# Variational Problems on Networks

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## Preface

These notes contain the topics dealt with in the 20-hour PhD course "Variational Problem on Networks" given at SISSA, Trieste, and partly online, in the period January-February 2021, and in the online "Winter School on Analysis and Applied Mathematics" at Münster, Germany, on 22-26 February 2021. The course is thought to be an introduction to the study of variational problems with interfacial energies on networks with a growing number of nodes, mainly parameterized on lattices but with a final section on general abstract graphs. The content of the notes, with more proofs, a general abstract compactness and integral-representation approach, a part on homogenization of stochastic lattices, and an analysis of frustrated systems will be included in the forthcoming book [2].

I gladly acknowledge the kind invitation of Gianni Dal Maso to give the course at SISSA, and the organization of the Winter School by Manuel Friedrich and Leonard Kreutz. The notes much owe to the critical insight and advices of Margherita Solci, who also contributed to their redaction, and to the ability of Andrea Causin in the pictorial representations. Without them the notes would not have been written.

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# Contents

# 1 Setting and model problems

The environment where our problems will be set will consist of a *graph* composed of a set of *nodes* $\mathcal{L}$ with *connections* $\mathcal{E}$; i.e., a symmetric subset of $\mathcal{L} \times \mathcal{L}$.

We will deal with

(a) (portions of) regular lattices: e.g., given $\Omega \subset \mathbb{R}^d$, we consider $\Omega \cap \mathbb{Z}^d$ and a set $\mathcal{E}$ of connections between these points. The model case is that of *nearest-neighbour* connections, or connections up to some given range;

(b) stochastic lattices, in which the positions of the nodes and/or the connections are random, so that we have to confront with a less regular geometry of the environment;

(c) dense graphs, where the cardinality of $\mathcal{E}$ is of the order of the cardinality of $\mathcal{L} \times \mathcal{L}$. In this case the geometry will be less important, and combinatoric arguments will be more relevant.

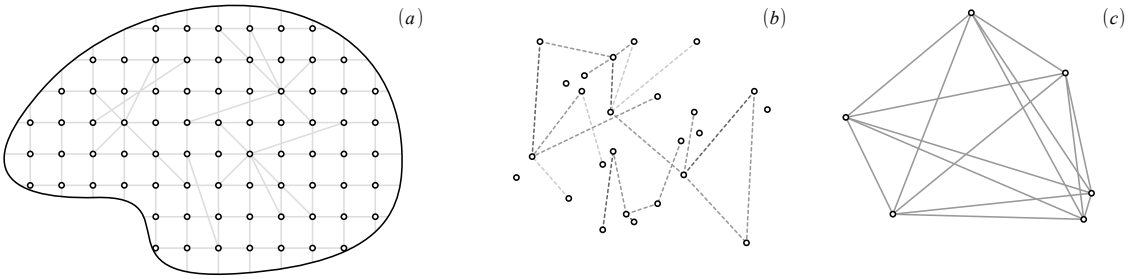


Figure 1: different types of graphs.

Our model problem is that of the *minimal cut* of a graph; that is, the subdivision of $\mathcal{L}$ into two subsets of given cardinality so as to minimize the number of connections between the two sets. This problem can be described as an energy minimization of the functional depending on sets $A$ defined by

$$E(A) = \#\{(i,j) \in \mathcal{E}, i \in A, j \notin A\}.$$

Equivalently, we may rewrite this problem as defined on *spin functions*; that is, functions

$u\colon \mathcal{L} \to \{-1, 1\}$.[1] In this case, with an abuse of notation, the energy can be written as

$$E(u) = \sum_{i,j \in \mathcal{L}:(i,j)\in\mathcal{E}} (u_i - u_j)^2.$$

The value $E(u)$ corresponds to the value $E(A)$, where $A = \{i \in \mathcal{L} : u(i) = 1\}$ and $u_i = u(i)$; that is, $u_i = 1$ if $i \in A$ and $u_i = -1$ if $i \notin A$, up to a factor 8. This factor is due to the fact that $(u_i - u_j)^2 \in \{0, 4\}$ and that if $(i, j) \in \mathcal{E}$, then $(j, i) \in \mathcal{E}$.[2]

We are interested in describing approximately such a minimal-cut problem as the number of nodes diverges; i.e., $\#\mathcal{L} >> 1$. This will be done by constructing approximate continuum problems by a $\Gamma$-convergence approach (and will heavily depend on some characteristics of the set $\mathcal{E}$ related to $\mathcal{L}$). The scopes of this process are many; in particular

- on the one hand connect discrete problems with a PDE formulation for which we may use techniques characteristic of continuum problems;
- on the other hand "validate" continuum models by "molecular" or "atomistic" arguments.

A further illustrative problem in this framework is connected to Image Processing. The target function $u$ represents a black and white image with the value $-1$ corresponding to white and 1 to black at a given pixel, identified with a point $i \in \mathcal{L} = \Omega \cap \mathbb{Z}^2$. The connections are given by *nearest neighbours*; that is,

$$\mathcal{E} = \{(i, j) : i, j \in \Omega \cap \mathbb{Z}^2, \|i - j\| = 1\},$$

and the problem is to minimize the energy given by

$$\sum_{i,j \in \mathcal{L}:(i,j)\in\mathcal{E}} (u_i - u_j)^2 + \lambda \sum_{i \in \mathcal{L}} |u_i - g_i|^2,$$

[1] The choice of $\{-1, 1\}$ as parameters comes from Statistical Mechanics and is coherent with a more general notation involving vector spin functions. Equivalently (and some times more handy) we might consider $u\colon \mathcal{L} \to \{0, 1\}$ making the connection between functions and (characteristic functions of) sets more evident. Energies defined on spin functions are often referred to as Ising systems.

[2] Note that the choice of the square is arbitrary since only the values at $u_i \in \{\pm 1\}$ are considered. Equivalently we could use

$$E(u) = 2 \sum_{i,j \in \mathcal{L}:(i,j)\in\mathcal{E}} |u_i - u_j|.$$

Again, the way of writing $E$ is due to traditional choices and the fact that developing squares is easier. Note that $(u_i - u_j)^2 = 2(1 - u_i u_j)$ so that if $\Omega$ is bounded, up to a constant the energy can be rewritten as

$$E(u) = - \sum_{i,j \in \mathcal{L}:(i,j)\in\mathcal{E}} u_i u_j,$$

which is the usual way of writing Ising systems.

where $\lambda$ is a real parameter and $g\colon \mathcal{L} \to [-1,1]$ is a given *input* function representing a corrupted black and white image to be reconstructed. Here, we do not have a cardinality constraint but we minimize with a perturbation given by an $L^2$-distance from a given datum.

# 2 Limits on regular lattices and sets of finite perimeter

We first consider the case when $\mathcal{L}$ is a (portion of a) Bravais lattice in $\mathbb{R}^d$. It will not be restrictive to suppose that the reference lattice be $\mathbb{Z}^d$.

## 2.1 Scaling

In order to describe the limit of energies as $\#\mathcal{L} >> 1$, we use a scaling parameter. Let $\Omega \subset \mathbb{R}^d$ be an open set.

• *Modeling of the lattice* $\mathcal{L}$. We introduce a space scale $\varepsilon > 0$ (where $\varepsilon$ is a "small" parameter) and consider the scaled lattice

$$\mathcal{L}_\varepsilon = \mathcal{L}_\varepsilon(\Omega) = \Omega \cap \varepsilon\mathbb{Z}^d,$$

whose cardinality is of the order $\frac{1}{\varepsilon^d}$.

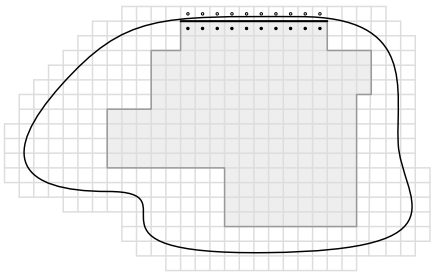

Figure 2: error due to missing interactions at the boundary.

• *Scaling of the energies.* We consider as a model case that of interactions between nearest neighbors only; that is,

$$\mathcal{E} = \mathcal{E}_\varepsilon = \{(i,j) : \varepsilon i, \varepsilon j \in \mathcal{L}_\varepsilon, \|i-j\| = 1\}. \tag{1}$$

We use the notation $\langle i,j\rangle$ to indicate pairs in $\mathbb{Z}^d$ such that $(i,j) \in \mathcal{E}$. The energies are then scaled according to a "surface scaling"; namely,

$$E_\varepsilon(u) = \sum_{\langle i,j\rangle} \varepsilon^{d-1} \frac{(u_i - u_j)^2}{8}, \tag{2}$$

where $u\colon \mathcal{L}_\varepsilon \to \{-1,1\}$ and $u_i = u(\varepsilon i)$. Note (again) the factor $\frac{1}{8}$.

Up to an "error" close to the boundary of $\Omega$ (see Figure 2) the value of $E_\varepsilon(u)$ can be interpreted as the perimeter of the set

$$A_\varepsilon(u) := \bigcup_{\substack{\varepsilon i \in \mathcal{L}_\varepsilon \\ u_i = 1}} \Big( \Big[ -\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \Big]^d + \varepsilon i \Big), \tag{3}$$

which is determined by the spin function $u$.

For a general Bravais lattice $\mathcal{L}$ in $\mathbb{R}^d$ generated by independent vectors $v_1, \dots, v_d$, we can give the corresponding identification of spin functions $u$ defined on $\varepsilon\mathcal{L}$ with sets $A_\varepsilon(u)$ upon taking the elementary cell

$$\mathcal{C} = \Big\{ \sum_{j=1}^{d} t_j v_j : |t_j| \le \frac{1}{2} \Big\} \tag{4}$$

in the place of $[-\frac{1}{2}, \frac{1}{2}]^d$.

The surface scaling guarantees the equicoerciveness of the energies, in the sense that if $\{u_\varepsilon\}$ is a family of functions such that $E_\varepsilon(u_\varepsilon) \le C$, then for all $\Omega'$ regular open subset of $\mathbb{R}^d$ with $\Omega' \subset\subset \Omega$, for $\varepsilon$ small enough the perimeter of $\Omega' \cap A_\varepsilon(u_\varepsilon)$ is equibounded. This gives precompactness of such sets thanks to the properties of sets of equibounded perimeter as explained in the following section.

## 2.2 Sets of finite perimeter

As we have seen, functions taking two values (i.e., spin functions) can be identified with characteristic functions and then with sets. We then recalls the notion of sets of finite perimeter, which is particularly useful when dealing with energies that can be interpreted as surface energies on sets and provide compactness properties.

A good notion of perimeter of a set from the standpoint of the Calculus of Variations is the maximal extension of the usual notion of perimeter for polytopes (polyhedral sets in $d$ dimension) lower semicontinuous with respect to the convergence in measure.

**Definition 1** (perimeter)**.** *If $A$ is a polytope in $\mathbb{R}^d$, then the* perimeter of $A$*, denoted by* $\mathrm{Per}(A)$*, is elementarily defined.*

*Let $A \subset \mathbb{R}^d$; the* perimeter *of $A$ is defined by*

$$\mathrm{Per}(A) = \inf\Big\{ \liminf_n \mathrm{Per}(A_n) : |A_n \triangle A| \to 0, A_n \text{ polytope} \Big\}.$$

*The set $A$ is a* set of finite perimeter *if* $\mathrm{Per}(A) < +\infty$.

**Remark 2** (distributional definition of the perimeter)**.**

(i) An alternative way to define the perimeter of $A$ is in a distributional way as

$$\mathrm{Per}(A) = \sup\Big\{ \int_A \operatorname{div} \varphi \, dx : \varphi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d), \|\varphi\|_\infty \le 1 \Big\},$$

which states that the characteristic function of $A$ is a function of bounded variation. This characterization allows to prove functional-analytic properties of the space of sets of finite perimeter.

(ii) If $A$ is a "regular" subset of $\mathbb{R}^d$ whose surface element is denoted by $d\Sigma$, then $\int_{\partial A} \langle \varphi, \nu_A \rangle \, d\Sigma = -\int_A \operatorname{div} \varphi \, dx$ for any $\varphi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$. Taking $\varphi$ equal to (an approximation of) $-\nu$ on $\partial A$ we obtain its surface area; hence, the definition of the perimeter is an extension of the usual definition.

**Remark 3** (perimeter in $\Omega$)**.** The notion of perimeter can be localized on open subsets $\Omega$ of $\mathbb{R}^d$, setting

$$\mathrm{Per}(A; \Omega) = \sup\Big\{ \int_E \operatorname{div} \varphi \, dx : \varphi \in C_0^\infty(\Omega; \mathbb{R}^d), \|\varphi\|_\infty \le 1 \Big\}.$$

Sets $A$ such that $\mathrm{Per}(A; \Omega) < +\infty$ will be called *sets of finite perimeter in* $\Omega$. If $\Omega$ is regular then sets of finite perimeter can be locally approximated by polytopes in $\Omega$.

The following theorem summarizes the main structure properties of sets of finite perimeter. We denote by $\mathcal{H}^{d-1}$ the $d-1$-dimensional Hausdorff measure (which generalizes the notion of surface area).

**Theorem 4** (reduced boundary and inner normal)**.** *If $A \subset \mathbb{R}^d$ has finite perimeter in $\Omega$, then there exists a set $\partial^* A \subseteq \partial A$, called the* reduced boundary *of $A$, such that*

- *the total variation of the distributional derivative of the characteristic function $\chi_A$ is given by $\mathcal{H}^{d-1} \llcorner \partial^* A$; in particular,*

$$\mathrm{Per}(A) = \mathcal{H}^{d-1}(\partial^* A);$$

- *$\partial^* A$ is rectifiable; that is, there exists $N \subset \mathbb{R}^d$ with $\mathcal{H}^{d-1}(N) = 0$ such that*

$$\partial^* A \subset \bigcup_{h \in \mathbb{N}} \Gamma_h \cup N$$

*where $(\Gamma_h)$ is a sequence of compact subsets of $C^1$-hypersurfaces.*

*Moreover (by the implicit-function theorem) there exists $\nu$, the* (inner) normal *to $\partial^* A$, defined $\mathcal{H}^{d-1}$-almost everywhere on $\partial^* A$ (corresponding to the (common) normal to the hypersurfaces $\Gamma_h$).*

**Remark 5** (blow up)**.** From the result above we deduce that there exists a function $\nu_A : \partial^* A \to S^{d-1}$, the *inner normal* to $A$ (coinciding almost everywhere with $\nu$ defined above), such that

$$\lim_{\rho \to 0^+} \frac{|B_\rho^+(x, \nu_A(x)) \setminus A|}{\rho^d} = 0 \quad \forall x \in \partial^* A,$$

where $B_\rho^+(x, \nu) := \{y \in \mathbb{R}^d : \|y - x\| < \rho,\ \langle y - x, \nu \rangle > 0\}$.

As a consequence if $A$ is a set of finite perimeter, then for $\mathcal{H}^{d-1}$-almost all $x_0 \in \partial^* A$ we have the convergence of the blown-up sets

$$\frac{1}{\varrho}(A - x_0) \to \{x \in \mathbb{R}^d : \langle x, \nu \rangle \geq 0\} \text{ in } \ L^1_{\mathrm{loc}}(\mathbb{R}^d) \text{ as } \ \varrho \to 0.$$

The introduction of the sets of finite perimeter is motivated by the following compactness theorem.

**Theorem 6** (compactness)**.** *Let $\{A_n\}$ be a sequence of sets of finite perimeter such that $\sup_{n \in \mathbb{N}} \mathrm{Per}(A_n; \Omega) < +\infty$. Then there exists a subsequence $\{A_{n_k}\}$ (locally) converging in measure to a set $A$ of finite perimeter; that is, for any $\Omega' \subset\subset \Omega$*

$$|(A_{n_k} \triangle A) \cap \Omega'| \to 0 \text{ as } \ k \to +\infty.$$

# 3 Asymptotic description of the energies $E_\varepsilon$: discrete-to-continuum $\Gamma$-convergence

The compactness theorem for sets of finite perimeter allows to deduce a compactness property for the discrete energies $E_\varepsilon$ defined in (2).

**Remark 7** (compactness of $E_\varepsilon$)**.** Let $\{u_\varepsilon\}$ be a family of spin functions defined on $\mathcal{L}_\varepsilon$ such that $E_\varepsilon(u_\varepsilon) \leq C$. Then, for any $\Omega' \subset\subset \Omega$

$$\mathcal{H}^{d-1}(\Omega' \cap A_\varepsilon(u_\varepsilon)) \leq E_\varepsilon(u_\varepsilon) \leq C$$

for $\varepsilon$ small enough. By Theorem 6 we obtain that there exist a subsequence (again denoted by $\{u_\varepsilon\}$) and a set of finite perimeter $A$ such that

$$A_\varepsilon(u_\varepsilon) \to A \text{ in } \ L^1_{\mathrm{loc}}(\Omega).$$

Now we can give a definition of convergence for spin functions.

**Definition 8** (convergence of discrete sets and of spin functions)**.** *For all $\varepsilon > 0$ let $u_\varepsilon$ be a spin function defined on $\mathcal{L}_\varepsilon$, and let $A$ be a subset of $\mathbb{R}^d$. Then*

$$u_\varepsilon \to A \;\Leftrightarrow\; A_\varepsilon(u_\varepsilon) \to A \text{ in } \; L^1_{\mathrm{loc}}(\mathbb{R}^d). \tag{5}$$

*Correspondingly, we say that a sequence of discrete sets $A_\varepsilon \subset \varepsilon\mathbb{Z}^d$ converges to $A$ if $u_\varepsilon = 2\chi_{A_\varepsilon} - 1 \to A$ (i.e., the interpolations of $A_\varepsilon$ converge to $A$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.*

*An analogous definition can be given for spin functions on arbitrary Bravais lattices with sets constructed starting from the elementary cells defined in* (4)*.*

We can now specialize the definition of $\Gamma$-convergence to the discrete setting as follows.

**Definition 9** (discrete-to-continuum $\Gamma$-convergence)**.** *For any $\varepsilon > 0$, let $E_\varepsilon$ be a functional defined on the set $\{u\colon \mathcal{L}_\varepsilon \to \{-1; 1\}\}$, and $F$ be a functional defined on the family of sets of finite perimeter. The sequence $\{E_\varepsilon\}$ $\Gamma$-converges to $F$ at $A$ with respect to the convergence defined in* (5) *if and only if*

(i) (liminf inequality) *for all $u_\varepsilon$ converging to $A$ we have $F(A) \le \liminf\limits_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon)$;*

(ii) (existence of a recovery sequence) *there exist $u_\varepsilon$ converging to $A$ such that*[3] *$F(A) = \lim\limits_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon)$ (a such family $\{u_\varepsilon\}$ is called a* recovery sequence *for $F(A)$).*

*If this holds for any $A$ of finite perimeter, we say that $F$ is the* $\Gamma$-limit of $E_\varepsilon$ *and write*

$$F = \Gamma\text{-}\lim_{\varepsilon\to 0} E_\varepsilon.$$

**Remark 10.** Using the fundamental theorem of $\Gamma$-convergence we may prove that problems for energies $E_\varepsilon$ can be approximated using analogous continuum problems related to the limit energy $F$. If $E_\varepsilon$ $\Gamma$-converge to $F$, then

- the minimal cut problems for the energies $E_\varepsilon$ with the constraint $\varepsilon^d \#\{u_i = 1\} = m_\varepsilon$ converge to the constrained minimum problem

$$\min\{F(A) : |A| = m\},$$

where $m = \lim\limits_{\varepsilon\to 0} m_\varepsilon$;

[3]Note that if (i) holds, in order to prove (ii) it suffices to show that for all $\eta > 0$ there exists $u_\varepsilon^\eta$ converging to $A$ such that $F(A) \ge \limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon^\eta) - \eta$. Indeed by a diagonal argument then we obtain a sequence $u_\varepsilon$ converging to $A$ that satisfies $F(A) \ge \limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon)$, which is a recovery sequence by (i).

- the minimum problem for $E_\varepsilon(u)+\lambda\sum_{i\in\mathcal{L}}\varepsilon^d|u_i-g_i|^2$ converge to the minimum problem
$$\min\{F(A)+\lambda\int_\Omega(\chi_A-g)^2\,dx\}.$$

Theorem 4 allows to define perimeter functionals as integrals on the reduced boundary as in the following definition. Such perimeters will be obtained as limits of energies $E_\varepsilon$ under general conditions. Note that the usual perimeter is obtained by taking the constant 1 as $\varphi$.

**Definition 11** (perimeter functionals)**.** Let $\Omega$ be an open subset of $\mathbb{R}^d$ and let $X=\{A : A \text{ of finite perimeter in } \Omega\}$. A functional $F\colon X\to\mathbb{R}$ is a (homogeneous) *perimeter functional* if it is given by
$$F(A)=\int_{\Omega\cap\partial^* A}\varphi(\nu(x))\,d\mathcal{H}^{d-1}(x), \tag{6}$$
where $\nu(x)$ is the inner normal to $\partial^*A$ at $x$ and $\varphi\colon S^{d-1}\to[0,+\infty)$ is a continuous function.

Since $\Gamma$-limits are lower semicontinuous with respect to the convergence in which they are computed, we may restrict our analysis to lower-semicontinuous functionals, characterized in the following theorem.

**Theorem 12** (semicontinuity of perimeter functionals)**.** *The (homogeneous) perimeter functional $F$ defined in* (6) *is lower semicontinuous with respect to the $L^1_{\mathrm{loc}}$ convergence if and only if $\varphi$ defines a norm; i.e., (the one-homogeneous extension of) $\varphi$ is convex; that is, the function (again denoted by $\varphi$) defined in $\mathbb{R}^d$ by*
$$\varphi(z)=\begin{cases}\|z\|\varphi\big(\frac{z}{\|z\|}\big) & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases}$$
*is convex. Furthermore, if $\Omega$ is a Lipschitz set then for all $A$ there exists a sequence of polytopes $A_k$ such that $F(A_k)\to F(A)$.*

**Remark 13** (representation of perimeter functionals; Wulff shapes)**.** Let $\varphi\colon S^{d-1}\to\mathbb{R}$ be such that $\varphi\geq c>0$, extended to $\mathbb{R}^d$ by one-homogeneity.

- We can characterize the perimeter functional $F$ given by (6) by the set
$$B_\varphi=\{z\in\mathbb{R}^d : \varphi(z)\leq 1\},$$
which is a convex set if $F$ is lower semicontinuous. Identifying $F$ with $B_\varphi$ is a handy way to describe the limit energies in a pictorial way.

• Another equivalent characterization of $F$ is the identification with a *Wulff shape* of $\varphi$; that is, a (convex) set $W_\varphi$ maximizing[4]

$$\max\{|A| : F(A) \le 1\}.$$

The sets $W_\varphi$ van be obtained from $B_\varphi$ by a duality argument. Note that minimizers $W_\varphi$ satisfy $F(A) = 1$ by the homogeneity of $F$. If $\varphi$ is even; that is, $\varphi(\nu) = \varphi(-\nu)$, then we can choose uniquely $W_\varphi$ by centering it at 0.

**Example 14** ($\Gamma$-limit of nearest neighbour energies)**.** We consider the discrete energies defined in (2) and compute the $\Gamma$-limit with respect to the convergence (5).

• *Optimization of the lower bound.* Let $\{u_\varepsilon\}$ be such that $E_\varepsilon(u_\varepsilon)$ is equibounded. The compactness ensures that (up to subsequences) $u_\varepsilon \to A$ where $A$ is a set of finite perimeter. We note that for $\varepsilon$ small enough

$$E_\varepsilon(u_\varepsilon) \ge \mathcal{H}^{d-1}(\Omega' \cap \partial A_\varepsilon(u_\varepsilon)), \tag{7}$$

for all $\Omega' \subset\subset \Omega$. We may optimize lower estimate (7) by observing that the inner normal to $\partial A_\varepsilon(u_\varepsilon)$, denoted by $\nu_\varepsilon$, may only take the values $\pm e_k$, $k = 1, \dots, d$, and then, for all $\Omega' \subset\subset \Omega$, we have

$$E_\varepsilon(u_\varepsilon) \ge \int_{\Omega' \cap \partial A_\varepsilon(u_\varepsilon)} \varphi(\nu_\varepsilon)\, d\mathcal{H}^{d-1}, \tag{8}$$

for every norm $\varphi$ such that $\varphi(e_k) \le 1$ for all $k = 1, \dots, d$. The largest such norm is

$$\|\nu\|_1 := \sum_{k=1}^{d} |\nu_k|, \quad \nu = (\nu_1, \dots, \nu_d).$$

Hence, thanks to (8) and the lower semicontinuity of the perimeter functionals, we have

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \ge \sup_{\Omega' \subset\subset \Omega} \int_{\Omega' \cap \partial^* A} \|\nu\|_1\, d\mathcal{H}^{d-1} = \int_{\Omega \cap \partial^* A} \|\nu\|_1\, d\mathcal{H}^{d-1}. \tag{9}$$

• *Upper bound by density.* As for the $\Gamma$-lim sup inequality, it is sufficient to construct a recovery sequence for a polytope $A$; indeed, by Theorem 12, they are (strongly) dense in the space of characteristic functions of sets of finite perimeter, and we can use a diagonal argument.

[4]Equivalently, we may define Wulff shapes as $W_\varphi = B/(F(B))^{\frac{1}{d-1}}$, where $B$ is a solution of $\min\{F(B) : |B| \ge 1\}$. Conversely, minimizers of this last problem are given by $W_\varphi/|W_\varphi|^{\frac{1}{d}}$. The equivalence of the two problems is obtained by using the fact that $\mathcal{H}^{d-1}$ is positively $d-1$-homogeneous as set function.

We now construct a recovery sequence for a polytope $A$. Note that by localization we can consider separately any edge of the polytope. The sequence $\{u_\varepsilon\}$ defined by $u_\varepsilon(\varepsilon i) := -1 + 2\chi_A(\varepsilon i)$ can be chosen as a recovery sequence, since

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = \int_{\Omega\cap\partial A} \|\nu\|_1 \, d\mathcal{H}^{d-1}.$$

Hence, we have

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E_\varepsilon(A) = \int_{\Omega\cap\partial^* A} \|\nu\|_1 \, d\mathcal{H}^{d-1}. \tag{10}$$

The functional $F(A) = \int_{\Omega\cap\partial^* A} \|\nu\|_1 \, d\mathcal{H}^{d-1}$ is the so-called 1-*crystalline perimeter* of $A$.

The Wulff shape of the 1-crystalline norm is the $d$-dimensional coordinate cube with measure 1.

# 4 A class of pairwise ferromagnetic homogeneous systems

With Example 14 in mind, we study the asymptotic behaviour of functionals of the form

$$E_\varepsilon(u) = \sum_{\varepsilon i,\varepsilon j\in\mathcal{L}_\varepsilon} \varepsilon^{d-1} \, a_{ij}(u_i - u_j)^2, \tag{11}$$

where $\mathcal{L}_\varepsilon = \mathcal{L}_\varepsilon(\Omega) = \varepsilon\mathbb{Z}^d \cap \Omega$, $u\colon \varepsilon\mathbb{Z}^d \to \{-1, 1\}$, $u_i = u(\varepsilon i)$ and $a_{ij}$ is a family of non-negative coefficients describing the set $\mathcal{E}$ of the connections, each considered with a possible weight. In particular, we consider the following problem:

- in which conditions on $a_{ij}$ the $\Gamma$-limit is a perimeter functional of the form (6).

As an example, we can consider the set of connections in $\mathbb{Z}^2$ given by the nearest and next-to-nearest neighbours; that is,

$$\mathcal{E} = \{(i, j) : \|i - j\| \le \sqrt{2}\}.$$

The set of such connections is indicated by $\langle\langle i, j\rangle\rangle$. This corresponds in (11) to the choice $a_{ij} = 1$ if $\|i-j\| \le \sqrt{2}$, and $a_{ij} = 0$ otherwise. The network of interactions is represented in Fig. 3, where, in the picture on the right-hand side, we have highlighted the sites interacting with a given site.

We assume that the coefficients $a_{ij}$ in (11) satisfy the following hypotheses.

- (ferromagnetic energies) $a_{ij} \ge 0$ for any $i, j$; this corresponds to having uniform ground states; i.e., $u$ identically 1 or $-1$;
- (coerciveness of nearest-neighbour interactions) $a_{ij} \ge c > 0$ if $\|i - j\| = 1$;

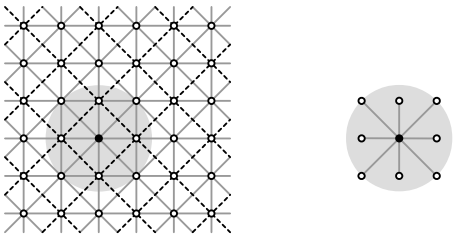

Figure 3: square lattice with next-to-nearest neighbour interactions and connections of a given point.

- (homogeneity) $a_{ij} = \alpha_{i-j} = \alpha_{j-i}$ for any $i, j$; note that the symmetry can be assumed without loss of generality by possibly choosing coefficients $\frac{a_{ij}+a_{ji}}{2}$.

Moreover, we suppose (as a working condition) that the coefficients $a_{ij}$ be uniformly of *finite range*; that is,

- (finite range) there exists $R > 0$ such that $a_{ij} = 0$ if $\|j - i\| > R$.

**Remark 15** (coerciveness and domain of the $\Gamma$-limit)**.** If a sequence $(u_\varepsilon)$ is such that $E_\varepsilon(u_\varepsilon) \le C$, then the coerciveness of the nearest-neighbour interactions ensures that the corresponding sets $A_\varepsilon(u_\varepsilon)$ have equibounded perimeter, so that they are precompact thanks to Theorem 6. Up to subsequences, we can assume that $u_\varepsilon \to A$ for a set $A$ of finite perimeter. Since the coefficients are of finite range, by choosing $u_\varepsilon = -1 + 2\chi_A$ on $\varepsilon\mathbb{Z}^d$ we obtain $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \le C_R \mathcal{H}^{d-1}(\Omega \cap \partial A)$; hence, the domain of the $\Gamma$-limit of $E_\varepsilon$ is the family of sets with finite perimeter.

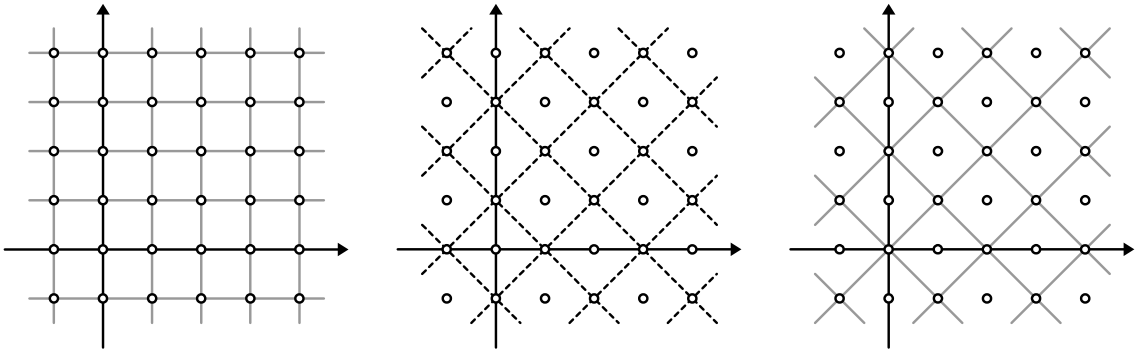

Figure 4: decomposition of a square lattice with next-to-nearest neighbour interactions.

S **Computation of the $\Gamma$-limit by "superposition"**

We start with a "pictorial" example to describe the superposition of lattices. Again, we consider the lattice $\mathbb{Z}^2$ with the set of connections given by the nearest and next-to-nearest

interactions. This corresponds to assuming that the range of $a_{ij}$ is $R = \sqrt{2}$. We can decompose the lattice in three lattices as pictured in Fig. 4.

We separately estimate the contribution of the energy on the different sublattices of the decomposition, noting that the restriction of converging sequences on the whole lattice still converge to the same limit when considered on each such sublattice.

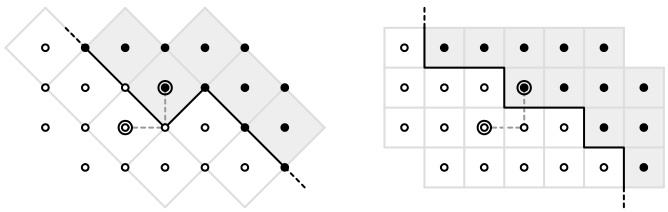

Figure 5: coerciveness of next-to-neighbour interactions.

**Remark 16** (coerciveness and convergence on sublattices)**.** Let $\{u_\varepsilon\}$ be an equibounded sequence. We can assume that $A_\varepsilon := A_\varepsilon(u_\varepsilon) \to A$, noting that the convergence is strong and that the perimeters of $A_\varepsilon$ are equibounded. If $\tilde{\mathcal{L}}$ is a sublattice of $\mathbb{Z}^d$, whose elementary cell is denoted by $\mathcal{C}$, we define $\tilde{u}_\varepsilon$ as the restriction of $u_\varepsilon$ to $\Omega \cap \varepsilon\tilde{\mathcal{L}}$, and let $\tilde{A}_\varepsilon$ denote the set corresponding by interpolation to $\tilde{u}_\varepsilon$ in $\Omega \cap \varepsilon\tilde{\mathcal{L}}$ (see definition (3) with $\mathcal{C}$ in the place of $[-\frac{1}{2}, \frac{1}{2}]^d$). We omit the dependence on $\Omega$ in the following notation. Then

$$\tilde{A}_\varepsilon \to A \quad \text{strongly.}$$

Indeed, $\mathcal{H}^{d-1}(\Omega \cap \tilde{A}_\varepsilon) \leq C\mathcal{H}^{d-1}(\Omega \cap \partial^* A_\varepsilon)$, since to a change of sign of $\tilde{u}_\varepsilon$ between $i$ and $j$ corresponds (at least) a change of sign of $u_\varepsilon$ along a path joining $i$ and $j$ (see Fig. 5); then $\tilde{A}_\varepsilon$ strongly converges to a set of finite perimeter $\tilde{A}$.

Moreover, by an argument of "weak-strong" convergence, we note that

$$\tilde{A}_\varepsilon \cap \varepsilon\tilde{\mathcal{L}} \rightharpoonup c_{\tilde{\mathcal{L}}}\chi_{\tilde{A}} \text{ and } \quad A_\varepsilon \cap \varepsilon\tilde{\mathcal{L}} \rightharpoonup c_{\tilde{\mathcal{L}}}\chi_A,$$

where $c_{\tilde{\mathcal{L}}}$ is a positive constant depending on the lattice. Since $\tilde{A}_\varepsilon \cap \varepsilon\tilde{\mathcal{L}} = A_\varepsilon \cap \varepsilon\tilde{\mathcal{L}}$, we deduce that $\tilde{A} = A$.

Now we compute the $\Gamma$-limit of the sequence $E_\varepsilon$.

• (lower bound) We can write $E_\varepsilon$ as the sum of functionals depending on a parameter $k \in \mathbb{Z}^d$; that is,

$$E_\varepsilon(u) = \sum_{k\in\mathbb{Z}^d} \sum_{\varepsilon i \in \mathcal{L}_\varepsilon} \varepsilon^{d-1}\alpha_k(u_{i+k} - u_i)^2 =: \sum_{k\in\mathbb{Z}^d} E_\varepsilon^k(u).$$

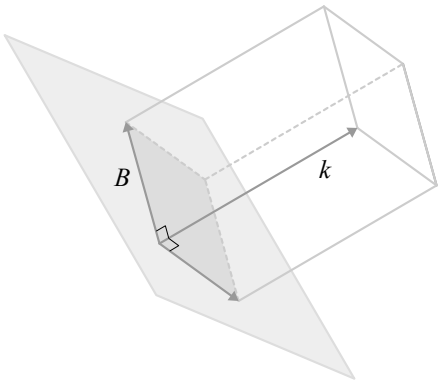


Figure 6: the elementary cell of $\mathcal{L}^k$.

For a fixed $k \in \mathbb{Z}^d$ we consider a lattice $\mathcal{L}^k$ obtained by using $k$ and a basis of the orthogonal space ($B$ denotes the corresponding cell in this $d-1$ subspace, as pictured in Fig. 6), and define the functional

$$E_\varepsilon^{\mathcal{L}^k}(u) = \sum_{i \in \mathcal{L}^k \cap \frac{1}{\varepsilon}\Omega} \varepsilon^{d-1} \alpha_k (u_{i+k} - u_i)^2,$$

which is a nearest-neighbour energy in the lattice $\mathcal{L}^k$, where all the interactions in the directions orthogonal to $k$ have coefficient equal to 0. Recalling Remark 16, we have that if $u_\varepsilon \to A$ then the restriction of $u_\varepsilon$ to the sublattice $\varepsilon\mathcal{L}^k \cap \Omega$ (again denoted by $u_\varepsilon$) converges strongly to the same $A$. Denoting the interpolation set of $u_\varepsilon$ on the sublattice $\varepsilon\mathcal{L}^k \cap \Omega$ by $A_\varepsilon^k(u_\varepsilon)$, we can write for any open set $\Omega' \subset\subset \Omega$ and for $\varepsilon$ small enough

$$E_\varepsilon^{\mathcal{L}^k}(u_\varepsilon) \geq \frac{4}{\mathcal{H}^{d-1}(B)} \int_{\Omega' \cap \partial A_\varepsilon^k(u_\varepsilon)} \alpha_k \left| \left\langle \frac{k}{\|k\|}, \nu \right\rangle \right| d\mathcal{H}^{d-1} = \frac{4}{|\mathcal{C}^k|} \int_{\Omega' \cap \partial A_\varepsilon^k(u_\varepsilon)} \alpha_k |\langle k, \nu \rangle| \, d\mathcal{H}^{d-1},$$

where $\mathcal{C}^k$ is the fundamental cell of $\mathcal{L}^k$ as in (4). By lower semicontinuity we then obtain

$$\liminf_{\varepsilon \to 0} E_\varepsilon^{\mathcal{L}^k}(u_\varepsilon) \geq \frac{4}{|\mathcal{C}^k|} \int_{\Omega' \cap \partial^* A} \alpha_k |\langle k, \nu \rangle| \, d\mathcal{H}^{d-1}.$$

Taking the supremum over all sets $\Omega'$ compactly contained in $\Omega$, we get

$$\liminf_{\varepsilon \to 0} E_\varepsilon^{\mathcal{L}^k}(u_\varepsilon) \geq F^k(A) := \frac{4}{|\mathcal{C}^k|} \int_{\Omega \cap \partial^* A} \alpha_k |\langle k, \nu \rangle| \, d\mathcal{H}^{d-1}.$$

Now, we have to count how many different (translated) lattices we have with one side of the elementary cell equal to $k$; we note that $\mathbb{Z}^d$ contains $|\mathcal{C}^k|$ disjoint copies of the lattice $\mathcal{L}^k$; hence, going back to the functionals $E_\varepsilon^k$

$$\liminf_{\varepsilon \to 0} E_\varepsilon^k(u_\varepsilon) \geq 4 \int_{\Omega \cap \partial^* A} \alpha_k |\langle k, \nu \rangle| \, d\mathcal{H}^{d-1},$$

since the sum of the lim inf is lower than the lim inf of the sum. We conclude that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0} \sum_{k\in\mathbb{Z}^d} E^k_\varepsilon(u_\varepsilon) \geq 4 \sum_{k\in\mathbb{Z}^d} \alpha_k \int_{\Omega\cap\partial^* A} |\langle k,\nu\rangle|\, d\mathcal{H}^{d-1}.$$

Note that the factor 4 (instead of 8) is due to the fact that the vectors $k$ and $-k$ are both accounted for.

The "candidate" limit energy density is then

$$\varphi(\nu) = 4 \sum_{k\in\mathbb{Z}^d} \alpha_k |\langle k,\nu\rangle|. \tag{12}$$

• (upper bound) Note that in general the Γ-limit of a sum does not coincide with the sum of the Γ-limits, since the recovery sequences can be different. In the present case, we obtain the upper inequality by noting that the same recovery sequence can be used for all $k$. As in the case of nearest-neighbour interactions, we prove the lim sup inequality for a polytope $A$ and conclude by using a density argument.

If $A$ is a polytope, then locally it is a portion of a half-space, with boundary orthogonal to a direction $\nu$. Then, by choosing $u_\varepsilon = -1 + 2\chi_{A\cap\varepsilon\mathbb{Z}^d}$, we obtain that the restriction to each $\varepsilon\mathcal{L}^k$ is a recovery sequence for $|\mathcal{C}^k|F^k(A)$. Indeed, if we consider these restrictions; that is, we limit the interactions to the sublattice $\varepsilon\mathcal{L}^k$, we have that the measure of the boundary of the set $A^k_\varepsilon(u_\varepsilon)$ is the measure of the projection on the hyperplane orthogonal to $\nu$, and then proportional to $|\langle k,\nu\rangle|$. This implies that $\{u_\varepsilon\}$ is a recovery sequence for all sequences of functionals $E^k_\varepsilon$ of the decomposition, and we have an upper bound with the same $\varphi$.

**Remark 17.** Following the proof above, we note that we can drop the hypothesis of finite range: if the range of the coefficients is not finite, but

$$\sum_{k\in\mathbb{Z}^d} \alpha_k \|k\| < +\infty, \tag{13}$$

the same proof can be repeated (almost) word for word. Indeed, we can estimate $E_\varepsilon(u_\varepsilon)$ from below by the same energy limited to the interactions with range less than $R$; that is, $\|i-j\| \leq R$, indicated by $E^{(R)}_\varepsilon(u_\varepsilon)$. Since $E^{(R)}_\varepsilon$ is of finite range, the sequence Γ-converges to $F^{(R)}(A) = \int_{\Omega\cap\partial^* A} \varphi_R(\nu)\, d\mathcal{H}^{d-1}$, where

$$\varphi_R(\nu) = 4 \sum_{\|k\|\leq R} \alpha_k |\langle k,\nu\rangle|.$$

Noting that $\varphi_R(\nu)$ increasingly converges to $\varphi(\nu)$, the lower bound can be optimized by taking the supremum over $R>0$, obtaining

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq \sup_R \liminf_{\varepsilon\to 0} E^{(R)}_\varepsilon(u_\varepsilon) \geq \sup_R \int_{\Omega\cap\partial^* A} \varphi_R(\nu)\, d\mathcal{H}^{d-1} = \int_{\Omega\cap\partial^* A} \varphi(\nu)\, d\mathcal{H}^{d-1}.$$

As for the upper bound, the same recovery sequence $\{u_\varepsilon\}$ as in the finite-range case can still be used, since the infinite sum $\sum_{k\in\mathbb{Z}^d} E^k_\varepsilon(u_\varepsilon)$ is uniformly convergent by hypothesis (for $A$ a portion of a half-space, it is bounded by the perimeter of the boundary of $A$ times $4\sum_{k\in\mathbb{Z}^d}\alpha_k\|k\|$).

**Remark 18** (crystalline norms)**.** Note that if the range of $\alpha_k$ is finite, then $B_\varphi = \{z : \varphi(z) \le 1\}$ is a polytope, since it is the intersection of half-spaces and the vectors $k$ for which $\alpha_k \neq 0$ span $\mathbb{Z}^d$. Since the coefficients are symmetric, $B_\varphi$ is symmetric with respect to the origin. Then, also the Wulff shape $W_\varphi$ is a polytope symmetric with respect to the origin. A norm $\varphi$ such that $W_\varphi$ satisfies this property is called a *crystalline norm*.

If the range is not finite, we can have energy densities $\varphi$ that are not crystalline; in particular, suitably choosing the coefficients we may obtain $\varphi$ constant (i.e., the limit perimeter proportional to the euclidean one).

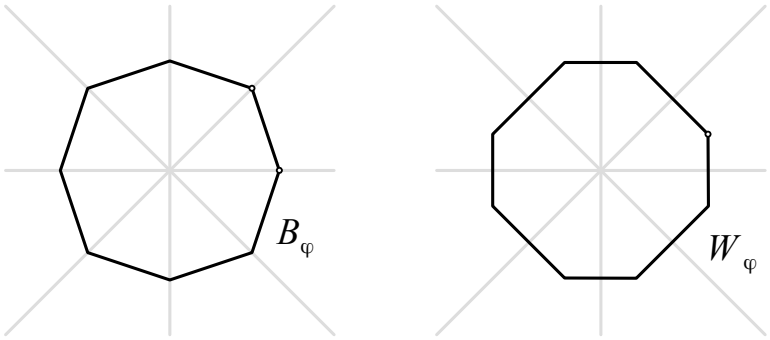


Figure 7: $B_\varphi$ and $W_\varphi$ for next-to-nearest systems.

**Example 19** (next-to-nearest neighbours in $\mathbb{Z}^2$)**.** We apply this result to the case of nearest and next-to-nearest interactions in $\mathbb{Z}^2$ with weighted coefficients; that is, we fix

$$\alpha_k = \begin{cases} \alpha & \text{if } \ k \in \{\pm e_1, \pm e_2\} \\ \beta & \text{if } \ k \in \{\pm(e_1+e_2), \pm(e_1-e_2)\} \\ 0 & \text{otherwise.} \end{cases}$$

with $\alpha, \beta > 0$. The energy density is

$$\begin{aligned} \varphi(\nu) &= 8\Big(\alpha(|\langle e_1,\nu\rangle| + |\langle e_2,\nu\rangle|) + \beta(|\langle e_1+e_2,\nu\rangle| + \langle e_1-e_2,\nu\rangle)\Big) \\ &= 8(\alpha\|\nu\|_1 + 2\beta\|\nu\|_\infty). \end{aligned}$$

The Wulff shape $W_\varphi$ is an octagon as in Fig. 7.

**Remark 20** (interactions on general Bravais lattices)**.** The Γ-convergence result can be generalized by considering a general ($d$-dimensional) Bravais lattice $\mathcal{L}$ in $\mathbb{R}^d$. In this case, the limit energy density depends on the volume of the cell of the lattice; we have

$$\varphi(\nu) = 4c_{\mathcal{L}} \sum_{k \in \mathbb{Z}^d} \alpha_k |\langle k, \nu \rangle|, \tag{14}$$

where $c_{\mathcal{L}} = |\mathcal{C}|^{-1}$ and $\mathcal{C}$ is the fundamental cell of the lattice $\mathcal{L}$ as in (4).

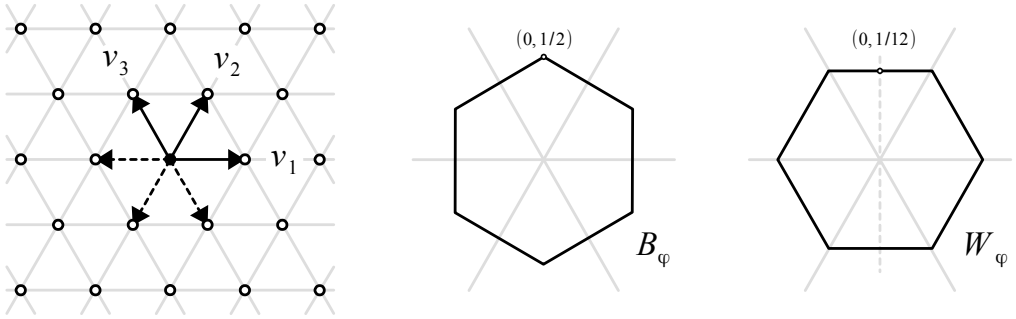


Figure 8: $B_\varphi$ and $W_\varphi$ for the triangular lattice.

**Example 21** (Nearest-neighbour interactions on the triangular lattice)**.** As an example we consider a triangular lattice in 2 dimensions. Let $v_n = \big(\cos((n-1)\frac{\pi}{3}), \sin((n-1)\frac{\pi}{3})\big)$ for $n = 1, 2, 3$, and let $\mathcal{L}$ be the Bravais lattice given by

$$\mathcal{L} = \mathbb{Z}v_1 + \mathbb{Z}v_2.$$

We consider the nearest-neighbour interactions; that is, the coefficients $a_{ij}$ are given by

$$a_{ij} = \begin{cases} 1 & \text{if } i - j \in \{\pm v_1, \pm v_2, \pm v_3\} \\ 0 & \text{otherwise.} \end{cases}$$

Since $c_{\mathcal{L}} = \frac{2}{\sqrt{3}}$, by (14) we get

$$\varphi(\nu) = \frac{8}{\sqrt{3}} \sum_{n=1}^{3} |\langle \nu, v_n \rangle|.$$

The corresponding Wulff shape is an hexagon as in Fig. 8.

# 5 Homogenization of ferromagnetic energies

We now consider non-homogeneous systems; i.e., energies with interaction coefficients $a_{ij}$ that are not invariant by a common translation of the indices. In this section we consider periodic arrangements of $a_{ij}$ of period $K$. This gives an invariance by $\varepsilon K$ translations of the energies at a discrete level, and eventually homogeneous (i.e., translation-invariant) perimeter functionals in the limit. This process (the derivation of homogeneous functionals from non-homogeneous, possibly periodic, energies) is called *homogenization*.

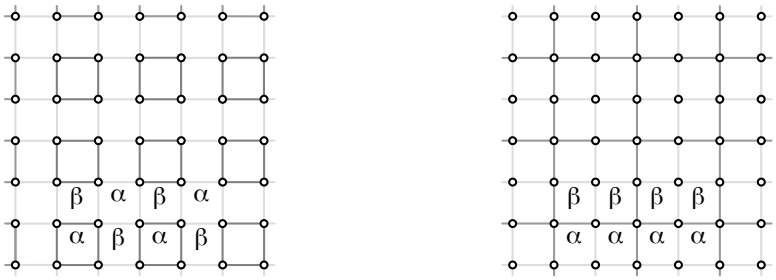


Figure 9: two non-homogeneous systems in $\mathbb{Z}^2$.

The homogenization process is already interesting when we have only nearest-neighbour interactions. In Fig. 9 we picture two two-dimensional nearest-neighbour systems with $a_{ij}$ taking only the values $\alpha$ and $\beta$. On the left-hand side picture the connections are located in series, on the right-hand side picture in parallel. While the connections are in equal number their geometrical properties are different; in particular, in the systems on the left-hand side connections with strength $\beta$ form isolated loops (if $\beta > \alpha$ such loops are sometime called *hard inclusions*). Both systems have the property of being periodic of period 2 in each direction.

We now formalize the hypothesis on the system of coefficients. Let $\Omega \subset \mathbb{R}^d$ be an open set, and $\mathcal{L}_\varepsilon = \Omega \cap \varepsilon\mathbb{Z}^d$; we consider, as in Section 4, the family of energies given by

$$E_\varepsilon(u) = \sum_{\varepsilon i, \varepsilon j \in \mathcal{L}_\varepsilon} \varepsilon^{d-1} a_{ij}(u_i - u_j)^2, \tag{15}$$

with $a_{ij}$ satisfying the following hypotheses

- (ferromagnetic interactions) $a_{ij} \geq 0$ (and symmetric);
- (coerciveness on nearest-neighbour interactions) $a_{ij} \geq c > 0$ if $\|i - j\| = 1$.

As for homogeneous energies, we make the working assumption that the range of the interactions is finite; that is,

- (finite range) there exists $R > 0$ such that $a_{ij} = 0$ if $\|i - j\| > R$.

The hypothesis of homogeneity is substituted by the following

- (periodicity) there exists a period $K \in \mathbb{N}$ such that for any $i, j \in \mathbb{Z}^d$

$$a_{ij} = a_{i+Ke_n\, j+Ke_n} \quad \text{for any} \quad n = 1, \dots, d.$$

  Note that if $K = 1$ we have homogeneity.

As for the homogeneous energies, the hypothesis of coerciveness of the nearest-neighbour interactions gives the compactness of sequences with equibounded energies. This condition and the finite-range assumption ensure that also in this case the domain of the $\Gamma$-limit (which exists up to subsequences) is given by sets of finite perimeter. Moreover, the $\Gamma$-limit is bounded from above and from below by (multiples of) the perimeter, so that we can conjecture that it is concentrated on the reduced boundary of $A$.

More precisely, we face the problem whether there exists a homogeneous limit energy density $\varphi_{\rm hom}$ such that

$$E_\varepsilon \xrightarrow{\Gamma} \int_{\partial^* A \cap \Omega} \varphi_{\rm hom}(\nu)\, d\mathcal{H}^{d-1} \quad \text{as} \quad \varepsilon \to 0$$

if the hypothesis of homogeneity of the coefficients $a_{ij}$ is replaced by the hypothesis of periodicity.

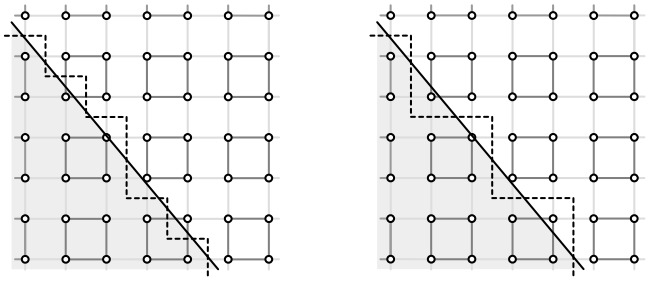

Figure 10: discretization of a half-space and a corresponding recovery sequence.

The arguments used in the proof of the $\Gamma$-convergence result for homogeneous networks do not work for non-homogeneous energies. This can be checked in the systems of interactions represented in Fig. 9, where recovery sequences follow least-energy paths. While in the right-hand side example the $\Gamma$-limit is a 1-crystalline perimeter with coefficient $(\alpha+\beta)/2$, corresponding to the trivial recovery sequences obtained by discretizing the target set, the hard-inclusion system on the left-hand side leads to a 1-crystalline perimeter with coefficient $\alpha$ (if $\alpha < \beta$), with recovery sequences avoiding the stronger $\beta$-connections (see Fig. 10).

To prove a $\Gamma$-convergence result for energies (15) we use the so-called *blow-up method* introduced by Fonseca and Müller. In general, this method is used to prove a lower-bound inequality for energies $F_\varepsilon$, showing that if $u_\varepsilon \to u$ then

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq F(u) = \int_\Omega f(x)\, d\lambda,$$

for some $f$ that can be characterized in terms of local quantities depending on $u$ only (e.g., $u(x)$ or $\nabla u(x)$, etc.). The idea is to interpret $F_\varepsilon(u_\varepsilon)$ as a sequence of (total variations of) equibounded measures (the sequence $u_\varepsilon$ being fixed); that is, $F_\varepsilon(u_\varepsilon) = \lambda_\varepsilon(\Omega)$. Since $\lambda_\varepsilon(\Omega) \leq C$, we get that up to subsequences $\lambda_\varepsilon \overset{*}{\rightharpoonup} \mu$. Then, to prove the lower estimate it is sufficient to show that

$$\frac{d\mu}{d\lambda}(x) \geq f(x) \quad \text{for} \quad \lambda\text{-almost all} \quad x \in \Omega$$

and then integrate with respect to the measure $\lambda$.

Now we specialize this abstract method to the case of the discrete energies $E_\varepsilon$, where the measure $\lambda$ is $\mathcal{H}^{d-1}$ restricted to the reduced boundary of the limit $A$ of $u_\varepsilon$, and $f(x)$ is $\varphi_{\hom}(\nu(x))$.

**A lower bound by blow-up**

Let $\{u_\varepsilon\}$ be a sequence with equibounded energy, and $u_\varepsilon \to A$ as $\varepsilon \to 0$. We define the sequence of measures $\{\mu_\varepsilon\}$ given by

$$\mu_\varepsilon = \sum_{\varepsilon i\in\Omega} \Big( \sum_{\varepsilon j\in\Omega} \varepsilon^{d-1} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 \Big) \delta_{\varepsilon i},$$

where $\delta_x$ is the Dirac measure concentrated at $x$; that is, for a Borel set $B$

$$\mu_\varepsilon(B) = \sum_{\varepsilon i\in B} \sum_{\varepsilon j\in\Omega} \varepsilon^{d-1} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2.$$

The value $\mu_\varepsilon(B)$ takes into account of interactions between nodes in $B$ and nodes in the whole $\Omega$, but, due to the finite-range hypothesis indeed the latter can be limited to an $\varepsilon R$-neighbourhood of $B$.

Since $\mu_\varepsilon(\Omega) = E_\varepsilon(u_\varepsilon)$, the measures are equibounded and up to subsequences we can assume

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu.$$

We want to show that there exists a function $\varphi_{\hom}$ such that

$$\frac{d\mu}{d\mathcal{H}^{d-1} \llcorner \partial^* A}(x_0) \geq \varphi_{\hom}(\nu(x_0)) \quad \text{for } \mathcal{H}^{d-1}\text{-almost every } x_0 \in \partial^* A. \tag{16}$$

Indeed, if (16) holds for some $\varphi_{\hom}$ (independent of the subsequence and of $\Omega$) we obtain the lower bound for the $\Gamma$-limit, since

$$\begin{aligned}\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon\to 0} \mu_\varepsilon(\Omega) \geq \mu(\Omega) \geq \int_{\partial^* A\cap\Omega} \frac{d\mu}{d\mathcal{H}^{d-1}\llcorner \partial^* A}(x_0)\, d\mathcal{H}^{d-1} \\ &\geq \int_{\partial^* A\cap\Omega} \varphi_{\hom}(\nu(x_0))\, d\mathcal{H}^{d-1}.\end{aligned}$$

Recalling Remark 5, we have that for almost all $x_0 \in \partial^* A$ the blown-up sets converge to the half space $\Pi^\nu = \{x \in \mathbb{R}^d : \langle x, \nu\rangle \geq 0\}$; that is,

$$\frac{1}{\varrho}(A - x_0) \to \Pi^\nu \quad \text{for } \mathcal{H}^{d-1}\text{-almost every } x_0 \in \partial^* A. \tag{17}$$

We restrict to points $x_0 \in \partial^* A$ such that the measure-theoretical derivative of the limit measure $\mu$ with respect to $\mathcal{H}^{d-1}\llcorner \partial^* A$ exists, and (17) holds. To obtain a lower estimate of the derivative of the measure $\mu$, we note that we can write

$$\frac{d\mu}{d\mathcal{H}^{d-1}\llcorner \partial^* A}(x_0) = \lim_{\varrho\to 0} \frac{\mu(Q^\nu_\varrho(x_0))}{\varrho^{d-1}},$$

where $Q^\nu_\varrho(x_0)$ is a $d$ dimensional cube with center $x_0$, a face orthogonal to $\nu$ and side length $\varrho$. For any $\varrho > 0$ such that $\mu(\partial Q^\nu_\varrho(x_0)) = 0$ (hence, for all $\varrho$ except a countable number) we have that $\mu_\varepsilon(Q^\nu_\varrho(x_0)) \to \mu(Q^\nu_\varrho(x_0))$, which implies

$$\frac{d\mu}{d\mathcal{H}^{d-1}\llcorner \partial^* A}(x_0) = \lim_{\varrho\to 0}\lim_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q^\nu_\varrho(x_0))}{\varrho^{d-1}}.$$

Hence, we can choose a sequence $\varrho_\varepsilon \to 0$ as $\varepsilon \to 0$ such that $\varrho_\varepsilon >> \varepsilon$, and

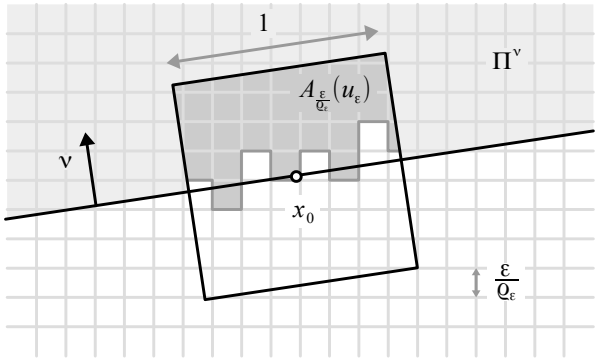


Figure 11: scaling to a cube of side-length 1.

$$\frac{d\mu}{d\mathcal{H}^{d-1} \llcorner \partial^* A}(x_0) = \lim_{\varrho \to 0} \frac{\mu(Q^\nu_\varrho(x_0))}{\varrho^{d-1}} = \lim_{\varepsilon \to 0} \frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon}(x_0))}{\varrho_\varepsilon^{d-1}}.$$

The problem is now to estimate $\frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon}(x_0))}{\varrho_\varepsilon^{d-1}}$ by using the definition of $E_\varepsilon$ (up to now, we have only used the fact that the measures $\mu_\varepsilon$ are equibounded). The idea is to obtain a lower bound by minimizing the effect of the sequence $(u_\varepsilon)$ with the given condition of being "close to a hyperplane orthogonal to $\nu$". We start by re-scaling the cube $Q^\nu_{\varrho_\varepsilon}(x_0)$ to a cube with side length 1 (see Fig. 11). From now on, we make some simplifying assumptions (not restrictive): we suppose that $x_0 = 0$ and $\nu = e_d$, and set $Q^\nu_\varrho = Q^\nu_\varrho(0)$. Since the coefficients $a_{ij}$ are positive, for the lower bound we can consider only the interactions between points inside the cube, and we obtain

$$\frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon})}{\varrho_\varepsilon^{d-1}} \quad \geq \quad \sum_{\frac{\varepsilon}{\varrho_\varepsilon} i, \frac{\varepsilon}{\varrho_\varepsilon} j \in Q^\nu_1} \frac{\varepsilon^{d-1}}{\varrho_\varepsilon^{d-1}} a_{ij} (u^\varepsilon_i - u^\varepsilon_j)^2.$$

We make now another simplification, supposing that we have only nearest-neighbour interactions; that is, $a_{ij} = 0$ if $\|i - j\| > 1$. This assumption allows to represent easier the interactions, which for nearest neighbour correspond to interfaces on the reference lattice. Note that we can generalize the proof to the case of finite-range interactions with some additional technical arguments.

Since giving a boundary condition is easier to handle than that of "being close to $\Pi^\nu$", we show that we can modify the sequence $\{u_\varepsilon\}$ near the boundary of the cube $Q^\nu_1$ without essentially modifying the energies.

**Lemma 22** (variation of boundary data)**.** *There exists $\tilde{u}_\varepsilon$ such that $A_{\frac{\varepsilon}{\varrho_\varepsilon}}(\tilde{u}_\varepsilon) \cap \frac{\varepsilon}{\varrho_\varepsilon}\mathbb{Z}^d = \Pi^\nu \cap \frac{\varepsilon}{\varrho_\varepsilon}\mathbb{Z}^d$ close to $\partial Q^\nu_1$, and the corresponding energy is not larger than the energy of $u_\varepsilon$ up to a term which goes to 0 as $\varepsilon \to 0$.*

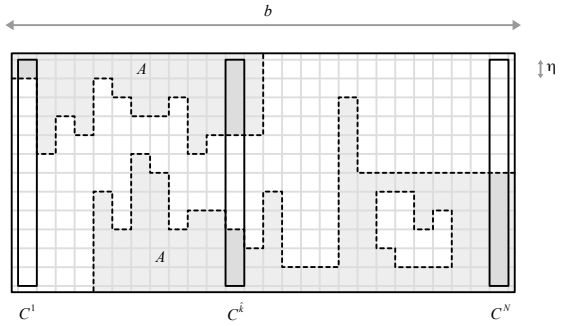


Figure 12: an illustration of Remark 23.

**Remark 23** (a discrete coarea argument)**.** The proof of Lemma 22 relies on a discrete coarea argument, which we briefly describe in a simple case in dimension $d = 2$. Let $R \subset \mathbb{R}^2$ be a coordinate rectangle and $b$ denote the length of the basis. Let $A$ be a subset of $R$ given by the union of coordinate squares with side length $\eta$ and centered at points of $\eta\mathbb{Z}^2$. We consider the columns of squares in $A$ such that the centers have the same first coordinate, and indicate them by $C^1, \dots, C^N$ with $N = \lfloor \frac{b}{\eta} \rfloor - 1$ (see Fig. 12).

For any $k = 1, \dots, N$ we have that

$$\mathcal{H}^1(\partial C^k) \leq \#(\eta\mathbb{Z}^2 \cap C^k) 4\eta \leq \frac{4|C^k|}{\eta}.$$

Since $\sum_{k=1}^N |C^k| \leq |A|$, there exists an index $\hat{k}$ such that $|C^{\hat{k}}| \leq \frac{|A|}{N}$, hence

$$\mathcal{H}^1(\partial C^{\hat{k}}) \leq C\frac{|A|}{b}.$$

*Proof of Lemma* 22. We set $\eta_\varepsilon = \frac{\varepsilon}{\varrho_\varepsilon}$. Let $A_\varepsilon := A_{\eta_\varepsilon}(u_\varepsilon)$, and $\Pi^\nu_\varepsilon$ denote the set given by the union of cubes $Q^\nu_{\eta_\varepsilon}(x)$ with $x \in \eta_\varepsilon\mathbb{Z}^d \cap \Pi^\nu$. We define the set $\tilde{A}_\varepsilon$ (and the corresponding discrete function $\tilde{u}_\varepsilon$) by setting

$$\tilde{A}_\varepsilon = \begin{cases} A_\varepsilon & \text{in } \ Q^\nu_{(2k+1)\eta_\varepsilon} \\ \Pi^\nu_\varepsilon & \text{otherwise in } \ Q^\nu_1 \end{cases}$$

where $k \in \mathbb{N}$ is such that $\frac{1-2\delta}{2\eta_\varepsilon} - \frac{1}{2} < k < \frac{1}{2\eta_\varepsilon} - \frac{1}{2}$ and $\delta \in (0, \frac{1}{4})$.

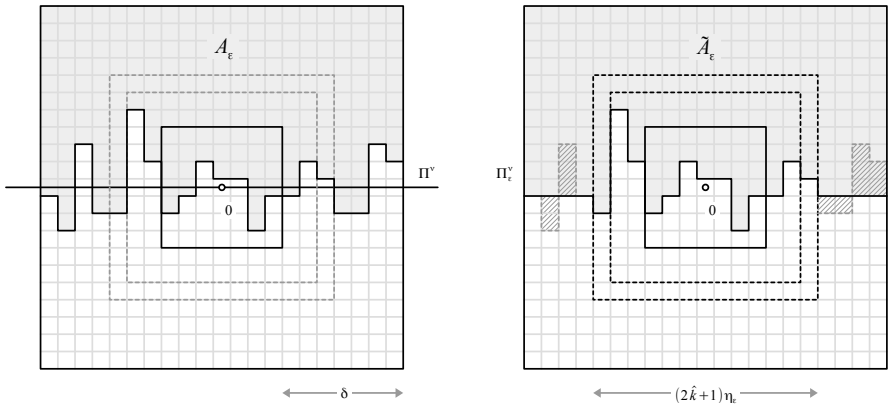


Figure 13: modification of $A_\varepsilon$ close to the boundary of $Q^\nu_1$.

We estimate the energy $E_\varepsilon(\tilde{u}_\varepsilon)$ inside the cube $Q_1^\nu$, obtaining

$$\sum_{\eta_\varepsilon i,\eta_\varepsilon j\in Q_1^\nu}\eta_\varepsilon^{d-1}a_{ij}(\tilde{u}_i^\varepsilon-\tilde{u}_j^\varepsilon)^2 \;\leq\; \sum_{\eta_\varepsilon i,\eta_\varepsilon j\in Q_1^\nu}\eta_\varepsilon^{d-1}a_{ij}(u_i^\varepsilon-u_j^\varepsilon)^2+C\delta+r(\varepsilon,k)$$

where $C\delta$ estimates the contribution of the boundary of $\Pi^\nu$ in $Q_1^\nu\setminus Q_{1-2\delta}^\nu$ and $r(\varepsilon,k)$ is the contribution of the additional boundary of $\tilde{A}_\varepsilon$ in $\partial Q^\nu_{(2k+1)\eta_\varepsilon}$ (see Fig. 13). To prove that we can choose $k$ such that this term is infinitesimal as $\varepsilon\to 0$, we use a discrete coarea argument as described in Remark 23 with $d=2$. Setting

$$C^k=(Q^\nu_{(2k+1)\eta_\varepsilon}\setminus Q^\nu_{(2k-1)\eta_\varepsilon})\cap A_\varepsilon$$

for all admissible $k$, which are less than $\frac{\delta}{\eta_\varepsilon}$, we have that there exists $\hat{k}$ such that

$$|C^{\hat{k}}|\leq\frac{\eta_\varepsilon}{\delta}|A_\varepsilon\triangle\Pi_\varepsilon^\nu|.$$

As in Remark 23 applied to $A_\varepsilon\triangle\Pi_\varepsilon^\nu$, we deduce that

$$\mathcal{H}^{d-1}\big(A_\varepsilon\cap\partial Q^\nu_{(2\hat{k}+1)\eta_\varepsilon}\big)\leq C\#(\eta_\varepsilon\mathbb{Z}^d\cap C^{\hat{k}})\eta_\varepsilon^{d-1}\leq C\frac{|C^{\hat{k}}|}{\eta_\varepsilon}\leq\frac{C}{\delta}|A_\varepsilon\triangle\Pi_\varepsilon^\nu|,$$

where $C$ is a positive constant depending on $d$. Hence, by choosing $k=\hat{k}$, we obtain

$$\sum_{\eta_\varepsilon i,\eta_\varepsilon j\in Q_1^\nu}\eta_\varepsilon^{d-1}a_{ij}(\tilde{u}_i^\varepsilon-\tilde{u}_j^\varepsilon)^2 \;\leq\; \sum_{\eta_\varepsilon i,\eta_\varepsilon j\in Q_1^\nu}\eta_\varepsilon^{d-1}a_{ij}(u_i^\varepsilon-u_j^\varepsilon)^2+C\delta+\frac{C}{\delta}|A_\varepsilon\triangle\Pi_\varepsilon^\nu|;$$

since $|A_\varepsilon\cap\Pi_\varepsilon^\nu|\to 0$ as $\varepsilon\to 0$ and $\delta$ is arbitrarily small, we get

$$\liminf_{\varepsilon\to 0}\frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon})}{\varrho_\varepsilon^{d-1}}\geq\liminf_{\varepsilon\to 0}\sum_{\frac{\varepsilon}{\varrho_\varepsilon}i,\frac{\varepsilon}{\varrho_\varepsilon}j\in Q_1^\nu}\Big(\frac{\varepsilon}{\varrho_\varepsilon}\Big)^{d-1}a_{ij}(\tilde{u}_i^\varepsilon-\tilde{u}_j^\varepsilon)^2. \tag{18}$$

concluding the proof. □

**Remark 24** (finite-range interactions)**.** If we consider the general case of interactions beyond the nearest neighbours, the proof of Lemma 22 can be repeated with some modifications. Indeed, in the estimate of the contribution of the interactions due to additional interfaces, we can apply a variant of Remark 23 considering the sum of the interfaces on the boundary of a finite number of consecutive cubic annuli, instead of only one. Note moreover that in the proof we can choose $\hat{k}$ greater than a fixed $k_0>R$, where $R$ is the range of the interactions. Hence the boundary data are fixed in a $\eta_\varepsilon R$-neighbourhood of the boundary of the cube $Q_1^\nu$.

Now, thanks to (18) and to Remark 24, we can estimate (up to an infinitesimal term as $\varepsilon \to 0$)

$$\frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon})}{\varrho_\varepsilon^{d-1}} \geq \min\Big\{ \sum_{\frac{\varepsilon}{\varrho_\varepsilon} i, \frac{\varepsilon}{\varrho_\varepsilon} j \in Q_1^\nu} \Big(\frac{\varepsilon}{\varrho_\varepsilon}\Big)^{d-1} a_{ij}(v_i - v_j)^2 : \ v = -1 + 2\chi_{\Pi^\nu} \text{ in } \ Q_1^\nu \setminus Q^\nu_{1-2\frac{\varepsilon}{\varrho_\varepsilon}R}$$
$$v\colon \frac{\varepsilon}{\varrho_\varepsilon}\mathbb{Z}^d \cap Q_1^\nu \to \{-1,+1\}\Big\}$$

where $R$ is the range of the interactions. Note that the minimum problem above in principle depends on the choice of the point $x_0$, which we have supposed for simplicity to be 0. By the perioditicity, the problems are invariant by translations in $\varepsilon K\mathbb{Z}^d$. We now set

$$\begin{aligned}\varphi_{\rm hom}(\nu) &= \liminf_{\eta\to 0} \min\Big\{ \sum_{\eta i, \eta j \in Q_1^\nu} \eta^{d-1} a_{ij}(v_i - v_j)^2 : \ v = -1 + 2\chi_{\Pi^\nu} \text{ in } \ Q_1^\nu \setminus Q^\nu_{1-2\eta R},\\ &\qquad v\colon \eta\mathbb{Z}^d \cap Q_1^\nu \to \{-1,+1\}\Big\}\\ &= \liminf_{T\to+\infty} \frac{1}{T^{d-1}} \min\Big\{ \sum_{i,j\in Q_T^\nu} a_{ij}(v_i - v_j)^2 : \ v = -1 + 2\chi_{\Pi^\nu} \text{ in } \ Q_T^\nu \setminus Q^\nu_{T-2R},\\ &\qquad v\colon \mathbb{Z}^d \cap Q_T^\nu \to \{-1,+1\}\Big\}\end{aligned}$$

where we have re-scaled by setting $T = \frac{1}{\eta}$. By the invariance by $\varepsilon K\mathbb{Z}^d$ translations, we can replace the cube $Q_T^\nu$ by any cube $Q_T^\nu(x_T)$ for arbitrary $x_T \in \mathbb{R}^d$ up to translating also the boundary conditions. The definition of $\varphi_{\rm hom}$, corresponding to a minimization over all oscillations close to a flat interface orthogonal to $\nu$, does not depend on the subsequence and on the choice of the center of the cube, and depends only on the normal $\nu$.

We then have a lower bound for the energies $E_\varepsilon$

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq \int_{\Omega\cap\partial^* A} \varphi_{\rm hom}(\nu)\, d\mathcal{H}^{d-1}. \tag{19}$$

**The asymptotic homogenization formula**

We now prove the existence of the limit in the definition of $\varphi_{\rm hom}$; that is,

$$\begin{aligned}\varphi_{\rm hom}(\nu) &= \lim_{T\to+\infty} \frac{1}{T^{d-1}} \min\Big\{ \sum_{i,j\in Q_T^\nu} a_{ij}(v_i - v_j)^2 : \ v = -1 + 2\chi_{\Pi^\nu} \text{ in } \ Q_T^\nu \setminus Q^\nu_{T-2R},\\ &\qquad v\colon \mathbb{Z}^d \cap Q_T^\nu \to \{-1,+1\}\Big\}.\end{aligned} \tag{20}$$

To that end, we define

$$\begin{aligned}g(T,\nu) &= \min\Big\{ \sum_{i,j\in Q_T^\nu} a_{ij}(v_i - v_j)^2 : \ v = -1 + 2\chi_{\Pi^\nu} \text{ in } \ Q_T^\nu \setminus Q^\nu_{T-2R},\\ &\qquad v\colon \mathbb{Z}^d \cap Q_T^\nu \to \{-1,+1\}\Big\}\end{aligned}$$

and show that there exists the limit

$$\lim_{T\to+\infty} \frac{g(T,\nu)}{T^{d-1}}.$$

To prove the existence of the limit we use a *subadditivity argument*. Given an optimal set $A_T$ (corresponding to a minimizer $v^T$) in the cube $Q_T^\nu$, we want to construct an almost optimal set in a larger cube $Q_S^\nu$, with $S >> T$.

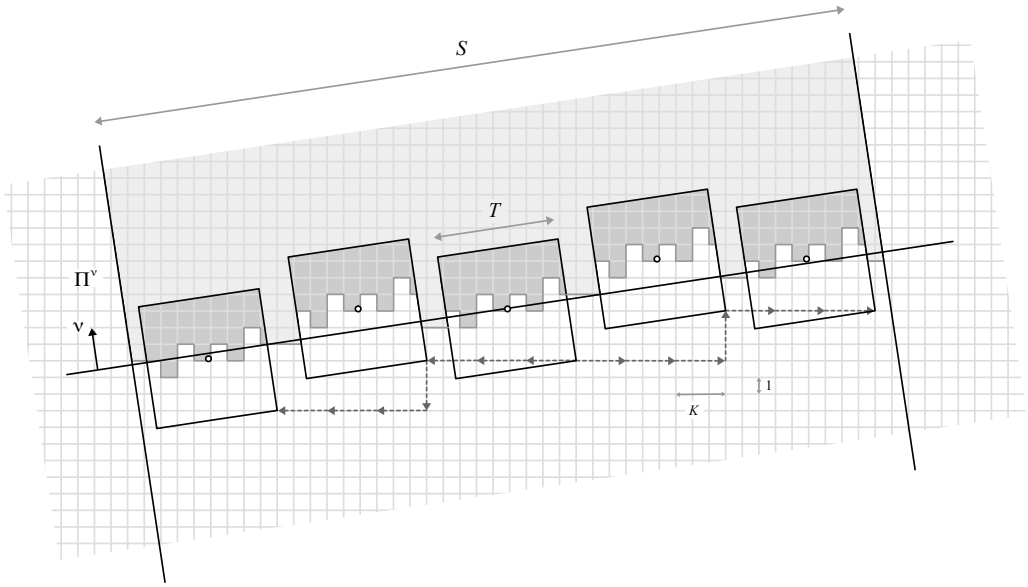


Figure 14: a pictorial proof of the homogenization formula.

Note that we have translational invariance of the optimal set if we move the cube $Q_T^\nu$ by any vector $Kw$, where $K$ is the period and $w \in \mathbb{Z}^d$. The elements of $K\mathbb{Z}^d$ are referred to as admissible translations. The translation moving $Q_T^\nu$ to an adjacent cube sharing a $d-1$ face and centered on the hyperplane orthogonal to $\nu$ is in general not admissible for arbitrary $\nu$ and $T$. However, we can find an admissible translation such that the distance of the center of the translated cube from the hyperplane $\partial\Pi^\nu$ is of order $K$ and the distance between the centers of the cubes is of order $T+K$, as pictured (for $d=2$) in Fig. 14. Then the distance between the cubes is uniformly bounded by a constant $C$, and the same holds for the distance of the centers from $\partial\Pi^\nu$. We repeat this construction by "invading" $\partial\Pi^\nu \cap Q_S^\nu$ with non overlapping cubes until the distance of the translated cubes from the boundary of $Q_S^\nu$ is less than $2T$, and define a set given by the translation of $A_T$ in each of these cubes, and by the discretization of the half-space $\Pi^\nu$ otherwise in $Q_S^\nu$. Hence, since the number of the cubes is less than $(\frac{S}{T})^{d-1}$ we have

$$\frac{g(S,\nu)}{S^{d-1}} \le \frac{1}{S^{d-1}}\Big(\frac{S^{d-1}}{T^{d-1}}g(T,\nu) + \frac{S^{d-1}}{T^{d-1}}CT^{d-2} + (S^{d-1} - (S-2T)^{d-1})\Big),$$

where the term $CT^{d-2}$ estimates the additional interface close to each translated cube with side length $T$, and the last term in the sum estimates the additional interface close to the boundary of $Q_S^\nu$. By taking the lim sup as $S \to +\infty$ we obtain

$$\limsup_{S\to+\infty} \frac{g(S,\nu)}{S^{d-1}} \le \frac{g(T,\nu)}{T^{d-1}} + \frac{C}{T}$$

since $S^{d-1} - (S-2T)^{d-1} = o(S^{d-1})_{S\to+\infty}$; now, taking the lim inf as $T \to +\infty$, it follows that

$$\limsup_{S\to+\infty} \frac{g(S,\nu)}{S^{d-1}} \le \liminf_{T\to+\infty} \frac{g(T,\nu)}{T^{d-1}}$$

which implies that the lim inf in the definition of $\varphi_{\rm hom}$ is in fact a limit.

**Upper bound**

We note that the construction of the test set in $Q_S^\nu$ used to prove the existence of the limit is in fact the construction of a recovery sequence for the $\Gamma$-limit of $E_\varepsilon$ for a half-space. Indeed, let $v^T$ denote the minimizer of the energies in the cube $Q_T^\nu$ with $v^T = -1 + 2\chi_{\Pi^\nu}$ close to the boundary of $Q_T^\nu$; that is,

$$g(T,\nu) = \sum_{i,j\in Q_\nu^T} a_{ij}(v_i^T - v_j^T)^2.$$

We indicate by $\tilde v^T$ the function constructed by translating the cube $Q_T^\nu$, as in the proof of the existence of the limit, to "almost invade" the whole hyperplane $\partial\Pi^\nu$, which corresponds to letting $S$ go to $+\infty$. If $A$ is a polytope, we localize the construction of the recovery sequence by considering $A$ as a portion of a half-space with boundary orthogonal to $\nu$ (we leave to the reader the details of the construction for a general polytope, noting that close to the boundary of each face this construction can not be performed, and the recovery sequence can be simply taken as the discretization of the polytope itself). We define $u^\varepsilon$ by setting $u_i^\varepsilon = u^\varepsilon(\varepsilon i) = \tilde v_i^T$. Then

$$\begin{aligned}
\limsup_{\varepsilon\to0} E_\varepsilon(u^\varepsilon) &= \limsup_{\varepsilon\to0} \sum_{\varepsilon i,\varepsilon j\in\mathcal{L}_\varepsilon} \varepsilon^{d-1} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 \\
&\le \limsup_{\varepsilon\to0} \varepsilon^{d-1} \frac{\mathcal{H}^{d-1}(\Omega\cap\partial A)}{\varepsilon^{d-1}T^{d-1}} (g(T,\nu) + CT^{d-2}) \\
&\le \mathcal{H}^{d-1}(\Omega\cap\partial A)\varphi_{\rm hom}(\nu) + o(1)_{T\to+\infty},
\end{aligned}$$

which shows that $u^\varepsilon$ is a recovery sequence up to a term arbitrarily small. As we noticed in the definition of $\Gamma$-convergence, this is sufficient to give an upper bound for the $\Gamma$-limit.

We can use again the density of polyhedral sets to conclude that for any $A$ with finite perimeter

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E_\varepsilon(A) \le \int_{\Omega\cap\partial^* A} \varphi_{\hom}(\nu)\, d\mathcal{H}^{d-1}.$$

This concludes the proof of the following result.

**Theorem 25** (Homogenization of periodic networks)**.** *The sequence of functionals $E_\varepsilon$ defined in* (15) *$\Gamma$-converges with respect to the convergence* (5) *to the functional $F$ defined on sets of finite perimeter by*

$$F(A) = \int_{\Omega\cap\partial^* A} \varphi_{\hom}(\nu)\, d\mathcal{H}^{d-1},$$

*where $\varphi_{\hom}$ satisfies the asymptotic formula* (20)*.*

# 6 A discrete-to-continuum localization method

We now consider the (more) general case allowing to obtain a non-homogeneous energy density in the limit.

We start by describing a one-dimensional model example. Note that for $d = 1$ sets of finite perimeter are simply (sets equivalent to) finite unions of intervals, whose endpoints are the (reduced) boundary. In this case the dependence on the normal is trivial, which makes it more convenient to use functions as parameters instead of sets, and the corresponding notation for the convergence. Given a sequence of spin functions $u_\varepsilon\colon \varepsilon\mathbb{Z} \to \{-1,+1\}$ converging to a set of finite perimeter $A$ in the sense of (5), we say that $u_\varepsilon$ converges to $u = -1 + 2\chi_A$, and instead of $\partial^* A$ we use the notation $S(u)$; that is, $S(u)$ is the set of discontinuity points of the piecewise-constant function $u$.

**Example 26** (Non-homogeneous limit energies)**.** Let $a\colon \mathbb{R} \to [0,+\infty)$ be a continuous function such that $a(x) \ge c > 0$ for any $x \in \mathbb{R}$. We fix $\Omega = \mathbb{R}$, and consider a discretization of the function $a$; that is, we consider interaction coefficients given by

$$a\Big(\frac{\varepsilon i + \varepsilon j}{2}\Big) =: a^\varepsilon_{ij}.$$

We define the nearest-neighbour energies

$$E_\varepsilon(u) = \sum_{|i-j|=1} a^\varepsilon_{ij}(u_i - u_j)^2.$$

The hypothesis $a(x) \ge c > 0$ gives compactness; hence, if $E_\varepsilon(u_\varepsilon)$ is equibounded there exists $u\colon \mathbb{R} \to \{-1,1\}$ such that $u_\varepsilon \to u$. For any $x \in S(u)$, where there is a change of sign

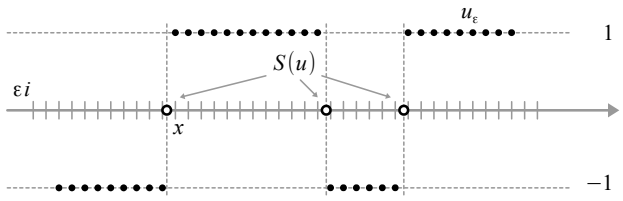


Figure 15: discretization of piecewise constant $u$.

of the limit function, for $\varepsilon$ small enough there is also a change of sign of $u_\varepsilon$ at some $i = i_\varepsilon$ with $\varepsilon i_\varepsilon \to x$, which gives a contribution to the discrete energy of $8a^\varepsilon_{i\,i+1} = 8a(x) + o(1)_{\varepsilon\to 0}$ by the continuity of $a$, as pictured in Figure 15. Hence, we have that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E_\varepsilon(u) = 8 \sum_{x\in S(u)} a(x),$$

which is a one-dimensional version of a non-homogeneous perimeter functional.

This elementary case shows that even in a simple discretization argument we have to introduce coefficients depending on $\varepsilon$.

The general case of non-homogeneous energies can be then formalized as follows. We fix $\Omega \subset \mathbb{R}^d$ and look at energies given by

$$E_\varepsilon(u) = \sum_{\varepsilon i, \varepsilon j \in \mathcal{L}_\varepsilon} \varepsilon^{d-1} a^\varepsilon_{ij} (u_i - u_j)^2 \tag{21}$$

where $\mathcal{L}_\varepsilon = \varepsilon\mathbb{Z}^d \cap \Omega$, $u\colon \mathcal{L}_\varepsilon \to \{-1, +1\}$.

The problem is now to give conditions on the coefficients $a^\varepsilon_{ij}$ such that, as in Example 26, the sequence $E_\varepsilon$ defined in (21) $\Gamma$-converge (possibly up to subsequences) to an integral functional of the form

$$F(A) = \int_{\Omega\cap\partial^* A} \varphi(x, \nu)\, d\mathcal{H}^{d-1}$$

for some $\varphi\colon \Omega \times S^{d-1} \to [0, +\infty)$.

We now formalize the hypotheses on the family of the coefficients $a^\varepsilon_{ij}$, keeping in mind the conditions already taken into account for homogeneous and for periodic systems. We consider the following assumptions:

(H1) (ferromagnetic energies) $a^\varepsilon_{ij} \geq 0$ (and the non-restrictive symmetry condition $a^\varepsilon_{ij} = a^\varepsilon_{ji}$). This ensures that ground states are the constant states $-1$ and 1;

(H2) (uniform coerciveness of nearest-neighbours interactions) there exists $c$ such that $a^\varepsilon_{ij} \geq c > 0$ if $\|i - j\| = 1$. This condition implies that the domain of the $\Gamma$-limit is included in the family of sets of finite perimeter.

As in the homogeneous and periodic case, if we want the limit to be finite on sets of finite perimeter, we have to (locally) test the energies on half-spaces and require a bound, which in this case has to be uniform. We assume that

(H3) (finiteness condition) there exists $c > 0$ such that for any $\varepsilon > 0$ and for any $i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega$

$$\sum_{j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega} a_{ij}^{\varepsilon} \|i - j\| \leq \frac{1}{c}.$$

By using the trivial recovery sequence, we get that the $\Gamma$-limit is finite on the intersection of half-spaces and bounded sets, hence on polyhedral sets, and then by density on sets of finite perimeter.

Hypothesis (H3) is not sufficient to ensure that the limit be local, as shown by the following example.

**Example 27** (A non-local $\Gamma$-limit)**.** Let $d = 1$, $\Omega = \mathbb{R}$ and the family $a_{ij}^{\varepsilon}$ be defined by

$$a_{ij}^{\varepsilon} = \begin{cases} 1 & \text{if } |i-j| = 1 \\ \varepsilon & \text{if } |i-j| = \lfloor \frac{1}{\varepsilon} \rfloor \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

By separating the interactions between points at distance $\varepsilon$ and points at distance of order 1, we write the energy as

$$E_{\varepsilon}(u) = \sum_{|i-j|=1} (u_i - u_j)^2 + 2\varepsilon \sum_{i \in \mathbb{Z}} \Big( u\Big(\varepsilon i + \varepsilon \Big\lfloor \frac{1}{\varepsilon} \Big\rfloor\Big) - u(\varepsilon i) \Big)^2.$$

Note that the second sum can be interpreted as the integral of the piecewise-constant interpolations of $u$, and it continuously converges to the (continuum) functional

$$2 \int_{\mathbb{R}} (u(x+1) - u(x))^2 \, dx.$$

Since $\Gamma$-convergence is stable under continuously converging perturbations, the $\Gamma$-limit is given by

$$F(u) = 8 \# S(u) + 2 \int_{\mathbb{R}} (u(x+1) - u(x))^2 \, dx,$$

which is the sum of a perimeter functional and a non-local term. This example implies that condition (H3) is not sufficient to ensure that the limit is local, since the finiteness assumption is satisfied by the coefficients $a_{ij}^{\varepsilon}$ defined in (22). Indeed, for any $\varepsilon > 0$ and $i \in \mathbb{Z}$,

$$\sum_{j \in \mathbb{Z}} a_{ij}^{\varepsilon} |i - j| = 2 + 2\varepsilon \Big\lfloor \frac{1}{\varepsilon} \Big\rfloor \leq 4.$$

To ensure the locality of the limit, we have to require that the "tails" of the sums of interactions are uniformly small. We assume:

(H4) (locality) for any $\delta$ there exist $R_\delta > 0$ and $\varepsilon_\delta > 0$ such that for any $i \in \mathbb{Z} \cap \frac{1}{\varepsilon}\Omega$ and for any $0 < \varepsilon < \varepsilon_\delta$

$$\sum_{\{j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega : \|i-j\| \geq R_\delta\}} a_{ij}^\varepsilon \|i-j\| < \delta.$$

Note that in Example 27 this assumption is not satisfied; indeed, the term $\varepsilon \lfloor \frac{1}{\varepsilon} \rfloor$ is always of order 1; more precisely, for any $R > 1$ we have $\sum_{\{j : \|i-j\| \geq R\}} a_{ij}^\varepsilon \|i-j\| \geq 1 - \varepsilon$ for any $\varepsilon$ such that $\lfloor \frac{1}{\varepsilon} \rfloor \geq R$.

**Theorem 28** (Compactness and integral representation)**.** *Let $\Omega$ be a Lipschitz subset of $\mathbb{R}^d$. If* (H1)–(H4) *hold, then for any $\varepsilon_j \to 0$ there exists a subsequence $\varepsilon_{j_k}$ and a Borel function $\varphi \colon \Omega \times S^{d-1} \to [0, +\infty)$ such that $E_{\varepsilon_{j_k}}$ $\Gamma$-converge to the functional $F$ given by*

$$F(A) = \int_{\Omega \cap \partial^* A} \varphi(x, \nu)\, d\mathcal{H}^{d-1}.$$

We are not going to give the proof of this theorem in all the details, which can be achieved by using the localization method of $\Gamma$-convergence. Before giving a hint of the proof we make a comparison with an analog result on Sobolev spaces.

**Remark 29** (Comparison with the compactness and representation result for integral functionals)**.** We consider functionals defined by

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(x, \nabla u)\, dx$$

in the Sobolev space $W^{1,p}(\Omega)$, with $f_\varepsilon$ Borel functions. A general condition on the family $f_\varepsilon$ to have a compactness and integral representation theorem are the following

$$c|\xi|^p - \frac{1}{c} \leq f_\varepsilon(x, \xi) \leq \frac{1}{c}(|\xi|^p + 1), \tag{23}$$

which we can compare with the hypotheses of Theorem 28. The bound (with a constant) from below is equivalent to the positiveness assumption (H1) on the coefficients $a_{ij}^\varepsilon$, and the $p$-growth condition from below in the gradient variable, ensuring the coerciveness in $W^{1,p}(\Omega)$, corresponds to hypothesis (H2). The estimate from above with the gradient variable is a locality condition as (H4), and the $p$-growth assumption from above ensures that the limit is finite on $W^{1,p}(\Omega)$, which corresponds in our case to the finiteness condition (H3). If (23) holds, then (up to subsequences) the functionals $F_\varepsilon$ $\Gamma$-converge to an integral functional of the form

$$F(u) = \int_\Omega f(x, \nabla u)\, dx$$

for some $f$.

Note that in the Sobolev case it can be proven that a necessary and sufficient condition for the lower semicontinuity of the integral functional is that the function $f(x,\cdot)$ be convex for almost all $x \in \Omega$ (or quasi-convex in the vector case $u \in W^{1,p}(\Omega;\mathbb{R}^n)$). Necessary and sufficient conditions on the limit density $\varphi$ in Theorem 28 are more delicate since surfaces of dimension $d-1$ are involved in the energies. In order not to overburden the notes we do not give general characterizations of the energy density $\varphi$, which anyhow will not be used in the sequel.

*Outline of the proof of Theorem* 28. The proof follows a localization method introduced by De Giorgi, adapted to the discrete-to-continuum framework. The idea is to introduce in the energies the dependence on a set variable; that is, for all $U \subset \Omega$ we define

$$E_\varepsilon(u;U) = \sum_{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^d \cap U} \varepsilon^{d-1} a_{ij}^\varepsilon (u_i - u_j)^2. \tag{24}$$

The steps of the *discrete-to-continuum localization method* are the following.

1. (Compactness) For any $\varepsilon_j$, there exists a (not-relabelled) subsequence such that

   $$E_{\varepsilon_j}(\cdot\,;U) \xrightarrow{\Gamma} F(\cdot\,;U)$$

   for any $U$ in a countable dense class $\mathcal{U}$ of open Lipschitz subsets of $\Omega$. Since the family of the sets $U$ is countable, this first step just relies on the compactness of the $\Gamma$-convergence on separable metric spaces and on a diagonal argument.

2. (Inner regularity and measure criterion) For any $A$ set of finite perimeter in $\Omega$, we define a set-function $\mu_A$ on the open sets of $\Omega$ by inner approximation with sets in $\mathcal{U}$, setting

   $$\mu_A(U) = \sup\{F(A;V) : V \subset\subset U,\ V \in \mathcal{U}\}.$$

   We prove that the set-function $\mu_A$ is the restriction of a finite Borel measure to the family of open sets of $\Omega$ by an application of the De Giorgi-Letta Criterion for measures. Note that condition (H4) in particular ensures the additivity of $\mu_A$ on sets with positive distance.

3. (Representation) Since $\mu_A$ is a measure, by using localization and density arguments based on the properties of sets finite perimeter, we prove that there exists $\varphi$ such that for any $A$ of finite perimeter

   $$\mu_A(U) = \int_{U \cap \partial^* A} \varphi(x,\nu)\, d\mathcal{H}^{d-1}.$$

   Moreover, by inner regularity on Lipschitz sets

   $$\mu_A(U) = F(A;U)$$

   for any open Lipschitz subset of $\Omega$, and this concludes the proof by choosing $U = \Omega$.

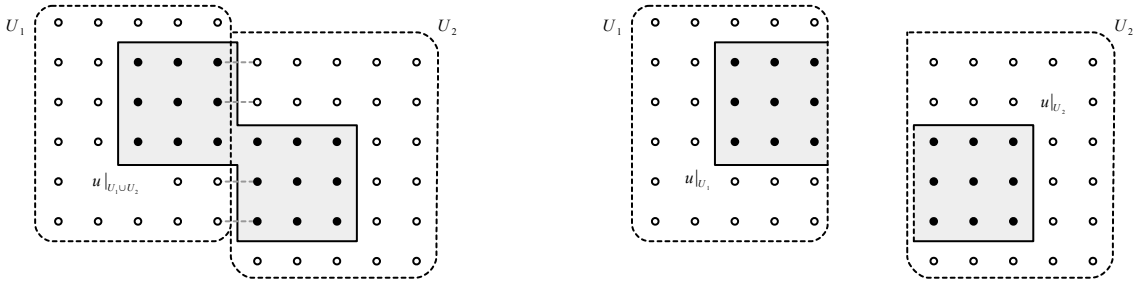


Figure 16: "non-subadditive" interactions through the boundary.

The method is the same as the one used to prove the compactness and representation theorem in the case of Sobolev functions, with some difference in the technical details. We note only that while for integral functionals the subadditivity is immediate, here the functionals $E_\varepsilon(u,\cdot)$ defined in (24) are not subadditive. Indeed, if we consider disjoint $U_1, U_2$ such that $\mathcal{H}^{d-1}(\partial^* U_1 \cap \partial^* U_2) > 0$, then the energy $E_\varepsilon(u, U_1 \cup U_2)$ may take into account the interactions across the common boundary, and $E_\varepsilon(u, U_1 \cup U_2) > E_\varepsilon(u, U_1) + E_\varepsilon(u, U_2)$ (see Fig. 16). This fact justifies the requirement for $\Omega$ to be a Lipschitz set, which allows to avoid the interactions crossing the boundary. □

After examining the role of hypotheses (H3) and (H4), in particular their necessity in order to have a local limit, we turn to hypotheses (H1) and (H2).

If (H1) does not hold, we allow $a_{ij}^\varepsilon < 0$ for some values of $i, j$. In this case, we have non-constant ground states; the minimization can favour alternating values, giving rise to "microstructures", and in general it is not possible to directly give the description of the limit in terms of sets of finite perimeter. Systems governed by energies with negative coefficients are called *antiferromagnetic systems*. We give some examples in which we can provide a limit description. Such examples do not involve $\varepsilon$-depending coefficients, so that they can be also considered as examples for the homogeneous and the periodic case.

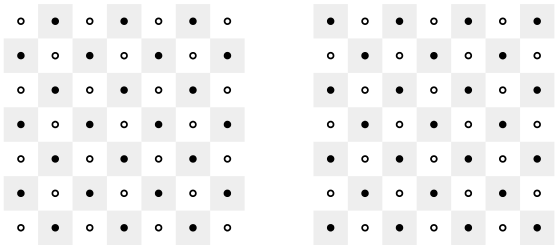

Figure 17: checkerboards with different parity.

**Example 30** (Ground states for antiferromagnetic energies in dimension $d = 2$)**.** We consider some discrete energies in the lattice $\mathbb{Z}^2$ with negative coefficients, and look at their ground states. Since we are interested in the shape of the minimizers, we can consider the non-scaled functionals with $\varepsilon = 1$.

- (Nearest neighbours) We set $a_{ij} = -1$ for any $i, j$ nearest neighbours, and 0 otherwise; that is, we consider energies $E = E_1$ given by

$$E(u) = -\sum_{\langle i,j\rangle}(u_i - u_j)^2.$$

  Minimizers alternate the values 1 and $-1$ for any pair of points at distance 1, so that we have two possible ground states given by $u_i = (-1)^{i_1+i_2}$ (where $i = (i_1, i_2)$) and by the translated checkerboard $u_i = -(-1)^{i_1+i_2}$, as pictured in Fig. 17.

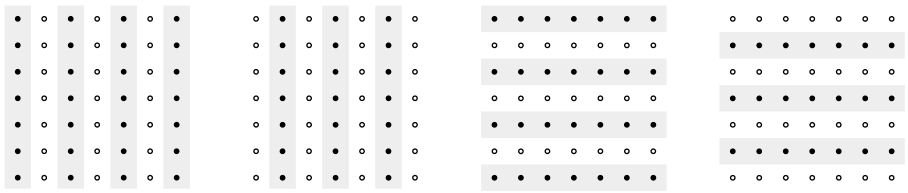

Figure 18: stripes with different orientations and parities.

- (Nearest and next-to-nearest neighbours) Given $\alpha > 0$, we set $a_{ij} = -1$ for any $i, j$ nearest neighbours, $a_{ij} = -\alpha$ if $\|i - j\| = \sqrt{2}$ and 0 otherwise; that is, we consider energies given by

$$E(u) = -\sum_{\langle i,j\rangle}(u_i - u_j)^2 - \sum_{\|i-j\|=\sqrt{2}} \alpha(u_i - u_j)^2.$$

  If $\alpha$ is large enough, the "diagonal bonds" are stronger, and the minimizers alternate the values 1 and $-1$ for any pair $(i, j)$ such that $\|i-j\| = \sqrt{2}$; hence, the corresponding minimizing sets are horizontal and vertical stripes, with two different parities each one (see Fig. 18), and we have four different ground states.

In the cases of the example above the $\Gamma$-limit can be described using the ground states themselves as parameters, with a representation in terms of partitions into sets of finite perimeter. This can be done when only a finite number of periodic ground states are present. Note that for arbitrary distributions of negative coefficients $a_{ij}$, the number of the ground states can be arbitrarily high, and their arrangements can be very complex. The representation in terms of partitions is not always possible, since some energies possess infinitely many (non-periodic) ground states as in the following example.

**Example 31** (Antiferromagnetic energy on the triangular lattice: *frustration*)**.** Let $\mathcal{L} = \mathbb{T}$ be the triangular lattice in $\mathbb{R}^2$ defined in Remark 21; that is, let $\mathbb{T}$ be the Bravais lattice given by

$$\mathbb{T} = \mathbb{Z}v_1 + \mathbb{Z}v_2$$

where $v_1 = (1, 0)$ and $v_2 = (\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3}))$. We consider the antiferromagnetic nearest-neighbour interaction given by

$$a_{ij} = \begin{cases} -1 & \text{if } \|i - j\| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In this case, any distribution of spins $u$ is forced to have in any triplet of nearest neighbours

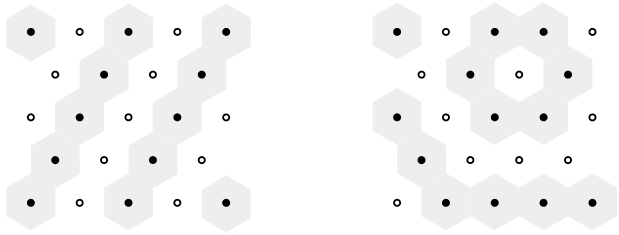

Figure 19: two minimizers in the triangular lattice.

at least a pair of sites with the same value, and hence not minimizing separately the energy. Such sites are called *frustrated* interactions. As a consequence of frustration, configurations minimizing the energy may have an almost arbitrary arrangement of $-1$ and 1 (see Fig. 19). Note that letting $\varepsilon \to 0$ any function $u$ with $u(x) \in [-\frac{1}{3}, \frac{1}{3}]$ can be obtained as a weak limit of piecewise-constant interpolations of minimal configurations.

Now we turn to the coerciveness hypothesis (H2); that is, $a^\varepsilon_{ij} \geq c > 0$ for any $i, j$ such that $\|i - j\| = 1$ and for any $\varepsilon > 0$. We consider two examples in dimension one where this condition does not hold.

**Example 32.**

- (decoupled media) Let $\Omega = \mathbb{R}$, and set $a_{ij} = 1$ if $|i - j| = 2$, and 0 otherwise; that is, the points of the lattice $\varepsilon\mathbb{Z}$ interact only with second neighbours. As pictured in

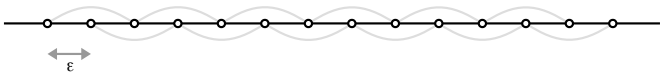


Figure 20: decoupled lattices.

Fig. 20, we have nearest-neighbour interactions on the lattice $2\varepsilon\mathbb{Z}$ (even lattice) and on the translated lattice $\varepsilon+2\varepsilon\mathbb{Z}$ (odd lattice), and they are independent. Applying the compactness argument for nearest-neighbour interactions separately on both lattices, we obtain that if a sequence $(u_\varepsilon)$ has equibounded energy then $u_\varepsilon$ converge to a pair of piecewise-constant functions $(u_{\rm e}, u_{\rm o})$, in the sense that the restrictions of $u_\varepsilon$ to the even and odd lattice converge to $u_{\rm e}$ and $u_{\rm o}$, respectively. The functionals $E_\varepsilon$ $\Gamma$-converge to

$$F(u_{\rm e}, u_{\rm o}) = 8(\#S(u_{\rm e}) + \#S(u_{\rm o}))$$

with the two functions completely independent.

- (double porosity) Let $\Omega = \mathbb{R}$, and modify the coefficients of the previous example by introducing a "weak" interaction between nearest neighbours as follows

$$a_{ij}^\varepsilon = \begin{cases} \varepsilon & \text{if } |i-j| = 1 \\ 1 & \text{if } |i-j| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that (H2) is still not satisfied, since the coefficient are strictly positive, but not uniformly strictly positive. The energies can be written as

$$\begin{aligned} E_\varepsilon(u) &= \sum_{|i-j|=2} (u_i - u_j)^2 + 2\sum_i \varepsilon(u_i - u_{i-1})^2 \\ &= \sum_{|i-j|=2} (u_i - u_j)^2 + 2\int_{\mathbb{R}} (u_{\rm e}^\varepsilon - u_{\rm o}^\varepsilon)^2\,dx, \end{aligned}$$

where $u_{\rm e}^\varepsilon$ and $u_{\rm o}^\varepsilon$ are the discretizations of $u$ on the even and odd lattice, respectively. The integral term continuously converges to $2\int_{\mathbb{R}}(u_{\rm e} - u_{\rm o})^2\,dx$; hence, the $\Gamma$-limit is

$$F(u_{\rm e}, u_{\rm o}) = 8(\#S(u_{\rm e}) + \#S(u_{\rm o})) + 2\int_{\mathbb{R}} (u_{\rm e} - u_{\rm o})^2\,dx.$$

Note that in these examples the limit depends on a vector (the pair $(u_{\rm e}, u_{\rm o})$) and not on a single function. The second example describes mixtures of media with highly different properties, the so-called *high-contrast media*. We note that similar problems can be seen also in the continuous framework, but in that case we can mimick the role of the two disjoint lattices only in dimension greater than 3, taking two unbounded disjoint connected components as a continuous equivalent of the lattices.

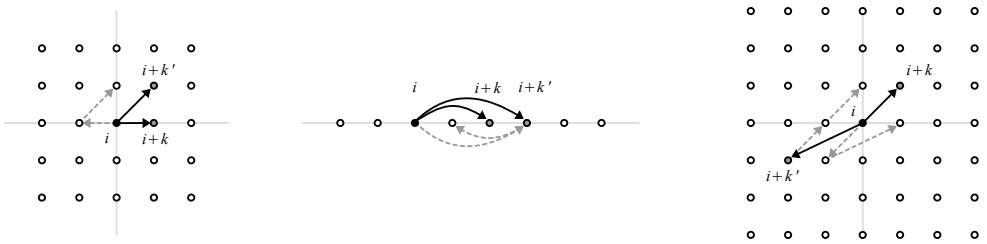


Figure 21: examples of coercive systems not satisfying (H2).

**Extensions**

*Generalization of* (H2). We can replace (H2) with the following condition

> there exists $R > 0$ such that for any $i \in \mathbb{Z}^d$
> the set $\{k \in \mathbb{Z}^d \cap B_0(R),\ a_{i\ i+k} \geq c > 0\}$ generates $\mathbb{Z}^d$ (with coefficients in $\mathbb{Z}$),

since in this case the strictly positive contribution to the energy due to nearest neighbours with $u$ of changing sign can be recovered using a cycle in the set above. In Fig. 21 some examples of this condition are pictured.

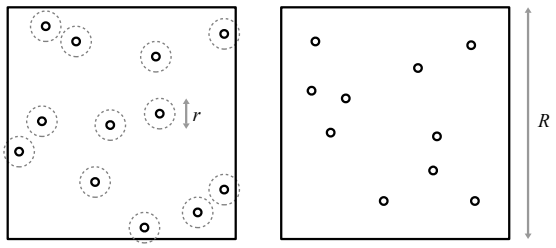


Figure 22: a "disordered" lattice.

*"Disordered" lattices.* A minimal requirement on the lattice $\mathcal{L}$ in order to have a compactness and representation theorem is to be not too "sparse" and not too "dense", while the precise geometric structure of $\mathcal{L}$ is not essential to prove those results. Indeed, instead of $\mathbb{Z}^d$ (or a periodic lattice), we can consider a lattice $\mathcal{L}$ such that:

(a) there exist $R > 0$ and $c > 0$ such that for any $x \in \mathbb{R}^d$

$$c \leq \#(\mathcal{L} \cap Q_R(x)) \leq \frac{1}{c},$$

where $Q_R(x)$ is any cube centered in $x$ with side length $R$;

(b) there exists $r > 0$ such that $\|x - y\| \geq r$ for any $x, y \in \mathcal{L}$

(see Fig. 22). If $\mathcal{L}$ satisfies these properties, it is possible to repeat the steps of the localization method and prove the theorem. The only point to be precised is the notion of nearest neighbours in this setting. To this end, for any $i \in \mathcal{L}$ we define the set

$$C_i = \{x \in \mathbb{R}^d : \|x - i\| \leq \|x - j\| \quad \forall j \in \mathcal{L}\}.$$

These sets are called *Voronoi cells* of the lattice.

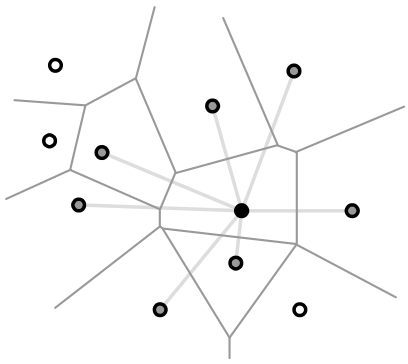

Figure 23: nearest neighbours in a "disordered" lattice.

Two points $i$ and $j$ in $\mathcal{L}$ are *nearest neighbours* if the corresponding Voronoi cells share a $d-1$-dimensional face; that is, $\mathcal{H}^{d-1}(\partial C_i \cap \partial C_j) > 0$ (see Fig. 23).

*Many-body interactions.* We can enlarge the class of energies $E_\varepsilon$ to functionals of the form

$$E_\varepsilon(u) = \sum_{i \in \mathcal{L}_\varepsilon} \varepsilon^{d-1} \phi_i^\varepsilon(\{u_j\}),$$

where $\phi_i^\varepsilon$ describes the way a point $i$ interacts with all the other points in the lattice. In the energies considered above

$$\phi_i^\varepsilon(\{u_j\}) = \sum_{j \in \mathcal{L}_\varepsilon} a_{ij}^\varepsilon (u_i - u_j)^2.$$

In this way, we can look at the possibility to extend the compactness and representation theorem to energies with a very general form of the interaction term $\phi_i^\varepsilon(\{u_j\})$, up to generalizing the assumptions (H1)-(H4) to this abstract setting. This approach allows, for instance, to include the analysis of interactions of many-body type.

**Example 33** (Three-point interactions)**.** We consider in $\mathbb{R}^2$ the triangular lattice $\mathbb{T}$ as in Example 31, and a generic triangular cell with vertices in $\mathbb{T}$ labelled as $i, j$ and $k$.

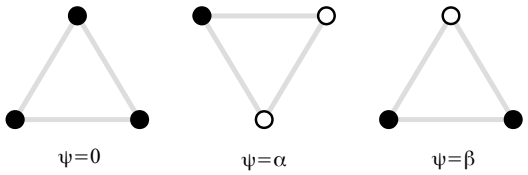


Figure 24: three-point interactions.

Given a spin function $u\colon \mathbb{T} \to \{-1, 1\}$ such that the values $u_i, u_j, u_k$ on the vertices of the (considered) triangle are not all equal, we can assume that the two equal values are $u_i$ and $u_j$ and we have two possibilities, $i - j = \pm e_1$ or $i - j \neq \pm e_1$.

Now, we define a function $\psi$ depending on the set $\{u_i, u_j, u_k\}$ by setting

$$\psi(\{u_i, u_j, u_k\}) = \begin{cases} 0 & \text{if } u_i = u_j = u_k \\ \alpha & \text{if } u_i = u_j \neq u_k \text{ and } i - j \neq \pm e_1 \\ \beta & \text{if } u_i = u_j \neq u_k \text{ and } i - j = \pm e_1 \end{cases}$$

where $0 < \alpha < \beta$ and we assume, as above, that the two equal values in the second and third cases are labelled as $i$ and $j$ (see Fig. 24). This interaction cannot be immediately written as a sum of two-point interactions; it can be proven, in fact, that it is not equivalent to a pairwise interaction. If we consider energies defined by functions $\psi_i^\varepsilon$ involving the interaction term $\psi$, we note that the configurations where the two coinciding values are on the horizontal side of the triangle are penalized, since $\beta > \alpha$, and this corresponds in the limit to a penalization of the horizontal interfaces.

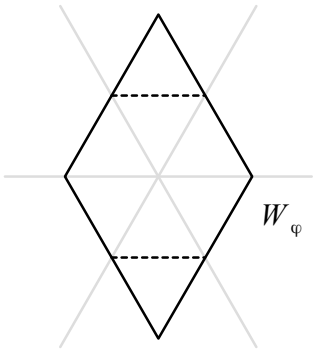


Figure 25: Wulff shape for three-point interactions.

It can be proven that the Γ-limit of these energies is given by

$$\int_{\Omega \cap \partial^* A} \varphi(\nu)\, d\mathcal{H}^1,$$

where the Wulff shape $W_\varphi$ is a rhombus, as pictured in Fig. 25. Note that $W_\varphi$ differs from the hexagonal Wulff shape corresponding to the nearest-neighbour interactions since in its boundary we cannot have horizontal edges.

*Different ranges of interactions.* Conditions (H3) and (H4) ensure that essentially (up to asymptotically negligible terms) the range of the interactions is finite. Anyway, also for some families of energies with long-range interactions we can prove a compactness result and obtain in the limit a perimeter functional, as in the following one-dimensional example.

**Example 34** (Coarse graining in dimension one)**.** Let $\Omega = \mathbb{R}$ and consider an energy of the form

$$G_\varepsilon(u) = \sum_{|i-j|\le R_\varepsilon} (u_i - u_j)^2$$

where the sum is over $i, j \in \mathbb{Z}$ and $R_\varepsilon$ satisfies the asymptotic properties

$$R_\varepsilon \to +\infty, \quad \varepsilon R_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Note that (H4) holds by the second condition, while (H3) does not hold since $R_\varepsilon \to +\infty$. Now, we look for a scaling of $G_\varepsilon$ such that in the limit we may have an interfacial energy. If we consider a discretization $u_\varepsilon$ of the function $u = -1 + 2\chi_{(0,+\infty)}$, the energy $G_\varepsilon(u_\varepsilon)$ is

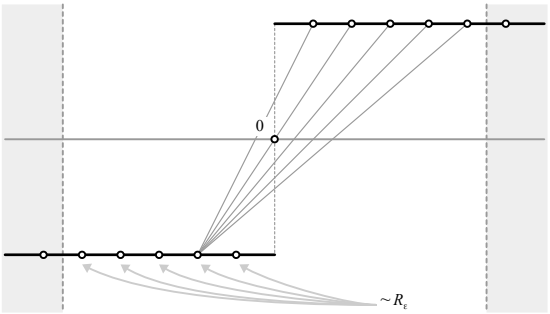


Figure 26: estimate of the energy of an interface.

of order $R_\varepsilon^2$, since each point interacts with all points at a distance less than $R_\varepsilon$, and we have to take into account an $\varepsilon R_\varepsilon$-neighbour of 0; that is, a number of interacting points of order $R_\varepsilon$ (see Fig. 26). Hence, to obtain equibounded energies for sequences approximating a jump, we scale $G_\varepsilon$ by a factor $\frac{1}{R_\varepsilon^2}$ and define

$$E_\varepsilon(u) = \sum_{|i-j|\le R_\varepsilon} \frac{1}{R_\varepsilon^2}(u_i - u_j)^2;$$

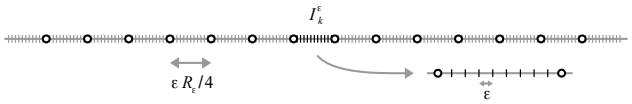


Figure 27: coarse graining of the lattice $\varepsilon\mathbb{Z}$.

the $\Gamma$-limit is then finite on piecewise-constant $u$ even though condition (H3) is not satisfied. In order to prove that the $\Gamma$-limit is finite exactly on this set of functions, we have to show a compactness result, and this can be done by a *coarse-graining* argument.

We subdivide the domain $\mathbb{R}$ in intervals $I_k^\varepsilon$, parametrized by $k \in \mathbb{Z}$, such that each interval contains a number of order $R_\varepsilon$ of points of the lattice $\varepsilon\mathbb{Z}$. Since we want all points in each interval to interact with all points in the nearest intervals, we fix $\frac{\varepsilon R_\varepsilon}{4}$ as the size of $I_k^\varepsilon$ (see Fig. 27).

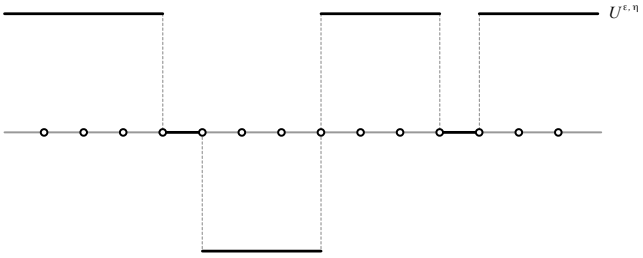


Figure 28: a coarse-grained function.

Let $u^\varepsilon$ be such that $E_\varepsilon(u^\varepsilon) \le c < +\infty$. We fix $\eta > 0$ and define a *coarse-grained function* $U^{\varepsilon,\eta}\colon \mathbb{Z} \to \{-1, 0, 1\}$ by setting

$$U^{\varepsilon,\eta}(k) = U_k^{\varepsilon,\eta} = \begin{cases} 1 & \text{if } \dfrac{4}{R_\varepsilon} \displaystyle\sum_{\varepsilon i \in I_k^\varepsilon} u_i^\varepsilon > 1 - \eta \\ -1 & \text{if } \dfrac{4}{R_\varepsilon} \displaystyle\sum_{\varepsilon i \in I_k^\varepsilon} u_i^\varepsilon < -1 + \eta \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

(see Fig. 28). Note that if $U_k^{\varepsilon,\eta} = 0$ then there exists a positive constant $C_\eta$ such that

$$\sum_{\varepsilon i, \varepsilon j \in I_k^\varepsilon} \frac{1}{R_\varepsilon^2} (u_i^\varepsilon - u_j^\varepsilon)^2 \ge C_\eta;$$

hence, the equiboundedness of $E_\varepsilon(u^\varepsilon)$ gives the estimate

$$\#\{k \in \mathbb{Z} : U_k^{\varepsilon,\eta} = 0\} \le \frac{\sup_\varepsilon E_\varepsilon(u^\varepsilon)}{C_\eta} \le C'_\eta. \tag{26}$$

Moreover, if $\eta < \frac{1}{2}$ we have that if $U_k^{\varepsilon,\eta} = 1$ and $U_{k+1}^{\varepsilon,\eta} = -1$ (and conversely, by exchanging 1 and $-1$) then

$$\sum_{\varepsilon i, \varepsilon j \in I_k^\varepsilon \cup I_{k+1}^\varepsilon} \frac{1}{R_\varepsilon^2}(u_i^\varepsilon - u_j^\varepsilon)^2 \geq C$$

for some $C > 0$ independent on $\varepsilon$ and $\eta$. Again using the uniform bound on $E_\varepsilon(u^\varepsilon)$, we get the estimate

$$\#\big\{k \in \mathbb{Z} : U_k^{\varepsilon,\eta}, U_{k+1}^{\varepsilon,\eta} \in \{-1,+1\} \text{ and } U_k^{\varepsilon,\eta} \neq U_{k+1}^{\varepsilon,\eta}\big\} \leq C'. \tag{27}$$

Estimates (26) and (27) ensure that, up to subsequences,

$$U^{\varepsilon,\eta} \to U^\eta \text{ as } \varepsilon \to 0,$$

where $U^\eta$ is a piecewise-constant function assuming only the values $-1$ and $+1$, with a finite number of jump points. By using the definition of $U_k^{\varepsilon,\eta}$ as the average of $u^\varepsilon$ in $I_k^\varepsilon$ (up to a finite number of intervals), it follows that for any $T > 0$

$$\int_{[-T,T]} |U^{\varepsilon,\eta} - u^\varepsilon|\, dx \leq O(\varepsilon R_\varepsilon)_{\varepsilon\to 0} + cT\eta, \tag{28}$$

since the total size of the union of the intervals where $U^{\varepsilon,\eta}$ vanishes is of order $\varepsilon R_\varepsilon$, and in each one of the other intervals the integral is less than $\frac{\varepsilon R_\varepsilon}{8}\eta$. In order to deduce the convergence of $u^\varepsilon$ in $L^1_{\mathrm{loc}}(\mathbb{R})$, we have to prove that in fact the limit $U^\eta$ does not depend on $\eta$. Indeed, if we choose $\eta' < \eta$, by the definition of $U^{\varepsilon,\eta}$ we get the following monotonicity properties for the limit functions $U^\eta$ and $U^{\eta'}$

$$\{U^{\eta'} = 1\} \subseteq \{U^\eta = 1\} \quad \text{and} \quad \{U^{\eta'} = -1\} \subseteq \{U^{\eta'} = -1\}. \tag{29}$$

Since both $\{U^{\eta'} = 1\} \cup \{U^{\eta'} = -1\}$ and $\{U^{\eta'} = 1\} \cup \{U^\eta = 1\}$ are equal to the whole $\mathbb{R}$, then the inclusions in (29) are in fact equalities; that is, $U^{\eta'} = U^\eta = U$. This allows to deduce by (28) that (up to subsequences) $u^\varepsilon \to U$, concluding the proof of the compactness.

To compute the $\Gamma$-limit, let $u^\varepsilon \to u$ and consider, for any $r \in \mathbb{N}$, $r \leq R_\varepsilon$ the restrictions of $u^\varepsilon$ to the $r$ disjoint sublattices $\mathcal{L}_\varepsilon^{k,r} = \varepsilon k + \varepsilon r\mathbb{Z}$, for $k = 1, \dots, r$. We localize the energies in a (sufficiently small) neighbourhood $I(x)$ of a point $x \in S(u)$, and note that for $r \leq R_\varepsilon$ (up to an arbitrary small fraction, less than $\delta R_\varepsilon$ for an arbitrarily small $\delta > 0$) the restrictions of $u^\varepsilon$ to the lattices $\varepsilon k + \varepsilon r\mathbb{Z}$ have a change of sign. Hence

$$E_\varepsilon(u^\varepsilon; I(x)) \geq \frac{1}{R_\varepsilon^2} \sum_{r \leq R_\varepsilon} r \sum_{\varepsilon i, \varepsilon j \in \mathcal{L}_\varepsilon^{k,r} \cap I(x)} (u_i^\varepsilon - u_j^\varepsilon)^2 \geq \frac{1}{R_\varepsilon^2} \sum_{r \leq R_\varepsilon} (8r) - c\delta \geq 4 - c\delta.$$

Then

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u^\varepsilon) \geq 4 \# S(u).$$

The upper estimate can be proven by using the restriction to $\varepsilon\mathbb{Z}$ of $u = -1 + 2\chi_{(x,+\infty)}$ as a recovery sequence.

The model described in Example 34 can be extended to dimension $d$ and to more general coefficients. We give only the statement of the result.

**Remark 35** (Coarse graining in dimension $d$)**.** Let $a\colon \mathbb{R}^d \to [0,+\infty)$ be a continuous function and $R_\varepsilon > 0$ be such that

$$R_\varepsilon \to +\infty, \quad \varepsilon R_\varepsilon \to 0 \ \text{ as } \ \varepsilon \to 0.$$

Noting that the right scaling of the energies in dimension $d$ is $\frac{1}{R_\varepsilon^{d+1}}$, we define

$$E_\varepsilon(u) = \sum_{\|i-j\|\le R_\varepsilon} \varepsilon^{d-1} a_{ij}^\varepsilon (u_i - u_j)^2, \quad \text{where } \ a_{ij}^\varepsilon = \frac{1}{R_\varepsilon^{d+1}} a\Big(\frac{i-j}{R_\varepsilon}\Big). \tag{30}$$

the sum taken over $i,j \in \mathbb{Z}^d \cap \Omega$. The $\Gamma$-limit is an integral functional given by

$$4 \int_{\Omega\cap\partial^* A} \int_{\mathbb{R}^d} a(\xi) |\langle \xi, \nu\rangle| \, d\xi \, d\mathcal{H}^{d-1}.$$

The limit density

$$\varphi(\nu) = 4 \int_{\mathbb{R}^d} a(\xi) |\langle \xi, \nu\rangle| \, d\xi$$

generalizes the formula we have seen in the homogeneous case, where $\varphi(\nu) = 4\sum_{k\in\mathbb{Z}^d} \alpha_k |\langle k, \nu\rangle|$; in this case, the effect of long-range interactions gives in the limit an integral term instead of a sum. Note that the corresponding Wulff shape may be not crystalline. In particular it is not crystalline in the case when $a = \chi_{B_1}$, for which $\varphi$ equals the constant

$$\tau := 4 \int_{B_1} |\xi_1| \, d\xi$$

and the limit is simply $\tau\, \mathcal{H}^{d-1}(\Omega \cap \partial^* A)$.

# 7 Optimal design by homogenization

We consider the set $V = \{e_1, \dots, e_d\} \subset \mathbb{Z}^d$, and we fix $\alpha$ and $\beta$ with $0 < \alpha < \beta$. We will consider systems of coefficients $a_{ij}^\varepsilon$, which are assumed to satisfy the *design constraint*

$$a_{i\,i+e_l}^\varepsilon \in \{\alpha, \beta\}$$

for all $i \in \mathbb{Z}^d$ and $l = 1, \dots, d$. Note that the systems are not assumed to be periodic, so that we have arbitrary mixtures of two types of nearest-neighbour connections.

Up to subsequences, we can suppose that for every $l = 1, \dots, d$ the *percentage of $\beta$-connections* has a weak limit $\theta_l \in L^1(\Omega)$ (that we call the *limit percentage of $\beta$-connections*); that is, the measures

$$\mu_\varepsilon^l = \sum_{i \in I_\varepsilon^l} \varepsilon^d \delta_{\varepsilon i} \text{ , where } I_\varepsilon^l = \Big\{ i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega : a^\varepsilon_{i\, i+e_l} = \beta \Big\}, \tag{31}$$

converge weakly* to a limit measure $\mu^l$, which is absolutely continuous with respect to the Lebesgue measure, with a density which we denote by $\theta_l$. Constraints on the percentage of $\beta$-connections (and hence of $\alpha$-connections) can be translated into constraints on $\theta_l$. Using Theorem 28 we can state the design problem as follows.

• *Design problem*: determine all the energy densities $\varphi = \varphi(x, \nu)$ that can be obtained as limits of nearest-neighbour energies

$$E_\varepsilon(u) := \sum_{\langle i,j \rangle} \varepsilon^{d-1} a^\varepsilon_{ij} (u_i - u_j)^2 \,, \tag{32}$$

where as usual $\langle i, j \rangle$ denotes the set of pair of nearest neighbours in $\mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega$, with given limit percentages $\theta_l$ of $\beta$-connections for all $e_l \in V$.

We call this a 'design problem' in that it requires to design optimal geometries that generate a given $\varphi$ which we think itself is optimal for some problem that has to be solved under a global design constraint.

The solution of this problem requires some technical results on the limit of perimeter energies on the continuum that are beyond the scope of these notes. We are going to state the result only, remarking the arguments that rely on the lattice structure. The key point is that the analysis can be reduced locally to periodic systems.

Let $a_{ij}$ be a system of periodic coefficients with period $K$ and such that $a_{ij} = 0$ if $\|i - j\| > 1$. Moreover, we assume that $a_{i\, i+e_l} \in \{\alpha, \beta\}$ for all $i \in \mathbb{Z}^d$ and $l = 1, \dots, d$. We define the *volume fraction* of $\beta$-bonds and the *total volume fraction of $\beta$-bonds* by setting

$$\begin{aligned} \theta_l(\{a_{i\, i+e_l}\}) &= \frac{1}{K^d} \#\{i \in \mathbb{Z}^d : i \in \{1, \dots, K\}^d, a_{i\, i+e_l} = \beta\}, \\ \theta(\{a_{i\, i+e_l}\}) &= \frac{1}{d} \sum_{l=1}^d \theta_l(\{a_{i\, i+e_l}\}), \end{aligned} \tag{33}$$

respectively. Now, given $\theta_l \in [0, 1]$ for all $e_l \in V$, we define the set $H(\{\theta_l\})$ as the closure of the limits of periodic systems with percentages of $\beta$-connections tending to $\theta_l$. More precisely, we give the following definition.

**Definition 36.** *Let $\theta_l \in [0,1]$ be given for all $l = 1, \dots, d$. The set of* homogenized energy densities of mixtures of $\alpha$ and $\beta$ bonds corresponding to $V = \{e_1, \dots, e_d\}$ with volume fractions $\theta_l$ *(of $\beta$ bonds) is defined as*

$$H(\{\theta_l\}) = \big\{\varphi\colon \mathbb{R}^d \to [0,+\infty) : \text{there exist } \theta_l^k \to \theta_l, \varphi^k \to \varphi \text{ and } \{a_{ij}^k\} \text{ periodic such that } \theta_l(\{a_{ij}^k\}) = \theta_l^k \text{ and } \varphi_{\hom}^k \text{ homogenized energy density of } \{a_{ij}^k\}\big\}.$$

Correspondingly, we define the set of *homogenized energy densities of mixtures of $\alpha$ and $\beta$ bonds corresponding to $V$ with volume fraction $\theta$* (of $\beta$ bonds), denoted by $H(\theta)$.

The first result is a localization theorem that characterizes the bounds of the energy densities in the design problem.

**Theorem 37** (a "Dal Maso-Kohn localization principle")**.** *Let $\{a_{ij}^\varepsilon\}$ be given, with limit percentages of $\beta$-connections $\theta_l \in L^1(\Omega)$ for all $l = 1, \dots, d$. Then we have $\varphi(x,\cdot) \in H(\{\theta_l(x)\})$ for almost every $x \in \Omega$ and for all $l = 1, \dots, d$.*

The main argument in the proof of this result is noting that the minimum problems in the blow-up formula giving $\varphi(x_0, \nu)$ can be interpreted as those in a homogenization formula related to a periodic discrete lattice with percentage of $\beta$-connections close to $\theta_l(x_0)$ for almost every $x_0$.

The following theorem gives bounds on the elements of the set $H(\{\theta_l\})$.

**Theorem 38** (bounds)**.** *The elements of the set $H(\{\theta_l\})$ are even and convex positively homogeneous functions of degree one $\varphi\colon \mathbb{R}^d \to [0,+\infty)$ such that*

$$\sum_{l=1}^{d} \alpha |\langle \nu, e_l \rangle| \le \varphi(\nu) \le \sum_{l=1}^{d} (\theta_l \beta + (1-\theta_l)\alpha) |\langle \nu, e_l \rangle|. \tag{34}$$

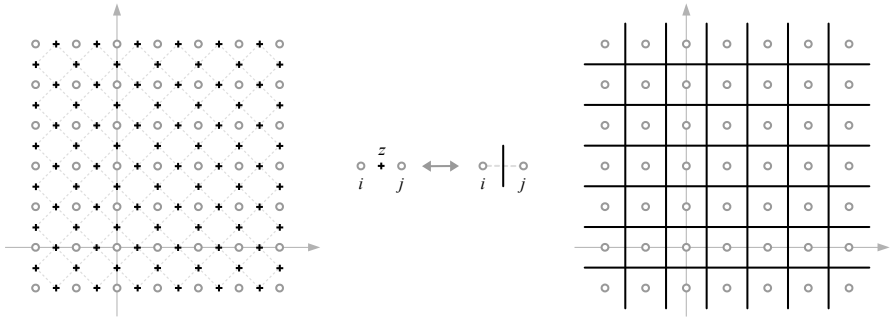


Figure 29: the lattice $\mathcal{Z}$ (cross nodes) and the identification of $z$ and the segment $[i,j]^\perp$.

We will give a description of the arguments in a two-dimensional setting, where the proofs are easier to visualize and formalize since we can characterize the homogenized energy density $\varphi_{\text{hom}}$ defined in (20) by a *path-minimization formula*. We postpone the proof after introducing the necessary notation.

Noting that each pair of nearest neighbours $(i,j) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ can be identified with the midpoint $\frac{i+j}{2}$, we consider the dual lattice of size $\frac{1}{\sqrt{2}}$ given by

$$\mathcal{Z} = \Big\{ \frac{i+j}{2} : i,j \in \mathbb{Z}^2 \ \text{ such that } \|i-j\| = 1 \Big\}. \tag{35}$$

We also identify $z \in \mathcal{Z}$ with the closed unit segment centered at $z$ and orthogonal to $j-i$, denoted by $[i,j]^\perp$ (see Fig. 29). Now we give the definition of *path* in the scaled lattice $\varepsilon\mathcal{Z}$.

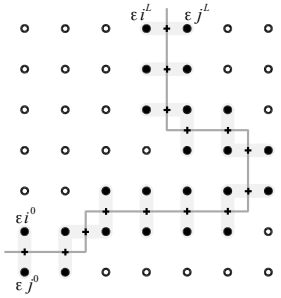


Figure 30: a path in $\varepsilon\mathcal{Z}$.

**Definition 39** (paths in $\varepsilon\mathcal{Z}$)**.** *A* path in $\varepsilon\mathcal{Z}$ *is a finite union*

$$\sigma = \{\varepsilon z_l\} = \bigcup_{l=1}^{L} \varepsilon [i^l, j^l]^\perp$$

*of segments with length $\varepsilon$ parametrized by a sequence of points $z_l \in \mathcal{Z}$ (equivalently, by a sequence of pairs of nearest neighbours $(i^l, j^l) \in \mathbb{Z}^2 \times \mathbb{Z}^2$), such that the segment labelled by $l$ has a common endpoint with the one labelled by $l+1$. The points $\varepsilon z_0$ and $\varepsilon z_L$ are referred to as the* endpoints *of the path (see Fig.* 30*). If $\varepsilon = 1$, we say that $\sigma$ is a* path in $\mathcal{Z}$*.*

It is useful to define a projection from $\mathbb{R}^2$ to the lattice $\mathcal{Z}$, which is a Bravais lattice generated by $\frac{e_1+e_2}{2}$ and $\frac{e_2-e_1}{2}$. In general, given a Bravais lattice $\mathcal{L} = \mathbb{Z}v_1 + \mathbb{Z}v_2$ we define the half-open cells corresponding to $\mathcal{L}$ by setting

$$C_k^{\mathcal{L}} = k + \Big\{ tv_1 + sv_2 : t,s \in \Big[-\frac{1}{2}, \frac{1}{2}\Big) \Big\}, \ k \in \mathcal{L}$$

as in (4). Moreover, we define the projection on a Bravais lattice $\mathcal{L}$ as follows. Given $x \in \mathbb{R}^2$, $\pi_{\mathcal{L}}(x)$ denotes the (unique) $k \in \mathcal{L}$ such that $x \in C_k^{\mathcal{L}}$; we say that $\pi_{\mathcal{L}}(x)$ is the *projection of $x$ on $\mathcal{L}$*.

**Remark 40** (A path-minimization formula for $d = 2$)**.** In dimension 2, in the definition of $\varphi_{\rm hom}$ given in (20) we can consider only functions $v\colon \mathbb{Z}^2 \cap Q_T^\nu \to \{-1, +1\}$ such that $v = -1 + 2\chi_{H^\nu}$ in $Q_T^\nu \setminus Q_{T-2R}^\nu$ and such that both the sets

$$A = \bigcup_{\{i:v_i=1\}} \Big(i + \Big[-\frac{1}{2}, \frac{1}{2}\Big]^2\Big)$$

and its complement are connected. Hence, the boundary of $A$ in the interior of the square $Q_T^\nu$ is (the restriction of) a path in $\mathcal{Z}$. Since the boundary condition corresponds to the fact that the endpoints of the path are the projections of $-\frac{T}{2}\nu^\perp$ and $\frac{T}{2}\nu^\perp$ on $\mathcal{Z}$, we can minimize over such paths, and get

$$\begin{aligned}\varphi_{\rm hom}(\nu) = \lim_{T\to+\infty} \frac{8}{T} \min\Big\{ \sum_{l=1}^{L} a_{z_l} : \{z_l\} \text{ path in } \mathcal{Z} \text{ with endpoints} \\ \pi_{\mathcal{Z}}\big(-\tfrac{T}{2}\nu^\perp\big), \pi_{\mathcal{Z}}\big(\tfrac{T}{2}\nu^\perp\big); L \in \mathbb{N}\Big\}\end{aligned} \tag{36}$$

where $a_z = a_{ij}$ if $z \in \mathcal{Z}$ corresponds to the pair $(i, j)$.

*Proof of Theorem* 38 *in dimension two.* We have to show that an element $\varphi \in H(\{\theta_1, \theta_2\})$ satisfies

$$\varphi(\nu) \le 8(\theta_1\beta + (1-\theta_1)\alpha)|\langle \nu, e_1\rangle| + 8(\theta_2\beta + (1-\theta_2)\alpha)|\langle \nu, e_2\rangle|, \tag{37}$$

the lower bound being trivial.

Let $\varphi_{\rm hom}$ be the homogenized energy density of a $K$-periodic system of coefficients $a_{ij}$ satisfying the design constraint; then the averages on horizontal and vertical bond are given by

$$\theta_1\beta + (1-\theta_1)\alpha = \frac{1}{K^2}\sum_{k=1}^{K} \sum_{\substack{z\in[0,K)^2\cap\mathcal{Z}\\ \langle z,e_2\rangle=k}} a_z \quad \text{and} \quad \theta_2\beta + (1-\theta_2)\alpha = \frac{1}{K^2}\sum_{k=1}^{K} \sum_{\substack{z\in[0,K)^2\cap\mathcal{Z}\\ \langle z,e_2\rangle=k}} a_z,$$

respectively. Let $n_1, n_2 \in \{1, \dots, K\}$ be such that, setting

$$z_1^k = \Big(n_1 - \frac{1}{2}, k\Big), \qquad z_2^k = \Big(k, n_2 - \frac{1}{2}\Big),$$

we have

$$\frac{1}{K}\sum_{k=1}^{K} a_{z_1^k} \le \frac{1}{K^2}\sum_{k=1}^{K} \sum_{\substack{z\in[0,K)^2\cap\mathcal{Z}\\ \langle z,e_2\rangle=k}} a_z \qquad \text{and} \qquad \frac{1}{K}\sum_{k=1}^{K} a_{z_2^k} \le \frac{1}{K^2}\sum_{k=1}^{K} \sum_{\substack{z\in[0,K)^2\cap\mathcal{Z}\\ \langle z,e_1\rangle=k}} a_z.$$

Recalling (36), the upper bound for $\varphi_{\rm hom}$ is then obtained by considering only sets whose boundary lies in $(n_1 - \frac{1}{2}, n_2 - \frac{1}{2}) + K\mathbb{Z}^2$, and hence give minimal paths in $\mathcal{Z}$. □

The following result states that Theorem 37 is sharp.

**Theorem 41** (design theorem)**.** *For all $l = 1, \dots, d$ let $\theta_l \colon \Omega \to [0,1]$ be measurable and let $\varphi \colon \Omega \times \mathbb{R}^d \to [0, +\infty)$ be positively 1-homogeneous and even in the second variable, such that the bounds* (34) *are satisfied for almost all $x \in \Omega$, and the functional $F$ given by*

$$F(A) = \int_{\Omega \cap \partial^* A} \varphi(x, \nu(x)) d\mathcal{H}^{d-1}$$

*is lower semicontinuous. Then, there exist $\{a_{ij}^\varepsilon\}$ such that $E_\varepsilon$ in* (32) *$\Gamma$-converge to $F$ given by and $\theta_l(\{a_{ij}^\varepsilon\})$ converge to $\theta_l$ as $\varepsilon \to 0$ for all $l = 1, \dots, d$.*

Note that this result is the same as the optimality of the bounds in Theorem 38 if the function $x \mapsto \varphi(x, \nu)$ and the limit percentages $\theta_l$ are constant.

# 8 Homogenization of random networks

In this section we compute the $\Gamma$-limit of an energy where we randomly mix coefficients. To this end we have to introduce some notions of Percolation Theory for what is called the *bond-percolation model* (i.e., when the random choice is thought to be performed on the connections). As a result we will consider coefficients $a_{ij} = a_{ij}^\omega$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ that depend on the realization of a random variable. A different model, that can be treated similarly, is the *site-percolation model*. In our intuition it would correspond to choosing weak and strong nodes – and to define a weak connection as a connection between two nodes of which at least one is a weak node.

## 8.1 Random mixtures

From now on we will restrict to the two-dimensional case $d = 2$. We recall the definition of the dual lattice of $\mathbb{Z}^2$ given in (35)

$$\mathcal{Z} = \Big\{ \frac{i+j}{2} : i, j \in \mathbb{Z}^2 \text{ such that } \|i - j\| = 1 \Big\}.$$

We identify each point $z = \frac{i+j}{2} \in \mathcal{Z}$ with the pair $(i, j) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ or with the segment $[i, j]^\perp$ orthogonal to $[i, j]$ and with middle point $z$.

Let $0 < \alpha < \beta < +\infty$. We consider the simplest case of two kinds of nearest-neighbour connections with weights $\alpha$ and $\beta$. A choice of connections between nodes of $\mathbb{Z}^2$ is a function

$$\omega : \mathcal{Z} \to \{\alpha, \beta\}.$$

We set $\Sigma = \{\alpha, \beta\}^{\mathcal{Z}}$. For any $\omega \in \Sigma$ we then define the coefficients

$$a_{ij}^\omega = \omega\Big(\frac{i+j}{2}\Big).$$

Let $\Omega$ be an open bounded and Lipschitz subset of $\mathbb{R}^2$, and consider for any $\varepsilon > 0$ the lattice $\mathcal{L}_\varepsilon = \varepsilon\mathbb{Z}^2 \cap \Omega$. For any $\omega \in \Sigma$ we define the energies

$$E_\varepsilon^\omega(u) = \sum_{\langle i,j\rangle} \varepsilon a_{ij}^\omega (u_i - u_j)^2 \tag{38}$$

where as usual $\langle i, j\rangle$ indicates the set of pairs in $\mathbb{Z}^2$ such that $\varepsilon i, \varepsilon j \in \mathcal{L}_\varepsilon$ and $\|i - j\| = 1$.

We will consider $\omega$ with the property that

$$\omega\Big(\frac{i+j}{2}\Big) = a_{ij}^\omega = \begin{cases} \alpha & \text{with probability } 1-p \\ \beta & \text{with probability } p \end{cases} \tag{39}$$

with $p \in [0, 1]$. This can be done rigorously by introducing some 'independent identically distributed' random variables. For our presentation it suffices to describe the 'almost-sure' properties of such $\omega$.

**Remark 42.** For a fixed realization $\omega$ of the random variable, the functionals $E_\varepsilon^\omega$ are exactly of the form which we considered in the corresponding deterministic setting. Hence, by the representation theorem we obtain that there exists a subsequence $(\varepsilon_k)$ such that

$$\Gamma\text{-}\lim_{k\to+\infty} E_{\varepsilon_k}^\omega(A) = F^\omega(A) := \int_{\Omega\cap\partial^* A} \varphi^\omega(x,\nu)\, d\mathcal{H}^1. \tag{40}$$

Since $\omega \in \{\alpha, \beta\}^{\mathcal{Z}}$ and the probabilities of $\alpha$ and $\beta$-connections are $1-p$ and $p$, respectively, it follows that, with the notation of Section 7 (see Definition 36),

$$\varphi^\omega(x, \cdot) \in \mathrm{H}(p) \quad \text{for } \mathcal{H}^1\text{-almost all } x \in \partial^* A,$$

where the local density $\theta(x)$ of $\beta$ is almost surely equal to the constant value $p$.

We are going to improve this description showing that the function $\varphi^\omega$ itself is almost surely depending only on $p$ and is homogeneous.

Now, we want to prove a more precise statement about the $\Gamma$-limit of the energies defined in (38).

**Theorem 43** (First-passage percolation)**.** *Let $a_{ij}^\omega = \omega(\frac{i+j}{2})$ with $\omega$, $\alpha, \beta$ and $p$ as in* (39)*. Then, setting*

$$m^\omega(x, y) = 8 \min\Big\{ \sum_{l=1}^{L} a_{z_l}^\omega : \{z_l\} \text{ path in } \mathcal{Z} \text{ joining } \pi_{\mathcal{Z}}(x), \pi_{\mathcal{Z}}(y); L \in \mathbb{Z}\Big\}, \tag{41}$$

*almost surely in $\omega$ there exists the limit*

$$\lim_{T\to+\infty} \frac{m^\omega(0, T\nu^\perp)}{T} := \varphi^\omega(\nu)$$

*and*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E^\omega_\varepsilon(A) = \int_{\Omega\cap\partial^* A} \varphi^\omega(\nu)\, d\mathcal{H}^1$$

*on sets $A$ of finite perimeter. Moreover, almost surely $\varphi^\omega$ depends only on $p$.*

The formula defining $\varphi^\omega$ is usually called *first-passage percolation formula* of the system.

To prove Theorem 43, in order to characterize almost surely the limit energy density we will use the following percolation result, which ensures the existence of the limit energy density $\varphi^\omega$.

**Theorem 44.** *Let $a^\omega_{ij} = \omega(\frac{i+j}{2})$ with $\omega$, $\alpha, \beta$ and $p$ as in* (39)*. Then the function $m^\omega$ defined in* (41) *satisfies the following properties:*

*(a) almost surely there exists the limit*

$$\lim_{T\to+\infty} \frac{m^\omega(0, T\nu^\perp)}{T} =: \varphi^\omega(\nu);$$

*(b) (translation invariance) almost surely there exists the limit*

$$\varphi^\omega(\nu) = \lim_{T\to+\infty} \frac{m^\omega(x_T, x_T + T\nu^\perp)}{T} \tag{42}$$

*for any sequence $(x_T)$ such that $\|x_T\| \le T^2$;*

*(c) (the limit is deterministic) there exists a function $\varphi_p$ such that almost surely*

$$\varphi^\omega(\cdot) = \varphi_p(\cdot)$$

*independently of $\omega$.*

Here and in the following, we only give a hint of the main arguments behind Percolation results and to differences and correspondences with the periodic setting, referring to the literature for more details (see the last section of these notes). As for the result above, we only mention that the existence of the limit comes from a *stationarity property* of the function $m^\omega$ seen as a subadditive "point process", for which a number of asymptotic results hold. Indeed, we have

$$m^\omega(x, y) \le m^\omega(x, z) + m^\omega(z, y)$$

for any $x, y, z \in \mathbb{R}^2$. Note that subadditivity is a key property also in the proof of the existence of the limit in the periodic homogenization case.

Note that translation invariance is different from the periodic case. Indeed, in that case it is clearly possible to translate by a (multiple of the) period, and we can approximate

any translation with an error of the order of the period, then negligible when $T$ diverges. Hence, the invariance holds without constraints on the growth of $\|x_T\|$. In the case of random coefficients, the same invariance property may not be true for any choice of $x_T$. A heuristic argument is that for any $T$ there is a positive probability of having a large set where all connections are $\alpha$-connections. Then, if we translate the minimum problem for $m^\omega(0, T\nu^\perp)$ to that set by a suitable $x_T$, we obtain $\alpha\|\nu\|_1$ as minimum value. The same holds for $\beta$-connections, which contradicts translation invariance. The statement says that we may suppose almost surely that these sets be "far away" from 0.

Finally, the fact that the limit $\varphi^\omega$ is in fact deterministic depends on the *ergodicity property* of the point process $m^\omega$.

Now, we can give the proof of the $\Gamma$-convergence result of Theorem 43.

*Proof of Theorem* 43. We divide the proof in two steps, showing a lower and an upper estimate.

*Lower bound by blow up at $x_0 \in \partial^* A$.* We prove the lower estimate by applying the blow-up method as in Section 5. Let $\{u^\varepsilon\}$ be such that $E^\omega_\varepsilon(u^\varepsilon)$ is equibounded and converges to a set $A$ of finite perimeter, and such that the corresponding sequence of measures $\{\mu^\omega_\varepsilon\}$ weak$^*$ converge to a measure $\mu^\omega$. We restrict to points $x_0 \in \partial^* A$ suitable for the blow up of $A$ and such that the measure-theoretical derivative of the limit measure $\mu^\omega$ with respect to $\mathcal{H}^1 \llcorner \partial^* A$ exists, and follow the steps of the blow-up procedure in the case of periodic energies, obtaining

$$\lim_{\varepsilon\to 0} \frac{\mu^\omega_\varepsilon(Q^\nu_{\varrho_\varepsilon}(x_0))}{\varrho_\varepsilon} \;\geq\; \liminf_{\varepsilon\to 0} \frac{8}{\varrho_\varepsilon} \min\Big\{ \sum_{l=1}^{L} \varepsilon a^\omega_{z_l} : \{\varepsilon z_l\} \text{ path in } \varepsilon\mathcal{Z} \text{ joining } \pi_{\varepsilon\mathcal{Z}}\big(x_0 - \frac{\varrho_\varepsilon}{2}\nu^\perp\big) \text{ and } \pi_{\varepsilon\mathcal{Z}}\big(x_0 - \frac{\varrho_\varepsilon}{2}\nu^\perp\big);\ L \in \mathbb{Z}\Big\}$$

(with points $z \in \mathcal{Z}$ identified with segments in the definition of a path), where $\pi_{\varepsilon\mathcal{Z}}$ denotes the projection of $\mathbb{R}^2$ on $\varepsilon\mathcal{Z}$ and $a^\omega_z = a^\omega_{ij}$ if $z = \frac{i+j}{2} \in \mathcal{Z}$. We used the fact that in dimension 2 the lower bound can be described by optimizing over paths with endpoints close to $x_0 \pm \frac{\varrho_\varepsilon}{2}\nu^\perp$. Note that there exists a scale $\overline{\varrho}_\varepsilon \to 0$ such that in the blow-up procedure we can choose any infinitesimal sequence $\varrho_\varepsilon$ satisfying $\varrho_\varepsilon \geq \overline{\varrho}_\varepsilon$.

By scaling $\varrho_\varepsilon$ to $T_\varepsilon = \frac{\varrho_\varepsilon}{\varepsilon}$, we get

$$\lim_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon}(x_0))}{\varrho_\varepsilon} \geq \liminf_{\varepsilon\to 0} \frac{m^\omega(x_\varepsilon, x_\varepsilon + T_\varepsilon\nu^\perp)}{T_\varepsilon} \tag{43}$$

where $x_\varepsilon = \frac{x_0}{\varepsilon} - \frac{T_\varepsilon}{2}\nu^\perp = \frac{T_\varepsilon}{\varrho_\varepsilon} - \frac{T_\varepsilon}{2}\nu^\perp$ and $m^\omega$ is defined in (41).

Until this point, we used only the blow-up method and the fact that we can see the minimum problem with Dirichlet boundary conditions as a minimum problem over paths in $\mathcal{Z}$ with fixed endpoints. In order to characterize almost surely the limit energy density,

we apply the percolation result of Theorem 44 to the sequence $\frac{m^\omega(x_\varepsilon, x_\varepsilon + T_\varepsilon \nu^\perp)}{T_\varepsilon}$ to obtain a lower bound. Note that in the blow-up procedure we can choose the infinitesimal sequence $\varrho_\varepsilon$ large enough so as to have

$$\|x_\varepsilon\| = T_\varepsilon \frac{\|x_0\|}{\varrho_\varepsilon} \leq T_\varepsilon^2.$$

Hence, by Theorem 44 we get that almost surely there exists the limit

$$\lim_{\varepsilon \to 0} \frac{m^\omega(x_\varepsilon, x_\varepsilon + T_\varepsilon \nu^\perp)}{T_\varepsilon} = \varphi_p(\nu).$$

By (43), we obtain the lower bound

$$\lim_{\varepsilon \to 0} \frac{\mu_\varepsilon^\omega(Q^\nu_{\varrho_\varepsilon}(x_0))}{\varrho_\varepsilon} \geq \varphi_p(\nu) \text{ for } \mathcal{H}^1 \text{ a.a. } x_0 \in \partial^* A \cap \Omega$$

for a set of $\omega$ with probability 1. Concluding the blow-up procedure by integrating, we deduce that for a set of realizations $\omega$ with probability 1 the following estimate holds

$$\liminf_{\varepsilon \to 0} E_\varepsilon^\omega(u^\varepsilon) \geq \int_{\Omega \cap \partial^* A} \varphi_p(\nu)\, d\mathcal{H}^1.$$

*Upper bound.* By density, it is sufficient to construct the recovery sequence for a polyhedral set $A$.

We first assume that $\Omega \cap \partial A = L$, where $L$ is a segment. Let $\nu$ be the normal to $\partial A$ on $L$. Note that to apply Theorem 44(b) we have to take into account a boundedness requirement on the centres of the involved squares. Hence, differently from the periodic case, it is more convenient to subdivide the construction of the recovery sequence in two steps.

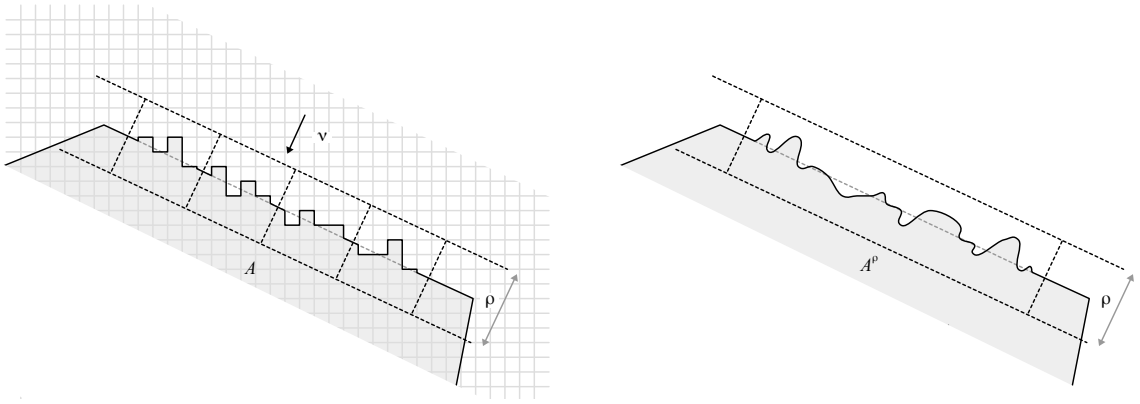


Figure 31: construction of the recovery sequence.

First, we fix $\varrho > 0$ and consider a finite number of squares with side-length $\varrho$ covering the segment $L$ except for a small neighbourhood of the endpoints. We define a recovery sequence $u_\varepsilon^\varrho$ by considering in each square $Q_\varrho^\nu(x_r^\varrho)$, $r = 1, \dots, R^\varrho$, (the scaling of) a solution of the minimum problem for $m^\omega(\frac{x_r^\varrho}{\varepsilon} - \frac{\varrho}{2\varepsilon}\nu^\perp, \frac{x_r^\varrho}{\varepsilon} + \frac{\varrho}{2\varepsilon}\nu^\perp)$. In the neighbourhood of the endpoints we set $u_\varepsilon^\varrho$ as the discretization of $A$. By construction it follows that

$$\begin{aligned} E_\varepsilon^\omega(u_\varepsilon^\varrho) &= \varrho \sum_{r=1}^{R^\varrho} \frac{m^\omega(\frac{x_r^\varrho}{\varepsilon} - \frac{\varrho}{2\varepsilon}\nu^\perp, \frac{x_r^\varrho}{\varepsilon} + \frac{\varrho}{2\varepsilon}\nu^\perp)}{\frac{\varrho}{\varepsilon}} + c\varrho \\ &= \mathcal{H}^1(L)\varphi_p(\nu) + c\varrho + o(1)_{\varepsilon\to 0}. \end{aligned}$$

If $A$ is an arbitrary polyhedral set, then we can repeat the construction for each of its sides (see Fig. 31), obtaining a family of spin functions again denoted by $u_\varepsilon^\varrho$. The characteristic functions of the union of the corresponding sets $A_\varepsilon(u_\varepsilon^\varrho)$ converge (up to subsequences) to some $A^\varrho$. For any fixed $\varrho > 0$ we get

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} E_\varepsilon^\omega(A^\varrho) \le \int_{\Omega\cap\partial A} \varphi_p(\nu_A)\, d\mathcal{H}^1 + o(1)_{\varrho\to 0}.$$

This gives an upper bound for the upper $\Gamma$-limit on $A^\varrho$.

Now, we let $\varrho \to 0$. By semicontinuity, since $A^\varrho \to A$ as $\varrho \to 0$, we get the same upper bound on $A$, and then conclude by density for sets of finite perimeter. □

**Remark 45.** We can extend the result of Theorem 43 to the case where the coefficients $a_{ij}^\omega$ are allowed to take values not only in $\{\alpha, \beta\}$, but in the interval $[\alpha, \beta]$. In this case, the notation becomes more complex since the result depends on the probability distribution and not only on the value $p$.

## 8.2 Extreme cases: rigid and dilute spin systems

In order to treat the cases when $\alpha = 0$ or $\beta = +\infty$ we need some more refined geometric properties of path-connected sets of connections from Percolation Theory.

Given $\alpha$ and $\beta$ such that $0 \le \alpha < \beta \le +\infty$, a realization $\omega \in \{\alpha, \beta\}^{\mathcal{Z}}$ and the corresponding system of coefficients $a_{ij}^\omega$, we use the notation

$$\mathcal{Z}_\gamma = \bigcup_{a_{ij}^\omega = \gamma} [i, j]^\perp \tag{44}$$

for $\gamma \in \{\alpha, \beta\}$. We again identify the segment $[i, j]^\perp$ with its center $z = \frac{i+j}{2} \in \mathcal{Z}$, so that $\mathcal{Z}_\alpha$ and $\mathcal{Z}_\beta$ can be viewed as subsets of $\mathcal{Z}$.

We will use the following result on the connected components of $\mathcal{Z}_\alpha$ and $\mathcal{Z}_\beta$ related to the cases $p < \frac{1}{2}$ and $p > \frac{1}{2}$, respectively (the case $p = \frac{1}{2}$ will not be dealt with explicitly).

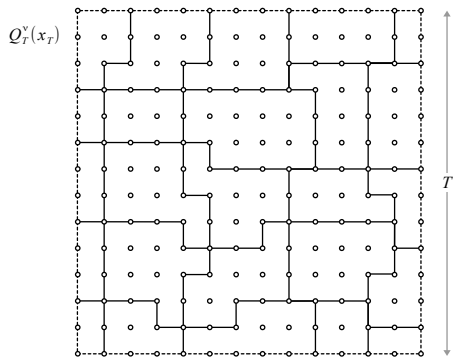


Figure 32: a grid of paths in the weak (or strong) cluster.

**Theorem 46** (bond percolation)**.**

*(a) If $p < \frac{1}{2}$, then almost surely (in $\omega$) there exists a unique infinite connected component of the set $\mathcal{Z}_\alpha$, which is denoted by $W^\omega$ (called the* weak cluster*).*

*If $p > \frac{1}{2}$, then almost surely (in $\omega$) there exists a unique infinite connected component of the set $\mathcal{Z}_\beta$, which is denoted by $S^\omega$ (called the* strong cluster*).*

*(b) If $p < \frac{1}{2}$, there exist $T_0$ and $\eta$ such that for any $T > T_0$ and for any square $Q_T^\nu(x_T)$ such that $\|x_T\| \leq T^2$ there exist $\eta T$ disjoint paths in $Q_T^\nu(x_T) \cap W^\omega$ connecting pairs of opposite sides of $Q_T^\nu(x_T)$ (see Fig.* 32*).*

*The corresponding result hold if $p > \frac{1}{2}$ in $Q_T^\nu(x_T) \cap S^\omega$.*

This result ensures that, almost surely, if we consider a large enough square $Q_T^\nu(x_T)$ then we can think of the weak cluster (or the strong cluster if $p > \frac{1}{2}$) as a "grid" that we can use, in a sense, as a generalization of the grid given by the regular square lattice.

### 8.2.1 Random rigid spin systems

As in the previous case, we consider nearest-neighbour connections with weights $\alpha$ and $\beta$ with independently assigned probability, but here we let the value $\beta$ to be "very large" and $\alpha$ normalized to 1. Instead of assuming $\beta = +\infty$, which can be read as a constraint in the lattice, we introduce a dependence on $\varepsilon$ allowing to consider "very large" - finite or infinite - values of the strong coefficients; that is, for fixed $p \in [0, 1]$ we assume for $i, j$ nearest neighbours in $\mathbb{Z}^2$

$$a_{ij}^\omega = a_{ij}^{\varepsilon,\omega} = \begin{cases} \alpha = 1 & \text{with probability } 1 - p \\ \beta = \beta_\varepsilon & \text{with probability } p, \end{cases} \tag{45}$$

and 0 otherwise, with $\beta_\varepsilon \in [1, +\infty]$ such that $\beta_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We call this a *rigid system.*

We now consider the energies defined by

$$E_\varepsilon^\omega(u) = \sum_{\langle i,j \rangle} \varepsilon a_{ij}^\omega (u_i - u_j)^2, \tag{46}$$

with the coefficients $a_{ij}^\omega$ as in (45). We use the convention that $0 \cdot +\infty = 0$, so that, if $a_{ij}^\omega = \beta_\varepsilon$, the choice $\beta_\varepsilon = +\infty$ forces $u_i = u_j$ in order to have a bounded energy $E_\varepsilon^\omega(u)$.

In order to apply the blow-up method to obtain a lower bound for these energies, we define a pseudometric by considering only paths which lie in the weak cluster. We set

$$\begin{aligned} m_{\text{weak}}^\omega(x,y) &= 8 \min \Big\{ \sum_{l=1}^M a_{z_l}^\omega : \{z_l\}_{l=1}^M \text{ path in } \mathcal{Z} \text{ joining } \pi_w(x) \text{ and } \pi_w(y);\ M \in \mathbb{N} \Big\} \\ &= 8 \min \Big\{ M : \{z_l\}_{l=1}^M \text{ path in } W^\omega \text{ joining } \pi_w(x) \text{ and } \pi_w(y) \Big\}, \end{aligned} \tag{47}$$

where $a_z^\omega = a_{ij}^\omega$ if $z = \frac{i+j}{2}$, and $\pi_w$ is the projection on $\big((\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2\big) \cap W^\omega$, where the weak cluster is understood as a union of segments. Note that this is the projection on the endpoints of the elements in $W^\omega$. Moreover, we let $L(\gamma) = M$ for a path $\gamma = \{z_l\}_{l=1}^M$. The following result holds.

**Theorem 47** (existence of the asymptotic chemical distance)**.** *There exists a function $\varphi = \varphi_p$ such that almost surely*

$$\varphi_p(\nu) = \lim_{T \to +\infty} \frac{m_{\text{weak}}^\omega(x_T, x_T + T\nu^\perp)}{T} \tag{48}$$

*if $\|x_T\| \le T^2$. This function is called the* asymptotic chemical distance*.*

Now, we can state the $\Gamma$-convergence result for rigid spin systems.

**Theorem 48** ($\Gamma$-convergence for rigid spin systems)**.** *Let $E_\varepsilon^\omega$ be defined as in* (46) *with the coefficients $a_{ij}^\omega = a_{ij}^{\varepsilon,\omega}$ given by* (45)*, and let $\beta_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Then*

- *if $p > \frac{1}{2}$, almost surely we have that*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} E_\varepsilon^\omega(A) = \begin{cases} 0 & \text{if } A = \emptyset \text{ or } A = \Omega \\ +\infty & \text{otherwise;} \end{cases} \tag{49}$$

- *if $p < \frac{1}{2}$, almost surely we have that*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} E_\varepsilon^\omega(A) = \int_{\Omega \cap \partial^* A} \varphi_p(\nu)\, d\mathcal{H}^1,$$

*where $\varphi_p$ is the asymptotic chemical distance defined in* (48)*.*

We only give a sketch of the proof of Theorem 48, considering separately the two cases $p > \frac{1}{2}$ and $p < \frac{1}{2}$, since we deal with different geometries.

*Case $p > \frac{1}{2}$.* In this case, we expect to have only isolated sets with finite energy; indeed, if $\beta_\varepsilon = +\infty$ we have an infinite connected component of bonds with value $+\infty$, and we show that if these sets have equibounded perimeters they necessarily converge to a set with perimeter equal to 0, so that the $\Gamma$-limit is finite (and it is equal to 0) only if $A = \emptyset$ or $A = \Omega$. In the general case, we consider a set of finite perimeter $A$ such that $\mathcal{H}^1(\Omega \cap \partial^* A) \neq 0$. Theorem 46(b) gives an estimate on the number of disjoint paths in the strong cluster intersecting the boundary of the set $A$ in a square. Hence, the blow-up method gives the estimate

$$\lim_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon}(x_0))}{\varrho_\varepsilon} \geq \liminf_{\varepsilon\to 0} \frac{1}{T_\varepsilon} \eta \frac{\varrho_\varepsilon}{\varepsilon} \beta_\varepsilon = \lim_{\varepsilon\to 0} \eta\beta_\varepsilon = +\infty,$$

which gives the claim.

*Case $p < \frac{1}{2}$.* If $\beta_\varepsilon = +\infty$, we note that the strong connections are never used in an optimal path, so that $m^\omega(x, y) = m^\omega_{\text{weak}}(x, y)$. Hence, we can exactly repeat the blow-up procedure obtaining the lower estimate by Theorem 47. The upper estimate can be given exactly as in the proof of Theorem 43. If the strong connections have a finite value $\beta_\varepsilon < +\infty$, in the minimization of the energy it could be in principle more convenient to have a "short-cut" path containing some segment $[i, j]^\perp$ such that $a^\omega_{ij} = \beta_\varepsilon$ instead of a longer path of weak connections. To show that this does not happen, we have to use another Percolation result.

**Lemma 49.** *For any $\delta > 0$ there exist $\eta > 0$ and $T_0$ such that for any $T > T_0$ and $x_T$ satisfying $\|x_T\| \leq T^2$ the following property holds: if $\gamma$ is a path between $x_T$ and $x_T + T\nu^\perp$ with $8L(\gamma) < (\varphi_p(\nu) - \delta)T$, then there exist $\eta T$ strong connections in $\gamma$.*

Now, we apply the blow-up procedure at a suitable $x_0 \in \partial^* A$. We fix $\delta > 0$ and assume by contradiction that

$$\liminf_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon}(x_0))}{\varrho_\varepsilon} < \varphi_p(\nu) - \delta.$$

If we consider a sequence of paths $\gamma_\varepsilon$ such that

$$\liminf_{\varepsilon\to 0} \frac{8L(\gamma_\varepsilon)}{\varrho_\varepsilon} \leq \liminf_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon}(x_0))}{\varrho_\varepsilon},$$

scaling as usual $\varrho_\varepsilon$ to $T_\varepsilon = \frac{\varrho_\varepsilon}{\varepsilon}$, the result of the lemma above ensures that in each (scaled) path there are at least $\eta\frac{\varrho_\varepsilon}{\varepsilon}$ strong connections with coefficient $\beta_\varepsilon$. Then the energy of each path $\gamma_\varepsilon$ is again greater than $\eta\beta_\varepsilon$, giving a contradiction.

### 8.2.2 Random dilute spin systems

For a fixed realization $\omega$ of the random variable, we consider the nearest-neighbour energies defined in (46) with coefficients $a_{ij}^{\omega}$ given by

$$a_{ij}^{\omega} = \begin{cases} 0 & \text{with probability } 1-p \\ 1 & \text{with probability } p. \end{cases} \tag{50}$$

The energies are defined as usual on the set of spin functions $u\colon \varepsilon\mathbb{Z}^2 \cap \Omega \to \{-1,1\}$, where $\Omega$ is a bounded Lipschitz subset of $\mathbb{R}^2$.

With the choice $\alpha = 0$ it is particularly important to examine the geometry of the connections, since this allows to understand whether we have coerciveness or not. In this context, we say that a subset of $\mathcal{Z}$ is connected if the union of the corresponding segments is connected. With the notation 44, the weak cluster $W^{\omega}$ given by Theorem 46 is a subset of $\mathcal{Z}_0$, and the strong cluster $S^{\omega}$ is a subset of $\mathcal{Z}_1$.

The following $\Gamma$-convergence result holds.

**Theorem 50** ($\Gamma$-convergence for dilute spin systems)**.** *Let $E_{\varepsilon}^{\omega}$ be defined as in* (46) *with the coefficients $a_{ij}^{\omega}$ given by* (50)*. Then*

- *if $p < \frac{1}{2}$, almost surely we have that for all $A$ set of finite perimeter*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E_{\varepsilon}^{\omega}(A) = 0;$$

- *if $p > \frac{1}{2}$, almost surely we have that*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E_{\varepsilon}^{\omega}(A) = \int_{\Omega\cap\partial^* A} \varphi_p(\nu)\, d\mathcal{H}^1,$$

*where $\varphi_p$ is given by the first-passage percolation formula in Theorem* 44*.*

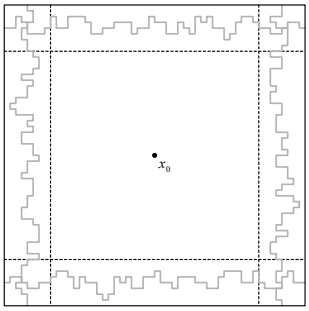


Figure 33: construction of a recovery sequence in a square in the dilute case.

Also in this case we only give a sketch of the proof, considering the two cases $p < \frac{1}{2}$ and $p > \frac{1}{2}$ separately, highlighting the arguments where we overcome the lack of coerciveness issues.

*Case* $p < \frac{1}{2}$. We will use the geometric properties of the weak cluster stated in Theorem 46. In particular, from a slight generalization of the second claim of Theorem 46 we deduce that, for a fixed rectangle with controlled centre and sides, we can always find a path with energy equal to 0; that is, there exists a spin function $u$ such that the interface in the rectangle between the regions where $u = 1$ and $u = -1$ has energy 0. This allows to construct a recovery sequence for a square $Q$. Indeed, we can construct a path $\gamma_\varepsilon$ in the weak cluster contained in a small neighbourhood of the boundary of the scaled square; the recovery sequence is then given by $u^\varepsilon = -1 + 2\chi_{A_\varepsilon}$, where $A_\varepsilon$ is the sequence of connected sets whose boundary is $\gamma_\varepsilon$ (see Fig. 33). Hence, the energy of each $u^\varepsilon$ vanishes, and $u^\varepsilon \to Q$. Since the energies are positive, the $\Gamma$-limit of $E^\omega_\varepsilon$ is 0 for any square, and hence for any finite union of squares. Then, by density, the claim follows.

*Case* $p > \frac{1}{2}$. In this case, once the compactness is proved, the proof of the $\Gamma$-convergence result can be given exactly as in the case $\alpha > 0$ for random mixtures in Theorem 43.

**Coerciveness in the case** $p > \frac{1}{2}$

We note that Theorem 43 in the case $p > \frac{1}{2}$ must be complemented with a coerciveness result. We again use Theorem 46, in this case describing the geometry of the strong cluster. Since $\alpha = 0$, we do not have coerciveness on the nearest-neighbour interactions. We already noticed in Section 6 that if $a_{ij} = 0$ for a pair $(i, j)$ of nearest neighbours we can recover coerciveness if we can find a path (with controlled length) joining $i$ and $j$ such that the bonds along this path have strictly positive coefficients. We can look at the problem of the asymptotic behaviour of energies $E^\omega_\varepsilon$ by considering its deterministic counterpart; that is, the corresponding problem in the periodic framework.

We recall some definitions. Given a set $\mathcal{A} \subset \mathbb{Z}^2$, we say that it is *connected* if for $i, j \in \mathcal{A}$ there exists a path of nearest neighbours in $\mathcal{A}$ joining $i$ and $j$; that is, there exist $i_0, i_1, \dots, i_n \in \mathcal{A}$ such that $i_0 = i$, $i_n = j$ and $\|i_k - i_{k-1}\| = 1$ for any $k = 1, \dots, n$. The boundary of $\mathcal{A}$ is defined as

$$\partial\mathcal{A} = \{i \in \mathcal{A} : \operatorname{dist}(i, \mathbb{Z}^2 \setminus \mathcal{A}) = 1\}.$$

Moreover, we define the external boundary $\partial^{\text{ext}}\mathcal{A}$ as the boundary of its complement.

**Remark 51** (Perforated domains in the periodic framework in $\mathbb{Z}^2$: compactness)**.** We consider the energies given by

$$E_\varepsilon(u) = \sum_{\langle i,j\rangle} \varepsilon a_{ij}(u_i - u_j)^2,$$

with the (symmetric) coefficients $a_{ij}$ assuming values 0 and 1 and periodic with a period $K$. We set

$$\mathcal{L} = \{i \in \mathbb{Z}^2 : j \in \mathbb{Z}^2 \text{ exists such that } \|i-j\| = 1 \text{ and } a_{ij} = 1\}$$

and assume that $\mathcal{L}$ is connected and infinite. Note that in dimension $d = 2$ this implies that each connected component of the complement is finite.

The lattice $\mathcal{L}$ satisfies our definition of disordered lattice, and we could analyze the problem by using the notion of convergence of spin function given by the Voronoi cells of the lattice; that is, identify a spin function $u$ defined on $\mathcal{L}$ with its piecewise-constant interpolation on the corresponding Voronoi cells. In this case, we prefer to use a different approach, which is more suitable for generalizations. The idea is to consider the missing connections as "holes" inside the domain, and, given a sequence $\{u_\varepsilon\}$ with equibounded energy, to modify the sequence in the holes in such a way that we can prove compactness, and such that the energy is controlled by the energies of $\{u_\varepsilon\}$.

Let $\{u^\varepsilon\}$ be such that $E_\varepsilon(u^\varepsilon)$ is equibounded. We consider, in the set of the connected components of the complement of $\varepsilon\mathcal{L}$, the connected components where $u^\varepsilon$ is not constant in the external boundary; let $\mathcal{C}^0_\varepsilon$ denote the set of such components, and $K^0_\varepsilon = \#\mathcal{C}^0_\varepsilon$. We modify $u^\varepsilon$ outside $\varepsilon\mathcal{L}$ by setting $\tilde{u}^\varepsilon_i = 1$ if $\varepsilon i$ belongs to a connected component such that $u^\varepsilon$ is identically equal to 1 in the external boundary, and $\tilde{u}^\varepsilon_i = -1$ otherwise. With this definition, the energy does not increase. Now we show that the perimeters of the sets $A_\varepsilon(\tilde{u}^\varepsilon)$ are equibounded. Indeed, it is sufficient to estimate $K^0_\varepsilon$. For each $C \in \mathcal{C}^0_\varepsilon$ there exist $\varepsilon i, \varepsilon j$ in the external boundary of $C$ such that $u^\varepsilon_i \neq u^\varepsilon_j$. Since $\mathcal{L}$ is connected, up to enlarging the period we can assume that each pair of points in $\mathcal{L} \cap [0,K]^2$ is connected by a path contained in $\mathcal{L} \cap [-K, 2K]^2$. Hence, for each $C \in \mathcal{C}^0_\varepsilon$ there exist $i', j'$ nearest neighbours in the path connecting $i$ and $j$ with $a_{i'j'}(u^\varepsilon_i - u^\varepsilon_j)^2 = 4$, shared with an equibounded number of other such connected components. We then obtain $K^0_\varepsilon \leq cE_\varepsilon(u^\varepsilon)$. Since the additional contribution of each connected component $C$ is at most $4\varepsilon K^2$, we get

$$\mathcal{H}^1(A_\varepsilon(\tilde{u}^\varepsilon)) \leq E_\varepsilon(u^\varepsilon) + c'K^2E_\varepsilon(u^\varepsilon) \leq c''.$$

This implies that, up to subsequences, the convergence of $\tilde{u}^\varepsilon$ to a set $A$ with finite perimeter.

In the random case we can proceed similarly to the periodic case, but, since the connected components in the complement of the strong cluster may have arbitrarily large size, we have to provide a more refined estimate.

We introduce the set

$$\mathcal{L}^\omega = \{i \in \mathbb{Z}^2 : j \in \mathbb{Z}^2 \text{ exists such that } [i,j]^\perp \subset S^\omega\},$$

and consider the connected components of the complement of $\mathcal{L}^\omega$. Since $p > \frac{1}{2}$, we have that almost surely in $\omega$ each of these connected components is finite.

Now, let $\{u^\varepsilon\}$ be such that $E^\omega_\varepsilon(u^\varepsilon) \le S < +\infty$ for all $\varepsilon$, where the energies are computed for nearest-neighbour pairs in the bounded Lipschitz open set $\Omega$. As in the periodic framework, if we change the values of $u^\varepsilon$ by setting $\tilde{u}^\varepsilon = 1$ in each connected component of the complement of $\varepsilon\mathcal{L}^\omega$ such that $u^\varepsilon = 1$ on its boundary, and $-1$ otherwise in the complement of $\varepsilon\mathcal{L}^\omega$, we do not change the values on $\varepsilon\mathcal{L}^\omega$ and the energy does not increase. Hence, we can assume that $u^\varepsilon$ is constant on each such connected component.

*Step 1. Estimate on the number of the maximal connected components of the set* $\{i : u^\varepsilon_i = 1\}$ *and of the set* $\{i : u^\varepsilon_i = -1\}$. For each maximal connected component $I$ of $\{i : u^\varepsilon_i = 1\}$ or of $\{i : u^\varepsilon_i = -1\}$, there exist $j$ in the external boundary $\partial^{\text{ext}} I$ and $i \in I$ such that $a^\omega_{ij} = 1$. Otherwise, we would have either a connected component of the complement of $\varepsilon\mathcal{L}^\omega$ where the constant value on the external boundary is different from the internal value, or a pair $(i, j)$ of nearest neighbours in the same connected component of the complement of $\varepsilon\mathcal{L}^\omega$ with different values of $u^\varepsilon$. Hence, by the equiboundedness of the energies, we get the estimate

$$\#M^+_\varepsilon \le \frac{S}{\varepsilon}, \quad \#M^-_\varepsilon \le \frac{S}{\varepsilon}$$

where $M^\pm_\varepsilon$ denotes the number of the maximal connected components where $u^\varepsilon_i = \pm 1$.

*Step 2. Estimate for components with size less than* $\frac{1}{\varepsilon}$. Identifying as usual a subset of $\varepsilon\mathbb{Z}^2$ with the union of the corresponding $\varepsilon$-squares, we show that the total measure of the union of the connected component with size (i.e. number of nodes) less than $\frac{1}{\varepsilon}$ is negligible as $\varepsilon \to 0$. We fix $\delta > 0$ and note that each component of size $O(\varepsilon^{-1+\delta})$ has measure $O(\varepsilon^{1+\delta})$, so that the total measure of such components is $O(\varepsilon^\delta)$. As for the components with size much larger than $\varepsilon^{-1+\delta}$ and not greater than $\frac{1}{\varepsilon}$, we use a well-known fact from Percolation Theory; that is, the fact that each connected set of points $z \in \mathcal{Z}$ with $a^\omega_z = 0$ has size at most $O(|\log\varepsilon|)$. Noting that the set of the boundary connections of such a component is of size much larger than $\varepsilon^{-\frac{1}{2}+\frac{\delta}{2}}$, we then deduce that the energy contribution is much larger than

$$\varepsilon^{\frac{1}{2}+\frac{\delta}{2}} |\log\varepsilon|^{-1}.$$

This gives an upper estimate on the total number of the components with size much larger than $\varepsilon^{-1+\delta}$. Since we are considering components with size at most $\frac{1}{\varepsilon}$, the measure of each such component is at most $\varepsilon$, and the measure of their union is less than

$$S|\log\varepsilon|\varepsilon^{\frac{1}{2}-\frac{\delta}{2}},$$

so that it is negligible as $\varepsilon \to 0$.

Hence, as the $L^1$-convergence of $u^\varepsilon$ is concerned, we can assume that all connected components in the sets $\{u^\varepsilon = 1\}$ and $\{u^\varepsilon = -1\}$ have size at least $\frac{1}{\varepsilon}$.

*Step 3. Estimate of the energy for components with size greater than* $\frac{1}{\varepsilon}$. To conclude, we have to estimate the perimeter of the "large" connected components. To this end, we will use the following Percolation result, which we state without proof.

**Lemma 52** (percolation animal)**.** *Fixed $M > 0$, almost surely there exist a deterministic positive constant $\kappa$ and $\varepsilon_0 = \varepsilon_0(\omega) > 0$ such that for all connected sets contained in the square $\left[-\frac{M}{\varepsilon}, \frac{M}{\varepsilon}\right]^2$ and of size larger than $\varepsilon^{-\frac{1}{2}}$ with $\varepsilon < \varepsilon_0$, the proportion of strong links (such that $a_{ij}^{\omega} = 1$) in each such a set is at least $\kappa$.*

Lemma 52 ensures that for each $\Omega' \subset\subset \Omega$ we have

$$\mathcal{H}^1(\partial A_\varepsilon(u^\varepsilon) \cap \Omega') \leq \frac{1}{\kappa} E_\varepsilon^\omega(u^\varepsilon),$$

which gives the compactness of $\{u^\varepsilon\}$ (restricted to the strong cluster), concluding the proof.

# 9 Random lattices: Poisson clouds

In this section we consider an example of homogenization on random lattices not satisfying the definition of disordered lattice given in Section 6. Namely, we consider a *Poisson random set* $\mathcal{L}$ with *intensity* $\lambda$ in $\mathbb{R}^2$, defined on a probability space $(\mathcal{O}, \mathcal{F}, \mathbf{P})$, characterized by the properties:

• for any bounded Borel set $B \subset \mathbb{R}^2$ the number of points in $B \cap \mathcal{L}$ has a Poisson law with parameter $\lambda|B|$; i.e.,

$$\mathbf{P}(\{\#(B \cap \mathcal{L}) = n\}) = e^{-\lambda|B|} \frac{(\lambda|B|)^n}{n!};$$

• for any collection of bounded disjoint Borel subsets in $\mathbb{R}^2$ the random variables defined as the number of points of $\mathcal{L}$ in these subsets are independent.

We do not enter in the details of the definition of such a random set. As a technical note, the probability space $\mathcal{O}$ is equipped with a dynamical system $T_x : \mathcal{O} \mapsto \mathcal{O}$, for $x \in \mathbb{R}^2$, such that for any bounded Borel set $B$ and any $x \in \mathbb{R}^2$ we have $\#\big((B+x) \cap \mathcal{L}\big)(\omega) = \#\big(B \cap \mathcal{L}\big)(T_x\omega)$. We suppose that $T_x$ is a group of measurable measure-preserving transformations in $\mathcal{O}$ and is ergodic.

As usual, we define the ferromagnetic energy of $\mathcal{L}$ as

$$E^{\mathcal{L}}(u) = \sum_{\langle i,j \rangle} (u(i) - u(j))^2,$$

for $u : \mathcal{L} \to \{-1, 1\}$, where $\langle i, j \rangle$ denotes nearest-neighbour pairs in the sense of the Voronoi cells. The corresponding scaled energies are given by

$$E_\varepsilon^{\mathcal{L}}(u) = \sum_{\langle i,j \rangle} \varepsilon (u_i - u_j)^2, \tag{51}$$

for $u \colon \varepsilon\mathcal{L} \to \{-1, 1\}$, where $u_i = u(\varepsilon i)$.

We note that, contrary to a disordered lattice as defined in Section 6,

• $\mathcal{L}$ is isotropic since the properties of Poisson random sets are invariant under (translations and) rotations;

• $\mathcal{L}$ is not regular: we have pairs of points of $\mathcal{L}$ arbitrarily close, and squares of arbitrary size not containing points of $\mathcal{L}$.

As a consequence, given a family of Voronoi cells in $\mathcal{L}$, it is not possible to estimate the perimeter of its union by the number of edges, which is the underlying argument to prove compactness properties for sequences $u^\varepsilon$ with equibounded energies. Nevertheless, compactness properties are possible through the use of the following Percolation lemma, which uses a covering of Voronoi cells by *polyominos* (unions of squares). If $P$ is a finite connected union of Voronoi cells of $\mathcal{L}$ we set

$$\mathbf{A}(P) = \{z \in \mathbb{Z}^2 : (z + (0,1)^2) \cap P \neq \emptyset\}.$$

**Lemma 53** (polyomino lemma)**.** *Let $R > 0$ and $\gamma > 0$. Then there exists a deterministic constant $C$ such that for almost all $\omega$ there exists $\varepsilon_0 = \varepsilon_0(\omega) > 0$ such that if $P$ is a finite connected union of Voronoi cells of $\mathcal{L}$ and $\varepsilon < \varepsilon_0$ satisfy*

$$P \cap \frac{R}{\varepsilon}(-1,1)^2 \neq \emptyset, \qquad \max\Big\{\#\{i : C_i \subset P\}, \#\mathbf{A}(P)\Big\} \geq \varepsilon^{-\gamma}$$

*then we have*

$$\frac{1}{C}\#\{i : C_i \subset P\} \leq \#\mathbf{A}(P) \leq C\,\#\{i : C_i \subset P\}.$$

This lemma states that a large connected family of Voronoi cells can be identified with a union of a comparable number of unit squares. It can be used to prove compactness properties for sequences $u^\varepsilon$ with equibounded $E_\varepsilon^{\mathcal{L}}(u^\varepsilon)$ as follows. Let

$$V_\varepsilon(u^\varepsilon) = \bigcup_{\{i : u_i^\varepsilon = 1\}} \varepsilon C_i.$$

Then the argument is that we can write

$$V_\varepsilon(u^\varepsilon) = (A_\varepsilon \cup B'_\varepsilon) \setminus B''_\varepsilon,$$

where $|B'_\varepsilon| + |B''_\varepsilon| \to 0$ (using either an isoperimetric argument or the Polyomino Lemma), and $\{A_\varepsilon\}$ is a family of sets with equibounded perimeter.

**Lemma 54** (compactness of Voronoi sets)**.** *Let $u^\varepsilon$ be such that $\sup_\varepsilon E_\varepsilon(u^\varepsilon) < +\infty$. Then there exists a set of finite perimeter $A$ such that $\chi_{V_\varepsilon(u^\varepsilon)}$ converge to $\chi_A$ in $L^1_{\mathrm{loc}}(\mathbb{R})$.*

*Proof.* Since we reason locally, we may asume that all $V_\varepsilon(u^\varepsilon)$ are contained in a fixed cube.

We fix $\gamma > 0$ small enough. We subdivide $\partial V_\varepsilon(u^\varepsilon)$ into its connected components. We denote by $\mathcal{C}_\varepsilon^{\gamma,+}$ the family of such connected components $S$ with

$$\#\{i \in \mathcal{L} : u_{\varepsilon i} = 1, \varepsilon C_i \cap S \neq \emptyset\} \geq \varepsilon^{-\gamma}. \tag{52}$$

Note that each such connected component can be identified with the set

$$P = P(S) = \bigcup \Big\{ C_i : u_{\varepsilon i} = 1, \varepsilon C_i \cap S \neq \emptyset \Big\}. \tag{53}$$

We denote by $\mathcal{C}_\varepsilon^{\gamma,-}$ the family of the remaining connected components.

The first step will be to identify the small sets $B'_\varepsilon$ and $B''_\varepsilon$ as the 'interior' of contours in $\mathcal{C}_\varepsilon^{\gamma,-}$ where the inner trace of $\chi_{V_\varepsilon(u^\varepsilon)}$ is 0 and 1, respectively. In this way the remaining set will have a boundary only composed of 'large' components from $\mathcal{C}_\varepsilon^{\gamma,+}$.

For each $S \in \mathcal{C}_\varepsilon^{\gamma,-}$, let $P$ be defined from $S$ by (53). We have two cases, whether $\varepsilon P$ is interior to $S$ or not. We denote by $\mathcal{C}_{1,\varepsilon}^{\gamma,-}$ the first family, by $\mathcal{C}_{2,\varepsilon}^{\gamma,-}$ the second one, and define $B_\varepsilon$ as the union of the $\varepsilon C_i$ in the interior of $S$ for some $S \in \mathcal{C}_{1,\varepsilon}^{\gamma,-}$ and such that $u_i^\varepsilon = 1$, and $B''_\varepsilon$ as the union of the $\varepsilon C_i$ in the interior of $S$ for some $S \in \mathcal{C}_{2,\varepsilon}^{\gamma,-}$ and such that $u_i^\varepsilon = -1$. If we set

$$V_\varepsilon = (V_\varepsilon(u^\varepsilon) \setminus B_\varepsilon) \cup B''_\varepsilon$$

then $\partial V_\varepsilon$ consists only of components in $\mathcal{C}_\varepsilon^{\gamma,+}$, and $|B_\varepsilon \cup B''_\varepsilon| \leq C\varepsilon^{1-2\gamma}$.

We now write $V_\varepsilon = A_\varepsilon \cup A'_\varepsilon$, where

$$A_\varepsilon = \bigcup \{(\varepsilon z + \varepsilon Q) : \varepsilon z + \varepsilon Q \subset A_\varepsilon\}, \qquad \text{and} \qquad A'_\varepsilon = V_\varepsilon \setminus A_\varepsilon.$$

By Lemma 53 we have

$$\mathcal{H}^1(\partial \mathbf{A}(P(S))) \leq C \#\{i \in \mathcal{L} : u_{\varepsilon i} = 1, \varepsilon C_i \cap S \neq \emptyset\}$$

Summing up over all $S \in \mathcal{C}_\varepsilon^{\gamma,+}$ we obtain

$$\mathcal{H}^1(\partial A_\varepsilon) \leq C\, E_\varepsilon(u^\varepsilon).$$

Hence, $\{A_\varepsilon\}$ is a family of sets with equibounded perimeter, and the functions $\chi_{A_\varepsilon}$ are locally precompact in $L^1(\mathbb{R}^2)$. Again by Lemma 53 we have

$$|A'_\varepsilon| \leq C\varepsilon^2 \sum_{S \in \mathcal{C}_\varepsilon^{\gamma,+}} \#\mathbf{A}(P(S)) \leq C\varepsilon E_\varepsilon(u^\varepsilon).$$

This shows that $|A'_\varepsilon| \to 0$, and proves the claim, upon setting $B'_\varepsilon = A'_\varepsilon \cup B_\varepsilon$. □

This lemma defines a convergence $u^\varepsilon \to A$; moreover, it gives the first variational ingredient in order to prove the existence of the $\Gamma$-limit; the second one regards the possibility to change boundary values in order to prove a homogenizion formula. To this end we would like to use a discrete coarea argument. In general again this is not possible since from an estimate on the length of an interface we cannot deduce an estimate on the energy of such an interface (since $\mathcal{L}$ is not regular we can have arbitrarily small edges of Voronoi cells). However, we can prove that we can limit our interfaces to lie on the boundary of regular Voronoi cells defined as follows.

For $\alpha > 0$ we set

$$\mathcal{L}^0_\alpha = \Big\{ i \in \mathcal{L} : C_i \text{ contains a ball of radius } \alpha,\ \operatorname{diam} C_i \le \frac{1}{\alpha}, \#\text{edges of } C_i \le \frac{1}{\alpha} \Big\} \tag{54}$$

the family of *regular Voronoi cells with parameter* $\alpha$. The following lemma describes some geometrical features of regular Voronoi tessellations.

**Lemma 55** (a channel property of $\mathcal{L}^0_\alpha$)**.** *Let $\delta > 0$. For every $T \in \mathbb{R}$, $\nu \in S^1$ and $x \in \mathbb{R}^2$ we define*

$$R^\nu_{T,\delta}(x) = \Big\{ x : |\langle x - x_i, \nu_i \rangle| \le \delta T, |\langle x - x_i, \nu_i^\perp \rangle| \le \frac{1}{2} T \Big\}.$$

*Then there exist $\alpha_0, C_\delta > 0$ such that a.s. there exists $T_0(\omega) > 0$ such that for all $T > T_0(\omega)$ the rectangle $R^\nu_{T,\delta}(x)$ contains at least $C_\delta T$ disjoint paths of Voronoi cells $C_i$ with $i \in \mathcal{L}^0_\alpha$ connecting the two opposite sides of $R^\nu_{T,\delta}(x)$ parallel to $\nu$. This property is uniform as $x/T$ vary on a bounded set of $\mathbb{R}^2$.*

This lemma can be proven using combinatoric arguments that allow to use well-known results for the Bernoulli bond-percolation model. Its use allows to fix boundary values in the blow-up argument, which, in this two-dimensional setting involves minimal-path problems. To that end we first let $\mathcal{L}^*$ denote the dual lattice to $\mathcal{L}$; i.e., the set of all endpoints of edges of Voronoi cells, and let

$$\pi_0(x) = \text{ closest point of } \mathcal{L}^* \text{ to } x.$$

For almost all $x$ this point is uniquely defined. For the remaining points we choose one of the closest points of $\mathcal{L}^*$ to $x$.

By the isotropy of the Poission set it is sufficient to characterize the limit surface tension when the normal is e.g. $\nu_2$.

**Proposition 56** (surface tension)**.** *Almost surely there exists the limit*

$$\begin{aligned} \tau \;=\; \lim_{t\to+\infty} \frac{1}{t} \min\{ \#\{e_i\} : \{e_i\}_i \text{ path of edges of Voronoi cells} \\ \text{with endpoints } \pi_0(0,0) \text{ and } \pi_0(t,0) \} \end{aligned}$$

*and is a deterministic quantity.*

Note that $\tau$ depends on $\lambda$, and a scaling argument allows to conclude that $\tau = \sqrt{\lambda}\,\tau_1$ ($\tau_1$ being the surface tension for a Poisson random set with intensity 1). The homogenization theorem reads as follows.

**Theorem 57** (homogenization on random Poisson sets in the plane)**.** *Let $\mathcal{L}$ be a Poisson random set in $\mathbb{R}^2$ with intensity $\lambda$ and let $E_\varepsilon^{\mathcal{L}}$ be defined by* (51)*. Then almost surely $E_\varepsilon^{\mathcal{L}}$ $\Gamma$-converges to $8\sqrt{\lambda}\tau_1\mathcal{H}^1(\partial^* A)$ with $\tau_1$ the deterministic constant given by Proposition* 56 *when $\lambda = 1$.*

In the following remark we note that we can also treat interactions depending on the distance of points in the lattice $\mathcal{L}$. Note that the corresponding ferromagnetic energies cannot be directly compared with the nearest-neighbour energies.

**Remark 58** (an extension: finite-range systems)**.** We can use the properties of $\mathcal{L}_\alpha$ and the arguments leading to the theorem above to prove that for $R > 0$ large enough (corresponding to $\alpha$ small enough) the energies

$$E_\varepsilon^{\mathcal{L},R}(u) = \sum_{\{i,j:\|i-j\|\le R\}} \varepsilon(u_i - u_j)^2$$

almost surely $\Gamma$-converge to an isotropic energy $\tau_R\sqrt{\lambda}\,\mathcal{H}^1(\partial^* A)$.

# 10 Graphons

We now consider the minimal-cut problem in arbitrary (abstract) graphs as the number of nodes diverges.

Given a sequence of graphs

$$G_k = (\mathcal{N}_k, \mathcal{E}_k),$$

where $\mathcal{N}_k$ denotes the set of nodes and $\mathcal{E}_k$ is the set of the connections; i.e., a symmetric subset of $\mathcal{N}_k \times \mathcal{N}_k$. In order to quantify the density of a sequence of graphs, as a parameter we use the number of connections with respect to the total number of possible connections (which is of order $(\#\mathcal{N}_k)^2$). We are interested in the case $\#\mathcal{N}_k \to +\infty$ as $k \to +\infty$.

**Definition 59** (dense and sparse (sequences of) graphs)**.** *Let $\{G_k\}$ be a sequence of graphs with $G_k = (\mathcal{N}_k, \mathcal{E}_k)$ and $\#\mathcal{N}_k \to +\infty$ as $N_k \to +\infty$. The sequence is* dense *(or the graphs are dense) if*

$$\liminf_{k\to+\infty} \frac{\#\mathcal{E}_k}{(\#\mathcal{N}_k)^2} \ge c > 0.$$

*Otherwise, the sequence is* sparse *(or the graphs are sparse).*

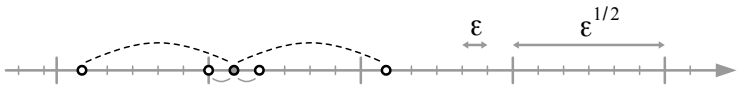


Figure 34: connections of nearest neighbours and points at distance $\frac{1}{\sqrt{\varepsilon}}$.

In the following example we study the ferromagnetic energies on a sequence of sparse graphs in $\mathbb{Z}$ whose limit is not described by a sharp interface, but by a parameter taking all values in $[-1,1]$. Correspondingly, optimal sequences exhibit microscopic "diffuse" interfaces.

**Example 60** (a diffuse interface for a sparse graph sequence)**.** We consider the set of nodes given by $\mathcal{N}_\varepsilon = [0,1] \cap \varepsilon\mathbb{Z}$ and the set of connections $\mathcal{E}_\varepsilon$ given by nearest neighbours and pairs of points at distance $\lfloor \frac{1}{\sqrt{\varepsilon}} \rfloor$; that is,

$$\mathcal{E}_\varepsilon = \Big\{ (\varepsilon i, \varepsilon j) \in \mathcal{N}_\varepsilon \times \mathcal{N}_\varepsilon : \|i-j\| = 1 \text{ or } \|i-j\| = \Big\lfloor \frac{1}{\sqrt{\varepsilon}} \Big\rfloor \Big\},$$

as pictured in Fig. 34. Without loss of generality, we can suppose $\frac{1}{\sqrt{\varepsilon}} \in \mathbb{N}$.

We consider the energies

$$E_\varepsilon(u) = \frac{\alpha_\varepsilon}{8} \sum_{(\varepsilon i, \varepsilon j) \in \mathcal{E}_\varepsilon} (u_i - u_j)^2$$

defined for $u \colon \mathcal{N}_\varepsilon \to \{-1,1\}$, where the scaling factor $\alpha_\varepsilon$ is to be determined such that interfaces have finite (and non-vanishing) energy; i.e., the limit is finite on piecewise-constant functions. We test the energies on $u = -1 + 2\chi_{[0,+\infty)}$. The number of interacting pairs giving a contribution to the energy is $\frac{1}{\sqrt{\varepsilon}} + 1$, hence, if $u^\varepsilon$ denotes the discretization of $u$, we have

$$E_\varepsilon(u^\varepsilon) = \alpha_\varepsilon \Big( \frac{1}{\sqrt{\varepsilon}} + 1 \Big)$$

which is equibounded if $\alpha_\varepsilon$ is of order $\sqrt{\varepsilon}$. Then, the energies we consider are scaled as

$$E_\varepsilon(u) = \frac{1}{8} \sum_{(\varepsilon i, \varepsilon j) \in \mathcal{E}_\varepsilon} \sqrt{\varepsilon} (u_i - u_j)^2. \tag{55}$$

We are going to 'lift' the energies to $\mathbb{Z}^2$. For all $u \colon [0,1] \cap \varepsilon\mathbb{Z} \to \{-1,1\}$ we define a corresponding function

$$v_u \colon [0,1]^2 \cap \sqrt{\varepsilon}\,\mathbb{Z}^2 \to \{-1,1\}$$

given by

$$v_u(\sqrt{\varepsilon}\, i_1, \sqrt{\varepsilon}\, i_2) = \begin{cases} u(\sqrt{\varepsilon}\, i_1 + \varepsilon i_2) & \text{if } i_2 < \frac{1}{\sqrt{\varepsilon}} \\ v_u(\sqrt{\varepsilon}\, i_1, 0) = u(\sqrt{\varepsilon}\, i_1) & \text{if } \sqrt{\varepsilon}\, i_2 = 1. \end{cases}$$

Note that the definition in the second line above allows to identify the upper and lower sides of the square $[0,1]^2$. In this way the 'lifted energy' can be thought of as a nearest-neighbour energy taking into account a periodic vertical extension. We define $G_\varepsilon(v_u) = E_\varepsilon(u)$, which is a nearest-neighbour energy in $\mathbb{Z}^2$.

The sequence $\{G_\varepsilon\}$ $\Gamma$-converges as $\varepsilon \to 0$ to the crystalline perimeter given by

$$G(A) = \int_{[0,1]^2 \cap \partial^*(A^\#)} \|\nu\|_1 \, d\mathcal{H}^1$$

with respect to the convergence in $L^1_{\mathrm{loc}}$ of the piecewise-constant extensions of the spin functions. In this formula, given $A \subset [0,1] \times [0,1)$, $A^\#$ denotes the periodic extension of $A$ to $[0,1] \times \mathbb{R}$.

Let $\{u^\varepsilon\}$ be a sequence with equibounded energy, and let $u$ be the $L^\infty$-weak$^*$ limit of $u^\varepsilon$ (which we can assume exists up to subsequences). The corresponding functions $v^\varepsilon = v_{u^\varepsilon}$ converge, up to subsequences, to a set $A$ of finite perimeter. In order to characterize $u$ in terms of $A$ we modify the sequence $u^\varepsilon$ so that the limit $u$ is the same and the corresponding $\widehat{v}^\varepsilon$ converge to a set $\widehat{A}$ which is the subgraph of $u$, writing $u(x) = \int_0^1 v(x,y)\, dy$, where $v = -1 + 2\chi_{\widehat{A}}$. Hence, $u \in BV(0,1)$.

Note that $G(\widehat{A}) \le G(A)$, so that it suffices to rewrite $G(\widehat{A})$ in terms of $u$ in order to obtain a lower bound. The crystalline perimeter of $\widehat{A}$ can be written as

$$E(u) = 2\mathcal{H}^1(\{-1 < u(x) < 1\}) + |Du|(0,1).$$

In order to check this, it suffices to consider the case in which $\widehat{A}$ is a polyrectangle, in which case the horizontal segments of the boundary give the first term and the vertical segments correspond to the second one.

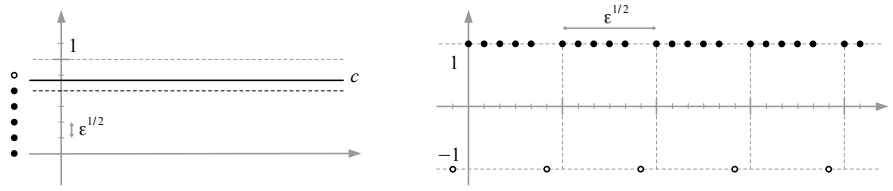


Figure 35: diffuse interface.

As for the construction of a recovery sequence, we only consider a constant function $u(x) = c$ with $c \in (-1,1)$. A general $u$ can be then approximated by piecewise-constant

functions so that the total variations converge. For such approximating $u^\varepsilon$ for any interval $[\sqrt{\varepsilon}\,(k-1), \sqrt{\varepsilon}\,k)$ we have an interface in $\varepsilon\mathbb{Z}$ (see Fig. 35), which can be viewed as a *diffuse interface*.

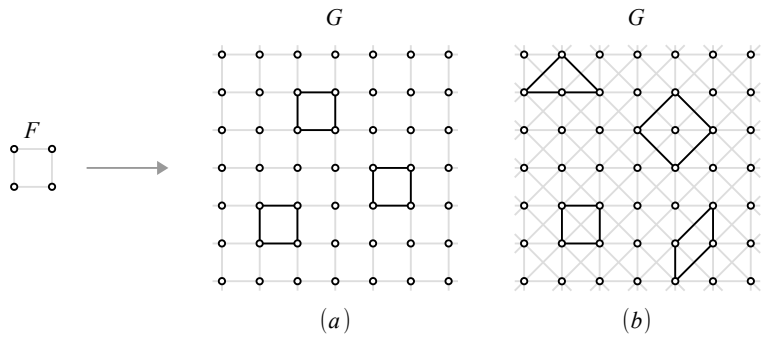


Figure 36: two examples of immersions of a cycle of order 4.

Example 60 highlights that the choice of the "natural" framework where to embed a graph is in a sense connected to the dimension of "cycles" of interactions (or edges) between nodes of the graph. In that case, even though we deal with a one-dimensional set of nodes $\mathbb{Z}$, the graph contains cycles of order 4. This shows that a more "natural" setting for the same graph is $\mathbb{Z}^2$, with nearest-neighbour connections.

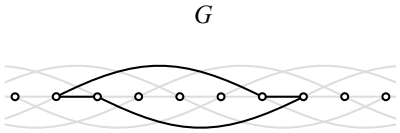


Figure 37: a cycle of order 4 in the graph of Example 60.

In Fig. 36 possible embeddings of a cycle of order 4, denoted by $F$, in the graph given by $\mathbb{Z}^2$ with nearest-neighbour connections and next-to-nearest neighbour connections are pictured in figure (a) and (b), respectively. In Fig. 37 we picture a cycle of order 4 in the one-dimensional graph of Example 60.

The problem of embedding an arbitrary graph (nodes and edges) in a proper $d$-dimensional space with a metric structure is not of a simple solution, and makes sense for graphs with "few" connections. For dense graph sequences the analysis of cycles (or simple graphs) embedded in a given graph leads to a notion of topological convergence, that we briefly describe.

Given $F = (\mathcal{N}(F), \mathcal{E}(F))$ and $G = (\mathcal{N}(G), \mathcal{E}(G))$ graphs, we consider the set $\hom(F, G)$ of the homomorphisms of $F$ on $G$; that is, the set of functions $\Phi\colon \mathcal{N}(F) \to \mathcal{N}(G)$ which are adjacency-preserving. The structure of a graph $G$ can be examined by considering the density of the homomorphisms of simple graphs $F$ on $G$; we introduce the quantity

$$t(F, G) = \frac{\#\hom(F, G)}{(\#\mathcal{N}(G))^{\#\mathcal{N}(F)}}, \tag{56}$$

noting that $(\#\mathcal{N}(G))^{\#\mathcal{N}(F)}$ is the number of all possible functions between the set of nodes of $F$ and the set of nodes of $G$.

**Definition 61** (left convergence for sequence of graphs)**.** *A sequence of graphs $G_k$ is said to be* left convergent *if there exists*

$$\lim_{n\to+\infty} t(F, G_k)$$

*for any $F$ simple graph.*

Note that this definition is meaningful only for dense sequences of graphs, otherwise the limit is always 0.

As dense graph sequences are concerned, the structure of $d$-dimensional Enclidean spaces for the description of the geometry of edges is abandoned, and the problem of the asymptotic behaviour of the associated energies is set in dimension 1. As we will see, this makes the choice of the labelling of graphs essential.

Given a graph $G = (\mathcal{N}, \mathcal{E})$, we define the *energy of* $G$ by setting

$$E^G(u) = \frac{1}{\#\mathcal{N}} \sum_{i,j\in\mathcal{N}} a_{ij}(u_i - u_j)^2 \tag{57}$$

for $u\colon \mathcal{N} \to \{-1, 1\}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in \mathcal{E} \\ 0 & \text{if } (i, j) \in \mathcal{E}. \end{cases}$$

The matrix $A = (a_{ij})$ is called the *adjacency matrix* of the graph.

Let $n = \#\mathcal{N}$; then, we can use the set $\mathcal{N}_n = \{1, \dots, n\}$ as the set of nodes, with a slight abuse of notation. Correspondingly, we will write the graph as $G_n = (\mathcal{N}_n, \mathcal{E}_n)$ and the adjacency matrix as $A^n = (a^n_{ij})$. Now, in view of analyzing the asymptotic behaviour of the energies $E^{G_n}$ as $n \to +\infty$, we introduce a "pixel representation" of the adjacency matrices on a common set of parameters $[0, 1]\times[0, 1]$. To that end, we consider the piecewise-constant *adjacency function* of the matrix $A^n$; that is, the function $a^n\colon [0, 1] \times [0, 1] \to \{0, 1\}$ defined as

$$a^n(x, y) = a^n_{ij} \quad \text{if} \quad (x, y) \in I^n_i \times I^n_j, \tag{58}$$

where $I_1^n = [0, \frac{1}{n}]$ and $I_k^n = (\frac{k-1}{n}, \frac{k}{n}]$ for $k = 2, \ldots, n$.

Note that for a sparse sequence of graphs the adjacency functions $a^n$ defined in (58) always converge to 0 strongly in $L^1$. The following examples show that the representation of the adjacency matrices and the corresponding weak limits are highly dependent on the choice of the labelling.

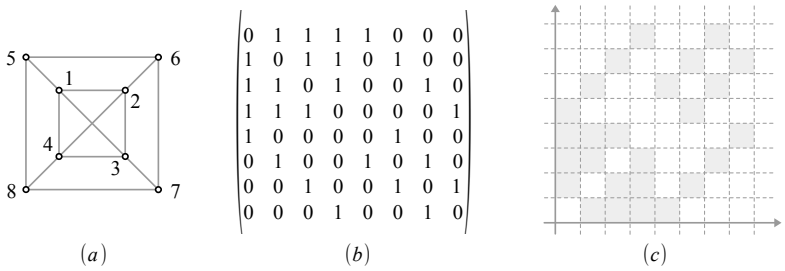


Figure 38: labelling of a graph, adjacency matrix and "pixel representation".

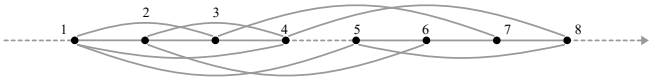


Figure 39: one-dimensional representation of the graph in Fig. 38.

**Example 62** (parametrization of a graph and adjacency matrix)**.** Let $G$ be the graph pictured in Fig. 38(a). The adjacency matrix $A = (a_{ij})$ corresponding to the labelling is represented in Fig. 38(b), and the corresponding adjacency function $a$ given by (58) is pictured in Fig. 38(c). Note the different position of the origin in the representation of $a$ and in the corresponding matrix. The same graph can be represented in dimension 1. In Fig. 39 it is pictured together with the corresponding connections.

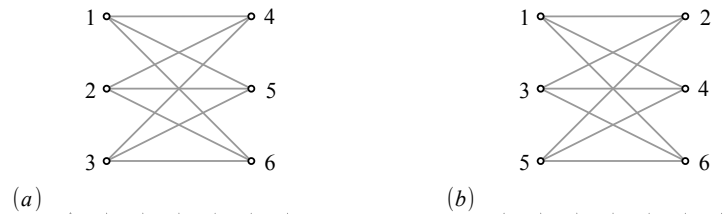


Figure 40: different labellings of the complete bipartite graph.

**Example 63** (different labellings of the complete bipartite graph)**.** We consider the *complete bipartite graph* with $2N$ nodes divided in two families of $N$ nodes each, highlighting the dependence of the adjacency matrix on the choice of the labelling. In Fig. 40 two different parametrizations for $N = 3$ are pictured. The corresponding adjacency matrices are

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \hat{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

for (a) and (b) respectively.

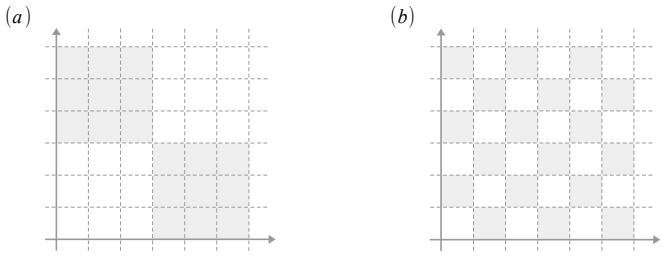


Figure 41: different pixel representations of the complete bipartite graph.

Now, we associate to each parametrization the corresponding "pixel representation" defined in (58) and analyze the asymptotic behaviour as the number of nodes diverges. Let $\widetilde{a}^{2N}$ and $\widehat{a}^{2N}$ denote the piecewise-constant representations of the adjacency matrices $\widetilde{A}^{2N}$ and $\widehat{A}^{2N}$ generalizing $\widetilde{A}$ and $\widehat{A}$ to the graphs with $2N$ nodes, respectively (see Fig. 41). We have that $\widetilde{a}^{2N}$ is actually constant, and equal to

$$\widetilde{a}(x,y) = \begin{cases} 0 & \text{in } \left[0,\frac{1}{2}\right]^2 \cup \left[\frac{1}{2},1\right]^2 \\ 1 & \text{otherwise,} \end{cases}$$

while $\widehat{a}^{2N} \overset{*}{\rightharpoonup} \widehat{a}(x,y) = \frac{1}{2}$ in $L^\infty$.

Contrary to the case studied until now, where the limit is represented as a local functional in $\mathbb{R}^d$ starting from a lattice graph in $\varepsilon\mathbb{Z}^d$ with essentially finite range connections, here we embed arbitrary graphs in one-dimensional lattices in $\frac{1}{n}\mathbb{Z}$, and in order to analyze the asymptotic behaviour of the energies defined on dense sequences of graphs, we give the definition of *graphon*. This concept extends the notion of (parametrized) graph, identified with its adjacency function $a$, to a "continuous" set of nodes given by $[0,1]$, and a corresponding notion of norm.

**Definition 64** (graphons and cut norm)**.** *A* graphon *is a bounded measurable function* $W\colon [0,1]\times[0,1]\to\mathbb{R}$ *that is symmetric. The* cut norm *of a graphon* $W$ *is defined as*

$$\|W\|_{\square} = \sup_{T,S\subset(0,1)} \Big| \int_S \int_T W(x,y)\,dx\,dy \Big| \tag{59}$$

*where the* sup *is taken over all measurable subsets of* $(0,1)$*.*

Note that in the definition of graphon we consider $\mathbb{R}$-valued functions, and not only taking values in $[0,1]$, since we need to use the linear structure of the space.

**Remark 65.** An equivalent definition of the cut norm is given by

$$\|W\|_{\square} = \sup\Big\{ \Big| \int_{(0,1)}\int_{(0,1)} W(x,y)f(x)g(y)\,dx\,dy \Big| : f,g\colon [0,1]\to[0,1] \text{ measurable} \Big\}. \tag{60}$$

**Remark 66** (cut norm and parametrizations of a graph)**.** Let $G$ be a graph. By using the definition of the adjacency function $a(x,y)$ in (58), to each parametrization of $G$ we can associate a graphon $W_G$ by setting

$$W_G(x,y) = a(x,y). \tag{61}$$

Note that if $W_G$ and $W'_G$ are associated to different parametrizations of the same $G$, then in general $\|W_G - W'_G\|_{\square} \neq 0$. If we consider the bipartite graph described in Example 63, and denote by $\widetilde{W}_{2N}$ and $\widehat{W}_{2N}$ the graphons associated to parametrizations (a) and (b) respectively, we get $\|\widetilde{W}_{2N} - \widehat{W}_{2N}\|_{\square} = \frac{1}{2} > 0$.

**Remark 67** (cut norm convergence, $L^1$-strong and $L^1$-weak convergence)**.**

(a) Let $\{W_n\}$ be a sequence of graphons such that $\|W_n\|_{\square}\to 0$ as $n\to+\infty$. Then, if $\{f_n\}$ and $\{g_n\}$ are equibounded sequences of measurable functions, it follows that

$$\lim_{n\to+\infty} \int_{(0,1)}\int_{(0,1)} W_n(x,y)f_n(x)g_n(y)\,dx\,dy = 0. \tag{62}$$

If $f_n$ and $g_n$ are positive, then the result follows by the (equivalent) definition of cut norm in (60) since the sequences are equibounded. Otherwise, we can write $f_ng_n$ as the difference of products of positive functions

$$f_ng_n = \frac{1}{2}(\|f_n\|_\infty + f_n)(\|g_n\|_\infty + g_n) + \frac{1}{2}(\|f_n\|_\infty - f_n)(\|g_n\|_\infty - g_n) - \|f_n\|_\infty\|g_n\|_\infty$$

and conclude.

(b) The convergence in the cut norm is stronger than the weak-$L^1$ convergence. Indeed, if $\|W_n - W\|_\square \to 0$ as $n \to +\infty$, then $W_n \rightharpoonup W$ in $L^1((0,1)\times(0,1))$ by using (60), which is uniform. Note that the weak convergence of $W_n$ is not sufficient to have (62) which will be a key property to analyze the convergence of energies on graphs. On the contrary, the $L^1$-strong convergence implies the convergence in the cut norm, but does not provide sufficient compactness properties.

We now introduce a notion of metric on the set of graphons which is independent of the parametrizations. A map $\phi\colon [0,1] \to [0,1]$ is *measure preserving* if $\phi^{-1}(A)$ is measurable for all measurable $A \subset [0,1]$, and $|\phi^{-1}(A)| = |A|$, where $|\cdot|$ denotes the 1-dimensional Lebesgue measure. Note that measure-preserving maps are a generalizations of relabelling of nodes of a graph, in the sense that a permutation in the set of nodes (and hence in the set of the intervals $I_k^n$) corresponds to a measure-preserving transformation of $[0,1]$.

Let $W$ be a graphon. For each measure-preserving map $\phi$ we define a corresponding graphon $W_\phi$ by setting

$$W_\phi(x,y) = W(\phi(x),\phi(y)). \tag{63}$$

**Definition 68** (cut metric)**.** *Let $W, W'$ be graphons. The* cut distance *between $W$ and $W'$ is defined as*

$$\delta_\square(W,W') = \inf_{\phi,\psi} \|W_\phi - W'_\psi\|_\square,$$

*where the* inf *is taken over all measure-preserving maps of $[0,1]$, and $W_\phi, W'_\psi$ are defined as in* (63)*.*

Note that, by definition, the cut distance is independent of the parametrization.

We introduce the set $\mathcal{W}_0$ of the graphons with image in $[0,1]$; that is,

$$\mathcal{W}_0 = \{W\colon [0,1]^2 \to [0,1] \text{ measurable and symmetric}\}.$$

Note that $\mathcal{W}_0$ is in fact the set of limits of graphs.

The topological properties of the cut metric and the cut norm, and their link with the left convergence, are widely studied. In particular, a fundamental theorem ensures some properties which will be important for the asymptotic analysis of the energies on graphs. In the following statements, $W_n$ denotes the graphon associated to the parameterized graph $G_n$.

(a) (completeness) Let $\{W_n\}$ be a sequence of graphons which is a Cauchy sequence with respect to the metric $\delta_\square$; then, there exists $W \in \mathcal{W}_0$ such that $\delta_\square(W, W_n) \to 0$.

(b) (existence of the parameterization) Let $\{W_n\}$ be a sequence of graphons and $W \in \mathcal{W}_0$ such that $\delta_\square(W, W_n) \to 0$, for all $n$ there exists a measure preserving map $\phi_n$ such that, setting $\widetilde{W}_n = (W_n)_{\phi_n}$,

$$\|\widetilde{W}_n - W\|_\square \to 0.$$

(c) (compactness) The convergence induced by the cut metric is compact.

It would be convenient to have a constructive way to choose the parameterization in (b), whose existence is only proved in abstract terms in the literature.s

**Remark 69** (Left convergence and convergence in the cut metric)**.** The cut norm translates in analytical terms the notion of topological convergence of graphs; that is, the left convergence. The relation with the convergence in the cut metric is given by the following result.

*Let $\{G_n\}$ be a sequence of parametrized graphs with $\#\mathcal{N}(G_n) = n$, and let $\{W_n\}$ be the sequence of corresponding graphons given by* (61)*. Then:*

*(a) the sequence $\{W_n\}$ is a Cauchy sequence with respect to the metric $\delta_\square$ if and only if the sequence $\{G_n\}$ is left convergent;*

*(b) if $W \in \mathcal{W}_0$ is such that $\delta_\square(W, W_n) \to 0$, then for any simple graph $F = (\mathcal{N}(F), \mathcal{E}(F))$*

$$t(F, G_n) \to \int_{[0,1]^k} \prod_{(i,j)\in\mathcal{E}(F)} W(x_i, x_j)\, dx_1 \dots\, dx_k,$$

*where $\mathcal{N}(F) = \{1, \dots, k\}$.*

Now, we go back to the asymptotic analysis of the energies on graphs. Let $G_n = (\mathcal{N}_n, \mathcal{E}_n)$ be a parametrized graph such that $\#\mathcal{N}_n = n$, with the set of nodes parametrized as $\mathcal{N}_n = \{1, \dots, n\}$. For each $u\colon \frac{1}{n}\mathbb{Z} \to \{-1, 1\}$, we define the energy by

$$E_n(u) = \frac{1}{n^2} \sum_{i,j=1}^{n} a_{ij}^n (u_i - u_j)^2,$$

where $u_i = u(\frac{i}{n})$ and $(a_{ij}^n)_{ij}$ is the adjacency matrix of the graph $G_n$. If we extend the spin function $u$ to a piecewise-constant function by setting

$$u(x) = u_i \quad \text{if} \quad x \in I_i^n,$$

the energies can be written as

$$E_n(u) = \int_{(0,1)} \int_{(0,1)} a^n(x, y)(u(x) - u(y))^2\, dx\, dy,$$

where $a^n$ is defined in (58) and $u \in X_n$ defined as

$$X_n = \{u\colon [0, 1] \to \{-1, 1\} : u \text{ constant in each } I_i^n\}.$$

We now consider the $\Gamma$-limit of the sequence $E_n$, the convergence of the corresponding graphons in the cut norm describing the asymptotic behaviour of the underlying environment.

**Theorem 70** (Γ-convergence and volume constraint)**.** *Let $\{G_n\}$ be a sequence of parametrized graphs such that the corresponding graphons $W_n$ converge to a graphon $W$ with respect to the cut norm. Then, the sequence $\{E_n\}$ Γ-converges, with respect to the $L^\infty$-weak$^*$ convergence, to the functional $F$ defined by*

$$F(u) = \begin{cases} 2\displaystyle\int_{(0,1)}\int_{(0,1)} W(x,y)(1-u(x)u(y))\,dx\,dy & \text{if } \|u\|_\infty \le 1 \\ +\infty & \text{otherwise in } L^\infty(0,1). \end{cases}$$

*Moreover, if $\lambda_n \to \lambda$ with $\lambda_n \in [-1,1] \cap \frac{2}{n}\mathbb{Z}$, and*

$$E_n^{\lambda_n}(u) = \begin{cases} E_n(u) & \text{if } \displaystyle\int_{(0,1)} u\,dx = \lambda_n \\ +\infty & \text{otherwise,} \end{cases}$$

*then the sequence $\{E_n^{\lambda_n}\}$ Γ-converges to the functional $F^\lambda$ given by*

$$F^\lambda(u) = \begin{cases} F(u) & \text{if } \displaystyle\int_{(0,1)} u\,dx = \lambda \\ +\infty & \text{otherwise.} \end{cases}$$

*Proof. Lower bound.* Let $u_n \overset{*}{\rightharpoonup} u$. Since $\|W_n - W\|_\square \to 0$, we apply Remark 67(a) obtaining

$$\lim_{n\to+\infty}\int_{(0,1)}\int_{(0,1)} (W_n(x,y) - W(x,y))u_n(x)u_n(y)\,dx\,dy = 0. \tag{64}$$

Since $u_n(x) \in \{-1,1\}$, it follows that $(u_n(x)-u_n(y))^2 = 2 - 2u_n(x)u_n(y)$, so that

$$\begin{aligned} \liminf_{n\to+\infty} E_n(u_n) &= 2\liminf_{n\to+\infty}\int_{(0,1)}\int_{(0,1)} W_n(x,y)(1-u_n(x)u_n(y))\,dx\,dy \\ &= 2\int_{(0,1)}\int_{(0,1)} W(x,y)(1-u_n(x)u_n(y))\,dx\,dy, \end{aligned}$$

where we used (64) and the convergence

$$\int_{(0,1)}\int_{(0,1)} W_n(x,y)\,dx\,dy \to \int_{(0,1)}\int_{(0,1)} W(x,y)\,dx\,dy,$$

obtained by using as test functions $f(x) = g(x) \equiv 1$. Since

$$v_n(y) = \int_{(0,1)} W(x,y)u_n(x)\,dx \to v(y) = \int_{(0,1)} W(x,y)u(x)\,dx$$

strongly in $L^1(0,1)$ and $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(0,1)$, it follows that

$$\begin{aligned}\liminf_{n\to+\infty} \int_{(0,1)}\int_{(0,1)} W(x,y)u_n(x)u_n(y)\,dx\,dy &= \liminf_{n\to+\infty} \int_{(0,1)} v_n(y)u_n(y)\,dy \\ &= \int_{(0,1)}\int_{(0,1)} W(x,y)u(x)u(y)\,dx\,dy,\end{aligned}$$

concluding the proof of the lower estimate.

*Upper bound.* Let $u\colon (0,1)\to[-1,1]$ and $n_h = h^2$ for any $h\in\mathbb{N}$. We consider a partition of $(0,1)$ in intervals with length $\frac{1}{h}$, so that the scale is larger than $\frac{1}{n_h} = \frac{1}{h^2}$. We set $J_k^h = (\frac{k-1}{h}, \frac{k}{h})$ and define

$$\tilde{u}_h = c_k^h = h\int_{J_k^h} u(x)\,dx \quad \text{in} \quad J_k^h.$$

We define the recovery sequence $u_{n_h}$ by setting, in each interval $J_k^h$,

$$u_{n_h}(x) = \begin{cases} 1 & \text{if } x \in \Big(\dfrac{k-1}{h}, \dfrac{k-1}{h} + \dfrac{t_k^h}{h^2}\Big) \\ -1 & \text{otherwise in } J_k^h, \end{cases}$$

where $t_k^h = \frac{h}{2}(c_k^h + 1)$. This is a good definition since $u(x)\in[-1,1]$ implies $1 \le t_k^h \le h$, and, up to a small error, we can assume that $t_k^h$ is an integer.

The sequence $(u_{n_h})$ weak* converges to $u$ in $L^\infty(0,1)$. By using (64), we have

$$\lim_{h\to+\infty} E_{n_h}(u_{n_h}) = 2\lim_{h\to+\infty}\int_{(0,1)}\int_{(0,1)} W(x,y)(1-u_{n_h}(x)u_{n_h}(y))\,dx\,dy,$$

and again by an argument of strong-weak convergence as in the proof of the lower bound we obtain

$$\lim_{h\to+\infty} E_{n_h}(u_{n_h}) = 2\int_{(0,1)}\int_{(0,1)} W(x,y)(1-u(x)u(y))\,dx\,dy.$$

Finally, we note that a slight variation of the construction of the recovery sequence is compatible with the volume constraint. □

**Example 71** (minimal-cut problem for the half-graph)**.** Let $G_{2k}$ be the half-graph with $2k$ nodes, with the "natural" labelling of the nodes described in Fig. 42(a). The corresponding sequence of graphons $W_{2k}$ is pictured in Fig. 42(b). Since $W_{2k}$ strongly converges in $L^1((0,1)\times(0,1))$ to the function

$$W(x,y) = \begin{cases} 0 & \text{if } |x-y| \le \dfrac{1}{2} \\ 1 & \text{otherwise in } (0,1)^2, \end{cases}$$

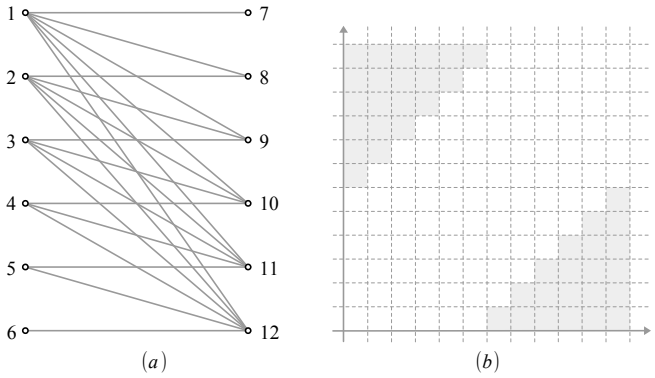


Figure 42: the half-graph with 12 nodes and its pixel representation.

it follows that $\|W_{2k} - W\|_{\square} \to 0$ as $k \to +\infty$. Fixing $\lambda_{2k} = 0$ for all $k$, we can apply Theorem 70 and obtain that the sequence of the energies converges to the functional

$$F^0(u) = 2\int_0^{\frac{1}{2}} \int_{\frac{1}{2}+x}^1 (1 - u(x)u(y))\,dx\,dy + 2\int_{\frac{1}{2}}^1 \int_0^{x-\frac{1}{2}} (1 - u(x)u(y))\,dx\,dy. \tag{65}$$

We consider the minimal-cut problem for the limit functional $F$; that is,

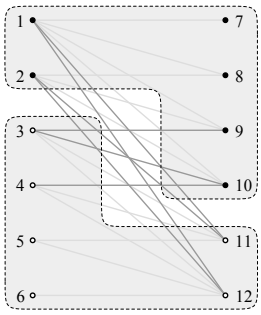


Figure 43: minimal cut of the half-graph.

$$\min\Big\{F^0(u) : u\colon (0,1) \to [-1,1] \ \text{ such that } \int_0^1 u(x)\,dx = 0\Big\}.$$

It can be proven that the minimum is attained at a function $u$ such that $u(x) \in \{-1, 1\}$; then, it follows that a minimizer is given by

$$u(x) = \begin{cases} 1 & \text{in } \left[0, \dfrac{1}{6}\right] \cup \left[\dfrac{1}{2}, \dfrac{5}{6}\right] \\ -1 & \text{otherwise.} \end{cases}$$

In Fig. 43 we give a pictorial representation of the corresponding discrete solution of the minimal-cut problem in the case $k = 6$.

# 11 A minimal bibliography

A review of interfacial problems on networks can be found in [10]. We also refer to the forthcoming book [2] and to the lecture notes [20]. We only provide some essential bibliography for the content of these notes, to which we refer for further information.

**Section 2.** An account of the properties of sets of finite perimeter can be found in [7] and [6].

**Section 3.** Classical references for $\Gamma$-convergence are [8] and [23]. Functionals on sets of finite perimeter have been studied in [5]. The study of spin systems is a classical subject in Statistical Mechanics, even though the types of problems somewhat differ from the ones studied here; a first treatment of spin systems by variational methods in the spirit of these notes is in [21], followed by [1].

**Section 4.** The results on spin systems actually can be seen as a particular case of earlier results on systems with bulk and surface parts [12]. Here we simplify the proof in the case without bulk part. More examples in the planar case can be found in [20].

**Section 5.** The homogenization of ferromagnetic energies has been performed in [16]. Here we give a different proof based on the Fonseca-Müller blow-up method [24] adapted to homogenization [14].

**Section 6.** A compactness and integral-representation result for ferromagnetic energies is found in [3]. Antiferromagnetic energies on a square lattice are dealt with in [1]. Coarse-graining has been analyzed in [19].

**Section 7.** The design of networks is the subject of [13]. The Dal Maso-Kohn principle refers to an unpublished result showing the analog for mixtures of quadratic energies (see also [9]). The homogenization method for shape optimization is the subject of the book [4] (see also [31]).

**Section 8.** The homogenization of networks with random positive coefficients is performed in [16], for hard inclusions in [30] (see also [15]), for dilute systems in [17]. A classical reference for Percolation Theory is [26]. In random homogenization problems an important tool are results of sudadditive processes [27]. The analysis of dilute system is an active research topic [22].

**Section 9.** The homogenization of Poisson random sets in the plane is contained in [18]. The polyomino lemma is due to Pimentel [29].

**Section 10.** Minimal-cut problems for dense-graph sequences are studied in [11]. For the theory of graphons we refer to [25] and [28].

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