved that the continuity of f implies the continuity of a solution u and, in the case of convex domains, the modulus of continuity is also inherited globally by the solution. Eventually, in [32], Porretta was able to remove the restriction on the space-dimension.

For the variational problems associated with the nonlocal total variation, Aubert and Kornprobst in [4] and Gilboa and Osher in [22, 23] have proposed the methods for approximating the solutions to (1.3) with a sequence of nonlocal total variations associated with non-singular smooth kernels. However, as far as we know, there are no results on the regularity of minimizers of the functional $\mathcal{F}_{K,f}$. Thus, in this paper, we consider the local Hölder regularity of the minimizers of (1.2) in dimension 2 as an analogy of the regularity results shown in [12, 13]. Precisely, we prove the following result:

Theorem 1.1. *Let $n = 2$, $s \in (0, 1)$, $K(x) = |x|^{-(2+s)}$ and $f \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. If f is locally β-Hölder continuous with $\beta \in (1 - s, 1]$, then the minimizer u of the functional $\mathcal{F}_{K,f}$ is also locally β-Hölder continuous in $\mathbb{R}^2$.*

We point out that we cannot show the regularity result in any dimension due to the appearance of singularities on the boundary of the levelsets of minimizers. However, the two-dimensional case is of particular interest for the application to image denoising.

As discussed in the case of the denoising problem in [12, 13, 14], our regularity result is based on the following observation: if u is a minimizer of the functional $\mathcal{F}_{K,f}$, then the super-levelset $\{u > t\}$ for each $t \in \mathbb{R}$ is also a minimizer of the functional associated with the prescribed nonlocal mean curvature problem

$$\min\{\mathcal{E}_{K,f,t}(E) \mid E \subset \mathbb{R}^n\colon \text{measurable}\} \tag{1.6}$$

where we define the functional $\mathcal{E}_{K,f,t}$ by

$$\mathcal{E}_{K,f,t}(E) \coloneqq P_K(E) + \int_E (t - f(x))\,dx$$

for any measurable set $E \subset \mathbb{R}^n$ and $t \in \mathbb{R}$. Here P_K is the *nonlocal perimeter* associated with the kernel K, which is given as

$$P_K(E) \coloneqq \int_E \int_{E^c} K(x-y)\,dx\,dy$$

for any measurable set $E \subset \mathbb{R}^n$ (see Section 2 for the detail). If E_t is a minimizer of $\mathcal{E}_{K,f,t}$ for each t and ∂E_t is smooth (C^2-regularity is sufficient), then we can obtain that the boundary ∂E_t satisfies the following *prescribed nonlocal mean curvature equation*

$$H^K_{E_t}(x) + t - f(x) = 0$$

for any $x \in \partial E_t$. Here $H^K_{E_t}$ is the so-called *nonlocal mean curvature* defined by

$$H^K_{E_t}(x) \coloneqq \text{p.v.} \int_{\mathbb{R}^n} K(x-y)(\chi_{E_t^c}(x) - \chi_{E_t}(y))\,dy \tag{1.7}$$

for any $x \in \mathbb{R}^n$, where we mean by "p.v." the Cauchy principal value and E^c means the complement of a set E.

The idea to show the local Hölder regularity of a minimizer u is based on the observation that the distance between the boundaries of the two super-levelsets $\{u > t\}$ and $\{u > t'\}$ for t, $t' \in \mathbb{R}$ with $t \neq t'$ should not be too close. To observe this, we compare between the nonlocal mean curvatures of $\partial\{u > t\}$ and $\partial\{u > t'\}$ at the points where the smallest distance between the boundaries $\partial\{u > t\}$ and $\partial\{u > t'\}$ is attained. The comparison can be done thanks to the computations of the first variation of the nonlocal mean curvature shown in [17, 24]. Thus we are able to derive some local estimate to assert the local Hölder regularity with the assumption of the local Hölder regularity of f.

The organization of this paper is as follows: In Section 2, we will introduce the notation related to the nonlocal total variations. In Section 3.1, we will show the correspondence between the minimizers of $\mathcal{F}_{K,f}$ in (1.2) and the solutions to the nonlocal 1-Laplace equation. In Section 3.2, we will give a sort of comparison principle for the minimizers. As a result of this claim, we will show that, if a datum f is bounded, then the minimizer of $\mathcal{F}_{K,f}$ is also bounded. In Section 3.3, we will show that each super-levelset of a minimizer of $\mathcal{F}_{K,f}$ is also a minimizer of $\mathcal{E}_{K,f,t}$ for $t \in \mathbb{R}$. In Section 3.4, we will show the boundedness of each super-levelset of the minimizer and, moreover, this set can be uniformly bounded whenever the minimizer is bounded from below. Finally, by using all the previous results, in Section 4 we prove the main theorem in this paper on the Hölder regularity of minimizers in two dimensions.

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2 Notation

In this section, we give several definitions and properties of the space of functions with finite nonlocal total variations. First of all, we define the space $BV_K(\Omega)$ of functions with nonlocal bounded variations associated with the kernel K by

$$BV_K(\Omega) \coloneqq \{u \in L^1(\Omega) \mid [u]_K(\Omega) < \infty\} \tag{2.1}$$

where we set, for any measurable function u,

$$[u]_K(\Omega) \coloneqq \frac{1}{2}\int_{\Omega}\int_{\Omega} K(x-y)\,|u(x)-u(y)|\,dx\,dy. \tag{2.2}$$

We observe that the space $BV_K(\Omega)$ coincides with the fractional Sobolev space $W^{s,1}(\Omega)$ when the kernel K is given as $K(x) = |x|^{-(n+s)}$ with $s \in (0, 1)$ (see, for instance, [31]).

Secondly, if we set $\Omega = \mathbb{R}^n$ and substitute a characteristic function χ_E of a set $E \subset \mathbb{R}^n$ in (2.2), then we obtain the so-called *nonlocal perimeter*. Namely, we define the nonlocal perimeter of a set $E \subset \mathbb{R}^n$ associated with the kernel K by

$$P_K(E) \coloneqq \int_{E}\int_{E^c} K(x-y)\,dx\,dy. \tag{2.3}$$

In the case that $K(x) = |x|^{-(n+s)}$ for $s \in (0, 1)$, we call P_K the *s-fractional perimeter*, and we denote it by P_s. This notion was introduced by Caffarelli, Roquejoffre, and Savin in [10]. After their work, any problems involving not only the s-fractional perimeter but also the nonlocal perimeter with the kernel K were studied by many authors. We leave here a short list of papers, which are related to our problems, for those who are interested in the variational problems involving the nonlocal perimeter [4, 5, 9, 16, 20, 28, 29] and the references are therein.

Next we can consider a localized version of the nonlocal perimeter P_K as follows: let $\Omega \subset \mathbb{R}^n$ be any domain. Then the nonlocal perimeter in Ω associated with the kernel K is given by

$$\begin{aligned} P_K(E;\Omega) \coloneqq &\int_{\Omega\cap E}\int_{\Omega\cap E^c} K(x-y)\,dx\,dy + \int_{\Omega\cap E}\int_{\Omega^c\cap E^c} K(x-y)\,dx\,dy \\ &+ \int_{\Omega\cap E^c}\int_{\Omega^c\cap E} K(x-y)\,dx\,dy \end{aligned}$$

for any $E \subset \mathbb{R}^n$. Secondly, we give the definition of solutions to the so-called *nonlocal 1-Laplace equations* associated with the kernel K.

Definition 2.1. Let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a measurable function in $L^2(\mathbb{R}^n \times \mathbb{R})$. We say that $u \in BV_K \cap L^2(\mathbb{R}^n)$ is a solution to the nonlocal equation

$$\Delta_1^K u(x) = F(x, u(x)) \quad \text{for a.e. } x \in \mathbb{R}^n \tag{2.4}$$

if there exists a function $z : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with $|z| \le 1$ a.e. in $\mathbb{R}^n \times \mathbb{R}^n$ and $z(x, y) = -z(y, x)$ for a.e. $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x-y)\, z(x, y)(v(x) - v(y))\, dx\, dy = \int_{\mathbb{R}^n} F(x, u(x))\, v(x)\, dx \tag{2.5}$$

for every $v \in BV_K \cap L^2(\mathbb{R}^n)$ with a compact support and

$$z(x, y) \in \operatorname{sgn}(u(y) - u(x)) \quad \text{for a.e. } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$$

where $\operatorname{sgn}(x)$ is a generalized sign function satisfying that

$$\operatorname{sgn}(x) \in [-1, 1], \quad \operatorname{sgn}(x)\, x = |x| \quad \text{for any } x \in \mathbb{R}.$$

In the present paper, we only consider the case that $F(x, u(x)) = u(x) - f(x)$ for a given datum f. The concept of the definition is motivated by the Euler-Lagrange equation of the functional

$$\mathcal{I}_K(u) := \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x-y)|u(x) - u(y)|\, dx\, dy.$$

Indeed, when we assume that u is a minimizer of $\mathcal{I}_K$ and consider the first variation of the functional $\mathcal{I}_K$, namely, the quantity $\frac{d}{d\varepsilon}\lfloor_{\varepsilon=0}\mathcal{I}_K(u+\varepsilon\phi)$ for any suitable test function ϕ, we can formally obtain

$$\frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x-y) \frac{u(x) - u(y)}{|u(x) - u(y)|} (\phi(x) - \phi(y))\, dx\, dy = 0.$$

However, it is quite problematic for us to give a rigorous meaning to the ratio $\frac{u(x)-u(y)}{|u(x)-u(y)|}$. To overcome this difficulty, we may apply Definition 2.1 and this can be regarded as one of the proper treatments for this issue. Indeed, in Definition 2.1, we may consider the condition that $z(x, y)(u(y) - u(x)) = |u(y) - u(x)|$ for a.e. $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $u(x) \neq u(y)$ as a natural requirement. Note that the framework of solutions in the sense of Definition 2.1 has been originally developed by, for instance, Mazón, Rossi, and Toledo in [27] and can be seen as a nonlocal counterpart of the framework given in [3] and [26].

3 Preliminary results

3.1 Euler-Lagrange equation for $\mathcal{F}_{K,f}$

In this section, we show the necessary and sufficient condition for the minimizers of the functional $\mathcal{F}_{K,f}$ in $\mathbb{R}^n$. Before stating the claim, we give some conditions on the kernel K which we will assume in the sequel.

(K1) $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is a non-negative measurable function.

(K2) K is symmetric with respect to the origin, namely $K(-x) = K(x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

We observe that a typical example of the kernel K is given as $K(x) = |x|^{-(n+s)}$ with $s \in (0, 1)$ and this function satisfies all the above assumptions.

In the following lemma, we show that the minimizer of $\mathcal{F}_{K,f}$ satisfies a prescribed nonlocal mean curvature equation. This equation can be regarded as the Euler-Lagrange equation. Moreover, we show that the converse statement is also valid.

Lemma 3.1. *Assume that the kernel K satisfies (K1) and (K2) and a given datum f is $L^2(\mathbb{R}^2)$. If $u \in BV_K \cap L^2(\mathbb{R}^n)$ is a minimizer of the functional $\mathcal{F}_{K,f}$, then u satisfies the equation*

$$\Delta_1^K u = u - f \quad \text{in } \mathbb{R}^n \tag{3.1}$$

in the sense of Definition 2.1. Conversely, if $u \in BV_K \cap L^2(\mathbb{R}^n)$ is a solution of the equation (3.1) *in the sense of Definition 2.1, then u is a minimizer of $\mathcal{F}_{K,f}$.*

Proof. First, we recall the definition of the functional $\mathcal{I}_K$ and the non-negativity of K and thus, find that $\mathcal{I}_K$ is convex, lower semi-continuous, and positive homogeneous of degree one. Then, by using the same argument as in [28, 29], we can show the characterization of the sub-differential of $\mathcal{I}_K(u)$ as follows:

$$\partial\mathcal{I}_K(u) = \left\{v \in L^2(\mathbb{R}^n) \mid u \text{ satisfies the equation } \Delta_1^K u = v \text{ in } \mathbb{R}^n\right\}. \tag{3.2}$$

Here we recall the definition of the sub-differential $\partial\mathcal{E}(u)$ for $u \in X$ of the functional $\mathcal{E} : X \to \mathbb{R} \cup \{+\infty\}$ where X is the Hilbert space with the inner product $(\cdot, \cdot)_X$. We say that $v \in X$ belongs to $\partial\mathcal{E}(u)$ for each $u \in X$ if it holds that, for any $w \in X$,

$$\mathcal{E}(w) - \mathcal{E}(u) \geq (w,\, v)_X.$$

Note that $u \in X$ is a minimizer of $\mathcal{E}$ if and only if $0 \in \partial\mathcal{E}(u)$. Then, from the general theory on the sub-differential, we can also show the identity

$$\partial\mathcal{F}_{K,f}(u) = \partial\mathcal{I}_K(u) + u - f. \tag{3.3}$$

for any $u \in L^2$. Indeed, if $v \in \partial\mathcal{F}_{K,f}(u)$, then we can compute the functional of u as follows; for any $w \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
\mathcal{I}_K(w) - \mathcal{I}_K(u) &= \mathcal{F}_{K,f}(w) - \mathcal{F}_{K,f}(u) + \frac{1}{2}\int_{\mathbb{R}^n} (u-f)^2\,dx - \frac{1}{2}\int_{\mathbb{R}^n} (w-f)^2 \\
&\geq \int_{\mathbb{R}^n} v(w-u)\,dx - \frac{1}{2}\int_{\mathbb{R}^n} (w-u)(w+u-2f)\,dx \\
&= \int_{\mathbb{R}^n} (v-u+f)(w-u)\,dx + \int_{\mathbb{R}^n} (u-f)(w-u)\,dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^n} (w-u)(w+u-2f)\,dx \\
&= \int_{\mathbb{R}^n} (v-u+f)(w-u)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} (w-u)^2\,dx \\
&\geq \int_{\mathbb{R}^n} (v-u+f)(w-u)\,dx.
\end{aligned} \tag{3.4}$$

Therefore we obtain $v - u + f \in \partial\mathcal{I}_K(u)$. On the other hand, if $v \in \partial\mathcal{I}_K(u) + u - f$, then we can compute in the following manner; for any $w \in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}
\mathcal{F}_{K,f}(w) - \mathcal{F}_{K,f}(u) &= \mathcal{I}_K(w) - \mathcal{I}_K(u) + \frac{1}{2}\int_{\mathbb{R}^n} (w-f)^2\,dx - \frac{1}{2}\int_{\mathbb{R}^n} (u-f)^2 \\
&\geq \int_{\mathbb{R}^n} (v-u+f)(w-u)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} (w-u)(w+u-2f)\,dx \\
&= \int_{\mathbb{R}^n} v(w-u)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} (w-u)^2\,dx \\
&\geq \int_{\mathbb{R}^n} v(w-u)\,dx,
\end{aligned} \tag{3.5}$$

and thus we have that $v \in \partial \mathcal{F}_{K,f}(u)$. Therefore, from the computations (3.4) and (3.5), we conclude that the first part of the claim is valid. Then from (3.3), we can easily obtain the equity

$$\partial \mathcal{F}_{K,f}(u) = \left\{ v + u - f \in L^2(\mathbb{R}^n) \mid u \text{ satisfies the equation } \Delta_1^K u = v \right\}. \tag{3.6}$$

We can readily see that $0 \in \partial \mathcal{F}_{K,f}(u)$ whenever u is a minimizer of $\mathcal{F}_{K,f}$ Therefore, we conclude that, if u is a minimizer of $\mathcal{F}_{K,f}$, then u is a solution of the equation (3.1).

Conversely, if u is a solution of the equation (3.1), then from (3.6) we have that 0 belongs to the set in the right-hand side of (3.6), and thus we obtain $0 \in \partial \mathcal{F}_{K,f}(u)$. □

3.2 Comparison between minimizers

In this section, we prove a comparison principle for the minimizers of $\mathcal{F}_{K,f}$. We assume that K satisfies the assumptions (K1) and (K2) shown in Section 3.1 and the data f_1 and f_2 satisfy that $f_1 \le f_2$. Then we show that the minimizers u_1 and u_2 associated with f_1 and f_2, respectively, preserves the inequality. Precisely, we prove the following result:

Lemma 3.2. *Let f_i be in $L^2(\mathbb{R}^n)$ for each $i \in \{1, 2\}$ and $u_i \in BV_K \cap L^2(\mathbb{R}^n)$ be a minimizer of $\mathcal{F}_{K,f_i}$ for each $i \in \{1, 2\}$. Assume that the kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ satisfies (K1) and (K2). If $f_1 \le f_2$ $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$, then $u_1 \le u_2$ $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$.*

Proof. Let $u_1, u_2 \in BV_K(\mathbb{R}^n)$ be minimizers of $\mathcal{F}_{K,f}$ associated with given data $f_1, f_2 \in L^2(\mathbb{R}^n)$, respectively. First of all, we prove the following inequality:

$$[u_+]_K(\mathbb{R}^n) + [u_-]_K(\mathbb{R}^n) \le [u_1]_K(\mathbb{R}^n) + [u_2]_K(\mathbb{R}^n). \tag{3.7}$$

Indeed, setting

$$u_+(x) \coloneqq \max\{u_1(x), u_2(x)\}, \quad u_-(x) \coloneqq \min\{u_1(x), u_2(x)\} \tag{3.8}$$

for any $x \in \mathbb{R}^n$ and by the co-area formula, we have that

$$\begin{aligned} [u_i]_K(\mathbb{R}^n) &= \int_{-\infty}^{\infty} \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x-y)\, |\chi_{\{u_i > t\}}(x) - \chi_{\{u_i > t\}}(y)| \, dx\, dy\, dt \\ &= \int_{-\infty}^{\infty} P_K(\{u_i > t\})\, dt \end{aligned} \tag{3.9}$$

for any $i \in \{1, 2, +, -\}$. We recall that the nonlocal perimeter P_K is sub-modular, namely, it holds that

$$P_K(E \cup F) + P_K(E \cap F) \le P_K(E) + P_K(F) \tag{3.10}$$

for any $E, F \subset \mathbb{R}^n$. Therefore from (3.10) and the definitions of u_+ and u_-, we obtain the claim.

Now from the general theory of calculus of variations, the minimizer of $\mathcal{F}_{K,f}$ is unique in $L^2(\mathbb{R}^n)$ and thus, it is sufficient to prove that

$$\mathcal{F}_{K,f_2}(u_+) \le \mathcal{F}_{K,f_2}(u_2)$$

where u_+ is defined in (3.8) to obtain the lemma. From a simple computation, we can easily see that the inequality

$$(u_- - f_1)^2 + (u_+ - f_2)^2 \le (u_1 - f_1)^2 + (u_2 - f_2)^2 \tag{3.11}$$

in $\mathbb{R}^n$. From the minimality of u_i for $i \in \{1, 2\}$, we have

$$\mathcal{F}_{K,f_1}(u_1) + \mathcal{F}_{K,f_2}(u_2) \le \mathcal{F}_{K,f_1}(u_-) + \mathcal{F}_{K,f_2}(u_+). \tag{3.12}$$

On the other hand, from (3.7) and (3.11), we have

$$\begin{aligned}
&\mathcal{F}_{K,f_1}(u_-) + \mathcal{F}_{K,f_2}(u_+) \\
&\le [u_-]_K(\mathbb{R}^n) + \frac{1}{2}\int_{\mathbb{R}^n} (u_- - f_1)^2\,dx + [u_+]_K(\mathbb{R}^n) + \frac{1}{2}\int_{\mathbb{R}^n} (u_+ - f_2)^2\,dx \\
&= [u_1]_K(\mathbb{R}^n) + \frac{1}{2}\int_{\mathbb{R}^n} (u_1 - f_1)^2\,dx + [u_2]_K(\mathbb{R}^n) + \frac{1}{2}\int_{\mathbb{R}^n} (u_2 - f_2)^2\,dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} (u_- - f_1)^2\,dx - \frac{1}{2}\int_{\mathbb{R}^n} (u_1 - f_1)^2\,dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} (u_+ - f_2)^2\,dx - \frac{1}{2}\int_{\mathbb{R}^n} (u_2 - f_2)^2\,dx \\
&\le \mathcal{F}_{K,f_1}(u_1) + \mathcal{F}_{K,f_2}(u_2).
\end{aligned} \tag{3.13}$$

Thus from (3.12) and (3.13), we obtain

$$\mathcal{F}_{K,f_1}(u_1) + \mathcal{F}_{K,f_2}(u_2) = \mathcal{F}_{K,f_1}(u_-) + \mathcal{F}_{K,f_2}(u_+) \tag{3.14}$$

Now suppose by contradiction that $\mathcal{F}_{K,f_2}(u_+) > \mathcal{F}_{K,f_2}(u_2)$. Then from (3.14) we have

$$\mathcal{F}_{K,f_1}(u_1) > \mathcal{F}_{K,f_1}(u_-)$$

which contradicts the minimality of u_1. Thus we obtain the inequality $\mathcal{F}_{K,f_2}(u_+) \le \mathcal{F}_{K,f_2}(u_2)$. Therefore, by the uniqueness of the minimizer u_2, this implies that $u_+ = u_2$ a.e. in $\mathbb{R}^n$, which implies that $u_2 \ge u_1$ a.e. in $\mathbb{R}^n$. □

Corollary 3.3. *Assume that the kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions (K1) and (K2) in Section 3.1. If a datum $f \in L^2(\mathbb{R}^n)$ is non-negative a.e. in $\mathbb{R}^n$, then the minimizer $u \in BV_K \cap L^2(\mathbb{R}^n)$ is also non-negative a.e. in $\mathbb{R}^n$.*

Proof. Since it holds that

$$\mathcal{F}_{K,0}(0) = 0 \le \mathcal{F}_{K,0}(v)$$

for every $v \in BV_K \cap L^2(\mathbb{R}^n)$, we have that the unique solution of the problem

$$\inf\{\mathcal{F}_{K,0}(v) \mid v \in BV_K \cap L^2\}$$

is $v = 0$. Hence, by applying Lemma 3.2 to the case that $f_1 = 0$ and $f_2 = f$, we obtain that $0 \le u$ a.e. in $\mathbb{R}^n$. □

Finally, we show a sort of comparison property of minimizers under the assumption that a datum f is bounded in $\mathbb{R}^n$. We do not derive the following proposition directly from Lemma 3.2 but from a simple computation.

Proposition 3.4. *Let $u \in BV_K \cap L^2(\mathbb{R}^n)$ be a minimizer of $\mathcal{F}_{K,f}$ with a datum $f \in L^2(\mathbb{R}^n)$. Assume that the kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is non-negative measurable function. If there exists a constant $C > 0$ such that $|f(x)| \le C$ for a.e. $x \in \mathbb{R}^n$, then $|u(x)| \le C$ for a.e. $x \in \mathbb{R}^n$ with the same constant C.*

Proof. It is sufficient to show that, if $f \le C$ a.e. in $\mathbb{R}^n$ with some constant $C > 0$, then $u \le C$ a.e. in $\mathbb{R}^n$ with the same constant C because we only repeat the same argument as we show in this proof. We define $v(x) \coloneqq \min\{u(x), C\}$ for $x \in \mathbb{R}^n$. It is sufficient to show that $u = v$ for a.e. in $\mathbb{R}^n$. From the definition, we can show the claim that $|v(x) - v(y)| \le |u(x) - u(y)|$ for $x, y \in \mathbb{R}^n$. Indeed, if $u(x) \le C$ and $u(y) \le C$ or $u(x) > C$ and $u(y) > C$, then we can readily obtain the claim. If $u(x) \le C$ and $u(y) > C$, then we have

$$\begin{aligned} |u(x) - u(y)|^2 - |v(x) - v(y)|^2 &= u^2(y) - C^2 - 2u(x)\,u(y) + 2u(x)\,C \\ &= (u(y) - C)(u(y) + C - 2u(x)) \ge 0. \end{aligned}$$

In the same way, we can prove the claim if $u(x) > C$ and $u(y) \le C$. Moreover, we can show that $(v - f)^2 \le (u - f)^2$ in $\mathbb{R}^n$. Therefore we compute the functional associated with v as follows:

$$\begin{aligned} \mathcal{F}_{K,f}(v) &= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\,|v(x) - v(y)|\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^n}(v - f)^2\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\,|u(x) - u(y)|\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^n}(u - f)^2\,dx \\ &= \mathcal{F}_{K,f}(u). \end{aligned}$$

Thus, from the uniqueness of the minimizer of the functional $\mathcal{F}_K$, we obtain $v = u$ a.e. in $\mathbb{R}^n$ and this concludes the proof. □

3.3 Characterization of minimizers for $\mathcal{F}_{K,f}$

In this section, we show the following claim which gives a relation between the minimizers of $\mathcal{F}_{K,f}$ and $\mathcal{E}_{K,f,t}$ for $t \in \mathbb{R}$. Recall that $\mathcal{E}_{K,f,t}(E)$ as

$$\mathcal{E}_{K,f,t}(E) \coloneqq P_K(E) + \int_E (t - f(x))\,dx \tag{3.15}$$

for every measurable set $E \subset \mathbb{R}^n$ where we assume that $f \in L^2(\mathbb{R}^n)$ is a given datum and $t \in \mathbb{R}$ is any number.

Lemma 3.5. *Assume that the kernel $K(x) = |x|^{-(n+s)}$ for $x \in \mathbb{R}^n \setminus \{0\}$ and a datum $f \in L^2 \cap L^\infty(\mathbb{R}^n)$. If $u \in BV_K \cap L^2(\mathbb{R}^n)$ be a minimizer of $\mathcal{F}_{K,f}$, then the set $\{x \in \mathbb{R}^n \mid u(x) > t\}$ is also a minimizer of $\mathcal{E}_{K,f,t}(E)$ for every $t \in \mathbb{R}$ among measurable sets $E \subset \mathbb{R}^n$.*

Proof. Let $F \subset \mathbb{R}^n$ be any measurable set. We may assume that $P_K(F) < \infty$; otherwise this set cannot minimize the functional $\mathcal{E}_{K,f,t}$. Moreover, we may assume that $\|\chi_F\|_{L^1} = |F| < \infty$ because of the nonlocal isoperimeteric inequality. Then it suffices to show that the super-level set $\{u > t\}$ for each $t \in \mathbb{R}$ satisfies the inequality

$$P_K(\{u > t\}) + \int_{\{u>t\}} (t - f(x))\,dx \le P_K(F) + \int_F (t - f(x))\,dx. \tag{3.16}$$

From Lemma 3.1 and the assumption that u is a minimizer of the functional $\mathcal{F}_{K,f}$, we have that u is also a solution of the equation

$$\Delta_1^K u = u - f \quad \text{in } \mathbb{R}^n. \tag{3.17}$$

Thus, by definition, there exists a function $z_u \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $|z_u| \le 1$ and z_u being antisymmetric such that

$$\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\, z_u(x,\, y)\, (w(x)-w(y))\, dx\, dy = \int_{\mathbb{R}^n} (u-f)\, w(x)\, dx \tag{3.18}$$

for any $w \in BV_K \cap L^2(\mathbb{R}^n)$ with a compact support and moreover

$$z_u(x,\, y)(u(y)-u(x)) = |u(y)-u(x)| \tag{3.19}$$

for a.e. $x,\, y \in \mathbb{R}^n$. From the co-area formula, we have the following two identities:

$$|u(x)-u(y)| = \int_{-\infty}^{+\infty} |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|\, dt \tag{3.20}$$

and

$$(u(x)-u(y)) = \int_{-\infty}^{+\infty} (\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y))\, dt \tag{3.21}$$

for any measurable $u : \mathbb{R}^n \to \mathbb{R}$ and a.e. $x,\, y \in \mathbb{R}^n$. Thus from (3.19), (3.20), and (3.21), we obtain

$$z_u(x,\, y)(\chi_{\{u>t\}}(y) - \chi_{\{u>t\}}(x)) = |\chi_{\{u>t\}}(y) - \chi_{\{u>t\}}(x)| \tag{3.22}$$

for a.e. $t \in \mathbb{R}$. Now we fix $t \in \mathbb{R}$ such that (3.22) holds. From the specific choice of $K(x) = |x|^{-(n+s)}$, the function space $BV_K(\mathbb{R}^n)$ coincides with the fractional Sobolev space $W^{s,1}(\mathbb{R}^n)$. Recall that the space $C_c^\infty(\mathbb{R}^n)$ of smooth functions with compact supports is dense in $W^{s,1}(\mathbb{R}^n)$ (see [1] for the detail). Hence, from the fact that $P_K(\{u>t\})$ and $P_K(F)$ are finite, we can choose sequences $\{\eta_l^u\}_{l\in\mathbb{N}}$ and $\{\eta_l^F\}_{l\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^n)$ such that

$$\eta_l^u \xrightarrow[l\to\infty]{} \chi_{\{u>t\}}, \quad \eta_l^F \xrightarrow[l\to\infty]{} \chi_F \quad \text{in } W^{s,1}(\mathbb{R}^n). \tag{3.23}$$

From the choice of the approximation, we notice that the difference function $\eta_l^u - \eta_l^F$ is also in $W^{s,1} \cap L^2(\mathbb{R}^n)$ and has a compact support for each $l \in \mathbb{N}$. Hence, from the definition of solutions to the equation (3.17), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} (u-f)\, (\eta_l^u - \eta_l^F)\, dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\, z_u(x,\, y)\, [(\eta_l^u - \eta_l^F)(y) - (\eta_l^u - \eta_l^F)(x)]\, dx\, dy \\
&= -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\, z_u(x,\, y)\, (\eta_l^u(y) - \eta_l^u(x))\, dx\, dy \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\, z_u(x,\, y)\, (\eta_l^F(y) - \eta_l^F(x))\, dx\, dy.
\end{aligned} \tag{3.24}$$

By applying Proposition 3.4 and from the assumption that $f \in L^\infty(\mathbb{R}^n)$, we have that the minimizer u is also in $L^\infty(\mathbb{R}^n)$ and thus

$$\left| \int_{\mathbb{R}^n} (u-f)(\eta_l^u - \eta_l^F)\, dx - \int_{\mathbb{R}^n} (u-f)(\chi_{\{u>t\}} - \chi_F)\, dx \right| \xrightarrow[l\to\infty]{} 0. \tag{3.25}$$

Hence by applying the dominated convergence theorem and from (3.23), (3.24), and (3.25), we obtain that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} (u-f)(\chi_{\{u>t\}} - \chi_F)\, dx \\
&= \lim_{l\to\infty} \int_{\mathbb{R}^n} (u-f)\,(\eta_l^u - \eta_l^F)\, dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\, z_u(x,\, y)\, (\chi_{\{u>t\}}(y) - \chi_{\{u>t\}}(x))\, dx\, dy \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\, z_u(x,\, y)\, (\chi_F(y) - \chi_F(x))\, dx\, dy. \qquad (3.26)
\end{aligned}
$$

From the definition of z_u, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\, z_u(x,\, y)\, (\chi_F(x) - \chi_F(y))\, dx\, dy \\
&\le \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)|\chi_F(x) - \chi_F(y)|\, dx\, dy = P_K(F). \qquad (3.27)
\end{aligned}
$$

Taking into account (3.22), (3.26), and (3.27), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n} (u-f)\,(\chi_{\{u>t\}} - \chi_F)\, dx \\
&\le -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)|\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|\, dx\, dy + P_K(F). \qquad (3.28)
\end{aligned}
$$

Regarding the left-hand side of (3.28), we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} (u-f)\,(\chi_{\{u>t\}} - \chi_F)\, dx &= \int_{\mathbb{R}^n} (u-t+t-f)\,(\chi_{\{u>t\}} - \chi_F)\, dx \\
&\ge \int_{\{u>t\}\cap F^c} (t-f)\, dx - \int_{\{u\le t\}\cap F} (u-f)\, dx \\
&\ge \int_{\{u>t\}\cap F^c} (t-f)\, dx - \int_{\{u\le t\}\cap F} (t-f)\, dx \\
&= \int_{\mathbb{R}^n} (t-f)\,(\chi_{\{u>t\}} - \chi_F)\, dx \qquad (3.29)
\end{aligned}
$$

for a.e. $t \in \mathbb{R}$. Hence, from (3.28) and (3.29), we have

$$
\begin{aligned}
&P_K(\{u>t\}) + \int_{\mathbb{R}^n} (t-f)\,\chi_{\{u>t\}}\, dx \\
&= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)|\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|\, dx\, dy + \int_{\mathbb{R}^n} (t-f)\,\chi_{\{u>t\}}\, dx \\
&\le P_K(F) + \int_{\mathbb{R}^n} (t-f)\,\chi_F\, dx \qquad (3.30)
\end{aligned}
$$

for a.e. $t \in \mathbb{R}$. Therefore we conclude that the inequality (3.16) holds for a.e. $t \in \mathbb{R}$. Notice that, for any $t \in \mathbb{R}$ such that (3.22) does not hold, we can choose a sequence $\{t_j\}_{j\in\mathbb{N}}$ such that $t_j \to t$ as $j \to \infty$ and (3.22) holds for any t_j; otherwise we can choose a constant $\delta > 0$

such that $B_\delta(t) \subset \{t \in \mathbb{R} \mid (3.22) \text{ is not true}\}$. Since the condition (3.22) holds true for a.e. $t \in \mathbb{R}$, we have that

$$0 < 2\delta = |B_\delta(t)| \leq |\{t \in \mathbb{R} \mid (3.22) \text{ is not true}\}| = 0,$$

which is a contradiction. Thus from the lower semi-continuity of P_K and the continuity of the map $t \mapsto |\{u > t\}|$, we conclude that (3.16) holds for every $t \in \mathbb{R}$. $\square$

3.4 Boundedness of super-levelsets of minimizers

Let $u \in BV_K \cap L^2(\mathbb{R}^n)$ be a minimizer of $\mathcal{F}_{K,f}$ with a datum $f \in L^p(\mathbb{R}^n)$ with $p \in (\frac{n}{s}, \infty]$. In this section, we show that the super-levelset $\{u > t\}$ for each $t \in \mathbb{R}$ is bounded up to negligible sets. Precisely, we prove

Lemma 3.6. *Let $n \geq 1$ and $s \in (0, 1)$. Assume that the kernel $K(x) = |x|^{-(n+s)}$ for $x \in \mathbb{R}^n \setminus \{0\}$ and $f \in L^p(\mathbb{R}^n)$ with $p \in (\frac{n}{s}, \infty]$. If E_T is a minimizer of $\mathcal{E}_{K,f,T}$ among sets with finite volumes for any $T \in \mathbb{R}$, then there exists a constant $R_T > 0$ such that $|E_T \setminus B_{R_T}| = 0$.*

Proof. We basically follow the proof shown in [16, Proposition 3.2]. Suppose by contradiction that $|E_T \setminus B_r| > 0$ for any $r > 0$. By setting $\phi_T(r) \coloneqq |E_T \setminus B_r|$ for any $r > 0$, we have

$$(\phi_T)'(r) = -\mathcal{H}^{n-1}(E_T \cap \partial B_r)$$

for a.e. $r > 0$. We fix any $R > 1$. From the minimality of E_T, we have

$$\mathcal{E}_{K,f,T}(E_T) \leq \mathcal{E}_{K,f,T}(E_T \cap B_r). \tag{3.31}$$

Since it holds that

$$P_K(A \cup B) = P_K(A) + P_K(B) - 2\int_A \int_B K(x-y)\,dx\,dy$$

for sets A, $B \subset \mathbb{R}^n$ with $A \cap B = \emptyset$, we have

$$P_K(E_T \setminus B_r) \leq 2\int_{E_T \cap B_r} \int_{E_T \setminus B_r} K(x-y)\,dx\,dy - \int_{E_T \setminus B_r} (T - f(x))\,dx. \tag{3.32}$$

From the isoperimetric inequality for the nonlocal perimeter, we can have the following lower bound of the term of the left-hand side in (3.32) (see for instance [21]):

$$P_K(E_T \setminus B_r) \geq \frac{P_K(B_1)}{|B_1|^{\frac{n-s}{n}}}\,|E_T \setminus B_r|^{\frac{n-s}{n}} = C(n,s)\,\phi_T^{\frac{n-s}{n}}(r) \tag{3.33}$$

for $r \geq R$, where we set $C(n,s) \coloneqq |B_1|^{-\frac{n-s}{n}}\, P_K(B_1)$. Secondly, from Fubini-Tonelli's theorem and the co-area formula, we can compute the first term of the right-hand side in (3.32) as follows:

$$
\begin{aligned}
\int_{E_T\cap B_r}\int_{E_T\setminus B_r} K(x-y)\,dx\,dy &\le \int_{E_T\setminus B_r}\int_{B_{|y|-r}(y)} \frac{1}{|x-y|^{n+s}}\,dx\,dy\\
&= \int_{E_T\setminus B_r} |\mathbb{S}^{n-1}| \int_{|y|-r}^{\infty} \frac{1}{r^{1+s}}\,dr\,dy\\
&\le \frac{|\mathbb{S}^{n-1}|}{s}\int_{E_T\setminus B_r} (|y|-r)^{-s}\,dy\\
&= \frac{|\mathbb{S}^{n-1}|}{s}\int_r^{+\infty} \frac{\mathcal{H}^{n-1}(E_T\cap\partial B_\sigma)}{(\sigma-r)^s}\,d\sigma\\
&= -\frac{|\mathbb{S}^{n-1}|}{s}\int_r^{+\infty} \frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,d\sigma \qquad (3.34)
\end{aligned}
$$

for any $r \ge R$. Finally, regarding the second term of the right-hand side in (3.32), from the assumption of f and Cauchy-Schwartz inequality (if $p \neq \infty$), we have

$$
\begin{aligned}
\int_{E_T\setminus B_r} (-T+f(x))\,dx &\le T\,|E_T\setminus B_r| + \|f\|_{L^p(\mathbb{R}^n)}\,|E_T\setminus B_r|^{\frac{1}{q}}\\
&= T\,\phi_T(r) + \|f\|_{L^p(\mathbb{R}^n)}\,\phi_T^{\frac{1}{q}}(r) < \infty \qquad (3.35)
\end{aligned}
$$

for any $r \ge R > 1$ where $q \ge 1$ satisfies $p^{-1}+q^{-1}=1$. By combining all the computations (3.33), (3.34), and (3.35) with (3.32), we obtain

$$
C(n,s)\,\phi_T^{\frac{n-s}{n}}(r) \le -C_1\int_r^{+\infty}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,d\sigma + T\,\phi_T(r) + \|f\|_{L^p(\mathbb{R}^n)}\,\phi_T^{\frac{1}{q}}(r) \qquad (3.36)
$$

for any $r \ge R$ where we set $C_1 := \frac{|\mathbb{S}^{n-1}|}{s}$. Since $\phi_T(r)$ vanishes as $r\to\infty$ and $\frac{1}{q} > \frac{n-s}{n}$, we can have that

$$
2T\,\phi_T(r) + 2\|f\|_{L^p(\mathbb{R}^n)}\phi_T^{\frac{1}{q}}(r) \le C(n,s)\,\phi_T^{\frac{n-s}{n}}(r)
$$

for sufficiently large $r \ge R$. Hence, by integrating the both sides of (3.36) over $r \in (R,\infty)$, we obtain

$$
\frac{C(n,s)}{2}\int_R^{\infty}\phi_T^{\frac{n-s}{n}}(r)\,dr \le -C_1\int_R^{\infty}\int_r^{+\infty}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,d\sigma\,dr. \qquad (3.37)
$$

By exchanging the order of the integration, we have

$$
\int_R^{\infty}\int_r^{+\infty}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,d\sigma\,dr = \int_R^{\infty}\int_R^{\sigma}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,dr\,d\sigma. \qquad (3.38)
$$

Then by employing the similar computation shown in [16], we obtain

$$
\int_R^{\infty}\int_R^{\sigma}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,dr\,d\sigma \ge -\frac{\phi_T(R)}{1-s} - \int_{R+1}^{\infty}\frac{\phi_T(r)}{(\sigma-R)^s}\,d\sigma.
$$

Therefore, from (3.37), we have

$$
\begin{aligned}
\frac{C(n,s)}{2}\int_R^{\infty}\phi_T^{\frac{n-s}{n}}(r)\,dr &\le C_1\frac{\phi_T(R)}{1-s} + C_1\int_{R+1}^{\infty}\frac{\phi_T(\sigma)}{(\sigma-R)^s}\,d\sigma\\
&\le C_1\frac{\phi_T(R)}{1-s} + C_1\int_{R+1}^{\infty}\phi_T(\sigma)\,d\sigma.
\end{aligned}
$$

Again, by choosing R sufficiently large so that the inequality

$$C_1 \int_{R+1}^{\infty} \phi_T(r)\,dr \le \frac{C(n,s)}{4} \int_R^{\infty} \phi_T^{\frac{n-s}{n}}(r)\,dr$$

holds, we have

$$\int_R^{\infty} \phi_T^{\frac{n-s}{n}}(r)\,dr \le \frac{4C_1}{C(n,s)(1-s)}\,\phi_T(R).$$

Then by applying the method shown in, for instance, [15, 16], we obtain the contradiction to the assumption that $\phi_T(r) > 0$ for any $r > 0$. Therefore, we conclude the essential boundedness of the set E_T. □

We assume that $u \in BV_K \cap L^2(\mathbb{R}^n)$ is a minimizer of the functional $\mathcal{F}_{K,f}$ and u is bounded from below with the constant $c \in \mathbb{R}$. Then, since the super-levelset $\{u > c\}$ is also a minimizer of $\mathcal{E}_{K,f,c}$, we may obtain from Lemma 3.6 that there exists a constant $R_c > 1$ such that $|\{u > c\} \setminus B_{R_c}| = 0$. In addition to this, we have the inclusion of the super-levelsets that $\{u > t'\} \subset \{u > t\}$ for any $t' > t$. Thus, we conclude that the following corollary holds.

Corollary 3.7. *Assume that the kernel $K(x) = |x|^{-(n+s)}$ for $x \in \mathbb{R}^n \setminus \{0\}$ with $s \in (0,1)$. Let $u \in BV_K(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ be a minimizer of $\mathcal{F}_{K,f}$. If a datum f is in $L^p(\mathbb{R}^n)$ with $p \in (\frac{n}{s}, \infty]$ and $u \ge c$ a.e. in $\mathbb{R}^n$ for some $c \in \mathbb{R}$, then the super-levelset $\{u > t\}$ is uniformly bounded with respect to $t \ge c$. Namely, there exists $R_c > 0$, independent of t, such that $\{u > t\} \subset B_{R_c}$ for any $t \ge c$.*

4 Hölder regularity of minimizers

First of all, we prove that, if the boundary of $\{u > t\}$ is regular, then u is continuous.

Lemma 4.1. *Assume that K is non-negative measurable function and the datum f is in $L^\infty(\mathbb{R}^n)$. Let $u \in BV_K(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ be a minimizer of $\mathcal{F}_{K,f}$, and assume that $\partial\{u > t\}$ is of class $C^{1,\alpha}$ with $\alpha \in (0,1)$ for each $t \in \mathbb{R}$. Then u is continuous in $\mathbb{R}^n$.*

Proof. From Lemma 3.5, we have that the set $E_t \coloneqq \{u > t\}$ is a minimizer of $\mathcal{E}_{K,f,t}$ for each $t \in \mathbb{R}$. Suppose by contradiction that u is not continuous in $\mathbb{R}^n$. Then there exist a point $x_0 \in \mathbb{R}^n$ and $-\infty < t' < t < \infty$ such that $x_0 \in \partial E_t \cap \partial E_{t'}$. Indeed, if u is not continuous at x_0, then it holds that $t_+ \coloneqq \limsup_{x \to x_0} u(x) > \liminf_{x \to x_0} u(x) =: t_-$. Note that $t_+ \ge u(x_0) \ge t_-$ by definition. Setting $\delta \coloneqq t_+ - t_- > 0$ and the definition of t_+, we can choose a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_k \to x_0$ and $u(x_k) > t_+ - \frac{\delta}{2^k}$ for any $k \in \mathbb{N}$ with $k \ge 1$. If $u(x_0) = t_+$, then we have that $x_k \in \{u > u(x_0) - \frac{\delta}{2}\}$ for large $k \in \mathbb{N}$. Thus we obtain that $x_0 \in \overline{\{u > u(x_0) - \frac{\delta}{2}\}}$. However, from the definition of δ, x_0 cannot be a interior point of $\{u > u(x_0) - \frac{\delta}{2}\}$; otherwise we can choose a sequence $\{y_k\}_{k\in\mathbb{N}}$ such that

$$u(x_0) - \frac{\delta}{2} < u(y_k) < t_- + \frac{\delta}{2^k} \tag{4.1}$$

for any large k. From the definition of δ and the fact that $u(x_0) = t_+$, we obtain a contradiction. Thus we may assume that $u(x_0) < t_+$. Setting $\tilde{\delta} \coloneqq t_+ - u(x_0) > 0$ and since $u(x_k) > t_+ - \frac{\delta}{2^k}$ for any $k \in \mathbb{N}$, we have that $u(x_k) > u(x_0) + \frac{1}{2}\tilde{\delta}$ for any $k \in \mathbb{N}$ with $k \ge (2\delta)^{-1}\tilde{\delta}$ and that $x_k \in \{u > u(x_0) + \frac{1}{2}\tilde{\delta}\}$ for large $k \in \mathbb{N}$. Hence, recalling that $x_k \to x_0$

as $k \to \infty$, we obtain that $x_0 \in \partial\{u > u(x_0) + \frac{1}{2}\tilde{\delta}\}$. In the same way, we can show that $x_0 \in \partial\{u > u(x_0) + \frac{3}{4}\tilde{\delta}\}$. Therefore, we conclude that, if u is not continuous at x_0, we can find distinct constants t, $t' \in \mathbb{R}$ such that $x_0 \in \partial\{u > t\} \cap \partial\{u > t'\}$.

Since E_t and $E_{t'}$ are the minimizers of $\mathcal{E}_{K,f,t}$ and $\mathcal{E}_{K,f,t'}$, respectively, and we assume the $C^{1,\alpha}$-regularity of the boundaries of E_t and $E_{t'}$, we obtain the Euler-Lagrange equations

$$H^K_{E_t}(x) + t - f(x) = 0 \tag{4.2}$$

and

$$H^K_{E_{t'}}(x) + t' - f(x) = 0 \tag{4.3}$$

for each $x \in \partial E_t \cap \partial E_{t'}$. We recall that $H^{\partial E}_K$ is the nonlocal mean curvature of a set $E \subset \mathbb{R}^2$ defined in (1.7). Notice that, if $E \subset F$ and $\partial E \cap \partial F \neq \emptyset$, then the nonlocal mean curvature of the boundary ∂F cannot exceed that of the boundary ∂E, namely, it holds that $H^K_E \geq H^K_F$ at the points of $\partial E \cap \partial F$. Indeed, by definition of the nonlocal mean curvature, we have

$$\begin{aligned} H^K_E(x) - H^K_F(x) &= \text{p.v.} \int_{\mathbb{R}^2} K(x-y)(\chi_E(x) - \chi_E(y))\, dy \\ &\quad - \text{p.v.} \int_{\mathbb{R}^2} K(x-y)(\chi_F(x) - \chi_F(y))\, dy \\ &= \text{p.v.} \int_{\mathbb{R}^2} K(x-y)(\chi_E(x) - \chi_F(x) - \chi_E(y) + \chi_F(y))\, dy \end{aligned} \tag{4.4}$$

for any $x \in \partial E \cap \partial F$. Since $E \subset F$, it holds $\chi_E \leq \chi_F$ in $\mathbb{R}^2$ and $\chi_E(x) = \chi_F(x)$ for any $x \in \partial E \cap \partial F$. Thus from (4.4) and the non-negativity of K, we obtain the claim.

Therefore, from (4.2), (4.3), and the fact that $H^K_{E_{t'}} \geq H^K_{E_t}$, we obtain

$$t' - f(x_0) \geq t - f(x_0)$$

and it turns out that $t' \geq t$. This contradicts the fact that $t' < t$. □

4.1 Regularity of boundaries of super-levelsets for minimizers

We show some regularity results of the boundary of the set $\{u > t\}$ for each t under suitable assumptions on the datum f, where u is a minimizer of the functional $\mathcal{F}_{K,f}$. To see this, we apply the results shown by Caputo and Guillen in [11] and Barrios, Figalli, and Valdinoci in [6] (we also refer to [21]).

First, we prove $C^{1,\alpha}$-regularity of the boundaries of minimizers. Before stating the lemma, we recall the two results; one is the C^1-regularity result of "almost" nonlocal minimal surfaces given by Caputo and Guillen in [11] and the other is the "improvement-of-flatness" result given by Figalli et.al. in [21]. First of all, we state the regularity result proved by Caputo and Guillen in [11, Theorem 1.1] as follows:

Theorem 4.2 ([11]). *Let $s \in (0, 1)$ and $\delta > 0$, and let $\Omega \subset \mathbb{R}^n$ be any bounded domain with Lipschitz boundary. Suppose E is a $(P_s, \rho, \delta,)$-minimal in Ω, where $\rho : (0, \delta) \to \mathbb{R}$ is a non-decreasing and bounded function with some growth condition. Here we mean by $(P_s, \rho, \delta,)$-minimal in B_R for some $R > 0$ that for any $x_0 \in \partial E$, a measurable set $F \subset \mathbb{R}^n$, and $0 < r < \min\{\delta, \operatorname{dist}(x_0, \partial B_R)\}$ with $E \triangle F \subset B_r(x_0)$, we have*

$$P_s(E; B_R) \leq P_s(F; B_R) + \rho(r)\, r^{n-s}.$$

Then ∂E is of class C^1 in $B_{\frac{R}{2}}$, except a closed set of $\mathcal{H}^{n-2}$-dimension.

Remark 4.3. Note that we may choose, for instance, a function $r \mapsto C\, r^{\beta}$ with $0 < \beta \leq s$ where $C > 0$ is some constant, as the function ρ.

Secondly, we recall another regularity result of the boundary of minimizers, namely, the improvement of flatness proved in [21, Theorem 3.4] by using the method developed in [11]. This result implies $C^{1,\alpha}$-regularity of almost minimal surfaces except some singular set.

Theorem 4.4 ([11, 21])**.** *Let $s_0 \in (0,\, 1)$, $\Lambda > 0$, and $x_0 \in \partial E$. Then there exist τ, η, $q \in (0,\, 1)$ depending only on n, s_0, and Λ with the following property: assume that E is a Λ-minimizer of the s-perimeter for some $s \in [s_0,\, 1)$, namely, we mean E is a bounded measurable set in $\mathbb{R}^n$ satisfying the condition that, for any bounded set $F \subset \mathbb{R}^n$,*

$$P_s(E) \leq P_s(F) + \frac{\Lambda}{1-s}|E \Delta F|.$$

Then, if

$$\partial E \cap B_1(x_0) \subset \{y \mid |(y - x_0) \cdot e| < \tau\}$$

for some $e \in \mathbb{S}^{n-1}$, then there exists $e_0 \in \mathbb{S}^{n-1}$ such that

$$\partial E \cap B_{\eta}(x_0) \subset \{y \mid |(y - x_0) \cdot e_0| < q\, \tau, \eta\}.$$

Originally, the regularity of nonlocal minimal surfaces was obtained by Caffarelli, Roquejoffre, and Savin in [10]. Precisely they proved that every s-minimal surface is locally $C^{1,\alpha}$ except the singular sets of $\mathcal{H}^{n-2}$-dimension. Moreover, thanks to the result by Savin and Valdinoci in [34], the singular set of s-minimal surfaces has the Hausdorff dimension up to $n-3$. Hence we obtain that s-minimal surfaces in $\mathbb{R}^2$ are fully $C^{1,\alpha}$-regular.

As a consequence of these regularity results, we obtain

Lemma 4.5 ($C^{1,\alpha}$-regularity of boundary of super-levelset of minimizers)**.** *Let $s \in (0,\, 1)$ and let $f \in L^{\infty}(\mathbb{R}^n)$ be non-negative. Assume that $K(x) = |x|^{-(n+s)}$ for $x \in \mathbb{R}^n \setminus \{0\}$ and $u \in BV_K \cap L^2(\mathbb{R}^n)$ is a minimizer of the functional $\mathcal{F}_{K,f}$. Then, for each $t \in \mathbb{R}$, the boundary of the super-level set of u is of class $C^{1,\alpha}$ with $0 < \alpha < 1$, except a closed set of $\mathcal{H}^{n-3}$-dimension.*

Proof. We fix $t \in \mathbb{R}$. Let $x_0 \in \partial\{u > t\}$ and $r > 0$ be any number. First, from the assumption on f and Lemma 3.6 in Section 3.2, u is non-negative and there exists a constant $R_0 > 0$ such that $E_t \coloneqq \{u > t\} \subset B_{\frac{R_0}{2}}$ for any $t \geq 0$. In order to apply the regularity and improvement flatness result to our case, it is sufficient to show that the set E_t is an almost minimizer in B_{R_0} in the sense of Theorem 4.4. Note that this also indicates that E_0 is a minimizer in the sense of Theorem 4.2. From Lemma 3.5, we know that $\{u > t\}$ is a solution to the problem

$$\min\{\mathcal{E}_{K,f,t}(E) \mid |E| < \infty\}$$

for each $t \in \mathbb{R}$. Hence, from the minimality and boundedness of E_t, we have that

$$\mathcal{E}_{K,f,t}(E_t) \leq \mathcal{E}_{K,f,t}(F) \tag{4.5}$$

for any $F \subset \mathbb{R}^n$ and $E_t \triangle F \subset B_r(x_0)$. From the definition of $P_s(\cdot\,; B_{R_0})$ and the fact that $E_t \subset B_{\frac{R_0}{2}}$, we have the identity that $P_K(E_t; B_{R_0}) = P_K(E_t)$. Hence, from (4.5), we can compute as follows: for any set F and $r > 0$ with $E_t \triangle F \subset B_r(x_0) \subset B_{\frac{R_0}{2}}$, we have

$$\begin{aligned} P_K(E_t; B_{R_t}) - P_K(F; B_{R_t}) &= \mathcal{E}_{K,f,t}(E_t) - \int_{E_t} (t - f(x))\, dx - \mathcal{E}_{K,f,t}(F) + \int_F (t - f(x)\, dx \\ &\le \int_{\mathbb{R}^n} |\chi_{E_t} - \chi_F|\, |t - f(x)|\, dx \\ &\le \int_{B_r(x_0)} |t - f(x)|\, dx. \end{aligned} \tag{4.6}$$

Since we assume that $f \in L^\infty(\mathbb{R}^n)$, we have

$$\int_{B_r(x_0)} |t - f(x)|\, dx \le (t + \|f\|_{L^\infty(\mathbb{R}^n)})\, |E_t \Delta F|. \tag{4.7}$$

Hence, from (4.6) and (4.7), we have

$$P_K(E_t; B_{R_0}) \le P_K(F; B_{R_0}) + (t + \|f\|_{L^\infty(\mathbb{R}^n)})\, |E_t \Delta F|$$

or equivalently, since $E_t \cup F \subset B_{\frac{R_0}{2}}$, we have

$$P_K(E_t) \le P_K(F) + (t + \|f\|_{L^\infty(\mathbb{R}^n)})\, |E_t \Delta F|$$

for any $F \subset \mathbb{R}^n$ with $E_t \triangle F \subset B_r(x_0)$. Therefore, we apply the regularity result in [11], the improvement flatness result in [21], and the regularity result of nonlocal minimal cones in [34] to conclude that the claim is valid. □

In addition, we employ another result of the regularity for solutions via the bootstrap argument. This result are obtained by Barrios, Figalli, and Valdinoci in [6]. The authors in [6, Theorem 1.6] proved the following regularity theorem for the solutions to a certain differential equation. Note that, for simplicity, we write the following theorem in a less rigorous manner than the original one.

Theorem 4.6. *Let $v \in L^\infty(\mathbb{R}^{n-1})$ be a solution (in the viscosity sense) to the integro-differential equation*

$$\int_{\mathbb{R}^{n-1}} k_r(y', x') \frac{v(x'+y') + v(x'-y') - 2v(x')}{|y'-x'|^{(n-1)+(1+s)}}\, dy' = G(x', v(x'))$$

for any $x' \in B'_r(0) \subset \mathbb{R}^{n-1}$ with some "nicely behaving" function k_r with $\operatorname{spt} k_r(\cdot, x') \subset B'_r(0)$ and $G \in C^{0,\beta}(B'_r(0))$ and $\beta \in (1-s, 1]$. Then, we have that $v \in C^{1+s+\alpha}(B'_{\frac{r}{2}}(0))$ with $1 - s < \alpha < \beta$.

Taking into account all the above arguments, we can obtain the local $C^{2,\gamma}$-regularity of the boundary of minimizers under the same Hölder regularity of a datum f. Precisely, we prove

Lemma 4.7. *Let $s \in (0, 1)$. Assume that $K(x) = |x|^{-(n+s)}$ for $x \in \mathbb{R}^n \setminus \{0\}$ and $u \in BV_K \cap L^2(\mathbb{R}^n)$ is a minimizer of the functional $\mathcal{F}_{K,f}$. If a datum f is in $C^{0,\beta}_{loc} \cap L^\infty(\mathbb{R}^n)$ with $\beta \in (1-s, 1]$ and non-negative, then for each $t \in \mathbb{R}$, the boundary of the super-level set $\{u > t\}$ is of class $C^{2,s+\alpha-1}$ with $1 - s < \alpha < \beta \le 1$ except a closed set of $\mathcal{H}^{n-3}$-dimension.*

Proof. From Lemma 4.5 and the assumption that $f \in C^{0,\beta}_{loc} \cap L^\infty(\mathbb{R}^n)$ with $\beta \in (1-s,\, 1]$, the boundary of the set $\{u > t\}$ has global $C^{1,\gamma}$-regularity with $\gamma \in (0,\, 1)$ and thus we can represent the boundary as a graph of some $C^{1,\gamma}$-function v_t locally. By employing the computation shown in [6], we may have that v_t is actually a solution in the viscosity sense to the equation

$$\int_{\mathbb{R}^{n-1}} k(y',\, x') \frac{v(x'+y') + v(x'-y') - 2v(x')}{|y'-x'|^{(n-1)+(1+s)}}\, dy' = G(x',\, v(x')) + t - f(x',\, v_t(x'))$$

in some open bounded domain of v_t, where k and G are "good" functions precisely given in [6]. Then since $f \in C^{0,\beta}_{loc}(\mathbb{R}^n)$, we may apply the result in [6, Theorem 1.6] with $k = 0$ and $\sigma = 2s$ to conclude that the local regularity of v_t is improved up to $C^{2,s+\alpha-1}$ with $1-s < \alpha < \beta \leq 1$. From the compactness of the boundary of $\{u > t\}$ and by the standard covering argument, we obtain the $C^{2,s+\alpha-1}$-regularity of entire $\partial\{u > t\}$ with $\alpha \in (1-s,\, \beta)$. □

4.2 Proof of the main regularity result

By combining the regularity result in Lemma 4.7 with Lemma 4.5, we are now ready to prove the main result of this paper.

Proof of Theorem 1.1. From Lemma 4.1, we first recall that the minimizer u is continuous in $\mathbb{R}^2$ because of the $C^{2,s+\alpha-1}$-regularity of $\partial\{u > t\}$ shown in Lemma 4.7. Let $d_t := d_{E_t}$ for $t \in [0,\, \infty)$ be a signed distance function from $\partial\{u > t\}$, which is negative inside $\{u > t\}$. We set $E_t := \{x \mid u(x) > t\}$ for any t. Since $n = 2$, from Lemma 4.7 it follows that all the points on ∂E_t are regular points. Thus, the signed distance function d_t is of class $C^{2,s+\alpha-1}$ in a neighborhood of ∂E_t with $1-s < \alpha < \beta$ (see, for instance, [35, 18, 19, 7] for the relation between the distance function and regularity of surfaces).

Now we take any $t_1 \in [0,\, \infty)$ and set $E_1 := E_{t_1}$. Then we can choose a neighborhood $U_1 \subset \mathbb{R}^2$ of the boundary ∂E_1 such that $d_1 := d_{t_1} \in C^{2,s+\alpha-1}(U_1)$. Moreover, we take any $t_2 \in \mathbb{R}$ be with $t_2 > t_1$ such that $E_2 \cap U_1 \neq \emptyset$. From the assumption on f and from Proposition 3.4, we have that $\|u\|_{L^\infty} \leq \|f\|_{L^\infty} < \infty$. Hence from Lemma 3.7, we obtain that there exists a constant $R_c > 0$ independent of t_1 and t_2 such that $E_2 \subset E_1 \subset B_{R_c}$.

Now we set $\tilde{\delta} := \mathrm{dist}\,(\partial E_1,\, \partial E_2)$. To show the Hölder regularity, we first prove the inequality

$$t_2 - t_1 \leq ([f]_\beta + C\, \tilde{\delta}^{1-\beta})\, \tilde{\delta}^\beta$$

where $C > 0$ is the constant depending only on s and d_1. Thanks to the boundedness of both ∂E_1 and ∂E_2 from Lemma 3.6, we can choose points $x_1 \in \partial E_1$ and $x_2 \in \partial E_2$ such that

$$\tilde{\delta} := \mathrm{dist}\,(\partial E_1,\, \partial E_2) = |x_1 - x_2|.$$

Notice that we may assume that $\tilde{\delta} \neq 0$. Indeed, if $\tilde{\delta} = 0$, then we have that $\tilde{x} := x_1 = x_2 \in \partial E_1 \cap \partial E_2$. From the inclusion that $E_2 \subset E_1$ and the definition of the nonlocal mean curvature, we can show the inequality

$$H^K_{E_1}(\tilde{x}) \leq H^K_{E_2}(\tilde{x}). \tag{4.8}$$

Moreover, from the minimality of the sets E_1 and E_2 and the $C^{2,s+\alpha-1}$-regularity of the minimizers of $\mathcal{E}_{K,f,t_i}$ for each $i \in \{1,\, 2\}$, we are bale to compute by hand the Euler-Lagrange equations

$$H^K_{E_1}(\tilde{x}) + t_1 - f(\tilde{x}) = 0 \tag{4.9}$$

and

$$H^K_{E_2}(\tilde{x}) + t_2 - f(\tilde{x}) = 0. \tag{4.10}$$

Therefore, combining (4.9) and (4.10) with (4.8), we obtain

$$t_2 - t_1 = t_2 - f(\tilde{x}) - (t_1 - f(\tilde{x})) = -H^K_{E_2}(\tilde{x}) + H^K_{E_1}(\tilde{x}) \leq 0,$$

which contradicts the assumption that $t_2 > t_1$. In the sequel, we always assume that $\tilde{\delta} > 0$.

Now we try to prove the local Hölder regularity of the minimizer u of the functional $\mathcal{F}_{K,f}$. To see this, first we define a set E_1^δ as

$$E_1^\delta \coloneqq \{x \in E_1 \mid \mathrm{dist}\,(x,\, \partial E_1) \leq \delta\}$$

for any $\delta \in (0,\, \tilde{\delta}]$. Then the boundary of E_1^δ is described as $\partial E_1^\delta = \{x - \delta \nabla d_1(x) \mid x \in \partial E_1\}$ where ∇d_1 is the outer unit normal vector of ∂E_1. Note that we can readily see that $E_2 \subset E_1^\delta$ and $x_2 \in \partial E_2 \cap \partial E_1^{\tilde{\delta}}$. As we show in (4.8), we have the comparison inequality of the nonlocal mean curvatures that

$$H^K_{E_1^{\tilde{\delta}}}(x_2) \leq H^K_{E_2}(x_2). \tag{4.11}$$

From the choice of x_1 and x_2, we have $x_2 = x_1 - \delta\, \nabla d_1(x_1)$. Now we compare the two nonlocal curvatures $H^K_{E_1^\delta}(x_2)$ and $H^K_{E_1}(x_1)$. To do this, we employ the computation shown by Dávila, del Pino, and Wei in [17] (see also [24]). This computation is on the variation of the nonlocal mean curvature. Precisely, we have that, for any set $E \subset \mathbb{R}^2$ with a smooth boundary (at least C^2), it holds that

$$\begin{aligned} -\frac{d}{d\delta}\bigg|_{\delta=0} H^K_{E_{\delta h}}(x - \delta h(x) \nabla d_E(x)) &= 2 \int_{\partial E} \frac{h(y) - h(x)}{|y - x|^{2+s}}\, d\mathcal{H}^{n-1}(y) \\ &\quad + 2 \int_{\partial E} \frac{(\nabla d_E(y) - \nabla d_E(x)) \cdot \nabla d_E(x)}{|y - x|^{2+s}}\, d\mathcal{H}^{n-1}(y) \end{aligned} \tag{4.12}$$

for $x \in \partial E$ where $h \in C^\infty \cap L^\infty(\partial E)$ and we define the set $E_{\delta h}$ in such a way that its boundary is given by $\partial E_{\delta h} \coloneqq \{x - \delta\, h(x)\, \nabla d_E(x) \mid x \in \partial E\}$ for any $\delta > 0$. Then from (4.12) and by some computation, we have the estimate of the variation of the nonlocal mean curvature $H^s_{E_1^\delta}$ for small $\delta > 0$. Precisely we can obtain that there exist constants $C > 0$ and $\delta_0 > 0$, which depends on the space-dimension $n = 2$, s, and the L^∞-norm of $\nabla^2 d_1$ (equivalently the second fundamental form of ∂E_1), such that

$$-\frac{d}{d\delta} H^K_{E_1^\delta}(\Psi_\delta(x_1)) \leq C \int_{\partial E_1} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{n-1}(y) \tag{4.13}$$

for any $\delta \in (0,\, \delta_0)$ where we set $\Psi_\delta(x_1) \coloneqq x - \delta\, \nabla d_1(x)$. Indeed, choosing any smooth cut-off function η_ε such that $\mathrm{spt}\, \eta_\varepsilon \subset B^c_\varepsilon(0)$, $\eta_\varepsilon \equiv 1$ in $B^c_{2\varepsilon}(0)$, and $0 \leq \eta_\varepsilon \leq 1$, we can write the nonlocal curvature as follows:

$$\begin{aligned} &- H^s_{E_1^\delta}(\Psi_\delta(x_1)) \\ &= \int_{\mathbb{R}^2} \frac{\chi_{E_1^\delta}(y) - \chi_{(E_1^\delta)^c}(y)}{|y - \Psi_\delta(x_1)|^{2+s}} \eta_\varepsilon(y - \Psi_\delta(x_1))\, dy \\ &\quad + \int_{\mathbb{R}^2} \frac{\chi_{E_1^\delta}(y) - \chi_{(E_1^\delta)^c}(y)}{|y - \Psi_\delta(x_1)|^{2+s}} (1 - \eta_\varepsilon(y - \Psi_\delta(x_1)))\, dy \\ &\eqqcolon A_\varepsilon(\delta) + B_\varepsilon(\delta). \end{aligned} \tag{4.14}$$

Then we can compute the derivative of $A_\varepsilon(\delta)$ in (4.14) for small $\delta > 0$ in the following manner:

$$\begin{aligned}
&\frac{d}{d\delta}\int_{\mathbb{R}^2}\frac{\chi_{E_1^\delta}(y)-\chi_{(E_1^\delta)^c}(y)}{|y-\Psi_\delta(x_1)|^{2+s}}\eta_\varepsilon(y-\Psi_\delta(x_1))\,dy \\
&=\int_{\partial E_1^\delta}\frac{\eta_\varepsilon(y-\Psi_\delta(x_1))}{|y-\Psi_\delta(x_1)|^{2+s}}\,d\mathcal{H}^{n-1}(y)+\int_{\partial (E_1^\delta)^c}\frac{\eta_\varepsilon(y-\Psi_\delta(x_1))}{|y-\Psi_\delta(x_1)|^{2+s}}\,d\mathcal{H}^{n-1}(y) \\
&\quad-(2+s)\int_{\mathbb{R}^2}\frac{\chi_{E_1^\delta}(y)-\chi_{(E_1^\delta)^c}(y)}{|y-\Psi_\delta(x_1)|^{2+s+2}}(y-x_1+\delta\nabla d_1(x_1))\cdot\nabla d_1(x_1)\,\eta_\varepsilon(y-\Psi_\delta(x_1))\,dy \\
&\qquad+\int_{\mathbb{R}^2}\frac{\chi_{E_1^\delta}(y)-\chi_{(E_1^\delta)^c}(y)}{|y-\Psi_\delta(x_1)|^{2+s}}\nabla\eta_\varepsilon(y-\Psi_\delta(x_1))\cdot\nabla d_1(x_1)\,dy
\end{aligned}\tag{4.15}$$

for any $\delta \in (0,\,1)$ with $\Psi_\delta(x_1) \in U_1$. Then by using the Gauss-Green theorem, we have

$$\begin{aligned}
&-(2+s)\int_{\mathbb{R}^2}\frac{\chi_{E_1^\delta}(y)-\chi_{(E_1^\delta)^c}(y)}{|y-\Psi_\delta(x_1)|^{2+s+2}}(y-x_1+\delta\nabla d_1(x_1))\cdot\nabla d_1(x_1)\,\eta_\varepsilon(y-\Psi_\delta(x_1))\,dy \\
&=\int_{\mathbb{R}^2}(\chi_{E_1^\delta}(y)-\chi_{(E_1^\delta)^c}(y))\nabla_y\left(\frac{1}{|y-\Psi_\delta(x_1)|^{2+s}}\right)\cdot\nabla d_1(x_1)\,\eta_\varepsilon(y-\Psi_\delta(x_1))\,dy \\
&=\int_{\partial E_1^\delta}\frac{\nabla d_1(x_1)\cdot\nabla d_{E_1^\delta}(y)}{|y-\Psi_\delta(x_1)|^{2+s}}\eta_\varepsilon(y-\Psi_\delta(x_1))\,d\mathcal{H}^{n-1} \\
&\quad-\int_{\partial (E_1^\delta)^c}\frac{\nabla d_1(x_1)\cdot(-\nabla d_{E_1^\delta}(y))}{|y-\Psi_\delta(x_1)|^{2+s}}\eta_\varepsilon(y-\Psi_\delta(x_1))\,d\mathcal{H}^{n-1} \\
&\qquad-\int_{\mathbb{R}^2}\frac{\chi_{E_1^\delta}(y)-\chi_{(E_1^\delta)^c}(y)}{|y-\Psi_\delta(x_1)|^{2+s}}\nabla\eta_\varepsilon(y-\Psi_\delta(x_1))\cdot\nabla d_1(x_1)\,dy.
\end{aligned}\tag{4.16}$$

Thus from (4.15) and (4.16), we obtain

$$\begin{aligned}
\frac{d}{d\delta}A_\varepsilon(\delta)&=\int_{\partial E_1^\delta}\frac{2-2(\nabla d_1(x_1)\cdot\nabla d_{E_1^\delta}(y))}{|y-\Psi_\delta(x_1)|^{2+s}}\eta_\varepsilon(y-\Psi_\delta(x_1))\,d\mathcal{H}^{n-1}(y) \\
&=\int_{\partial E_1^\delta}\frac{|\nabla d_1(x_1)-\nabla d_{E_1^\delta}(y)|^2}{|y-\Psi_\delta(x_1)|^{2+s}}\eta_\varepsilon(y-\Psi_\delta(x_1))\,d\mathcal{H}^{n-1}(y)
\end{aligned}$$

for any small $\delta > 0$ with $\Psi_\delta(x_1) \in U_1$. Hence from the change of variables, we obtain

$$\frac{d}{d\delta}A_\varepsilon(\delta)=\int_{\partial E_1}\frac{|\nabla d_1(x_1)-\nabla d_1(y)|^2}{|\Psi_\delta(y)-\Psi_\delta(x_1)|^{2+s}}\eta_\varepsilon(\Psi_\delta(y)-\Psi_\delta(x_1))\,J_{\partial E_1}\Psi_\delta(y)\,d\mathcal{H}^{n-1}(y)$$

where $J_{\partial E_1}\Psi_\delta(y)$ is the tangential Jacobian of ∂E_1 at y. As is shown in [17], we can have that there exist constants $c' > 0$ and $\delta' > 0$, depending on the space-dimension $n = 2$ and s but independent of $\varepsilon > 0$, such that $|\frac{d}{d\delta}B_\varepsilon(\delta)| \le c'\varepsilon^{1-s}$ for any $\delta \in (0,\,\delta')$ and $\varepsilon \in (0,\,1)$. Therefore, we conclude that

$$\begin{aligned}
-\frac{d}{d\delta}H^s_{E_1^\delta}(\Psi_\delta(x_1))&=\lim_{\varepsilon\downarrow 0}\left(\frac{d}{d\delta}A_\varepsilon(\delta)+\frac{d}{d\delta}B_\varepsilon(\delta)\right) \\
&=\int_{\partial E_1}\frac{|\nabla d_1(x_1)-\nabla d_1(y)|^2}{|\Psi_\delta(y)-\Psi_\delta(x_1)|^{2+s}}J_{\partial E_1}\Psi_\delta(y)\,d\mathcal{H}^{n-1}(y)
\end{aligned}$$

for any $\delta \in (0,\, \delta_0')$ where $\delta_0' > 0$ is a constant depending on the space-dimension $n = 2$, s, and the L^∞-norm of $\nabla^2 d_1$. From the definition of Ψ_δ, we have that there exists a constant $C_0 > 0$ depending on the space-dimension $n = 2$, s, and the L^∞-norm of $\nabla^2 d_1$, such that

$$\frac{J_{\partial E_1} \Psi_\delta(y)}{|\Psi_\delta(y) - \Psi_\delta(x_1)|^{2+s}} \leq \frac{C_0}{|y - x_1|^{2+s}}$$

for any $y \in \partial E_1$ and $\delta \in (0,\, \delta_0')$. Therefore we obtain that there exist constants $C > 0$ and $\delta_0 > 0$, depending on the space-dimension $n = 2$, s, and the second derivative of d_1 but independent of δ, such that the inequality (4.13) with the constant C holds for any $\delta \in (0,\, \delta_0)$. Thus, from the fundamental theorem of calculus and (4.13), we obtain that

$$\begin{aligned} -H^K_{E_1^\delta}(x - \delta\, \nabla d_1(x)) &= -H^K_{E_1}(x_1) - \delta \int_0^1 \frac{d}{d\delta} H^K_{E_1^\delta}(x - \lambda\delta\, \nabla d_1(x))\, d\lambda \\ &\leq -H^K_{E_1}(x_1) + C\,\delta \int_{\partial E_1} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{n-1}(y) \end{aligned} \tag{4.17}$$

for any $\delta \in (0,\, \delta_0)$. Now we show that the integral

$$\int_{\partial E_1} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{n-1}(y)$$

is uniformly bounded for any $x_1 \in V$ and any open set $V \subsetneq U_1$. Indeed, we define the set $U_1^r := \{x \in U_1 \mid \operatorname{dist}(x,\, \partial U_1) > r\}$ for any $r > 0$ satisfying that $B_{2r}(x) \subset U_1$ for any $x \in U_1$. Then we can compute the integral as follows: for any $x_1 \in U_1^r$, it holds that

$$\begin{aligned} &\int_{\partial E_1} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{n-1}(y) \\ &= \int_{\partial E_1 \cap B_r(x_1)} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{n-1}(y) \\ &\quad + \int_{\partial E_1 \cap B_r^c(x_1)} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{n-1}(y) \\ &\leq \int_{\partial E_1 \cap B_r(x_1)} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^2} \frac{1}{|y - x_1|^{n-2+s}}\, d\mathcal{H}^{n-1}(y) \\ &\quad + \int_{\partial E_1 \cap B_r^c(x_1)} \frac{4}{|y - x_1|^{2+s}}\, d\mathcal{H}^{n-1}(y). \end{aligned} \tag{4.18}$$

From the fundamental theorem of calculus and the fact that $B_r(x_1) \subset U_1$ for any $x_1 \in U_1^r$, we have that

$$\frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^2} \leq \|\nabla^2 d_1\|^2_{L^\infty(B_r(x_1))} \tag{4.19}$$

for any $y \in B_r(x_1)$. Thus from (4.18) and (4.19) and noticing that $x_1 \in U_1^r$ and $E_t \subset B_{R_c}$ holds uniformly in $t \geq c$ where $c := -\|u\|_{L^\infty} > -\infty$, we obtain

$$\int_{\partial E_1} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{n-1}(y) \leq c_1\, \|\nabla^2 d_1\|^3_{L^\infty(B_r(x_1))}\, r^{1-s} + \frac{c_2\, \|\nabla^2 d_1\|_{L^\infty(U_1)}}{r^s} \tag{4.20}$$

where $c_1 > 0$ and $c_2 > 0$ are constants depending on the space-dimension $n = 2$ and s. Since we choose any r in such a way that $B_r(x_1) \subset U_1$, we conclude the claim is valid. Thus, from (4.17) and (4.20), we finally obtain the inequality

$$-H^K_{E_1^\delta}(x_1 - \delta\,\nabla d_1(x)) \le -H^K_{E_1}(x_1) + C(n, s, R_c)\,\delta \tag{4.21}$$

for any $\delta \in (0,\, \delta_0)$ where $C(n, s, R_c) > 0$ ($n = 2$ is the space-dimension) and $\delta_0 > 0$ are some constants, which also depend on the L^∞-norm of $\nabla^2 d_1$. Note that the constant δ_0 can be bounded by the inverse of the L^∞-norm of $\nabla^2 d_1$. Thus from (4.11) and (4.21), we have that, for any $\delta \in (0,\, \delta_0)$,

$$-H^K_{E_2}(x_2) \le -H^K_{E_1}(x_1) + C(n, s, R_c)\,\delta. \tag{4.22}$$

Now we consider the following two cases:

Case 1: $0 < \tilde{\delta} < \delta_0$. In this case, we simply substitute $\delta = \tilde{\delta}$ with (4.22) and obtain

$$-H^K_{E_2}(x_2) \le -H^K_{E_1}(x_1) + C(n, s, R_c)\,\tilde{\delta}$$

where $\tilde{\delta} = \mathrm{dist}\,(\partial E_1,\, \partial E_2)$.

Case 2: $\tilde{\delta} \ge \delta_0$. In this case, there exists a number $N \in \mathbb{N}$ such that $\frac{\tilde{\delta}}{N} < \|\nabla^2 d_1\|^{-1}_{L^\infty(U_1)}$. Then setting $\tilde{\delta}_k := \frac{k}{N}\tilde{\delta}$ for each $k \in \{1, \cdots, N\}$ and taking into account all the above arguments, we obtain the inequality that

$$-H^K_{E_1^{\tilde{\delta}_k}}(x_1^{\tilde{\delta}_k}) \le -H^K_{E_1^{\tilde{\delta}_{k-1}}}(x_1^{\tilde{\delta}_{k-1}}) + C(n, s, R_c)\,\frac{\tilde{\delta}}{N} \tag{4.23}$$

for each $k \in \{1, \cdots, N\}$ where we understand the notation $x_1^{\tilde{\delta}_0} = x_1$ and $E_1^{\tilde{\delta}_0} = E_1$. Thus by summing the inequality (4.23) for all $i \in \{1, \cdots, N\}$, we obtain

$$\begin{aligned}
-H^K_{E_1^{\tilde{\delta}}}(x_2) &= -H^K_{E_1^{\tilde{\delta}_N}}(x_1^{\tilde{\delta}_N}) \\
&\le -H^K_{E_1^{\tilde{\delta}_0}}(x_1^{\tilde{\delta}_0}) + N\,C(n, s, R_c)\,\frac{\tilde{\delta}}{N} = -H^K_{E_1}(x_1) + C(n, s, R_c)\,\tilde{\delta}
\end{aligned}$$

where $\tilde{\delta} = \mathrm{dist}\,(\partial E_1,\, \partial E_2)$. In both cases, we finally obtain the inequality

$$-H^K_{E_2}(x_2) \le -H^K_{E_1}(x_1) + C(n, s, R_c)\,\tilde{\delta}.$$

Then since both E_1 and E_2 are the minimizers of $\mathcal{E}_{K,f,t_1}$ and $\mathcal{E}_{K,f,t_2}$, respectively, and from the Euler-Lagrange equations, we obtain

$$t_2 - t_1 \le f(x_2) - f(x_1) + C(n, s, R_c)\,\tilde{\delta}.$$

Recalling the definition of x_2, the Hölder continuity of f, and the fact that $E_t \subset B_{R_c}$ for any $t \ge c$, we conclude that

$$t_2 - t_1 \le ([f]_\beta(B_{R_c}) + C(n, s, R_c)\,\tilde{\delta}^{1-\beta})\,\tilde{\delta}^\beta \tag{4.24}$$

where $[f]_\beta(B_{R_c})$ is the Hölder constant of f in B_{R_c} given as

$$[f]_\beta(B_{R_c}) := \sup_{x,\,y \in B_{R_c},\, x \ne y} \frac{|f(x) - f(y)|}{|x - y|^\beta}$$

and the constant $\tilde{\delta}$ is defined as $\tilde{\delta} \coloneqq \mathrm{dist}\,(\partial E_1,\, \partial E_2)$. Note that the constant $C(n, s, R_c) > 0$ also depends on the L^∞-norm of $\nabla^2 d_1$.

We are now prepared to prove the local Hölder continuity of u. Let $B_{r_0}(x_0) \subset \mathbb{R}^2$ be any open ball of radius r_0 with $x_0 \in \{u = t_0\}$ for a number $t_0 \geq c \coloneqq -\|u\|_{L^\infty}$. We take any points $x, y \in B_{r_0}(x_0)$ with $x \neq y$ and set $t_1, t_2 \in \mathbb{R}$ as $t_1 \coloneqq u(x)$ and $t_2 \coloneqq u(y)$. We may assume that $t_1 > t_2 \geq c$ because we only repeat the same argument in the case of $t_1 < t_2$. In addition to this, we also assume that $t_1 > t_0 > t_2$. Indeed, in the case of $t_1 > t_2 \geq t_0$ or $t_0 \geq t_1 > t_2$, it is sufficient to take another point $x_0' \in B_{r_0}(x_0)$ and $t_0' \in \mathbb{R}$ such that $x_0' \in \{u = t_0'\}$ and $t_1 > t_0' > t_2$, and do the argument that we will show below. Moreover, since we only observe the local regularity of u, it is sufficient to consider the case that $B_{r_0}(x_0) \subset U_0$ where U_0 is a neighborhood of $\partial\{u > t_0\}$ such that the signed distance function from $\partial\{u > t_0\}$ is of class $C^{1+s+\alpha}(U_0)$. Indeed, if $x \in B_{r_0}(x_0) \setminus U_0$ and $y \in B_{r_0}(x_0)$, then, from the continuity of u, we can choose a point z_0 in $B_{r_0}(x_0)$ and close to x such that the estimate $|u(x) - u(z_0)| \leq |x - y|^\beta$ holds and $t_1 = u(x) > u(z_0) \geq u(y) = t_2$. In the case of $z_0 \in U_0$, we just apply the argument that we will show below with (4.24) for z_0, x_0, and y; otherwise we can repeat the above argument until we have the point belonging to U_0.

Now we choose sufficiently small $\varepsilon > 0$ such that $t_1 - \varepsilon > t_0$ and $t_0 - \varepsilon > t_2$ and then we have that $x \in \{u > t_1 - \varepsilon\}$, $y \in \{u > t_2 - \varepsilon\}$, and $x_0 \in \{u > t_0 - \varepsilon\}$. Hence, from (4.24) and the fact that $x, y \in B_{r_0}(x_0)$, we obtain the two inequalities

$$\begin{aligned} u(x) - u(x_0) = t_1 - \varepsilon - (t_0 - \varepsilon) &\leq ([f]_\beta(B_{R_c}) + C(n, s, R_c)\, \tilde{\delta}_1^{1-\beta})\, \tilde{\delta}_1^\beta \\ &\leq ([f]_\beta(B_{R_c}) + C(n, s, R_c)\, r_0^{1-\beta})\, \tilde{\delta}_1^\beta. \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} u(x_0) - u(y) = t_0 - \varepsilon - (t_2 - \varepsilon) &\leq ([f]_\beta(B_{R_c}) + C(n, s, R_c)\, \tilde{\delta}_2^{1-\beta})\, \tilde{\delta}_2^\beta \\ &\leq ([f]_\beta(B_{R_c}) + C(n, s, R_c)\, r_0^{1-\beta})\, \tilde{\delta}_2^\beta \end{aligned} \tag{4.26}$$

where we set $\tilde{\delta}_1 \coloneqq \mathrm{dist}\,(\partial E_{t_0}, \partial E_{t_1})$ and $\tilde{\delta}_2 \coloneqq \mathrm{dist}\,(\partial E_{t_0}, \partial E_{t_2})$. Note that the constant $C(n, s, R_c) > 0$ also depends on the L^∞-norm of $\nabla^2 d_{t_0}$, which can be uniformly bounded in $B_{r_0}(x_0)$. Notice that the inequality

$$\tilde{\delta}_1 + \tilde{\delta}_2 = \mathrm{dist}\,(\partial E_{t_0},\, \partial E_{t_1}) + \mathrm{dist}\,(\partial E_{t_0},\, \partial E_{t_2}) \leq \mathrm{dist}\,(\partial E_{t_1},\, \partial E_{t_2}) \leq |x - y|$$

holds because of the fact that $E_{t_1} \subset E_{t_0} \subset E_{t_2}$. Therefore from (4.25) and (4.26), we obtain that there exists a constant $C = C(n, s, f, R_c, r_0, x_0) > 0$ (we have assumed that the space-dimension n is two) such that

$$\begin{aligned} |u(x) - u(y)| &= |u(x) - u(x_0) + u(x_0) - u(y)| \\ &\leq C\,(\tilde{\delta}_1^\beta + \tilde{\delta}_2^\beta) \leq C\, 2^{1-\beta}(\tilde{\delta}_1 + \tilde{\delta}_2)^\beta \leq 2^{1-\beta} C\, |x - y|^\beta. \end{aligned}$$

Here, in the second inequality, we have used the fact that $2^{1-\beta}(x+1)^\beta \geq x^\beta + 1$ for any $x \geq 1$ and $\beta \in (0,\, 1)$ and applied this fact with $x = \tilde{\delta}_1\, \tilde{\delta}_2^{-1}$ if $\tilde{\delta}_1 \geq \tilde{\delta}_2$ or $x = \tilde{\delta}_2\, \tilde{\delta}_1^{-1}$ if $\tilde{\delta}_1 < \tilde{\delta}_1$. □

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